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Abstract

We give a short proof of Theorem 1.2(i) from [5]. We show that the expected size of the intrinsic ball of radius r is at most Cr if the susceptibility exponent γ is at most 1. In particular, this result follows if the so-called triangle condition holds.

Let $G = (V, E)$ be an infinite connected bounded degree graph. We consider independent bond percolation on G. For $p \in [0, 1]$, each edge of G is open with probability p and closed with probability $1 - p$ independently for distinct edges. The resulting product measure is denoted by $\mathbb{P}_p$. For two vertices $x, y \in V$ and an integer n, we write $x \leftrightarrow y$ if there is an open path from x to y, and we write $x \overset{\leq n}{\longleftrightarrow} y$ if there is an open path of at most n edges from x to y. Let $C(x)$ be the set of all $y \in V$ such that $x \leftrightarrow y$. For $x \in V$, the *intrinsic ball* of radius n at x is the set $B_I(x, n)$ of all $y \in V$ such that $x \overset{\leq n}{\longleftrightarrow} y$. Let $p_c = \inf\{p : \mathbb{P}_p(|C(x)| = \infty) > 0\}$ be the critical percolation probability. Note that p_c does not depend on a particular choice of $x \in V$, since G is a connected graph. For general background on Bernoulli percolation we refer the reader to [2].

In this note we give a short (and slightly more general) proof of Theorem 1.2(i) from [5].

Theorem 1. *Let $x \in V$. If there exists a finite constant C_1 such that $\mathbb{E}_p|C(x)| \leq C_1(p_c - p)^{-1}$ for all $p < p_c$, then there exists a finite constant C_2 such that for all n,*

$$\mathbb{E}_{p_c}|B_I(x, n)| \leq C_2 n.$$

Before we proceed with the proof of this theorem, we discuss examples of graphs for which the assumption of Theorem 1 is known to hold. This assumption can be interpreted as the mean-field bound $\gamma \leq 1$, where γ is the susceptibility exponent. It is well known that for vertex-transitive graphs this assumption is satisfied if the triangle condition holds at p_c [1]: For $x \in V$,

$$\sum_{y,z \in V} \mathbb{P}_{p_c}(x \leftrightarrow y)\mathbb{P}_{p_c}(y \leftrightarrow z)\mathbb{P}_{p_c}(z \leftrightarrow x) < \infty.$$

This condition holds on certain Euclidean lattices [3, 4] including the nearest-neighbor lattice $\mathbb{Z}^d$ with $d \geq 19$ and sufficiently spread-out lattices with $d > 6$. It also holds for a rather general class of non-amenable transitive graphs [6, 8, 9, 10]. It has been shown in [7] that for vertex-transitive graphs, the triangle condition is equivalent to the open triangle condition. The latter is often used instead of the triangle condition in studying the mean-field criticality.

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Proof of Theorem 1. Since G is a bounded degree graph, it is sufficient to prove the result for $n \geq 2/p_c$. Let $p < p_c$. We consider the following coupling of percolation with parameter p and with parameter p_c. First delete edges independently with probability $1 - p_c$, then every present edge is deleted independently with probability $1 - (p/p_c)$. This construction implies that for $x, y \in V$, $p < p_c$, and an integer n,

$$\mathbb{P}_p(x \stackrel{\leq n}{\longleftrightarrow} y) \geq \left(\frac{p}{p_c}\right)^n \mathbb{P}_{p_c}(x \stackrel{\leq n}{\longleftrightarrow} y).$$

Summing over $y \in V$ and using the inequality $\mathbb{P}_p(x \stackrel{\leq n}{\longleftrightarrow} y) \leq \mathbb{P}_p(x \leftrightarrow y)$, we obtain

$$\mathbb{E}_{p_c}|B_I(x,n)| \leq \left(\frac{p_c}{p}\right)^n \mathbb{E}_p|C(x)|.$$

The result follows by taking $p = p_c - \frac{1}{n}$. ☐

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