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ERGODICITY OF THE 3D STOCHASTIC NAVIER-STOKES EQUATIONS DRIVEN BY MILDLY DEGENERATE NOISE

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Abstract. We prove that the any Markov solution to the 3D stochastic Navier-Stokes equations driven by a mildly degenerate noise (i.e. all but finitely many Fourier modes are forced) is uniquely ergodic. This follows by proving strong Feller regularity and irreducibility.

1. Introduction

The well-posedness of three dimensional Navier-Stokes equations is still an open problem, in both the deterministic and stochastic cases (see [9] for a general introduction to the deterministic problem and [14] for the stochastic one). Although the existence of global weak solutions have been proven in both cases ([18], [10]), the uniqueness is still unknown. Inspired by the Hadamard definition of well-posedness for Cauchy problems, it is natural to ask if there are ways to find a good selection among the weak solutions to obtain additional properties, such as Markovianity or continuity with respect to the initial data.

Da Prato and Debussche proved in [3] that there exists a continuous selection (i.e. the selection is strong Feller) with unique invariant measure by studying the Kolmogorov equation associated to the stochastic Navier-Stokes equations (SNSE). Later Debussche and Odasso [6] proved that this selection is also Markovian. However, their approach essentially depends on the non-degeneracy of the driving noise. A different and slightly more general approach to Markov solutions, which includes the cases of degenerate noise and even deterministic equations, was introduced in [14]. Under the assumption of non-degeneracy and regularity of the covariance, the authors also proved that every Markov solution is strong Feller. Under the same assumptions every such dynamics is uniquely ergodic and exponentially mixing ([22]). However, both approaches rely on the non-degeneracy of the driving noise to obtain the strong Feller property, and consequently
ergodicity.

The strong Feller property and ergodicity of SPDEs driven by degenerate noise have been intensively studied in recent years (see for instance [8],[16], [7], [17], [21]). For the two dimensional case there are several results on ergodicity, among which the most remarkable one is by Hairer and Mattingly [16]. They prove that the 2D stochastic dynamics has a unique invariant measure as long as the noise forces at least two linearly independent Fourier modes. In this respect the three dimensional case is still open (only partial results are known, see the aforementioned [3], [14], [22], see also [21], [20]) and this paper tries to partly fill this gap. More precisely, we will study the three dimensional Navier-Stokes equations

$$\begin{align*}
\dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= \dot{\eta}, \\
\text{div } u &= 0, \\
 u(0) &= x,
\end{align*}$$

(1.1)
on the torus $[0, 2\pi]^3$ with periodic boundary conditions and forced by a Gaussian noise $\dot{\eta}$. We assume that all except finitely many Fourier modes are driven by the noise, and prove that any Markov solution to the problem is strong Feller and ergodic.

Essentially, our approach combines the Malliavin calculus developed in [8] and the weak-strong uniqueness principle of [14]. Comparing with well-posed problems, the dynamics here exists only in the weak martingale sense and the standard tools of stochastic analysis are not available. Hence, the computations are made on an approximate cutoff dynamics (see Section 2.3), which equals any dynamics up to a small time. On the other hand, due to the degeneracy of the noise, the Bismut-Elworthy-Li formula cannot directly be applied to prove the strong Feller property. To fix this problem, we divide the dynamics into high and low frequencies, applying the formula only to the dynamics of high modes (thanks to the essential non-degeneracy of the noise).

Finally, we remark that, at least with the approach presented here, general results such as the truly hypoelliptic case in [16] seem to be hardly achievable. Here (as well as in [14]) the strong Feller property is essential to propagate smoothness from small times (where trajectories are regular with high probability) to all times. To overcome this difficulty and understand how to study the general case, the second author (with one of his collaborator) is proving in a work in progress ([1]) some results similar to those in this paper, via the Kolmogorov equation approach originally used in [3].

The paper is organized as follows. Section 2 gives a detailed description of the problem, the assumptions on the noise and the main results (Theorems 2.4 and 2.5). Section 3 contains the proof of strong Feller regularity, while Section 4 applies Malliavin calculus to prove the crucial Lemma 3.3. Section 5 shows the irreducibility of the dynamics, the appendix contains additional details and the proofs of some technical results.
2. Description of the problem and main results

Before stating the main results of the paper, we recast the problem in an abstract form, give the assumption on the noise and recall a few known results.

2.1. Settings and notations.

Let us start by writing (1.1) in an abstract form, using the standard formalism for the equations (see Temam [26] for details). Let $\mathbb{T}^3 = [0, 2\pi]^3$ be the three-dimensional torus, let $H$ be the subspace of $L^2(\mathbb{T}^3; \mathbb{R}^3)$ of mean-zero divergence-free vector fields and let $P$ be the projection from $L^2(\mathbb{T}^3; \mathbb{R}^3)$ onto $H$. Denote by $A$ the Stokes operator (that is, $A = -P\Delta$ is the projection on $H$ of the Laplace operator) and by $B(u, v) = P(u \cdot \nabla)v$ the projection of the nonlinearity. Following Temam [26], we consider the spaces $V_\alpha = D(A^{\alpha/2})$ and in particular we set $V = V_1$. Problem (1.1) is recast in the following form,

\[
\begin{aligned}
du + [\nu Au + B(u, u)] \, dt &= Q \, dW_t, \\
u(0) &= x.
\end{aligned}
\]

where $Q$ is a bounded operator on $H$ satisfying suitable assumptions (see below) and $W$ is a cylindrical Brownian motion on $H$. In the rest of the paper we shall assume $\nu = 1$, as its exact value will play no essential role.

Consider on $H$ the Fourier basis $(e_k)_{k \in \mathbb{Z}^3}$ defined in (A.1) and, given $N \geq 1$, let $\pi_N : H \to H$ be the projection onto the subspace of $H$ generated by all modes $k$ such that $|k|_\infty := \max |k_i| \leq N$.

**Assumption 2.1** (Assumptions on $Q$). The operator $Q : H \to H$ is linear bounded and there are $\alpha_0 > \frac{1}{2}$ and an integer $N_0 \geq 1$ such that

- [A1] (diagonality) $Q$ is diagonal on the Fourier basis $(e_k)_{k \in \mathbb{Z}^3}$,
- [A2] (finite degeneracy) $\pi_{N_0}Q = 0$ and $\ker((Id - \pi_{N_0})Q) = \{0\}$,
- [A3] (regularity) $(Id - \pi_{N_0})A^{\alpha_0 + 3/4}Q$ is bounded invertible (with bounded inverse) on $(Id - \pi_{N_0})H$.

Further details can be found in Subsection A.1. We only remark that [A3] is essentially the same as in [14] (we restrict here to $\alpha_0 > \frac{1}{2}$ for simplicity), while [A2] is the main assumption. The restriction $\pi_{N_0}Q = 0$ in [A2] (as well as property [A1]) has been taken to simplify the exposition.

2.2. Markov solutions.

Following the framework introduced in [14] (to which we refer for further details), we define the weak martingale solutions to problem (2.1) (cfr. Definition 3.3, [14]).

**Definition 2.2** (Weak martingale solutions). Given a probability measure $\mu$ on $H$, a solution $P$ to problem (2.1) with initial condition $\mu$ is a probability measure on $\Omega = C([0, \infty); D(A'))$ such that

1. the marginal at time $t = 0$ of $P$ is equal to $\mu$,
2. $P[L^\infty_{loc}([0, \infty); H) \cap L^2_{loc}([0, \infty); V)] = 1$,
A regularized cut-off problem.

Let us make this rough observation more precise.

Thanks to (2.2), for every \( x \in H \), there is a Lebesgue null-set \( T_x \subset (0, \infty) \) such that \( P_t \phi(x) = P_s P_t \phi(x) \) for all \( t \geq 0 \) and all \( s \notin T_x \).

2.3. A regularized cut-off problem. The dynamics (1.1) is dissipative, hence it is possible to prove existence of a unique solution up to a small random time. Within this time, the solution to the following equation (2.3) coincides with any Markov solution.

Let \( \chi : [0, \infty) \to [0, 1] \) be a smooth function such that \( \chi(r) \equiv 1 \) for \( r \leq 1 \) and \( \chi(r) \equiv 0 \) for \( r \geq 2 \). Set

\[
W = V_{\alpha_0 + \frac{1}{2}}, \quad W' = V_{-(\alpha_0 + \frac{1}{2})}, \quad \tilde{W} = V_{\alpha_0 + \frac{1}{2}},
\]

(where \( \alpha_0 \) is the constant in the Assumption 2.1). Given \( \rho > 0 \), and \( x \in W \), consider

\[
\begin{cases}
du^\rho + [Au^\rho + B(u^\rho, u^\rho)\chi(\frac{|u^\rho|}{\rho})]dt = Q(u^\rho)dW_t \\
u^\rho(0) = x,
\end{cases}
\]

where

\[
Q(u) = Q + (1 - \chi(\frac{|u|}{\rho}))\overline{Q}
\]

and \( \overline{Q} \) is a non-degenerate operator on \( \pi_{W_0}H \) (see (A.2) for a detailed definition). It is easy to see that \( Q(u) \) is non-degenerate as \( |u|_W \leq \rho \).

**Theorem 2.3** (Weak-strong uniqueness). For every \( x \in W \), there exists a unique weak solution to (2.3) so that the associated distribution \( P_x^\rho \) satisfies \( P_x^\rho[C(0, \infty);W] = 1 \). Moreover, given \( \rho \geq 1 \), define \( \tau_\rho : \Omega \to [0, \infty] \) by

\[
\tau_\rho(\omega) = \inf\{t \geq 0 : |\omega(t)|_W \geq \rho\},
\]
(and \(\tau_\rho(\omega) = \infty \) if the set is empty). If \(x \in \mathcal{W}\) and \(|x|_\mathcal{W} < \rho\), then on \([0, \tau_\rho]\), \(P^\rho_t\) coincides with any Markov solution \((P_x)_{x \in \mathcal{W}}\) of (2.1), i.e., for all \(t > 0\) and \(\phi \in \mathcal{B}_b(H)\),

\[
E^{P^\rho_x}[\phi(\xi_t)1_{\{\tau_\rho \geq t\}}] = E^{P^\rho_x}[\phi(\xi_t)1_{\{\tau_\rho \geq t\}}].
\]

Finally, if \(|x|_\mathcal{W} < \rho\), then

\[
\lim_{\epsilon \to 0} P^\rho_{x+h}[\tau_\rho \geq \epsilon] = 1,
\]

uniformly for \(h\) in any closed subset of \(\{h \in \mathcal{W} : |x + h|_\mathcal{W} < \rho\}\).

Proof. Existence and uniqueness for problem (2.3) are standard, since the nonlinearity and the operator \(Q(u^\rho)\) are Lipschitz. Let \(\tilde{w}^\rho\) be the solution to problem (2.3) with \(Q(u^\rho)\) replaced by \(Q\), then \(\tau_\rho(\tilde{w}^\rho) = \tau_\rho(w^\rho)\). By pathwise uniqueness, \(w^\rho(t) = \tilde{w}^\rho(t)\) on \([0, \tau_\rho]\). This immediately implies (2.4) and (2.5) by Theorem 5.12 of [14].

2.4. Main results. The strong Feller and ergodicity results of [14], [13], [22] are obtained under a strong non-degeneracy assumption on the covariance. This paper relaxes this assumption, as shown by the following results.

Theorem 2.4. Assume Assumption 2.1. Let \((P_x)_{x \in \mathcal{H}}\) be a Markov solution to (2.1), and let \((P_t)_{t \geq 0}\) be the associated transition semigroup. Then \((P_t)_{t \geq 0}\) is strong Feller in \(\mathcal{W}\).

Proof. The theorem is a straightforward application of Theorem 5.4 of [14], once Theorems 2.3 and 3.1 are taken into account.

Theorem 2.5. Under the same assumptions of the previous theorem, every Markov solution \((P_x)_{x \in \mathcal{H}}\) to (2.1) is uniquely ergodic and strongly mixing. Moreover, the (unique) invariant measure \(\mu\) corresponding to a given Markov solution is fully supported on \(\mathcal{W}\), i.e., \(\mu(\mathcal{W}) = 1\) and \(\mu(U) > 0\) for every open set \(U\) of \(\mathcal{W}\).

Proof. Given a Markov solution \((P_x)_{x \in \mathcal{H}}\), there exists at least one invariant measure (Theorem 3.1, [22]). Uniqueness follows from Doob’s theorem (Theorem 4.2.1 of [4]), since by Theorem 2.4 and Proposition 5.1 the system is both strong Feller and irreducible. The claim on the support follows again from Proposition 5.1.

Remark 2.6. The strong Feller estimate on the transition semigroup can be made more quantitative with the same method used in [13], but unfortunately this only gives a Lipschitz estimate for the semigroup up to a logarithmic correction (compare with [3]).

Moreover, by Theorem 3.3 of [22], the convergence to the invariant measure is exponentially fast, if the Markov solutions satisfy an almost sure version of the energy inequality (see [22], [24]). The theorem in [22] is proved under an assumption of non-degeneracy of the noise, but the only arguments really used are that the dynamics is strong Feller and irreducible.

3. Strong Feller property of cutoff dynamics

This section will mainly prove the following theorem:

Theorem 3.1. There is \(\rho_0 > 0\) (depending only on \(N_0\) and \(Q\)) such that for \(\rho \geq \rho_0\) the transition semigroup \(P^\rho_t\) associated to equation (2.3) is strong Feller.
Fix $N \geq N_0$ (whose value will be suitably chosen later in Proposition 4.5). In this and the following section we shall denote with the superscript $L$ the quantities projected onto the modes smaller than $N$ and with the superscript $H$ those projected onto the modes larger than $N$. We divide the equation (2.3) into the low and high frequency parts (dropping the $\rho$ for simplicity),

$$
\begin{align*}
&\left\{\begin{array}{ll}
\frac{du^L}{dt} + [A_{u}^L + B^L(u,u)\chi(\frac{|u|^2}{\delta^2})] dt = Q^L(u) dW_t^L \\
\frac{du^H}{dt} + [A_{u}^H + B^H(u,u)\chi(\frac{|u|^2}{\delta^2})] dt = Q^H dW_t^H
\end{array}\right.
\end{align*}
$$

where $u^L = \pi_N u$, $u^H = (I - \pi_N)u$, $W^L = \pi_N W$, $W^H = (I - \pi_N)W$, $B_L = \pi_N B$, $B_H = (I - \pi_N)B$, $Q_L(u) = Q(u)\pi_N$ and $Q_H = Q(u)(I - \pi_N)$. In particular, $Q_H$ is independent of $u$.

With the above separation for the dynamics, it is natural to define the Frechet derivatives for their low and high frequency parts. More precisely, for any stochastic process $X(t,x)$ on $H$ with $X(0,x) = x$, the Frechet derivative $D_hX(t,x)$ is defined by

$$
D_hX(t,x) := \lim_{\epsilon \to 0} \frac{X(t,x + \epsilon h) - X(t,x)}{\epsilon}, \quad h \in H,
$$

provided the limit exists. Moreover, it is natural to define the linear map $D^X(t,x) : H \to H$ by

$$
D^X(t,x)h = D_hX(t,x), \quad h \in H.
$$

One can easily define $D_L(t,x)$, $D_HX(t,x)$, $D_LX^H(t,x)$, $D_HX^L(t,x)$ and so on in a similar way, for instance, $D_HX^L(t,x) : H^H \to H^L$ is defined by

$$
D_HX^L(t,x)h = D_hX^L(t,x), \quad h \in H^H
$$

with $D_hX^L(t,x) = \frac{1}{\epsilon} \lim_{\epsilon \to 0}[X^L(t,x + \epsilon h) - X^L(t,x)]$.

Let $C^k_b(W)$ be the set of functions on $W$ with bounded $0$-th, $\ldots$, $k$-th order derivatives. Given a $\psi \in C^k_b(W)$, for any $h \in W$, the derivative of $\psi(x)$ along $h$, denoted by $D_h\psi(x)$, is defined by

$$
D_h\psi(x) = \lim_{\epsilon \to 0} \frac{\psi(x + \epsilon h) - \psi(x)}{\epsilon}.
$$

Clearly, the map $D\psi(x) : W \to \mathbb{R}$, defined by $D\psi(x)h = D_h\psi(x)$ for all $h \in W$, is linear bounded. Hence $D\psi(x) \in W'$. Similarly, $D_L\psi(x)$ and $D_H\psi(x)$ can be defined (e.g. $D_L\psi(x)h = \lim_{\epsilon \to 0}[\psi(x + \epsilon h) - \psi(x)]/\epsilon$, $h \in W_L$).

To prove Theorem 3.1, we need to approximate (3.1) by the following more regular dynamics:

$$
\begin{align*}
&\left\{\begin{array}{ll}
\frac{du^{\delta,\rho}}{dt} + [A_{u}^{\delta,\rho} + e^{-A_H^{\delta,\rho}}B(u^{\delta,\rho},u^{\delta,\rho})\chi(\frac{|u^{\delta,\rho}|^2}{\delta^2})] dt = Q(u^{\delta,\rho}) dW_t \\
u^{\delta,\rho}(0) = x
\end{array}\right.
\end{align*}
$$

where $\delta > 0$ and $A_H = (I - \pi_N)A$ (the existence and uniqueness of weak solution to equation (3.2) is standard). The reason for introducing this approximation, roughly speaking, is that one cannot prove $B(u,v) \in \text{Ran}(Q)$ but easily has $e^{-A_H^{\delta,\rho}}B(u,v) \in$
\( Ran(Q) \), which is the key point for finding a suitable direction for the Malliavin derivatives (see Section 4).

Define two maps \( \Phi_t(\cdot) \) and \( \Phi_t^\delta(\cdot) \) from \( H \) to \( H \) by
\[
\Phi_t(x) := u^\rho(t) \quad \text{and} \quad \Phi_t^\delta(x) := u^{\delta, \rho}(t),
\]
where \( u^\rho(t) \), \( u^{\delta, \rho}(t) \) are the solutions to (2.3) and (3.2) respectively. The following proposition shows that \( \Phi_t \) is the limit of \( \Phi_t^\delta \) as \( \delta \to 0^+ \) in the same sense, and will be proven in the appendix.

**Proposition 3.2.** For every \( T > 0 \) and \( p \geq 2 \), there exist some constant \( C_i = C_i(p, \rho, \alpha_0) > 0 \), \( i = 1, 2 \) such that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Phi_t - \Phi_t^\delta|^p \right] \leq C_1 e^{C_1 T} |e^{-A\delta} - I|_W^p, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |D\Phi_t - D\Phi_t^\delta|^p \right] \leq C_2 e^{C_2 T} |e^{-A\delta} - I|_W^p.
\]

For any \( \psi \in C^1_b(H) \), \( h \in W \) and \( t > 0 \),
\[
\lim_{\delta \to 0^+} |D_h \mathbb{E}[\psi(\Phi_t)] - D_h \mathbb{E}[\psi(\Phi_t^\delta)]| = 0.
\]

The main ingredients of the proof of Theorem 3.1 are the following two lemmas, i.e. Lemmas 3.3 (proved in Section 4) and 3.4 (proved in the appendix, see page 22).

**Lemma 3.3.** There exists some constant \( p > 1 \) (possibly large) such that such that for every \( x \in \overline{W} \), \( h \in W^L \), \( \psi \in C^1_b(H) \) and \( t \geq t_0 \),
\[
\mathbb{E}[|D_L \psi(\Phi_t^\delta(x))|D_h \Phi_t^{\delta, L}(x)|] \leq \frac{C_6 e^{C_1 T} (1 + |x|_{\overline{W}})^p}{p^p} \|\psi\|_\infty |h|_W,
\]
where \( C = C(\rho, \alpha_0) > 0 \).

**Lemma 3.4.** For any \( T > 0 \), \( p \geq 2 \) and \( \delta \geq 0 \), there exist some \( C_i = C_i(p, \alpha_0, \rho) \), \( i = 1, \ldots, 7 \), such that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Phi_t^\delta(x)|_W^p \right] \leq C_1 e^{C_1 T} |x|_W^p, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Phi_t^\delta(x)|_W^p \right] \leq C_2 e^{C_2 T} |x|_W^p, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |t^{1/2} \Phi_t^\delta(x)|_W^p \right] \leq C_3 e^{C_3 T} |x|_W^p, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |D_h \Phi_t^\delta(x)|_W^p \right] \leq C_4 e^{C_4 T} |h|_W^p, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |D_h \Phi_t^{\delta, L}(x)|_W^p \right] \leq C_5 e^{C_5 T} |h^L|_W^p, \\
\mathbb{E}\left[ \int_0^T |A^{1/2} D_h \Phi_t^\delta(x)|_W^2 ds \right] \leq C_6 e^{C_6 T} |h^2|_W, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |D_h \Phi_t^{\delta, H}(x)|_W^p \right] \leq (T^{p/2} + T^{p/8}) C_7 e^{C_7 T} |h^H|_W^p, \\
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |D_h \Phi_t^{\delta, H}(x)|_W^p \right] \leq (T^{p/2} + T^{p/8}) C_7 e^{C_7 T} |h^H|_W^p.
Proof of Theorem 3.1. Here we follow the idea in the proof of Proposition 5.2 of [8]. Set $S_t \psi(x) = \mathbb{E}[\psi(\Phi_t^x)]$ for any $\psi \in C_b^2(\mathcal{W})$, we prove the theorem in the following two steps.

**Step 1.** Estimate $D S_t \psi(x)$ for all $x \in \mathcal{W}$: By Assumption 2.1, the operator $A_{t}^{3/4 + \rho}\Phi^x_{t,t_2}$ is bounded invertible on $H$, we know by (3.10) that $y_t^H = Q_H^{-1} D_{t}^{H} \Phi_t^x \otimes H \in H^H dt \times dP$ a.s., hence we can proceed as in the proof of Proposition 5.2 of [8] (more precisely, formula (5.8)) to get

$$ D_{t} H S_t \psi(x) = \frac{2}{t} \mathbb{E} \left[ \psi(\Phi_t^x) \int_{\mathcal{W}} \langle y_s^H, dW_s^H \rangle_H \right] + \frac{2}{t} \int_{\mathcal{W}} \mathbb{E} \left[ |D_t S_{t-s} \psi(\Phi_{t-s}^x) D_{t}^{H} \Phi_{t-s}^x|_H^2 \right] \, ds $$

Hence, by Burkholder-Davis-Gundy’s inequality,

$$ |D_{t} H S_t \psi(x)| \leq \frac{2}{t} \|\psi\|_\infty \left( \int_{\mathcal{W}} \mathbb{E} \|y_s^H\|_H^2 \, ds \right)^{1/2} + \frac{2}{t} \int_{\mathcal{W}} \mathbb{E} \left[ |D_t S_{t-s} \psi(\Phi_{t-s}^x)|_{H} |D_{t}^{H} \Phi_{t-s}^x|_H^2 \right] \, ds $$

with $C_1 = C_1(\rho, \alpha, \rho)$, since by (3.10),

$$ \int_{\mathcal{W}} \mathbb{E} \|y_s^H\|_H^2 \, ds = \int_{\mathcal{W}} \mathbb{E} \|Q_H^{-1} D_{t}^{H} \Phi_t^x\|_H^2 \, ds \leq c \int_{\mathcal{W}} \mathbb{E} \|A_{t}^{1/2} D_{t}^{H} \Phi_t^x\|_H^2 \, ds \leq C e^{t^2} \|H\|_H^2. $$

For the low frequency part, according to Lemma 3.3, there exists $C_2 = C_2(\alpha, \rho)$ such that

$$ |D_{t} L S_t \psi(x)| = |D_{t} L S_{t/2}(S_{t/2} \psi) (x)| = \mathbb{E} \left[ |D_t L S_{t/2} \psi(\Phi_{t/2}^x) D_{t}^{H} \Phi_{t/2}^x|_H^2 \right] + \mathbb{E} \left[ |D_{t} L S_{t/2} \psi(\Phi_{t/2}^x)|_{H} |D_{t}^{H} \Phi_{t/2}^x|_H^2 \right] $$

where $p > 1$ is the constant in Lemma 3.3.

Fix $0 < T < 1$, denote

$$ \psi_T = \sup_{x \in \mathcal{W}, 0 \leq t \leq T} \frac{t^p |DS_t \psi(x)|_W}{(1 + |x|_W^p)^p}, $$

combine (3.13) and (3.14), then for every $t \in (0, T]$,

$$ |D_{t} S_t \psi(x)| \leq C_1 \frac{e^{t^2 T}}{t} \|\psi\|_\infty \|H\|_W + C_2 \frac{e^{t^2 T}}{t} \|\psi\|_\infty \|H\|_W $$

$$ + \psi_T \left[ \frac{2}{t} \int_{\mathcal{W}} \frac{1}{(t-s)^p} \mathbb{E} \left[ |(\Phi_s^x|_W^p)|D_{t}^{H} \Phi_{t}^x|_H^2 \right] \, ds + \frac{2}{t} \mathbb{E} \left[ |(\Phi_{t/2}^x|_W^p)|D_{t}^{H} \Phi_{t/2}^x|_H^2 \right] \right]. $$
thus (noticing \(0 < T < 1\))
\[
\frac{t^p|D_hS_t\psi(x)|}{(1 + |x|_W^2)^p} \leq C_3 e^{C_3T} \|\psi\|_\infty |h|_W + \psi_T e^{C_4T} T^{1/8} |h|_W,
\]
where \(C_i = C_i(p, \alpha_0, \rho) > 0\) (i=3,4) and the previous inequality is due to
\[
(E[(1 + |\Phi_s^\delta|_W)^p |D_h^\delta \Phi_s^L|_W])^2 \leq E[\sup_{0 \leq s \leq T} (1 + |\Phi_s^\delta|_W)^{2p}] E[\sup_{0 \leq s \leq T} |D_h^\delta \Phi_s^L|_W^2] \leq T^{1/4} C e^{C T} |h|_W^2 (1 + |x|_W)^2p.
\]
Hence
\[
\psi_T \leq C_3 e^{C_3T} \|\psi\|_\infty |h|_W + \psi_T e^{C_4T} T^{1/8} |h|_W.
\]
From the above inequality, as \(T\) is sufficiently small, we have
\[
\psi_T \leq C_5 \|\psi\|_\infty
\]
with \(C_5 = C_5(T, \rho, \alpha_0) > 0\), thus for \(0 < t \leq T,\)
\[
(3.15) \quad |DS_t \psi(x)|_W \leq \frac{C_5 (1 + |x|_W^2)^p}{t^p} \|\psi\|_\infty.
\]

**Step 2. Strong Feller property of \(P_t^\delta\).** Applying Cauchy-Schwartz inequality, (3.15), (3.9) and (3.8) in order, for any \(h \in W\) and any \(0 < t \leq T\), we have
\[
|D_hS_t^\delta \psi(x)|^2 = |E[DS_t^\delta \psi(\Phi_t^\delta)D_h^\delta \Phi_t^\delta]|^2 \leq E[|DS_t^\delta \psi(\Phi_t^\delta)|^2 |W]|E[|D_h^\delta \Phi_t^\delta|^2 |W] \leq \frac{C}{t^{2p}} \|\psi\|_\infty^2 E[(1 + |\Phi_t^\delta|_W^2)^{3p} |h|_W^2 \leq \frac{C}{t^{2p}} \|\psi\|_\infty^2 (1 + |x|_W)^{3p} |h|_W^2
\]
where \(C = C(\alpha_0, \rho, T)\). Let \(\delta \to 0^+\), we have by (3.5)
\[
(3.16) \quad |D_hP_t^\delta \psi(x)| \leq \frac{C}{t^{2p/8}} \|\psi\|_\infty |x|_W |h|_W, \quad 0 < t \leq T.
\]
Clearly, (3.16) implies that \((P_t^\delta)_{t \in (0, T)}\) is strong Feller ([4]). The extension of the strong Feller property to arbitrary \(T > 0\) is standard. \(\square\)

4. Malliavin Calculus and Proof of Lemma 3.3

In this section, we will only study the equation (3.2), following the idea in [8] to prove Lemma 3.3. A very important point is that all the estimates in lemmas 4.2 and 4.3 are independent of \(\delta\) (thanks to the cutoff and to that our Malliavin calculus is essentially on low frequency part of \(\Phi_t^\delta\)). We will simply write \(\Phi_t = \Phi_t^\delta\) throughout this section.

4.1. Proof of Lemma 3.3. Given \(v \in L^2_{\text{loc}}(R_+, H)\), the Malliavin derivative of \(\Phi_t\) in direction \(v\), denoted by \(D_v \Phi_t\), is defined by
\[
D_v \Phi_t = \lim_{\epsilon \to 0} \frac{\Phi_t(W + \epsilon V, x) - \Phi_t(W, x)}{\epsilon}
\]
where \(V(t) = \int_0^t v(s) \, ds\). The direction \(v\) can be random and is adapted to the filtration generated by \(W\). The Malliavin derivatives on the low and high frequency parts, denoted
by $D_v \Phi_t^L$ and $D_v \Phi_t^H$, can be defined in a similar way. $D_v \Phi_t^L$ and $D_v \Phi_t^H$ satisfies the following two SPDEs respectively:

$$dD_v \Phi^L + [AD_v \Phi^L + DL(B_L(\Phi, \Phi)\chi(\frac{|\Phi|_W}{3\rho}))D_v \Phi^L + DH(B_L(\Phi, \Phi)\chi(\frac{|\Phi|_W}{3\rho}))D_v \Phi^H]dt = [DLQ_L(\Phi)D_v \Phi^L + DHQ_L(\Phi)D_v \Phi^H]dW^L_t + Q_L(\Phi)v^L_t dt,$$

(4.1)

$$dD_v \Phi^H + [AD_v \Phi^H + DL(e^{-AH^}\delta B_H(\Phi, \Phi)\chi(\frac{|\Phi|_W}{3\rho}))D_v \Phi^H + DH(e^{-AH^}\delta B_H(\Phi, \Phi)\chi(\frac{|\Phi|_W}{3\rho}))D_v \Phi^H]dt = Q_H v^H dt,$$

(4.2)

with $D_v \Phi_0^L = 0$ and $D_v \Phi_0^H = 0$.

Define the derivative flow of $\Phi_t^L(x)$ between $s$ and $t$ by $J_{s,t}(x)$, $s \leq t$, which satisfies the following equation: for all $h \in H^L$

$$dJ_{s,t}h + [AJ_{s,t}h + DL(B_L(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}))J_{s,t}h]dt = DLQ_L(\Phi_t)J_{s,t}hdW^L_t,$$

with $J_{s,s}(x) = Id \in \mathcal{L}(H^L, H^L)$. The inverse $J_{s,t}^{-1}(x)$ satisfies

$$dJ_{s,t}^{-1}h - J_{s,t}^{-1}\left[Ah + DL(B_L(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}))h - Tr((DLQ_L(\Phi_t))2)h\right]dt = -J_{s,t}^{-1}DLQ_L(\Phi_t)hdW^L_t$$

with $Tr((DLQ_L(\Phi_t))2)h = \sum_{k \in Z(N)} \sum_{i=1}^2 D[q_k(\Phi_t)e_k^i]D[q_k(\Phi_t)e_k^i]h$ and $q_k(x) = (1 - \chi(|x|_W/\rho))q_k$ (recall the notations in Appendix A.1). Simply writing $J_t = J_{0,t}$, clearly,

$$J_{s,t} = J_t J_{s,t}^{-1}.$$

We follow the ideas in Section 6.1 of [8] to develop a Malliavin calculus for (3.2). One of the key points for this approach is to find an adapted process $v \in L^2_{loc}(\mathbb{R}^+, H)$ so that

$$Q_H v^H(t) = DL(e^{-AH^}\delta B_H(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}))D_v \Phi_t^L,$$

which implies that $D_v \Phi_t^H = 0$ for all $t > 0$ (hence, the Malliavin calculus is essentially restricted in low frequency part). More precisely,

**Proposition 4.1.** There exists $v \in L^2_{loc}(\mathbb{R}^+; H)$ satisfying (4.4), and

$$D_v \Phi_t^L = J_t \int_0^t J_{s,t}^{-1}Q_L(\Phi_s)v^L(s)ds \quad \text{and} \quad D_v \Phi_t^H = 0.$$

**Proof.** We first claim that

$$DL(e^{-AH^}\delta B_H(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}))D_v \Phi_t^L \in (D(A_0^{\alpha + 3/4}))^H.$$
Indeed, $\Phi_t \in \tilde{W}$ from (3.8). Since $\mathcal{D}_v \Phi^L_t$ is finite dimensional, $\mathcal{D}_v \Phi^L_t \in \tilde{W}$. It is easy to see
\[
\mathcal{D}_L(e^{-Ah\delta}B_H(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}))\mathcal{D}_v \Phi^L_t = e^{-Ah\delta}B_H(\mathcal{D}_v \Phi^L_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}) + \\
e^{-Ah\delta}B_H(\Phi_t, \mathcal{D}_v \Phi^L_t)\chi(\frac{|\Phi_t|_W}{3\rho}) + e^{-Ah\delta}B_H(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}) + e^{-Ah\delta}B_H(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho}) + e^{-Ah\delta}B_H(\Phi_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho})
\]

The three terms on the right hand of the above equality can all be bounded in the same way, for instance, applying (A.6) with $\beta = \alpha_0 + 1/8$, the first term is bounded by
\[
|e^{-Ah\delta}B_H(\mathcal{D}_v \Phi^L_t, \Phi_t)\chi(\frac{|\Phi_t|_W}{3\rho})|_{\mathcal{D}(A^{\alpha_0} + \frac{1}{8})} = |A^\frac{2}{3}e^{-Ah\delta}A^{\alpha_0 - \frac{2}{3}}B_H(\mathcal{D}_v \Phi^L_t, \Phi_t)|_{H} \leq \frac{C_1}{\delta^4}|\mathcal{D}_v \Phi^L_t|_{W}|\Phi_t|_{\tilde{W}},
\]
and (4.5) follows immediately. Hence, by Assumption [A3] for $Q$, there exists at least one $v \in L^2_{lo}(\mathbb{R}^d; H)$ so that $v^H$ satisfies (4.4) (we will see in (4.6) that $\mathcal{D}_v \Phi^L_t$ does not depend on $v^H$). Thus equation (4.2) is a homogeneous linear equation and has a unique solution
\[
\mathcal{D}_v \Phi^H_t = 0,
\]
for all $t > 0$. Hence, equation (4.1) now reads
\[
d\mathcal{D}_v \Phi^L + [A \mathcal{D}_v \Phi^L + \mathcal{D}_L(B_L(\Phi, \Phi)\chi(\frac{|\Phi|_W}{3\rho}))\mathcal{D}_v \Phi^L] dt = \mathcal{D}_LQ_L(\Phi)\mathcal{D}_v \Phi^L dW^L_t + Q_L(\Phi) v^L dt,
\]
with $\mathcal{D}_v \Phi^L_0 = 0$, which is solved by
\[
\mathcal{D}_v \Phi^L_t = \int_0^t J_{s,t}Q_L(\Phi_s)v^L(s) ds = J_t \int_0^t J_{s}^{-1}Q_L(\Phi_s)v^L(s) ds
\]

Let $N \geq N_0$ be the integer fixed at the beginning of Section 3 and consider $M = 2(2N+1)^3-2$ vectors $v_1, \ldots, v_M \in L^2_{lo}(\mathbb{R}^d; H)$, with each of them satisfying Proposition 4.1 (notice that $M$ is the dimension of $H^L = \pi_N H$). Set
\[
v = [v_1, \ldots, v_M],
\]
we have
\[
\mathcal{D}_v \Phi^H_t = 0, \quad \mathcal{D}_v \Phi^L_t = J_t \int_0^t J_{s}^{-1}Q_L(\Phi_s)v^L(s) ds,
\]
where $Q_L$ is defined in (3.1). Choose
\[
v^L(s) = (J_{s}^{-1}Q_L(\Phi_s))^d
\]
and define the Malliavin matrix
\[
\mathcal{M}_t = \int_0^t J_{s}^{-1}Q_L(\Phi_s)(J_{s}^{-1}Q_L(\Phi_s))^* ds.
\]
Since $J_t^{-1} \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$ and $Q_L \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$, it follows that $\mathcal{M}_t \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$. By Parseval identity (using the notation in Section A.1).

$$\langle \mathcal{M}_t \eta, \eta \rangle >_W = \int_0^t |(J_s^{-1} Q_L(\Phi_s))^* \eta|_W^2 \, ds = \sum_{k \in Z_L(N), i=1,2} \frac{1}{|k|^{4q_0+1}} \int_0^t |(J_s^{-1} Q_L(\Phi_s) e_k^i, \eta)_W|^2 \, ds \tag{4.9}$$

where $q_k(\Phi_s) = q_k(1-\chi(\frac{|\Phi_s|_W}{p}))$ for $k \in Z_L(N_0)$ and $q_k(\Phi_s) = q_k$ for $k \in Z_L(N_0) \setminus Z_L(N_0)$.

The following two lemmas are crucial for the proof of Lemma 3.3. The first one will be proven in the appendix (see page 25), while the other in Section 4.3.

**Lemma 4.2.** For any $T > 0$ and $p \geq 2$, there exist some $C_i = C_i(p, \rho, \alpha_0) > 0$ ($i = 1, 2, 3, 4$) such that

$$\mathbb{E}(\sup_{0 \leq t \leq T} |J_t(x) h^L|_W^p) \leq C_1 e^{C_1 T} |h_L|_V^p, \tag{4.10}$$

$$\mathbb{E}(\sup_{0 \leq t \leq T} |J_t^{-1}(x) h^L|_W^p) \leq C_2 e^{C_2 T} |h_L|_W^p, \tag{4.11}$$

$$\mathbb{E}(\sup_{0 \leq t \leq T} |J_t^{-1}(x) h^L - h^L|_W^p) \leq T^{p/2} C_3 e^{C_3 T} |h_L|_W^p, \tag{4.12}$$

$$\mathbb{E}(\sup_{0 \leq t \leq T} |\Phi_t(x) - e^{-At} x|_W^p) \leq (T^{p/8} \lor T^{p/2}) C_4 e^{C_4 T}. \tag{4.13}$$

Suppose that $v_1$, $v_2$ satisfy Proposition 4.1 and $p \geq 2$, then

$$\mathbb{E}(\sup_{0 \leq t \leq T} |D_{v_1} \Phi_L^T(x)|_W^p) \leq C_5 e^{C_5 T} \mathbb{E}[\int_0^T |v_1^T(s)|_W^p \, ds], \tag{4.14}$$

$$\mathbb{E}(\sup_{0 \leq t \leq T} |D_{v_1 v_2} \Phi_L^T(x)|_W^p) \leq C_6 e^{C_6 T} \mathbb{E}[\int_0^T |v_1^T(s)|_W^p \, ds]^1/2 \mathbb{E}[\int_0^T |v_2^T(s)|_W^p \, ds]^1/2 \tag{4.15}$$

$$\mathbb{E}(\sup_{0 \leq t \leq T} |D_{v_1} D_h \Phi_L^T(x)|_W^p) \leq C_7 e^{C_7 T} |h|_W^p \mathbb{E}[\int_0^T |v_1^T(s)|_W^p \, ds]^1/2 \tag{4.16}$$

with $h \in \mathcal{W}$ and $C_i = C_i(p, \rho, \alpha_0) > 0$, $i = 5, 6, 7$.

**Lemma 4.3.** Suppose that $\Phi_t$ is the solution to equation (3.2) with initial data $x \in \mathcal{W}$. Then $\mathcal{M}_t \in \mathcal{L}(\mathcal{W}^L, \mathcal{W}^L)$ is invertible almost surely. Denote $\lambda_{\min}(t)$ the smallest eigenvalue of $\mathcal{M}_t$. then there exists some $q > 1$ (possibly large) such that for every $p > 0$, there is some $C = C(p, \rho, \alpha_0)$ such that

$$P[|1/\lambda_{\min}(t)| \geq 1/e^t] \leq \frac{Ce^{p/8}(1+|x|_W^p)}{tp} \tag{4.17}$$

Now let us combine the previous two lemmas to prove Lemma 3.3.
Proof of Lemma 3.3. Under an orthonormal basis of $W^L$, the operators $J_t, M_t, D_v \Phi^L_t$ with $v$ defined in (4.7), and $D_L \Phi^L_t$ can all be represented by $M \times M$ matrices, where $M$ is the dimension of $W^L$. Let us consider

$$\psi_{ik}(\Phi_t) = \psi(\Phi_t) \sum_{j=1}^{M} [(D_v \Phi^L_t - 1)_{ij}]_{jk} \quad i, k = 1, \ldots, M.$$Given any $h \in W^L$, by (4.8), it is easy to see that

$$D_L \psi_{ik}(\Phi_t) D_v \Phi^L_t h = D_L \psi(\Phi_t) (D_v \Phi^L_t h) \sum_{j=1}^{M} [(D_v \Phi^L_t - 1)_{ij}]_{jk}$$

$$+ \psi(\Phi_t) \sum_{j=1}^{M} D_v h \{ [(D_v \Phi^L_t - 1)_{ij}]_{jk} \}$$

(4.18)

where $v = v(t)$ is defined by (4.7) with $v^L(t) = (J_t^{-1} Q^L(\Phi_t))^*$. Note that $W^L$ is isomorphic to $R^M$, given the standard orthonormal basis $\{ h_i : i = 1, \ldots, M \}$ of $R^M$, it can be taken as a presentation of the orthonormal basis of $W^L$. Setting $h = h_i$ in (4.18), summing over $i$ and noticing the identity $D_v \Phi^L_t = J_t M_t$, we obtain

$$\mathbb{E} (D_L \psi(X(t)) D_{h_k} \Phi^L_t)$$

$$= \mathbb{E} \left( \sum_{i=1}^{M} D_v h_i \psi_{ik}(\Phi_t) \right) - \mathbb{E} \left( \sum_{i,j=1}^{M} \psi(\Phi_t) D_v h_i \{ [(D_v \Phi^L_t - 1)_{ij}]_{jk} \} \right)$$

(4.19)

Let us estimate the first term on the right hand of (4.19) as follows. By Bismut formula and the identity $D_v \Phi^L_t = J_t M_t$ (see the argument below (4.7)),

$$\left| \mathbb{E} \left[ \sum_{i=1}^{M} D_L \psi_{ik}(\Phi_t) D_{v h_i} \Phi^L_t \right] \right|$$

$$\leq \sum_{i,j=1}^{M} \left| \mathbb{E} \left[ \psi(\Phi_t) [J_t^{-1} M_t^{-1}]_{ij} D_L \Phi^L_t \right] \int_{0}^{t} \langle v^L h_i, dW_s \rangle \right|$$

$$\leq ||\phi||_{\infty} \sum_{i,j=1}^{M} \mathbb{E} \left( \frac{1}{\lambda_{\min}} |J_t^{-1} h_j| \left| D_{h_k} \Phi^L_t \right| \right) \int_{0}^{t} \langle v^L h_i, dW_s \rangle \right),$$

(4.20)
morover, by Hölder’s inequality, Burkholder-Davis-Gundy’s inequality, (4.11), (3.9) and the inequality (see $c^L_k$ in the appendix)

$$\mathbb{E} \left[ \langle v^L(s) h_i, \cdot \rangle^2 \right] = \mathbb{E} \left[ \langle (J_s^{-1} Q^L)^* h_i, \cdot \rangle^2 \right] \leq C \sum_{j=1}^{M} \sum_{k=1}^{2} \mathbb{E} \left[ \langle h_i, J_s^{-1} Q^L c^L_k \rangle^2 \right] \leq C e^{Ct}$$
in order, we have
\begin{equation}
\mathbb{E}\left(\frac{1}{\lambda_{\text{min}}} |J_t^{-1}h_j|_{W} |D_{h_j} \Phi_t^{L} |_{W} \right) \int_0^t \langle u_t^i h_i , dW_t \rangle)
\end{equation}
\begin{align*}
& \leq \left[ \mathbb{E}\left(\frac{1}{\lambda_{\text{min}}} \right) \right]^{\frac{1}{2}} \left[ \mathbb{E}\left(|J_t^{-1}h_j|_{W}^6 \right) \right]^{\frac{1}{6}} \left[ \mathbb{E}\left(|D_{h_j} \Phi_t^{L} |_{W}^6 \right) \right]^{\frac{1}{6}} \left[ \mathbb{E}\left(\int_0^t |(J_s^{-1}Q_t^{L})^* h_i |^2 ds \right) \right]^{\frac{1}{2}} \\
& \leq C e^{C t} (1 + |x|_{W})^p
\end{align*}
where $p > 48q + 1$ and $C = C(p, Q, \alpha_0, \rho) > 0$. Combining (4.21) and (4.20), one has
\begin{equation}
\left| \mathbb{E}\left(\sum_{i=1}^M D_L \psi_{ik}^{(\Phi_t(x))} D_{e_{h_i} \Phi_t^{L}}(x) h_k \right) \right| \leq ||\psi||_{\infty} \frac{C e^{C t} (1 + |x|_{\overbar{W}})^p}{t^p}.
\end{equation}
By a similar argument but with more complicated calculation, we can have the same bounds for the second term on the r.h.s. of (4.19). Hence,
\begin{equation}
\left| \mathbb{E}\left(D_L \psi^{(\Phi_t(x))} D_L \Phi_t^{L}(x) h_k \right) \right| \leq \frac{C_1 e^{C t} (1 + |x|_{\overbar{W}})^p}{t^p} ||\psi||_{\infty}
\end{equation}
where $C_1 = C_1(p, \rho, \alpha_0, Q) > 0$. Since the above argument is in the framework of $W^L$ with the orthonormal base $\{h_k; 1 \leq k \leq M\}$, we have
\begin{equation}
\left| \mathbb{E}\left(D_L \psi^{(\Phi_t(x))} D_L \Phi_t^{L}(x) \right) \right| \leq \frac{C_1 e^{C t} (1 + |x|_{\overbar{W}})^p}{t^p} ||\psi||_{\infty} |h|_{W},
\end{equation}
for every $h \in W^L$ and $t > 0$.

4.2. Hörmander’s systems. This is an auxiliary subsection for the proof of Lemma 4.3 given in the next subsection and we use the notations detailed in Section A.1 (in particular Subsection A.1.1). Let us consider the SPDE for $u^L_t$ in Stratonovich form as
\begin{equation}
du^L + [Au^L + B_L(u, u) \chi(\frac{|u|_{W}}{3\rho}) - \frac{1}{2} \sum_{k \in Z_L(N_0)} D_{q_k(u)} e_k q_k(u) e_k^*] dt = \sum_{k \in Z_L(N_0)} q_k(u) \circ dw_k(t) e_k
\end{equation}
where $q_k(u) = (1 - \chi(\frac{|u|_{W}}{\rho})) q_k$ for $k \in Z_L(N_0)$ and $q_k(u) = q_k$ for $k \in Z_L(N) \setminus Z_L(N_0)$. For any $x \in W$, it is clear that if $k \in Z_L(N_0)$ and $i = 1, 2$,
\begin{equation}
D_{q_k(x)} e_k q_k(x) e_k^* = -\frac{1}{\rho} \chi(\frac{|x|_{W}}{\rho})(1 - \chi(\frac{|x|_{W}}{\rho})) \frac{\langle x, e_k^* \rangle_{W}}{|x|_{W}}.
\end{equation}

For any two Banach spaces $E_1$ and $E_2$, denote by $P(E_1, E_2)$ the set of all $C^\infty$ functions $E_1 \to E_2$ with all orders derivatives being polynomially bounded. If $K \in P(H, H^L)$ and $X \in P(H, H)$, define $[X, K]_L$ by
\begin{equation}
[X, K]_L(x) = DK(x)X(x) - D_L X^L(x) K(x), \quad x \in H.
\end{equation}
For instance, \([A, K]_L \in P(D(A), H^L)\) with \([A, K]_L(x) = DK(x)Ax - A_LK(x)\). Define
\[
X^0(x) = Ax + \chi(\frac{|x|}{\rho}) e^{-\delta A_H} B(x, x) + \frac{1}{2\rho} \sum_{k \in Z_L(N_0), i = 1, 2} \chi'(\frac{|x|}{\rho})(1 - \chi(\frac{|x|}{\rho})) \langle \langle x, e_i^j \rangle \psi \rangle e_k^i
\]
The brackets \([X^0, K]_L\) and \([A, K]_L\) will appear when applying the Itô formula on \(J_t^{-1}q_k^i(\Phi_t)\) (see (??)) in the proof of Lemma 4.3.

**Definition 4.4.** The Hörmänder’s system \(K\) for equation (4.22) is defined as follows: given any \(y \in \mathcal{W}\), define
- \(K_0(y) = \{q_k(y)e_k^i : k \in Z_L(N), i = 1, 2\}\)
- \(K_1(y) = \{[X^0(y), q_k(y)e_k^i]_L : k \in Z_L(N), i = 1, 2\}\)
- \(K_2(y) = \{[q_k(y)e_k^i, K(y)]_L : K \in K_1(y), k \in Z_L(N), i = 1, 2\}\)
and \(K(y) = K_0(y) \cup K_1(y) \cup K_2(y)\).

**Proposition 4.5.** There exist \(\mathcal{M} > 0\) and \(\mathcal{N} \geq N_0\) (which depend only on \(N_0\) and \(Q\)) such that if \(\rho \geq \mathcal{M}\) and \(\mathcal{N} \geq \mathcal{N}\), then the following property holds: for every \(x \in \mathcal{W}\) and \(h \in H^L\) there exist \(\sigma > 0\) and \(R > 0\) such that
\[
\inf_{\delta > 0} \sup_{K \in K} \inf_{|y - x|_\mathcal{W} \leq R} |\langle K(y), h \rangle_\mathcal{W}| \geq \sigma |h|_\mathcal{W}.
\]
**Proof.** We are going to show that there are \(\sigma > 0\) and \(R > 0\) (independent of \(\delta\)) such that for every \(x \in \mathcal{W}\) and \(h \in \mathcal{W}^L\),
\[
\sup_{K \in K} \inf_{|y - x|_\mathcal{W} \leq R} |\langle K(y), h \rangle_\mathcal{W}| \geq \sigma |h|_\mathcal{W}.
\]
To this end, it is sufficient to show that there is a (finite) set \(\widetilde{K} \subset K(y)\) for every \(y\), such that span(\(\widetilde{K}\)) = \(H^L\). We choose \(R \leq \frac{1}{4}\rho\).

**Case 1:** \(|x|_\mathcal{W} \geq R + 2\rho\). Hence \(|y|_\mathcal{W} \geq 2\rho\) for every \(y\) such that \(|x - y|_\mathcal{W} \leq R\) and \(q_k(y) = q_k\) for all \(k\). So we can take \(\widetilde{K} = K_0\) which spans the whole \(H^L\) thanks to (A.2).

**Case 2:** \(|x| \leq \rho - R\). Hence \(|y|_\mathcal{W} \leq \rho\) for every \(y\) such that \(|x - y|_\mathcal{W} \leq R\) and \(q_k(y) = 0\) for all \(k \in Z_L(N_0)\). In particular, \(X^0(y) = Ay + e^{-\delta A_H} B(y, y)\) and so for \(l, m \in Z_L(N_0) \setminus Z_L(N_0)\) and \(i, j = 1, 2\) (cfr. Subsection A.1.2),
\[
[q_0e_i^j, [X^0, q_m e_m^j]]_L = \pi_N B(q_0e_i^j, q_m e_m^j) + \pi_N B(q_m e_m^j, q_0e_i^j)
\]
(which are independent of \(\delta\), thus providing the uniformity in \(\delta\) we need). The proof that the vectors \([q_0e_i^j, [X^0, q_m e_m^j]]_L\), where \(l, m \) run over \(Z_L(N_0) \setminus Z_L(N_0)\) and \(i, j = 1, 2\), span \(H^L\) follows exactly as in [21] (using (A.3)-(A.4), since the only difference is that here we use the Fourier basis (A.1) rather than the complex exponentials). Hence, thanks to Lemma 4.2 of [21], it is sufficient to choose \(N \geq N_0\) large enough so that for every \(k \in Z_L(N_0)\) there are \(l, m \in Z_L(N) \setminus Z_L(N_0)\) such that \(|l| \neq |m|\), \(l\) and \(m\) are linearly independent and \(k = l + m\) (or \(k = l - m\)). Take \(\widetilde{K} = K_0 \cup K_2\).
4.3. Proof of Lemma 4.3. The key points for the proof are Proposition 4.5 and the following Norris’ Lemma (Lemma 4.1 of [19]).

**Lemma 4.6 (Norris’ Lemma).** Let \( a, y \in \mathbb{R} \). Let \( \beta_t, \gamma_t = (\gamma_1^t, \ldots, \gamma_m^t) \) and \( u_t = (u_1^t, \ldots, u_m^t) \) be adapted processes. Let

\[
a_t = a + \int_0^t \beta_s ds + \int_0^t \gamma_s^i dw^i_s, \quad Y_t = y + \int_0^t a_s ds + \int_0^t u_s^i dw^i_s,
\]

where \((w^1_s, \ldots, w^m_s)\) are i.i.d. standard Brownian motions. Suppose that \( T < t_0 \) is a bounded stopping time such that for some constant \( C < \infty \):

\[
|\beta_t|, |\gamma_t|, |a_t|, |u_t| \leq C \quad \text{for all } t \leq T.
\]

Then for any \( r > 8 \) and \( \nu > \frac{r-8}{9} \) there is \( C = C(T, q, \nu) \) such that

\[
P\left[ \int_0^T Y_t^2 dt < \epsilon, \int_0^T (|a_t|^2 + |u_t|^2) dt \geq \epsilon \right] < Ce^{-\frac{\epsilon}{\nu}}.
\]

**Proof of Lemma 4.3.** We follow the lines of the proof of Theorem 4.2 of [19]. Denote \( \mathcal{S}^L = \{ \eta \in \mathcal{W}^L; |\eta|_{\mathcal{W}^L} = 1 \} \). It is sufficient to show the inequality (4.17), which is by (4.9) equivalent to

(4.24) \[P\left[ \inf_{\eta \in \mathcal{S}^L} \sum_{k \in Z_L(N), i=1,2} \frac{1}{|k|^{4a_0+1}} \int_0^t |(S^{-1}q_k(\Phi_s), \eta)_\mathcal{W}|^2 ds \leq \epsilon^2 \right] \leq \frac{C\epsilon^{p/8}(1 + |x|_W^p)}{t^p}\]

for all \( p > 0 \), where \( q_k(\Phi_s) = q_k(\Phi_s)c_k^1 \) with \( q_k(\Phi_s) = q_k(1 - \chi(\frac{|\Phi_s|_{\mathcal{W}}}{\rho})) \) for \( k \in Z_L(N_0) \) and \( q_k(\Phi_s) = q_k \) for \( k \in Z_L(N) \setminus Z_L(N_0) \).
Formula (4.24) is implied by
(4.25)
\[ D_\theta \sup_{j} \sup_{\eta \in \mathcal{N}_j} P \left[ \int_0^t \sum_{k \in Z_L(N)_i} \frac{1}{|k|^\alpha_0 + 1} |\langle J_s^{-1} q_k^j(\Phi_s), \eta \rangle_W \rangle^2 ds \leq \epsilon^q \right] \leq \frac{C e^{p/8}(1 + |x|_\infty)^p}{t^p}, \]
for all \( p > 0 \), where \( \{\mathcal{N}_j\}_j \) is a finite sequence of disks of radius \( \theta \) covering \( S^L \), \( D_\theta = \#\{\mathcal{N}_j\} \) and \( \theta \) is sufficiently small. Define a stopping time \( \tau \) by
\[ \tau = \inf\{s > 0 : |\Phi_s(x) - x|_W > R, |J_s^{-1} - Id|_L(W) > \epsilon^c\}. \]
where \( R > 0 \) is the same as in (4.23) and \( c > 0 \) is sufficiently small. It is easy to see that (4.25) holds as long as for any \( \eta \in S^L \), we have some neighborhood \( \mathcal{N}(\eta) \) of \( \eta \) and some \( k \in Z_L(N), i \in \{1, 2\} \) so that
(4.26)
\[ \sup_{\eta' \in \mathcal{N}(\eta)} P \left( \int_0^{1/8} |\langle J_s^{-1} q_k^j(\Phi_s), \eta' \rangle_W \rangle^2 ds \leq \epsilon^q \right) = \frac{C e^{p/8}(1 + |x|_\infty)^p}{t^p}. \]
The key point of the proof is to bound \( P(\tau \leq \epsilon) \). By (4.13) and the easy fact \( |e^{-At}x - x|_W \leq C t^{1/8}|x|_W \), we have for any \( p \geq 2 \)
(4.27)
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Phi_t - x|_W^p \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |e^{-At}x - x|_W \right] + \sup_{0 \leq t \leq T} |\Phi_t(x) - e^{-At}x|_W \]
\[ \leq C_1(1 + |x|_W^p)(T^{p/8} \vee T^{p/2}) \]
where \( C_1 = C_1(\alpha_0, p, \rho) \). Combining (4.27) and (4.12), we have
(4.28)
\[ P(\tau \leq \epsilon) = C_1 e^{p/8}(1 + |x|_W^p) \]
for all \( p > 0 \).

Let us prove (4.26). According to Definition 4.4 and Proposition 4.5, given a fixed \( x \in W \), for any \( \eta \in S^L \), there exists a \( K \in \mathbf{K} \) such that
\[ \sup_{K \in \mathbf{K}} \inf_{|y - x|_W \leq R} |\langle K(y), \eta \rangle_W | \geq \sigma |\eta|_W. \]
Without loss of generality, assume that \( K \in \mathbf{K}_2 \), so there exists some \( q_k^j e_k \) and \( q_l^j e_l \) such that
\[ K_0(y) := q_k^j(y)e_k, K_1(y) := [X^0(y), q_k^j(y)e_k], K = K_2 := [q_l^j(y)e_l, K_1(y)]. \]
Now one can follow the same but more simple argument as in Proof of Claim 2 in [19] (page 127) to show that
\[ P \left( \int_0^{1/8} |\langle J_s^{-1} q_k^j(\Phi_s), \eta' \rangle_W \rangle^2 ds \leq \epsilon^q \right) = \frac{C e^{p/8}(1 + |x|_\infty)^p}{t^p}, \]
(where the power \( r^2 \) is because one needs to use Norris’ Lemma two times).

Hence, take the neighborhood \( \mathcal{N}(\eta) \) small enough and \( q = r^2 \), by the continuity, we have (4.26) immediately from the previous inequality. \( \square \)
5. Controllability and Support

The following proposition describes the support of the distribution associated to a Markov solution.

**Proposition 5.1.** Let \((P_x)_{x \in H}\) be a Markov solution. For every \(x \in W\) and \(T > 0\), the following properties hold,

- \(P_x[|\xi_T| \in W] = 1\),
- for every \(W\)-open set \(U \subset W\), \(P_x[|\xi_T| \in U] > 0\).

The proof of the above proposition relies on the following control problem (see [25] for a general result on the same lines).

**Lemma 5.2.** Given any \(T > 0\), \(x, y \in W\) and \(\epsilon > 0\), there exist \(\rho_0 = \rho_0(|x|_W, |y|_W, T)\), \(u\) and \(w\) such that

- \(w \in L^2([0, T]; H)\) and \(u \in C([0, T]; W)\),
- \(u(0) = x\) and \(|u(T) - y|_W \leq \epsilon\),
- \(\sup_{t \in [0, T]} |u(t)|_W \leq \rho_0\),

and \(u, w\) solve the following problem,

\[
\partial_t u + Au + B(u, u) = Qw, \tag{5.1}
\]

where \(Q\) is defined in Assumption 2.1.

**Proof.** Let \(z \in D(A^{\alpha_0 + 7/4})\) such that \(|y - z|_W \leq \frac{\epsilon}{2}\), it suffices to show that there exist \(u, w\) satisfying the conditions of the lemma and

\[
|u(T) - z|_W \leq \frac{\epsilon}{2}. \tag{5.2}
\]

Decompose \(u = u^H + u^L\) where \(u^H = (I - \pi_{N_0})u\) and \(u^L = \pi_{N_0}u\) and \(N_0\) is the number in Assumption 2.1, then equation (5.1) can be written as

\[
\partial_t u^L + Au^L + B_L(u, u) = 0, \tag{5.3}
\]

\[
\partial_t u^H + Au^H + B_H(u, u) = Qw. \tag{5.4}
\]

We split \([0, T]\) into the pieces \([0, T_1]\), \([T_1, T_2]\), \([T_2, T_3]\) and \([T_3, T]\), with the times \(T_1\), \(T_2\), \(T_3\) to be chosen along the proof, and prove that (5.2) holds in the following four steps, provided \(\rho_0\) is chosen large enough (depending on \(|x|_W, |y|_W\) and \(T\)).

**Step 1:** regularization of the initial condition. Set \(w \equiv 0\) in \([0, T_1]\), using (A.5), one obtains

\[
\frac{d}{dt} |u|^2_W + 2|A^{3/2}u|^2_W \leq 2\langle A^{3/2 - \alpha_0}u, A^{\alpha_0 - 1/2}B(u, u)\rangle_H \leq |A^{3/2}u|^2_W + c|u|^4_W. \tag{5.5}
\]

It is easy to see, by solving a differential inequality, that \(|u(t)|^2_W + \int_0^t |A^{1/2}u|^2_W ds \leq 2|x|^2_W\) for \(t \leq t_0 := (2c|x|^2_W)^{-1}\). In particular \(u(t) \in D(A^{\alpha_0 + 3/4})\) for a.e. \(t \in [0, t_0]\). An energy estimate similar to the one above, this time in \(D(A^{\alpha_0 + 3/4})\) and with initial condition \(u(t_0/2)\) (w.l.o.g. assume \(u(t_0/2) \in D(A^{\alpha_0 + 3/4})\), implies that \(u(t) \in D(A^{\alpha_0 + 5/4})\) a.e. for \(t \in [t_0/2, t_0]\). By repeating the argument, we can finally find a time \(T_1 \leq \frac{4}{\epsilon} \wedge t_0\) such
that \( u(T_1) \in D(A^{\alpha_0+7/4}) \).

**Step 2: high modes led to zero.** Choose a smooth function \( \psi \) on \([T_1, T_2]\) such that 
\( 0 \leq \psi \leq 1 \), \( \psi(T_1) = 1 \) and \( \psi(T_2) = 0 \), and set \( u^H(t) = \psi(t)u^H(T_1) \) for \( t \in [T_1, T_2] \). An estimate similar to (5.5) yields
\[
\frac{d}{dt}[u^L]_W^2 + |A^4 u^L]_W^2 \leq c(|u^L]_W^2 + |u^H]_W^2)_2^2,
\]
and \( |u(t)]_W^2 \leq |u^L(t)]_W^2 + |u^H(T_1)]_W^2 \leq 4|x]_W^2 \) for \( T_1 \leq t \leq T_2 := \frac{T}{2} \wedge (T_1 + (4c|x]_W^2)^{-1}) \). Plug \( u^L \) in (5.4), take
\[
w(t) = \psi(t)Q^{-1}u^H(T_1) + \psi(t)Q^{-1}Au^H(T_1) + Q^{-1}B_H(u(t), u(t)).
\]
By the previous step \( u(T_1) \in D(A^{\alpha_0+7/4}), |Q^{-1}Au^H(T_1)| < \infty \); by (A.5), \( |Q^{-1}B_H(u(t), u(t))| \leq c|Au(t)]_W^2 \leq 2cNL_0^2(|Au^H(T_1)]_W^2 + |u^L(t)]_W^2 \) for \( t \in [T_1, T_2] \). Hence, \( w \in L^2([T_1, T_2], H) \).

**Step 3: low modes close to 0.** Let \( u^L(t) \) be the linear interpolation between \( u^L(T_2) \) and \( z^L \) for \( t \in [T_2, T_3] \). Write \( u(t) = \sum u_k(t)e_k \), then (5.3) in Fourier coordinates is given by
\[
\dot{u}_k + |k|^2u_k + B_k(u, u) = 0, \quad k \in Z_L(N_0),
\]
where \( B_k(u, u) = B_k(u^L, u^L) + B_k(u^L, u^L) + B_k(u^H, u^L) + B_k(u^L, u^H) \). Let us choose a suitable \( u^H \) to simplify the above \( B_k(u, u) \). To this end, consider the set \( \{(l_k, m_k) : k \in Z_L(N_0)\} \) such that
1. If \( k \in Z_L(N_0)_+ \), then \( l_k, -m_k \in Z_H(N_0)_+ \) and \( l_k + m_k = k \).
2. If \( k \in Z_L(N_0)_- \), then \( l_k, m_k \in Z_H(N_0)_- \) and \( l_k - m_k = k \).
3. \( |l_k| \neq |m_k| \) and \( l_k \not\parallel m_k \) for all \( k \in Z_L(N_0) \).
4. For every \( k \in Z_L(N_0) \), \( |l_k|, |m_k| \geq 2(2N_0+1)^3 \).
5. If \( k_1 \neq k_2 \), then \( |l_k_1 \pm l_k_2|, |m_k_1 \pm m_k_2|, |l_k_1 \pm l_k_2|, |m_k_1 \pm l_k_2| \geq 2(2N_0+1)^3 \).

Define
\[
u^H(t) = \sum_{k \in Z_L(N_0)} u_{l_k}(t)e_{l_k} + u_{m_k}(t)e_{m_k},
\]
with \( u_{l_k}(t) \) and \( u_{m_k}(t) \) to be determined by equation (5.7) below. Using the formulas (A.3)-(A.4) in Section A.1.2, it is easy to see that
- by (4), \( B_k(u^L, u^H) = B_k(u^H, u^L) = 0 \),
- by (5), \( B_k(u_{l_k}, u_{l_k}) = B_k(u_{m_k}, u_{m_k}) = B_k(u_{m_k}, u_{l_k}) = B_k(u_{l_k}, u_{m_k}) = 0 \).

Hence, using again the computations of Section A.1.2, equation (5.6) is simplified to the following equation
\[
\begin{cases}
(m_k \cdot X)P_kY \pm (l_k \cdot Y)P_kX + 2G_k(t) = 0, \\
X \cdot l_k = 0, \quad Y \cdot m_k = 0, \quad l_k \pm m_k = k,
\end{cases}
\]
for each \( k \in Z_L(N_0)_\pm \), where \( G_k = \dot{u}_k + |k|^2u_k + B_k(u^L, u^L) \) is a polynomial in \( t \) and clearly \( G_k \cdot k = 0 \). In order to see that the above equation has a solution, consider for instance the case \( k \in Z_L(N_0)_+ \). Let \( \{k, g_1, g_2\} \) be an orthonormal basis of \( \mathbb{R}^3 \) such that
there exist \( \eta \in \mathbb{R}^{14} \) and \( \bar{k} = \frac{k}{|k|} \). Let \( X = x_0\bar{k} + x_1g_1 + x_2g_2 \) and \( Y = y_0\bar{k} + y_1g_1 + y_2g_2 \). A simple computation yields
\[
(X \cdot m_k)(P_kY) + (Y \cdot l_k)(P_kX) = |k|(x_0y_2 + x_2y_0) - |k|c_kx_0y_0g_1,
\]
where \( c_k = \frac{|l_k|^2 - |m_k|^2}{\sqrt{|k|^2|l_k|^2 - (l_k \cdot m_k)^2}} \). One can for instance set \( x_0 = 1, x_2 = 1 \) and solve the problem in the unknown \( y_0, y_2 \) (notice that \( x_1, y_1 \) can be determined by the divergence free constraint).

In conclusion the solution \( u^H(t) \) is smooth in \( t \) and by this construction the dynamics \( u = u^L + u^H \) is finite dimensional. Hence \( u(t) \) is smooth in space and time for \( t \in [T_2, T_3] \) and \( \sup|u(t)|_W \) can be bounded only in terms of \( |u^L(T_2)|, z^L \) and \( T_3 - T_2 \). We finally set \( w = Q^{-1}[u^H + Au^H + B_H(u, u)] \).

**Step 4:** high modes close to \( z \). In the interval \( [T_3, T] \) we choose \( u^H \) as the linear interpolation between \( u^H(T_3) \) and \( z^H \). Let \( u^L \) be the solution to equation (5.3) on \( [T_3, T] \) with the choice of \( u^H \) given above. Since \( u(T_3) \in D(A^{a_0+7/4}) \) and \( u^L(T_3) = z^L \) from step 3, by the continuity of the dynamics, \( \sup_{T_3 \leq t \leq T} |u^L(t) - z^L|_W \leq \frac{\epsilon}{2} \) if \( T - T_3 \) is small enough (recall that we can choose an arbitrary \( T_3 \in (T_2, T) \) in the third step). Thus (5.2) holds and, as in the second step, we can find \( w \in L^2([T_3, T], H) \) solving (5.4). It is clear from the above construction that \( \sup_{T_3 \leq t \leq T} |u(t)|_W \leq C|z|_W + C|u(T_3)|_W \).

**Proof of Proposition 5.1.** The first property follows from Theorem 6.3 of [14] (which only uses strong Feller). For the second property, fix \( x \in W \) and \( T > 0 \), then it is sufficient to show that for every \( \eta \in W \) and \( \epsilon > 0 \), \( P_x[|\xi_T - \eta|_W \leq \epsilon] > 0 \). Consider \( \rho > \rho_0 \) (where \( \rho_0 \) is the constant provided by Lemma 5.2), then by Theorem 2.3,
\[
P_x[|\xi_T - \eta|_W \leq \epsilon] \geq P_x[|\xi_T - \eta|_W \leq \epsilon, \tau_\rho > T] = P_x^\rho[|\xi_T - \eta|_W \leq \epsilon, \tau_\rho > T].
\]
By Lemma 5.2 there exist \( \bar{\eta} \) and \( \bar{\eta} \) such that \( \bar{\eta} \) is the solution to the control problem (5.1) connecting \( x \) at 0 with \( y \) at \( T \) corresponding to the control \( \partial_{\bar{\eta}} \bar{\eta} \). Choose \( s \in (0, \frac{1}{2}) \), \( p > 1 \) and \( \beta > \frac{3}{2} \) such that \( s - \frac{1}{p} > 0 \) and \( \beta + \frac{1}{p} - s < \frac{1}{2} \), then by Lemma C.3 of [14] (which does not rely on non-degeneracy of the covariance), there is \( \delta > 0 \) such that for all \( \eta \) in the \( \delta \)-ball \( B_\delta(\eta) \) centred at \( \bar{\eta} \) in \( W^{s,p}([0, T]; D(A_H^\beta)) \), we have that \( |u(T, \eta) - \eta|_W \leq \epsilon \) and \( \sup_{[0, T]} |u(t, \eta)|_W \leq \rho_0 \), where \( u(., \eta) \) is the solution to the control problem with control \( \partial_{\bar{\eta}} \eta \). By proceeding as in the proof of Proposition 6.1 of [14], it follows that in conclusion the probability \( P_x^\rho[|\xi_T - \eta|_W \leq \epsilon, \tau_\rho > T] \) is bounded from below by the (positive) measure of \( B_\delta(\eta) \) with respect to the Wiener measure corresponding to the cylindrical Wiener process on \( H \). 

**Appendix A. Appendix**

A.1. Details on the geometry of modes. Here we reformulate the problem in Fourier coordinates and explain in full details the conditions of Assumption 2.1.
Define $Z^3_+ = Z^3 \setminus \{(0,0,0)\}$, $Z^3_{++} = \{k \in Z^3 : k_1 > 0\} \cup \{k \in Z^3 : k_1 = 0, k_2 > 0\} \cup \{k \in Z^3 : k_1 = 0, k_2 = 0, k_3 > 0\}$ and $Z^3_{++} = -Z^3_{++}$, and set
\begin{align}
e_k(x) = \begin{cases} \cos k \cdot x & k \in Z^3_+, \\ \sin k \cdot x & k \in Z^3_. \end{cases}
\end{align}

Fix for every $k \in Z^3_+$ an arbitrary orthonormal basis $(x_k^1, x_k^2)$ of the subspace $k^\perp$ of $\mathbb{R}^3$ and set $e_k^1 = x_k^1 e_k(x)$ and $e_k^2 = x_k^2 e_k(x)$, then $\{e_k^i : k \in Z^3_+, \ i = 1, 2\}$ is an orthonormal basis of $H$. In particular, $\pi_N H = \text{span}(\{e_k^i : 0 < |k|_\infty \leq N, \ i = 1, 2\})$. Denote moreover, for any $N > 0$, $Z_L(N) = [-N, N]^3 \setminus (0,0,0)$ and $Z_H(N) = Z^3_+ \setminus Z_L(N)$.

A.1.1. Assumptions on the covariance. Under the Fourier basis of $H$, the diagonal-ity assumption $[A1]$ means that for each $k \in Z^3_+$, there exists some linear operator $q_k : k^+ \to k^\perp$ such that $Q(y e_k) = (q_k y) e_k$ for $y \in k^\perp$. The finite degeneracy assumption $[A2]$ says that $q_k$ is invertible on $k^\perp$ if $k \in Z_H(N_0)$ and $q_k = 0$ otherwise. If $W$ is a cylindrical Wiener process on $H$, then $Q dW = \sum_{k \in Z_H(N_0)} e_k q_k \, dw_k$, where $(w_k)_{k \in Z_H(N_0)}$ is a sequence of independent 2d Brownian motions and each $w_k \in k^\perp$.

The $\overline{Q}$ in (2.3) is a non-degenerate operator on $\pi_{N_0} H$, which is defined under the Fourier basis by
\begin{align}
\overline{Q} = \sum_{k \in Z_L(N_0)} e_k q_k \langle \cdot, e_k \rangle_H,
\end{align}
where, for each $k \in Z_L(N_0)$, $q_k$ is an invertible operator on $k^\perp$.

A.1.2. The nonlinearity. In Fourier coordinates, equation (2.1) can be represented under the Fourier basis by
\begin{align}
\begin{cases}
du_k + [k^2 u_k + B_k(u,u)] \, dt = q_k \, dw_k(t), & k \in Z_H(N_0) \\
du_k + [k^2 u_k + B_k(u,u)] \, dt = 0, & k \in Z_L(N_0) \\
u_k(0) = x_k, & k \in Z^3_+,
\end{cases}
\end{align}
where $u = \sum u_k e_k$, $u_k \in k^\perp$ for all $k \in Z^3_+$ and $B_k(u,u)$ is the Fourier coefficient of $B(u,u)$ corresponding to $k$. To be more precise,
\begin{align}
B(u,u) = \sum_{l,m \in Z^3_+} B(u_l e_l, u_m e_m)
\end{align}
and if, for instance, $l, -m, l + m \in Z^3_+$,
\begin{align}
B(u_l e_l, u_m e_m) = \mathcal{P}((u_l e_l) u_m e_{l+m}) = \frac{1}{2} \left[ (u_l e_l) \mathcal{P}_{l+m} u_m e_{l+m} + (u_l e_l) \mathcal{P}_{l-m} u_m e_{l-m} \right],
\end{align}
where $\mathcal{P}_k$ is the projection of $\mathbb{R}^3$ onto $k^\perp$, given by $\mathcal{P}_k \eta = \eta - \frac{k \cdot \eta}{|k|^2} k$, then, clearly,
\begin{align}
B_{l+m}(u_l e_l, u_m e_m) = \frac{1}{2} (u_l \cdot m) \mathcal{P}_{l+m} u_m e_{l+m},
\end{align}
\begin{align}
B_{l-m}(u_l e_l, u_m e_m) = \frac{1}{2} (u_l \cdot m) \mathcal{P}_{l-m} u_m e_{l-m},
\end{align}
and $B_k(u_l e_l, u_m e_m) = 0$ otherwise. For the other cases (of $l, m$), similar formulas hold.
A.2. Proofs of the auxiliary results. The key points for the proofs of this section are the following two inequalities and Lemma A.1 below. Given \( \beta > \frac{1}{2} \), there exist constants \( C_1 > 0 \), \( C_2 > 0 \) such that for every \( u, v \in D(\mathcal{A}^{\beta+1/4}) \),

\[
(A.5) \quad |A^{\beta-\frac{1}{4}}B(u, v)|_H \leq C_1 |A^{\beta+\frac{1}{4}}u|_H |A^{\beta+\frac{1}{4}}v|_H,
\]

\[
(A.6) \quad |A^{\beta+\frac{1}{4}}e^{-A(t-s)}B(u, v)|_H \leq \frac{C_2}{\sqrt{t}} |A^{\beta+\frac{1}{4}}u|_H |A^{\beta+\frac{1}{4}}v|_H.
\]

The first inequality is given by Lemma D.2. in [14], the second follows from the standard estimate \( |A^{1/2}e^{-A(t-s)}B|_H \leq C t^{-1/2} \) for analytical semigroups. The other basic tool is the following Lemma which is a straightforward modification of Proposition 7.3 of [4].

**Lemma A.1.** Let \( Q : H \to H \) be a linear bounded operator such that \( A^{\alpha_0+3/4}Q \) is also bounded, and let \( W \) be a cylindrical Wiener process on \( H \). Then for any \( 0 < \beta < \frac{1}{4} \), \( p > 2 \) and \( \epsilon \in (0, \frac{1}{4} - \beta) \), there exists \( C > 0 \) such that

\[
E \left[ \sup_{0 \leq s \leq T} |A^\beta \int_0^t e^{-A(t-s)}Q DW_s|^p_{|W|} \right] \leq CT^{\frac{1}{2}-\epsilon-\beta p}|A^{-\frac{3}{2}-\epsilon}|_{HS}^p.
\]

**Proof of Lemma A.1.** We simply write \( \Phi_t = \Phi_t^\delta \) (with \( \delta \geq 0 \)) and prove (3.10) at the end. Clearly, \( \Phi_t(x) \) satisfies the following equation

\[
\Phi_t = e^{-At}x + \int_0^t e^{-A(t-s)}e^{-A\delta}B(\Phi_s, \Phi_s)\chi\left(\frac{|\Phi_s|_W}{3\rho}\right)ds + \int_0^t e^{-A(t-s)}Q(\Phi_s)DW_s.
\]

By inequality (A.6), the fact \( |e^{-A\delta}|_W \leq 1 \) and the inequality \( \chi(\frac{|\Phi_s|_W}{3\rho})|\Phi_s|_W \leq 3\rho \), it is easy to see that

\[
|\Phi_t|_W \leq |x|_W + \int_0^t |e^{-A(t-s)}B(\Phi_s, \Phi_s)|_W \chi\left(\frac{|\Phi_s|_W}{3\rho}\right)ds + \int_0^t |e^{-A(t-s)}Q(\Phi_s)DW_s|_W
\]

\[
\leq |x|_W + \int_0^t \frac{C\rho}{\sqrt{t-s}} |\Phi_s|_W \chi\left(\frac{|\Phi_s|_W}{3\rho}\right)ds + \int_0^t |e^{-A(t-s)}Q DW_s|_W
\]

\[
\leq |x|_W + C\rho t^{\frac{3}{2}} \sup_{0 \leq s \leq t} |\Phi_s|_W + \int_0^t e^{-A(t-s)}(1 - \chi(\frac{|\Phi_s|_W}{\rho}))Q DW_s|_W,
\]

and that for any \( p \geq 2 \), \( T > 0 \),

\[
E \left( \sup_{0 \leq t \leq T} |\Phi_t|^p_{|W|} \right) \leq |x|^p_{|W|} + C_1 T^{p/8} + C_1 T^{p/2} E \left( \sup_{0 \leq t \leq T} |\Phi_t|^p_{|W|} \right)
\]

by Lemma A.1 (with \( \epsilon = \frac{1}{8}, \beta = 0 \)) and some basic computation, with \( C_1 = C_1(p, \alpha_0, \rho) \).

For \( T \) small, \( E(\sup_{0 \leq t \leq T} |\Phi_t|^p_{|W|}) \leq \frac{|x|^p_{|W|} + C_1 T^{p/8}}{1-C_1 T^{p/2}} \). Now, by taking \( T, 2T, \ldots \) as initial times, by applying the same procedure on \( [T, 2T], [2T, 3T], \ldots \), respectively one can obtain similar estimates as the above on these time intervals. Inductively, the estimate (3.6) follows. The proof of (3.7) and (3.8) proceeds similarly.
For every $h \in W$, $D_h \Phi_t$ satisfies the following equation
\[
D_h \Phi_t = e^{-At}h + \int_0^t e^{-A(t-s)}(B(D_h \Phi_s, \Phi_s) + B(\Phi_s, D_h \Phi_s))\chi(\frac{|\Phi_s|_W}{3\rho}) + \\
e^{-A(t-s)}B(\Phi_s, \Phi_s)\chi'(\frac{|\Phi_s|_W}{3\rho})\frac{1}{3\rho} \langle D_h \Phi_s, \Phi_s \rangle_W ds + \\
- \int_0^t e^{-A(t-s)} \frac{|\Phi_s|_W}{\rho} \langle D_h \Phi_s, \Phi_s \rangle_W Q_L dW_s^L,
\]
By (A.6) and $\chi(\frac{|\Phi_s|_W}{3\rho})|\Phi_t|_W \leq 3\rho$,
\[
|D_h \Phi_t|_W \leq |h|_W + \int_0^t \frac{C}{\sqrt{t-s}}\left(\frac{|\Phi_s|_W}{3\rho}\right)|\Phi_s|_W + \frac{1}{3\rho} |\Phi_s|_W^2 \chi'(\frac{|\Phi_s|_W}{3\rho})|D_h \Phi_s|_W ds \\
+ \frac{1}{\rho} \int_0^t e^{-A(t-s)} \chi'(\frac{|\Phi_s|_W}{\rho}) \langle D_h \Phi_s, \Phi_s \rangle_W Q_L dW_s^L \\
\leq |h|_W + \int_0^t \frac{C\rho}{\sqrt{t-s}} |D_h \Phi_s|_W ds + \frac{1}{\rho} \int_0^t e^{-A(t-s)} \chi'(\frac{|\Phi_s|_W}{\rho}) \langle D_h \Phi_s, \Phi_s \rangle_W Q_L dW_s^L,
\]
by Lemma A.1 (with $\beta = 0$ and $\epsilon = \frac{1}{8}$),
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} \frac{|D_h \Phi_t|_W^p}{|h|_W^p} \right] \leq 1 + CT^\frac{p}{2} \mathbb{E}\left[\sup_{0 \leq t \leq T} \frac{|D_h \Phi_t|_W^p}{|h|_W^p} \right], \quad 0 \leq T \leq 1,
\]
where $C = C(\alpha_0, p, \rho) > 0$. For $T > 0$ small enough, $\mathbb{E}[\sup_{0 \leq t \leq T} |D_h \Phi_t|^p] \leq \frac{1}{1 - CT^\frac{p}{2}} |h|_W^p$.

For $|D_h u |\Phi_t|_W$, it is easy to see by a similar argument as in proving (3.9) that
\[
|D_h u |\Phi_t|_W \leq \int_0^t \frac{C\rho}{\sqrt{t-s}} |D_h u |\Phi_s|_W ds + \frac{1}{\rho} \int_0^t e^{-A(t-s)} \chi'(\frac{|\Phi_s|_W}{\rho}) \langle D_h u |\Phi_s, \Phi_s \rangle_W Q_L dW_s^L,
\]
so by Lemma A.1 and (3.9),
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} \frac{|D_h u |\Phi_t|_W^p}{|h|_W^p} \right] \leq T^\frac{p}{2} Ce^{CT} |h|_W^p, \quad 0 \leq T \leq 1,
\]
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} \frac{|D_h u |\Phi_t|_W^p}{|h|_W^p} \right] \leq T^\frac{p}{2} Ce^{CT} |h|_W^p, \quad T > 1,
\]
where $C = C(\alpha_0, p, \rho) > 0$. Similarly but more simply, we have (3.11).

Let us now prove (3.10). By Itô formula,
\[
\mathbb{E}|D_h \Phi_t|_W^2 + 2 \int_0^t \mathbb{E}|A^\frac{\alpha}{2} D_h \Phi_s|_W^2 ds \leq \\
\leq |h|_W^2 + C\rho \int_0^t \mathbb{E}\left[|A^\frac{\alpha}{2} D_h \Phi_s|_W^2 A^{\alpha_0 - \frac{1}{2}} D_h [e^{-A t} B(\Phi_s, \Phi_s) \chi(\frac{|\Phi_s|_W}{3\rho})]|H] ds.
\]
By (A.5) and Cauchy inequality, we have
\[
\mathbb{E}|D_h \Phi_t|_W^2 + \int_0^t \mathbb{E}|A^\frac{\alpha}{2} D_h \Phi_s|_W^2 ds \leq |h|_W^2 + C \int_0^t \mathbb{E}|D_h \Phi_s|_W^2 ds
\]
with $C = C(\alpha_0, \rho) > 0$, which easily implies (3.10) by Gronwall’s lemma. \qed
Proof of Proposition 3.2. Recall that the solutions to (2.3) and (3.2) are respectively denoted by $\Phi_t(x)$ and $\Phi_t^\delta(x)$. Denote $\Psi_t = \Phi_t - \Phi_t^\delta$, we have

\[(A.7)\]
$$\Psi_t = \int_0^t I_1 \, ds + \int_0^t I_2 \, dW_s$$

with
$$I_1 = e^{-A(t-s)} [B(\Phi_s, \Phi_s)\chi(\frac{[\Phi_s]_W}{\rho}) - e^{-A\delta} B(\Phi_s^\delta, \Phi_s^\delta)\chi(\frac{[\Phi_s^\delta]_W}{\rho})],$$
and
$$I_2 = e^{-A(t-s)} [Q(\Phi_s) - Q(\Phi_s^\delta)].$$

By fundamental calculus and Lemma A.10, we have

\[(A.8)\]
$$|I_1|_W \leq |I_d - e^{-A\delta} L(W)| e^{-A(t-s)} B(\Phi_s, \Phi_s)\sup_{s \leq t} |\Phi_s|_W$$

with $C_1 = C_1(\rho, \alpha_0)$ and $C_2 = C_2(\rho, \alpha_0)$, since

$$e^{-A(t-s)} B(\Phi_s, \Phi_s)\chi(\frac{[\Phi_s]_W}{\rho}) - e^{-A(t-s)} B(\Phi_s^\delta, \Phi_s^\delta)\chi(\frac{[\Phi_s^\delta]_W}{\rho})$$

$$= \left| \int_0^1 e^{-A(t-s)} \frac{d}{d\lambda} B(\lambda \Phi_s + (1 - \lambda) \Phi_s^\delta, \lambda \Phi_s + (1 - \lambda) \Phi_s^\delta) \chi(\frac{[\lambda \Phi_s + (1 - \lambda) \Phi_s^\delta]_W}{\rho}) d\lambda \right|_W$$

By fundamental calculus and Lemma A.1 (with $\beta = 0$ and $\epsilon = 1/8$),

$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| \int_0^t I_2 \, dW_s \right|^p \right] \leq \mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| \int_0^t e^{-A(t-s)} \left( \chi(\frac{[\Phi_s]_W}{\rho}) - \chi(\frac{[\Phi_s^\delta]_W}{\rho}) \right) Q_L \, dW_s^L \right|^p \right]$$

\[(A.9)\]
$$\leq \mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| \int_0^t e^{-A(t-s)} \frac{d}{d\lambda} \chi(\frac{[\lambda \Phi_s + (1 - \lambda) \Phi_s^\delta]_W}{\rho}) Q_L \, dW_s^L \right|^p \right]$$

with $p \geq 2$, $C_3 = C_3(\rho, \alpha_0, \rho)$ and $T > 0$. Combining (A.7), (A.8) and (A.9), we have

\[(A.10)\]
$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Psi_t|_W^p \right] \leq C_4 T^p |I_d - e^{-A\delta} Q_L| W + C_4 T^p \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Psi_t|_W^p \right]$$

with $C_4 = C_4(\rho, \alpha_0, \rho) > 0$. With the estimate of (A.10) and by the same induction argument as in the proof of Lemma 3.4, estimate (3.3) follows.

As for the estimate (3.4), differentiating both sides of (A.7) along directions $h \in W$, applying the same method as above but with a little more complicated computation, and noticing (A.9), we have

$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} |D_h \Psi_t|_W^p \right] \leq C_5 e^{C_5 |h|_W^p} |I_d - e^{-A\delta} Q_L| W |h|_W^p,$$
for all $h \in \mathcal{W}$, with $C_5 = C_5(\alpha_0, \rho, p)$. Formula (3.5) follows from the two estimates in the lemma immediately.

**Proof of Lemma 4.2.** That the constants of the estimates in the lemma are independent
of $\delta$ is due to the uniform estimates (in $\delta$) of the nonlinear term and to the fact that the
Malliavin derivatives $D_v \Phi_t$ do not depend on $v^H$.

The proofs of (4.10), (4.12) are classical since the SDEs for $J_t$, $J_t^{-1}$ are both finite
dimensional and have the cutoff. The proof of (4.13) is by the same procedure as for
(3.12). For the other estimates, we will apply the bootstrap argument in the proof of
(3.6) but omit the trivial induction argument.

As for (4.11), we consider the integral form of equation (4.3) and obtain by applying
some classical inequalities

$$3^{-p} |J_t^{-1} h^L |^p \leq |h^L |^p_W + t^{p/q} \int_0^t |J_s^{-1} |A_L + D_L (B_L (\Phi_s, \Phi_s) \chi(\frac{\Phi_s |W}{3 \rho})) - \text{Tr}((D_t Q_L (\Phi_t))^2) |h^L |^p_W dt$$

$$+ \int_0^t J_s^{-1} D_L Q_L (\Phi_s) h^L dW^L_s |^p_W.$$ 

Since all the operators in the above inequalities are finite dimensional, by (A.6), Doob’s
martingale inequality and Birkhold-Davis-Gundy inequality, one has

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |J_t^{-1} h^L |^p_W \right] \leq C_1 (1 + T^{p/q} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |J_t^{-1} |L(\mathcal{W}) \right]) |h^L |^p_W$$

where $C_1 = C_1(p, \rho, \alpha_0)$. When $T$ is small enough, we have $\mathbb{E}[\sup_{0 \leq t \leq T} |J_t^{-1} |L(\mathcal{W})] \leq 1 - C_1(1/p + T^{p/q} - 2)$.

Clearly, $D_v \Phi_t^L$ satisfies the following equation

$$D_v \Phi_t^L = \int_0^t e^{-A(t-s)} [-B_L (\Phi_s, D_v \Phi_s^L) - B_L (D_v \Phi_s^L, \Phi_s)] \chi(\frac{\Phi_s |W}{3 \rho}) ds$$

$$- \frac{1}{3 \rho} \int_0^t e^{-A(t-s)} B_L (\Phi_s, \Phi_s) \chi(\frac{\Phi_s |W}{3 \rho}) (D_v \Phi_s^L, \Phi_s) |\Phi_s | |W| ds$$

$$+ \int_0^t e^{-A(t-s)} (1 - \chi(\frac{\Phi_s |W}{\rho})) Q_L v^L ds - \frac{1}{\rho} \int_0^t e^{-A(t-s)} (\frac{\Phi_s |W}{\rho}) (D_v \Phi_s^L, \Phi_s) |\Phi_s | |W| Q_L dW^L_s$$

$$= J_1(t) + J_2(t) + J_3(t) + J_4(t)$$

By (A.6) and Lemma A.1, one has

$$|J_1(t)| \leq \int_0^t \frac{C_2}{\sqrt{t-s}} |D_v \Phi_s^L | |W| ds$$

$$|J_2(t)| \leq \int_0^t \frac{C_3}{\sqrt{t-s}} |D_v \Phi_s^L | |W| ds$$

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |J_3(t) |^p_W \right] \leq C_4 \mathbb{E} \left( \int_0^T |v^L (s) |^p_W ds \right)$$

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |J_4(t) |^p_W \right] \leq C_5 T^{p/8} \mathbb{E} \left( \sup_{0 \leq t \leq T} |D_v \Phi_t^L |^p_W \right), \quad 0 \leq T \leq 1.$$
Expanding the terms estimates. Noticing the fact very complex expressions which we omit them but point out the key points for their small enough.

Lemma for \( 0 < T \), with

\[
\|e^{A(t-s)}D_{v_1}D_{v_2}Q_{L}(\Phi_s)\|_{W} \leq C_T\|D_{v_1}D_{v_2}Q_{L}(\Phi_s)\|_{W}^2,
\]

Expanding the terms \( D_{v_1}D_{v_2}(B_L(\Phi_s, \Phi_s^L)\chi(\frac{\|\Phi_s\|}{3\rho})) \) and \( D_{v_1}D_{v_2}L_{i}(\Phi_s) \), we obtain two very complex expressions which we omit them but point out the key points for their estimates. Noticing the fact \( D_{v_2}\Phi_t = D_{v_2}\Phi_t |_{W} \), \( |\Phi_t|_{W} \chi(\frac{\|\Phi_t\|}{3\rho}) \leq 3\rho \), and using (A.6) and Lemma A.1, one has

\[
\left|e^{A(t-s)}D_{v_1}D_{v_2}L_{i}(\Phi_s)\right|_{W} \leq C_T\|D_{v_1}D_{v_2}L_{i}(\Phi_s)\|_{W}^2,
\]

and

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_0^t e^{-A(t-s)}D_{v_1}D_{v_2}Q_{L}(\Phi_s)\|D_{v_2}\Phi_t^L\|_{W}^p ds \right| \right) \leq \frac{C_9}{\sqrt{T - s}}\left(\|D_{v_1}D_{v_2}\Phi_t^L\|_{W}^2 + \|D_{v_1}\Phi_t^L\|_{W}\|D_{v_2}\Phi_t^L\|_{W}\right),
\]

for \( 0 < T \leq 1 \), with \( C_i = C_i(\rho, \alpha_0) \) (i = 7, 8) and \( C_9 = C_9(\rho, \alpha_0, p) \). Hence, when \( T \) is small

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |D_{v_1}D_{v_2}\Phi_t^L|_{W}^p \right) \leq \frac{C_9}{1 - C_9T^{p/8}} \mathbb{E}\left(\left|D_{v_1}\Phi_t^L\right|_{W}^p + \left|D_{v_2}\Phi_t^L\right|_{W}^p \right) \leq \left(\frac{C_{10}}{1 - C_{10}T^{p/8}}\right)^2 \left(1 + \mathbb{E}\left[\int_0^T |\Phi_t^L(s)|_{W}^{2p} ds\right]\right)^{\frac{1}{2}} \left(1 + \mathbb{E}\left[\int_0^T |\Phi_t^L(s)|_{W}^{2p} ds\right]\right)^{\frac{1}{2}},
\]

with \( C_{10} = C_{10}(\rho, \alpha_0, p) \). The proof of (4.16) is similar. \(\square\)

References
