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ON SOME TRACTABLE GROWTH COLLAPSE PROCESSES WITH RENEWAL COLLAPSE EPOCHS

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Abstract

In this paper we generalize existing results for the steady state distribution of growth collapse processes with independent exponential inter-collapse times to the case where they have a general distribution on the positive real line having a finite mean. In order to compute the moments of the stationary distribution, no further assumptions are needed. However, in order to compute the stationary distribution, the price that we are required to pay is the restriction of the collapse ratio distribution from a general one concentrated on the unit interval to minus-log-phase-type distributions. A random variable has such a distribution if the negative of its natural logarithm has a phase type distribution. Thus, this family of distributions is dense in the family of all distributions concentrated on the unit interval. The approach is to first study a certain Markov modulated shot-noise process from which the steady state distribution for the related growth collapse model can be inferred via level crossing arguments.

Keywords: Growth collapse; shot noise process; minus-log-phase-type distribution; Markov modulated.

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1. Introduction

Consider a growth collapse process that grows linearly at some given rate c. The collapses occur at renewal instants with inter-renewal time distribution function F with mean μ and Laplace-Stieltjes transform (LST) G. The remaining level (e.g., funds) after a given collapse form a random fraction of the level just before the collapse occurred. It is assumed that the sequence of random proportions $X_1, X_2, ...$ are i.i.d. and independent of the underlying renewal process. Of course, since these are proportions it is naturally assumed that $P[0 \le X_1 \le 1] = 1$ and, to avoid trivialities, that $0 < EX_1 < 1$. From, e.g., [14] it is known that this process is stable without any further conditions. We aim at identifying a relatively broad family of distributions of X_1, which is dense in the family of all distributions on $[0, 1]$, for which the stationary distribution of this process can be calculated.

The idea is to first consider an on/off process, where during *on* times the process increases linearly at rate c and during *off* times, whenever the process is at level x it decreases at the rate rx for some $r > 0$. As assumed, *on* times have some general distribution F while off times, denoted by $P_1, P_2, \ldots$, will be assumed to have a phase type distribution. If we restrict the process to *off* times, then what we obtain is a shot-noise type process with upward jumps having the distribution $F(\cdot/c)$ with mean $c\mu$, LST $G(c\alpha)$, and inter-arrival times which have a phase type distribution.

Given that at the beginning of an *off* time the level is given by some x, as long as the period does not end, the dynamics of the process is given via

$$W(t) = x - r\int_0^t W(s)\mathrm{d}s \tag{1}$$

where t is the time that has elapsed since the beginning of this period. Hence, as is well known, it follows that $W(t) = xe^{-rt}$ for $0 \le t \le P_i$ for some index i. Thus at the end of this period the level will be xe^{-rP_i}. Setting $X_i = e^{-rP_i}$ we see that when we restrict our process to *on* times, then it becomes the type of process that is described in the first paragraph. For example, when P_i are exponential with rate r then X_i are Uniform$(0, 1)$. If P_i have an Erlang distribution, then X_i are products of uniform random variables.

For a general phase-type distribution, X_i are (possibly infinite) mixtures of products of uniformly (on $(0, 1)$) distributed random variables. Since phase-type distributions are dense in the family of all distributions on $[0, \infty)$, then the family of distributions of the collapse ratio is dense in the family of all distributions on $[0, 1]$.

For the process at hand, it follows from [12] that if f_0 is the stationary density for the process restricted to on times and f_1 is the stationary density for the process restricted to off times, provided that they exist (note that starting from a positive state, the process as described never hits zero), then $cpf_0(x) = rx(1 - p)f_1(x)$ where $p = \frac{\mu}{\mu + EP_1}$ is the fraction of on times. Thus, studying the workload in the shot-noise-type model is in some sense equivalent with studying the workload in the growth collapse model. Also we note that the growth collapse model with growth rate 1 and inter-collapse times with LST $G(c\alpha)$ has the same stationary distribution as for the model initially proposed (with growth rate c) and thus we will, without loss of generality, assume from now on in order to simplify notations, that $c = 1$.

The paper is organized as follows. Regarding shot noise, in Section 2 we actually study a more general model and then restrict to the special case of the model proposed in this introduction. Section 3 relates the moments of the shot noise process to the moments of the original growth collapse process. Section 4 studies the steady-state behavior of the growth collapse process right after a collapse. Several distributions for the intercollapse times and the collapse proportions are being considered.

2. Shot noise type processes with phase type interarrival times

In [3] a shot-noise type process with Markov modulated release rate was considered. [16] studied a more general model where the input is a Markov additive process (MAP) and the release rate is Markov modulated as well. In the latter paper, the MAP is not the most general possible. In particular it did not include the additional jumps that can occur at state changes of the underlying Markov chain. This additional aspect, which we very much need here, can be included applying a technique from [4]. We will first write some general results regarding the most general setup, that is, the one dimensional version of [16] but with the possibility of additional jumps at state change epochs. We will then specialize to the case which we need to solve the problem of this paper. Thus, let $(X, J) = \{(X(t), J(t))|\ t \geq 0\}$ be a nondecreasing MAP (see [4]) with exponent matrix $F(\alpha) = Q \circ G(\alpha) + \mathrm{diag}\,(()\,\phi_1(\alpha)\ldots\phi_K(\alpha))$, where $Q \circ G(\alpha) \equiv (q_{ij}G_{ij}(\alpha))$, J is an irreducible finite state space continuous time Markov chain with states $1, \ldots, K$ and rate transition matrix $Q = (q_{ij})$ and stationary probability vector $\pi = \pi_i$ and $G_{ij}(\alpha)$ is the LST of the distribution of the (nonnegative) jump occuring when the Markov chain J changes states from i to j and $G_{ii}(\alpha) \equiv 1$ for all $\alpha \geq 0$ (LST of the constant 0). ϕ_i is the Laplace exponent of a nondecreasing Lévy process which is of the form

$$\phi_i(\alpha) = -c_i\alpha - \int_{(0,\infty)} (1 - e^{-\alpha x})\mathrm{d}\nu_i(x) \tag{2}$$

where ν_i is a Lévy measure satisfying $\int_{(0,\infty)} \min(x, 1)\mathrm{d}\nu(x) < \infty$. Moreover, we assume that $\mu(i, j) \equiv -G'_{ij}(0) < \infty$ and $\rho(i) \equiv -\phi'_i(0) = c_i + \int_{(0,\infty)} x\mathrm{d}\nu_i(x) < \infty$ for all i and j.

As in [4], we recall that the process X behaves like a nondecreasing Lévy process (subordinator) with exponent $\phi_i(\cdot)$ when J is in state i and when J switches from state i to a different state j, then in addition X jumps up by an independent amount which has a distribution having the LST $G_{ij}(\cdot)$.

Now consider the following Markov modulated linear dam process:

$$W(t) = W(0) + X(t) - \int_0^t r(J(s))W(s)\mathrm{d}s \tag{3}$$

where the input is the process X and the output rate is proportional to the content of the dam where the proportion $r(J(s))$, where $r(i) \geq 0$ for all i, is modulated by the Markov process J. Then we have the following:

Theorem 2.1. *If, in addition to the irreducibility of J and the assumption that $\rho_i < \infty$ and $\mu(i, j) < \infty$ $\forall i, j$ (see above), there is at least one i for which $r(i) > 0$, then a unique stationary distribution for the joint (Markov) process (W, J) exists; and it is also the limiting distribution, which is independent of initial conditions.*

Before we prove this result let us first show the following result concerning an alternating renewal process.

Lemma 2.1. *Let $\{(X_n, Y_n)|\ n \geq 1\}$ be independent pairs of nonnegative random variables which are identically distributed for $n \geq 2$, $P[Y_2 > 0] > 0$ and $EX_1, EX_2 <$*

∞. *Set* $S_0 = 0$, $S_n = \sum_{i=1}^n (X_i + Y_i)$ *for* $n \geq 1$ *and*

$$I(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{n=0}^{\infty} [S_n, S_n + X_{n+1}) \ , \\ 1 & \text{if } t \in \bigcup_{n=0}^{\infty} [S_n + X_{n+1}, S_{n+1}) \ , \end{cases} \tag{4}$$

$$Z(t) = \int_0^t I(s) \mathrm{d}s \ . \tag{5}$$

Then, for any positive constant r, $E \int_0^\infty e^{-rZ(t)} \mathrm{d}t < \infty$.

Proof. For simplicity, we prove this for the case where (X_1, Y_1) has the same distribution as for the rest of the sequence. The generalization to the case where the first pair has a different distribution is trivial. Since

$$\int_{[S_n, S_{n+1})} e^{-rZ(t)} \mathrm{d}t = e^{-rZ(S_n)} (X_{n+1} + r^{-1}(1 - e^{-rY_{n+1}})) \tag{6}$$

and $Z(S_n) = S_n^y$ where $S_0^y = 0$ and $S_n^y = \sum_{i=1}^n Y_i$ for $n \geq 1$, it follows that

$$\int_0^\infty e^{-rZ(t)} \mathrm{d}t = \sum_{n=0}^\infty e^{-rS_n^y} (X_{n+1} + r^{-1}(1 - e^{-rY_{n+1}})) = r^{-1} + \sum_{n=0}^\infty e^{-rS_n^y} X_{n+1} \ . \tag{7}$$

Thus, as $Ee^{-rY_1} < 1$ (since $P[Y_1 > 0] > 0$), it follows that

$$E \int_0^\infty e^{-rZ(t)} \mathrm{d}t = r^{-1} + \frac{EX_1}{1 - Ee^{-rY_1}} < \infty \ , \tag{8}$$

as required. ■

We note that in Lemma 2.1, the off-times ($Z(t) = 0$) must have a finite mean while, for $n \geq 2$, the on-times ($Z(t) = 1$) cannot be almost surely zero. These are the minimal assumptions in the sense that if one of them fails to hold then $E \int_0^\infty e^{-rZ(t)} \mathrm{d}t = \infty$. We note that it is possible that X_1, X_2 or Y_1 are a.s. zero.

Proof. (of Theorem 2.1) From (3) and Theorem 1 of [17] it follows that

$$W(t) = W(0) e^{-\int_0^t r(J(s)) \mathrm{d}s} + \int_{(0,t]} e^{-\int_u^t r(J(s)) \mathrm{d}s} \mathrm{d}X(u) \tag{9}$$

and thus if we start the system with two different initial conditions $W^1(0)$ and $W^2(0)$ then

$$W^1(t) - W^2(t) = (W^1(0) - W^2(0)) e^{-\int_0^t r(J(s)) \mathrm{d}s} \ . \tag{10}$$

Since there is at least one i for which $r(i) > 0$ and J is irreducible, it follows that a.s. $\int_0^\infty r(J(s))\mathrm{d}s = \infty$ so that the right side of (10) converges a.s. to zero as $t \to \infty$. Thus, if there is a limiting distribution for $(W(t), J(t))$, it does not depend on $W(0)$. It is standard that J can be coupled with its stationary version after an a.s. finite time. Since the value of W at this coupling time has no effect on the limiting distribution if it exists (for the same reasons as just explained for the initial conditions), we may assume without loss of generality that J is stationary. For this case the two dimensional process $\left\{\left(\int_0^t r(J(s))\mathrm{d}s, X_t\right) \mid t \geq 0\right\}$ has stationary increments in the strong sense that the distribution of $\left\{\left(\int_u^{t+u} r(J(s))\mathrm{d}s, X_{t+u} - X_u\right) \mid t \geq 0\right\}$ is independent of u. Thus, we can extend this process together with J to be a double sided process having these properties. From Theorem 2 of [17] it follows that to complete the proof it remains to show that a.s.

$$\int_{(-\infty,0]} e^{-\int_u^0 r(J(s))\mathrm{d}s}\mathrm{d}X(u) < \infty\ . \tag{11}$$

We will in fact show that

$$E\int_{(-\infty,0]} e^{-\int_u^0 r(J(s))\mathrm{d}s}\mathrm{d}X(u) < \infty\ . \tag{12}$$

Denoting $\bar{J}(t) = J(-t)$ and $\bar{X}(t) = -X(-t)$ for $t \geq 0$ we have that $\{(\bar{X}(t), \bar{J}(t))|\ t \geq 0\}$ is also a Markov additive process where $\bar{J}$ is stationary with transition rates $\bar{q}_{ij} = \pi_j q_{ji}/\pi_i$, $\bar{G}_{ij} = G_{ji}$ and $\bar{\phi}_i = \phi_i$. By the method of uniformization, let $\{\bar{N}(t)|\ t \geq 0\}$ be a Poisson process with some (finite) rate $\lambda \geq \max_i(-\bar{q}_{ii}) = \max_i(-q_{ii})$, in which arrival epochs we embed a (stationary) discrete time Markov chain $\{\bar{J}_n|\ n \geq 0\}$ with transition probabilities

$$\bar{p}_{ij} = \begin{cases} \frac{\bar{q}_{ij}}{\lambda}, & i \neq j\ , \\ \\ 1 + \frac{\bar{q}_{ii}}{\lambda}, & i = j\ . \end{cases} \tag{13}$$

Now, by conditioning on $\bar{J}$ we have that

$$E\int_{(-\infty,0]} e^{-\int_u^0 r(J(s))\mathrm{d}s}\mathrm{d}X(u) = E\int_{[0,\infty)} e^{-\int_0^u r(\bar{J}(s))\mathrm{d}s}\mathrm{d}\bar{X}(u)$$

$$= \quad E\int_{[0,\infty)} e^{-\int_0^u r(\bar{J}(s))\mathrm{d}s}\rho(J(t))\mathrm{d}t + E\int_{[0,\infty)} e^{-\int_0^t r(\bar{J}(s))\mathrm{d}s}\mathrm{d}\sum_{n=1}^{\bar{N}(t)} \mu(\bar{J}_{n-1}, \bar{J}_n),$$

where we recall that $\rho(i) = -\phi_i'(0) < \infty$ and $\mu(i,j) = -G_{ij}'(0) < \infty$. Denoting $\bar\rho = \max_i \rho(i)$ and $\bar\mu = \max_{ij} \mu(i,j)$ we have that the right hand side of (14) is bounded above by

$$\bar\rho E \int_{[0,\infty)} e^{-\int_0^t r(\bar J(s))\mathrm{d}s}\mathrm{d}t + \bar\mu E \int_{[0,\infty)} e^{-\int_0^t r(\bar J(s))\mathrm{d}s}\mathrm{d}\bar N(t)\ . \tag{14}$$

Since $\{\bar N(t) - \lambda t |\ t \ge 0\}$ is a zero mean right continuous Martingale and $e^{-\int_0^t (\bar J(s))\mathrm{d}s}$ is adapted, continuous and bounded, it follows that (14) is equal to

$$(\bar\rho + \lambda\bar\mu) E \int_{[0,\infty)} e^{-\int_0^t r(\bar J(s))\mathrm{d}s}\mathrm{d}t\ . \tag{15}$$

For any i such that $r(i) > 0$,

$$E \int_{[0,\infty)} e^{-\int_0^t r(\bar J(s))\mathrm{d}s}\mathrm{d}t \le E \int_{[0,\infty)} e^{-r(i)\int_0^t 1_{\{\bar J(s)=i\}}\mathrm{d}s}\mathrm{d}t \tag{16}$$

and the right side is finite by the irreducibility, hence positive recurrence, of $\bar J$ (due to that of J) and Lemma 2.1. ■

From [4], we have that the following is a zero mean martingale:

$$\begin{aligned} \int_0^t e^{-\alpha W(s)}\mathbf{1}_{J(s)}\mathrm{d}s F(\alpha) \quad &+ \quad e^{-\alpha W(0)}\mathbf{1}_{J(0)} - e^{-\alpha W(t)}\mathbf{1}_{J(t)} \\ &+ \quad \alpha \int_0^t e^{-\alpha W(s)}\mathbf{1}_{J(s)} r(J(s))W(s)\mathrm{d}s\ . \end{aligned} \tag{17}$$

Thus, if (W^*, J^*) has the stationary distribution associated with the process (W, J), then from (17) it follows that

$$Ee^{-\alpha W^*}\mathbf{1}_{J^*}F(\alpha) = \alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} Ee^{-\alpha W^*}\mathbf{1}_{J^*} r(J^*) \tag{18}$$

and thus, with $w_i(\alpha) = Ee^{-\alpha W^*}1_{\{J^*=i\}}$, $w(\alpha) = (w_i(\alpha))$ and with $D_r = \mathrm{diag}(r(1), \dots, r(K))$, we have that

$$w(\alpha)^T F(\alpha) = \alpha w'(\alpha)^T D_r \tag{19}$$

where $\pi_i = w_i(0)$ is the stationary distribution for the Markov chain J, summing to one and satisfying $\pi^T Q = 0$. We do not expect to be able to solve it analytically. Nevertheless, it can immediately be deduced from this equation by differentiation that

$$\sum_{k=0}^n \binom{n}{k} w^{(k)}(0)^T F^{(n-k)}(0) = n w^{(n)}(0)^T D_r \tag{20}$$

and since $F^{(0)}(0) = F(0) = Q$ we have that

$$\sum_{k=0}^{n-1} \binom{n}{k} w^{(k)}(0)^T F^{(n-k)}(0) = w^{(n)}(0)^T (nD_r - Q) \tag{21}$$

Thus, when $nD_r - Q$ is invertible, then we have a recursion formula that computes the moments $E(W^*)^n 1_{\{J^*=i\}}$.

Specializing to the problem we set out to solve, assume that instead of K states for the modulating Markov chain there are $K+1$ indexed by $0, \ldots, K$. We will first consider the modulated process with $r(0) = 0$, $r(1) = \ldots = r(K) = 1$, with $G_{i0}(\alpha) = G(\alpha)$ and for all other i, j, $G_{ij}(\alpha) = 1$. Finally, we assume that other than the jump that occurs when entering state 0 and the specified rates, nothing happens. That is, $\phi_0(\alpha) = \ldots = \phi_K(\alpha) = 0$. Thus we see that if we restrict the process to the intervals where the modulating Markov chain spends in the states $1, \ldots, K$ then we have a shot-noise process with phase-type inter-arrival times and general i.i.d. jumps.

It is easy to check that (19) becomes

$$\sum_{i=0}^{K} w_i(\alpha) q_{ij} = \alpha w_j'(\alpha) \tag{22}$$

for $j \neq 0$ and for $j = 0$ we have that

$$-q_0 w_0(\alpha) + G(\alpha) \sum_{i=1}^{K} w_i(\alpha) q_{i0} = 0 \tag{23}$$

where $q_0 = -q_{00}$ and by substitution we thus have that for $j \neq 0$

$$\sum_{i=1}^{K} w_i(\alpha) \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} G(\alpha) \right) = \alpha w_j'(\alpha) \tag{24}$$

with the initial conditions $w_i(0) = \pi_i$ where π is the stationary distribution for the modulating Markov chain. Thus the stationary LST for the shot-noise process with phase-type inter-arrival times and jumps with distribution having LST G is given by

$$w(\alpha) = \frac{\sum_{i=1}^{K} w_i(\alpha)}{1 - \pi_0}. \tag{25}$$

It is easy to check that

$$\sum_{j=1}^{K} \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} \right) = 0 \tag{26}$$

and that

$$\sum_{j=1}^{K} \pi_i \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} \right) = 0 \tag{27}$$

and thus (24) can be written as follows

$$\mu \frac{q_{0j}}{q_0} G_e(\alpha) \sum_{i=1}^{K} w_i(\alpha) q_{i0} = -\sum_{i=1}^{K} \frac{w_i(0) - w_i(\alpha)}{\alpha} \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} \right) - w_j'(\alpha) \tag{28}$$

where $G_e(\alpha) = \frac{1-G(\alpha)}{\alpha\mu}$ is the stationary residual lifetime LST associated with G. If we similarly denote

$$w_{e,i}(\alpha) = \frac{w_i(0) - w_i(\alpha)}{-w_i'(0)\alpha} = \frac{1 - E[e^{-\alpha W^*} | J^* = i]}{E[W^* | J^* = i]\alpha} \tag{29}$$

and we let $\mu_{n,i}^w = E[(W^*)^n | J^* = i]$ then

$$\mu \frac{q_{0j}}{q_0} G_e(\alpha) \sum_{i=1}^{K} w_i(\alpha) q_{i0} = -\sum_{i=1}^{K} \pi_i \mu_{1,i}^w w_{e,i}(\alpha) \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} \right) - w_j'(\alpha). \tag{30}$$

In particular, letting $\alpha \downarrow 0$,

$$\begin{aligned} \mu \pi_0 q_{0j} = \mu \frac{q_{0j}}{q_0} \sum_{i=1}^{K} \pi_i q_{i0} &= -\sum_{i=1}^{K} \pi_i \mu_{1,i}^w \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} \right) + \pi_j \mu_{1,j}^w \\ &= \sum_{i=1}^{K} \pi_i \mu_{1,i}^w \left(\delta_{ij} - \tilde{q}_{ij} \right), \end{aligned} \tag{31}$$

where $\tilde{q}_{ij} = q_{ij} + q_{i0} q_{0j} / q_0$. It follows from (26) and (27) that $\tilde{Q} = (\tilde{q}_{ij})_{1 \le i,j \le K}$ is a rate transition matrix with stationary distribution $\pi_i / (1 - \pi_0)$ for $i = 1, \ldots, K$.

The following is a straightforward exercise, but we include it for ease of reference.

Lemma 2.2. *If P is a stochastic matrix, D_1 and D_2 are nonnegative diagonal matrices and $D_1 + D_2$ has a strictly positive diagonal, then $D_1 - (D_2(P - I))$ is nonsingular.*

Proof. Note that $D_1 - (D_2(P - I)) = (D_1 + D_2)(I - (D_1 + D_2)^{-1} D_2 P)$ and thus it suffices to show that with $A = ((D_1 + D_2)^{-1} D_2 P)$, $A^n \to 0$ as $n \to \infty$, since then $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$. To show this, we note that $(D_1 + D_2)^{-1} D_2$ is a nonnegative diagonal matrix where the diagonal entries are all strictly less than one. Thus if we let d denote the maximum of these entries then $A^n \le d^n P^n$ and since the entries of P^n are bounded the result immediately follows. ■

Since a rate transition matrix of a finite state space continuous time Markov chain is of the form $D(P-I)$ for some nonnegative diagonal matrix D and some stochastic matrix P, it follows from Lemma 2.2 that $I-\tilde{Q}$ is nonsingular and thus (31) has a unique solution for the unknowns $\mu^w_{1,i}$. Denoting by μ_n the nth moment with respect to the jump distribution F (with LST G) then, since $\mu G_e^{(n-1)}(0) = (-1)^{n-1}\frac{\mu_n}{n}$ and similarly

$$\mu^w_{1,i} w^{(n-1)}_{e,i}(0) = (-1)^{n-1}\frac{\mu^w_{n,i}}{n} \tag{32}$$

then it is easy to check that after differentiating $n-1$ times and letting $\alpha \downarrow 0$, the following recursion holds:

$$\frac{q_{0j}}{q_0}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\mu_{k+1}}{k+1}\sum_{i=1}^{K}\pi_i\mu^w_{n-1-k,i}q_{i0} = \sum_{i=1}^{K}\pi_i\mu^w_{n,i}\left(\delta_{ij}-\frac{\tilde{q}_{ij}}{n}\right) \tag{33}$$

which upon multiplying by n, observing that $\binom{n-1}{k}\frac{n}{k+1} = \binom{n}{k+1}$ and making an obvious change of variables in the first sum gives

$$\frac{q_{0j}}{q_0}\sum_{k=1}^{n}\binom{n}{k}\mu_k\sum_{i=1}^{K}\pi_i\mu^w_{n-k,i}q_{i0} = \sum_{i=1}^{K}\pi_i\mu^w_{n,i}\left(n\delta_{ij}-\tilde{q}_{ij}\right) \ . \tag{34}$$

Finally, denoting by $\tilde{p}_0$ a vector with coordinates $p_{0j} = q_{0j}/q_0$, then with $\tilde{a}^T = \tilde{p}_0^T(I-\tilde{Q})^{-1}$ we have that

$$\pi_j\mu^w_{n,j} = \tilde{a}_j\sum_{k=1}^{n}\binom{n}{k}\mu_k\sum_{i=1}^{K}\pi_i\mu^w_{n-k,i}q_{i0}. \tag{35}$$

Thus, setting $m^w_n = \sum_{i=1}^{K}\pi_i\mu^w_{n,i}q_{i0}$ and

$$\tilde{a} = \sum_{i=1}^{K}\tilde{a}_i q_{i0} = \frac{1}{q_0}\sum_{i=1}^{K}\sum_{j=1}^{K}q_{0i}(I-\tilde{Q})^{-1}_{ij}q_{j0} \ , \tag{36}$$

we have that for $n \geq 1$

$$m^w_n = \tilde{a}\sum_{k=1}^{n}\binom{n}{k}\mu_k m^w_{n-k} \tag{37}$$

so that

$$\mu^w_{n,j} = \frac{\tilde{a}_j}{\pi_j\tilde{a}}m^w_n \tag{38}$$

and the unconditional moment is

$$\frac{1}{1-\pi_0}\sum_{j=1}^{K}\pi_j\mu^w_{n,j} = \frac{m^w_n}{(1-\pi_0)\tilde{a}}\sum_{j=1}^{K}\tilde{a}_j. \tag{39}$$

From (37) it follows that

$$m_n^w = \frac{\tilde{a}}{1+\tilde{a}} \sum_{k=0}^{n} \binom{n}{k} \mu_k m_{n-k}^w + \frac{1}{1+\tilde{a}} \delta_{0n} \tag{40}$$

and upon multiplying by $(-\alpha)^n$, dividing by $n!$ and summing and noting that if G is uniquely defined by its moments then $G(\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k}{k!} \alpha^k$, one obtains that

$$m^w(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n m_n^w}{n!} \alpha^n = \frac{\frac{1}{1+\tilde{a}}}{1 - \frac{\tilde{a}}{1+\tilde{a}} G(\alpha)} = \sum_{k=0}^{\infty} \frac{1}{1+\tilde{a}} \left(\frac{\tilde{a}}{1+\tilde{a}} \right)^k G^k(\alpha) \ . \tag{41}$$

This implies that if we let

$$H(x) = \sum_{k=0}^{\infty} \frac{1}{1+\tilde{a}} \left(\frac{\tilde{a}}{1+\tilde{a}} \right)^k F^{*k}(x) \ , \tag{42}$$

then m_n^w is the nth moment with respect to H. If S_n is a sum of n i.i.d. random variables distributed F ($S_0 = 0$) and N is an independent geometric random variable with probability of success $(1+\tilde{a})^{-1}$, counting only the number of failures, then H is the distribution of S_N. Although this seems nice, we should point out that the analysis is not complete as we have not shown that these moments define a unique distribution.

2.1. The case $K = 2$

With $a_{ij} = \frac{q_{i0} q_{0j}}{q_0}$, (24) reduces to

$$\begin{aligned} \alpha w_1'(\alpha) &= w_1(\alpha)(q_{11} + a_{11} G(\alpha)) + w_2(\alpha)(q_{21} + a_{21} G(\alpha)) \ , & (43) \\ \alpha w_2'(\alpha) &= w_1(\alpha)(q_{12} + a_{12} G(\alpha)) + w_2(\alpha)(q_{22} + a_{22} G(\alpha)) \ . & (44) \end{aligned}$$

Differentiating the second of these equations, using (43) for $w_1'(\alpha)$ and finally (44) to eliminate $w_1(\alpha)$, we obtain the following second order differential equation in $w_2(\alpha)$:

$$\begin{aligned} \alpha w_2''(\alpha) &= w_2'(\alpha) \left[-1 + q_{22} + a_{22} G(\alpha) + q_{11} + a_{11} G(\alpha) + \frac{\alpha a_{12} G'(\alpha)}{q_{12} + a_{12} G(\alpha)} \right] \\ &\quad + w_2(\alpha) \left[\frac{1}{\alpha} (q_{21} + a_{21} G(\alpha))(q_{12} + a_{12} G(\alpha)) + a_{22} G'(\alpha) \right. \qquad (45) \\ &\quad \left. - \frac{1}{\alpha} (q_{22} + a_{22} G(\alpha))(q_{11} + a_{11} G(\alpha)) + \frac{q_{22} + a_{22} G(\alpha)}{q_{12} + a_{12} G(\alpha)} a_{12} G'(\alpha) \right] . \end{aligned}$$

Now consider the special case of $\exp(\mu)$ jumps, i.e., $G(\alpha) = \frac{\mu}{\mu+\alpha}$, and the following Markov transition rates: $q_{12} = -q_{11} = \nu_1$, $q_{20} = -q_{22} = \nu_2$, and $q_{01} = -q_{00} = \nu_0$, i.e., the times in state i are $\exp(\nu_i)$ distributed, for $i = 0, 1, 2$. Then (45) reduces to

$$\alpha(\mu+\alpha)w_2''(\alpha) + (\nu_1+\nu_2+1)(\mu+\alpha)w_2'(\alpha) + \nu_1\nu_2 w_2(\alpha) = 0. \tag{46}$$

Taking $z = \mu + \alpha$, $a = \nu_1$, $b = \nu_2$ and $c = 0$ in the differential equation (15.5.1) on p. 562 of [1] reduces that differential equation to (46). Its solution is given by the hypergeometric functions 15.5.20 and 15.5.21 on p. 564 of [1].

3. Back to the growth collapse model

In this section we relate the nth moment of the stationary distribution of the growth collapse model to the $n+1$st moment of the stationary distribution of the shot noise model. Consider the growth collapse model with collapse ratio distribution being minus-log-phase-type. Let P denote a generic interarrival time of the corresponding shot-noise process; P is phase-type (cf. Section 1). An expression for EP is given via the system of equations for the t_i, which are mean interarrival times when the first phase is i:

$$t_i = \frac{1}{q_i} + \sum_{j\neq 0,i} \frac{q_{ij}}{q_i} t_j \tag{47}$$

or equivalently

$$\sum_{j\neq 0} q_{ij} t_j = -1 \tag{48}$$

which has a unique solution. EP is then a weighted average of $t_1, \ldots, t_K$ where the weights are the initial distribution of the phase-type distribution which is in our case chosen to be q_{0i}/q_0.

From [12] we recall that, for the on/off model of Section 1, the relationship between the stationary density during on times ($f_0(\cdot)$) and that during off times ($f_1(\cdot)$) is given by $pf_0(x) = (1-p)xf_1(x)$, where $p = \mu/(\mu + EP)$. Hence

$$\int_0^\infty e^{-\alpha x} f_0(x)\mathrm{d}x = \frac{EP}{\mu}\int_0^\infty e^{-\alpha x} x f_1(x)\mathrm{d}x = -\frac{EP}{\mu}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^\infty e^{-\alpha x} f_1(x)\mathrm{d}x \tag{49}$$

so that the nth moment of the stationary distribution for the growth collapse model is given by EP/μ times the $n+1$st moment (see (39)) for the shot noise model with phase type inter-arrival times.

4. The discrete time process embedded after collapse epochs

In this section we study the steady-state behavior of the growth collapse process right after the nth collapse, V_n, defined by

$$V_n = (V_{n-1} + Y_n)X_n, \tag{50}$$

where V_0 is the initial state.

As observed, for example in (2) of [14],

$$V_n = V_0 \prod_{j=1}^{n} X_j + \sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j. \tag{51}$$

We assume in the sequel that $\{X_i|i \geq 1\}$ and $\{Y_i|i \geq 1\}$ are independent sequences of i.i.d. random variables distributed like X and Y, where X and Y are independent and nonnegative. We shall initially assume that X has support $[0, 1]$. From [14] it follows that when $EX < 1$ and $EY < \infty$, a limiting distribution for the process $\{V_n|\ n \geq 0\}$ exists, which is independent of the initial condition V_0; and it has a unique stationary version. It is easy to check that this continues to hold when X is nonnegative but is not necessarily restricted to $[0, 1]$, as when $EX_1 < 1$, $\prod_{j=1}^{n} X_j \to 0$ a.s. as $n \to \infty$ and $\sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j$ is stochastically increasing (as it is distributed like $\sum_{i=1}^{n} Y_i \prod_{j=1}^{i} X_j$) and its mean is bounded above by $\frac{EYEX}{1-EX} < \infty$. Thus, throughout this section it is assumed that $EX < 1$ and $EY < \infty$.

The fact that when starting from $V_0 = 0$, V_n is stochastically increasing can also be used to justify the fact that the limiting distribution of V_n has a finite mth moment if and only if $EY^m < \infty$ and $EX^m < 1$, as in this case

$$(1 - EX^m)EV_n^m \leq EV_n^m - EX^m EV_{n-1}^m = EX^m \sum_{k=0}^{m-1} \binom{m}{k} EV_{n-1}^k EY^{m-k}, \tag{52}$$

where by induction, $EV_n^k < \infty$ and converges to the kth moment of the limiting distribution by monotone convergence (finite or infinite). Thus if the first $m - 1$ moments of the limiting distribution of V_n are finite, $EY^m < \infty$ and $EX^m < 1$ then the mth moment is finite as well. If either $EY^m = \infty$ or $EX^m \geq 1$ then (52) also implies that the mth moment of the limiting distribution of V_n is necessarily infinite.

Let V denote a random variable having this distribution, such that X, Y and V are independent. Then

$$V \stackrel{d}{=} (V + Y)X. \tag{53}$$

We might also focus on $Z := V + Y$, which has the limiting distribution of the state of the system right before collapses. This leads to

$$Z \stackrel{d}{=} ZX + Y. \tag{54}$$

Much of the literature on Equation (50) has concentrated on the existence of a limiting distribution, and on the tail behavior of that limiting distribution. In the present section we know that this limiting distribution exists provided that $EX < 1$. Our goal is to determine it, for a number of choices for the distributions of X and Y. We start with the following formula for the LST $\psi(\alpha)$ of V, with $\eta(\alpha)$ denoting the LST of Y,

$$\psi(\alpha) = E\psi(\alpha X)\eta(\alpha X) = \int_{[0,1]} \psi(\alpha x)\eta(\alpha x)\mathrm{d}F_X(x), \tag{55}$$

where $F_X(x) = P[X \le x]$. We shall also study EV^n in a number of cases, comparing it with the expression

$$EV^n = \frac{EX^n}{1 - EX^n} \sum_{j=0}^{n-1} \binom{n}{j} EV^j EY^{n-j}, \tag{56}$$

as in (34) of [14]. In the sequel we assume that all required moments of Y are finite, with the exception of an example of regular variation at the end of the section, and note that since X has support $[0,1]$ and $EX < 1$ then also $EX^n < 1$ for all $n \ge 1$.

We start with a case that has already been treated by Vervaat [21]. We review it here, as it is the basis for extensions later in this section.

4.1. Case 1: $X \sim \mathrm{Beta}(D, 1)$

In this case, X has distribution $F(x) = P(X \le x) = x^D$, $0 \le x \le 1$. When D is a positive integer, X is distributed like the maximum of D independent random variables distributed $U[0,1]$. From (55) we have

$$\psi(\alpha) = \int_0^\alpha \psi(u)\eta(u) D \left(\frac{u}{\alpha}\right)^{D-1} \frac{\mathrm{d}u}{\alpha}, \tag{57}$$

or

$$\alpha^D \psi(\alpha) = D \int_0^\alpha \psi(u)\eta(u)u^{D-1}\mathrm{d}u. \tag{58}$$

Differentiation yields, after some rearrangement, that

$$\psi'(\alpha) = -D\psi(\alpha)\frac{1 - \eta(\alpha)}{\alpha}, \tag{59}$$

so, since $\psi(0) = 1$,

$$\psi(\alpha) = e^{-D\int_{v=0}^{\alpha}\frac{1-\eta(v)}{v}\mathrm{d}v} = e^{-DEY\int_0^{\alpha}\eta_e(v)\mathrm{d}v} \tag{60}$$

where $\eta_e(v) = \frac{1-\eta(v)}{vEY}$ is the LST of the stationary residual lifetime distribution of Y.

Remark 4.1. For $D = 1$, $\psi(\alpha)$ is the LST of the classical shot-noise process, see [13]. For integer $D > 1$, V apparently is the sum of D independent shot-noise processes each having $D = 1$. This is not a coincidence. It follows from the fact that if

$$W_i(t) = X_i(t) - r\int_0^t W_i(s)\mathrm{d}s \tag{61}$$

for $i = 1, \ldots, D$, then

$$\sum_{i=1}^{D} W_i(t) = \sum_{i=1}^{D} X_i(t) - r\int_0^t \sum_{i=1}^{D} W_i(s)\mathrm{d}s \tag{62}$$

so that if $X_i(\cdot)$ are independent processes then also $W_i(\cdot)$ are independent shot-noise processes and $\sum_{i=1}^{D} W_i(\cdot)$ is itself a shot-noise process with driving process $\sum_{i=1}^{D} X_i(\cdot)$. In this particular case one may observe from the relationship discussed earlier between the shot-noise and growth collapse processes, that a uniformly distributed jump ratio for the growth collapse process corresponds to exponentially distributed inter-jump times for the shot noise process. Thus in this case for $D = 1$, $X_i(\cdot)$ are independent compound Poisson processes with arrival rate $\lambda = 1$ and jumps distributed like Y, so that $\sum_{i=1}^{D} X_i(\cdot)$ is also a compound Poisson process with arrival rate D and jumps distributed like Y.

4.1.1. *Moments* It follows from (59) after $k-1$ differentiations and denoting by Y_e a random variable with the stationary residual lifetime distribution associated with Y, that

$$EV^n = DEY\sum_{k=0}^{n-1}\binom{n-1}{k}EV^kEY_e^{n-1-k}, \tag{63}$$

from which recursion all moments of V can be obtained (starting with $EV = DEY$). The equivalence with (56) follows by observing that $EX^n = \frac{D}{D+n}$ and that $EY_e^n = \frac{EY^{n+1}}{(n+1)EY}$.

4.2. Case 2: $X \sim$ Beta(ζ_1, ζ_2) and $Y \sim$ Gamma(ζ_2, β)

In this case there is the following short-cut. It is well known (usually given as a standard exercise in a first year probability course when discussing multi-dimensional transformations and Jacobians) that if V is Gamma(ζ_1, β) distributed and Y is Gamma(ζ_2, β) distributed, V and Y are independent, then $\frac{V}{V+Y}$ is distributed Beta(ζ_1, ζ_2) and is independent of $V + Y$, which is distributed Gamma$(\zeta_1 + \zeta_2, \beta)$. Thus, the joint distribution of $(X, V + Y)$ is the same as that of $\left(\frac{V}{V+Y}, V + Y\right)$ which implies that $X(V + Y)$ is distributed like V, so that (53) is satisfied. As there is a unique limiting distribution for the recursion (50), it must be Gamma(ζ_1, β).

Remark 4.2. It is easily verified that, in the case of Y being exponentially distributed, i.e., Gamma$(1, \beta)$, and X being Beta$(D, 1)$ distributed as in Case 1, Formula (60) yields that $\psi(\alpha) = (\frac{1}{1+\beta\alpha})^D$; so indeed V has a Gamma(D, β) distribution. This particular case is mentioned on p. 765 of Vervaat [21].

4.2.1. *Moments* Since V has a Gamma(ζ_1, β) distribution, it immediately follows that

$$EV^n = \beta^n \frac{\Gamma(\zeta_1 + n)}{\Gamma(\zeta_1)}. \tag{64}$$

4.3. Case 3: X has an atom at zero

Suppose that $P(X = 0) = p > 0$, and that X assumes with probability $1 - p$ values on $(0, \infty)$ (so we do not necessarily restrict X to $[0, 1]$). It is easy to see that the limiting distribution of $\{V_n |\ n \geq 0\}$ always exists as zero is a regenerative state with geometrically distributed (hence, aperiodic finite mean) regeneration epochs. We shall study several subcases.

4.4. Case 3a: X has atoms at 0 and c

Assume that $P(X = 0) = 1 - P(X = c) = p$, with $p > 0$ and $c > 0$ (allowing also $c > 1$). From (50),

$$\psi(\alpha) = p + (1 - p)\psi(c\alpha)\eta(c\alpha), \tag{65}$$

of which repeated iterations yield

$$\psi(\alpha) = \sum_{j=0}^{\infty} p(1 - p)^j \prod_{i=1}^{j} \eta(c^i\alpha). \tag{66}$$

The sum obviously converges for all $0 < p \le 1$. Inversion of the LST reveals that

$$V \stackrel{d}{=} \sum_{i=1}^{N} c^i Y_i, \tag{67}$$

where N is geometrically distributed (counting only failures) with probability of success p and is independent of $\{Y_i |\ i \ge 1\}$. Indeed, this also follows directly by applying (50), in the form $V_n = c(V_{n-1} + Y_n)$, N times, with $V_0 = 0$. See also p. 762 of Vervaat [21] (where $c = 1$).

4.4.1. *Moments* From (67),

$$EV^n = \sum_{j=0}^{\infty} p(1-p)^j \sum_{\sum_{i=1}^{j} n_i = n} \frac{n!}{\prod_{i=1}^{j} n_i!} c^{\sum_{i=1}^{j} i n_i} \prod_{i=1}^{j} EY^{n_i} . \tag{68}$$

4.5. Case 3b: $X \sim$ mixture of an atom at 0 and Beta$(D, 1)$

Assume that $P(X \le x) = p + (1-p)x^D$, $0 \le x \le 1$, $p > 0$. In this case, (55) reduces to

$$\psi(\alpha) = p + (1-p) \int_0^1 \psi(\alpha x)\eta(\alpha x)\mathrm{d}x, \tag{69}$$

yielding after similar manipulations as in Case 1:

$$\psi'(\alpha) = \frac{pD}{\alpha} + \psi(\alpha)\frac{(1-p)D\eta(\alpha) - D}{\alpha} . \tag{70}$$

The solution of this first-order inhomogeneous differential equation is:

$$\begin{aligned}\psi(\alpha) &= C \exp\left[D \int_0^{\alpha} \frac{(1-p)D\eta(v) - D}{v}\mathrm{d}v\right] \\ &+ \int_0^{\alpha} \frac{pD}{z} \exp\left[\int_z^{\alpha} \frac{(1-p)D\eta(v) - D}{v}\mathrm{d}v\right] \mathrm{d}z.\end{aligned} \tag{71}$$

It is easily seen that the first integral on the right hand side of (71) diverges, so we have to take $C = 0$. By observing that $\frac{(1-p)D\eta(v)-D}{v}$ is bounded between $-D/v$ and $-pD/v$, and that hence the expression in the last line of (71) is bounded between $\int_0^\alpha \frac{pD}{z}(z/\alpha)^{pD}\mathrm{d}z$ and $\int_0^\alpha \frac{pD}{z}(z/\alpha)^{D}\mathrm{d}z$, it follows that the expression in the last line of (71) has a value between p and 1. When $\alpha \downarrow 0$, then the above bound $-pD/v$ becomes tight and the expression in the last line of (71) approaches 1.

4.5.1. *Moments* The most suitable approach to obtain EV^n via the LST here seems to be to multiply both sides of Formula (70) with α and differentiate $k-1$ times. However, the resulting calculation is not really easier then when starting from (56), and hence we omit it.

4.5.2. *Tail behavior* Suppose that the distribution of Y is regularly varying at infinity with index $-\nu$. Then application of Lemma 8.1.6 of [5] to (71) readily shows that V is also regularly varying, with the same index. We don't provide details, because considerably more general tail results can be obtained for (50); see Volkovitch and Litvak [20] for regularly varying Y, and Denisov and Zwart [8] for light-tailed Y.

4.6. Case 3c: X ~ mixture of an atom at 0 and a product of two i.i.d. $U[0,1]$

The density of the product of two i.i.d. random variables which are uniformly distributed on $[0,1]$ is $-\ln x$, $0<x<1$. Formula (55) now reduces to:

$$\psi(\alpha) = p - (1-p)\int_0^1 \psi(\alpha x)\eta(\alpha x)\ln x \mathrm{d}x, \tag{72}$$

so

$$\alpha\psi(\alpha) = p\alpha - (1-p)\int_0^\alpha \psi(u)\eta(u)\ln(u/\alpha)\mathrm{d}u, \tag{73}$$

which after two differentiations leads to

$$\alpha^2\psi''(\alpha) + 3\alpha\psi'(\alpha) + (1-(1-p)\eta(\alpha))\psi(\alpha) = p. \tag{74}$$

In the special case that $Y \sim \exp(\mu)$, hence $\eta(\alpha) = \frac{\mu}{\mu+\alpha}$, this equation simplifies into

$$\alpha^2(\mu+\alpha)\psi''(\alpha) + 3\alpha(\mu+\alpha)\psi'(\alpha) + (\mu+\alpha-(1-p)\mu)\psi(\alpha) = p. \tag{75}$$

$p=0$ gives a known case:

$$\alpha(\mu+\alpha)\psi''(\alpha) + 3(\mu+\alpha)\psi'(\alpha) + \psi(\alpha) = 0. \tag{76}$$

Note that this is the differential equation (46) for the case of $\nu_1 = \nu_2 = 1$, which makes sense: Y being exponential and X being a product of two independent $U[0,1]$ random variables corresponds to having an exponential on-time distribution and an Erlang-2 off-time distribution in the on-off model of Section 1 (that was directly related to the growth collapse model and the shot noise model). Slightly more generally, if $X = U_1^{1/\nu_1} U_2^{1/\nu_2}$, with U_1 and U_2 independent and $U[0,1]$ distributed, then one gets (46) with ν_1 and ν_2.

Remark 4.3. We note that the density of the product of $k \geq 2$ i.i.d. random variables which are uniformly distributed on $[0,1]$ is $\frac{(-\ln(x))^{k-1}}{(k-1)!}$, $0 < x < 1$, thus in a similar manner one may derive a kth order differential equation for $\psi(\alpha)$.

Remark 4.4. When $p = 0$ and $\eta(\alpha) = (\frac{\mu}{\mu+\alpha})^2$, i.e., Y is Erlang-2, then (74) becomes:

$$\psi''(\alpha) + \frac{3}{\alpha}\psi'(\alpha) + \frac{\alpha + 2\mu}{\alpha(\mu+\alpha)^2}\psi(\alpha) = 0. \tag{77}$$

When $p = 0$ and $\eta(\alpha) = b\frac{\mu_1}{\mu_1+\alpha} + (1-b)\frac{\mu_2}{\mu_2+\alpha}$ with $0 < b < 1$, i.e., Y is hyperexponentially distributed, then (74) becomes:

$$\psi''(\alpha) + \frac{3}{\alpha}\psi'(\alpha) + \frac{b(\mu_2+\alpha) + (1-b)(\mu_1+\alpha)}{\alpha(\mu_1+\alpha)(\mu_2+\alpha)}\psi(\alpha) = 0. \tag{78}$$

Both (77) and (78) are special cases of Heun's differential equation, cf. [10].

4.7. Case 4: $X \sim U[0,a]$

We are interested in studying a case in which X is not restricted to $[0,1]$. We assume that X is $U[0,a]$ distributed, with $EX = a/2 < 1$. As noted in the first paragraph of this section, together with $EY < \infty$ this implies stability. Formula (55) now becomes

$$\psi(\alpha) = \frac{1}{a}\int_0^a \psi(\alpha x)\eta(\alpha x)\mathrm{d}x = \frac{1}{a\alpha}\int_0^{a\alpha} \psi(u)\eta(u)\mathrm{d}u \ , \tag{79}$$

and differentiation gives (see (59))

$$\psi'(\alpha) = \psi(a\alpha)\frac{\eta(a\alpha)}{\alpha} - \frac{\psi(\alpha)}{\alpha}. \tag{80}$$

By introducing $\zeta(\alpha) \equiv \psi(e^\alpha)$ one can rewrite (80) as the differential-difference equation

$$\zeta'(\alpha) = \zeta(\alpha + c)\xi(\alpha + c) - \zeta(\alpha), \tag{81}$$

with $c = \ln a < 0$ and $\xi(\alpha) := \eta(e^\alpha)$. There is an extensive literature on differential-difference equations, see for example [11]. However, solutions of such equations are only known in special cases such as when $\xi(\alpha)$ is a constant. Below we consider (80) in the special case that $Y \sim \exp(\mu)$. Equation (80) then reduces to

$$(\mu + a\alpha)\alpha\psi'(\alpha) = \mu\psi(a\alpha) - (\mu + a\alpha)\psi(\alpha). \tag{82}$$

One might solve it by introducing the Taylor series expansion $\psi(\alpha) \equiv \sum_{n=0}^{\infty} f_n \alpha^n$, with $f_0 = \psi(0) = 1$, and solving the resulting recursion for f_n (which is $\frac{(-1)^n EV^n}{n!}$).

We prefer an alternative approach, starting from (56):

$$\begin{aligned} EV^n &= \frac{EX^n}{1 - EX^n} \sum_{k=0}^{n-1} \binom{n}{k} EV^k EY^{n-k} \\ &= n! \frac{a^n}{n + 1 - a^n} \sum_{k=0}^{n-1} \frac{EV^k}{k!} \frac{1}{\mu^{n-k}}. \end{aligned}$$

If we denote $B_n = \sum_{k=0}^{n} \frac{\mu^k EV^k}{k!}$ then $B_n - B_{n-1} = \frac{a^n}{n+1-a^n} B_{n-1}$ so that $B_n = \frac{n+1}{n+1-a^n} B_{n-1}$ which implies that $B_n = \frac{(n+1)!}{\prod_{i=1}^{n}(i+1-a^i)}$. Hence

$$\frac{\mu^n EV^n}{n!} = B_n - B_{n-1} = \frac{a^n n!}{\prod_{i=1}^{n}(i + 1 - a^i)}$$

and thus when $a < (1 + n)^{1/n}$:

$$EV^n = \frac{(a/\mu)^n (n!)^2}{\prod_{i=1}^{n}(i + 1 - a^i)} .$$

Remark 4.5. $a = 1$ yields $EV^n = \frac{n!}{(\mu)^n}$, corresponding to V being exponentially distributed; see already Remark 2. We further recall $EV^n < \infty$ if and only if $EX^n = \frac{a^n}{n+1} < 1$ (see (52) and discussion there). Note that $(1 + n)^{1/n}$ equals 2 for $n = 1$ and decreases to 1 as $n \to \infty$, so that for $0 < a \le 1$ all the moments exist but not for $1 < a < 2$.

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