Extreme value theory, Poisson-Dirichlet distributions and FPP on random networks

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Abstract

We study first passage percolation on the configuration model (CM) having power-law degrees with exponent $\tau \in [1, 2)$. To this end we equip the edges with exponential weights. We derive the distributional limit of the minimal weight of a path between typical vertices in the network and the number of edges on the minimal weight path, which can be computed in terms of the Poisson-Dirichlet distribution. We explicitly describe these limits via the construction of an infinite limiting object describing the FPP problem in the densely connected core of the network. We consider two separate cases, namely, the *original CM*, in which each edge, regardless of its multiplicity, receives an independent exponential weight, as well as the *erased CM*, for which there is an independent exponential weight between any pair of direct neighbors. While the results are qualitatively similar, surprisingly the limiting random variables are quite different.

Our results imply that the flow carrying properties of the network are markedly different from either the mean-field setting or the locally tree-like setting, which occurs as $\tau > 2$, and for which the hopcount between typical vertices scales as $\log n$. In our setting the hopcount is tight and has an explicit limiting distribution, showing that one can transfer information remarkably quickly between different vertices in the network. This efficiency has a down side in that such networks are remarkably fragile to directed attacks. These results continue a general program by the authors to obtain a complete picture of how random disorder changes the inherent geometry of various random network models, see [2, 4, 5].

1 Introduction

First passage percolation (FPP) was introduced by Hammersley and Welsh [12] to model the flow of fluid through random media. This model has evolved into one of the fundamental problems studied in modern probability theory, not just for its own sake but also due to the fact that it plays a crucial role in the analysis of many other problems in statistical physics, in areas such as the contact process, the voter model, electrical resistance problems and in fundamental stochastic models from evolutionary biology,

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see e.g. [9]. The basic model for FPP on (random) graph is defined as follows: We have some connected graph on n vertices. Each edge is given some random weight, assumed to be non-negative, independent and identically distributed (i.i.d.) across the edges. The weight on an edge has the interpretation of the *length* or *cost* of traversing this edge. Fixing two vertices in the network, we are then interested in the length and weight of the minimal weight path between these two vertices and the asymptotics of these statistics as the size of the network tends to infinity.

Most of the classical theorems about FPP deal with the d-dimensional integer lattice, where the connected network is the $[-r, r]^d$ box in the integer lattice and one is interested in asymptotics of various quantities as $n = (2r + 1)^d \to \infty$. In this context, probabilists are often interested in proving shape theorems, namely, for fixed distance t, showing that $\mathcal{C}_t/t$ converges to a deterministic limiting set as $t \to \infty$, where $\mathcal{C}_t$ is the cluster of all vertices within distance t from the origin. See e.g., [17] for a survey of results in this context.

In the modern context such problems have taken on a new significance. The last few years have witnessed an explosion in the amount of empirical data on networks, including data transmission networks such as the Internet, biochemical networks such as gene regulatory networks, spatial flow routing networks such as power transmission networks and transportation networks such as road and rail networks. This has stimulated an intense cross-disciplinary effort in formulating network models to understand the structure and evolution of such real-world networks. Understanding FPP in the context of these random models seems to be of paramount importance, with the minimal weight between typical vertices representing the cost of transporting flow between these vertices, while the hopcount, which is defined to be the number of edges on the minimal weight path between two typical vertices, representing the amount of time it takes for flow to be transported between these vertices.

In this study we shall analyze FPP problems on the Configuration Model (CM), a model of constructing random networks with arbitrary degree distributions. We shall defer a formal definition of this model to Section 2 and shall discuss related work in Section 4. Let it suffice to say that this model has arisen in myriad applied contexts, ranging from combinatorics, computer science, statistical physics, and epidemiology and seems to be one of the most widely used models in the modern networking community.

We shall consider FPP on the CM where the exponent τ of the degree distribution satisfies $\tau \in [1, 2)$ and each edge is given a random exponential edge weight. FPP for the case $\tau > 2$ was analyzed in [5] where the hopcount seems to exhibit a remarkably universal behavior. More precisely, the hopcount always scales as $\log n$ and central limit theorems (CLTs) with matching asymptotic means and variances hold. While these graphs are sparse and locally tree-like, what is remarkable is that the same fact also holds in the case of the most well-connected graph, namely the complete graph, for which the hopcount satisfies a CLT as the model with $\tau > 2$, with asymptotic mean and variance equal to $\log n$, as $n \to \infty$. See, e.g., [18] and [4] and the references therein.

When the degree exponent τ is in the interval $[1, 2)$ we shall find that CLTs do *not* hold, and that the hopcount remains uniformly bounded due to the remarkable shape of such networks, which we may think of as a collection of interconnected star networks, the centers of the stars corresponding to the vertices with highest degrees. We shall consider two models of the network topology, one where we look at the original CM and the second more realistic model called the *erased* model where we shall delete all self loops and merge all multiple edges from the original CM. In the resulting graph, each edge receives an independent exponential weight with rate 1. Thus, for the erased CM, the direct weight between two vertices connected by an edge is an exponential random variable with rate 1, while for the original CM, it is also exponential, but with rate equal to the number of edges between the pair of vertices. When $\tau > 2$, there is no essential difference between the original and the CM [5].

In both cases, we shall see that the hopcount is tight and that a limit distribution exists. More surprisingly, in the erased CM, this limiting distribution puts mass only on the *even* integers. We also exhibit a nice constructive picture of how this arises, which uses the powerful machinery of Poisson-Dirichlet distributions. We further find the distributional limit of the weight of the minimal weight path joining two typical vertices.

Since the hopcount remains tight, this model is remarkably efficient in transporting or routing flow between vertices in the network. However, a downside of this property of the network is its extreme fragility w.r.t. directed attacks on the network. More precisely, we shall show that there exists a simple algorithm deleting a bounded number of vertices such that the chance of disconnecting any two typical vertices is close to 1 as $n \to \infty$. At the same time we shall also show that these networks are relatively stable against random attacks.

This paper is organized as follows. In Section 2, we shall introduce the model and some notation. In Section 3, we state our main results. In Section 4, we describe connections to the literature and discuss our results. In Section 5, we give the proof in the original CM, and in Section 6, we prove the results in the erased CM.

2 Notation and definitions

In this section, we introduce the random graph model that we shall be working on, and recall some limiting results on i.i.d. random variables with infinite mean. We shall use the notation that $f(n) = O(g(n))$, as $n \to \infty$, if $|f(n)| \leq Cg(n)$, and $f(n) = o(g(n))$, as $n \to \infty$, if $|f(n)|/g(n) \to 0$. For two sequences of random variables X_n and Y_n, we write that $X_n = O_{\mathbb{P}}(Y_n)$, as $n \to \infty$, when $\{X_n/Y_n\}_{n \geq 1}$ is a tight sequence of random variables. We further write that $X_n = \Theta_{\mathbb{P}}(Y_n)$ if $X_n = O_{\mathbb{P}}(Y_n)$ and $Y_n = O_{\mathbb{P}}(X_n)$. Further, we write that $X_n = o_{\mathbb{P}}(Y_n)$, when $|X_n|/Y_n$ goes to 0 in probability ($|X_n|/Y_n \overset{\mathbb{P}}{\longrightarrow} 0$); equality in distribution is denoted by the symbol $\sim$. Throughout this paper, for a sequence of events $\{F_n\}_{n \geq 1}$, we say say that F_n *occurs with high probability* (**whp**) if $\lim_{n \to \infty} \mathbb{P}(F_n) = 1$.

Graphs: We shall typically be working with random graphs on n vertices, which have a giant component consisting of $n - o(n)$ vertices. Edges are given a random edge weight (sometimes alternatively referred to as cost) which in this study will always be assumed to be independent, exponentially distributed random variables with mean 1. We pick two vertices uniformly at random in the network. We let W_n be the random variable denoting the total weight of the minimum weight path between the two typical vertices and H_n be the number of edges on this path or *hopcount*.

Construction of the configuration model: We are interested in constructing a random graph on n vertices. Given a **degree sequence**, namely a sequence of n positive integers $\mathbf{D} = (D_1, D_2, \ldots, D_n)$ with the total degree

$$L_n = \sum_{i=1}^{n} D_i \tag{2.1}$$

assumed to be even, the CM on n vertices with degree sequence $\mathbf{D}$ is constructed as follows:

Start with n vertices and D_j stubs adjacent to vertex j. The graph is constructed by pairing up each stub to some other stub to form edges. Number the stubs from 1 to L_n in some arbitrary order. Then, at each step, two stubs (not already paired) are chosen uniformly at random among all the *free* stubs and are paired to form a single edge in the graph. These stubs are no longer free and removed from the list of free stubs. We continue with this procedure of choosing and pairing two stubs until all the stubs are paired.

Degree distribution: The above denoted the construction of the CM when the degree distribution is given and the total degree is even. Here we specify how we construct the actual degree sequence $\mathbf{D}$. We shall assume that each of the random variables $D_1, D_2, \ldots D_n$ are independent and identically distributed (i.i.d.) with distribution F. (Note that if the sum of stubs L_n is not even then we use the degree sequence with D_n replaced with $D_n + 1$. This will not effect our calculations).

We shall assume that the degree distribution F, with atoms $f_1, f_2, \ldots$ satisfies the property:

$$1 - F(x) = x^{-(\tau-1)}L(x), \tag{2.2}$$

for some slowly varying function $x \mapsto L(x)$. Here, the parameter τ, which we shall refer to as the *degree exponent*, is assumed to be in the interval $[1, 2)$, so that $\mathbb{E}[D_i] = \infty$. In some cases, we shall make stronger assumptions than (2.2).

Original model: We assign to each edge a random and i.i.d. exponential mean one edge weight. Throughout the sequel, the weighted random graph so generated will be referred to as the *original* model and we shall denote the random network so obtained as $\mathcal{G}_n^{\texttt{or}}$.

Erased model: This model is constructed as follows: Generate a CM as before and then erase all self loops and merge all multiple edges into a single edge. *After* this, we put independent exponential weights with rate 1 on the (remaining) edges. Thus, while the graph distances are not affected by the erasure, we shall see that the hopcount has a different limiting distribution. We shall denote the random network on n vertices so obtained by $\mathcal{G}_n^{\texttt{er}}$.

2.1 Poisson-Dirichlet distribution

Before describing our results, we shall need to make a brief detour into extreme value theory for heavy-tailed random variables. As in [10], where the graph distances in the CM with $\tau \in [1, 2)$ are studied, the relative sizes of the order statistics of the degrees play a crucial role in the proof. In order to describe the limiting behavior of the order statistics, we need some definitions.

We define a (random) probability distribution $P = \{P_i\}_{i \geq 1}$ as follows. Let $\{E_i\}_{i=1}^{\infty}$ be i.i.d. exponential random variables with rate 1, and define $\Gamma_i = \sum_{j=1}^{i} E_j$. Let $\{D_i\}_{i=1}^{\infty}$ be an i.i.d. sequence of random variables with distribution function F in (2.2), and let $D_{(n:n)} \geq D_{(n-1:n)} \geq \cdots \geq D_{(1:n)}$ be the order statistics of $\{D_i\}_{i=1}^{n}$. In the sequel of this paper, we shall label vertices according to their degree, so that vertex 1 has maximal degree, etc.

We recall [10, Lemma 2.1], that there exists a sequence u_n, with $u_n = n^{1/(\tau-1)}l(n)$, where l is slowly varying, such that

$$u_n^{-1}\left(L_n, \{D_{(n+1-i:n)}\}_{i=1}^{\infty}\right) \xrightarrow{d} \left(\sum_{j=1}^{\infty}\Gamma_j^{-1/(\tau-1)}, \{\Gamma_i^{-1/(\tau-1)}\}_{i=1}^{\infty}\right), \tag{2.3}$$

where $\xrightarrow{d}$ denotes convergence in distribution. We abbreviate $\xi_i = \Gamma_i^{-1/(\tau-1)}$ and $\eta = \sum_{j=1}^{\infty}\xi_j$ and let

$$P_i = \xi_i/\eta, \qquad i \geq 1, \tag{2.4}$$

so that, $P = \{P_i\}_{i \geq 1}$ is a *random* probability distribution. The sequence $\{P_i\}_{i \geq 1}$ is called the *Poisson-Dirichlet distribution* (see e.g., [26]). A lot is known about the probability distribution P. For example, [26, Eqn. (6)] proves that for any $f \colon [0, 1] \to \mathbb{R}$, and with $\alpha = \tau - 1 \in (0, 1)$,

$$\mathbb{E}\Big[\sum_{i=1}^{\infty} f(P_i)\Big] = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 f(u)u^{-\alpha-1}(1-u)^{\alpha-1}du. \tag{2.5}$$

For example, this implies that

$$\mathbb{E}\Big[\sum_{i=1}^{\infty} P_i^2\Big] = \frac{\Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} = 1 - \alpha = 2 - \tau. \tag{2.6}$$

3 Results

In this section, we state the main results of the paper, separating between the original CM and the erased CM.

3.1 Analysis of shortest-weight paths for the original CM

Before describing the results we shall need to construct a limiting infinite object $\mathcal{K}_\infty^{\text{or}}$ in terms of the Poisson-Dirichlet distribution $\{P_i\}_{i \geq 1}$ given in (2.4) and the sequence of random variables ξ_i and their sum η which arise in the representation of this distribution. This will be an infinite graph with weighted edges on the vertex set $\mathbb{Z}^+ = \{1, 2, \ldots\}$, where every pair of vertices (i, j) is connected by an edge which, conditionally on $\{\xi_i\}_{i \geq 1}$, are independent exponential random variables with exponential distribution with rate $\xi_i \xi_j / \eta$.

Let W_{ij}^{or} and H_{ij}^{or} denote the weight and number of edges of the minimal-weight path in $\mathcal{K}_\infty^{\text{or}}$ between the vertices $i, j \in \mathbb{Z}^+$. Our results will show that, in fact, the FPP problem on $\mathcal{K}_\infty^{\text{or}}$ is well defined (see Proposition 5.1. Let I^{or} and J^{or} be two vertices chosen independently at random from the vertex set $\mathbb{Z}^+$ with probability $\{P_i\}_{i \geq 1}$. Finally, recall that $\mathcal{G}_n^{\text{or}}$ is the random network on n vertices with exponential edge weights constructed in Section 2. We are now in a position to describe our limiting results for the original CM:

Theorem 3.1 (Asymptotics FPP for the original CM) *Consider the random network $\mathcal{G}_n^{\text{or}}$, with the degree distribution F satisfying (2.2) for some $\tau \in [1, 2)$.*
(a) Let W_n^{or} be the weight of the minimal weight path between two uniformly chosen vertices in the network. Then,

$$W_n^{\text{or}} \xrightarrow{d} V_1^{\text{or}} + V_2^{\text{or}}, \tag{3.1}$$

where V_i^{or}, $i = 1, 2$, are independent random variables with $V_i^{\text{or}} \sim E_i / D_i$, where E_i is exponential with rate 1 and D_1, D_2 are independent and identically distributed with distribution F, independently of E_1, E_2. More precisely, as $n \to \infty$,

$$u_n \big(W_n^{\text{or}} - (V_1^{\text{or}} + V_2^{\text{or}}) \big) \xrightarrow{d} W_{I^{\text{or}} J^{\text{or}}}^{\text{or}}, \tag{3.2}$$

where u_n is defined by

$$u_n = \sup\{u : 1 - F(u) \geq 1/n\}. \tag{3.3}$$

(b) Let H_n^{or} be the number of edges in the minimal weight path between two uniformly chosen vertices in the network. Then,

$$H_n^{\text{or}} \xrightarrow{d} 2 + H_{I^{\text{or}} J^{\text{or}}}^{\text{or}}. \tag{3.4}$$

Writing $\pi_k = \mathbb{P}(H_{I^{\text{or}} J^{\text{or}}}^{\text{or}} = k - 2)$, we have $\pi_k > 0$ for each $k \geq 2$, when $\tau \in (1, 2)$. The probability distribution π depends only on τ, and not on any other detail of the degree distribution F. Moreover,

$$\pi_2 = 2 - \tau. \tag{3.5}$$

Theorem 3.1 implies that, for $\tau \in [1, 2)$, the hopcount is uniformly bounded, as is the case for the typical graph distance obtained by taking the weights to be equal to 1 a.s. (see [10]). However, while for unit edge weights and $\tau \in (1, 2)$, the limiting hopcount is at most 3, for i.i.d. exponential weights the limiting hopcount can take all integer values greater than or equal to 2.

3.2 Analysis of shortest-weight paths for the erased CM

The results in the erased CM hold under a more restricted condition on the degree distribution F. More precisely, we assume that there exists a constant $0 < c < \infty$, such that

$$1 - F(x) = cx^{-(\tau - 1)}(1 + o(1)), \qquad x \to \infty, \tag{3.6}$$

and we shall often make use of the upper bound $1 - F(x) \le c_2 x^{-(\tau-1)}$, valid for all $x \ge 0$ and some constant $c_2 > 0$.

Before we can describe our limit result for the erased CM, we shall need an explicit construction of a limiting infinite network $\mathcal{K}_\infty^{\text{er}}$ using the Poisson-Dirichlet distribution described in (2.4). Fix a realization $\{P_i\}_{i \ge 1}$. Conditional on this sequence, let $f(P_i, P_j)$ be the probability

$$f(P_i, P_j) = \mathbb{P}(\mathcal{E}_{ij}), \tag{3.7}$$

of the following event $\mathcal{E}_{ij}$:

> Generate a random variable $D \sim F$ where F is the degree distribution. Conduct D independent multinomial trials where we select cell i with probability P_i at each stage. Then $\mathcal{E}_{ij}$ is the event that both cells i and j are selected.

More precisely, for $0 \le s, t \le 1$,

$$f(s, t) = 1 - \mathbb{E}[(1-s)^D] - \mathbb{E}[(1-t)^D] + \mathbb{E}[(1-s-t)^D]. \tag{3.8}$$

Now consider the following construction $\mathcal{K}_\infty^{\text{er}}$ of a random network on the vertex set $\mathbb{Z}^+$, where every vertex is connected to every other vertex by a single edge. Further, each edge (i, j) has a random weight l_{ij} where, given $\{P_i\}_{i \ge 1}$, the collection $\{l_{ij}\}_{1 \le i < j < \infty}$ are conditionally independent with distribution:

$$\mathbb{P}\left(l_{ij} > x\right) = \exp\left(-f(P_i, P_j)x^2/2\right). \tag{3.9}$$

Let W_{ij}^{er} and H_{ij}^{er} denote the weight and number of edges of the minimal-weight path in $\mathcal{K}_\infty^{\text{er}}$ between the vertices $i, j \in \mathbb{Z}^+$. Our analysis shall, in particular, show that the FPP on $\mathcal{K}_\infty^{\text{er}}$ is well defined (see Proposition 6.4).

Finally, construct the random variables D^{er} and Ier as follows: Let $D \sim F$ and consider a multinomial experiment with D independent trials where at each trial, we choose cell i with probability P_i. Let D^{er} be the number of *distinct* cells so chosen and suppose the cells chosen are $\mathcal{A} = \{a_1, a_2, \ldots, a_{D^{\text{er}}}\}$. Then let Ier be a cell chosen uniformly at random amongst $\mathcal{A}$. Now we are in a position to describe the limiting distribution of the hopcount in the erased CM:

Theorem 3.2 (Asymptotics FPP for the erased CM) *Consider the random network $\mathcal{G}_n^{\text{er}}$, with the degree distribution F satisfying (3.6) for some $\tau \in (1, 2)$.*
(a) Let W_n^{er} be the weight of the minimal weight path between two uniformly chosen vertices in the network. Then,

$$W_n^{\text{er}} \xrightarrow{d} V_1^{\text{er}} + V_2^{\text{er}}. \tag{3.10}$$

where V_i^{er}, $i = 1, 2$, are independent random variables with $V_i^{\text{er}} \sim E_i/D_i^{\text{er}}$, where E_i is exponential with rate 1 and $D_1^{\text{er}}, D_2^{\text{er}}$ are, conditionally on $\{P_i\}_{i \ge 1}$, independent random variables distributed as D^{er}, independently of E_1, E_2. More precisely, as $n \to \infty$,

$$\sqrt{n}\left(W_n^{\text{er}} - (V_1^{\text{er}} + V_2^{\text{er}})\right) \xrightarrow{d} W_{I^{\text{er}} J^{\text{er}}}^{\text{er}}. \tag{3.11}$$

(b) Let H_n^{er} be the number of edges in the minimal weight path between two uniformly chosen vertices in the network. Then,

$$H_n^{\text{er}} \xrightarrow{d} 2 + 2H_{I^{\text{er}} J^{\text{er}}}^{\text{er}}, \tag{3.12}$$

where $I^{\text{er}}, J^{\text{er}}$ are two copies of the random variable Ier described above, which are conditionally independent given $P = \{P_i\}_{i \ge 1}$. In particular, the limiting probability measure of the hopcount is supported only on the even integers.

We shall now present an intuitive explanation of the results claimed in Theorem 3.2, starting with (3.10). We let A_1 and A_2 denote two uniformly chosen vertices, note that they can be identical with probability $1/n$. We further note that both vertex A_1 and A_2 have a random degree which are close to independent copies of D. We shall informally refer to the vertices with degrees $\Theta_{\mathbb{P}}(n^{1/(\tau-1)})$ as *super vertices* (see (6.1) for a precise definition, and recall (2.3)). We shall frequently make use of the fact that normal vertices are, **whp**, exclusively attached to super vertices. The number of super vertices to which A_i, $i = 1, 2$, is attached to is equal to D_i^{er}, $i = 1, 2$, as described above. The minimal weight edge between A_i, $i = 1, 2$, and any of its neighbors is hence equal in distribution to the minimum of a total of D_i^{er} independent exponentially distributed random variables with mean 1. The shortest-weight path between two super vertices can pass through intermediate normal vertices, of which there are $\Theta_{\mathbb{P}}(n)$. This induces that the minimal weight between any pair of super vertices is of order $o_{\mathbb{P}}(1)$, so that the main contribution to W_n^{er} in (3.10) is from the two minimal edges coming out of the vertices $A_i, i = 1, 2$. This shows (3.10) on an intuitive level.

We proceed with the intuitive explanation of (3.11). We use that, **whp**, the vertices A_i, $i = 1, 2$, are only attached to super vertices. Thus, in (3.11), we investigate the shortest-weight paths between super vertices. Observe that we deal with the erased CM, so between any pair of vertices there exists only *one* edge having an exponentially distributed weight with mean 1. As before, we number the super vertices by $i = 1, 2, \ldots$ starting from the largest degree. We denote by N_{ij}^{er}, the number of common neighbors of the super vertices i and j, for which we shall show that N_{ij}^{er} is $\Theta_{\mathbb{P}}(n)$.

Each element in N_{ij}^{er} corresponds to a unique two-edge path between the super vertices i and j. Therefore, the weight of the minimal two-edge path between the super vertices i and j has distribution $w_{ij}^{(n)} \equiv \min_{s \in N_{ij}^{\mathrm{er}}}(E_{is} + E_{sj})$. Note that $\{E_{is} + E_{sj}\}_{s \in N_{ij}^{\mathrm{er}}}$ is a collection of N_{ij}^{er} i.i.d. Gamma(2,1) random variables. More precisely, N_{ij}^{er} behaves as $nf(P_i^{(n)}, P_j^{(n)})$, where $P_i^{(n)} = D_{(n+1-i:n)}/L_n$. Indeed, when we consider an *arbitrary* vertex with degree $D \sim F$, the conditional probability, conditionally on $\{P_i^{(n)}\}_{i \geq 1}$, that this vertex is both connected to super vertex i and super vertex j equals

$$1 - (1 - P_i^{(n)})^D - (1 - P_j^{(n)})^D + (1 - P_i^{(n)} - P_j^{(n)})^D.$$

Thus, the expected number of vertices connected to both super vertices i and j is, conditionally on $\{P_i^{(n)}\}_{i \geq 1}$, $N_{ij}^{\mathrm{er}} \approx nf(P_i^{(n)}, P_j^{(n)})$, and $f(P_i^{(n)}, P_j^{(n)})$ weakly converges to $f(P_i, P_j)$.

We conclude that the minimal two-edge path between super vertex i and super vertex j is the minimum of $nf(P_i^{(n)}, P_j^{(n)})$ Gamma(2,1) random variables Y_s, which are close to being independent. Since

$$\lim_{n \to \infty} \mathbb{P}(\sqrt{n} \min_{1 \leq s \leq \beta n} Y_s > x) = \mathrm{e}^{-\beta x^2/2}, \tag{3.13}$$

for any $\beta > 0$, (3.13), with $\beta = \beta_{ij} = f(P_i^{(n)}, P_j^{(n)}) \approx f(P_i, P_j)$ explains the weights l_{ij} defined in (3.9), and also explains intuitively why (3.11) holds.

The convergence in (3.12) is explained in a similar way. Observe that in (3.12) the first 2 on the right side originates from the 2 edges that connect A_1 and A_2 to the minimal-weight super vertex. Further, the factor 2 in front of H_n^{er} is due to the fact that shortest-weight paths between super vertices are concatenations of two-edge paths with random weights l_{ij}. We shall further show that two-edge paths, consisting of an alternate sequence of super and normal vertices, are the optimal paths in the sense of minimal weight paths between super vertices.

This completes the intuitive explanation of Theorem 3.2.

3.3   Robustness and fragility

The above results show that the hopcount H_n in both models converges in distribution as $n \to \infty$. Interpreting the hopcount as the amount of travel time it takes for messages to get from one typical vertex to another typical vertex, the above shows that the CM with $\tau \in (1, 2)$ is remarkably efficient in

routing flow between vertices. We shall now show that there exists a down side to this efficiency. The theorem is stated for the more natural erased CM but one could formulate a corresponding theorem for the original CM as well.

Theorem 3.3 (Robustness and fragility) *Consider the random weighted network $\mathcal{G}_n^{\mathrm{er}}$, where the degree distribution satisfies (3.6) for some $\tau \in (1, 2)$. Then, the following properties hold:*
(a) **Robustness:** *Suppose an adversary attacks the network via randomly and independently deleting each vertex with probability $1 - p$ and leaving each vertex with probability p. Then, for any $p > 0$, there exists a unique giant component of size $\Theta_{\mathbb{P}}(n)$.*
(b) **Fragility:** *Suppose an adversary attacks the network via deleting vertices of maximal degree. Then, for any $\varepsilon > 0$, there exists an integer $K_\varepsilon < \infty$ such that deleting the K_ε maximal degree vertices implies that, for two vertices A_1 and A_2 chosen uniformly at random from $\mathcal{G}_n^{\mathrm{er}}$,*

$$\limsup_{n \to \infty} \mathbb{P}\left(A_1 \leftrightarrow A_2\right) \leq \varepsilon. \tag{3.14}$$

where $A_1 \leftrightarrow A_2$ means that there exists a path connecting vertex A_1 and A_2 after deletion of the maximal vertices. Thus one can disconnect the network by deleting $O_{\mathbb{P}}(1)$ vertices.

Remark: As in much of percolation theory, one could ask for the size of the giant component in part (a) above when we randomly delete vertices. See Section 7, where we find the size of the giant component as $n \to \infty$, and give the idea of the proofs for the reported behavior.

4 Discussion and related literature

In this section, we discuss the literature and state some further open problems and conjectures.

The configuration model. The CM was introduced by Bender and Canfield [3], see also Bollobás [6]. Molloy and Reed [23] were the first to use specified degree sequences. The model has become quite popular and has been used in a number of diverse fields. See in particular [21, 22] for applications to modeling of disease epidemics and [24] for a full survey of various questions from statistical physics.

For the CM, the graph distance, i.e., the minimal number of edges on a path connecting two given vertices, is well understood. We refer to [15] for $\tau > 3$, [16, 25] for $\tau \in (2, 3)$ and [10] for $\tau \in (1, 2)$. In the latter paper, it was shown that the graph distance weakly converges, where the limit is either two or three, each with positive probability.

FPP on random graphs. Analysis of FPP in the context of modern random graph models has started only recently (see [4, 13, 14, 18, 27]). The particular case of the CM with degree distribution $1 - F(x) = x^{1-\tau} L(x)$, where $\tau > 2$, was studied in [5]. For $\tau > 2$, where , the hopcount remarkably scales as $\Theta(\log n)$ and satisfies a central limit theorem (CLT) with asymptotic mean and variance both equal to $\alpha \log n$ for some $\alpha > 0$ (see [5]), this despite the fact that for $\tau \in (2, 3)$, the graph distance scales as $\log \log n$. The parameter α belongs to $(0, 1)$ for $\tau \in (2, 3)$, while $\alpha > 1$ for $\tau > 3$ and is the only feature which is left over from the randomness of the random graph. As stated in Theorem 3.1 and 3.2, the behavior for $\tau \in (1, 2)$, where the hopcount remains bounded and weakly converges, is rather different from the one for $\tau > 2$.

Universality of $\mathcal{K}_\infty^{\mathrm{or}}$ and $\mathcal{K}_\infty^{\mathrm{er}}$. Although we have used exponential edge weights, we believe that one obtains the same result with any "similar" edge weight distribution with a density g satisfying $g(0) = 1$. More precisely, the hopcount result, the description of $\mathcal{K}_\infty^{\mathrm{or}}$ and $\mathcal{K}_\infty^{\mathrm{er}}$ and the corresponding limiting distributions in Theorems 3.1–3.2 will remain unchanged. The only thing that will change is the distribution of $(V_1^{\mathrm{or}}, V_2^{\mathrm{or}})$ and $(V_1^{\mathrm{er}}, V_2^{\mathrm{er}})$. In Section 8, Theorem 8.1, we state what happens when

the weight density g satisfies $g(0) = \zeta \in (0, \infty)$. When the edge weight density g satisfies $g(0) = 0$ or $g(0) = \infty$, then we expect that the hopcount remains tight, but that the weight of the minimal path W_n, as well as the limiting FPP problems, both for the original and erased CM, are different.

Robustness and fragility of random networks. The issue of robustness, yet fragility, of random network models has stimulated an enormous amount of research in the recent past. See [1] for one of the original statistical physics papers on this topic, and [7] for a rigorous derivation of this fact when the power-law exponent $\tau = 3$ in the case of the preferential attachment model. The following universal property is believed to hold for a wide range of models:

> If the degree exponent τ of the model is in $(1, 3]$, then the network is robust against random attacks but fragile against directed attacks, while for $\tau > 3$, under random deletion of vertices there exists a critical (model dependent p_c) such that for $p < p_c$ there is no giant component, while for $p > p_c$, there is a giant component.

Proving these results in a wide degree of generality is a challenging program in modern applied probability.

Load distributions on random networks. Understanding the FPP model on these networks opens the door to the analysis of more complicated functionals such as the load distribution on various vertices and edges of the network, which measure the ability of the network in dealing with congestion when transporting material from one part of the network to another. We shall discuss such questions in some more detail in Section 8.

Organization of the proofs and conventions on notation. The proofs in this paper are organized as follows. In Section 5 we prove the results for the original CM, while Section 6 contains the proofs for the erased CM. Theorem 3.3 is proved in Section 7, and we close with a conclusion and discussion in Section 8.

In order to simplify notation, we shall drop the superscripts `er` and `or` so that for example the minimal weight random variable W_n^{or} between two uniformly selected vertices will be denoted by W_n when proving facts about the original CM in Section 5, while W_n will be used to denote W_n^{er} when proving facts about the erased CM in Section 6.

5 Proofs in the original CM: Theorem 3.1

In this section, we prove Theorem 3.1. As part of the proof, we also prove that the FPP on $\mathcal{K}_\infty^{\text{or}}$ is well defined, as formalized in the following proposition:

Proposition 5.1 (FPP on $\mathcal{K}_\infty^{\text{or}}$ is well defined) *For any fixed $K \geq 1$ and for all $i, j < K$ in $\mathcal{K}_\infty^{\text{or}}$, we have $W_{ij}^{\text{or}} > 0$ for $i \neq j$ and $H_{ij}^{\text{or}} < \infty$. In particular, this implies that $H_{I^{\text{or}} J^{\text{or}}}^{\text{or}} < \infty$ almost surely, where we recall that I^{or} and J^{or} are two random vertices in $\mathbb{Z}_+$ chosen (conditionally) independently with distribution $\{P_i\}_{i \geq 1}$.*

Recall that we label vertices according to their degree. We let A_1 and A_2 denote two uniformly chosen vertices. Since the CM has a giant component containing $n - o(n)$ vertices, **whp**, A_1 and A_2 will be connected. We note that the edge incident to vertex A_1 with minimal weight has weight given by $V_i = E_i / D_{A_i}$, $i = 1, 2$, where D_{A_1} denotes the degree of vertex A_1. As a result, (V_1, V_2) has the same distribution as $(E_1/D_1, E_2/D_2)$, where (D_1, D_2) are two independent random variables with distribution function F. Further, by [10, Theorem 1.1], **whp**, the vertices A_1 and A_2 are not directly connected. When A_1 and A_2 are not directly connected, then $W_n \geq V_1 + V_2$, and V_1 and V_2 are independent, as they depend on the exponential weights of *disjoint* sets of edges, while, by construction, D_{A_1} and D_{A_2} are independent.

This proves the required lower bound in Theorem 3.1(a). For the upper bound, we further note that, by [10, Lemma 2.2], the vertices A_1 and A_2 are, **whp**, exclusively connected to so-called *super vertices*, which are the m_n vertices with the largest degrees, for any $m_n \to \infty$ arbitrarily slowly. Thus, the upper bound follows if any two of such super vertices are connected by an edge with weight which converges to 0 in distribution. Denote by $M_{i,j}$ the minimal weight of all edges connecting the vertices i and j. Then, conditionally on the number of edges between i and j, we have that $M_{i,j} \sim \mathrm{Exp}(N(i,j))$, where $N(i,j)$ denotes the number of edges between i and j, and where we use $\mathrm{Exp}(\lambda)$ to denote an exponential random variable with rate λ. We further denote $P_i^{(n)} = D_{(n+1-i:n)}/L_n$, so that $P^{(n)} = \{P_i^{(n)}\}_{i=1}^n$ converges in distribution to the Poisson-Dirichlet distribution. We will show that, conditionally on the degrees and **whp**,

$$N(i,j) = (1 + o_{\mathbb{P}}(1))L_n P_i^{(n)} P_j^{(n)}. \tag{5.1}$$

Indeed, we note that

$$N(i,j) = \sum_{s=1}^{D_i} I_s(i,j), \tag{5.2}$$

where $I_s(i,j)$ is the indicator that the sth stub of vertex i connects to j. We write $\mathbb{P}_n$ for the conditional distribution given the degrees, and $\mathbb{E}_n$ for the expectation w.r.t. $\mathbb{P}_n$. It turns out that we can even prove Theorem 3.1 *conditionally on the degrees*, which is stronger than Theorem 3.1 *averaged* over the degrees. For this, we note that, for $1 \le s_1 < s_2 \le D_i$,

$$\mathbb{P}_n(I_{s_1}(i,j) = 1) = \frac{D_j}{L_n - 1}, \qquad \mathbb{P}_n(I_{s_1}(i,j) = I_{s_2}(i,j) = 1) = \frac{D_j(D_j - 1)}{(L_n - 1)(L_n - 3)}, \tag{5.3}$$

which implies, further using that $D_j = D_{(n+1-j:n)}$ and thus $D_j/L_n \xrightarrow{d} P_j$, that

$$\mathrm{Var}_n(N(i,j)) \le C\frac{D_i^2 D_j}{L_n^2} = o_{\mathbb{P}}\Big(\frac{D_i^2 D_j^2}{L_n^2}\Big) = o_{\mathbb{P}}\big(\mathbb{E}_n[N(i,j)]^2\big). \tag{5.4}$$

As a result, $N(i,j)$ is concentrated, and thus (5.1) follows.

In particular, we see that the vector $\{N(i,j)/L_n\}_{i,j=1}^n$ converges in distribution to $\{P_i P_j\}_{i,j=1}^\infty$. Thus, for every i,j, and conditionally on the degrees, we have that $M_{i,j}$ is approximately equal to an exponential random variable with asymptotic mean $L_n P_i P_j$. This proves that, with J_1 and J_2 being two random variables, which are independent, conditionally on $P = \{P_i\}_{i=1}^\infty$, and with

$$\mathbb{P}(J_s = i|P) = P_i, \tag{5.5}$$

we have that

$$V_1 + V_2 \le W_n \le V_1 + V_2 + \mathrm{Exp}(L_n P_{J_1} P_{J_2}). \tag{5.6}$$

Consequently, $u_n\big(W_n - (V_1 + V_2)\big)$ is a tight random variable. Below, we shall prove that, in fact, $u_n\big(W_n - (V_1 + V_2)\big)$ converges weakly to a non-trivial random variable.

Recall the above analysis, and recall that the edges with minimal weight from the vertices A_1 and A_2 are connected to vertices J_1 and J_2 with asymptotic probability, conditionally on the degrees, given by (5.5). Then, $H_n = 2$ *precisely* when $J_1 = J_2$, which occurs, by the conditional independence of J_1 and J_2 given P, with asymptotic probability

$$\mathbb{P}_n(H_n = 2) = \sum_{i=1}^\infty (P_i^{(n)})^2 + o_{\mathbb{P}}(1). \tag{5.7}$$

Taking expectations and using (2.6) together with the bounded convergence theorem proves (3.5).

Recall that J_1 and J_2 are the vertices to which the edges with minimal weight from A_1 and A_2 are connected, and recall their distribution in (5.5). We now prove the weak convergence of H_n and of $u_n\big(W_n - (V_1 + V_2)\big)$ by constructing a shortest-weight tree in $\mathcal{K}_\infty^{\mathrm{or}}$.

We start building the shortest-weight tree from J_1, terminating when J_2 appears for the first time in this tree. We denote the tree of size l by T_l, and note that $T_1 = \{J_1\}$. Now we have the following recursive procedure to describe the asymptotic distribution of T_l. We note that, for any set of vertices A, the edge with minimal weight outside of A is a uniform edge pointing outside of A. When we have already constructed T_{l-1}, and we fix $i \in T_{l-1}, j \notin T_{l-1}$, then by (5.1) there are approximately $L_n P_i P_j$ edges linking i and j. Thus, the probability that vertex j is added to T_{l-1} is, conditionally on P, approximately equal to

$$p_{ij}(l) = \frac{L_n P_j \sum_{a \in T_{l-1}} P_a}{L_n \sum_{a \in T_{l-1}, b \notin T_{l-1}} P_a P_b} = \frac{P_j}{1 - P_{T_{l-1}}} \geq P_j, \tag{5.8}$$

where, for a set of vertices A, we write

$$P_A = \sum_{a \in A} P_a. \tag{5.9}$$

Denote by B_l the lth vertex chosen. We stop this procedure when $B_l = J_2$ for the first time, and denote this stopping time by S, so that, **whp**, $H_n = 2 + H(S)$, where $H(S)$ is the height of B_S in T_S. Also, $u_n\big(W_n - (V_1 + V_2)\big)$ is equal to W_S, which is the weight of the path linking J_1 and J_2 in $\mathcal{K}_\infty^{\mathrm{or}}$.

Note that the above procedure terminates in finite time, since $P_{J_2} > 0$ and at each time, we pick J_2 with probability at least P_{J_2}. This proves that H_n weakly converges, and that the distribution is given only in terms of P. Also, it proves that the FPP problem on $\mathcal{K}_\infty^{\mathrm{or}}$ is well defined, as formalized in Proposition 5.1.

Further, since the distribution of P only depends on $\tau \in [1, 2)$, and not on any other details of the degree distribution F, the same follows for H_n. When $\tau = 1$, then $P_1 = 1$ a.s., so that $\mathbb{P}_n(H_n = 2) = 1 + o_{\mathbb{P}}(1)$. When $\tau \in (1, 2)$, on the other hand, $P_i > 0$ a.s. for each $i \in \mathbb{N}$, so that, by the above construction, it is not hard to see that $\lim_{n\to\infty} \mathbb{P}_n(H_n = k) = \pi_k(P) > 0$ a.s. for each $k \geq 2$. Thus, the same follows for $\pi_k = \lim_{n\to\infty} \mathbb{P}(H_n = k) = \mathbb{E}[\pi_k(P)]$. It would be of interest to compute π_k for $k > 2$ explicitly, or even π_3, but this seems a difficult problem. $\blacksquare$

6 Proofs in the erased CM: Theorem 3.2

In this section, we prove the various results in the erased setup. We start by giving an overview of the proof.

6.1 Overview of the proof of Theorem 3.2

In this section, we formulate four key propositions, which, together, shall make the intuitive proof given below Theorem 3.2 precise, and which shall combine to a formal proof of Theorem 3.2.

As before, we label vertices by their (original) degree so that vertex i will be the vertex with the ith largest degree. Fix a sequence $\varepsilon_n \to 0$ arbitrarily slowly. Then, we define the set of *super vertices* $\mathcal{S}_n$ be the set of vertices with largest degrees, namely,

$$\mathcal{S}_n = \{i : D_i > \varepsilon_n n^{1/(\tau-1)}\}. \tag{6.1}$$

We shall refer to $\mathcal{S}_n^c$ as the set of *normal vertices*.

Recall the definition of the limiting infinite "complete graph" $\mathcal{K}_\infty^{\mathrm{er}}$ defined in Section 3.2 and for any fixed $k \geq 1$, let $(\mathcal{K}_\infty^{\mathrm{er}})^k$ denote the projection of this object onto the first k vertices (so that we retain only the first k vertices $1, 2, \ldots, k$ and the corresponding edges between these vertices). Then the following proposition says that we can move between the super vertices via two-edge paths which have weight $\Theta(1/\sqrt{n})$. For notational convenience, we write $[k] := \{1, 2 \ldots, k\}$.

Proposition 6.1 (Weak convergence of FPP problem) *Fix k and consider the subgraph of the CM formed by retaining the maximal k vertices and all paths connecting any pair of these vertices by a single intermediary normal vertex (i.e., two-edge paths). For any pair of vertices $i, j \in [k]$, let $l_{ij}^{(n)} = \sqrt{n} w_{ij}^{(2)}$, where $w_{ij}^{(2)}$ is the minimal weight of all two-edge paths between i and j (with $w_{ij}^{(2)} = \infty$ if they are not connected by a two-edge path). Consider the complete graph $\mathcal{K}_n^k$ on vertex set $[k]$ with edge weights $l_{ij}^{(n)}$. Then,*

$$\mathcal{K}_n^k \xrightarrow{d} (\mathcal{K}_\infty^{\text{er}})^k, \tag{6.2}$$

where $\xrightarrow{d}$ denotes the usual finite-dimensional convergence of the $\binom{k}{2}$ random variables $l_{ij}^{(n)}$.

The proof of Proposition 6.1 is deferred to Section 6.2. Proposition 6.1 implies that the FPP problem on the first k super vertices along the two-edge paths converges in distribution to the one on $\mathcal{K}_\infty^{\text{er}}$ restricted to $[k]$. We next investigate the structure of the minimal weights from a uniform vertex, and the tightness of recentered minimal weight:

Proposition 6.2 (Coupling of the minimal edges from uniform vertices) *Let (A_1, A_2) be two uniform vertices, and let $(V_1^{(n)}, V_2^{(n)})$ denote the minimal weight in the erased CM along the edges attached to (A_1, A_2).*
(a) Let $I^{(n)}$ and $J^{(n)}$ denote the vertices to which A_i, $i = 1, 2$, are connected, and let (I, J) be two random variables having the distribution specified right before Theorem 3.2, which are conditionally independent given $\{P_i\}_{i \geq 1}$. Then, we can couple $(I^{(n)}, J^{(n)})$ and (I, J) in such a way that

$$\mathbb{P}\big((I^{(n)}, J^{(n)}) \neq (I, J)\big) = o(1). \tag{6.3}$$

(b) Let $V_i = E_i / D_i^{\text{er}}$, where $(D_1^{\text{er}}, D_2^{\text{er}})$ are two copies of the random variable D^{er} described right before Theorem 3.2, which are conditionally independent given $\{P_i\}_{i \geq 1}$.
Then, we can couple $(V_1^{(n)}, V_2^{(n)})$ to (V_1, V_2) in such a way that

$$\mathbb{P}\big((V_1^{(n)}, V_2^{(n)}) \neq (V_1, V_2)\big) = o(1). \tag{6.4}$$

As a result, the recentered random variables $\sqrt{n}\big(W_n - (V_1 + V_2)\big)$ form a tight sequence.

The proof of Proposition 6.2 is deferred to Section 6.3. The following proposition asserts that the hopcount and the recentered weight between the first k super vertices are tight random variables, and, in particular, they remain within the first $[K]$ vertices, **whp**, as $K \to \infty$:

Proposition 6.3 (Tightness of FPP problem and evenness of hopcount) *Fix $k \geq 1$. For any pair of vertices $i, j \in [k]$, let $H_n(i, j)$ denote the number of edges of the minimal-weight path between i and j. Then,*
(a) $H_n(i, j)$ is a tight sequence of random variables, which is such that $\mathbb{P}(H_n(i, j) \notin 2\mathbb{Z}^+) = o(1)$;
(b) the probability that any of the minimal weight paths between $i, j \in [k]$, at even times, leaves the K vertices of largest degree tends to zero when $K \to \infty$;
(c) the hopcount H_n is a tight sequence of random variables, which is such that $\mathbb{P}(H_n \notin 2\mathbb{Z}^+) = o(1)$.

The proof of Proposition 6.3 is deferred to Section 6.4. The statement is consistent with the intuitive explanation given right after Theorem 3.2: the minimal weight paths between two uniform vertices consists of an *alternating* sequence of normal vertices and super vertices. We finally state that the infinite FPP on the erased CM is well defined:

Proposition 6.4 (Infinite FPP is well defined) *Consider FPP on $\mathcal{K}_\infty^{\text{er}}$ with weights $\{l_{ij}\}_{1 \leq i < j < \infty}$ defined in (3.9). Fix $k \geq 1$ and $i, j \in [k]$. Let $\mathcal{A}_K$ be the event that there exists a path of weight at most W connecting i and j, which contains a vertex in $\mathbb{Z}^+ \setminus [K]$, and which is of weight at most W. Then, there exists a $C > 0$ such that, for all K sufficiently large,*

$$\mathbb{P}(\mathcal{A}_K) \leq CWK^{-1}\mathrm{e}^{CW\sqrt{\log K}}. \tag{6.5}$$

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The proof of Proposition 6.4 is deferred to Section 6.5. With Propositions 6.1–6.4 at hand, we are able to prove Theorem 3.2:

Proof of Theorem 3.2 subject to Propositions 6.1–6.4. By Proposition 6.2(b), we can couple $(V_1^{(n)}, V_2^{(n)})$ to (V_1, V_2) in such a way that $(V_1^{(n)}, V_2^{(n)}) = (V_1, V_2)$ occurs **whp**. Further, **whp**, for k large, $I, J \leq k$, which we shall assume from now on, while, by Proposition 6.2(b), $\sqrt{n}(W_n - (V_1 + V_2))$ is a tight sequence of random variables.

By Proposition 6.3, the hopcount is a tight sequences of random variables, which is **whp** even. Indeed, it consist of an alternating sequence of normal and super vertices. We shall call the path of super vertices the *two-edge path*. Then, Proposition 6.3 implies that the probability that any of the two-edge paths between any of the first $[k]$ vertices leaves the first K vertices is small when K grows big. As a result, we can write $H_n = 2 + 2H_{I^{(n)} J^{(n)}}^{(n)}$, where $H_{I^{(n)} J^{(n)}}^{(n)}$ is the number of two-edge paths in $\mathcal{K}_n^{\mathrm{er}}$. By (6.3), we have that, **whp**, $H_{I^{(n)} J^{(n)}}^{(n)} = H_{IJ}^{(n)}$.

By Proposition 6.1, the FPP on the k vertices of largest degree in the CM weakly converges to the FPP on the first k vertices of $\mathcal{K}_\infty^{\mathrm{er}}$, for any $k \geq 1$. By Proposition 6.4, **whp**, the shortest-weight path between any two vertices in $[k]$ in $\mathcal{K}_\infty^{\mathrm{er}}$ does not leave the first K vertices, so that W_{IJ} and H_{IJ} are finite random variables, where W_{IJ} and H_{IJ} denote the weight and number of steps in the minimal path between I and J in $\mathcal{K}_\infty^{\mathrm{er}}$. In particular, it follows that $\sqrt{n}(W_n - (V_1^{(n)} + V_2^{(n)})) \xrightarrow{d} W_{IJ}$, and that $H_{ij}^{(n)} \xrightarrow{d} H_{ij}$ for every $i, j \in [k]$, which is the number of hops between $i, j \in [k]$ in $\mathcal{K}_\infty^{\mathrm{er}}$. Since, **whp**, $(V_1, V_2) = (V_1^{(n)}, V_2^{(n)})$, $\sqrt{n}(W_n - (V_1 + V_2))$ converges to the same limit. This completes the proof of Theorem 3.2 subject to Propositions 6.1–6.4. ∎

6.2 Weak convergence of the finite FPP problem to $\mathcal{K}_\infty^{\mathrm{er}}$: Proof of Proposition 6.1

In this section, we study the weak convergence of the FPP on $\mathcal{K}_n^k$ to the one on $(\mathcal{K}_\infty^{\mathrm{er}})^k$, by proving Proposition 6.1.

We start by proving some elementary results regarding the extrema of Gamma random variables. We start with a particularly simple case, and after this, generalize it to the convergence of all weights of two-edge paths in $\mathcal{K}_n^{\mathrm{er}}$.

Lemma 6.5 (Minima of Gamma random variables) *(a) Fix $\beta > 0$ and consider $n\beta$ i.i.d. Gamma(2,1) random variables Y_i. Let $T_n = \min_{1 \leq i \leq \beta n} Y_i$ be the minimum of these random variables. Then, as $n \to \infty$,*

$$\mathbb{P}(\sqrt{n} T_n > x) \to \exp\left(-\beta x^2 / 2\right). \tag{6.6}$$

(b) Let $\{X_i\}_{1 \leq i \leq m}, \{Y_i\}_{1 \leq i \leq m}$ and $\{Z_i\}_{1 \leq i \leq m}$ be all independent collections of independent exponential mean 1 random variables. Let

$$\eta_m = \sqrt{m} \min_{1 \leq i \leq m} (X_i + Y_i), \qquad \kappa_m = \sqrt{m} \min_{1 \leq i \leq m} (X_i + Z_i), \qquad and \qquad \rho_m = \sqrt{m} \min_{1 \leq i \leq m} (Y_i + Z_i). \tag{6.7}$$

Then, as $m \to \infty$,

$$(\eta_m, \kappa_m, \rho_m) \xrightarrow{d} (\zeta_1, \zeta_2, \zeta_3). \tag{6.8}$$

Here ζ_i are independent with the distribution in part (a) with $\beta = 1$.

We note that the *independence* claimed in part (b) is non-trivial, in particular, since the random variables $(\eta_m, \kappa_m, \rho_m)$ are all defined in terms of the *same* exponential random variables. We shall later see a more general version of this result.

Proof. Part (a) is quite trivial and we shall leave the proof to the reader and focus on part (b). Note that for any fixed x_0, y_0 and z_0 all positive and for X, Y, Z all independent exponential random variables, we have

$$\mathbb{P}(X + Y \leq x_0 / \sqrt{m}) = \frac{x_0^2}{2m} + O(m^{-3/2}), \tag{6.9}$$

and similar estimates hold for $\mathbb{P}(X + Z \leq y_0/\sqrt{m})$ and $\mathbb{P}(Y + Z \leq z_0/\sqrt{m})$. Further, we make use of the fact that, for $m \to \infty$,

$$\mathbb{P}\left(X + Y \leq x_0/\sqrt{m}, X + Z \leq y_0/\sqrt{m}\right) = \Theta(m^{-3/2}), \tag{6.10}$$

since $X + Y \leq x_0/\sqrt{m}, X + Z \leq y_0/\sqrt{m}$ implies that X, Y, Z are all of order $1/\sqrt{m}$. Then, we rewrite

$$\mathbb{P}\left(\eta_m > x_0, \kappa_m > y_0, \rho_m > z_0\right) = \mathbb{P}\left(\sum_{i=1}^m I_i = 0, \sum_{i=1}^m J_i = 0, \sum_{i=1}^m L_i = 0\right), \tag{6.11}$$

where $I_i = \mathbb{1}_{\{X_i + Y_i < x_0/\sqrt{m}\}}$, $J_i = \mathbb{1}_{\{X_i + Z_i < y_0/\sqrt{m}\}}$ and $L_i = \mathbb{1}_{\{Y_i + Z_i < z_0/\sqrt{m}\}}$, where we write $\mathbb{1}_A$ for the indicator of the event A. This implies, in particular, that

$$\begin{aligned}
\mathbb{P}(\eta_m > x_0, \kappa_m > y_0, \rho_m > z_0) &= (\mathbb{P}(I_1 = 0, J_1 = 0, L_1 = 0))^m \\
&= \left(1 - \mathbb{P}\left(\{I_1 = 1\} \cup \{J_1 = 1\} \cup \{L_1 = 1\}\right)\right)^m \\
&= \left[1 - \left(\frac{x_0^2}{2m} + \frac{y_0^2}{2m} + \frac{z_0^2}{2m} - \Theta(m^{-3/2})\right)\right]^m \\
&= e^{-(x_0^2/2 + y_0^2/2 + z_0^2/2)}(1 + o(1)),
\end{aligned} \tag{6.12}$$

as $m \to \infty$, where we use that

$$\begin{aligned}
\left|\mathbb{P}\left(\{I_1 = 1\} \cup \{J_1 = 1\} \cup \{L_1 = 1\}\right) - \mathbb{P}(I_1 = 1) - \mathbb{P}(J_1 = 1) - \mathbb{P}(L_1 = 1)\right| & \tag{6.13} \\
\leq \mathbb{P}(I_1 = J_1 = 1) + \mathbb{P}(I_1 = L_1 = 1) + \mathbb{P}(J_1 = L_1 = 1) = \Theta(m^{-3/2}). &
\end{aligned}$$

This proves the result. ∎

The next lemma generalizes the statement of Lemma 6.5 in a substantial way:

Lemma 6.6 (Minima of Gamma random variables on the complete graph) *Fix $k \geq 1$ and $n \geq k$. Let $\{E_{s,t}\}_{1 \leq s < t \leq n}$ be an i.i.d. sequence of exponential random variables with mean 1. For each $i \in [k]$, let $\mathcal{N}_i \subseteq [n] \setminus [k]$ denote deterministic sets of indices. Let $\mathcal{N}_{ij} = \mathcal{N}_i \cap \mathcal{N}_j$, and assume that, for each $i, j \in [k]$,*

$$|\mathcal{N}_{ij}|/n \to \beta_{ij} > 0. \tag{6.14}$$

Let

$$\eta_{ij}^{(n)} = \sqrt{n} \min_{s \in \mathcal{N}_{ij}} (E_{i,s} + E_{s,j}). \tag{6.15}$$

Then, for each k,

$$\{\eta_{ij}^{(n)}\}_{1 \leq i < j \leq k} \xrightarrow{d} \{\eta_{ij}\}_{1 \leq i < j \leq k}, \tag{6.16}$$

where the random variables $\{\eta_{ij}\}_{1 \leq i < j \leq k}$ are independent random variables with distribution

$$\mathbb{P}(\eta_{ij} > x) \to \exp\left(-\beta_{ij} x^2/2\right). \tag{6.17}$$

When $\mathcal{N}_i$ denote random sets of indices which are independent of the exponential random variables, then the same result holds when the convergence in (6.14) is replaced with convergence in distribution where the limits β_{ij} satisfy that $\beta_{ij} > 0$ holds a.s., and the limits $\{\eta_{ij}\}_{1 \leq i < j \leq k}$ are conditionally independent given $\{\beta_{ij}\}_{1 \leq i < j \leq k}$.

Proof. We follow the proof of Lemma 6.5 as closely as possible. For $i \in [k]$ and $s \in [n] \setminus [k]$, we define $X_{i,s} = E_{i,s}$, when $s \in \mathcal{N}_i$, and $X_{i,s} = +\infty$, when $s \notin \mathcal{N}_i$. Since the sets of indices $\{\mathcal{N}_i\}_{i \in [k]}$ are independent from the exponential random variables, the variables $\{X_{i,s}\}_{i \in [k], s \in [n] \setminus [k]}$ are, conditionally on $\{\mathcal{N}_i\}_{i \in [k]}$, *independent* random variables. Then, since $\mathcal{N}_{ij} = \mathcal{N}_i \cap \mathcal{N}_j$,

$$\eta_{ij}^{(n)} = \sqrt{n} \min_{s \in \mathcal{N}_{ij}} (E_{i,s} + E_{j,s}) = \sqrt{n} \min_{s \in [n] \setminus [k]} (X_{i,s} + X_{j,s}). \tag{6.18}$$

Let $\{x_{ij}\}_{1 \leq i < j \leq k}$ be a vector with positive coordinates. We note that

$$\mathbb{P}(\eta_{ij}^{(n)} > x_{ij}, \forall i, j \in [k]) = \mathbb{P}\Big(\sum_{s \in [n] \setminus [k]} J_{ij,s} = 0, \forall i, j \in [k] \Big), \tag{6.19}$$

where $J_{ij,s} = \mathbb{1}_{\{X_{i,s} + X_{j,s} < x_{ij}/\sqrt{n}\}}$. We note that the random vectors $\{J_{ij,s}\}_{s \in [n] \setminus [k]}$ are conditionally independent given $\{\mathcal{N}_i\}_{i \in [k]}$, so that

$$\mathbb{P}(\eta_{ij}^{(n)} > x_{ij}, \forall i, j \in [k]) = \prod_{s \in [n] \setminus [k]} \mathbb{P}(J_{ij,s} = 0, \forall i, j \in [k]). \tag{6.20}$$

Now, note that $J_{ij,s} = 0$ a.s. when $s \notin \mathcal{N}_{ij}$, while, for $s \in \mathcal{N}_{ij}$, we have, similarly to (6.9),

$$\mathbb{P}(J_{ij,s} = 1) = \frac{x_{ij}^2}{2n} + O(n^{-3/2}). \tag{6.21}$$

Therefore, we can summarize these two claims by

$$\mathbb{P}(J_{ij,s} = 1) = \mathbb{1}_{\{s \in \mathcal{N}_{ij}\}} \Big(\frac{x_{ij}^2}{2n} + \Theta(n^{-3/2}) \Big). \tag{6.22}$$

Similarly to the argument in (6.12), we have that

$$\mathbb{P}(J_{ij,s} = 0, \forall i, j \in [k]) = 1 - \sum_{1 \leq i < j \leq k} \mathbb{P}(J_{ij,s} = 1) + \Theta(n^{-3/2})$$

$$= \exp \Big\{ - \sum_{1 \leq i < j \leq k} \mathbb{1}_{\{s \in \mathcal{N}_{ij}\}} \Big(\frac{x_{ij}^2}{2n} + \Theta(n^{-3/2}) \Big) \Big\}. \tag{6.23}$$

We conclude that

$$\mathbb{P}(\eta_{ij}^{(n)} > x_{ij}, \forall i, j \in [k]) = \prod_{s \in [n] \setminus [k]} \mathbb{P}(J_{ij,s} = 0 \forall i, j \in [k]) \tag{6.24}$$

$$= \exp \Big\{ - \sum_{s \in [n] \setminus [k]} \sum_{1 \leq i < j \leq k} \mathbb{1}_{\{s \in \mathcal{N}_{ij}\}} \Big(\frac{x_{ij}^2}{2n} + \Theta(n^{-3/2}) \Big) \Big\}$$

$$= \exp \{ - \sum_{1 \leq i < j \leq k} x_{ij}^2 \beta_{ij}/2 \} (1 + o(1)),$$

as required. ∎

We shall apply Lemma 6.6 to $\mathcal{N}_i$ being the direct neighbors in $[n] \setminus [k]$ of vertex $i \in [k]$. Thus, by Lemma 6.6, in order to prove the convergence of the weights, it suffices to prove the convergence of the number of *joint* neighbors of the super vertices i and j, simultaneously, for all $i, j \in [k]$. That is the content of the following lemma:

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Lemma 6.7 (Weak convergence of N_{ij}^{er}/n) *The random vector $\{N_{ij}^{\text{er}}/n\}_{1\leq i<j\leq n}$, converges in distribution in the product topology to $\{f(P_i, P_j)\}_{1\leq i<j<\infty}$, where $f(P_i, P_j)$ is defined in (3.8), and $\{P_i\}_{i\geq 1}$ has the Poisson-Dirichlet distribution.*

Proof. We shall first prove that the random vector $\{N_{ij}^{\text{er}}/n - f(P_i^{(n)}, P_j^{(n)})\}_{1\leq i<j\leq n}$, converges in probability in the product topology to zero, where $P_i^{(n)} = D_{(n+1-i:n)}/L_n$ is the normalized i^{th} largest degree. For this, we note that

$$N_{ij}^{\text{er}} = \sum_{s=1}^{n} I_s(i,j), \qquad (6.25)$$

where $I_s(i,j)$ is the indicator that $s \in [n]$ is a neighbor of both i and j. Now, weak convergence in the product topology is equivalent to the weak convergence of $\{N_{ij}^{\text{er}}/n\}_{1\leq i<j<K}$ for any $K \in \mathbb{Z}^+$ (see [20, Theorem 4.29]). For this, we shall use a second moment method. We first note that $|N_{ij}^{\text{er}}/n - N_{\leq b_n}^{\text{er}}(i,j)/n| \leq \frac{1}{n}\sum_{s=1}^{n} \mathbb{1}_{\{D_s \geq b_n\}} \xrightarrow{\mathbb{P}} 0$, where $b_n \to \infty$ and

$$N_{\leq b_n}^{\text{er}}(i,j) = \sum_{s=1}^{n} I_s(i,j)\mathbb{1}_{\{D_s \leq b_n\}}. \qquad (6.26)$$

Take $b_n = n$ and note that when $i, j \leq K$, the vertices i and j both have degree of order $n^{1/(\tau-1)}$ which is at least n **whp**. Thus, the sum over s in $N_{\leq n}(i,j)$ involves different vertices than i and j. Next, we note that

$$\mathbb{E}_n[N_{\leq n}^{\text{er}}(i,j)/n] = \frac{1}{n}\sum_{s=1}^{n} \mathbb{1}_{\{D_s \leq n\}}\mathbb{P}_n(I_s(i,j)=1)$$

$$= \frac{1}{n}\sum_{s=1}^{n} \mathbb{1}_{\{D_s \leq n\}}[1-(1-P_i^{(n)})^{D_s}-(1-P_j^{(n)})^{D_s}+(1-P_i^{(n)}-P_j^{(n)})^{D_s}] + o_{\mathbb{P}}(1), \quad (6.27)$$

in a similar way as in (3.8). By dominated convergence, we have that, for every $s \in [0,1]$,

$$\frac{1}{n}\sum_{s=1}^{n} \mathbb{1}_{\{D_s \leq n\}}(1-s)^{D_s} \xrightarrow{a.s.} \mathbb{E}[(1-s)^D], \qquad (6.28)$$

which implies that

$$\mathbb{E}_n[N_{\leq n}^{\text{er}}(i,j)/n] - f(P_i^{(n)}, P_j^{(n)}) \xrightarrow{\mathbb{P}} 0. \qquad (6.29)$$

Further, the indicators $\{I_s(i,j)\}_{s=1}^{n}$ are close to independent, so that $\text{Var}_n(N_{\leq n}^{\text{er}}(i,j)/n) = o_{\mathbb{P}}(1)$, where Var_n denotes the variance w.r.t. $\mathbb{P}_n$. The weak convergence claimed in Lemma 6.7 follows directly from the above results, as well as the weak convergence of the order statistics in (2.3) and the continuity of $(s,t) \mapsto f(s,t)$. ∎

The following corollary completes the proof of the convergence of the rescaled minimal weight two-edge paths in $\mathcal{G}_n^{\text{er}}$:

Corollary 6.8 (Conditional independence of weights) *Let $l_{ij}^{(n)} = \sqrt{n}w_{ij}^{(2)}$, where $w_{ij}^{(2)}$ is the minimal weight of all two-edge paths between the vertices i and j (with $w_{ij}^{(2)} = \infty$ if they are not connected by a two-edge path). Fix $k \geq 1$. Then,*

$$\left(\{l_{ij}^{(n)}\}_{1\leq i<j\leq k}, \{D_i/L_n\}_{1\leq i\leq n}\right) \xrightarrow{d} \left(\{l_{ij}\}_{1\leq i<j\leq k}, \{P_i\}_{i\geq 1}\right), \qquad (6.30)$$

where, given $\{P_i\}_{i\geq 1}$ the random variables $\{l_{ij}\}_{1\leq i<j\leq k}$ are conditionally independent with distribution

$$\mathbb{P}(l_{ij} > x) \to \exp\left(-f(P_i, P_j)x^2/2\right). \qquad (6.31)$$

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Proof. The convergence of $\{D_m/L_n\}_{1 \leq m \leq n}$ follows from Section 2.1. Then we apply Lemma 6.6. We let $\mathcal{N}_i$ denote the set of neighbors in $[n] \setminus [k]$ of the super vertex $i \in [k]$. Then, $|\mathcal{N}_{ij}| = |\mathcal{N}_i \cap \mathcal{N}_j| = Ner_{ij}$, so that (6.14) is equivalent to the convergence in distribution of N_{ij}^{er}/n. The latter is proved in Lemma 6.7, with $\beta_{ij} = f(P_i, P_j)$. Since $P_i > 0$ a.s. for each $i \in [k]$, we obtain that $\beta_{ij} > 0$ a.s. for all $i, j \in [k]$. Therefore, Lemma 6.6 applies, and completes the proof of the claim. ∎

Now we are ready to prove Proposition 6.1:

Proof of Proposition 6.1. By Corollary 6.8, we see that the weights in the FPP problem $\mathcal{K}_n^k$ converge in distribution to the weights in the FPP on $(\mathcal{K}_\infty^{\mathrm{er}})^k$. Since the weights $W_{ij}^{(n)}$ of the minimal two-edge paths between $i, j \in [k]$ are continuous functions of the weights $\{l_{ij}^{(n)}\}_{1 \leq i < j \leq k}$, it follows that $\{W_{ij}^{(n)}\}_{1 \leq i < j \leq k}$ converges in distribution to $\{W_{ij}\}_{1 \leq i < j \leq k}$. Since the weights are continuous random variables, this also implies that the hopcounts $\{H_{ij}^{(n)}\}_{1 \leq i < j \leq k}$ in $\mathcal{K}_n^k$ converge in distribution to the hopcounts $\{H_{ij}\}_{1 \leq i < j \leq k}$ in $(\mathcal{K}_\infty^{\mathrm{er}})^k$. This proves Proposition 6.1. ∎

6.3 Coupling of the minimal edges from uniform vertices: Proof of Proposition 6.2

In this section, we prove Proposition 6.2. We start by noticing that the vertices A_i, $i = 1, 2$, are, **whp**, only attached to super vertices. Let $I^{(n)}$ and $J^{(n)}$ denote the vertices to which A_i, $i = 1, 2$, are connected and of which the edge weights are minimal. Then, by the discussion below (3.9), $(I^{(n)}, J^{(n)})$ converges in distribution to the random vector (I, J) having the distribution specified right before Theorem 3.2, and where the two components are conditionally independent, given $\{P_i\}_{i \geq 1}$.

Further, denote the weight of the edges attaching (A_1, A_2) to $(I^{(n)}, J^{(n)})$ by $(V_1^{(n)}, V_2^{(n)})$. Then, $(V_1^{(n)}, V_2^{(n)}) \xrightarrow{d} (V_1^{\mathrm{er}}, V_2^{\mathrm{er}})$ defined in Theorem 3.2. This in particular proves (3.10) since the weight between any two super vertices is $o_{\mathbb{P}}(1)$. Further, since $(I^{(n)}, J^{(n)})$ are discrete random variables that weakly converge to (I, J), we can couple $(I^{(n)}, J^{(n)})$ and (I, J) in such a way that (6.3) holds.

Let $(D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)})$ denote the erased degrees of the vertices (A_1, A_2) in $\mathcal{G}^{\mathrm{er}}$. The following lemma states that these erased degrees converge in distribution:

Lemma 6.9 (Convergence in distribution of erased degrees) *Under the conditions of Theorem 3.2, as $n \to \infty$,*

$$(D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)}) \xrightarrow{d} (D_1^{\mathrm{er}}, D_2^{\mathrm{er}}), \tag{6.32}$$

which are two copies of the random variable D^{er} described right before Theorem 3.2, and which are conditionally independent given $\{P_i\}_{i \geq 1}$.

Proof. We note that the degrees before erasure, i.e., (D_{A_1}, D_{A_2}), are i.i.d. copies of the distribution D with distribution function F, so that, in particular, (D_{A_1}, D_{A_2}) are bounded by K **whp** for any K sufficiently large. We next investigate the effect of erasure. We condition on $\{P_i^{(n)}\}_{i=1}^{m_n}$, the rescaled m_n largest degrees, and note that, by (2.3), $\{P_i^{(n)}\}_{i=1}^{m_n} = \{D_i/L_n\}_{i=1}^{m_n}$ converges in distribution to $\{P_i\}_{i \geq 1}$. We let $m_n \to \infty$ arbitrarily slowly, and note that, **whp**, the (D_{A_1}, D_{A_2}) half-edges incident to the vertices (A_1, A_2), are exclusively connected to vertices in $[m_n]$. The convergence in (6.32) follows when

$$\mathbb{P}\Big((D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)}) = (k_1, k_2) \mid \{P_i^{(n)}\}_{i=1}^{m_n}, (D_{A_1}, D_{A_2}) = (j_1, j_2)\Big) \tag{6.33}$$
$$= G_{k_1, j_1}(\{P_i^{(n)}\}_{i=1}^{m_n}) G_{k_2, j_2}(\{P_i^{(n)}\}_{i=1}^{m_n}) + o_{\mathbb{P}}(1),$$

for an appropriate function $G_{k,j} \colon \mathbb{R}_+^{\mathbb{N}} \to [0, 1]$, which, for every k, j, is *continuous* in the product topology. (By convention, for a vector with finitely many coordinates $\{x_i\}_{i=1}^m$, we let $G_{k_1, j_1}(\{x_i\}_{i=1}^m) = G_{k_1, j_1}(\{x_i\}_{i=1}^\infty)$, where $x_i = 0$ for $i > m$.)

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Indeed, from (6.33), it follows that, by dominated convergence,

$$\mathbb{P}\Big((D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)}) = (k_1, k_2)\Big) = \mathbb{E}\Big[\mathbb{P}\Big((D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)}) = (k_1, k_2) \mid \{P_i^{(n)}\}_{i=1}^{m_n}, (D_{A_1}, D_{A_2})\Big)\Big]$$

$$= \mathbb{E}\Big[G_{k_1, D_1}(\{P_i^{(n)}\}_{i=1}^{m_n})G_{k_2, D_2}(\{P_i^{(n)}\}_{i=1}^{m_n})\Big] + o(1)$$

$$\to \mathbb{E}\Big[G_{k_1, D_1}(\{P_i\}_{i\geq 1})G_{k_2, D_2}(\{P_i\}_{i\geq 1})\Big], \qquad (6.34)$$

where the last convergence follows from weak convergence of $\{P_i^{(n)}\}_{i=1}^{m_n}$ and the assumed continuity of G. The above convergence, in turn, is equivalent to (6.32), when $G_{k,j}(\{P_i\}_{i\geq 1})$ denotes the probability that k distinct cells are chosen in a multinomial experiment with j independent trials where, at each trial, we choose cell i with probability P_i. It is not hard to see that, for each k, j, $G_{k,j}$ is indeed a continuous function in the product topology.

To see (6.33), we note that, conditionally on $\{P_i^{(n)}\}_{i=1}^{m_n}$, the vertices to which the $D_{A_i} = j_i$ stubs attach are close to independent, so that it suffices to prove that

$$\mathbb{P}\big(D_{A_1}^{\mathrm{er}(n)} = k_1 \mid \{P_i^{(n)}\}_{i=1}^{m_n}, D_{A_1} = j_1\big) = G_{k_1, j_1}(\{P_i^{(n)}\}_{i=1}^{m_n}) + o_{\mathbb{P}}(1). \qquad (6.35)$$

The latter follows, since, again conditionally on $\{P_i^{(n)}\}_{i=1}^{m_n}$, each stub chooses to connect to vertex i with probability $D_i/L_n = P_i^{(n)}$, and the different stubs choose close to independently. This completes the proof of Lemma 6.9. ∎

By Lemma 6.9, we can also couple $(D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)})$ to $(D_1^{\mathrm{er}}, D_2^{\mathrm{er}})$ in such a way that

$$\mathbb{P}\big((D_{A_1}^{\mathrm{er}(n)}, D_{A_2}^{\mathrm{er}(n)}) \neq (D_1^{\mathrm{er}}, D_2^{\mathrm{er}})\big) = o(1). \qquad (6.36)$$

Now, $(V_1^{(n)}, V_2^{(n)})$ is equal in distribution to $(E_1/D_{A_1}^{\mathrm{er}(n)}, E_2/D_{A_2}^{\mathrm{er}(n)})$, where (E_1, E_2) are two independent exponential random variables with mean 1. Let $V_i = V_i^{\mathrm{er}} = E_i/D_i^{\mathrm{er}}$, where we use the *same* exponential random variables. Then (V_1, V_2) has the right distribution, and the above coupling also provides a coupling of $(V_1^{(n)}, V_2^{(n)})$ to (V_1, V_2) such that (6.4) holds.

By the above couplings, we have that $\sqrt{n}\big(W_n - (V_1^{(n)} + V_2^{(n)})\big) = \sqrt{n}\big(W_n - (V_1 + V_2)\big)$ **whp**. By construction, $\sqrt{n}\big(W_n - (V_1^{(n)} + V_2^{(n)})\big) \geq 0$ a.s., so that also, **whp**, $\sqrt{n}\big(W_n - (V_1 + V_2)\big) \geq 0$. Further, $\sqrt{n}\big(W_n - (V_1^{(n)} + V_2^{(n)})\big) \leq l_{I^{(n)}, J^{(n)}}^{(n)}$, which is the weight of the minimal two-edge path between the super vertices $I^{(n)}$ and $J^{(n)}$. Now, by (6.3), $(I^{(n)}, J^{(n)}) = (I, J)$ **whp**. Thus, **whp**, $l_{I^{(n)}, J^{(n)}}^{(n)} = l_{I,J}^{(n)}$, which, by Proposition 6.1, converges in distribution to l_{IJ}, which is a finite random variable. As a result, $l_{I^{(n)}, J^{(n)}}^{(n)}$ is a tight sequence of random variables, and, therefore, also $\sqrt{n}\big(W_n - (V_1^{(n)} + V_2^{(n)})\big)$ is. This completes the proof of Proposition 6.2. ∎

6.4 Tightness of FPP problem and evenness of hopcount: Proof of Proposition 6.3

In this section, we prove that the only possible minimal weight paths between the super vertices are two-edge paths. All other paths are much too costly to be used. We start by stating and proving a technical lemma about expectations of degrees conditioned to be at most x. It is here that we make use of the condition in (3.6):

Lemma 6.10 (Bounds on restricted moments of D) *Let D be a random variable with distribution function F satisfying (3.6) for some $\tau \in (1, 2)$. Then, there exists a constant C such that, for every $x \geq 1$,*

$$\mathbb{E}[D\mathbb{1}_{\{D\leq x\}}] \leq Cx^{2-\tau}, \quad \mathbb{E}[D^{\tau-1}\mathbb{1}_{\{D\leq x\}}] \leq C\log x, \quad \mathbb{E}[D^{\tau}\mathbb{1}_{\{D\leq x\}}] \leq Cx, \quad \mathbb{E}[D^{2(\tau-1)}\mathbb{1}_{\{D\leq x\}}] \leq Cx^{\tau-1}. \qquad (6.37)$$

Proof. We note that, for every $a > 0$, using partial integration,

$$\mathbb{E}[D^a \mathbb{1}_{\{D \leq x\}}] = -\int_{(0,x]} y^a \, d(1 - F(y)) \leq a \int_0^x y^{a-1}[1 - F(y)]dy \leq c_2 a \int_0^x y^{a-\tau} dy. \tag{6.38}$$

The proof is completed by considering the four cases separately and computing in each case the integral on the right-hand side of (6.38). ∎

The following lemma shows that paths of an *odd* length are unlikely:

Lemma 6.11 (Shortest-weight paths on super vertices are of even length) *Let the distribution function F of the degrees of the CM satisfy (3.6). Let $\mathcal{B}^{(n)}$ be the event that there exists a path between two super vertices consisting of all normal vertices and having an odd number of edges and of total weight $w_n/\sqrt{n}$. Then, for some constant C,*

$$\mathbb{P}(\mathcal{B}^{(n)}) \leq \frac{\varepsilon_n^{-2(\tau-1)}}{\sqrt{n \log n}} \mathrm{e}^{Cw_n\sqrt{\log n}}. \tag{6.39}$$

Proof. We will show that the probability that there exists a path between two super vertices consisting of all normal vertices and having an odd number of edges and of total weight $w_n/\sqrt{n}$ is small. For this, we shall use the first moment method and show that the expected number of such paths goes to 0 as $n \to \infty$. Fix two super vertices which will be the end points of the path and an *even* number $m \geq 0$ of normal vertices with indices $i_1, i_2, \ldots i_m$. Note that when a path between two super vertices consists of an even number of vertices, then the path has an odd number of edges.

Let $\mathcal{B}_m^{(n)}$ be the event that there exists a path between two super vertices consisting of exactly m intermediate normal vertices with total weight $w_n/\sqrt{n}$. We start by investigating the case $m = 0$, so that the super vertices are directly connected. Note that $|\mathcal{S}_n| = O_{\mathbb{P}}(\mathbb{E}[|\mathcal{S}_n|])$, by concentration, and that

$$\mathbb{E}[|\mathcal{S}_n|] = n\mathbb{P}(D_1 > \varepsilon_n n^{1/(\tau-1)}) = O(\varepsilon_n^{-(\tau-1)}),$$

Hence, there are $O_{\mathbb{P}}(\varepsilon_n^{-(\tau-1)})$ super vertices and thus $O_{\mathbb{P}}(\varepsilon_n^{-2(\tau-1)})$ edges between them. The probability that any one of them is smaller than $w_n/\sqrt{n}$ is of order $\varepsilon_n^{-2(\tau-1)}w_n/\sqrt{n}$, and it follows that $\mathbb{P}(\mathcal{B}_0^{(n)}) \leq \varepsilon_n^{-2(\tau-1)}w_n/\sqrt{n}$.

Let $M_m^{(n)}$ be the total number of paths connecting two specific super vertices and which are such that the total weight on the paths is at most $w_n/\sqrt{n}$, so that

$$\mathbb{P}(\mathcal{B}_m^{(n)}) = \mathbb{P}(M_m^{(n)} \geq 1) \leq \mathbb{E}[M_m^{(n)}]. \tag{6.40}$$

In the following argument, for convenience, we let $\{D_i\}_{i=1}^n$ denote the i.i.d. vector of degrees (i.e., below D_i is not the ith largest degree, but rather a copy of the random variable $D \sim F$ independently of the other degrees.)

Let $\vec{\imath} = (i_1, i_2, \ldots, i_m)$, and denote by $p_{m,n}(\vec{\imath})$ the probability that the m vertices $i_1, i_2, \ldots, i_m$ are normal and are such that there is an edge between i_s and i_{s+1}, for $s = 1, \ldots, m-1$. Further, note that with $S_{m+1} = \sum_{i=1}^{m+1} E_i$, where E_i are independent exponential random variables with mean 1, we have, for any $u \in [0, 1]$,

$$\mathbb{P}(S_{m+1} \leq u) = \int_0^u \frac{x^m \mathrm{e}^{-x}}{m!} \leq \frac{u^{m+1}}{(m+1)!}. \tag{6.41}$$

Together with the fact that there are $O_{\mathbb{P}}(\varepsilon_n^{-(\tau-1)})$ super vertices, this implies that

$$\mathbb{P}(\mathcal{B}_m^{(n)}) \leq \mathbb{E}[M_m^{(n)}] \leq \frac{C\varepsilon_n^{-2(\tau-1)}w_n^{m+1}}{(m+1)!n^{(m+1)/2}} \sum_{\vec{\imath}} p_{m,n}(\vec{\imath}), \tag{6.42}$$

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since (6.41) implies that the probability that the sum of $m+1$ exponentially distributed r.v.'s is smaller than $u_n = w_n/\sqrt{n}$ is at most $u_n^{m+1}/(m+1)!$.

By the construction of the CM, we have

$$p_{m,n}(\vec{\imath}) \leq \mathbb{E}\left[\prod_{j=1}^{m-1}\left(\frac{D_{i_j}D_{i_{j+1}}}{L_n - 2j + 1} \wedge 1\right)\mathbb{1}_{\mathcal{F}_m}\right] \leq \mathbb{E}\left[\prod_{j=1}^{m-1}\left(\frac{D_{i_j}D_{i_{j+1}}}{L_n} \wedge 1\right)\mathbb{1}_{\mathcal{F}_m}\right](1 + o(1)), \qquad (6.43)$$

where $\mathcal{F}_m$ is the event that $D_{i_j} < \varepsilon_n n^{1/(\tau-1)}$ for all $1 \leq j \leq m$. We shall prove by induction that, for every $\vec{\imath}$, and for m even,

$$p_{m,n}(\vec{\imath}) \leq \frac{(C \log n)^{m/2}}{n^{m/2}}. \qquad (6.44)$$

We shall initiate (6.44) by verifying it for $m = 2$ directly, and then advance the induction by relating $p_{m,n}$ to $p_{m-2,n}$.

We start by investigating expectations as in (6.43) iteratively. First, conditionally on $D_{i_{m-1}}$, note that

$$\mathbb{E}\left[\frac{D_{i_{m-1}}D_{i_m}}{L_n} \wedge 1 \Big| D_{i_{m-1}}\right] = \mathbb{P}\left(D_{i_m} > \frac{L_n}{D_{i_{m-1}}} \Big| D_{i_{m-1}}\right) + D_{i_{m-1}}\mathbb{E}\left[\frac{D_{i_m}}{L_n}\mathbb{1}_{\{D_{i_m} \leq L_n/D_{i_{m-1}}\}} \Big| D_{i_{m-1}}\right]$$

$$(6.45)$$

Furthermore,

$$\mathbb{P}\left(D_{i_m} > \frac{L_n}{D_{i_{m-1}}} \Big| D_{i_{m-1}}\right) \leq c_2(D_{i_{m-1}})^{\tau-1}\mathbb{E}\left[(L_n)^{1-\tau} \Big| D_{i_{m-1}}\right]. \qquad (6.46)$$

In a similar way, we obtain using the first bound in Lemma 6.10 together with the fact that $\{D_j\}_{j=1}^n$ is an i.i.d. sequence, that

$$D_{i_{m-1}}\mathbb{E}\left[\frac{D_{i_m}}{L_n}\mathbb{1}_{\{D_{i_m} \leq L_n/D_{i_{m-1}}\}} \Big| D_{i_{m-1}}\right] \leq C D_{i_{m-1}}\mathbb{E}\left[L_n^{-1}(L_n/D_{i_{m-1}})^{2-\tau} \Big| D_{i_{m-1}}\right]$$

$$= C(D_{i_{m-1}})^{\tau-1}\mathbb{E}\left[(L_n)^{1-\tau} \Big| D_{i_{m-1}}\right], \qquad (6.47)$$

where we reach an equal upper bound as above. Thus,

$$\mathbb{E}\left[\frac{D_{i_{m-1}}D_{i_m}}{L_n} \wedge 1 \Big| D_{i_{m-1}}\right] \leq C(D_{i_{m-1}})^{\tau-1}\mathbb{E}\left[(L_n)^{1-\tau} \Big| D_{i_{m-1}}\right]. \qquad (6.48)$$

Now, [8, Lemma 4.1(b)] implies that $\mathbb{E}[(L_n)^{1-\tau}|D_{i_{m-1}}] \leq \mathbb{E}[(L_n - D_{i_{m-1}})^{-(\tau-1)}] \leq c/n$, a.s. so that

$$\mathbb{E}\left[\mathbb{P}\left(D_{i_m} > \frac{L_n}{D_{i_{m-1}}}\mathbb{1}_{\{D_{i_{m-1}} \leq \varepsilon_n n^{1/(\tau-1)}\}} \Big| D_{i_{m-1}}\right)\right] \leq C \log n/n, \qquad (6.49)$$

where, in the inequality, we have used the second inequality in Lemma 6.10 together with the fact that $\{D_j\}_{j=1}^n$ is an i.i.d. sequence. The second term on the right-hand side of (6.45) can be treated similarly, and yields the same upper bound. Putting the two bounds together we arrive at

$$p_{2,n}(i_1, i_2) = \mathbb{E}\left[\left(\frac{D_{i_1}D_{i_2}}{L_n} \wedge 1\right)\mathbb{1}_{\mathcal{F}_2}\right] \leq C \log n/n. \qquad (6.50)$$

which is (6.44) for $m = 2$.

To advance the induction, we need to extend (6.45). Indeed, we use (6.48) to compute that

$$\mathbb{E}\left[\left(\frac{D_{i_{m-2}}D_{i_{m-1}}}{L_n}\wedge 1\right)\cdot\left(\frac{D_{i_{m-1}}D_{i_m}}{L_n}\wedge 1\right)\mathbb{1}_{\mathcal{F}_m}\Big|D_{i_{m-2}}\right] \tag{6.51}$$

$$\leq C'\mathbb{E}\left[\left(\frac{D_{i_{m-2}}D_{i_{m-1}}}{L_n}\wedge 1\right)\left(\frac{D_{i_{m-1}}}{L_n}\right)^{\tau-1}\mathbb{1}_{\mathcal{F}_m}\Big|D_{i_{m-2}}\right]$$

$$= C'\mathbb{E}\left[\left(\frac{D_{i_{m-1}}}{L_n}\right)^{\tau-1}\mathbb{1}_{\{D_{i_{m-1}}>\frac{L_n}{D_{i_{m-2}}}\}}\mathbb{1}_{\mathcal{F}_{m-1}}\Big|D_{i_{m-2}}\right]$$

$$+ C'\mathbb{E}\left[\left(\frac{D_{i_{m-1}}}{L_n}\right)^{\tau}D_{i_{m-2}}\mathbb{1}_{\{D_{i_{m-1}}<\frac{L_n}{D_{i_{m-2}}}\}}\mathbb{1}_{\mathcal{F}_{m-1}}\Big|D_{i_{m-2}}\right],$$

Now using Lemma 6.10 together with the fact that $\{D_j\}_{j=1}^n$ is an i.i.d. sequence, and simplifying, we obtain the following to hold *almost surely*,

$$\mathbb{E}\left[\left(\frac{D_{i_{m-2}}D_{i_{m-1}}}{L_n}\wedge 1\right)\cdot\left(\frac{D_{i_{m-1}}D_{i_m}}{L_n}\wedge 1\right)\mathbb{1}_{\mathcal{F}_m}\Big|D_{i_{m-2}},D_{i_m}\right]\leq C\log n/n. \tag{6.52}$$

This shows that $p_{m,n}\leq (C\log n/n)p_{m-2,n}$, and hence proves (6.44).

Using this estimate in (6.42), and summing over all even m, using the notation that $m=2\mathbb{Z}^+$, shows that

$$\begin{aligned}\mathbb{P}(\mathcal{B}^{(n)}) &\leq \sum_{m=2\mathbb{Z}^+}\mathbb{P}(\mathcal{B}_m^{(n)})\leq\sum_{m=2\mathbb{Z}^+}\frac{C\varepsilon_n^{-2(\tau-1)}w_n^{m+1}}{(m+1)!n^{(m+1)/2}}\sum_{\vec{\iota}}p_{m,n}(\vec{\iota})\\ &\leq \sum_{m=2\mathbb{Z}^+}\frac{C\varepsilon_n^{-2(\tau-1)}w_n^{m+1}}{(m+1)!n^{(m+1)/2}}(Cn\log n)^{m/2}=C\varepsilon_n^{-2(\tau-1)}n^{-1/2}\sum_{k=1}^{\infty}\frac{w_n^{2k+1}(C\log n)^k}{(2k+1)!}\\ &\leq C\frac{\varepsilon_n^{-2(\tau-1)}}{n^{1/2}\sqrt{\log n}}e^{Cw_n\sqrt{\log n}}. \end{aligned} \tag{6.53}$$

∎

Lemma 6.11 shows that with the correct choice of ε_n, we find that $\mathbb{P}(H_n\notin 2\mathbb{Z}^+)=o(1)$, and to prove Theorem 3.2, we shall show that the shortest-weight paths between any two specific super vertices alternate between super vertices and normal vertices. We will prove this statement, in Lemma 6.13 below, by showing that the probability that a vertex with index at least K is used at an even place, is for K large, quite small. This shows in particular that, **whp**, at all even places we have super vertices. In the following lemma, we collect the properties of the degrees and erased degrees that we shall make use of in the sequel. In its statement, we define

$$\mathcal{G}^{(n)}=\mathcal{G}_1^{(n)}\cap\mathcal{G}_2^{(n)}\cap\mathcal{G}_3^{(n)}, \tag{6.54}$$

where, for $a\in(0,1)$ and $C,C^{\mathbf{er}}>0$, we let

$$\mathcal{G}_1^{(n)}=\left\{L_nn^{-1/(\tau-1)}\in[a,a^{-1}]\right\}, \tag{6.55}$$

$$\mathcal{G}_2^{(n)}=\left\{C^{-1}(n/i)^{1/(\tau-1)}\leq D_{(n+1-i:n)}\leq C(n/i)^{1/(\tau-1)},\forall i\in[n]\right\}, \tag{6.56}$$

$$\mathcal{G}_3^{(n)}=\left\{D_i^{\mathbf{er}}\leq C^{\mathbf{er}}(n/i),\forall i\in[n]\right\}. \tag{6.57}$$

The event $\mathcal{G}^{(n)}$ is the *good event* that we shall work with. We shall first show that, if we take $a>0$ sufficiently small and $C,C^{\mathbf{er}}$ sufficiently large, then $\mathbb{P}(\mathcal{G}^{(n)})$ is close to 1:

Lemma 6.12 (The good event has high probability) *For every $\varepsilon > 0$, there exist $a > 0$ sufficiently small and C, C^{er} sufficiently large such that*

$$\mathbb{P}(\mathcal{G}^{(n)}) \geq 1 - \varepsilon. \tag{6.58}$$

Proof. We split

$$\mathbb{P}\big((\mathcal{G}^{(n)})^c\big) = \mathbb{P}\big((\mathcal{G}_1^{(n)})^c\big) + \mathbb{P}\big((\mathcal{G}_2^{(n)})^c\big) + \mathbb{P}\big(\mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)} \cap (\mathcal{G}_3^{(n)})^c\big), \tag{6.59}$$

and bound each term separately. We can make $\mathbb{P}\big((\mathcal{G}_1^{(n)})^c\big) \leq \varepsilon/3$ by choosing $a > 0$ sufficiently small by the weak convergence in (2.5).

To bound $\mathbb{P}\big(\mathcal{G}_1^{(n)} \cap (\mathcal{G}_2^{(n)})^c\big)$, we note that $D_{(n+1-l:n)} > C(n/l)^{1/(\tau-1)}$ is equivalent to the statement that the number of values i such that $D_i > C(n/l)^{1/(\tau-1)}$ is at least l. Since $\{D_i\}_{i=1}^n$ is an i.i.d. sequence, this number has a Binomial distribution with parameters n and success probability

$$q_{l,n} = [1 - F(C(n/l)^{1/(\tau-1)})] \leq c_2 C^{-(\tau-1)} l/n, \tag{6.60}$$

by (3.6). When the mean of this binomial, which is $c_2 C^{-(\tau-1)} l$ is much smaller than l, which is equivalent to $C > 0$ being large, the probability that this binomial exceeds l is exponentially small in l:

$$\mathbb{P}(D_{(n+1-l:n)} > C(n/l)^{1/(\tau-1)}) \leq \mathrm{e}^{-I(C)l}, \tag{6.61}$$

where $I(C) \to \infty$ when $C \to \infty$. Thus, by taking C sufficiently large, we can make the probability that there exists an l for which $D_{(n+1-l:n)} > C(n/l)^{1/(\tau-1)}$ small. In more detail,

$$\mathbb{P}\Big(\exists l : D_{(n+1-l:n)} > C(n/l)^{1/(\tau-1)}\Big) \leq \sum_{l \in [n]} \mathbb{P}(D_{(n+1-l:n)} > C(n/l)^{1/(\tau-1)}) \leq \sum_{l \in [n]} \mathrm{e}^{-I(C)l} \leq \varepsilon/3, \tag{6.62}$$

when we make $C > 0$ sufficiently large. In a similar way, we can show that the probability that there exists l such that $D_{(n+1-l:n)} \leq C^{-1}(n/l)^{1/(\tau-1)}$ is small when $C > 0$ is large.

In order to bound $\mathbb{P}\big(\mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)} \cap (\mathcal{G}_3^{(n)})^c\big)$ we need to investigate the random variable D_i^{er}. We claim that there exists a $R = R(a, C, C^{\mathrm{er}})$ with $R(a, C, C^{\mathrm{er}}) \to \infty$ as $C^{\mathrm{er}} \to \infty$ for each fixed $a, C > 0$, such that

$$\mathbb{P}\big(D_i^{\mathrm{er}} \geq C^{\mathrm{er}} j^{\tau-1} | D_i = j, \mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)}\big) \leq \mathrm{e}^{-Rj^{\tau-1}}. \tag{6.63}$$

Fix $C^{\mathrm{er}} > 0$. In order for $D_i^{\mathrm{er}} \geq C^{\mathrm{er}} j^{\tau-1}$ to occur, we must have that at least $C^{\mathrm{er}} j^{\tau-1}/2$ of the neighbors of vertex i have index at least $C^{\mathrm{er}} j^{\tau-1}/2$, where we recall that vertex i is such that $D_i = D_{(n+1-i:n)}$ is the ith largest degree. The j neighbors of vertex i are close to being independent, and the probability that any of them connects to a vertex with index at least k is, conditionally on the degrees $\{D_i\}_{i=1}^n$, equal to

$$\sum_{l \geq k} D_{(n+1-l:n)}/L_n. \tag{6.64}$$

When $\mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)}$ holds, then $D_{(n+1-l:n)}/L_n \leq (C/a) l^{-1/(\tau-1)}$, so that

$$\sum_{l \geq k} D_{(n+1-l:n)}/L_n \leq c(C/a) k^{-(2-\tau)/(\tau-1)}. \tag{6.65}$$

As a result, we can bound the number of neighbors of vertex i by a binomial random variable with $p = c' k^{-(2-\tau)/(\tau-1)}$, where $k = C^{\mathrm{er}} j^{\tau-1}/2$, i.e.,

$$\mathbb{P}\big(D_i^{\mathrm{er}} \geq C^{\mathrm{er}} j^{\tau-1} | D_i = j, \mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)}\big) \leq \mathbb{P}\big(\mathrm{Bin}(j, c(C/a) j^{-(2-\tau)}) \geq C^{\mathrm{er}} j^{\tau-1}/2\big). \tag{6.66}$$

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Next, we note that the mean of the above binomial random variable is given by $c(C/a)j^{1-(2-\tau)} = c(C/a)j^{\tau-1}$. A concentration result for binomial random variables [19], yields that for $C > 0$ sufficiently large,

$$\mathbb{P}\big(D_i^{\mathrm{er}} \geq C^{\mathrm{er}}j^{\tau-1}|D_i = j, \mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)}\big) \leq \mathrm{e}^{-Rj^{\tau-1}}. \tag{6.67}$$

This proves (6.63). Taking $C^{\mathrm{er}} > 0$ sufficiently large, we obtain that

$$\mathbb{P}\big(\mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)} \cap (\mathcal{G}_3^{(n)})^c\big) \leq \sum_{i=1}^{n}\sum_{j} \mathbb{P}\big(D_i^{\mathrm{er}} \geq C^{\mathrm{er}}j^{\tau-1}|D_i = j, \mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)}\big)\mathbb{P}(D_i = j, \mathcal{G}_1^{(n)} \cap \mathcal{G}_2^{(n)})$$

$$\leq \sum_{i=1}^{n} \mathrm{e}^{-RCn/i} \leq \varepsilon/3, \tag{6.68}$$

where we use the fact that $D_i \geq C^{-1}(n/i)^{1/(\tau-1)}$ since the event $\mathcal{G}_2^{(n)}$ occurs, and where, in the last step, we use the fact that, for each $a, C > 0$, we can make $R(a, C, C^{\mathrm{er}})$ large by taking C^{er} sufficiently large. ∎

Now we are ready to prove the tightness of the FPP problem. In the statement below, we let $\mathcal{A}_{m,K}^{(n)}(i,j)$ be the event that there exists a path of length $2m$ connecting i and j of weight at most $W/\sqrt{n}$ that leaves $[K]$, and we write for k fixed,

$$\mathcal{A}_{m,K}^{(n)} = \bigcup_{i,j\in[k]} \mathcal{A}_{m,K}^{(n)}(i,j), \qquad \mathcal{A}_K^{(n)} = \bigcup_{m=1}^{\infty} \mathcal{A}_{m,K}^{(n)}. \tag{6.69}$$

Lemma 6.13 (Tightness of even shortest-weight paths on the super vertices) *Fix $k, K \in \mathbb{Z}^+$ and $i, j \in [k]$. Then, there exists a $C > 0$ such that*

$$\mathbb{P}(\mathcal{A}_{m,K}^{(n)} \cap \mathcal{G}^{(n)}) \leq CWK^{-1}\mathrm{e}^{CW\sqrt{\log K}}. \tag{6.70}$$

Proof. We follow the same line of argument as in the proof of Lemma 6.11, but we need to be more careful in estimating the expected number of paths between the super vertices i and j. For $m \geq 2$ and $\vec{\iota} = (i_1, \ldots, i_{m-1})$, let $q_{m,n}(\vec{\iota})$ be the expected number paths with $2m$ edges ($2m$ step paths) such that the position of the path at time $2k$ is equal to i_k, where, by convention, $i_0 = i$ and $i_m = j$. Then, similarly as in (6.42) but note that now $q_{m,n}(\vec{\iota})$ is an expectation and not a probability, we have that

$$\mathbb{P}(\mathcal{A}_{m,K}^{(n)} \cap \mathcal{G}^{(n)}) \leq \frac{CW^{2m}}{(2m)!n^m} \sum_{\vec{\iota}} q_{m,n}(\vec{\iota}). \tag{6.71}$$

Observe that

$$q_{m,n}(\vec{\iota}) \leq \mathbb{E}[\prod_{s=1}^{m} N_{i_{s-1}i_s}^{\mathrm{er}} \mathbb{1}_{\{\mathcal{G}^{(n)}\}}]. \tag{6.72}$$

We further note that, by Lemma 6.12 and on $\mathcal{G}^{(n)}$,

$$N_{ij} \leq D_i^{\mathrm{er}} \wedge D_j^{\mathrm{er}} \leq D_{i\vee j}^{\mathrm{er}} \leq C^{\mathrm{er}}n/(i \vee j), \tag{6.73}$$

where we abbreviate, for $x, y \in \mathbb{R}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Thus, by (6.72), we arrive at

$$q_{m,n}(\vec{\iota}) \leq \prod_{s=1}^{m} C^{\mathrm{er}}n/(i_s \vee i_{s-1}), \tag{6.74}$$

and hence, after possibly enlarging C^{er},

$$\mathbb{P}(\mathcal{A}_{m,K}^{(n)} \cap \mathcal{G}^{(n)}) \leq \frac{CW^{2m}}{(2m)!n^m} \sum_{\vec{\iota}} \prod_{s=1}^{m} \frac{C^{\mathrm{er}}n}{i_s \vee i_{s-1}} = \frac{(C^{\mathrm{er}})^m W^{2m}}{(2m)!} \sum_{\vec{\iota}} \prod_{s=1}^{m} \frac{1}{i_s \vee i_{s-1}}, \tag{6.75}$$

where the sum over $\vec{\imath}$ is such that there exists at least one s such that $i_s > K$, because the path is assumed to leave $[K]$. We now bound (6.75). Let $1 \leq t \leq m$ be such that $i_t = \max_{s=1}^{m} i_s$, so that $i_t > K$. Then, using that both i_s and i_{s-1} are smaller than $i_s \vee i_{s-1}$, we can bound

$$\prod_{s=1}^{m} \frac{1}{i_s \vee i_{s-1}} = \Big(\prod_{s=1}^{t-1} \frac{1}{i_s \vee i_{s-1}}\Big) \frac{1}{i_{t-1} \vee i_t} \frac{1}{i_t \vee i_{t+1}} \Big(\prod_{s=t+2}^{m} \frac{1}{i_s \vee i_{s-1}}\Big) \leq \frac{1}{i_t^2} \prod_{s=1}^{t-1} \frac{1}{i_s} \prod_{s=t+2}^{m} \frac{1}{i_{s-1}}. \qquad (6.76)$$

Thus,

$$\sum_{\vec{\imath}} \prod_{s=1}^{m} \frac{1}{i_s \vee i_{s-1}} = \sum_{t=1}^{m} \sum_{i_t > K} \frac{1}{i_t^2} \sum_{i_1,\dots,i_{t-1} \leq i_t} \prod_{s=1}^{t-1} \frac{1}{i_s} \sum_{i_{t+1},\dots,i_{m-1} \leq i_t} \prod_{s=t+2}^{m} \frac{1}{i_{s-1}} = m \sum_{u > K} \frac{1}{u^2} h_u^{m-2}, \qquad (6.77)$$

where

$$h_u = \sum_{v=1}^{u} \frac{1}{v}. \qquad (6.78)$$

We arrive at

$$\mathbb{P}(\mathcal{A}_{m,K}^{(n)} \cap \mathcal{G}^{(n)}) \leq \frac{(C^{\mathrm{er}})^m W^{2m}}{(2m)!} m \sum_{u > K} \frac{1}{u^2} h_u^{m-1}. \qquad (6.79)$$

By Boole's inequality and (6.79), we obtain, after replacing C^{er} by C, that

$$\mathbb{P}(\mathcal{A}_K^{(n)} \cap \mathcal{G}^{(n)}) \leq \sum_{m=2}^{\infty} \mathbb{P}(\mathcal{A}_{m,K}^{(n)}) \leq \sum_{m=2}^{\infty} \frac{C^m W^{2m}}{(2m)!} m \sum_{u > K} \frac{1}{u^2} h_u^{m-1}$$

$$\leq W \sum_{u > K} \frac{1}{u^2} \sum_{m=2}^{\infty} h_u^{(2m-1)/2} \frac{C^{2m-1} W^{2m-1}}{(2m-1)!}$$

$$\leq CW \sum_{u > K} \frac{1}{u^2} e^{CW\sqrt{h_u}} \leq CWK^{-1} e^{CW\sqrt{\log K}}, \qquad (6.80)$$

where we used that

$$\sum_{u > K} \frac{1}{u^2} e^{CW\sqrt{h_u}} \leq \int_{\log K}^{\infty} e^{-y + CW\sqrt{y+c}}\, dy$$

$$\leq \int_{\log K}^{\infty} \exp\Big\{-y\Big(1 - \frac{2CW}{\sqrt{\log K}}\Big)\Big\}\, dy \leq K^{-1} e^{C'W\sqrt{\log K}}, \qquad (6.81)$$

for some $c, C' > 0$. This completes the proof of Lemma 6.13. ∎

Now we are ready to complete the proof of Proposition 6.3:

Proof of Proposition 6.3. We write $H_n(i,j)$ and $W_n(i,j)$ for the number of edges and weight of the shortest-weight path between the super vertices $i, j \in [k]$.

(a) The fact that $\mathbb{P}(H_n(i,j) \notin 2\mathbb{Z}^+) = o(1)$ for any super vertices i, j, follows immediately from Lemma 6.11, which implies that the even length path between i and j is a two-edge path **whp**. The tightness of $H_n(i,j)$ follows from part (b), which we prove next.

(b) By Proposition 6.1, the rescaled weight $\sqrt{n} W_n(i,j) \leq l_{ij}^{(n)}$ is a tight sequence of random variables, so that, for W large, it is at most W with probability converging to 1 when $W \to \infty$. Fix $\varepsilon > 0$ arbitrary. Then, fix $W > 0$ sufficiently large such that the probability that $\sqrt{n} W_n(i,j) > W$ is at most $\varepsilon/3$, $K > 0$ such that $CWK^{-1} e^{CW\sqrt{\log K}} < \varepsilon/3$, and, use Lemma 6.12 to see that we can choose a, C, C^{er} such that $\mathbb{P}(\mathcal{G}^{(n)}) \geq 1 - \varepsilon/3$. Then, by Lemma 6.13, the probability that this two-edge path leaves $[K]$ is at most $\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. This completes the proof of (b).

(c) The proof that $\mathbb{P}(H_n \notin 2\mathbb{Z}^+) = o(1)$ follows from (a) since, for k large, **whp**, A_1 and A_2 are exclusively attached to super vertices in $[k]$. The tightness of H_n also follows from this argument and (a).
∎

6.5 The FPP on $\mathcal{K}_\infty^{\mathrm{er}}$ is well defined: Proof of Proposition 6.4

In this section, we prove Proposition 6.4. For this, we start by investigating $f(P_i, P_j)$ for large i, j. The main result is contained in the following lemma:

Lemma 6.14 (Asymptotics for $f(P_i, P_j)$ for large i, j) *Let η be a stable random variable with parameter $\tau - 1 \in (0, 1)$. Then, there exists a constant $c > 0$ such that, as $i \wedge j \to \infty$,*

$$f(P_i, P_j) \leq \frac{c\eta^{1-\tau}}{i \vee j}, \quad a.s. \tag{6.82}$$

Proof. We note that, by (2.3) and (2.4) and the strong law of large numbers that, as $i \to \infty$,

$$\eta P_i i^{1/(\tau-1)} = (i/\Gamma_i)^{1/(\tau-1)} \xrightarrow{a.s.} 1. \tag{6.83}$$

Further, by (3.8),

$$f(s, t) = 1 - \mathbb{E}[(1-s)^D] - \mathbb{E}[(1-t)^D] + \mathbb{E}[(1-s-t)^D] \leq 1 - \mathbb{E}[(1-s)^D] \leq cs^{\tau-1}, \tag{6.84}$$

since, for $\alpha = \tau - 1$, and D in the domain of attraction of an α-stable random variable, we have that, as $u \downarrow 0$,

$$\mathbb{E}[\mathrm{e}^{-uD}] = \mathrm{e}^{-cu^\alpha(1+o(1))} = 1 - cu^\alpha(1 + o(1)). \tag{6.85}$$

Combing these asymptotics proves (6.82). ∎

Proof of Proposition 6.4. Let $\mathcal{A}_{m,K}$ be the event that there exists a path of length m and weight at most W connecting i and j and which contains a vertex in $\mathbb{Z}^+ \setminus [K]$. Then, by Boole's inequality and the conditional independence of the weights $\{l_{ij}\}_{1 \leq i \leq j < \infty}$, we obtain that

$$\mathbb{P}(\mathcal{A}_{m,K}) \leq \sum_{\vec{\iota}} \mathbb{P}(\sum_{s=1}^m l_{i_{s-1}i_s} \leq W), \tag{6.86}$$

where, as in the proof of Lemma 6.13, the sum over $\vec{\iota}$ is over $\vec{\iota} = (i_1, \ldots, i_{m-1})$, where, by convention, $i_0 = i$ and $i_m = j$, and $\max_{s=1}^m i_s \geq K$. Now, by the conditional independence of $\{l_{ij}\}_{1 \leq i < j < \infty}$,

$$\begin{aligned} \mathbb{P}(\sum_{s=1}^m l_{i_{s-1}i_s} \leq W | \{P_i\}_{i \geq 1}) &= \int_{x_1 + \cdots + x_m \leq W} \prod_{s=1}^m f(P_{i_{s-1}}, P_{i_s}) x_s \mathrm{e}^{-f(P_{i_{s-1}}, P_{i_s})x_s^2/2} dx_1 \cdots dx_m \\ &\leq \prod_{s=1}^m f(P_{i_{s-1}}, P_{i_s}) \int_{x_1 + \cdots + x_m \leq W} x_1 \cdots x_m dx_1 \cdots dx_m \\ &= \frac{W^{2m}}{(2m)!} \prod_{s=1}^m f(P_{i_{s-1}}, P_{i_s}), \end{aligned} \tag{6.87}$$

by [11, 4.634]. We have that $f(P_i, P_j) \leq c\eta^{1-\tau}(i \vee j)^{-1}$, *a.s.*, by Lemma 6.14. The random variable η has a stable distribution, and is therefore **whp** bounded above by C for some $C > 0$ sufficiently large. The arising bound is identical to the bound (6.75) derived in the proof of Lemma 6.13, and we can follow the proof to obtain (6.5). ∎

7 Robustness and fragility: Proof of Theorem 3.3

We start by proving Theorem 3.3(a), for which we note that whatever the value of $p \in (0, 1)$, **whp**, not all super vertices will be deleted. The number of undeleted vertices that are connected to a kept super vertex will be $\Theta_{\mathbb{P}}(n)$, which proves the claim. In fact, we now argue that a stronger result holds. We note that the size of the giant component is the same wether we consider $\mathcal{G}_n^{\mathrm{er}}$ or $\mathcal{G}_n^{\mathrm{or}}$. It is easy to prove the following result:

Theorem 7.1 (Giant component after random attack) *Consider either $\mathcal{G}_n^{\mathrm{or}}$ or $\mathcal{G}_n^{\mathrm{er}}$ and leave each vertex with probability p or delete it with probability $(1-p)$. The resulting graph (of vertices which are left) has a unique giant component $\mathcal{C}_n(p)$. Further, with $|\mathcal{C}_n(p)|$ denoting the number of vertices in $\mathcal{C}_n(p)$,*

$$\frac{\mathbb{E}[|\mathcal{C}_n(p)|]}{n} \longrightarrow p\mathbb{E}[1-(1-p)^{D^{\mathrm{er}}}] = \lambda(p), \qquad Var\left(\frac{|\mathcal{C}_n(p)|}{n}\right) \to \beta(p) > 0. \tag{7.1}$$

Unlike for other random graph models, (7.1) suggests that $|\mathcal{C}_n(p)|/n \xrightarrow{d} Z_p$, where Z_p is a *non-degenerate* random variable. We shall however not attempt to prove the latter statement here.

Sketch of proof: Note that we have the identity

$$\frac{\mathbb{E}[|\mathcal{C}_n(p)|]}{n} = \mathbb{P}(1 \in \mathcal{C}_n(p)),$$

where 1 is a uniformly chosen vertex in $\mathcal{G}_n^{\mathrm{er}}$. For large n, the vertex 1 being in the giant component is *essentially* equivalent to the following two conditions:

(i) Vertex 1 is not deleted; this happens with probability p.

(ii) Vertex 1 is attached to D^{er} super vertices. If one of those super vertices is not deleted, then the component of this super vertex is of order n and thus has to be the giant component. Thus at least one of the super vertices to which 1 is attached should remain undeleted; conditionally on D^{er}, this happens with probability $1-(1-p)^{D^{\mathrm{er}}}$. Combining (i) and (ii) gives the result. A calculation, using similar ideas as in the proof of Lemma 6.12, suggests that $\lambda(p) = \Theta(p^2)$ when $p \downarrow 0$. Further, the giant component is unique, since any pair of super vertices which are kept are connected top each other, and are each connected to $\Theta_{\mathbb{P}}(n)$ other vertices.

To prove the convergence of the variance, we note that

$$\mathrm{Var}(|\mathcal{C}_n(p)|) = \sum_{i,j} \left[\mathbb{P}(i,j \in \mathcal{C}_n(p)) - \mathbb{P}(i \in \mathcal{C}_n(p))\mathbb{P}(j \in \mathcal{C}_n(p)) \right]. \tag{7.2}$$

Thus,

$$\mathrm{Var}(|\mathcal{C}_n(p)|/n) = \mathbb{P}(1,2 \in \mathcal{C}_n(p)) - \mathbb{P}(1 \in \mathcal{C}_n(p))\mathbb{P}(2 \in \mathcal{C}_n(p)), \tag{7.3}$$

where $1,2$ are two independent uniform vertices in $[n]$. Now,

$$\mathbb{P}(1,2 \in \mathcal{C}_n(p)) = p^2\mathbb{P}(1,2 \in \mathcal{C}_n(p)|1,2 \text{ kept}) + o(1), \tag{7.4}$$

and

$$\mathbb{P}(1,2 \in \mathcal{C}_n(p)|1,2 \text{ kept}) = 1 - \mathbb{P}(\{1 \notin \mathcal{C}_n(p)\} \cup \{2 \notin \mathcal{C}_n(p)\}|1,2 \text{ kept}) \tag{7.5}$$
$$= 1 - \mathbb{P}(1 \notin \mathcal{C}_n(p)|1,2 \text{ kept}) - \mathbb{P}(2 \notin \mathcal{C}_n(p)|1,2 \text{ kept}) + \mathbb{P}(1,2 \notin \mathcal{C}_n(p)|1,2 \text{ kept})$$
$$= 1 - \mathbb{P}(1 \notin \mathcal{C}_n(p)|1 \text{ kept}) - \mathbb{P}(2 \notin \mathcal{C}_n(p)|2 \text{ kept}) + \mathbb{P}(1,2 \notin \mathcal{C}_n(p)|1,2 \text{ kept}) + o(1),$$

so that

$$\mathrm{Var}(|\mathcal{C}_n(p)|/n) = p^2\mathbb{P}(1,2 \notin \mathcal{C}_n(p)|1,2 \text{ kept}) - p^2\mathbb{P}(1 \notin \mathcal{C}_n(p)|1 \text{ kept})\mathbb{P}(2 \notin \mathcal{C}_n(p)|2 \text{ kept}) + o(1). \tag{7.6}$$

Then, we compute that

$$\mathbb{P}(1 \notin \mathcal{C}_n(p)|1 \text{ kept}) = \mathbb{E}[(1-p)^{D_1^{\mathrm{er}}}], \tag{7.7}$$

while

$$\mathbb{P}(1,2 \notin \mathcal{C}_n(p)|1,2 \text{ kept}) = \mathbb{E}[(1-p)^{D_1^{\mathrm{er}}+D_2^{\mathrm{er}}-N_{12}^{\mathrm{er}}}], \tag{7.8}$$

where $D_1^{\mathrm{er}}, D_2^{\mathrm{er}}$ are conditionally independent given $\{P_i\}_{i\geq 1}$, and N_{12}^{er} denotes the number of joint neighbors of 1 and 2, and we use that the total number of super vertices to which 1 and 2 are connected is equal to $D_1^{\mathrm{er}} + D_2^{\mathrm{er}} - N_{12}^{\mathrm{er}}$. As a result,

$$\mathrm{Var}(|\mathcal{C}_n(p)|/n) = p^2\left(\mathbb{E}[(1-p)^{D_1^{\mathrm{er}}+D_2^{\mathrm{er}}-N_{12}^{\mathrm{er}}}] - \mathbb{E}[(1-p)^{D_1^{\mathrm{er}}}]\mathbb{E}[(1-p)^{D_2^{\mathrm{er}}}] \right) + o(1), \tag{7.9}$$

which identifies
$$\beta(p) = p^2 \left(\mathbb{E}[(1-p)^{D_1^{\mathtt{er}}+D_2^{\mathtt{er}}-N_{12}^{\mathtt{er}}}] - \mathbb{E}[(1-p)^{D_1^{\mathtt{er}}}]\mathbb{E}[(1-p)^{D_2^{\mathtt{er}}}] \right). \tag{7.10}$$

To see that $\beta(p) > 0$, we note that $N_{12}^{\mathtt{er}} > 0$ with positive probability, so that

$$\mathbb{E}[(1-p)^{D_1^{\mathtt{er}}+D_2^{\mathtt{er}}-N_{12}^{\mathtt{er}}}] > \mathbb{E}[(1-p)^{D_1^{\mathtt{er}}+D_2^{\mathtt{er}}}] = \mathbb{E}\Big[\mathbb{E}[(1-p)^{D_1^{\mathtt{er}}+D_2^{\mathtt{er}}} \mid \{P_i\}_{i\geq 1}]\Big] = \mathbb{E}\Big[\mathbb{E}\big[(1-p)^{D_1^{\mathtt{er}}} \mid \{P_i\}_{i\geq 1}\big]^2\Big],$$

by the conditional independence of $D_1^{\mathtt{er}}$ and $D_2^{\mathtt{er}}$. Thus, $\beta(p) > 0$ by the Cauchy-Schwarz inequality, as claimed. ∎

To prove Theorem 3.3(b), we again use that a uniform vertex is, **whp**, only connected to a super vertex. Thus, there exists K_ε such that by deleting the K_ε vertices with largest degree, we shall isolate A_1 with probability at least ε. This proves (3.14). ∎

8   Conclusion

We conclude with a discussion about various extensions of the above results together with some further results without proof. Throughout the discussion we shall use $\mathcal{G}_n$ to denote either of $\mathcal{G}_n^{\mathtt{or}}$ and $\mathcal{G}_n^{\mathtt{er}}$, where the choice depends on the context under consideration.

(a) **Load distribution:** Understanding how random disorder changes the geometry of the network is crucial for understanding asymptotics of more complicated constructs such as the load distribution. More precisely, for any pair of vertices $i, j \in \mathcal{G}_n$, let $\boldsymbol{\pi}(i,j)$ denote the minimal weight path between the two vertices. For any vertex $v \in \mathcal{G}_n$, the *load* on the vertex is defined as

$$L_n(v) = \sum_{i \neq j} \mathbb{1}_{\{v \in \boldsymbol{\pi}(i,j)\}}.$$

For any fixed x, define the function $G_n(x)$ as

$$G_n(x) = \#\{v : L_n(v) > x\}.$$

Understanding such functions is paramount to understanding the flow carrying properties of the network and are essential for the study of *betweenness centrality* of vertices in a network. For example in social networks, such measures are used to rate the relative importance of various individuals in the network, while in data networks such as the World-Wide Web, such measures are used to rank the relative importance of web pages. An actual theoretical analysis of such questions is important but seems difficult in many relevant situations. It would be of interest to find asymptotics of such functions in terms of the infinite objects $\mathcal{K}_\infty^{\mathtt{or}}$ and $\mathcal{K}_\infty^{\mathtt{er}}$ constructed in this paper. See also [2] for an analysis of such questions in the mean-field setting.

(b) **Universality for edge weights:** In this study, to avoid technical complications we assumed that each edge weight in $\mathcal{G}_n$ has an exponential distribution. One natural question is how far do these results depend on this assumption. It is well known in probabilistic combinatorial optimization that in a wide variety of contexts, when considering problems such as those in this paper, the actual distribution of the edge weights is not that important, what is important is the value of the density at 0. More precisely, consider $\mathcal{G}_n^{\mathtt{er}}$ (i.e., the erased CM) where each edge is given an i.i.d. edge weight having a continuous distribution with density g and let $g(0) = \zeta \in (0, \infty)$. Similar to $\mathcal{K}_\infty^{\mathtt{er}}$ defined in Section 3.2, define $\mathcal{K}_\infty^{\mathtt{er}}(\zeta)$ to be the infinite graph on the vertex set $\mathbb{Z}^+$ where each edge l_{ij} has the distribution

$$\mathbb{P}(l_{ij} > x) = \exp\left(-f(P_i, P_j)\zeta^2 x^2/2\right). \tag{8.1}$$

Equation (8.1) can be proved along similar lines as in the proof of Lemma 6.5, and we leave this to the reader.

Let $I^{\mathtt{er}}$ be as defined in Section 3.2. Then we have the following modification of Theorem 3.2 which can be proved along the same lines:

Theorem 8.1 (Extension to other densities) *Theorem 3.2 continues to hold with the modification that the quantities $W_{ij}^{\mathrm{er}}, H_{ij}^{\mathrm{er}}$ arising in the limits are replaced by the corresponding quantities in $\mathcal{K}_\infty^{\mathrm{er}}(\zeta)$ instead of $\mathcal{K}_\infty^{\mathrm{er}}$, V_i^{er} is distributed as the minimum of D^{er} random variables having density g, while the distributions of Ier and J^{er} remain unchanged.*

A more challenging extension would be to densities for which either $g(0) = 0$, or for which $\lim_{x\downarrow 0} g(x) = \infty$. In this case, we believe the behavior to be entirely different from the one in Theorems 3.2 and 8.1, and it would be of interest to investigate whether a similar limiting FPP process arises.

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