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CONVERGENCE OF THE ALL-TIME SUPREMUM OF A LÉVY PROCESS IN THE HEAVY-TRAFFIC REGIME

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Abstract. In this paper we derive a technique of obtaining limit theorems for suprema of Lévy processes from their random walk counterparts. That is, we show that if $\{Y_n^{(k)} : n \geq 0\}$ is a sequence of independent identically distributed random variables and $\{X_t^{(k)} : t \geq 0\}$ is a sequence of Lévy processes such that $X_1^{(k)} \stackrel{d}{=} Y_1^{(k)}$, then, with $S_n^{(k)} = \sum_{i=1}^n Y_i^{(k)}$ and under some mild assumptions, $\Delta(k) \max_{n\geq 0} S_n^{(k)} \stackrel{d}{\to} \mathscr{R} \iff \Delta(k) \sup_{t\geq 0} X_t^{(k)} \stackrel{d}{\to} \mathscr{R}$, as $k \to \infty$, for some random variable $\mathscr{R}$ and normalizing sequence $\Delta(k)$. We utilize this result to present a number of limiting theorems for suprema of Lévy processes in heavy-traffic regime.

1. Introduction

Let $X \equiv \{X_t : t \geq 0\}$ be a Lévy process with $\mathbb{E}X_1 = 0$. Define a Lévy process with drift $X_t^{(a)}$ via $X_t^{(a)} = X_t - at$, for $a \geq 0$. Along with the Lévy process $X^{(a)}$ define $\bar{X}^{(a)} = \sup_{t\geq 0} X_t^{(a)}$. Since $X_t^{(a)}$ drifts to $-\infty$, the *all-time* supremum $\bar{X}^{(a)}$ is a proper random variable for each $a > 0$. However, $\bar{X}^{(a)} \to \infty$ in probability as $a \downarrow 0$. From this fact a natural question arises: How fast does $\bar{X}^{(a)}$ grow as $a \downarrow 0$?

The main purpose of this paper is to answer the above question by considering the discrete approximation of a Lévy process by a random walk. For a sequence of zero mean, independent and identically distributed random variables $\{Y_n, n \geq 0\}$, put $\bar{S}^{(a)} = \sup_{n\geq 0} S_n^{(a)}$, where $S_n^{(a)} = S_n - na$ and S_n is the nth partial sum $S_n = \sum_{i=1}^n Y_i$. We shall show that if Y_1 has the same distribution as X_1, then the limiting distribution of $\bar{X}^{(a)}$ can be derived from the limiting distribution of $\bar{S}^{(a)}$. In doing so we shall utilize a bound by Willekens [22]. Loosely speaking, this bound allows to derive certain properties of Lévy processes via their corresponding random walk approximations (see also Doney [9]). The advantage of this approach is that the problem on how fast does $\bar{S}^{(a)}$ grow as $a \downarrow 0$ has been treated extensively and various methods have been developed.

One major reason why the behaviour of $\bar{S}^{(a)}$ has been studied is that it is well-known that the stationary distribution of the waiting time of a customer in a single-server first-come-first-served $GI/GI/1$ queue coincides with the distribution of the maximum of a corresponding random walk. The condition on the mean of the random walk becoming small ($a \downarrow 0$) means in the context of a queue that the traffic load tends to 1. Thus, the problem under consideration (in the random walk setting) may be seen as the investigation of the growth rate of the stationary waiting-time distribution in a $GI/GI/1$ queue. This

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is one of the most important problems in queueing theory that is referred to as the heavy-traffic approximation problem. The question was first posed by Kingman (see [14] for an extensive discussion on the early results). It has been solved in various settings by, e.g., Prokhorov [16], Boxma and Cohen [5], Resnick and Samorodnitsky [18], Szczotka and Woyczyński [21] and many others.

Surprisingly, there are no results in the literature on the heavy-traffic limit theorems for Lévy-driven (*fluid*) queues. Our approach however allows to translate each single result in the random walk setting to its analogue in the Lévy setting, therefore providing a range of fluid heavy-traffic limit theorems. With the notation introduced above, our main result, Theorem 1, states that, under some mild conditions, for some random variable $\mathscr{R}$, $\bar{S}^{(a)}\Delta(a) \stackrel{d}{\to} \mathscr{R}$ if and only if $\bar{X}^{(a)}\Delta(a) \stackrel{d}{\to} \mathscr{R}$, where $\Delta(a)$ is some proper normalization. In fact, Theorem 1 allows to consider general sequences of Lévy processes $\{X_t^{(a)} : t \geq 0\}$, not only $X_t^{(a)} = X_t - at$ for a fixed process X.

The remainder of the paper is organized as follows. In Section 2 we fix notation and give some necessary preliminaries. Section 3 contains the main result of this paper, Theorem 1, and its proof. Instances of this theorem applied to the results by Boxma and Cohen [5], Shneer and Wachtel [19] and Szczotka and Woyczyński [20] (see also Czystołowski and Szczotka [8]) are presented in Section 4 and conclude the paper.

2. Preliminaries and notation

Let us begin by fixing the notation for Lévy processes. Let $X \equiv \{X_t : t \geq 0\}$ be a *nondeterministic* Lévy process with $X_0 = 0$ and Lévy characteristic exponent $\psi(u)$ so that $\mathbb{E}e^{iuX_t} = e^{-t\psi(u)}$, for all $u \in \mathbb{R}$. In this case, for some $\sigma > 0$ and $\delta \in \mathbb{R}$, ψ has the form

$$\psi(u) = i\delta u + \frac{1}{2}\sigma^2 u^2 + \int_{|x|<1} \left(1 - e^{iux} + iux\right) \nu(dx) + \int_{|x|\geq 1} \left(1 - e^{iux}\right) \nu(dx),$$

where ν is the Lévy measure (on $\mathbb{R} \setminus \{0\}$) satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$, noting that nondeterministic is synonymous with $\sigma^2 + \nu(\mathbb{R} \setminus \{0\}) > 0$. We say that: X is *centered* if $\mathbb{E}X_t = 0$ for all t; *spectrally positive* if $\nu(-\infty, 0) = 0$; *spectrally negative* if $\nu(0, \infty) = 0$. If X_1 has a stable distribution with index $\alpha \in (0, 2]$ then we say that X is an α-*stable* Lévy process and denote it by $\mathscr{L}^{(\alpha)}$. For more background on Lévy processes we refer the reader to Bertoin [2] and references therein.

In the sequel we will encounter the Mittag-Leffler distribution, see, e.g., [4, p. 329]. A positive random variable M is said to have a Mittag-Leffler distribution with parameter $\alpha \in (0, 1]$ if the Laplace-Stieltjes transform (LST) is given by

$$\mathbb{E}\exp(-sM) = \frac{1}{1 + s^\alpha}.$$

A random variable with this LST shall be denoted by $\mathscr{M}\mathscr{L}_\alpha$. Observe that $\mathscr{M}\mathscr{L}_1$ has the 1-exponential distribution.

We will also make use of some standard notation. For two functions f, g we shall write $f(x) \sim g(x)$ as $x \to x_0 \in [0, \infty]$ to mean $\lim_{x\to x_0} f(x)/g(x) = 1$. The class of regularly varying functions with index α shall be denoted by $\mathscr{R}\mathscr{V}_\alpha$.

In what follows we shall also write

$$\bar{X}^{(k)} = \sup_{t\geq 0} X_t^{(k)}, \quad \bar{S}^{(k)} = \max_{k\geq 0} S_n^{(k)},$$

where $\{X_t^{(k)} : t \geq 0\}$ is a sequence of Lévy processes and $S_n^{(k)} = \sum_{i=1}^n Y_i^{(k)}$ is the nth partial sum of a sequence of random variables $\{Y_n^{(k)} : n \geq 0\}$.

3. Main theorem

Theorem 1. *For any $k \geq 0$, let $\{Y_n^{(k)} : n \geq 0\}$ be a sequence of independent, identically distributed random variables and $\{X_t^{(k)} : t \geq 0\}$ be a sequence of Lévy processes. Moreover, assume that $Y_1^{(k)} \stackrel{d}{=} X_1^{(k)}$, for each k. Then, for some random variable $\mathscr{R}$,*

$$\Delta(k) \max_{n\geq 0} S_n^{(k)} \xrightarrow{d} \mathscr{R} \iff \Delta(k) \sup_{t\geq 0} X_t^{(k)} \xrightarrow{d} \mathscr{R}, \quad \text{as} \quad k \to \infty,$$

where $\{\Delta(k) : k \geq 0\}$ is a normalizing sequence such that $\Delta(k)X_1^{(k)} \to 0$.

Proof. Let R be the distribution function of $\mathscr{R}$ and let x be a continuity point of R. Assume that $\Delta(k)\bar{S}^{(k)} \xrightarrow{d} \mathscr{R}$, the converse implication follows in the same manner.
Observe that $\bar{S}^{(k)} \stackrel{d}{=} \max_{n\geq 0} X_n^{(k)}$. Thus, we trivially have

$$\mathbb{P}\left(\Delta(k)\bar{X}^{(k)} > x\right) \geq \mathbb{P}\left(\Delta(k)\bar{S}^{(k)} > x\right). \tag{3.1}$$

On the other hand, for any $x_0 > 0$,

$$\mathbb{P}\left(\Delta(k)\bar{X}^{(k)} > x\right) \leq \mathbb{P}\left(\Delta(k)\bar{S}^{(k)} > x - x_0\right) + \mathbb{P}\left(\Delta(k)\bar{X}^{(k)} > x,\, \Delta(k)\bar{S}^{(k)} \leq x - x_0\right).$$

Define a stopping time $\tau^{(k)}(x) = \inf\{t \geq 0 : \Delta(k)X_t^{(k)} \geq x\}$, then the second term on the right hand side of the above inequality can be bounded from above by

$$\begin{aligned}
&\mathbb{P}\left(\tau^{(k)}(x) < \infty, \Delta(k)\left(\inf_{t\in(\tau^{(k)}(x),\tau^{(k)}(x)+1]}\left(X_t^{(k)} - X_{\tau^{(k)}(x)}^{(k)}\right)\right) \leq -x_0\right) \\
&= \mathbb{P}\left(\tau^{(k)}(x) < \infty\right) \mathbb{P}\left(\Delta(k) \inf_{t\in(0,1]} X_t^{(k)} \leq -x_0\right),
\end{aligned}$$

where we used the strong Markov property in the last equality. Thus,

$$\mathbb{P}\left(\Delta(k)\bar{X}^{(k)} > x\right) \mathbb{P}\left(\Delta(k) \inf_{t\in(0,1]} X_t^{(k)} > -x_0\right) \leq \mathbb{P}\left(\Delta(k)\bar{S}^{(k)} > x - x_0\right). \tag{3.2}$$

Now $\Delta(k)X_1^{(k)} \to 0$ implies that $\{\Delta(k)X_t^{(k)} : t \in [0,1]\}$ converges to zero in $D[0,1]$, therefore by the continuous mapping theorem $\Delta(k)\inf_{t\in(0,1]} X_t^{(k)} \to 0$. Thus, combining formulas (3.1) and (3.2) we get

$$\bar{R}(x) \leq \liminf_{k\to\infty} \mathbb{P}\left(\Delta(k)\bar{X}^{(k)} > x\right) \leq \limsup_{k\to\infty} \mathbb{P}\left(\Delta(k)\bar{X}^{(k)} > x\right) \leq \bar{R}(x - x_0),$$

where $\bar{R}(x) = 1 - R(x)$. The thesis follows by letting $x_0 \to 0$. □

Remark 1. We shall use the *if* part of Theorem 1 to derive various limiting theorems for suprema of Lévy processes in the subsequent section. It is worth noting however that the *only if* part could be used as well to derive limiting theorems for suprema of random walks. A variation of this approach has been undertaken in [20], where first a heavy-traffic limit theorem is derived in continuous time and then this theorem is used to claim an analogue behaviour in discrete time.

4. Special instances

Theorem 1 provides a tool for translating limiting theorems for random walks to their analogues in the Lévy setting. In this section we shall focus our attention on some seminal results about the convergence of the maxima of random walks and reformulate them to the Lévy case. We illustrate each special case that we consider with a remark that explains an alternative way of obtaining the particular result via a direct approach undertaken in the literature. These remarks, albeit short, are rigorous enough to act as alternative proofs. Let us start with the case in which the underlying processes are spectrally positive, which is closely related to the queueing setting via the compound Poisson process.

4.1. **Spectrally positive processes.** For a sequence of zero mean, independent and identically distributed random variables $\{Y, Y_n, n \geq 0\}$, the question of how fast does $\bar{S}^{(a)} = \max_{n\geq 0}(S_n - na)$ grow as $a \downarrow 0$ was first posed by Kingman [12, 13]. Kingman in his proof assumed exponential moments of $|Y|$ and used Wiener-Hopf factorization to obtain the Laplace transform of $\bar{S}^{(a)}$. Prokhorov [16] generalized Kingman's result to the case when only the second moment of Y is finite. His approach was based on the functional Central Limit Theorem. These two approaches have become classical and have both been used to prove various heavy-traffic results. The analytical approach of Kingman was used by Boxma and Cohen [5] (see also Cohen [6]) to study the limiting behaviour of $\bar{S}^{(a)}$ in the case of infinite variance. They proved that if $\mathbb{P}(Y > x)$ is regularly varying at infinity with a parameter $\alpha \in (1, 2)$ (and under some additional assumptions), then there exists a function $\Delta(a)$ such that $\Delta(a)\bar{S}^{(a)}$ converges in law to a proper random variable.
Theorem 5.1 of Boxma and Cohen [5] acts as the first application of our main result. For a Lévy measure ν define

$$r(s) := \int_0^\infty \left(e^{-sx} - 1 + sx\right) \nu(dx).$$

For a Lévy process X, let F be the distribution function of X_1 and set $\bar{F} := 1 - F$. [4, Theorem 8.2.1] asserts that $\bar{F} \in \mathscr{RV}_\alpha$ if and only if $\nu(x, \infty) \in \mathscr{RV}_\alpha$, where ν is the Lévy measure of X, moreover $\bar{F}(x) \sim \nu(x, \infty)$, as $x \to \infty$. This combined with [5, Theorem 5.1] and Theorem 1 yields:

Theorem 2. *Let X be a spectrally positive Lévy process such that $\nu(x, \infty) \in \mathscr{RV}_{-\alpha}$ for $\alpha \in (1, 2)$. Set $\rho(a) = \mu/a$, where $\mu = \mathbb{E}X_1$, then*

$$\Delta(\rho(a)) \sup_{t\geq 0}(X_t - at) \xrightarrow{d} \mathscr{ML}_{\alpha-1}, \quad \text{as} \quad \rho(a) \uparrow 1,$$

where $\Delta(x) = d(x)/\mu$ and $d(x)$ is such that

$$r(d(x)) \sim d(x)\frac{1-x}{x}\mu^\alpha, \quad \text{as} \quad x \uparrow 1. \tag{4.1}$$

See also [5, 18] for possible refinements of the assumption on regular variation in this special case.

Remark 2. It is possible to prove Theorem 2 using a direct approach like the one in [5]. Let $X_t^{(a)} = X_t - at$, then the Pollaczek-Khinchine formula (see, e.g., [1] Chapter IX) yields

$$\mathbb{E}e^{-\lambda \bar{X}^{(a)}} = \frac{\lambda\varphi_a'(0)}{\varphi_a(\lambda)} < \infty, \quad \text{for} \quad \lambda > 0,$$

where $\varphi_a(\lambda) = \log \mathbb{E}\exp(-\lambda(X_1 - a))$. Substituting $\varphi_a(\lambda) = \lambda\varphi_a'(0) + r(\lambda)$ yields

$$\frac{\lambda\varphi_a'(0)}{\varphi_a(\lambda)} = \frac{1}{1 + \frac{r(\lambda)}{\lambda\varphi_a'(0)}},$$

where we assumed $\sigma = 0$ for simplicity. Let $\lambda = s\Delta(\rho(a))$ with $\Delta(\cdot)$ as in Theorem 2. Using [4, Theorem 8.1.6] one infers that, under the assumption $\nu(x,\infty) \in \mathscr{RV}_{-\alpha}$, r is a regularly varying function at 0 with index α. We necessary have $d(x) \downarrow 0$, as $x \uparrow 1$. Hence, as $\rho(a) \uparrow 1$,

$$\frac{r(\lambda)}{\lambda\varphi_a'(0)} \sim \left(\frac{s}{\mu}\right)^{\alpha-1} \frac{r(d(\rho(a)))}{d(\rho(a))(a-\mu)} = \frac{s^{\alpha-1}}{\mu^\alpha}\frac{r(d(\rho(a)))}{d(\rho(a))}\frac{\rho(a)}{1-\rho(a)} \sim s^{\alpha-1}.$$

4.2. **Regular variation.** Theorem 2 limits the class of Lévy processes under consideration to spectrally positive. Further improvements of the result from [5] by Furrer [10] and Resnick and Samorodnitsky [18] assumed that the random walk belongs to the domain of attraction of a spectrally positive stable law and relied on functional limit theorems. Shneer and Wachtel [19] relaxed this assumption to allow the random walk to belong to the domain of attraction of any stable law. The main result from [19] acts as the second instance of an application of Theorem 1.

Theorem 3. *Let X be a centred Lévy process such that the random variable X_1 belongs to the domain of attraction of a stable law $\mathscr{L}_1^{(\alpha)}$ with index $\alpha \in (1,2]$. That is, there exists a sequence $\{d(n) : n \geq 0\}$ such that*

$$\frac{X_n}{d(n)} \stackrel{d}{\to} \mathscr{L}_1^{(\alpha)}, \quad \text{as} \quad n \to \infty. \tag{4.2}$$

Then,

$$\Delta(a)\sup_{t\geq 0}(X_t - at) \stackrel{d}{\to} \sup_{t\geq 0}(\mathscr{L}_t^{(\alpha)} - t), \quad \text{as} \quad a \downarrow 0, \quad \text{where} \quad \Delta(a) = \frac{1}{d(n(a))}$$

and $n(a)$ is such that

$$an(a) \sim d(n(a)), \quad \text{as} \quad a \downarrow 0. \tag{4.3}$$

Remark 3. It is well known that the sequence $d(\cdot)$ in Theorem 3 is regularly varying with index $1/\alpha$. Therefore, Theorem 3 implies that, with $X_t^{(a)} = X_t - at$, $\bar{X}^{(a)}$ grows as a regularly varying function with index $-1/(\alpha-1)$ at zero. If $\mathscr{L}^{(\alpha)}$ is spectrally negative, then the limiting distribution of S is exponential, see, e.g. Bingham [3, Proposition 5]. If $\mathscr{L}^{(\alpha)}$ is spectrally positive, then, as seen in Theorem 2, the limiting random variable has a Mittag-Leffler distribution, see, e.g. Kella and Whitt [11, Theorem 4.2]. If $\mathscr{L}^{(\alpha)}$ is symmetric, then one can give the Laplace-Stieltjes transform of $\mathscr{R} = \sup_{t\geq 0}(\mathscr{L}_t^{(\alpha)} - t)$, see Szczotka and Woyczyński [20, Theorem 8]. In the other cases the explicit form of the distribution might be infeasible to compute, however, one can easily find its tail asymptotics $\mathbb{P}(\mathscr{R} > x) \sim Cx^{1-\alpha}$. For more details on the supremum distribution of a Lévy process see Szczotka and Woyczyński [20].

Remark 4. In Shneer and Wachtel [19] it is shown that both classical approaches, i.e., via Wiener-Hopf factorization and via a functional central limit theorem, can be applied to obtain their result. Moreover, the technical difficulties arising from these methods can be overcome using a generalization of Kolmogorov's inequality based on a result by Pruitt [17]. A similar result is also available for Lévy processes and can also be found in [17]. Let us introduce $V(x) = \int_{|y|\le x} y^2\nu(dy)$, the truncated second moment of the Lévy measure ν. Under the assumptions of Theorem 3, $V \in \mathscr{RV}_{2-\alpha}$. Moreover, [17, Section 3] asserts that there exists a constant C such that

$$\mathbb{P}\left(\sup_{s\le t} X_s \ge x\right) \le C\frac{tV(x)}{x^2}. \tag{4.4}$$

Using the regular variation of V, (4.4) and (4.3), for any fixed $T > 0$ there exist constants $C_1, C_2 > 0$, such that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\ge n(a)T}(X_t - at)\ge 0\right) &\le \sum_{k=0}^{\infty}\mathbb{P}\left(\sup_{t\le 2^{k+1}n(a)T} X_t \ge 2^k an(a)T\right)\\ &\le C_1\frac{V(an(a)T)}{a^2n(a)T}\sum_{k=0}^{\infty}(2^k)^{1-\alpha} \le C_2\frac{V(d(n(a)))}{c^2(n(a))}n(a)T^{1-\alpha}.\end{aligned} \tag{4.5}$$

The sequence (c_n) can be defined as $\inf\{t > 0 : V(t) \le t^2/n\}$, therefore the last expression tends to zero, uniformly in $a > 0$, as T tends to infinity. This combined with the classical functional limit theorem corresponding to (4.2) and the fact that, for a fixed $T > 0$, supremum on $[0, T]$ is a continuous map, yields the thesis of Theorem 3.
On the other hand, as a consequence of the Wiener-Hopf factorization (see [15, Chapter 6]), with $X_t^{(a)} = X_t - at$, the LST of $\bar{X}^{(a)}$ is given by,

$$\mathbb{E}e^{-\lambda\bar{X}^{(a)}} = \exp\left(-\int_0^\infty \frac{1}{t}\,\mathbb{E}\left(1 - e^{-\lambda(X_{n(a)t}-an(a)t)},\, X_{n(a)t} - an(a)t > 0\right)dt\right).$$

Plugging in $\lambda = \Delta(a)s$ for $s > 0$, from (4.2) and (4.3) it follows that, as $a \downarrow 0$, this expression tends to

$$\mathbb{E}e^{-s\mathscr{R}} = \exp\left(-\int_0^\infty \frac{1}{t}\,\mathbb{E}\left(1 - e^{-s(\mathscr{L}_t^{(\alpha)}-t)},\, \mathscr{L}_t^{(\alpha)} - t > 0\right)dt\right),$$

the LST of $\mathscr{R} = \sup_{t\ge0}(\mathscr{L}_t^{(\alpha)} - t)$, provided that we can interchange the limit with the integral. This follows by using the dominated convergence theorem. For big values of t, say $t > T$ and some $C_3, C_4 > 0$, we can estimate the integrand by (cf. (4.4) and (4.5))

$$\frac{1}{t}\mathbb{P}\left(X_{n(a)t} > an(a)t\right) \le C\frac{V(an(a)t)}{a^2n(a)t^2} \le C_3t^{-\alpha}\frac{V(d(n(a)))}{c^2(n(a))}n(a) \le C_4t^{-\alpha}.$$

For $t \le T$ and some $C_5 > 0$, one can simply bound the integrand by (cf. (4.4))

$$C_5st^{1/\alpha-1}\mathbb{E}(\mathscr{L}_1^{(\alpha)}, \mathscr{L}_1^{(\alpha)} > 0).$$

4.3. **Heavy-traffic invariance principle.** A general principle called *heavy-traffic invariance principle* has been established in Szczotka and Woyczyński [20], see also [7, 8, 21]. This principle asserts under what condition one can infer the limiting distributions of maxima of random walks from functional limit theorems. According to Theorem 1 this principle can be also reformulated to the Lévy setting. Therefore we conclude the paper with the following theorem:

Theorem 4 (Heavy-traffic invariance principle)**.** *For a sequence of Lévy processes* $\{X_t^{(k)} : t \geq 0\}$ *denote* $\mu^{(k)} = \mathbb{E}X_1^{(k)} < 0$ *and assume that* $\mu^{(k)} \uparrow 0$ *as* $k \to \infty$*. Moreover, assume that there exist sequences* $\{d(k) : k \geq 0\}$ *and* $\{\Delta(k) : k \geq 0\}$*, such that the following conditions hold:*

(I) $d(k)\Delta(k)|\mu^{(k)}| \to \beta \in (0, \infty)$*;*

(II) $\Delta(k)\{X_{d(k)t}^{(k)} - td(k)\mu^{(k)} : t \geq 0\} \xrightarrow{d} \{X_t : t \geq 0\}$ *in* $D[0, \infty)$ *equipped with the Skorokhod* J_1 *topology, where* X *is a Lévy process;*

(III) *The sequence* $\{\Delta(k)\bar{X}^{(k)} : k \geq 0\}$ *is tight.*

Then,

$$\Delta(k) \sup_{t \geq 0} X_t^{(k)} \xrightarrow{d} \sup_{t \geq 0} (X_t - \beta t) .$$

Remark 5. See Szczotka and Woyczyński [20, Theorem 2] for an extension to sequences of processes $X^{(k)}$ with stationary increments in the case X is stochastically continuous.

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