**Implied Liquidity
Towards stochastic liquidity
modeling and liquidity trading**

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August 3, 2010

Abstract

In this paper we introduce the concept of implied (il)liquidity of vanilla options. Implied liquidity is based on the fundamental theory of conic finance, in which the one-price model is abandoned and replaced by a two-price model giving bid and ask prices for traded assets. The pricing is done by making use of non-linear distorted expectations. We first recall a two parameter distortion function representing the notions of risk aversion and absence of gain enticement. After reviewing under the Black-Scholes setting the theory and numerics of the calculation of bid-ask prices under conic finance theory, we introduce the concept of implied liquidity. In a fixed market with no movement in the cone of acceptable risks and hence no change in liquidity as the market is then fixed, the bid ask spread moves around nonlinearly with maturity and/or volatility. Because the spread can move in a constant market with no change in liquidity, spread itself is not a perfect measure of liquidity. Implied liquidity can overcome this criticism. We illustrate the theory on SP500 and Dow Jones Index data. We show that for vanilla options we typically have for higher strikes (OTM) more implied illiquidity. We typically see not much term structure. Also, we perform a historical study, in which we clearly see a serious drying up of liquidity in the weeks post the Lehman bankruptcy. Next, we elaborate on stochastic liquidity behavior and potential liquidity contracts and modeling. Seen the evidence of changing liquidity in the recent past with a potentially very disruptive drying up of liquidity, these contracts could provide extra hedges for such circumstances. The above notion of implied liquidity leads toward a mean-reverting modeling of liquidity similar to stochastic volatility. We believe such stochastic liquidity modeling could be very useful in structured product pricing, delta-gamma-vega hedging studies and risk-management in general.

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1 Introduction

In business, economics or investment, market liquidity is an important quantity. It reflects the asset's ability to be sold without causing a significant movement in the price and with minimum loss of value. Liquidity goes hand in hand with bid and ask spreads; high liquid products have a small spread; illiquid assets have a high spread. The essential characteristic of a liquid market is that there are ready and willing buyers and sellers at all times. Some products are more liquid than other investments. Dow Jones index components are obviously much more liquid than real estate.

In transactions, investors sometimes apply liquidity discount and take in that way into account a reduced promised yield or expected return for such assets. Buyers know that other investors are not willing to buy back that easily an illiquid product. Hence to be prudent it has been argued that if one buys an asset at its ask price, it should be booked on the basis of its bid prices [3] (and hence a transaction immediately accounts for a loss, this bid ask spread).

Portfolio managers that oversee huge investment portfolios are subject to systematic and structural liquidity risk. In times of crisis, liquidity dries up and one can not easily unwind positions near theoretical prices. Fire-sale transactions are typically at much lower prices, due to huge bid-ask spread at such moments.

In this paper, we initiate stochastic liquidity models and introduce related liquidity derivatives providing hedges against liquidity changes. First, we will, by making use of conic finance theory, introduce the concept of implied risk-aversion and implied gain-enticement; these quantities immediately lead to bid-ask prices (and corresponding spreads) and hence reflect the liquidity situation of a certain asset at a certain point in time. We elaborate on the calculation of these numbers in the Black-Scholes world. Besides the implied volatility, which reflects the mid price of a vanilla, we now have hence at hand one more implied parameter giving us a fundamental understanding of the implied liquidity situation. The implied liquidity parameter is a unitless quantity that, as the implied volatility, makes comparison of liquidity over different assets and markets straightforward. We calculate these implied liquidity parameters over different strikes and maturities and hence come to implied liquidity parameter surfaces. We typically see an upsloping skew over strike; there is hardly any term structure. Next, we study the behavior of these quantities over time. More precisely we calculate these for the ATM levels over time. We clearly see the effect during the credit crisis; implied liquidity parameter spiked up during the weeks after Lehman collapsed, indicating a clear drying up of liquidity in major vanilla markets.

Finally, we are tempted to introduce some liquidity derivative contracts that could serve as potential hedge instruments against the drying up of liquidity. We come to the concept of realized liquidity and propose risk liquidity based swaps and options as new liquidity derivatives.

2 Conic Finance Bid and Ask Pricing

In this section, we summarize the basic conic finance techniques used. For more background see [1], [2] and [3]. We will discuss non-linear distorted expectation, acceptability and bid-ask pricing.

In this paper, we will make use of a distortion function from the minmaxvar family parameterized as given in Equation 1 by two parameters lambda λ and gamma γ.

$$\Phi(u; \lambda, \gamma) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\gamma} \tag{1}$$

Lambda determines the rate of loss aversion of the investor; gamma determines the absence of gain enticement. Securities are traded in their own markets and we model different markets using different levels of lambda and gamma to reflect the different preferences of investors in these markets. We actually assume that each asset has its one market and hence its very specific loss aversion and absence of gain enticement.

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We use a non-linear expectation to calculate (bid and ask) prices. The prices arise form the theory of acceptability. We say that a risk X is acceptable ($X \in \mathcal{A}$) if

$$E_Q[X] \geq 0 \text{ for all measures } Q \text{ in a convex set } \mathcal{M}.$$

The convex set is called a cone of measures; operational cones were defined by Cherney and Madan [1] and depend solely on the distribution function $G(x)$ of X and a distortion function Φ. $X \in \mathcal{A}$ if the distorted expectation is non-negative. More precisely, the distorted expectation of a random variable X with distribution function $G(x)$ relative to the distortion function Φ (we use the one given in Equation (1), but other distortion functions are also possible), is defined as

$$de(X; \lambda, \gamma) = \mathrm{E}^{\lambda, \gamma}[X] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(G(x); \lambda, \gamma). \tag{2}$$

Note that if $\lambda = \gamma = 0$, $\Phi(u; 0, 0) = u$ and hence $de(X; 0, 0) = \mathrm{E}[X]$ is the ordinary linear expectation.

The ask price of payoff X is determined as

$$ask(X) = -\exp(-rT)\mathrm{E}^{\lambda, \gamma}[-X].$$

This formula is derived by noting that the cash-flow of selling X at its ask price is acceptable in the relevant market: $ask(X) - X \in \mathcal{A}$

Similarly, the bid price of payoff X is determined as

$$bid(X) = \exp(-rT)\mathrm{E}^{\lambda, \gamma}[X].$$

Here the cash-flow of buying X at its bid price is acceptable in the relevant market : $X - bid(X) \in \mathcal{A}$.

One can prove that the bid and ask prices of a positive contingent claim X with distribution function $G(x)$ can be calculated as:

$$bid(X) = \exp(-rT) \int_0^{+\infty} x \mathrm{d}\Phi(G(x); \lambda, \gamma), \tag{3}$$

$$ask(X) = \exp(-rT) \int_0^{+\infty} x \mathrm{d}\Phi(1 - G(-x); \lambda, \gamma). \tag{4}$$

Suppose like in an usual market situation we have a bid and ask price for a European call. We then can calculate the mid price of that call option, as the average of the bid and ask prices. Out of this mid price we calculate the implied Black-Scholes volatility. Next, we can calculate an implied (λ, γ), pair such that conic bid and ask prices (using the implied vol as parameter) are perfectly matched with market prices.

Under the Black-Scholes framework, this comes down to the following calculations for a European call option with strike K and maturity T.

Because $P((S_T - K)^+ \geq x) = P(S_T \geq K + x)$ for $x \geq 0$, the distribution function value in point x of the call payoff, is nothing else than one minus the probability of finishing above $K + x$. Using the usual interpretation of the so-called $N(d_2)$-term of the Black-Scholes formula, we have

$$G(x) = 1 - \mathrm{N}\left(\frac{\log(S_0/(K+x)) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \qquad x \geq 0$$

where N is the cumulative distribution function of the standard normal law, σ is the implied vol determined on the basis of the mid price. For $x < 0$, $G(x) = 0$, since the payoff is a positive random variable. The above close-form solution for $G(x)$ in combination with Equation 3 and 4 give rise to very fast and accurate calculations of the bid and ask pricing.
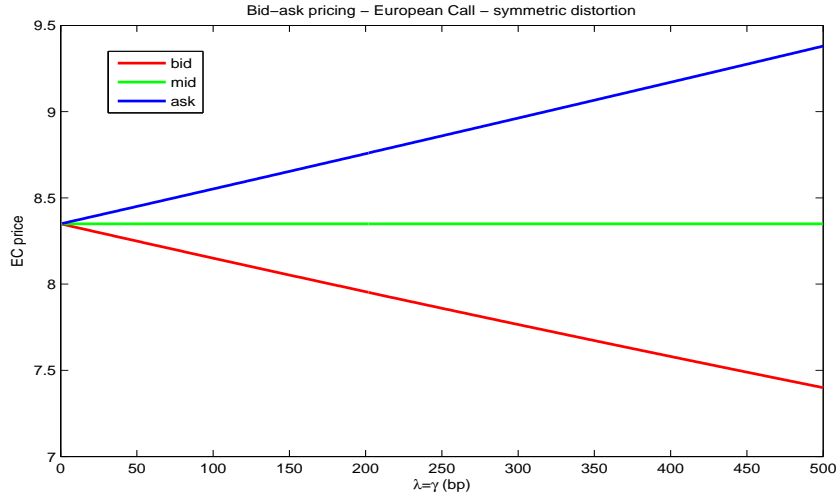
Figure 1: Bid-Ask Pricing - European Call - $\lambda = \gamma$

In Figure 1, one sees the bid, mid and ask prices for a range of distortions for which $\lambda = \gamma$. Values are graphed for a 1year ATM call option under a 20 % volatility.

In Figure 2, one sees the bid and ask prices around a market quote for the situation where $\lambda \neq \gamma$ (actually $\gamma = 5\gamma$). Values are graphed again for a 1year ATM call option under a 20 % volatility. Here one sees clearly that the theoretical Black-Scholes price is no longer the mid price.

Since in the sequel, we will investigate the bid-ask spread around the mid price, we will work from now on under the situation $\lambda = \gamma$. The theory can readily be extended to the more general case $\lambda \neq \gamma$.

3   Implied Liquidity

We will call the parameter, fitting the bid-ask spread (under a symmetric distortion) around the mid price, *the implied liquidity parameter*. Hence for the European Call option (strike K and maturity T) with given market bid (b) and ask (a) prices, the implied liquidity parameter is the specific $\lambda > 0$, such that:

$$a = -\exp(-rT)\mathrm{E}^\lambda[-(S_T - K)^+] \text{ and } b = \exp(-rT)\mathrm{E}^\lambda[(S_T - K)^+],$$

where we have written E^λ as short notation of the above $\mathrm{E}^{\lambda,\lambda}$ operator, because we work in a symmetric case.

As it can be seen from Figure 1, the smaller the implied liquidity parameter the more liquid the underlying and the smaller the bid-ask price. In the extremal case where the implied liquidity parameter equals 0, the bid price coincides with the ask price, and we do work again under the one-price framework.

In a fixed market with no movement in the cone of acceptable risks and hence no change in liquidity as the market is then fixed, the bid ask spread moves around nonlinearly with maturity and or volatility. So the spread can move in a constant market with no change in liquidity. Therefore spread itself is not a perfect measure of liquidity. Implied liquidity can overcome this criticism.
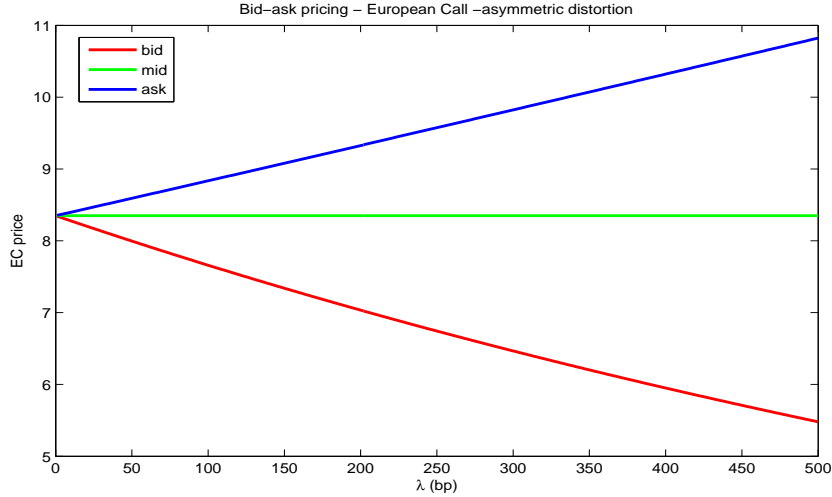
Figure 2: Bid-Ask Pricing - European Call - $\lambda \neq \gamma$

A new parameter, calls for also a new greek. We call the sensitivity of the bid and ask price with respect to a change in the liquidity parameter λ, the $lidip$[1]:

$$\text{lidip}_{bid} = \frac{\partial bid(X)}{\partial \lambda} \text{ and lidip}_{ask} = \frac{\partial ask(X)}{\partial \lambda}.$$

Lidip is negative for the bid and positive for the ask.

As a first illustration we have calculated the implied liquidity parameter for European Calls on the SP500 with a 1y maturity on the 1st of October 2009. In Figure 3, we observe an upsloping curve for increasing strike.

Secondly, we have calculated the implied liquidity parameter for ATM European Calls on the SP500 over maturity on the 1st of October 2009. In Figure 4, we observe a slightly upsloping curve on average for increasing maturities; some maturities are clearly more liquid than others. In the example $T = 0.043$, 0.293, 0.96 and 2.21 years are the most liquid ones.

In Figure 5 we graph, the implied liquidity parameter for ATM European calls with maturity (the closed to) 1 year over time for the SP500. We clearly see that liquidity is non constant over time and exhibits a mean-reverting behavior. The period ranges from the 3rd of January 2007 until the 30th of October 2009. We have estimated the (long run) average of the implied liquidity of the data set and over the period of the investigation this equals 67.11 bp. The highest value for the implied liquidity parameter was 283.1 bp on the 20th of October 2008. Around that day (and the week-end before) several European banks were rescued by government interventions. The graph indicates that implied liquidity behaves in a stochastic manner and apparently has a mean-reverting nature.

A similar graph can be found in Figure 6 for the 3 months to maturity ATM European Call on the SP500.

In Figure 7 we graph, the implied liquidity parameter for the ATM European call with maturity (the closed to) 1 year over time for the Dow Jones Index. The (long run) average of the implied liquidity of the data set over the period of the investigation equals 95.90 bp. The implied liquidity

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[1]The reader can see the reference to liquidity dip in this word; an educated reader can see it as an anagram.
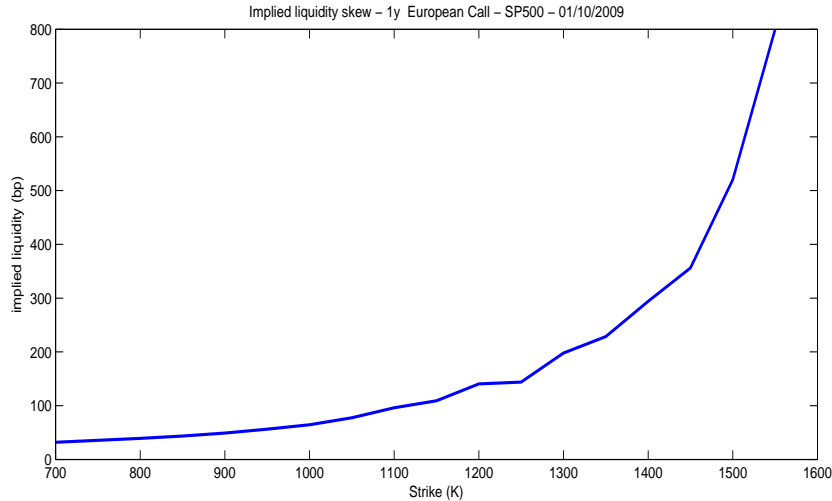
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Figure 3: Implied Liquidity Smile SP500 - 01/10/2009

parameter spikes in October 2008 to around 350 bp. Liquidity is hence on average and as well in distress situations lower than on the SP500.

A similar graph can be found in Figure 8 for the 3 months to maturity ATM European Call on the Dow Jones.

4 Liquidity Derivatives

Seen the utmost importance and the extreme dependence on liquidity of financial institutions, hedge funds and other financial players, claims contingent on liquidity may serve as potential hedges against drying up liquidity in critical times. In the subsections below we propose a few of liquidity derivatives, many other variations are possible.

4.1 Vanilla Liquidity Derivatives

We propose a contract that pays out a certain notional multiplied with the positive part of the difference of the implied liquidity on a certain day and a fixed strike liquidity. The fixing day can either be a fixed day, the maturity of the contract , in which case we deal with a European type of contract or a day during the lifetime of the contract (American Style). The European contract pays out at maturity :

$$payoff_{EC} = N \times (\lambda_T - K)^+,$$

where N is notional, K is the strike liquidity and λ_T is the implied ATM liquidity at maturity T. This contract can be useful for financial players who know that on a certain date a certain position needs to be liquidated and hence face liquidity risk at that date. The American style version is of interest for players where a liquidation will take place within a certain time window but the exact time is unsure when for the moment. Hedge fund facing redemption risk in periods of financial distress, which often comes along with less liquidity, could be potentially interested parties.
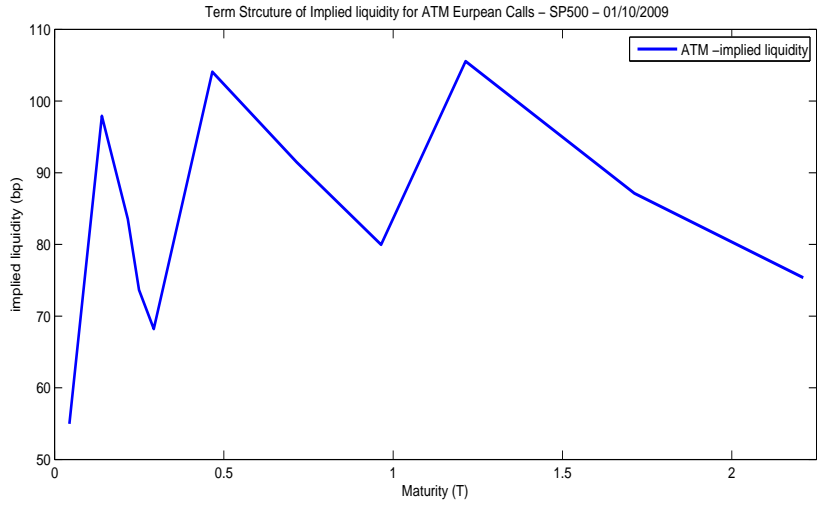
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Figure 4: Implied Liquidity Term SP500 - 01/10/2009

4.2 Average or Realized Liquidity Derivatives

A financial player who is exposed to liquidity risk all the time, because of the periodically adjustment of hedges or certain dynamics trading strategies (e.g. CPPIs), could be interested in a macro hedge against liquidity over the time-period of interest. Denoting the implied liquidity at time t with λ_t, an Asian type or realized liquidity swap could be of interest. More precisely, we propose a liquidity swap where "realized" liquidity (measured as average realized liquidity) is exchanged for a fixed strike liquidity. If liquidity dried up on average over the time period the swap will have a positive payoff; in a more liquid market than anticipated the swap will give a negative payoff:

$$payoff_{realizedliquidity} = N \times \left(\frac{\int_0^T \lambda_t \mathrm{d}t}{T} - K \right).$$

4.3 Exotic Liquidity Derivatives

Finally, we mention examples of exotic liquidity products which could be used as extra underlying clauses for structured products derivatives. The variations are endless. Imagine a principal protected note paying out either the initial investment or a certain percentage (participation rate) of the gain over a fixed time period of a certain underlier. Due to high volatility and low interest rates in the current times, the participation rate offered nowadays is quite low; this participation rate could be raised by making the payoff contingent on the behavior of liquidity. For example, one can have the regular payoff only if liquidity has never fallen/risen below/above a certain level.

References

[1] Cherny, A. and Madan, D. B. (2009) New Measures of Performance Evaluation, *Review of Financial Studies*, **22**, 2571-2606.

[2] Cherny, A. and Madan, D. B. (2009) Markets as a Counterparty: An Introduction to Conic Finance, *International Journal of Theoretical and Applied Finance*, to appear.
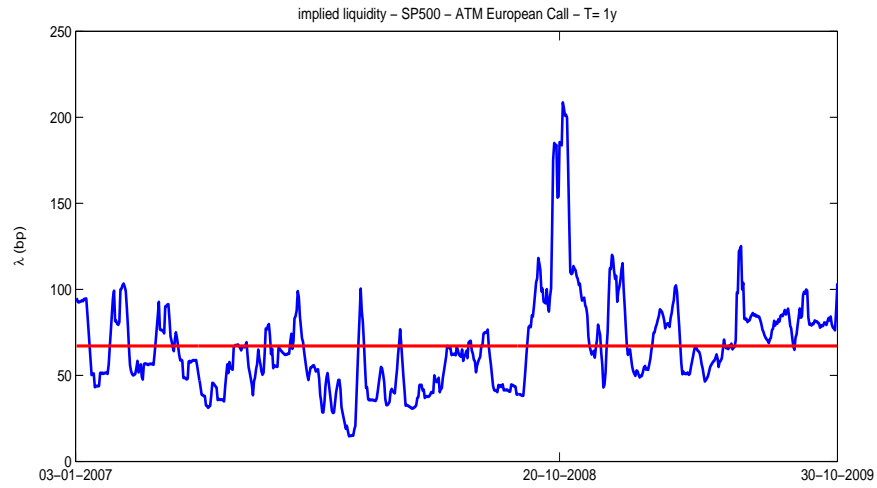
Figure 5: Implied Liquidity over time for ATM EC T=1y on SP500 - 03/01/2007 - 30/10/2009

[3] Madan, D. B. and Schoutens, W. (2010) Conic Financial Markets and Corporate Finance. *EURANDOM Report 2010*, Eindhoven. (`http://ssrn.com/abstract=1547022`).
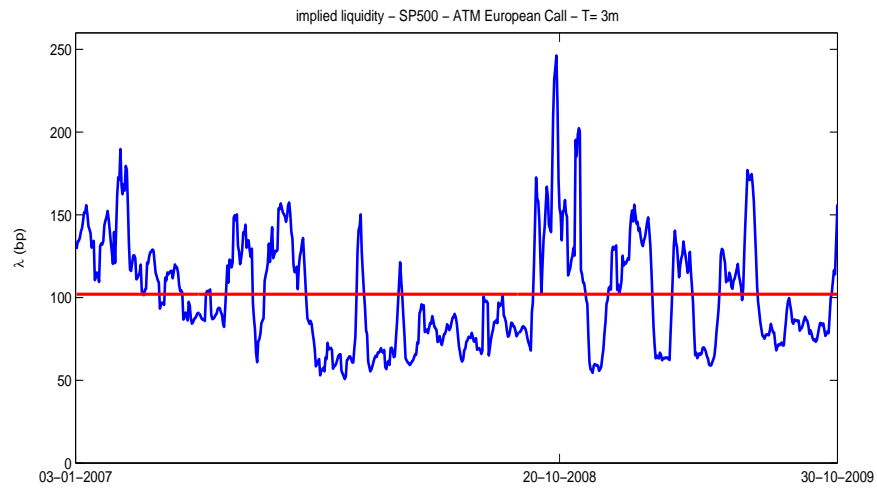
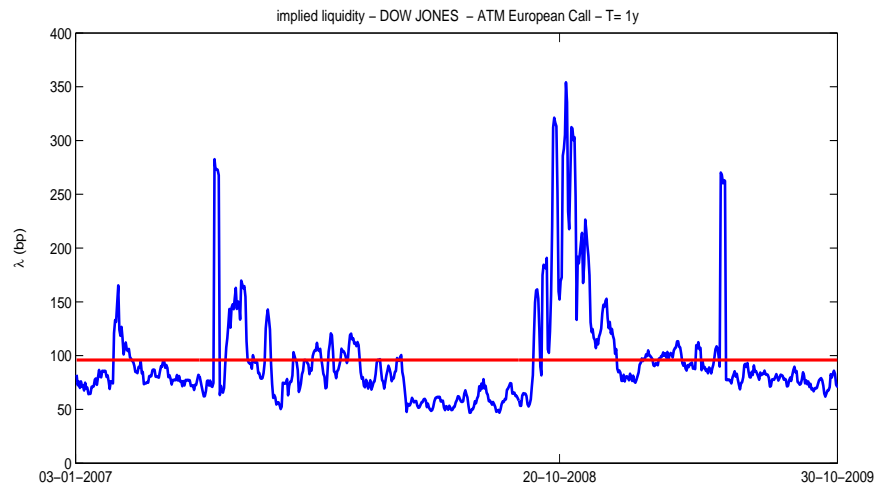Figure 6: Implied Liquidity over time for ATM EC T=3m on SP500 - 03/01/2007 - 30/10/2009

Figure 7: Implied Liquidity over time for ATM EC T=1y on Dow Jones - 03/01/2007 - 30/10/2009
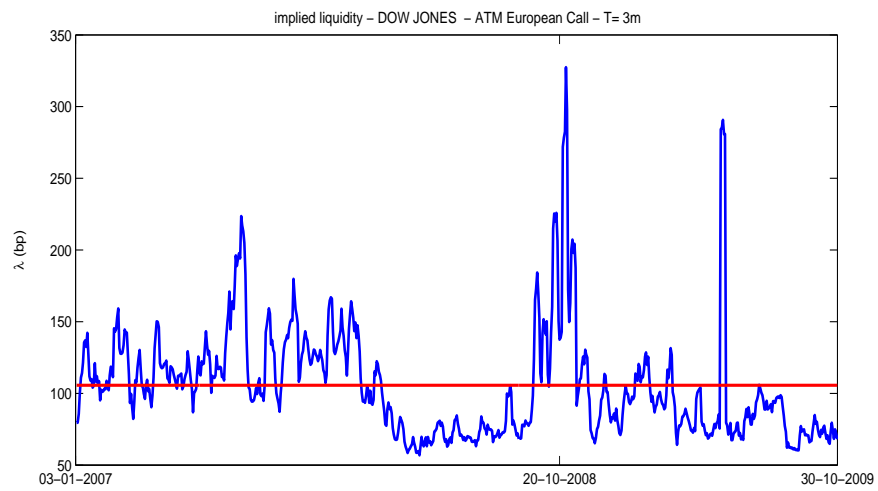
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Figure 8: Implied Liquidity over time for ATM EC T=3m on Dow Jones - 03/01/2007 - 30/10/2009