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The research described in this paper is joint work with M. Birkner (Mainz), E. Bolthausen (Zürich), D. Cheliotis (Athens) and A. Greven (Erlangen).

Abstract. In this paper we describe two large deviation principles for the empirical process of words cut out from a random sequence of letters according to a random renewal process: one where the letters are frozen ("quenched") and one where the letters are not frozen ("annealed"). We apply these large deviation principles to five classes of interacting stochastic systems: interacting diffusions, coupled branching processes, and three examples of a polymer chain in a random environment. In particular, we show how these large deviation principles can be used to derive variational formulas for the critical curves that are associated with the phase transitions occurring in these systems, and how these variational formulas can in turn be used to prove the existence of certain intermediate phases.

Mathematics Subject Classification (2000). Primary 60F10, 60G50, 60K35; Secondary 82D60.

Keywords. Large deviation principle, quenched vs. annealed, interacting stochastic systems, variational formulas, phase transitions, intermediate phases.

1. Large Deviation Principles

In Section 1 we describe two large deviation principles that were derived in Birkner, Greven and den Hollander [3]. In Sections 2–4 we apply these large deviation principles to five classes of interacting stochastic systems that exhibit a phase transition. In Section 5 we argue why these applications open up a new window of research, with a variational view, and we make a few closing remarks.

1.1. Letters, words and sentences. Let E be a Polish space (e.g. $E = \mathbb{Z}^d$, $d \geq 1$, with the lattice norm or $E = \mathbb{R}$ with the Euclidean norm). Think of E as an alphabet, i.e., a set of *letters*. Let $\widetilde{E} = \cup_{n\in\mathbb{N}} E^n$ be the set of finite *words* drawn from E, which is a Polish space under the discrete topology.

For ν a probability measure on E, let $X = (X_k)_{k\in\mathbb{N}_0}$ (with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) be i.i.d. with law ν. For ρ a probability measure on $\mathbb{N}$, let $\tau = (\tau_i)_{i\in\mathbb{N}}$ be i.i.d. with law ρ. Assume that X and τ are independent and write Pr to denote their joint law.

Given X and τ, define $Y = (Y^{(i)})_{i\in\mathbb{N}}$ by putting

$$T_0 = 0 \quad \text{and} \quad T_i = T_{i-1} + \tau_i, \quad i \in \mathbb{N}, \tag{1.1}$$

and

$$Y^{(i)} = \big(X_{T_{i-1}}, X_{T_{i-1}+2}, \ldots, X_{T_i - 1}\big), \quad i \in \mathbb{N}. \tag{1.2}$$

In words, Y is the infinite sequence of words cut out from the infinite sequence of letters X according to the renewal times τ (see Fig. 1). Clearly, under the law Pr, Y is i.i.d. with law $q_{\rho,\nu}^{\otimes\mathbb{N}}$ on $\widetilde{E}^{\mathbb{N}}$, the set of infinite *sentences*, where the marginal law $q_{\rho,\nu}$ on $\widetilde{E}$ is given by

$$q_{\rho,\nu}\big((x_1, \ldots, x_n)\big) = \rho(n)\,\nu(x_1)\cdots\nu(x_n), \quad n \in \mathbb{N},\, x_1, \ldots, x_n \in E. \tag{1.3}$$

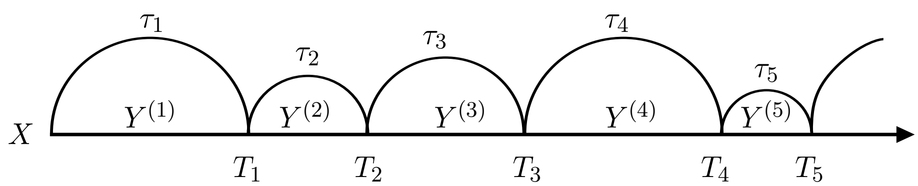

Figure 1. Cutting words out from a sequence of letters according to renewal times.

The reverse operation of *cutting* words out from a sequence of letters is *glueing* words together into a sequence of letters. Formally, this is done by defining a *concatenation* map κ from $\widetilde{E}^{\mathbb{N}}$ to $E^{\mathbb{N}}$. This map induces in a natural way a map κ from $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}(E^{\mathbb{N}})$, the sets of probability measures on $\widetilde{E}^{\mathbb{N}}$ and $E^{\mathbb{N}}$ (endowed with the topology of weak convergence). The concatenation $q_{\rho,\nu}^{\otimes\mathbb{N}} \circ \kappa^{-1}$ of $q_{\rho,\nu}^{\otimes\mathbb{N}}$ equals $\nu^{\otimes\mathbb{N}}$, as is evident from (1.3).

Note that in the above set-up three objects can be freely chosen: E (alphabet), ν (letter law) and ρ (word length law). In what follows we will assume that ρ has infinite support and satisfies

$$\lim_{\substack{n\to\infty\\ \rho(n)>0}} \frac{\log \rho(n)}{\log n} = -\alpha \qquad \text{for some } \alpha \in [1, \infty). \tag{1.4}$$

1.2. Annealed LDP. Let $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ be the set of probability measures on $\widetilde{E}^{\mathbb{N}}$ that are invariant under the left-shift $\widetilde{\theta}$ acting on $\widetilde{E}^{\mathbb{N}}$. For $N \in \mathbb{N}$, let $(Y^{(1)}, \ldots, Y^{(N)})^{\mathrm{per}}$ be the periodic extension of the N-tuple $(Y^{(1)}, \ldots, Y^{(N)}) \in \widetilde{E}^N$ to an element of $\widetilde{E}^{\mathbb{N}}$, and define

$$R_N = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\widetilde{\theta}^i (Y^{(1)}, \ldots, Y^{(N)})^{\mathrm{per}}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \tag{1.5}$$

This is the *empirical process of N-tuples of words* in Y. The following large deviation principle (LDP) is standard (see e.g. Dembo and Zeitouni [14], Corollaries 6.5.15 and 6.5.17). Let

$$H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) = \lim_{N\to\infty} \frac{1}{N} h\left(Q_{|\mathcal{F}_N} \;\middle|\; (q_{\rho,\nu}^{\otimes\mathbb{N}})_{|\mathcal{F}_N}\right) \in [0,\infty] \tag{1.6}$$

be the *specific relative entropy of Q w.r.t. $q_{\rho,\nu}^{\otimes\mathbb{N}}$*. Here, $\mathcal{F}_N = \sigma(Y^{(1)},\dots,Y^{(N)})$ is the sigma-algebra generated by the first N words, $Q_{|\mathcal{F}_N}$ is the restriction of Q to $\mathcal{F}_N$, and $h(\cdot \mid \cdot)$ denotes relative entropy.

Theorem 1.1. [Annealed LDP] *The family of probability distributions* $\Pr(R_N \in \cdot)$*, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with rate function $I^{\mathrm{ann}}\colon \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \to [0,\infty]$ given by*

$$I^{\mathrm{ann}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}). \tag{1.7}$$

The rate function I^{ann} is lower semi-continuous, has compact level sets, has a unique zero at $Q = q_{\rho,\nu}^{\otimes\mathbb{N}}$, and is affine.

Informally, Theorem 1.1 says that $\Pr(R_N \approx Q) \approx e^{-NI^{\mathrm{ann}}(Q)}$ as $N \to \infty$.

1.3. Quenched LDP.

To formulate the quenched analogue of Theorem 1.1, which is the main result in Birkner, Greven and den Hollander [3], we need some further notation. Let $\mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}})$ be the set of probability measures on $E^{\mathbb{N}}$ that are invariant under the left-shift θ acting on $E^{\mathbb{N}}$. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ such that $m_Q = E_Q[\tau_1] < \infty$ (where E_Q denotes expectation under the law Q and τ_1 is the length of the first word), define

$$\Psi_Q(\cdot) = \frac{1}{m_Q} E_Q\left[\sum_{k=0}^{\tau_1 - 1} \delta_{\theta^k \kappa(Y)}(\cdot)\right] \in \mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}}). \tag{1.8}$$

Think of Ψ_Q as the shift-invariant version of $Q \circ \kappa^{-1}$ obtained after *randomising* the location of the origin. This randomisation is necessary because a shift-invariant Q in general does not (!) give rise to a shift-invariant $Q \circ \kappa^{-1}$.

For $\mathrm{tr} \in \mathbb{N}$, let $[\cdot]_{\mathrm{tr}}\colon \widetilde{E} \to [\widetilde{E}]_{\mathrm{tr}} = \cup_{n=1}^{\mathrm{tr}} E^n$ denote the *word length truncation* map defined by

$$y = (x_1,\dots,x_n) \mapsto [y]_{\mathrm{tr}} = (x_1,\dots,x_{n\wedge \mathrm{tr}}), \qquad n \in \mathbb{N},\ x_1,\dots,x_n \in E, \tag{1.9}$$

i.e., $[y]_{\mathrm{tr}}$ is the word of length $\leq \mathrm{tr}$ obtained from the word y by dropping all the letters with label $> \mathrm{tr}$. This map induces in a natural way a map from $\widetilde{E}^{\mathbb{N}}$ to $[\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$, and from $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}([\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}})$. Note that if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, then $[Q]_{\mathrm{tr}}$ is an element of the set

$$\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\colon\ m_Q < \infty\}. \tag{1.10}$$

Theorem 1.2. [Quenched LDP] *For $\nu^{\otimes\mathbb{N}}$–a.s. all X, the family of regular conditional probability distributions* $\Pr(R_N \in \cdot \mid X)$, $N \in \mathbb{N}$, *satisfies the LDP on* $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ *with rate N and with deterministic rate function* $I^{\mathrm{que}}\colon \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \to [0,\infty]$ *given by*

$$I^{\mathrm{que}}(Q) = \begin{cases} I^{\mathrm{fin}}(Q), & \text{if } Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \\ \lim\limits_{\mathrm{tr}\to\infty} I^{\mathrm{fin}}\big([Q]_{\mathrm{tr}}\big), & \text{otherwise}, \end{cases} \tag{1.11}$$

where

$$I^{\mathrm{fin}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) + (\alpha-1)\, m_Q\, H(\Psi_Q \mid \nu^{\otimes\mathbb{N}}). \tag{1.12}$$

The rate function I^{que} is lower semi-continuous, has compact level sets, has a unique zero at $Q = q_{\rho,\nu}^{\otimes\mathbb{N}}$, and is affine.

Informally, Theorem 1.2 says that $\Pr(R_N \approx Q \mid X) \approx e^{-N I^{\mathrm{que}}(Q)}$ as $N \to \infty$ for $\nu^{\otimes\mathbb{N}}$-a.s. all X.

Note from (1.7) and (1.11–1.12) that I^{que} equals I^{ann} plus an additional term that quantifies the deviation of Ψ_Q, the randomised concatenation of Q, from the reference law $\nu^{\otimes\mathbb{N}}$ of the letter sequence. This term, which also depends on the exponent α in (1.4), is explicit when $m_Q < \infty$, but requires a truncation approximation when $m_Q = \infty$. Further note that if $\alpha = 1$, then the additional term vanishes and $I^{\mathrm{que}} = I^{\mathrm{ann}}$.

2. Collision local time of two random walks

In this section we apply Theorems 1.1–1.2 to study the collision local time of two random walks. The results are taken from Birkner, Greven and den Hollander [4]. In Section 3 we will use the outcome of this section to describe phase transitions in two interacting stochastic systems: interacting diffusions and coupled branching processes.

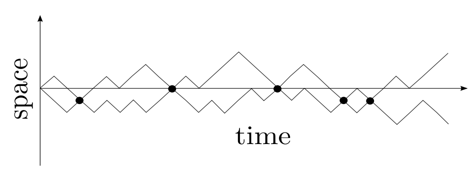

Figure 2. Two random walks that build up collision local time.

Let $S = (S_k)_{k\in\mathbb{N}_0}$ and $S' = (S'_k)_{k\in\mathbb{N}_0}$ be two independent random walks on $\mathbb{Z}^d$, $d \geq 1$, both starting at the origin and with an irreducible, symmetric and transient transition kernel $p(\cdot,\cdot)$. Write p^n for the n-th convolution power of p. Suppose that

$$\lim_{n\to\infty} \frac{\log p^{2n}(0,0)}{\log n} = -\alpha \qquad \text{for some } \alpha \in [1,\infty). \tag{2.1}$$

Write P to denote the joint law of S, S'. Let

$$V = V(S,S') = \sum_{k\in\mathbb{N}} 1_{\{S_k = S'_k\}} \tag{2.2}$$

be the *collision local time* of S, S' (see Fig. 2), which satisfies $P(V < \infty) = 1$ because $p(\cdot,\cdot)$ is transient. Define

$$\begin{aligned} z_1 &= \sup\left\{z \geq 1\colon\ E\left[\, z^V \mid S\,\right] < \infty \ \ S\text{-a.s.}\right\}, && (2.3)\\ z_2 &= \sup\left\{z \geq 1\colon\ E\left[\, z^V \,\right] < \infty\right\}. && (2.4)\end{aligned}$$

(The lower indices indicate the number of random walks being averaged over.) Note that, by the tail triviality of S, the range of z-values for which $E[\, z^V \mid S\,]$ converges is S-a.s. constant.

As shown in [4], Theorems 1.1–1.2 can be applied with the following choice of E, ν and ρ:

$$E = \mathbb{Z}^d, \quad \nu(x) = p(0,x), \quad \rho(n) = p^{2\lfloor n/2\rfloor}(0,0)/[2\bar{G}(0,0)-1], \tag{2.5}$$

where $\bar{G}(0,0) = \sum_{n\in\mathbb{N}_0} p^{2n}(0,0)$ is the Green function at the origin associated with $p^2(\cdot,\cdot)$, the transition kernel of $S - S'$. The following theorem provides variational formulas for z_1 and z_2. This theorem requires additional assumptions on $p(\cdot,\cdot)$:

$$\begin{aligned} &\sum_{x\in\mathbb{Z}^d} \|x\|^\delta p(0,x) < \infty \text{ for some } \delta > 0,\\ &\liminf_{n\to\infty} \frac{\log[\, p^n(0,S_n)/p^{2\lfloor n/2\rfloor}(0,0)\,]}{\log n} \geq 0 \qquad S-a.s., \\ &\inf_{n\in\mathbb{N}} E\big[\log[\, p^n(0,S_n)/p^{2\lfloor n/2\rfloor}(0,0)\,]\big] > -\infty.\end{aligned} \tag{2.6}$$

As shown in [4], the last two assumptions hold for a large class of random walks, including those that are in the domain of attraction of a normal law, respectively, a symmetric stable law. They potentially hold in full generality under a mild regularity condition on $p(\cdot,\cdot)$. [1]

Theorem 2.1. *Assume* (2.1) *and* (2.6). *Then* $z_1 = 1 + e^{-r_1}$, $z_2 = 1 + e^{-r_2}$ *with*

$$r_1 = \sup_{Q\in\mathcal{P}^{\mathrm{inv}}\left(\widetilde{\mathbb{Z}^d}^{\mathbb{N}}\right)} \left\{\int_{\widetilde{\mathbb{Z}^d}} (\pi_1 Q)(dy)\, \log f(y) - I^{\mathrm{que}}(Q)\right\} \in \mathbb{R}, \tag{2.7}$$

$$r_2 = \sup_{Q\in\mathcal{P}^{\mathrm{inv}}\left(\widetilde{\mathbb{Z}^d}^{\mathbb{N}}\right)} \left\{\int_{\widetilde{\mathbb{Z}^d}} (\pi_1 Q)(dy)\, \log f(y) - I^{\mathrm{ann}}(Q)\right\} \in \mathbb{R}, \tag{2.8}$$

where $\pi_1 Q$ *is the projection of* Q *onto* $\widetilde{\mathbb{Z}^d}$*, i.e., the law of the first word, and* $f\colon \widetilde{\mathbb{Z}^d} \to [0,\infty)$ *is given by*

$$f((x_1,\ldots,x_n)) = \frac{1}{\rho(n)}\, p^n(0, x_1 + \cdots + x_n), \quad n\in\mathbb{N},\, x_1,\ldots,x_n\in\mathbb{Z}^d. \tag{2.9}$$

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[1] The symmetry of $p(\cdot,\cdot)$ implies that $p^{2n}(0,0) > 0$ for all $n\in\mathbb{N}_0$ and $p^n(0,x)/p^{2\lfloor n/2\rfloor}(0,0) \leq 1$ for all $n\in\mathbb{N}_0$ and $x\in\mathbb{Z}^d$.

Remark: Since $P(V=k)=(1-\bar{F})\bar{F}^k$, $k\in\mathbb{N}_0$, with $\bar{F}=P\big(\exists\, k\in\mathbb{N}\colon\, S_k=S'_k\big)$, an easy computation gives $z_2=1/\bar{F}$. Since $\bar{F}=1-[1/\bar{G}(0,0)]$, we therefore have $z_2=\bar{G}(0,0)/[\bar{G}(0,0)-1]$. This simple formula reflects itself in the fact that the variational formula in (2.8) can be solved explicitly (see [4]). However, unlike for z_2, no closed form expression is known for z_1, because the variational formula in (2.7) cannot be solved explicitly.

Because $I^{\mathrm{que}}\geq I^{\mathrm{ann}}$, we have $r_1\leq r_2$, and hence $z_2\leq z_1$. The following corollary gives conditions under which strict inequality holds or not. Its proof in [4] relies on a comparison of the two variational formulas in (2.7–2.8).

Corollary 2.2. *Assume* (2.1) *and* (2.6).
(a) If $p(\cdot,\cdot)$ *is strongly transient, i.e.,* $\sum_{n\in\mathbb{N}} np^n(0,0)<\infty$, *then* $z_2<z_1$.
(b) If $\alpha=1$, *then* $z_1=z_2$.

Analogous results hold when we turn the discrete-time random walks S and S' into continuous-time random walks $\widetilde{S}=(S_t)_{t\geq 0}$ and $\widetilde{S'}=(\widetilde{S}'_t)_{t\geq 0}$ by allowing them to make steps at rate 1, while keeping the same transition kernel $p(\cdot,\cdot)$. Then the collision local time becomes

$$\widetilde{V}=\int_0^\infty 1_{\{\widetilde{S}_t=\widetilde{S}'_t\}}\,dt. \tag{2.10}$$

For the analogous quantities $\widetilde{z}_1$ and $\widetilde{z}_2$, variational formulas like in Theorem 2.1 can be derived, and a result similar to Corollary 2.2 holds:

Corollary 2.3. *Assume* (2.1) *and* (2.6).
(a) If $p(\cdot,\cdot)$ *is strongly transient, then* $\widetilde{z}_2<\widetilde{z}_1$.
(b) If $\alpha=1$, *then* $\widetilde{z}_1=\widetilde{z}_2$.

An easy computation gives $\log\widetilde{z}_2=2/G(0,0)$, where $G(0,0)=\sum_{n\in\mathbb{N}_0}p^n(0,0)$ is the Green function at the origin associated with $p(\cdot,\cdot)$. There is again no closed form expression for $\widetilde{z}_1$.

Recent progress on extending the gaps in Corollaries 2.2(a) and 2.3(a) to transient random walks that are not strongly transient (like simple random walk in $d=3,4$) can be found in Birkner and Sun [5], [6], and in Berger and Toninelli [1]. These papers require assumptions on the tail of $p(0,\cdot)$ and use fractional moment estimates rather than variational formulas.

3. Two applications without disorder

3.1. Interacting diffusions. Consider the following system of coupled stochastic differential equations:

$$dX_x(t)=\sum_{y\in\mathbb{Z}^d}p(x,y)[X_y(t)-X_x(t)]\,dt+\sqrt{qX_x(t)^2}\,dW_x(t),\quad x\in\mathbb{Z}^d,t\geq 0. \tag{3.1}$$

Here, $p(\cdot,\cdot)$ is a random walk transition kernel on $\mathbb{Z}^d$, $q \in (0,\infty)$ is a diffusion constant, and $W = (W(t))_{t\geq 0}$ with $W(t) = \{W_x(t)\}_{x\in\mathbb{Z}^d}$ is a collection of independent standard Brownian motions on $\mathbb{R}$. The initial condition is chosen such that $\{X_x(0)\}_{x\in\mathbb{Z}^d}$ is a shift-invariant and shift-ergodic random field taking values in $[0,\infty)$ with a positive and finite mean (the evolution in (3.1) preserves the mean).

It was shown in Greven and den Hollander [19] that if $p(\cdot,\cdot)$ is irreducible, symmetric and transient, then there exist $0 < q_2 \leq q_* < \infty$ such that the system in (3.1) locally dies out when $q > q_*$, but converges to a non-trivial equilibrium when $q < q_*$, and this equilibrium has an *infinite second moment* when $q \geq q_2$ and a *finite second moment* when $q < q_2$. It was conjectured in [19] that $q_2 < q_*$. Since it was shown in [19] that

$$q_* = \log \widetilde{z}_1, \qquad q_2 = \log \widetilde{z}_2, \tag{3.2}$$

Corollary 2.3(a) settles this conjecture when $p(\cdot,\cdot)$ satisfies (2.1) and (2.6) and is strongly transient.

3.2. Coupled branching processes. Consider a spatial population model on $\mathbb{Z}^d$ evolving as follows:

(1) Each individual migrates at rate 1 according to $p(\cdot,\cdot)$.
(2) Each individual gives birth to a new individual at the same site at rate q.
(3) Each individual dies at rate $q(1-r)$.
(4) All individuals at the same site die simultaneously at rate qr.

(3.3)

Here, $p(\cdot,\cdot)$ is a random walk transition kernel on $\mathbb{Z}^d$, $q \in (0,\infty)$ is a birth-death rate, and $r \in [0,1]$ is a coupling parameter. The case $r = 0$ corresponds to a critical branching random walk, for which the average number of individuals per site is preserved. The case $r > 0$ is challenging because the individuals descending from different ancestors are no longer independent.

For the case $r = 0$, the following *dichotomy* holds (where for simplicity we restrict to an irreducible and symmetric $p(\cdot,\cdot)$): if the initial configuration is drawn from a shift-invariant and shift-ergodic random field taking values in $\mathbb{N}_0$ with a positive and finite mean, then the system in (3.3) locally dies out when $p(\cdot,\cdot)$ is *recurrent*, but converges to a non-trivial equilibrium when $p(\cdot,\cdot)$ is *transient*, both irrespective of the value of q. In the latter case, the equilibrium has the same mean as the initial distribution and has all moments finite.

For the case $r > 0$, the situation is more subtle. It was shown in Greven [17], [18] that there exist $0 < r_2 \leq r_* \leq 1$ such that the system in (3.3) locally dies out when $r > r_*$, but converges to a non-trivial equilibrium when $r < r_*$, and this equilibrium has an *infinite second moment* when $r \geq r_2$ and a *finite second moment* when $r < r_2$. It was conjectured in [18] that $r_2 < r_*$. Since it was shown in [18] that

$$r_* \geq 1 \wedge (q^{-1} \log \widetilde{z}_1), \qquad r_2 = 1 \wedge (q^{-1} \log \widetilde{z}_2), \tag{3.4}$$

Corollary 2.3(a) settles this conjecture when $p(\cdot,\cdot)$ satisfies (2.1) and (2.6) and is strongly transient, and $q > \log \widetilde{z}_2 = 2/G(0,0)$.

4. Three applications with disorder

4.1. A polymer in a random potential.

Path measure. Let $S = (S_k)_{k\in\mathbb{N}_0}$ be a random walk on $\mathbb{Z}^d$, $d \geq 1$, starting at the origin and with transition kernel $p(\cdot,\cdot)$. Write P to denote the law of S. Let $\omega = \{\omega(k,x)\colon\, k \in \mathbb{N}_0,\, x \in \mathbb{Z}^d\}$ be an i.i.d. field of $\mathbb{R}$-valued non-degenerate random variables with marginal law μ_0, playing the role of a *random environment*. Write $\mathbb{P} = (\mu_0)^{\otimes[\mathbb{N}_0\times\mathbb{Z}^d]}$ to denote the law of ω. Assume that

$$M(\lambda) = \mathbb{E}\big(e^{\lambda\omega(0,0)}\big) < \infty \qquad \forall\, \lambda \in \mathbb{R}. \tag{4.1}$$

For fixed ω and $n \in \mathbb{N}$, define

$$\frac{dP_n^{\beta,\omega}}{dP}\big((S_k)_{k=0}^n\big) = \frac{1}{Z_n^{\beta,\omega}}\, e^{-H_n^{\beta,\omega}\left((S_k)_{k=0}^n\right)} \tag{4.2}$$

with

$$H_n^{\beta,\omega}\big((S_k)_{k=0}^n\big) = -\beta\sum_{k=1}^n \omega(k,S_k), \tag{4.3}$$

i.e., $P_n^{\beta,\omega}$ is the Gibbs measure on the set of paths of length $n \in \mathbb{N}$ associated with the Hamiltonian $H_n^{\beta,\omega}$. Here, $\beta \in [0,\infty)$ plays the role of *environment strength* (or "inverse temperature"), while $Z_n^{\beta,\omega}$ is the normalising partition sum. In this model, ω represents a space-time medium of "random charges" with which a directed polymer, described by the space-time path $(k,S_k)_{k=0}^n$, is interacting (see Fig. 3).

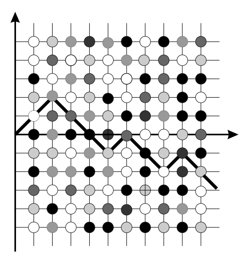

Figure 3. A directed polymer sampling random charges in a halfplane.

Weak vs. strong disorder. Let $\chi_n(\omega) = Z_n^{\beta,\omega}\, e^{-n\log M(\beta)}$, $n \in \mathbb{N}_0$. It is well known that $\chi(\omega) = (\chi_n(\omega))_{n\in\mathbb{N}_0}$ is a non-negative martingale with respect to the family of sigma-algebras $\mathcal{F}_n = \sigma(\omega(k,x),\, 0 \leq k \leq n,\, x \in \mathbb{Z}^d)$, $n \in \mathbb{N}_0$. Hence $\lim_{n\to\infty}\chi_n(\omega) = \chi_\infty(\omega) \geq 0$ ω-a.s., with $\mathbb{P}(\chi_\infty(\omega) = 0) = 0$ or 1. This leads to two phases:

$$\begin{aligned}\mathcal{W} &= \{\beta \in [0,\infty)\colon\, \chi_\infty(\omega) > 0\ \omega - a.s.\},\\ \mathcal{S} &= \{\beta \in [0,\infty)\colon\, \chi_\infty(\omega) = 0\ \omega - a.s.\},\end{aligned} \tag{4.4}$$

which are referred to as the *weak disorder phase* and the *strong disorder phase*, respectively. It was shown in Comets and Yoshida [13] that there is a unique critical value $\beta_* \in [0,\infty]$ (depending on d, $p(\cdot,\cdot)$ and μ_0) such that weak disorder holds for $0 \le \beta < \beta_*$ and strong disorder holds for $\beta > \beta_*$. Moreover, in the weak disorder phase the paths have a Gaussian scaling limit under the Gibbs measure, while this is not the case in the strong disorder phase. In the strong disorder phase the path tends to localise around the highest values of ω in a narrow space-time tube.

Suppose that $p(\cdot,\cdot)$ is irreducible, symmetric and transient. Abbreviate $\Delta(\beta) = \log M(2\beta) - 2\log M(\beta)$. Note that $\beta \mapsto \Delta(\beta)$ is strictly increasing. Bolthausen [9] observed that

$$\mathbb{E}\left[\chi_n(\omega)^2\right] = \mathbb{E}\Big[e^{\Delta(\beta)\,V_n}\Big] \quad \text{with} \quad V_n = \sum_{k=1}^{n} 1_{\{S_k = S'_k\}}, \tag{4.5}$$

where S and S' are two independent random walks with transition kernel $p(\cdot,\cdot)$, and concluded that $\chi(\omega)$ is L^2-bounded if and only if $\beta < \beta_2$ with $\beta_2 \in (0,\infty]$ the unique solution of

$$\Delta(\beta_2) = \log z_2 \tag{4.6}$$

(with $\beta_2 = \infty$ whenever $\Delta(\infty) \le \log z_2$). Since

$$\mathbb{P}(\chi_\infty(\omega) > 0) \ge \mathbb{E}[\chi_\infty(\omega)]^2 / \mathbb{E}[\chi_\infty(\omega)^2], \qquad \mathbb{E}[\chi_\infty(\omega)] = \chi_0(\omega) = 1, \tag{4.7}$$

it follows that $\beta < \beta_2$ implies weak disorder, i.e., $\beta_* \ge \beta_2$. By a stochastic representation of the size-biased law of $\chi_n(\omega)$, it was shown in Birkner [2] that in fact weak disorder holds if $\beta < \beta_1$ with $\beta_1 \in (0,\infty]$ the unique solution of

$$\Delta(\beta_1) = \log z_1, \tag{4.8}$$

i.e., $\beta_* \ge \beta_1$. Since $\beta \mapsto \Delta(\beta)$ is strictly increasing for any non-degenerate μ_0 satisfying (4.1), it follows from (4.6–4.8) and Corollary 2.2(a) that $\beta_1 > \beta_2$ when $p(\cdot,\cdot)$ satisfies (2.1) and (2.6) and is strongly transient, provided μ_0 is such that $\beta_2 < \infty$. In that case the weak disorder phase contains a subphase for which $\chi(\omega)$ is *not* L^2-bounded. This disproves a conjecture of Monthus and Garel [21], who argued that $\beta_2 = \beta_*$.

For further details, see den Hollander [20], Chapter 12. Main contributions in the mathematical literature towards understanding the two phases have come from M. Birkner, E. Bolthausen, A. Camanes, P. Carmona, F. Comets, B. Derrida, M.R. Evans, Y. Hu, J.Z. Imbrie, O. Mejane, M. Petermann, M.S.T. Piza, T. Shiga, Ya.G. Sinai, T. Spencer, V. Vargas and N. Yoshida.

4.2. A polymer pinned at an interface.

Path measure. Let $S = (S_k)_{k\in\mathbb{N}_0}$ be a recurrent Markov chain on a countable state space starting at a marked point 0. Write P to denote the law of S. Let K denote the law of the first return time of S to 0, which is assumed to satisfy

$$\lim_{n\to\infty} \frac{\log K(n)}{\log n} = -\alpha \qquad \text{for some } \alpha \in [1,\infty). \tag{4.9}$$

Let $\omega = (\omega_k)_{k\in\mathbb{N}_0}$ be an i.i.d. sequence of $\mathbb{R}$-valued non-degenerate random variables with marginal law μ_0, again playing the role of a *random environment*. Write $\mathbb{P} = \mu_0^{\otimes\mathbb{N}_0}$ to denote the law of ω. Assume that

$$M(\lambda) = \mathbb{E}(e^{\lambda\omega_0}) < \infty \qquad \forall\,\lambda \in \mathbb{R}. \tag{4.10}$$

Without loss of generality we take: $\mathbb{E}(\omega_0) = 0$, $\mathbb{E}(\omega_0^2) = 1$.

For fixed ω and $n \in \mathbb{N}$, define, in analogy with (4.2–4.3),

$$\frac{dP_n^{\beta,h,\omega}}{dP}\big((S_k)_{k=0}^n\big) = \frac{1}{Z_n^{\beta,h,\omega}}\, e^{-H_n^{\beta,h,\omega}\big((S_k)_{k=0}^n\big)} \tag{4.11}$$

with

$$H_n^{\beta,h,\omega}\big((S_k)_{k=1}^n\big) = -\sum_{k=1}^n (\beta\omega_k - h)\, 1_{\{S_k=0\}}, \tag{4.12}$$

where $\beta \in [0,\infty)$ again plays the role of *environment strength*, and $h \in [0,\infty)$ the role of *environment bias*. This models a directed polymer interacting with "random charges" at an interface (see Fig. 4). A key example is when S is simple random walk on $\mathbb{Z}$, which corresponds to the case $\alpha = \frac{3}{2}$.

The *quenched free energy per monomer* $f^{\mathrm{que}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n}\log Z_n^{\beta,h,\omega}$ is constant ω-a.s. (a property called self-averaging), and has two phases

$$\begin{aligned} \mathcal{L} &= \big\{(\beta,h)\colon\ f^{\mathrm{que}}(\beta,h) > 0\big\},\\ \mathcal{D} &= \big\{(\beta,h)\colon\ f^{\mathrm{que}}(\beta,h) = 0\big\}, \end{aligned} \tag{4.13}$$

which are referred to as the *localised phase* and the *delocalised phase*. These two phases are the result of a competition between entropy and energy: by staying close to the interface the polymer looses entropy, but at the same time it gains energy because it can more easily pick up large charges at the interface. The lower bound comes from the strategy where the path spends all its time above the interface, i.e., $S_k > 0$ for $1 \le k \le n$. Indeed, in that case $H_n^{\beta,h,\omega}((S_k)_{k=0}^n) = 0$, and since $\log[\sum_{m>n} K(m)] \sim -(\alpha-1)\log n$ as $n \to \infty$, the cost of this strategy under P is negligible on an exponential scale.

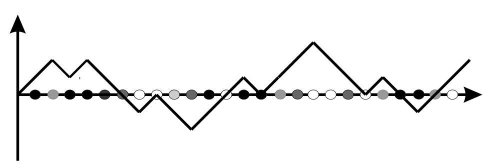

Figure 4. A directed polymer sampling random charges at an interface.

The associated *quenched critical curve* is

$$h_c^{\mathrm{que}}(\beta) = \inf\{h\colon\ f^{\mathrm{que}}(\beta,h) = 0\}, \quad \beta \in [0,\infty). \tag{4.14}$$

Both f^{que} and h_c^{que} are unknown. However, their *annealed* counterparts

$$f^{\mathrm{ann}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n}\log \mathbb{E}(Z_n^{\beta,h,\omega}), \qquad h_c^{\mathrm{ann}}(\beta) = \inf\{h\colon\ f^{\mathrm{ann}}(\beta,h) = 0\}, \tag{4.15}$$

can be computed explicitly, because they correspond to the degenerate case where $\omega_k = (1/\beta)\log M(\beta)$, $k \in \mathbb{N}_0$. In particular, $h_c^{\mathrm{ann}}(\beta) = \log M(\beta)$. Since $f^{\mathrm{que}} \leq f^{\mathrm{ann}}$, it follows that $h_c^{\mathrm{que}} \leq h_c^{\mathrm{ann}}$.

Disorder relevance vs. irrelevance. For a given choice of K, μ_0 and β, the disorder is said to be *relevant* when $h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{ann}}(\beta)$ and *irrelevant* when $h_c^{\mathrm{que}}(\beta) = h_c^{\mathrm{ann}}(\beta)$. Various papers have appeared in the literature containing various conditions under which relevant disorder, respectively, irrelevant disorder occurs, based on a variety of different estimation techniques. Main contributions in the mathematical literature have come from K. Alexander, B. Derrida, G. Giacomin, H. Lacoin, V. Sidoravicius, F.L. Toninelli and N. Zygouras. For overviews, see Giacomin [16], Chapter 5, and den Hollander [20], Chapter 11.

In work in progress with D. Cheliotis [12] a different view is taken. Namely, with the help of Theorems 1.1–1.2 for the choice

$$E = \mathbb{R}, \quad \nu = \mu_0, \quad \rho = K, \tag{4.16}$$

the following variational formulas are derived for h_c^{que} and h_c^{ann}.

Theorem 4.1. *For all $\beta \in [0,\infty)$,*

$$\begin{aligned} h_c^{\mathrm{que}}(\beta) &= \sup_{Q\in\mathcal{C}}[\beta\Phi(Q) - I^{\mathrm{que}}(Q)], \\ h_c^{\mathrm{ann}}(\beta) &= \sup_{Q\in\mathcal{C}}[\beta\Phi(Q) - I^{\mathrm{ann}}(Q)], \end{aligned} \tag{4.17}$$

where

$$\mathcal{C} = \left\{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\mathbb{R}}^{\mathbb{N}})\colon \int_{\mathbb{R}} |x|\,(\pi_{1,1}Q)(dx) < \infty\right\}, \qquad \Phi(Q) = \int_{\mathbb{R}} x\,(\pi_{1,1}Q)(dx), \tag{4.18}$$

with $\pi_{1,1}Q$ the projection of Q onto $\mathbb{R}$, i.e., the law of the first letter of the first word.

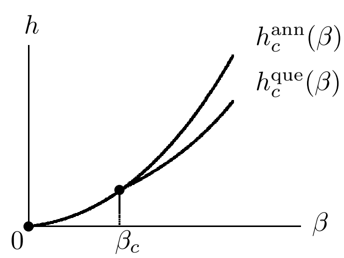

Figure 5. Critical curves for the pinned polymer

It is shown in [12] that a comparison of the two variational formulas in Theorem 4.1 yields the following *necessary and sufficient* condition for disorder relevance.

Corollary 4.2. *For every $\beta \in [0,\infty)$,*

$$h_c^{\rm que}(\beta) < h_c^{\rm ann}(\beta) \quad \Longleftrightarrow \quad I^{\rm que}(Q_\beta) > I^{\rm ann}(Q_\beta), \tag{4.19}$$

where $Q_\beta = q_{K,\beta}^{\otimes\mathbb{N}}$ is the unique maximiser of the annealed variational formula in (4.17)*, given by*

$$q_{K,\beta}((x_1,\dots,x_n)) = K(n)\,\mu_\beta(x_1)\cdots\mu_\beta(x_n), \quad n\in\mathbb{N},\ x_1,\dots,x_n\in\mathbb{R}, \tag{4.20}$$

with μ_β the law obtained from μ_0 by tilting:

$$d\mu_\beta(x) = \frac{1}{M(\beta)}\,e^{\beta x} d\mu_0(x), \qquad x\in\mathbb{R}. \tag{4.21}$$

As shown in [12], an immediate consequence of the variational characterisation in Corollary 4.2 is that there is a unique critical temperature (see Fig. 5).

Corollary 4.3. *For all μ_0 and K there exists a $\beta_c = \beta_c(\mu_0,K)\in[0,\infty]$ such that*

$$h_c^{\rm que}(\beta) \begin{cases} = h_c^{\rm ann}(\beta) & \text{if } \beta\in[0,\beta_c], \\ < h_c^{\rm ann}(\beta) & \text{if } \beta\in(\beta_c,\infty). \end{cases} \tag{4.22}$$

Moreover, necessary and sufficient conditions on μ_0 and K can be derived under which $\beta_c = 0$, $\beta_c\in(0,\infty)$, respectively, $\beta_c=\infty$, providing a *full classification of disorder relevance.*

4.3. A copolymer near a selective interface.

Path measure. Let S be a recurrent random walk on $\mathbb{Z}$. Keep (4.9–4.11), but change the Hamiltonian in (4.12) to

$$H_n^{\beta,h,\omega}\big((S_k)_{k=1}^n\big) = -\beta\sum_{k=1}^n(\omega_k+h)\,\mathrm{sign}(S_k). \tag{4.23}$$

This model was introduced in Garel, Huse, Leibler and Orland [15]. For the special case where $\mu_0 = \frac12(\delta_{-1}+\delta_{+1})$, it models a copolymer consisting of a random concatenation of hydrophobic and hydrophilic monomers (representated by ω), living in the vicinity of a linear interface that separates oil (above the interface) and water (below the interface) as solvents. The polymer is modelled as a two-dimensional directed path $(k,S_k)_{k\in\mathbb{N}_0}$. The Hamiltonian in (4.23) is such that hydrophobic monomers in oil ($\omega_k=+1$, $S_k>0$) and hydrophilic monomers in water ($\omega_k=-1$, $S_k<0$) receive a negative energy, while the other two combinations receive a positive energy.

The *quenched free energy per monomer*, $f^{\rm que}(\beta,h) = \lim_{n\to\infty}\frac1n\log Z_n^{\beta,h,\omega}$ ω-a.s., again has two phases (see Fig. 6)

$$\begin{aligned} \mathcal{L} &= \{(\beta,h)\colon\ g^{\rm que}(\beta,h)>0\},\\ \mathcal{D} &= \{(\beta,h)\colon\ g^{\rm que}(\beta,h)=0\}, \end{aligned} \tag{4.24}$$

where $g^{\text{que}}(\beta,h) = f^{\text{que}}(\beta,h) - \beta h$. These two phases are again the result of a competition between entropy and energy: by staying close to the interface the copolymer looses entropy, but it gains energy because it can more easily switch between the two sides of the interface in an attempt to place as many monomers as possible in their preferred solvent. The lower bound again comes from the strategy where the path spends all its time above the interface, i.e., $S_k > 0$ for $1 \leq k \leq n$. Indeed, in that case $\text{sign}(S_k) = +1$ for $1 \leq k \leq n$, resulting in $H_n^{\beta,h,\omega}((S_k)_{k=0}^n) = -\beta h n[1+o(1)]$ ω-a.s. as $n \to \infty$ by the strong law of large numbers for ω. Since $\log[\sum_{m>n} K(m)] \sim -(\alpha-1)\log n$ as $n \to \infty$, the cost of this strategy under P is again negligible on an exponential scale.

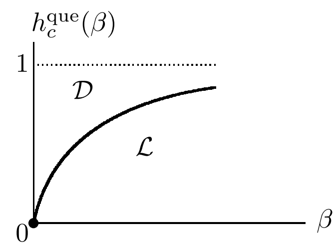

Figure 6. Quenched critical curve for the copolymer.

The associated *quenched critical curve* is

$$h_c^{\text{que}}(\beta) = \inf\{h\colon\ g^{\text{que}}(\beta,h) = 0\}, \quad \beta \in [0,\infty). \tag{4.25}$$

Both g^{que} and h_c^{que} are unknown. Their *annealed* counterparts $g^{\text{ann}}(\beta,h)$ and $h_c^{\text{ann}}(\beta) = \inf\{h\colon\ g^{\text{ann}}(\beta,h) = 0\}$ can again be computed explicitly.

The copolymer model is *much harder* than the pinning model described in Section 4.2, because the disorder ω is felt not just *at* the interface but along the *entire* polymer chain. The following bounds are known:

$$\left(\tfrac{2}{\alpha}\beta\right)^{-1} \log M\left(\tfrac{2}{\alpha}\beta\right) \leq h_c^{\text{que}}(\beta) \leq h_c^{\text{ann}}(\beta) = (2\beta)^{-1}\log M(2\beta) \qquad \forall\,\beta > 0. \tag{4.26}$$

The upper bound was proved in Bolthausen and den Hollander [10], and comes from the observation that $f^{\text{que}} \leq f^{\text{ann}}$. The lower bound was proved in Bodineau and Giacomin [7], and comes from strategies where the copolymer dips below the interface (into the water) during rare stretches in ω where the empirical density is sufficiently biased downwards (i.e., where the polymer is sufficiently hydrophilic).

Main contributions in the mathematical literature towards understanding the two phases have come from M. Biskup, T. Bodineau, E. Bolthausen, F. Caravenna, G. Giacomin, M. Gubinelli, F. den Hollander, H. Lacoin, N. Madras, E. Orlandini, A. Rechnitzer, Ya.G. Sinai, C. Soteros, C. Tesi, F.L. Toninelli, S.G. Whittington and L. Zambotti. For overviews, see Giacomin [16], Chapters 6–8, and den Hollander [20], Chapter 9.

Strict bounds. Toninelli [22] proved that the upper bound in (4.26) is strict for μ_0 with unbounded support and large β. This was later extended by Bodineau, Giacomin, Lacoin and Toninelli [8] to arbitrary μ_0 and β. The latter paper also proves that the lower bound in (4.26) is strict for small β. The proofs are based on fractional moment estimates of the partition sum and on finding appropriate localisation strategies.

In work in progress with E. Bolthausen [11], Theorems 1.1–1.2 are used, for the same choice as in (4.16), to obtain the following characterisation of the critical curves.

Theorem 4.4. *For every $\beta \in [0, \infty)$,*

$$h = h_c^{\mathrm{que}}(\beta) \qquad \Longleftrightarrow \qquad S^{\mathrm{que}}(\beta, h) = 0, \tag{4.27}$$

$$h = h_c^{\mathrm{ann}}(\beta) \qquad \Longleftrightarrow \qquad S^{\mathrm{ann}}(\beta, h) = 0, \tag{4.28}$$

with

$$S^{\mathrm{que}}(\beta, h) \quad = \sup_{Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{\mathbb{R}}^{\mathbb{N}})} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)], \tag{4.29}$$

$$S^{\mathrm{ann}}(\beta, h) \quad = \sup_{Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{\mathbb{R}}^{\mathbb{N}})} [\Phi_{\beta,h}(Q) - I^{\mathrm{ann}}(Q)], \tag{4.30}$$

where

$$\Phi_{\beta,h}(Q) = \int_{\widetilde{\mathbb{R}}} (\pi_1 Q)(dy) \, \log \phi_{\beta,h}(y), \qquad \phi_{\beta,h}(y) = \tfrac{1}{2} \left(1 + e^{-2\beta h \, \tau(y) - 2\beta \, \sigma(y)} \right), \tag{4.31}$$

with $\tau(y)$ and $\sigma(y)$ the length, respectively, the sum of the letters in the word y.

The variational formulas in Theorem 4.4 are more involved than those in Theorem 4.1 for the pinning model. The annealed variational formula in (4.30) can again be solved explicitly, the quenched variational formula in (4.29) cannot.

In [11] the strict upper bound in (4.26), which was proved in [8], is deduced from Theorem 4.4 via a criterion analogous to Corollary 4.2.

Corollary 4.5. *$h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{ann}}(\beta)$ for all μ_0 and $\beta > 0$.*

We are presently trying to prove that also the lower bound in (4.26) holds in full generality.

Weak interaction limit. A point of heated debate has been the slope of the quenched critical curve at $\beta = 0$,

$$\lim_{\beta \to \infty} \frac{1}{\beta} \, h_c^{\mathrm{que}}(\beta) = K_c, \tag{4.32}$$

which is believed to be *universal*, i.e, to only depend on α and to be robust against small perturbations of the interaction Hamiltonian in (4.23). The existence of the limit was proved in Bolthausen and den Hollander [10]. The bounds in (4.26) imply that $K_c \in [\alpha^{-1}, 1]$, and various claims were made in the literature arguing in favor of $K_c = \alpha^{-1}$, respectively, $K_c = 1$. In Bodineau, Giacomin, Lacoin and Toninelli [8] it is shown that $K_c \in (\alpha^{-1}, 1)$ under some additional assumptions on the excursion length distribution $K(\cdot)$ satisfying (4.9). We are presently trying to extend this result to arbitrary $K(\cdot)$ with the help of a space-time continuous version of the large deviation principles in Theorems 1.1–1.2.

5. Closing remarks

The large deviation principles in Theorems 1.1–1.2 are a powerful new tool to analyse the large space-time behaviour of interacting stochastic systems based on excursions of random walks and Markov chains. Indeed, they *open up a window with a variational view*, since they lead to explicit variational formulas for the critical curves that are associated with the phase transitions occurring in these systems. They are flexible, but at the same time technically demanding.

A key open problem is to find a good formula for $I^{\mathrm{que}}(Q)$ when $m_Q = \infty$ (recall (1.11–1.12)), e.g. when Q is Gibbsian.

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