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L. Xu
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ERGODICITY OF INFINITE WHITE α-STABLE SYSTEMS WITH LINEAR AND BOUNDED INTERACTIONS

LIHU XU

Abstract. We proved the existence of an infinite dimensional stochastic system driven by white α-stable noises ($1 < \alpha \le 2$), and prove this system is strongly mixing. Our method is by perturbing Ornstein-Uhlenbeck α-stable processes.

Key words and phrases. Ergodicity, Ornstein-Uhlenbeck α-stable processes, spin systems, finite speed of propagation of interactions.
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1. Introduction

We shall study an infinite dimensional spin system with linear and bounded interactions, driven by (white) α-stable noises. More precisely, our system is described by the following SDEs: for every $i \in \mathbb{Z}^d$,

$$\begin{cases} dX_i(t) = [\sum_{j \in \mathbb{Z}^d} a_{ij} X_j(t) + U_i(X(t))]dt + dZ_i(t) \\ X_i(0) = x_i \end{cases} \tag{1.1}$$

where $X_i, x_i \in \mathbb{R}$, $\{Z_i; i \in \mathbb{Z}^d\}$ are a sequence of i.i.d. standard symmetric α-stable process with $1 < \alpha \le 2$, and the assumptions for the a and U are specified in Assumption 1.2.

When $Z(t)$ is Wiener noise, the equation (1.1) has been intensively studied to model some phenomena in physics such as quantum spin systems since the 90s of last century ([1], [2], [6], [8], [3],[9], [10], [11], $\cdots$). The other motivation to study (1.1) is from the work by Zegarlinski on interacting unbounded spin systems driven by Wiener noise ([15]). In that paper, the author proved the following uniform ergodicity $||P_t f - \mu(f)||_\infty \le C(f)e^{-mt}$, where P_t is the semigroup generated by a reversible generator and μ is the ergodic measure of P_t. This type of ergodicity is very strong, and obtained by a logarithmic Sobolev inequality (LSI). Unfortunately, the LSI is not available in our set-up, however, we can find a *gradient* decay estimates, which is crucial in the proof of ergodic theorem. We should point out that our ergodicity result is strongly mixing type, which is much weaker than the uniform one.

The approach of this paper is via perturbing the Ornstein-Uhlenbeck α-stable process, this needs one to know some exact formula of this process. Comparing with the above perturbation approach, the main tools in [14] are some iterations under the framework of probability, one only uses the α-stable property and the moments of the stable processes. Hence, we can think that [14] is some *qualitative* analysis, while this paper is some *quantitative* one.

The organization of the paper is as follows. The introduction includes notations, main results and some preliminary about Ornstein-Uhlenbeck α-stable processes, the second and third sections prove the existence and ergodicity results respectively. The short appendix gives a simple but interesting derivation of (1.5).

1.1. **Notations, Assumptions and Main Results.** We shall study the system (1.1) on $\mathbb{B} \subset \mathbb{R}^{\mathbb{Z}^d}$, which is defined by

$$\mathbb{B} = \bigcup_{R>0,\rho>0} B_{R,\rho}$$

where for any $R, \rho > 0$

$$B_{R,\rho} = \{x = (x_i)_{i\in\mathbb{Z}^d}; |x_i| \le R(|i|^\rho + 1)\} \quad \text{with} \quad |i| = \sum_{k=1}^{d} |i_k|.$$

We shall see that given any initial data $x = (x_i)_{i\in\mathbb{Z}^d} \in \mathbb{B}$, the dynamics $X(t)$ defined in (1.1) evolves in $\mathbb{B}$ almost surely.

Remark 1.1. One can also check that the distributions of the standard white α-stable processes $(Z_i(t))_{i\in\mathbb{Z}^d}$ $(0 < \alpha \le 2)$ at any fixed time t are supported on $\mathbb{B}$. From the form of the equation (1.1), one can expect that the distributions of the system at any fixed time t is similar to those of α-stable processes but with some (complicated) shifts. Hence, it is *natural* to study (1.1) on $\mathbb{B}$.

Let us first list some notations which will be frequently used in the paper and then give the detailed assumption on a and U.

- Define $|i-j| = \sum_{1\le k\le d} |i_k - j_k|$ for any $i, j \in \mathbb{Z}^d$, define $|\Lambda| = \sharp\Lambda$ for any finite sublattice $\Lambda \subset\subset \mathbb{Z}^d$.
- For the national simplicity, we shall write $\partial_i := \partial_{x_i}$, $\partial_{ij} := \partial^2_{x_i x_j}$ and $\partial^\alpha_i := \partial^\alpha_{x_i}$. It is easy to see that $[\partial^\alpha_i, \partial_j] = 0$ for all $i, j \in \mathbb{Z}^d$.
- For any finite sublattice $\Lambda \subset\subset \mathbb{Z}^d$, let $C_b(\mathbb{R}^\Lambda, \mathbb{R})$ be the bounded continuous function space from $\mathbb{R}^\Lambda$ to $\mathbb{R}$, denote $\mathcal{D} = \bigcup_{\Lambda\subset\subset\mathbb{Z}^d} C_b(\mathbb{R}^\Lambda, \mathbb{R})$ and

  $$\mathcal{D}^k = \{f \in \mathcal{D}; f \text{ has bounded } 0, \cdots, \ kth \text{ order derivatives}\}.$$
- For any $f \in \mathcal{D}$, denote $\Lambda(f)$ the localization set of f, i.e. $\Lambda(f)$ is the smallest set $\Lambda \subset \mathbb{Z}^d$ such that $f \in C_b(\mathbb{R}^\Lambda, \mathbb{R})$.
- For any $f \in C_b(\mathbb{B}, \mathbb{R})$, define $||f|| = \sup_{x\in\mathbb{B}} |f(x)|$. For any $f \in \mathcal{D}^1$, define $|\nabla f(x)|^2 = \sum_{i\in\mathbb{Z}^d} |\partial_i f(x)|^2$.
- For any $f \in \mathcal{D}^1$, define $|\nabla f(x)|^2 = \sum_{i\in\mathbb{Z}^d} |\partial_i f(x)|^2$.
- $||\cdot||$ is the uniform norm, i.e. for any $f \in C_b(\mathbb{B}, \mathbb{R})$, $||f|| = \sup_{x\in\mathbb{B}} |f(x)|$. The seminorms $|||\cdot|||_1$ and $|||\cdot|||_2$ are respectively defined by

  $$|||f|||_1 = \sum_{i\in\mathbb{Z}^d} ||\partial_i f|| \qquad f \in \mathcal{D}^1,$$

and

$$|||f|||_2 = \sum_{j,k\in\mathbb{Z}^d} ||\partial_{jk} f|| \qquad f \in \mathcal{D}^2.$$

- $\mathcal{B}_b(H,\mathbb{R})$ is the function space including the bounded measurable functions from some topological space H to $\mathbb{R}$,

Assumption 1.2 (Assumptions of a and U)**.** *The a and U in* (1.1) *satisfies the following conditions:*

(1) (Linear interactions) $a_{ij} \geq 0$ *for all* $i \neq j$, $a_{ii} = -1$ *for all* $i \in \mathbb{Z}^d$.

(2) (Bounded interactions) $U_i \in \mathcal{D}^2$ *for all* $i \in \mathbb{Z}^d$, $\sup_{i\in\mathbb{Z}^d} ||U_i|| < \infty$.

(3) (Finite range property) *There exists some* $K \in \mathbb{N}$ *such that, for all* $i, j \in \mathbb{Z}^d$ *with* $|i-j| > K$, *one has* $a_{ij} = 0$ *and* $\partial_j U_i(x) = 0$ *for all* $x \in \mathbb{B}$.

(4) $\eta < \infty$ *with* $\eta := \sup_{j\in\mathbb{Z}^d}\left(\sum_{i\in\mathbb{Z}^d, i\neq j} a_{ij} + |||U_j|||_1\right)$, *and* $\sup_{j\in\mathbb{Z}^d} |||U_j|||_2 < \infty$.

The main results of this paper are the following two theorems

Theorem 1.3. *There exists a Markov semigroup P_t on the space $\mathcal{B}_b(\mathbb{B},\mathbb{R})$ generated by the system* (1.1)*.*

Theorem 1.4. *We have some constant $c > 0$ such that as $\eta < c$, there exists a probability measure μ supported on $\mathbb{B}$ so that for all $x \in \mathbb{B}$,*

$$\lim_{t\to\infty} P_t^* \delta_x = \mu \quad \text{weakly.}$$

Remark 1.5. The convergence in Theorem 1.4 implies that P_t is strongly mixing (see [6]).

1.2. Ornstein-Uhlenbeck α-stable Processes.

1.2.1. *One dimensional Ornstein-Uhlenbeck α-stable process.* This process is described by the following stochastic differential equation (SDE)

$$\begin{cases} dX(t) = -X(t)dt + dZ(t) \\ X(0) = x \end{cases} \tag{1.2}$$

where $Z(t)$ is a symmetric α-stable process $(0 < \alpha \leq 2)$ with infinitesimal generator ∂_x^α defined by

$$\partial_x^\alpha f(x) = \frac{1}{C_\alpha}\int_{\mathbb{R}\setminus\{0\}} \frac{f(y+x)-f(x)}{|y|^{\alpha+1}}dy, \quad C_\alpha = -\int_{\mathbb{R}\setminus\{0\}} (cosy-1)\frac{dy}{|y|^{1+\alpha}}. \tag{1.3}$$

as $0 < \alpha < 2$, and by $\frac{1}{2}\Delta$ as $\alpha = 2$ ([4]). Moreover, if f has Fourier transform $\hat{f}$, then

$$\partial_x^\alpha f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |\lambda|^\alpha \hat{f}(\lambda) e^{i\lambda x} d\lambda.$$

The Kolmogorov backward equation of (1.2) is

$$\begin{cases} \partial_t u = \partial_x^\alpha u - x\partial_x u, \\ u(0) = f, \end{cases} \tag{1.4}$$

which is solved by

$$u(t,x) = \mathbb{E}_x[f(X(t))] = \int_{\mathbb{R}} p\left(\frac{1-e^{-\alpha t}}{\alpha}; e^{-t}x, y\right) f(y)dy, \tag{1.5}$$

where $X(t)$ is the solution to (1.2) and

$$p(t;x,y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-t|\lambda|^\alpha + i(x-y)\lambda} d\lambda. \tag{1.6}$$

One can refer to the Appendix for a formal derivation of (1.5) and (1.6), or refer to [13] for the rigorous one. From (1.5), one can easily see that as $t \to \infty$

$$u(t,x) \to \int_{\mathbb{R}} p(1/\alpha; 0, y) f(y) dy. \tag{1.7}$$

Hence, the law of $X(t,x)$ weakly convergence to the measure $p(1/\alpha; 0, y)dy$, which is independent of the initial data x. Hence, $X(t,x)$ is ergodic and strongly mixing ([6]).

Define $S_t^0 f(x) := \mathbb{E}_x[f(X_t)]$, which is the Ornstein-Uhlenbeck α-stable semigroup generated by the operator $\partial_x^\alpha - x\partial_x$.

Lemma 1.6. *Let* $1 \le \beta < \alpha$, *then*

$$S_t^0[|x|^\beta](0) \le C(\beta). \tag{1.8}$$

Proof. Since the symmetric α-stable process $(1 < \alpha \le 2)$ has β order moments with $1 \le \beta < \alpha$, by (1.5) and (1.6), we have

$$\begin{aligned} S_t^0[|x|^\beta](0) &= \int_{\mathbb{R}} p\left(\frac{1-e^{-\alpha t}}{\alpha}; 0, y\right) |y|^\beta dy \\ &= \left[\frac{1-e^{-\alpha t}}{\alpha}\right]^{\beta/\alpha} \int_{\mathbb{R}} p(1;0,y)|y|^\beta dy \le C(\beta) \end{aligned}$$

□

1.2.2. *Infinite dimensional Ornstein-Uhlenbeck α-stable processes.* Consider the white symmetric α-stable noises $(Z_i(t))_{i\in\mathbb{Z}^d}$, i.e. $(Z_i(t))_{i\in\mathbb{Z}^d}$ are i.i.d. symmetric α-stable processes, and define the infinite dimensional Ornstein-Uhlenbeck α-stable processes by the following SDEs: for all $i \in \mathbb{Z}^d$

$$dX_i(t) = -X_i(t)dt + dZ_i(t).$$

Clearly, $X_i(t)$ at each $i \in Z^d$ is an Ornstein-Uhlenbeck α-stable process, which is independent of the processes on the other sites. By (1.7), $(X_i(t))_{i\in Z^d}$ is ergodic and has a unique invariant measure $(p(1/\alpha; 0, y_i)dy_i)_{i\in\mathbb{Z}^d}$ with p defined by (1.6).

Moreover, it is easy to see that the infinitesimal generator of $(X_i(t))_{i\in\mathbb{Z}^d}$ is given by

$$\mathcal{L} := \sum_{i\in\mathbb{Z}^d} (\partial_i^\alpha - x_i\partial_i), \tag{1.9}$$

which is well defined on $\mathcal{D}^\infty$. Clearly, $\mathcal{L}$ generates a Markov semigroup S_t on $\mathcal{D}$, which is the product of one dimensional Ornstein-Uhlenbeck α-stable semigroup, more precisely, for any function $f \in \mathcal{D}$, $S_t f$ is defined by

$$S_t f(x) = \int_{\mathbb{R}^{\Lambda(f)}} \prod_{j\in\Lambda(f)} p\left(\frac{1-e^{-\alpha t}}{\alpha}; e^{-t}x_j, y_j\right) f(y) \prod_{j\in\Lambda(f)} dy_j \tag{1.10}$$

2. Existence of Infinite Dimensional Interacting α-stable Systems

2.1. **Galerkin approximation of the interacting systems.** Let $\Gamma_N = [-N,N]^d$ be the cube of $\mathbb{Z}^d$. We approximate the infinite dimensional system by

$$\begin{cases} dX_i^N(t) = [\sum_{j\in\Gamma_N} a_{ij}X_j^N(t) + U_i^N(X^N(t))]dt + dZ_i(t), \\ X_i^N(0) = x_i, \end{cases} \tag{2.1}$$

for all $i \in \Gamma_N$, where $U_i^N(x^N) = U_i(x^N, 0)$ with $x^N = (x_i)_{i\in\Gamma_N}$. Since the coefficients in (2.1) are Lipschitz, by [5] (chapter 5), (2.1) has a unique global strong solution. Moreover, for any differentiable f, $P_t^N f$ is also differentiable (c.f. chapter 5 of [5]).

For any $f \in \mathcal{D}^\infty$ with $\Lambda(f) \subset \Gamma_N$, define

$$P_t^N f(x) = E_x^N[f(X^N(t))],$$

then P_t^N is a Markov semigroup, moreover, it satisfies

$$\begin{cases} \partial_t u(t) = \mathcal{L}_N u(t) \\ u(0) = f \end{cases} \tag{2.2}$$

where $\mathcal{L}_N$ is the generator of P_t^N defined by

$$\mathcal{L}_N = \sum_{i\in\Gamma_N} \partial_i^\alpha + \sum_{i\in\Gamma_N} [\sum_{j\in\Gamma_N} a_{ij}x_j + U_i^N(x)]\partial_i \tag{2.3}$$

$$= \sum_{i\in\Gamma_N} [\partial_i^\alpha - x_i\partial_i] + \sum_{i\in\Gamma_N} [\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j + U_i^N(x)]\partial_i. \tag{2.4}$$

From the form (2.4) of $\mathcal{L}_N$, by Du Hamel principle, we have

$$P_t^N f = S_t f + \int_0^t S_{t-s}\left\{\sum_{i\in\Gamma_N} [\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j + U_i^N(x)]\partial_i P_s^N f\right\} ds \tag{2.5}$$

where S_t is defined by (1.10).

2.2. **Auxiliary Lemmas.** The following relation (2.7) is usually called finite speed propagation of interactions ([7]), which roughly means that the effects of the initial condition (i.e. f in our case) need a long time to be propagated (by interactions) far away. The main reason for this phenomenon is that the interactions are finite range.

Lemma 2.1.
1. For any $f \in \mathcal{D}^\infty$, we have

$$|||P_t^N f|||_1 \le e^{(\eta+1)t}|||f|||_1. \tag{2.6}$$

2. (Finite speed of propagation of the interactions) *Given any $f \in \mathcal{D}^1$ and $k \notin \Lambda(f)$, for any $A > 0$, there exists some $B \ge 1$ such that when $n_k > Bt$, we have*

$$||\partial_k P_t^N f|| \le e^{-At-An_k}|||f|||_1 \tag{2.7}$$

where $n_k = [\frac{dist(k,\Lambda(f))}{K}]$ and $K \in \mathbb{N}$ is defined in Assumption 1.2.

Proof. For the notational simplicty, we drop the index N of the quantities if no confusions arise. By Markov property of P_t and the easy fact

$$\frac{d}{ds}P_{t-s}\partial_k P_s f = P_{t-s}[\partial_k, \mathcal{L}_N]P_s f$$

where $[\partial_k, \mathcal{L}_N] = \partial_k \mathcal{L}_N - \mathcal{L}_N \partial_k = \sum_{i\in\Gamma_N}[a_{ik} + \partial_k U_i]\partial_i$, we have

$$||\partial_k P_t f|| \le ||\partial_k f|| + \int_0^t \sum_{i\in\Gamma_N}(|a_{ik}| + ||\partial_k U_i||)||\partial_i P_s f||ds. \tag{2.8}$$

To prove (2.6), summing the index k of the above inequality over $\mathbb{Z}^d$, by Assumption 1.2, we have

$$|||P_t f|||_1 \le |||f|||_1 + (1+\eta)\int_0^t |||P_s f|||_1 ds,$$

which immediately implies (2.6).

Now let us show (2.7). Denote $c_{ik} = ||\partial_k U_i||$ and δ_{ik} Krockner's function, we have by iterating (2.8)

$$||\partial_k P_t f|| \le \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{i\in\Lambda(f)}[(c+a+\delta)^n]_{ik}||\partial_i f||$$

$$= \sum_{n=0}^{n_k}\frac{t^n}{n!}\sum_{i\in\Lambda(f)}[(c+a+\delta)^n]_{ik}||\partial_i f|| + \sum_{n=n_k+1}^{\infty}\frac{t^n}{n!}\sum_{i\in\Lambda(f)}[(c+a+\delta)^n]_{ik}||\partial_i f||$$

The first term of the last line is zero, since $(\delta+a+c)_{ik} = 0$ for all $n \le n_k$ by the definition of n_k. As for the second term, we can easily have

$$\sum_{n=n_k+1}^{\infty}\frac{t^n}{n!}\sum_{i\in\Lambda(f)}[(c+a+\delta)^n]_{ik}||\partial_i f|| \le \frac{t^{n_k}}{n_k!}e^{(1+\eta)t}|||f|||_1.$$

Hence,

$$||\partial_k P_t^N f|| \le \frac{t^{n_k}}{n_k!}e^{(1+\eta)t}|||f|||_1 \tag{2.9}$$

For any $A > 0$, choosing $B \geq 1$ such that

$$2 - logB + log(1+\eta) + \frac{1+\eta}{B} \leq -2A,$$

as $n > Bt$, one has

$$\frac{t^n(1+\eta)^n}{n!} e^{(1+\eta)t} \leq \exp\{n \log \frac{1+\eta}{B} + 2n + (1+\eta)\frac{n}{B}\}$$
$$\leq \exp\{-2An\} \leq \exp\{-An - At\}.$$

Replacing n by n_k, we conclude the proof. □

The following lemma roughly means that if the initial data is in a ball $B_{R,\rho}$, then the dynamics will not go far from this ball in finite time. Note that the following $P_t^N f_k(x)$ equals to $\mathbb{E}_x[|X_k^N(t)|]$.

Lemma 2.2. *Let $f_k(y) = |y_k|$ with $k \in \Gamma_N$, then for all $x \in B_{R,\rho}$*

$$P_t^N f_k(x) \leq C(1+|k|)^\rho e^{\rho d(1+\eta)t}. \tag{2.10}$$

where $C = C(\rho, R, \eta, d, K)$.

Proof. For the notional simplicity, we shall drop the index N of the quantities if no confusions arise. By (2.5), we have

$$P_t f_k(x) = S_t f_k(x) + \int_0^t S_{t-s}[\sum_{i\in\Gamma_N}(\sum_{i\in\Gamma_N\setminus i} a_{ij}y_j + U_i(y))\partial_i P_s f_k\ ](x)ds.$$

By (1.10), (1.8), and that of $B_{R,\rho}$, one has

$$\begin{aligned} |S_t f_k(x)| &\leq S_t[|y_k - e^{-t}x_k| + e^{-t}|x_k|] \\ &= \int_{\mathbb{R}} p\left(\frac{1-e^{-\alpha t}}{\alpha}, 0, y_k\right)|y_k|dy_k + e^{-t}|x_k| \\ &\leq C + R(1+|k|)^\rho. \end{aligned} \tag{2.11}$$

By (2.6) and the easy fact $|||f_k|||_1 = 1$, we have

$$\left|\int_0^t S_{t-s}\left[\sum_{i\in\Gamma_N} U_i(y)\partial_i P_s f_k\right]ds\right| \leq \sup_i ||U_i|| \int_0^t |||P_s^N f_k|||_1 ds \leq Ce^{(\eta+1)t}.$$

Moreover, by the same argument as in (2.11),

$$\begin{aligned} &\left|\int_0^t S_{t-s}\left[\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}y_j\partial_i P_s f_k\right](x)ds\right| \\ &\leq \int_0^t \sum_{i\in\Gamma_N} ||\partial_i P_s f_k|| \sum_{j\in\Gamma_N\setminus i} a_{ij}S_{t-s}\left[|y_j - e^{-(t-s)}x_j| + e^{-(t-s)}|x_j|\right](x)ds \\ &\leq C\int_0^t \sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}||\partial_i P_s f_k||ds + \int_0^t \sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}||\partial_i P_s^N f_k|||x_j|ds. \end{aligned} \tag{2.12}$$

For the first term in the last line, by Assumption 1.2 (in particular, $a_{ij} \leq 1+\eta$ and $a_{ij} = 0$ if $|i-j| > K$) and (2.6), one can easily have

$$\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}||\partial_i P_s f_k|| \leq K^d(1+\eta)|||P_s f_k|||_1 \leq K^d(1+\eta)e^{(1+\eta)s} \tag{2.13}$$

As for the second term, let us first estimate the double summations therein '$\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i}\cdots$', the idea is to split the first sum '$\sum_{i\in\Gamma_N}$' into two pieces '$\sum_{i:|i-k|>Bs}$', '$\sum_{i:|i-k|\leq Bs}$' and control them by (2.7) and (2.6). More precisely, for any $A > 0$, let $B > 1$ be chosen as in Lemma 2.1, by Assumption 1.2 (in particular, $a_{ij} = 0$ for $|i-j| > K$), and the fact $|x_j| \leq R(1+|j|)^\rho$, the piece '$\sum_{i:|i-k|>Bs}$'can be estimated by

$$\begin{aligned}\sum_{i:|i-k|>Bs} ||\partial_i P_s f_k|| \sum_{j\in\Gamma_N\setminus i} a_{ij}|x_j| &\leq \sum_{i\in\Gamma_N} e^{-A|i-k|/K-As} R\left(1+|i|+K^d\right)^\rho \\ &\leq C(K,d,\rho,R,\eta)\sum_{i\in\Gamma_N} e^{-A|i-k|/K-As}[(1+|k|)^\rho + |k-i|^\rho] \\ &\leq C(K,d,\rho,R,\eta,A)(1+|k|)^\rho.\end{aligned}$$

As for the other piece, by Assumption 1.2 again, we have

$$\begin{aligned}\sum_{i:|i-k|\leq Bs} ||\partial_i P_s f_k|| \sum_{j\in\Gamma_N\setminus i} a_{ij}|x_j| &\leq \sum_{i:|i-k|\leq Bs} ||\partial_i P_s f_k|| K^d(1+\eta)R(1+|i|+K^d)^\rho \\ &\leq |||P_s f_k|||_1 K^d(1+\eta)R\left(1+|k|+(Bs)^d+K^d\right)^\rho \\ &\leq C(R,K,d,\eta,B,\rho)s^{\rho d}e^{(1+\eta)s}(1+|k|)^\rho\end{aligned}$$

Combining the above two inequalities, one immediately has

$$\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}||\partial_i P_s^N f_k|||x_j| \leq C(R,d,K,\rho,\eta)(1+s)^{\rho d}e^{(1+\eta)s}(1+|k|)^\rho.$$

Hence, plugging the above estimates and (2.13) into (2.12), we have

$$\left|\int_0^t S_{t-s}\left[\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}y_j\partial_i P_s f_k\right](x)ds\right| \leq C(d,\eta,R,\rho,K)e^{\rho d(1+\eta)t}(1+|k|)^\rho$$

□

2.3. **Proof of Theorem 1.3.**

Proof. We shall firstly prove that $\lim_{N\to\infty} P_t^N f(x)$ exists for any $f \in \mathcal{D}^\infty$, $t > 0$ and $x \in \mathbb{B}$. It suffices to show that $\{P_t^N f(x)\}_N$ is a Cauchy sequence pointwisely in one $B_{R,\rho}$.

Take $x \in B_{R,\rho}$, for any $\Gamma_M \supset \Gamma_N \supset \Lambda(f)$ with $M > N$, we have

$$
\begin{aligned}
|P_t^M f(x) - P_t^N f(x)| &\leq \left| \int_0^t \frac{d}{ds} P_{t-s}^M \left[P_s^N f\right](x) ds \right| \\
&\leq \left| \int_0^t P_{t-s}^M \left[(\mathcal{L}_M - \mathcal{L}_N) P_s^N f\right](x) ds \right| \\
&\leq \left| \int_0^t P_{t-s}^M \Big[\sum_{i \in \Gamma_M \setminus \Gamma_N} \big(\sum_{j \in \Gamma_M} a_{ij} y_j + U_i^M(y) \big) \partial_i P_s^N f \ \Big](x) ds \right| \\
&\quad + \left| \int_0^t P_{t-s}^M \Big[\sum_{i \in \Gamma_N} \left(U_i^M(y) - U_i^N(y)\right) \partial_i P_s^N f \ \Big](x) ds \right|.
\end{aligned} \tag{2.14}
$$

By (2.10), (3) of Assumption 1.2 and (2.7),

$$
\begin{aligned}
& P_{t-s}^M \Big[\sum_{i \in \Gamma_M \setminus \Gamma_N} \sum_{j \in \Gamma_M \setminus i} a_{ij} |y_j| \cdot ||\partial_i P_s^N f|| \Big](x) \\
& \leq \sum_{i \in \Gamma_M \setminus \Gamma_N} ||\partial_i P_s^N f|| \sum_{j \in \Gamma_M} a_{ij} P_{t-s}^N [|y_j|](x) \\
& \leq C(t-s, R, d, \rho, \eta) \sum_{i \in \Gamma_M \setminus \Gamma_N} K^d (1+\eta)(1+|i|+K^d)^\rho ||\partial_i P_s^N f|| \\
& \to 0 \quad (M, N \to \infty).
\end{aligned} \tag{2.15}
$$

Similarly, by (2.7) again, as $M, N \to \infty$,

$$
\sum_{i \in \Gamma_M \setminus \Gamma_N} |P_{t-s}^M [\ U_i^M(y) \partial_i P_s^N f \ ](x)| \leq \sup_i ||U_i|| \sum_{i \in \Gamma_M \setminus \Gamma_N} ||\partial_i P_s^N f|| \to 0.
$$

By (3) of Assumption 1.2, the definition of U_i^N and (2.7), the term $'|\cdots|'$ in the last line of (2.14) can be bounded by

$$
\begin{aligned}
|\cdots| &= \left| \int_0^t P_{t-s}^M \Big[\sum_{i \in \partial_K(\Gamma_N)} \left(U_i^M(y) - U_i^N(y)\right) \partial_i P_s^N f \ \Big](x) ds \right| \\
&\leq 2 \sup_i ||U_i|| \int_0^t \sum_{i \in \partial_K(\Gamma_N)} ||\partial_i P_s^N f|| ds \\
&\to 0 \qquad (N \to \infty)
\end{aligned} \tag{2.16}
$$

where $\partial_K(\Gamma_N) = \{i \in \Gamma_N; dist(i, \partial\Gamma_N) \leq K\}$ and $\partial\Gamma_N$ is the boundary of Γ_N.

Injecting all the above inequalities into (2.14), we conclude that $\{P_t^N f(x)\}_N$ is a Cauchy sequence for any $t > 0$, $x \in B_{\rho,R}$ and $f \in D^\infty$.

For any $f \in \mathcal{D}^\infty$ and $x \in \mathbb{B}$, denote

$$
P_t f(x) := \lim_{N \to \infty} P_t^N f(x). \tag{2.17}
$$

Since $\mathcal{D}^\infty$ is dense in $\mathcal{B}_b(\mathbb{B},\mathbb{R})$ (under the product topology), we can extend the domain of P_t from $\mathcal{D}^\infty$ to $\mathcal{B}_b(\mathbb{B},\mathbb{R})$. Thanks to (2.6) and similar arguments as above, one can pass to the limit on the both sides of $P^N_{t_1+t_2}f(x) = P^N_{t_1}P^N_{t_2}f(x)$ and obtain the semigroup property of P_t, i.e. $P_{t_1+t_2}f(x) = P_{t_1}P_{t_2}f(x)$. It is easy to see that $P_t(\mathbf{1})(x) = 1$ for all $x \in \mathbb{B}$ and $P_tf(x) \geq 0$. Hence P_t is a Markov semigroup on $\mathcal{B}_b(\mathbb{B},\mathbb{R})$. □

3. Ergodic Theorem 1.4

3.1. **Auxiliary Lemmas.** To prove the ergodicity result, we need the following three auxiliary lemmas.

Let S_t be the product semigroup defined in (1.10), for any $f \in \mathcal{D}^\infty$, by replacing Λ_0 in (3.1) by $\Lambda(f)$, one can easily have

$$\begin{aligned}&|S_{t_2}f(0) - S_{t_1}f(0)|\\ &\leq |\Lambda(f)|\cdot ||f|| \int_{\mathbb{R}} \left| p\left(\frac{1-e^{-\alpha t_2}}{\alpha}, 0, y\right) - p\left(\frac{1-e^{-\alpha t_1}}{\alpha}, 0, y\right)\right| dy\\ &\to 0 \quad (t_1, t_2 \to \infty).\end{aligned}$$

However, the above convergence speed depends on $|\Lambda(f)|$, the size of f. This type of convergence is not enough to control some limit in the interacting system. Alternatively, we shall use the information of $f \in \mathcal{D}^1$, i.e. $|||f|||_1 < \infty$, and prove the convergence speed *asymptotically* depends on $\Lambda(f)$, this will make some room to uniformly control the second term on the r.h.s. of (3.18). More precisely, we have

Lemma 3.1. *Let S_t be the product semigroup generated by (1.9). Given any $f \in \mathcal{D}^\infty$ and $\Lambda_0 \subset \Lambda(f)$, for any $t_2, t_1 \geq 0$, we have*

$$\begin{aligned}|S_{t_2}f(0) - S_{t_1}f(0)| \leq |\Lambda_0|\cdot ||f|| \int_{\mathbb{R}} |p\left(\frac{1-e^{-\alpha t_2}}{\alpha}, 0, y\right) - p\left(\frac{1-e^{-\alpha t_1}}{\alpha}, 0, y\right)|dy\\ + C\sum_{i\in\Lambda(f)\setminus\Lambda_0} ||\partial_i f||.\end{aligned}$$

where $C > 0$ is some constant only depending on the parameter α.

Proof. We can easily have

$$\begin{aligned}&|S_{t_2}f(0) - S_{t_1}f(0)| \\ &\leq \left|\int_{\mathbb{R}^{\Lambda_0}}\left[\prod_{i\in\Lambda_0} p\left(\frac{1-e^{-\alpha t_2}}{\alpha}, 0, y_i\right) - \prod_{i\in\Lambda_0} p\left(\frac{1-e^{-\alpha t_1}}{\alpha}, 0, y_i\right)\right] f(y^{\Lambda_0}, 0)dy\right| \\ &\quad + \int_{\mathbb{R}^{\Lambda(f)}} \prod_{i\in\Lambda(f)} p\left(\frac{1-e^{-\alpha t_1}}{\alpha}, 0, y_i\right)|f(y) - f(y^{\Lambda_0}, 0)|dy \\ &\quad + \int_{\mathbb{R}^{\Lambda(f)}} \prod_{i\in\Lambda(f)} p\left(\frac{1-e^{-\alpha t_2}}{\alpha}, 0, y_i\right)|f(y) - f(y^{\Lambda_0}, 0)|dy \\ &= I_0 + I_1 + I_2\end{aligned} \tag{3.1}$$

It is easy to see

$$I_0 \le |\Lambda_0| \cdot ||f|| \int_{\mathbb{R}} \left| p\left(\frac{1-e^{-\alpha t_2}}{\alpha}, 0, y\right) - p\left(\frac{1-e^{-\alpha t_1}}{\alpha}, 0, y\right)\right| dy.$$

I_1 and I_2 can be bounded in the same way as the following: By (1.8),

$$I_1 \le \sum_{i \in \Lambda(f)\setminus\Lambda_0} ||\partial_i f|| \int_{\mathbb{R}} p\left(\frac{1-e^{-\alpha t_1}}{\alpha}, 0, y_i\right) |y_i| dy_i \le C \sum_{i \in \Lambda(f)\setminus\Lambda_0} ||\partial_i f||.$$

□

The next lemma claims that if η is sufficiently small, then the gradient of $P_t^N f$ uniformly decays in an exponential speed.

Lemma 3.2. *There exists some $c > 0$ such that if $\eta < c$, then, for any $m \in \mathbb{N}$, we have*

$$|\nabla P_t^N f|^{2m} \le e^{-2m\beta t} P_t^N |\nabla f|^{2m} \quad \forall\ f \in \mathcal{D}^\infty \tag{3.2}$$

where $\beta = \beta(a, U) > 0$. In particular,

$$||\partial_i P_t^N f|| \le e^{-\beta t} |||f|||_1. \tag{3.3}$$

Proof. For the notational simplicity, we drop the index of the quantities if no confusions arise. For any $f \in \mathcal{D}^\infty$ and any $m \in \mathbb{N}$, we have the following calculation

$$\begin{aligned}
\frac{d}{ds} P_s |\nabla P_{t-s} f|^{2m} &= P_s[\mathcal{L}_N |\nabla P_{t-s} f|^{2m} - 2m|\nabla P_{t-s} f|^{2(m-1)} \nabla P_{t-s} f \cdot \mathcal{L}_N \nabla P_{t-s} f] \\
&\quad + 2m P_s(|\nabla P_{t-s}|^{2(m-1)} \nabla P_{t-s} f \cdot [\mathcal{L}_N, \nabla] P_{t-s} f) \\
&\ge 2m P_s(|\nabla P_{t-s}|^{2(m-1)} \nabla P_{t-s} f \cdot [\mathcal{L}_N, \nabla] P_{t-s} f),
\end{aligned}$$

since we have $P_t F^{2m} \ge (P_t F)^{2m}$ and therefore $\lim_{t\to 0+} \frac{P_t F^{2m} - F^{2m}}{t} \ge \lim_{t \to 0+} \frac{(P_t F)^{2m} - F^{2m}}{t}$, i.e. $\mathcal{L}_N F^{2m} - 2m F^{2(m-1)} F \mathcal{L}_N F \ge 0$.

Hence,

$$\begin{aligned}
&\frac{d}{ds} P_s |\nabla P_{t-s} f|^{2m} \\
&\ge -2m P_s \left\{ |\nabla P_{t-s} f|^{2(m-1)} \sum_{i \in \Gamma_N} \sum_{j \in \Gamma_N \setminus i} (a_{ij} + \partial_j U_i)(\partial_i P_{t-s} f)(\partial_j P_{t-s} f) \right\} \\
&\quad + 2m P_s \left\{ |\nabla P_{t-s} f|^{2(m-1)} \sum_{i \in \Gamma_N} (1 - \partial_i U_i) |\partial_i P_{t-s} f|^2 \right\}.
\end{aligned}$$

Define the quadratic form

$$Q(\xi, \xi) := -\sum_{i \in \Gamma_N} \sum_{j \in \Gamma_N \setminus i} (a_{ij} + \partial_j U_i)\xi_i \xi_j + \sum_{i \in \Gamma_N} (1 - \partial_i U_i) \xi_i^2$$

for any $\xi \in \mathbb{R}^{\Gamma_N}$. As η in (4) of Assumption 1.2 is small enough, it is strictly positive definite, that is, we have some constant $c, \beta > 0$ such that as $\eta < c$,

$$Q(\xi, \xi) \ge \beta |\xi|^2.$$

Therefore,

$$\frac{d}{ds}P_s|\nabla P_{t-s}f|^{2m} \geq 2m\beta P_s\left(|\nabla P_{t-s}f|^{2m}\right),$$

from which we immediately obtain (3.2). (3.3) is an immediate corollary of (3.2). □

The last lemma can be taken as the second order finite speed propagation of interactions, which is also due to the finite range interactions.

Lemma 3.3. *There exists some constant $C = C(a, U) > 0$ so that*

$$|||P_t^N f|||_2 \leq e^{Ct}(|||f|||_1 + |||f|||_2). \tag{3.4}$$

Moreover, there exists some constant $B = B(a, U) \geq 1$ so that as

$$dist(j, \Lambda(f)), dist(j, \Lambda(f)) \geq Bt,$$

we have

$$||\partial_{jk}P_t^N f|| \leq h_{jk}(|||f|||_1 + |||f|||_2). \tag{3.5}$$

where $\{h_{jk}\}_{j,k\in\mathbb{Z}^d}$ is a constant sequence only depending on a and U, and

$$h_{jk} \geq 0, \sum_{j,k\in\mathbb{Z}^d} h_{jk} < \infty.$$

Proof. Furthermore, by differentiating $P_{t-s}^N \partial_{kj} P_s^N f$ (on s), we have

(3.6)

$$\begin{aligned}
||\partial_{kj}P_t^N f|| &\leq ||\partial_{kj}f|| + \int_0^t ||[\partial_{kj}, \mathcal{L}_N]P_s^N f||ds \\
&\leq ||\partial_{kj}f|| + \int_0^t \sum_{i\in\Gamma_N}[(|a_{ik}| + ||\partial_k U_i||)||\partial_{ji}P_s^N f|| + (|a_{ij}| + ||\partial_j U_i||)||\partial_{ki}P_s^N f||]ds \\
&\quad + \int_0^t \sum_{i\in\Gamma_N}||\partial_{jk}U_i||||\partial_i P_s^N f||ds
\end{aligned}$$

By some iteration similar to that for (2.9), we have some convergent sequence $\{h_{jk}\}_{j,k\in\mathbb{Z}^d}$ only depending on a and U so that $||\partial_{kj}P_t^N f||_{j,k\in\mathbb{Z}^d} \leq h_{jk}(|||f|||_1 + |||f|||_2)$ (The h_{jk} here plays a similar role as that of e^{-At-An_k} in (2.7)). Moreover, Summering i, k over $\mathbb{Z}^d$ in (3.6), we immediately have (3.4). □

3.2. **Proof of Theorem.** The proof of Theorem is lengthy, we first prove the following Proposition 3.4, which is the crucial step.

Proposition 3.4. *For any $f \in \mathcal{D}^\infty$, the limit $\lim_{t\to\infty} P_t f(0)$ exists.*

Proof. To prove the proposition, it suffices to show that for arbitrary $\varepsilon > 0$, there exists some constant $T > 0$ such that as $t_2 \geq t_1 \geq T$

$$|P_{t_2}f(0) - P_{t_1}f(0)| \leq 4\varepsilon. \tag{3.7}$$

Let us show this inequality in the following three steps:

Step 1: Proof of (3.7). For any $t_2 \geq t_1 \geq 0$, by triangle inequality, we have

$$|P_{t_2} f(0) - P_{t_1} f(0)| \leq |P_{t_2} f(0) - P_{t_2}^N f(0)| \\ + |P_{t_2}^N f(0) - P_{t_1}^N f(0)| + |P_{t_1} f(0) - P_{t_1}^N f(0)|.$$

By (2.17), there exists some $N_0 = N_0(t_1, t_2) \in \mathbb{N}$ such that as $N > N_0$

$$|P_{t_2} f(0) - P_{t_2}^N f(0)| + |P_{t_1} f(0) - P_{t_1}^N f(0)| < \varepsilon. \tag{3.8}$$

Hence, to conclude the proof, we only need to show that, there exists some constant $T > 0$, which is independent of N, such that as $t_2, t_1 > T$

$$|P_{t_2}^N f(0) - P_{t_1}^N f(0)| < 3\varepsilon. \tag{3.9}$$

By (2.5), the l.h.s. of (3.9) can be split into the following *two* pieces:

$$\begin{aligned} |P_{t_2}^N f(0) - P_{t_1}^N f(0)| \leq\ & |S_{t_2} f(0) - S_{t_1} f(0)| \\ & + \Big| \int_0^{t_2} S_{t_2 - s} \Big[\sum_{i \in \Gamma_N} \big(\sum_{j \in \Gamma_N \setminus i} a_{ij} x_j + U_{i,N} \big) \partial_i P_s^N f \Big] (0) ds \\ & - \int_0^{t_1} S_{t_1 - s} \Big[\sum_{i \in \Gamma_N} \big(\sum_{j \in \Gamma_N \setminus i} a_{ij} x_j + U_{i,N} \big) \partial_i P_s^N f \Big] (0) ds \Big|. \end{aligned} \tag{3.10}$$

By the fact $f \in \mathcal{D}^\infty$ and (1.10), we have some constant $T_0 = T_0(|\Lambda(f)|, ||f||) > 0$ such that as $t_1, t_2 \geq T_0$

$$|S_{t_2} f(0) - S_{t_1} f(0)| \leq \varepsilon. \tag{3.11}$$

From the above inequality, to conclude the proof of (3.9), it suffices to bound the second term '$|\cdots|$' on the r.h.s. of (3.10) by 2ε. To this end, we split it into three pieces as the following

$$J_1 = \int_L^{t_2} S_{t_2 - s} \Big[\sum_{i \in \Gamma_N} \big(\sum_{j \in \Gamma_N \setminus i} a_{ij} x_j + U_i^N \big) \partial_i P_s^N f \Big] (0) ds$$

$$J_2 = \int_L^{t_1} S_{t_1 - s} \Big[\sum_{i \in \Gamma_N} \big(\sum_{j \in \Gamma_N \setminus i} a_{ij} x_j + U_i^N \big) \partial_i P_s^N f \Big] (0) ds$$

$$J_3 = \int_0^L (S_{t_2 - s} - S_{t_1 - s}) \Big[\sum_{i \in \Gamma_N} \big(\sum_{j \in \Gamma_N \setminus i} a_{ij} x_j + U_i^N \big) \partial_i P_s^N f \Big] (0) ds$$

where $0 < L < t_1$ is some large number to be determined later, and show that there exists some constant $T > 0$, which is independent of N and larger than L, such that as $t_1, t_2 \geq T$

$$|J_1| + |J_2| + |J_3| \leq 2\varepsilon. \tag{3.12}$$

Step 2: Proof of (3.12). As for $|J_1|$, choose some $m \in \mathbb{N}$ with $2m/(2m-1) <$

α, by Höder's inequality, (1.8), (3) of Assumption 1.2 and (3.2),

$$\begin{aligned}
&\left|\int_L^{t_2} S_{t_2-s}\left[\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j\partial_i P_s^N f\right](0)ds\right| \\
\leq &\left|\int_L^{t_2}\sum_{i\in\Gamma_N}\sum_{j\in\Gamma_N\setminus i} a_{ij}\{S_{t_2-s}[|x_j|^{\frac{2m}{2m-1}}](0)\}^{\frac{2m-1}{2m}}\{S_{t_2-s}[|\partial_i P_s^N f|^{2m}](0)\}^{\frac{1}{2m}}ds\right| \\
\leq &C(m)\eta K^d\left|\int_L^{t_2}\{S_{t_2-s}[|\nabla P_s^N f|^{2m}](0)\}^{\frac{1}{2m}}ds\right| \\
\leq &C(m,\eta,K,d)|||f|||\ (e^{-L\beta}-e^{-t_2\beta}).
\end{aligned}$$

By a similar arguments, we have

$$\begin{aligned}
\left|\int_L^{t_2} S_{t_2-s}\left[\sum_{i\in\Gamma_N} U_i^N\partial_i P_s^N f\right](0)ds\right| \leq &\sup_i ||U_i||\int_L^{t_2}\left\{S_{t_2-s}[|\nabla P_s^N f|^2](0)\right\}^{1/2}ds \\
\leq &C|||f|||\ (e^{-\beta L}-e^{-\beta t_2}).
\end{aligned}$$

Hence,

$$|J_1| \leq C(m,\eta,K,d)|||f|||\ (e^{-L\beta}-e^{-t_2\beta}).$$

By the same method as for J_1, one has

$$|J_2| \leq C(m,\eta,K,d)|||f|||\ (e^{-L\beta}-e^{-t_1\beta}).$$

Therefore, there exists some constant $L>0$, which is independent of N, such that as $t_1,t_2\geq L$,

$$|J_1|+|J_2|\leq\varepsilon \tag{3.13}$$

To bound $|J_3|$, choose some cube $\Lambda\subset\mathbb{Z}^d$, which is centered at 0, such that $\Lambda(f)\subset\Lambda\subset\Gamma_N$ and split $|J_3|$ into three pieces:

$$\begin{aligned}
|J_3| \leq &\left|\int_0^L S_{t_2-s}\left[\sum_{i\in\Gamma_N\setminus\Lambda}(\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j+U_i^N)\partial_i P_s^N f\right](0)ds\right| \\
&+\left|\int_0^L S_{t_2-s}\left[\sum_{i\in\Gamma_N\setminus\Lambda}(\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j+U_i^N)\partial_i P_s^N f\right](0)ds\right| \\
&+\left|\int_0^L (S_{t_2-s}-S_{t_1-s})\left[\sum_{i\in\Lambda}(\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j+U_i^N)\partial_i P_s^N f\right](0)ds\right| \\
&=I_1+I_2+I_3.
\end{aligned}$$

where I_1, I_2 and I_3 denotes the three terms on the r.h.s. of the above inequality in order.

By Lemma 2.1 and (1.8), for any $A>0$, choose $B\geq 1$ as in Lemma 2.1, as Λ is sufficiently large, which is chosen independent of N, so that

$dist(\Lambda^c, \Lambda(f)) \geq BL$ and the following '$\sum_{i\in\mathbb{Z}^d\setminus\Lambda} e^{-n_i A}$' is sufficiently small, by (3) of Assumption 1.2 and (1.8), we have

$$\begin{aligned}
&\left|S_{t_2-s}\left[\sum_{i\in\Gamma_N\setminus\Lambda}\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j\partial_i P_s^N f\right](0)\right| \\
&\leq \sum_{i\in\Gamma_N\setminus\Lambda} e^{-n_iA-As}|||f|||_1 \sum_{j\in\Gamma_N\setminus i} a_{ij}S_{t_2-s}[|x_j|](0) \\
&\leq CK^d(1+\eta)\sum_{i\in\Gamma_N\setminus\Lambda} e^{-n_iA}|||f|||_1 \leq \frac{\varepsilon}{8L}
\end{aligned} \tag{3.14}$$

where $n_i = [dist(i, \Lambda(f))/K]$. Similarly, choose some sufficiently large Λ independent of N, one has

$$\left|S_{t_2-s}\left[\sum_{i\in\Gamma_N\setminus\Lambda} U_i^N(y)\partial_i P_s^N f\right](0)\right| \leq \frac{\varepsilon}{8L}.$$

Therefore, $I_1 \leq \frac{\varepsilon}{4L}$, one has $I_2 \leq \frac{\varepsilon}{4L}$ by the same method.

Hence, as Λ is sufficiently large, which is independent of N, one has

$$I_1 + I_2 \leq \frac{\varepsilon}{2L}. \tag{3.15}$$

In the following Step 3, we shall prove that there exists some constant $T > L$, which is *independent* of N but *depends* on $|\Lambda|, L$, so that as $t_1, t_2 \geq T$

$$I_3 \leq \frac{\varepsilon}{2L}, \tag{3.16}$$

thus $|J_3| \leq \varepsilon$. Combining this with (3.13), we conclude the proof of (3.12).

Step 3: Proof of (3.16). Let us first consider the linear part of I_3, it can be split into two pieces:

$$\begin{aligned}
&\left|(S_{t_2-s} - S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in\Gamma_N\setminus i} a_{ij}x_j\partial_i P_s^N f\ ](0)\right| \\
&\leq \left|(S_{t_2-s} - S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in\Gamma_N\setminus 2\Lambda} a_{ij}x_j\partial_i P_s^N f\ ](0)\right| \\
&\quad + \left|(S_{t_2-s} - S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in 2\Lambda\setminus i} a_{ij}x_j\partial_i P_s^N f\ ](0)\right|,
\end{aligned} \tag{3.17}$$

where 2Λ is the cube centered at 0, whose edge length is two times as that of Λ. By (3) of Assumption 1.2, if Λ is large enough, the first term on the r.h.s. of (3.17) are zero since $a_{ij} = 0$ for all i, j therein.

As for the last term on the r.h.s. of (3.17), we need some delicate analysis as the following: Take some large number $H > 0$ (to be determined later) and split the term into two pieces, and control them by some uniform integrability and Lemma 3.1, more precisely, we have

$$
\begin{aligned}
&\left|(S_{t_2-s}-S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in 2\Lambda\backslash i} a_{ij}x_j\partial_i P_s^N f\ ](0)\right| \\
\leq &\left|(S_{t_2-s}-S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in 2\Lambda\backslash i} a_{ij}x_j\left(1-\chi(\frac{|x_j|}{H})\right)\partial_i P_s^N f\ ](0)\right| \\
&+\left|(S_{t_2-s}-S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in 2\Lambda\backslash i} a_{ij}x_j\chi(\frac{|x_j|}{H})\partial_i P_s^N f\ ](0)\right|
\end{aligned}
\tag{3.18}
$$

where $\chi:[0,\infty)\to[0,\infty)$ is some smooth function such that $\chi(x)=1$ as $0\leq x\leq 1$ and $\chi(x)=0$ as $x\geq 2$. By (1.8) with $1<\beta<\alpha$ therein, the fact $a_{ij}\leq 1+\eta$ and (2.6), (note that $s\leq L$), the first term on the r.h.s. of (3.18) can be bounded by

$$
\begin{aligned}
&2^d|\Lambda|(1+\eta)|||P_s^N f|||_1\left(S^0_{t_2-s}+S^0_{t_1-s}\right)\left[|x|\left(1-\chi(\frac{|x|}{H})\right)\right] \\
\leq\ &2^d|\Lambda|(1+\eta)e^{L(1+\eta)}|||f|||_1\left(S^0_{t_2-s}+S^0_{t_1-s}\right)\left[|x|\left(1-\chi(\frac{|x|}{H})\right)\right] \\
\leq\ &\frac{\varepsilon}{12L}
\end{aligned}
$$

as $H>H_0$ if $H_0=H_0(d,|\Lambda|,L,\eta)>0$ is sufficiently large.

As for the second term of (3.18), by (2.6), the fact $s\leq L$ and Lemma 3.1, one can choose some cube $\tilde\Lambda\supset 2\Lambda$ (to be determined later), so that

$$
\begin{aligned}
&\left|(S_{t_2-s}-S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in 2\Lambda\backslash i} a_{ij}x_j\chi(\frac{|x_j|}{H})\partial_i P_s^N f\ ](0)\right| \\
\leq\ &C(H,L,\eta,|\Lambda|,|\tilde\Lambda|,d)|||f|||_1\int_{\mathbb{R}}\left|p\left(\frac{1-e^{-\alpha(t_2-s)}}{\alpha},0,y\right)-p\left(\frac{1-e^{-\alpha(t_1-s)}}{\alpha},0,y\right)\right|dy \\
&+C(H,|\Lambda|)\sum_{j\in\Gamma_N\backslash\tilde\Lambda}||\partial_{ij}P_s^N f||.
\end{aligned}
$$

Therefore, by Lemma 3.3 and the fact $s\leq L$, choose some sufficiently large $\tilde\Lambda$ (independent of N), the second term on the r.h.s. of the above inequality can be uniformly bounded by $\frac{\varepsilon}{12L}$. Moreover, since $s\leq L$, there exists some constant $T=T(H,L,\eta,|\Lambda|,|\tilde\Lambda|,d)>0$ so that as $t_1,t_2\geq T$ the first term is also bounded by $\frac{\varepsilon}{12L}$.

Combining the above estimates with (3.17), we have some constant $T>0$, which is independent of N, so that

$$
\left|(S_{t_2-s}-S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in\Gamma_N\backslash i} a_{ij}x_j\partial_i P_s^N f\ ](0)\right|\leq\frac{\varepsilon}{4L}
\tag{3.19}
$$

By the same arguments, one has

$$\left|(S_{t_2-s}-S_{t_1-s})[\sum_{i\in\Lambda}\sum_{j\in\Gamma_N\setminus i}\partial_j U_i^N\partial_i P_s^N f\ ](0)\right|\le\frac{\varepsilon}{4L}. \tag{3.20}$$

Combining the above two inequalities, we immediately conclude this step. □

Proof of Theorem 1.4. Denote $\ell(f)=\lim_{t\to\infty}P_tf(0)$, which is the limit in Proposition 3.4. We shall split the proof of the theorem into the following two steps.

Step 1: $\lim_{t\to\infty}P_tf(x)=\ell(f)\ \ \forall\ x\in\mathbb{B}$. It suffices to show that the limit is true on one ball $B_{R,\rho}$. By triangle inequality,

$$\begin{aligned}|P_tf(x)-\ell(f)|\le&\,|P_tf(x)-P_t^Nf(x)|+|P_t^Nf(x)-P_t^Nf(0)|\\&+|P_t^Nf(0)-P_tf(0)|+|P_tf(0)-\ell(f)|\end{aligned} \tag{3.21}$$

For arbitrary $\varepsilon>0$, by Proposition 3.4, there exists some $T_0>0$ such that, as $t>T_0$,

$$|P_tf(0)-\ell(f)|<\varepsilon. \tag{3.22}$$

By Theorem 1.3, there exists some $N(t)\in\mathbb{N}$ such that as $N>N(t)$

$$|P_tf(x)-P_t^Nf(x)|<\varepsilon,\ |P_t^Nf(0)-P_tf(0)|<\varepsilon. \tag{3.23}$$

We remain to show that there exists some constant $T_1>0$, which is independent of N, so that as $t\ge T_1$,

$$|P_t^Nf(x)-P_t^Nf(0)|\le\varepsilon. \tag{3.24}$$

By fundamental calculus, one has

$$\begin{aligned}\left|P_t^Nf(x)-P_t^Nf(0)\right|&=\left|\int_0^1\frac{d}{d\lambda}P_t^Nf(\lambda x)d\lambda\right|\\&\le\int_0^1\sum_{i\in\Gamma_N}|\partial_iP_t^Nf(\lambda x)||x_i|d\lambda\le\sum_{i\in\Gamma_N}||\partial_iP_t^Nf|||x_i|\end{aligned}$$

For any constant $A>0$, choose $B\ge1$ as in Lemma 2.1, and also a cube $\Lambda(f)\subset\Lambda\subset\Gamma_N$ such that $dist(\Gamma_N\setminus\Lambda,\Lambda(f))=[BtK]+1$ (up to some $O(1)$ correction), therefore, by (3.3) and (2.7),

$$\begin{aligned}\sum_{i\in\Gamma_N}||\partial_iP_t^Nf|||x_i|\le&\sup_{i\in\Lambda}|x_i||\Lambda|e^{-\beta t}|||f|||_1\\&+\sum_{i\in\Gamma_N\setminus\Lambda}e^{-An_i-At}|||f|||_1R(1+|i|)^\rho\\&\le e^{-\beta t}|||f|||_1R(|\Lambda(f)|+BKt+1)^{\rho+d}+C(A,R)e^{-At}|||f|||_1\end{aligned}$$

where n_i is defined in Lemma 2.1, since $\sum_{i\in\Gamma_N\setminus\Lambda}e^{-An_i}R(1+|i|)^\rho<\infty$ uniformly on N, thus there exists some constant $T_1=T_1(A,K,R,\rho,\Lambda,f)$

so that as $t \geq T_1$,

$$\sum_{i \in \Gamma_N} ||\partial_i P_t^N f|||x_i| \leq \varepsilon.$$

Take $T = \max\{T_0, T_1\}$, as $t \geq T$, we immediately have

$$|P_t f(x) - \ell(f)| \leq 4\varepsilon.$$

Step 2: P_t is strongly mixing. From the definition of $\ell(f)$, it is clear that $\ell(\cdot)$ is a linear functional on $\mathcal{D}^\infty$. Since $\mathbb{B}$ is locally compact, by Riesz representation theorem for linear functional (see a nice introduction in [12]), there exists some unsigned Randon measure μ on $\mathbb{B}$. From the easy fact $P_t(\mathbf{1}) = 1$, we have $\mu(\mathbb{B}) = 1$, thus μ is a probability measure. On the other hand, by the Riesz representation theorem again, for each fixed t, $P_t f(x)$ also admits a probability measure $P_t^* \delta_x$. By Step 1 and the fact that $\mathcal{D}^\infty$ is dense in $\mathcal{B}_b(\mathbb{B}, \mathbb{R})$ (under product topology), we immediately have $P_t^* \delta_x \to \mu$ in weak sense, which also implies that the semigroup P_t is strongly mixing. □

4. Appendix: Formal derivation of (1.5)

Suppose that the Fourier transforms for the solution $u(t)$ and f exist, then the Fourier transform of the equation (1.4) is

$$\begin{cases} \partial_t \hat{u} = -|\lambda|^\alpha \hat{u} + \hat{u} + \lambda \partial_\lambda \hat{u} \\ \hat{u}(0) = \hat{f} \end{cases} \tag{4.1}$$

where '$\hat{\ }$' denotes the Fourier transform of functions. Suppose $\lambda > 0$, set $\nu = \ln \lambda, \quad \hat{v} = e^{-t} \hat{u}(e^\nu), \quad \hat{g}(\nu) = \hat{f}(e^\nu)$, we have

$$\begin{cases} \partial_t \hat{v} = -e^{\alpha \nu} \hat{v} + \partial_\nu \hat{v} \\ \hat{v}(0) = \hat{g}(\nu) \end{cases} \tag{4.2}$$

Suppose $\hat{g}$ is positive, set $\hat{w} = ln \hat{v}$, the equation for $\hat{w}$ is

$$\begin{cases} \partial_t \hat{w} = -e^{\alpha \nu} + \partial_\nu \hat{w} \\ \hat{v}(0) = ln \hat{g}(\nu) \end{cases} \tag{4.3}$$

It is easy to solve the above equation by $\hat{w}(t) = ln \hat{g}(\nu + t) - e^{\alpha \nu} \frac{e^{\alpha t} - 1}{\alpha}$, thus

$$\hat{w}(t) = \hat{g}(\nu + t) \exp\{-e^{\alpha \nu} \frac{e^{\alpha t} - 1}{\alpha}\}$$

and

$$\hat{u}(t) = \hat{g}(\nu + t) \exp\{t - e^{\alpha \nu} \frac{e^{\alpha t} - 1}{\alpha}\} = \hat{f}(e^t \lambda) \exp\{t - |\lambda|^\alpha \frac{e^{\alpha t} - 1}{\alpha}\}.$$

Hence, by Parseval's Theorem, we have

$$\begin{aligned} u(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(e^t \lambda) \exp\{t - |\lambda|^\alpha \frac{e^{\alpha t} - 1}{\alpha}\} e^{i\lambda x} d\lambda \\ &= \int_{\mathbf{R}} \hat{f}(\lambda) \frac{1}{\sqrt{2\pi}} \exp\{-|\lambda|^\alpha \frac{1 - e^{-\alpha t}}{\alpha} + i\lambda e^{-t} x\} d\lambda \\ &= \int_{\mathbf{R}} p\left(\frac{1 - e^{-\alpha t}}{\alpha}; e^{-t} x, y\right) f(y) dy \end{aligned}$$

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PO Box 513, EURANDOM, 5600 MB Eindhoven. The Netherlands
E-mail address: xu@eurandom.tue.nl