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Quantitative assessment of securitisation deals

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Preface

Securitisation deals have come into focus during the recent years due to the challenges in their assessments and their role in the recent credit crises. These deals are created by the pooling of assets and the tranching of liabilities. The later are backed by the collateral pools. Tranching makes it possible to create liabilities of a variety of seniorities and risk-return profiles.

The assessment of a securitisation deal is based on qualitative and quantitative assessments of the risks inherent in the transaction and how well the structure manages to mitigate these risks. Example of risks related to the performance of a transaction are credit risk, prepayment risk, market risk, liquidity risk, counterparty risk, operational risk and legal risk.

In the light of the recent credit crisis, model risk and parameter uncertainty have come in focus. Model risk refers to the fact that the outcome of the assessment of a securitisation transaction can be influenced by the choice of the model used to derive defaults and prepayments. The uncertainties in the parameter values used as input to these models add to the uncertainty of the output of the assessment.

The aim of this report is to give an overview of recent performed research on model risk and parameter sensitivity of asset backed securities ratings.

The outline of the text is as follows.[3] In Chapter 1, an introduction to asset-backed securities (ABSs) is given. We describe, for example, key securitisation parties, structural characteristics and credit enhancements.

The cashflow modelling of ABS deals can be divided into two parts: (1) the modelling of the cash collections from the asset pool and the distribution of these collections to the note holders, discussed in Chapter 2, and (2) the modelling of defaults and prepayments. Deterministic models to generate default and prepayment scenarios are presented in Chapter 3; a collection of stochastic models is presented in Chapter 4. In Chapter 5, two of the major rating agencies quantitative methodologies for ABS rating are discussed.

Next, the model risk in rating ABSs is discussed and we elaborate on the parameter sensitivity of ABS ratings. More precisely, in Chapter 6 we look at how the choice of default model influences the ratings of an ABS structure. We illustrate this using a two tranche ABS. Furthermore, we also investigate the influence of changing some of the input parameters one at a time. A more systematic parameter sensitivity analysis is presented in Chapter 7. In this chapter we

[3] An earlier version of parts of this text was presented Jönsson, H. and Schoutens, W. *Asset backed securities: Risks, Ratings and Quantitative Modelling*, EURANDOM Report 2009-50, www.eurandom.nl.

introduce global sensitivity analysis techniques, which allow us to systematically analyse how the uncertainty in each input parameter's value contributes to the uncertainty of the expected loss and the expected average life of the notes and hence the rating. The report concludes with an summary of the findings in Chapter 8.

Chapter 1

Introduction to Asset Backed Securities

1.1 Introduction

Asset-Backed Securities (ABSs) are structured finance products backed by pools of assets. ABSs are created through a securitisation process, where assets are pooled together and the liabilities backed by these assets are tranched such that the ABSs have different seniority and risk-return profiles. The Bank for International Settlements defined structured finance through the following characterisation (BIS (2005), p. 5):

- Pooling of assets;
- Tranching of liabilities that are backed by these collateral assets;
- De-linking of the credit risk of the collateral pool from the credit risk of the originator, usually through the use of a finite-lived, standalone financing vehicle.

In the present chapter we are introducing some of the key features of ABSs followed by a discussion on the main risks inherent in these securitisation deals.

1.2 Asset Backed Securities Features

1.2.1 Asset Classes

The asset pools can be made up of almost any type of assets, ranging from common automobile loans, student loans and credit cards to more esoteric cash flows such as royalty payments ("Bowie bonds"). A few typical asset classes are listed in Table 1.1.

In this project we have performed case study analysis of SME loans ABSs.

There are several ways to distinguish between structured finance products according to their collateral asset classes: cash flow vs. synthetic; existing assets vs. future flows; corporate related vs. consumer related.

Auto leases	Auto loans
Commercial mortgages	Residential mortgages
Student loans	Credit cards
Home equity loans	Manufactured housing loans
SME loans	Entertainment royalties

Table 1.1: *Some typical ABS asset classes.*

- Cash flow: The interest and principal payments generated by the assets are passed through to the notes. Typically there is a legal transfer of the assets.
- Synthetic: Only the credit risk of the assets are passed on to the investors through credit derivatives. There is no legal transfer of the underlying assets.
- Existing assets: The asset pool consists of existing assets, e.g., loan receivables, with already existing cash flows.
- Future flows: Securitisation of expected cash flows of assets that will be created in the future, e.g., airline ticket revenues and pipeline utilisation fees.
- Corporate related: e.g., commercial mortgages, auto and equipment leases, trade receivables;
- Consumer related: e.g., automobile loans, residential mortgages, credit cards, home equity loans, student loans.

Although it is possible to call all types of securities created through securitisation asset backed securities it seems to be common to make a few distinctions. It is common to refer to securities backed by mortgages as mortgage backed securities (MBSs) and furthermore distinguish between residential mortgages backed securities (RMBS) and commercial mortgages backed securities (CMBS). Collateralised debt obligations (CDOs) are commonly viewed as a separate structured finance product group, with two subcategories: corporate related assets (loans, bonds, and/or credit default swaps) and resecuritisation assets (ABS CDOs, CDO-squared). In the corporate related CDOs can two sub-classes be distinguished: collateralised loan obligations (CLO) and collateralised bond obligations (CBO).

1.2.2 Key Securitisation Parties

The following parties are key players in securitisation:

- Originator(s): institution(s) originating the pooled assets;
- Issuer/Arranger: Sets up the structure and tranches the liabilities, sell the liabilities to investors and buys the assets from the originator using the proceeds of the sale. The Issuer

is a finite-lived, standalone, bankruptcy remote entity referred to as a special purpose vehicle (SPV) or special purpose entity (SPE);

- Servicer: collects payments from the asset pool and distribute the available funds to the liabilities. The servicer is also responsible for the monitoring of the pool performance: handling delinquencies, defaults and recoveries. The servicer plays an important role in the structure. The deal has an exposure to the servicer's credit quality; any negative events that affect the servicer could influence the performance and rating of the ABS. We note that the originator can be the servicer, which in such case makes the structure exposed to the originator's credit quality despite the de-linking of the assets from the originator.

- Investors: invests in the liabilities;

- Trustee: supervises the distribution of available funds to the investors and monitors that the contracting parties comply to the documentation;

- Rating Agencies: Provide ratings on the issued securities. The rating agencies have a more or less direct influence on the structuring process because the rating is based not only on the credit quality of the asset pool but also on the structural features of the deal. Moreover, the securities created through the tranching are typically created with specific rating levels in mind, making it important for the issuer to have an iterative dialogue with the rating agencies during the structuring process. We point here to the potential danger caused by this interaction. Because of the negotiation process a tranche rating, say 'AAA', will be just on the edge of 'AAA', i.e., it satisfies the minimal requirements for the 'AAA' rating without extra cushion.

- Third-parties: A number of other counterparties can be involved in a structured finance deal, for example, financial guarantors, interest and currency swap counterparties, and credit and liquidity providers.

1.2.3 Structural Characteristics

There are many different structural characteristics in the ABS universe. We mention here two basic structures, *amortising* and *revolving*, which refer to the reduction of the pool's aggregated outstanding principal amount.

Each collection period the aggregated outstanding principal of the assets can be reduced by scheduled repayments, unscheduled prepayments and defaults. To keep the structure fully collateralized, either the notes have to be redeemed or new assets have to be added to the pool.

In an *amortising structure*, the notes should be redeemed according to the relevant priority of payments with an amount equal to the note redemption amount. The *note redemption amount* is commonly calculated as the sum of the principal collections from scheduled repayments and unscheduled prepayments over the collection period. Sometimes the recoveries of defaulted loans are added to the note redemption amount. Another alternative, instead of adding the recoveries to the redemption amount, is to add the total outstanding principal amount of the loans defaulting in the collection period to the note redemption amount (see Loss allocation).

In a *revolving structure*, the Issuer purchases new assets to be added to the pool to keep the structure fully collateralized. During the revolving period the Issuer may purchase additional assets offered by the Originator, however these additional assets must meet certain eligibility criteria. The eligibility criteria are there to prevent the credit quality of the asset pool to deteriorate. The revolving period is most often followed by an amortisation period during which the structure behaves as an amortising structure. The *replenishment amount*, the amount available to purchase new assets, is calculated in a similar way as the note redemption amount.

1.2.4 Priority of Payments

The allocation of interest and principal collections from the asset pool to the transaction parties is described by the *priority of payments* (or *waterfall*). The transaction parties that keeps the structure functioning (originator, servicer, and issuer) have the highest priorities. After these senior fees and expenses, the interest payments on the notes could appear followed by pool replenishment or note redemption, but other sequences are also possible.

Waterfalls can be classified either as *combined waterfalls* or as *separate waterfalls*. In a combined waterfall, all cash collections from the asset pool are combined into available funds and the allocation is described in a single waterfall. There is, thus, no distinction made between interest collections and principal collections. However, in a separate waterfall, interest collections and principal collections are kept separated and distributed according to an interest waterfall and a principal waterfall, respectively. This implies that the available amount for note redemption or asset replenishment is limited to the principal cashflows.

A revolving structure can have a revolving waterfall, which is valid as long as replenishment is allowed, followed by an amortising waterfall.

In an amortising structure, principal is allocated either *pro rata* or *sequential*. *Pro rata* allocation means a proportional allocation of the note redemption amount, such that the redemption amount due to each note is an amount proportional to the note's fraction of the total outstanding principal amount of the notes on the closing date.

Using *sequential* allocation means that the most senior class of notes is redeemed first, before any other notes are redeemed. After the most senior note is redeemed, the next note in rank is redeemed, and so on. That is, principal is allocated in order of seniority.

It is important to understand that "pro rata" and "sequential" refer to the allocation of the note redemption amount, that is, the amounts *due* to be paid to each class of notes. It is not describing the amounts actually being paid to the notes, which is controlled by the priority of payments and depends on the amount of available funds at the respectively level of the waterfall.

One more important term in connection with the priority of payments is *pari passu*, which means that two or more parties have equal right to payments.

A simple example of a waterfall is given in 2.3.1.

1.2.5 Loss Allocation

At defaults in the asset pool, the aggregate outstanding principal amount of the pool is reduced by the defaulted assets outstanding principal amount. There are basically two different ways to distribute these losses in the pool to the note investors: either direct or indirect. In a structure where losses are directly allocated to the note investors, the losses are allocated according to *reverse order of seniority*, which means that the most subordinated notes are first suffering reduction in principal amount. This affects the subordinated note investors directly in two ways: loss of invested capital and a reduction of the coupon payments, since the coupon is based on the note's outstanding principal balance.

On the other hand, as already mentioned above in the description of structural characteristics, an amount equal to the principal balance of defaulted assets can be added to the note redemption amount in an amortising structure to make sure that the asset side and the liability side is at par. In a revolving structure, this amount is added to the replenishment amount instead. In either case, the defaulted principal amount to be added is taken from the excess spread (see Credit enhancement subsection below).

In an amortising structure with sequential allocation of principal, this method will reduce the coupon payments to the senior note investors while the subordinated notes continue to collect coupons based on the full principal amount (as long as there is enough available funds at that level in the priority of payments). Any potential principal losses are not recognised until the final maturity of the notes.

1.2.6 Credit Enhancement

Credit enhancements are techniques used to improve the credit quality of a bond and can be provided both internally as externally.

The *internal* credit enhancement is provided by the originator or from within the deal structure and can be achieved through several different methods: *subordination*, *reserve fund*, *excess spread*, *over-collateralisation*. The *subordination structure* is the main internal credit enhancement. Through the tranching of the liabilities a subordination structure is created and a priority of payments (the waterfall) is setup, controlling the allocation of the cashflows from the asset pool to the securities in order of seniority.

Over-collateralisation means that the total nominal value of the assets in the collateral pool is greater than the total nominal value of the asset backed securities issued, or that the assets are sold with a discount. Over-collateralisation creates a cushion which absorbs the initial losses in the pool.

The *excess spread* is the difference between the interest and revenues collected from the assets and the senior expenses (for example, issuer expenses and servicer fees) and interest on the notes paid during a month.

Another internal credit enhancement is a *reserve fund*, which could provide cash to cover interest or principal shortfalls. The reserve fund is usually a percentage of the initial or outstanding aggregate principal amount of the notes (or assets). The reserve fund can be funded at closing by proceeds and reimbursed via the waterfall.

When a third party, not directly involved in the securitisation process, is providing guarantees on an asset backed security we speak about an *external* credit enhancement. This could be, for example, an insurance company or a monoline insurer providing a surety bond. The financial guarantor guarantees timely payment of interest and timely or ultimate payment of principal to the notes. The guaranteed securities are typically given the same rating as the insurer. External credit enhancement introduces counterparty risk since the asset backed security now relies on the credit quality of the guarantor. Common monoline insurers are Ambac Assurance Corporation, Financial Guaranty Insurance Company (FGIC), Financial Security Assurance (FSA) and MBIA, with the in the press well documented credit risks and its consequences (see, for example, KBC's exposure to MBIA).

1.3 ABS Risk A-B-C

Due to the complex nature of securitisation deals there are many types of risks that have to be taken into account. The risks arise from the collateral pool, the structuring of the liabilities, the structural features of the deal and the counterparties in the deal.

The main types of risks are *credit risk*, *prepayment risk*, *market risks*, *reinvestment risk*, *liquidity risk*, *counterparty risk*, *operational risk* and *legal risk*.

1.3.1 Credit Risk

Beginning with credit risk, this type of risk originates from both the collateral pool and the structural features of the deal. That is, both from the losses generated in the asset pool and how these losses are mitigated in the structure.

Defaults in the collateral pool results in loss of principal and interest. These losses are transferred to the investors and allocated to the notes, usually in reverse order of seniority either directly or indirectly, as described in Section 1.2.5.

In the analysis of the credit risks, it is very important to understand the underlying assets in the collateral pool. Key risk factors to take into account when analyzing the deal are:

- asset class(-es) and characteristics: asset types, payment terms, collateral and collateralisation, seasoning and remaining term;
- diversification: geographical, sector and borrower;
- asset granularity: number and diversification of the assets;
- asset homogeneity or heterogeneity;

An important step in assessing the deal is to understand what kind of assets the collateral pool consists of and what the purpose of these assets are. Does the collateral pool consist of short term loans to small and medium size enterprizes where the purpose of the loans are working capital, liquidity and import financing, or do we have in the pool residential mortgages? The asset types and purpose of the assets will influence the overall behavior of the pool and the ABS.

If the pool consists of loan receivables, the loan type and type of collateral is of interest for determining the loss given default or recovery. Loans can be of unsecured, partially secured and secured type, and the collateral can be real estates, inventories, deposits, etc. The collateralisation level of a pool can be used for the recovery assumption.

A few borrowers that stands for a significant part of the outstanding principal amount in the pool can signal a higher or lower credit risk than if the pool consisted of a homogeneous borrower concentration. The same is true also for geographical and sector concentrations.

The granularity of the pool will have an impact on the behavior of the pool and thus the ABS, and also on the choice of methodology and models to assess the ABS. If there are many assets in the pool it can be sufficient to use a top-down approach modeling the defaults and prepayments on a portfolio level, while for a non-granular portfolio a bottom-up approach, modeling each individual asset in the pool, can be preferable. From a computational point of view, a bottom-up approach can be hard to implement if the portfolio is granular. (Moody's, for example, are using two different methods: factor models for non-granular portfolios and Normal Inverse default distribution and Moody's ABSROM™ for granular, see Section 5.2.)

1.3.2 Prepayment Risk

Prepayment is the event that a borrower prepays the loan prior to the scheduled repayment date. Prepayment takes place when the borrower can benefit from it, for example, when the borrower can refinance the loan to a lower interest rate at another lender.

Prepayments result in loss of future interest collections because the loan is paid back prematurely and can be harmful to the securities, specially for long term securities.

A second, and maybe more important consequence of prepayments, is the influence of unscheduled prepayment of principal that will be distributed among the securities according to the priority of payments, reducing the outstanding principal amount, and thereby affecting their weighted average life. If an investor is concerned about a shortening of the term we speak about *contraction risk* and the opposite would be the *extension risk*, the risk that the weighted average life of the security is extended.

In some circumstances, it will be borrowers with good credit quality that prepay and the pool credit quality will deteriorate as a result. Other circumstances will lead to the opposite situation.

1.3.3 Market Risk

The market risks can be divided into: *cross currency risk* and *interest rate risk*.

The collateral pool may consist of assets denominated in one or several currencies different from the liabilities, thus the cash flow from the collateral pool has to be exchanged to the liabilities' currency, which implies an exposure to exchange rates. This risk can be hedged using currency swaps.

The interest rate risk can be either *basis risk* or interest rate *term structure risk*. *Basis risk* originates from the fact that the assets and the liabilities may be indexed to different benchmark indexes. In a scenario where there is an increase in the liability benchmark index that is not followed by an increase in the collateral benchmark index there might be a lack of interest collections from the collateral pool, that is, interest shortfall.

The interest rate *term structure risk* arise from a mismatch in fixed interest collections from the collateral pool and floating interest payments on the liability side, or vice versa.

The basis risk and the term structure risk can be hedge with interest rate swaps.

Currency and interest hedge agreements introduce counterparty risk (to the swap counterparty), discussed later on in this section.

1.3.4 Reinvestment Risk

There exists a risk that the portfolio credit quality deteriorates over time if the portfolio is replenished during a revolving period. For example, the new assets put into the pool can generate lower interest collections, or shorter remaining term, or will influence the diversification (geographical, sector and borrower) in the pool, which potentially increases the credit risk profile.

These risks can partly be handled through eligibility criteria to be compiled by the new replenished assets such that the quality and characteristics of the initial pool are maintained. The eligibility criteria are typically regarding diversification and granularity: regional, sector and borrower concentrations; and portfolio characteristics such as the weighted average remaining term and the weighted average interest rate of the portfolio.

Moody's reports that a downward portfolio quality migration has been observed in asset backed securities with collateral pools consisting of loans to small and medium size enterprizes where no efficient criteria were used (see Moody's (2007d)).

A second common feature in replenishable transactions is a set of early amortisation triggers created to stop replenishment in case of serious delinquencies or defaults event. These triggers are commonly defined in such a way that replenishment is stopped and the notes are amortized when the cumulative delinquency rate or cumulative default rate breaches a certain level. More about performance triggers follow later.

1.3.5 Liquidity Risk

Liquidity risk refers to the timing mismatches between the cashflows generated in the asset pool and the cashflows to be paid to the liabilities. The cashflows can be either interest, principal or both. The timing mismatches can occur due to maturity mismatches, i.e., a mismatch between scheduled amortisation of assets and the scheduled note redemptions, to rising number of delinquencies, or because of delays in transferring money within the transaction. For interest rates there can be a mismatch between interest payment dates and periodicity of the collateral pool and interest payments to the liabilities.

1.3.6 Counterparty Risk

As already mentioned the servicer is a key party in the structure and if there is a negative event affecting the servicer's ability to perform the cash collections from the asset pool, distribute the cash to the investors and handling delinquencies and defaults, the whole structure is put under pressure. Cashflow disruption due to servicer default must be viewed as a very severe event, especially in markets where a replacement servicer may be hard to find. Even if a replacement servicer can be found relatively easy, the time it will take for the new servicer to start performing will be crucial.

Standard and Poor's consider scenarios where the servicer may be unwilling or unable to perform its duties and a replacement servicer has to be found when rating a structured finance transaction. Factors that may influence the likelihood of a replacement servicer's availability and willingness to accept the assignment are: *"... the sufficiency of the servicing fee to attract a substitute servicer, the seniority of the servicing fee in the transaction's payment waterfall, the availability of alternative servicers in the sector or region, and specific characteristics of the assets and servicing platform that may hinder an orderly transition of servicing functions to another party."*[1]

Originator default can cause severe problems to a transaction where replenishment is allowed, since new assets cannot be put into the collateral pool.

Counterparty risk arises also from third-parties involved in the transaction, for example, interest rate and currency swap counterparties, financial guarantors and liquidity or credit support facilities. The termination of a interest rate swap agreement, for example, may expose the issuer to the risk that the amounts received from the asset pool might not be enough for the issuer to meet its obligations in respect of interest and principal payments due under the notes. The failure of a financial guarantor to fulfill its obligations will directly affect the guaranteed note. The downgrade of a financial guarantor will have an direct impact on the structure, which has been well documented in the past years.

To mitigate counterparty risks, structural features, such as, rating downgrade triggers, collateralisation remedies, and counterparty replacement, can be present in the structure to (more or less) de-link the counterparty credit risk from the credit risk of the transaction.

The rating agencies analyse the nature of the counterparty risk exposure by reviewing both the counterparty's credit rating and the structural features incorporated in the transaction. The rating agencies analyses are based on counterparty criteria frameworks detailing the key criteria to be fulfilled by the counterparty and the structure.[2]

[1] Standard and Poor's (2007b) p. 4.

[2] See Standard and Poor's (2007a), Standard and Poor's (2008a), Standard and Poor's (2009c), and Moody's (2007c).

1.3.7 Operational Risk

This refers partly to reinvestment risk, liquidity risk and counterparty risk, which was already discussed earlier. However, operational risk also includes the origination and servicing of the assets and the handling of delinquencies, defaults and recoveries by the originator and/or servicer.

The rating agencies conducts a review of the servicer's procedures for, among others, collecting asset payments, handling delinquencies, disposing collateral, and providing investor reports.[3]

The originator's underwriting standard might change over time and one way to detect the impact of such changes is by analysing trends in historical delinquency and default data.[4] Moody's remarks that the underwriting and servicing standards typically have a large impact on cumulative default rates and by comparing historical data received from two originators active in the same market over a similar period can be a good way to assess the underwriting standard of originators: *"Differences in the historical data between two originators subject to the same macro-economic and regional situation may be a good indicator of the underwriting (e.g. risk appetite) and servicing standards of the two originators."*[5]

1.3.8 Legal Risks

The key legal risks are associated with the transfer of the assets from the originator to the issuer and the bankruptcy remoteness of the issuer. The transfer of the assets from the originator to the issuer must be of such a kind that an originator insolvency or bankruptcy does not impair the issuer's rights to control the assets and the cash proceeds generated by the asset pool. This transfer of the assets is typically done through a "true sale".

The bankruptcy remoteness of the issuer depends on the corporate, bankruptcy and securitisation laws of the relevant legal jurisdiction.

[3] Moody's (2007b) and Standard and Poor's (2007b)
[4] Moody's (2005b) p. 8.
[5] Moody's (2009a) p. 7.

Chapter 2

Cashflow modelling

2.1 Introduction

The modelling of the cash flows in an ABS deal consists of two parts: the modelling of the cash collections from the asset pool and the distribution of the collections to the note holders and other transaction parties.

The first step is to model the cash collections from the asset pool, which depends on the behaviour of the pooled assets. This can be done in two ways: with a top-down approach, modelling the aggregate pool behaviour; or with a bottom-up approach modelling each individual loan. For the top-down approach one assumes that the pool is homogeneous, that is, each asset behaves as the average representative of the assets in the pool (a so called *representative line analysis* or *repline analysis*). For the bottom-up approach one can chose to use either the representative line analysis or to model each individual loan (so called *loan level analysis*). If a top-down approach is chosen, the modeller has to choose between modelling defaulted and prepaid assets or defaulted and prepaid principal amounts, i.e., to count assets or money units.

On the liability side one has to model the waterfall, that is, the distribution of the cash collections to the note holders, the issuer, the servicer and other transaction parties.

In this section we make some general comments on the cash flow modelling of ABS deals. The case studies presented later in this report will highlight the issues discussed here.

2.2 Asset Behaviour

The assets in the pool can be categorised as performing, delinquent, defaulted, repaid and prepaid. A *performing asset* is an asset that pays interest and principal in time during a collection period, i.e. the asset is current. An asset that is in arrears with one or several interest and/or principal payments is *delinquent*. A delinquent asset can be cured, i.e. become a performing asset again, or it can become a *defaulted* asset. Defaulted assets goes into a recovery procedure and after a time lag a portion of the principal balance of the defaulted assets are recovered. A defaulted asset is never cured, it is once and for all removed from the pool. When an asset is fully amortised according to its amortisation schedule, the asset is *repaid*. Finally, an asset is *prepaid* if it is fully amortised prior to its amortisation schedule.

The cash collections from the asset pool consist of *interest collections* and *principal collections* (both scheduled repayments, unscheduled prepayments and recoveries). There are two parts of the modelling of the cash collections from the asset pool. Firstly, the modelling of performing assets, based on asset characteristics such as initial principal balance, amortisation scheme, interest rate and payment frequency and remaining term. Secondly, the modelling of the assets becoming delinquent, defaulted and prepaid, based on assumptions about the delinquency rates, default rates and prepayment rates together with recovery rates and recovery lags.

The characteristics of the assets in the pool are described in the Offering Circular and a summary can usually be found in the rating agencies pre-sale or new issue reports. The aggregate pool characteristics described are among others the total number of assets in the pool, current balance, weighted average remaining term, weighted average seasoning and weighted average coupon. The distribution of the assets in the pool by seasoning, remaining term, interest rate profile, interest payment frequency, principal payment frequency, geographical location, and industry sector are also given. Out of this pool description the analyst has to decide if to use a representative line analysis assuming a homogeneous pool, to use a loan-level approach modelling the assets individually or take an approach in between modelling sub-pools of homogeneous assets. In this report we focus on large portfolios of assets, so the homogeneous portfolio approach (or homogeneous sub-portfolios) is the one we have in mind.

For a homogeneous portfolio approach the average current balance, the weighted average remaining term and the weighted average interest rate (or spread) of the assets are used as input for the modelling of the performing assets. Assumptions on interest payment frequencies and principal payment frequencies can be based on the information given in the offering circular.

Assets in the pool can have fixed or floating interest rates. A *floating interest rate* consists of a base rate and a margin (or spread). The base rate is indexed to a reference rate and is reset periodically. In case of floating rate assets, the weighted average margin (or spread) is given in the offering circular. Fixed interest rates can sometimes also be divided into a base rate and a margin, but the base rate is fixed once and for all at the closing date of the loan receivable.

The scheduled repayments, or amortisations, of the assets contribute to the principal collections and has to be modelled. Assets in the pool might amortise with certain payment frequency (monthly, quarterly, semi-annually, annually) or be of bullet type, paying back all principal at the scheduled asset maturity, or any combination of these two (soft bullet).

The modelling of non-performing assets requires default and prepayment models which takes as input assumptions about delinquency, default, prepayment and recovery rates. These assumptions have to be made on the basis of historical data, geographical distribution, obligor and industry concentration, and on assumptions about the future economical environment. Several default and prepayment models will be described in the next chapter.

We end this section with a remark about delinquencies. *Delinquencies* are usually important for a deal's performance. A delinquent asset is usually defined as an asset that has failed to make one or several payments (interest or principal) on scheduled payment dates. It is common that delinquencies are categorised in time buckets, for example, in 30+ (30-59), 60+ (60-89), 90+ (90-119) and 120+ (120-) days overdue. However, the exact timing when a loan becomes delinquent and the reporting method used by the servicer will be important for the classification of an asset to be current or delinquent and also for determining the number of payments past due, see Moody's (2000a).

2.2.1 Example: Static Pool

As an example of cashflow modelling we look at the cashflows from a static, homogeneous asset pool of loan receivables.

We model the cashflow monthly and denote by t_m, $m = 0, 1, \ldots, M$ the payment date at the end of month m, with $t_0 = 0$ being the closing date of the deal and $t_M = T$ being the final legal maturity date.

The cash collections each month from the asset pool consists of interest payments and principal collections (scheduled repayments and unscheduled prepayments). These collections constitutes, together with the principal balance of the reserve account, available funds.

The number of performing loans in the pool at the end of month m will be denoted by $N(t_m)$. We denote by $n_D(t_m)$ and $n_P(t_m)$ the number of defaulted loans and the number of prepaid loans, respectively, in month m.

The first step is to generate the scheduled outstanding balance of and the cash flows generated by a performing loans. After this is done one can compute the aggregate pool cash flows.

Defaulted Principal

Defaulted principal is based on previous months ending principal balance times number of defaulted loans in current month:

$$P_D(t_m) = B(t_{m-1}) \cdot n_D(t_m),$$

where $B(t_m)$ is the (scheduled) outstanding principal amount at time t_m of an individual loan and $B(0)$ is the initial outstanding principal amount.

Interest Collections

Interest collected in month m is calculated on performing loans, i.e., previous months ending number of loans less defaulted loans in current month:

$$I(t_m) = (N(t_{m-1}) - n_D(t_m)) \cdot B(t_m) \cdot r_L,$$

where $N(0)$ is the initial number of loans in the portfolio and r_L is the loan interest rate. It is assumed that defaulted loans pay neither interest nor principal.

Principal Collections

Scheduled repayments are based on the performing loans from the end of previous month less defaulted loans:

$$P_{SR}(t_m) = (N(t_{m-1}) - n_D(t_m)) \cdot B_A(t_m),$$

where $B_A(t_m)$ is scheduled principal amount paid from one single loan.

Prepayments are equal to the number of prepaid loans times the ending loan balance. This means that we first let all performing loans repay their scheduled principal and then we assume that the prepaying loans pay back the outstanding principal after scheduled repayment has taken place:

$$P_{UP}(t_m) = B(t_m) \cdot n_P(t_m),$$

where $B(t_m) = B(t_{m-1}) - B_A(t_m)$

Recoveries

We will recover a fraction of the defaulted principal after a time lag, T_{RL}, the recovery lag:

$$P_{Rec}(t_m) = P_D(t_m - T_{RL}) \cdot RR(t_m - T_{RL}),$$

where RR is the Recovery Rate.

Available Funds

The available funds in each month, assuming that total principal balance of the cash reserve account (B_{CR}) is added, is:

$$I(t_m) + P_{SR}(t_m) + P_{UP}(t_m) + P_{Rec}(t_m) + B_{CR}(t_m).$$

In this example we combine all positive cash flows from the pool into one single available funds assuming that these funds are distributed according to a combined waterfall. In a structure with separate interest and principal waterfalls we instead have interest available funds and principal available funds.

Total Principal Reduction

The total outstanding principal amount of the asset pool has decreased with:

$$P_{Red}(t_m) = P_D(t_m) + P_{SR}(t_m) + P_{UP}(t_m),$$

and to make sure that the Notes remain fully collateralised we have to reduce the outstanding principal amount of the notes with the same amount.

2.2.2 Revolving Structures

A revolving period adds an additional complexity to the modelling because new assets are added to the pool. Typically each new subpool of assets should be handled individually, modelling defaults and prepayments separately, because the assets in the different subpools will be in different stages of their default history. Default and prepayment rates for the new subpools might also be assumed to be different for different subpools.

Assumptions about the characteristics of each new subpool of assets added to the pool have to be made in view of interest rates, remaining term, seasoning, and interest and principal payment frequencies. To do this, the pool characteristics at closing together with the eligibility criteria for new assets given in the offering circular can be of help.

2.3 Structural Features

The key structural features discussed earlier in Chapter 1: structural characteristics, priority of payments, loss allocation, credit enhancements, and triggers, all have to be taken into account when modelling the liability side of an ABS deal. So does the basic information on the notes legal final maturity, payment dates, initial notional amounts, currency, and interest rates. The structural features of a deal are detailed in the *offering circular*.

In the following example a description of the waterfall in a transaction with two classes of notes is given.

2.3.1 Example: Two Note Structure

Assume that the asset pool described earlier in this chapter is backing a structure with three classes of notes: A (senior) and B(junior). The class A notes constitutes 80% of the initial amount of the pool and the class B notes 20%.

The waterfall of the structure is presented in Table 2.1. The waterfall is a so called combined waterfall where the available funds at each payment date constitutes of both interest and principal collections.

1) Senior Expenses

On the top of the waterfall are the senior expenses that are payments to the transaction parties that keeps the transaction functioning, such as, servicer and trustee. In out example we assume that the first item consists of only the servicing fee, which is based on the ending asset pool principal balance in previous month multiplied by the servicing fee rate, plus any shortfall in the servicing fee from previous months multiplied with the servicing fee shortfall rate. After the servicing fee has been paid we update available funds, which is either zero or the initial available funds less the servicing fee paid, which ever is greater.

Waterfall	
Level	Basic amortisation
1)	Senior expenses
2)	Class A interest
3)	Class B interest
4)	Class A principal
5)	Class B principal
6)	Reserve account reimburs.
7)	Residual payments

Table 2.1: *Example waterfall.*

2) Class A Interest

The *Class A Interest Due* is the sum of the outstanding principal balance of the A notes at the beginning of month m (which is equal to the ending principal balance in month $m-1$) plus any shortfall from previous month multiplied by the A notes interest rate. We assume the interest rate on shortfalls is the same as the note interest rate. The *Class A Interest Paid* is the minimum of available funds from level 1 and the Class A Interest Due. If there was not enough available funds to cover the interest payment, the shortfall is carried forward to the next month. After the Class A interest payment has been made we update available funds. If there is a shortfall, the available funds are zero, otherwise it is available funds from level 1 less Class A Interest Paid.

3) Class B Interest

The Class B interest payment is calculated in the same way as the Class A interest payment.

4) Class A Principal

The principal payment to the Class A Notes and the Class B Notes are based on the *note replenishment amount*. How this amount is distributed depends on the allocation method used. If *pro rata* allocation is applied, the notes share the principal reduction in proportion to their fraction of the total initial outstanding principal amount. In our case, 80% of the available funds should be allocated to the Class A Notes. The *Class A Principal Due* is this allocated amount plus any shortfall from previous month.

On the other hand if we apply *sequential* allocation, the Class A Principal Due is the minimum of the outstanding principal amount of the A notes and the sum of the note redemption amount and any Class A Principal Shortfall from previous month, that is, we should first redeem the A notes until zero before we redeem the B notes.

The *Class A Principal Paid* is the minimum of the available funds from level 3 and the Class

A Principal Due. The available funds after principal payment to Class A is zero or the difference between available funds from level 3 and Class A Principal Paid, which ever is greater. Note that if there is a shortfall available funds equal zero.

5) Class B Principal

If *pro rata* allocation is applied, the *Class B Principal Due* is the allocated amount (20% of the available funds in our example) plus any shortfall from previous month.

The Class B Principal Due under a *sequential* allocation scheme is zero as long as the Class A Notes are not redeemed completely. After that the Class B Principal Due is the minimum of the outstanding principal amount of the B notes and the sum of the principal reduction of the asset pool and any principal shortfall from previous month.

The *Class B Principal Paid* is the minimum of the available funds from level 4 and the Class B Principal Due. The available funds after principal payment to note B is zero or the difference between available funds from level 4 and Class B Principal Paid, which ever is greater. Note that if there is a shortfall available funds equal zero.

6) Reserve Account Reimbursement

The principal balance of the reserve account at the end of the month must be restored to the target amount, which in our example is 5% of the outstanding balance of the asset pool. If enough available funds exists after the Class B principal payment, the reserve account is fully reimbursed, otherwise the balance of the reserve account is equal to the available funds after level 5 and a shortfall is carried forward.

7) Residual Payments

Whatever money that is left after level 6 is paid as a residual payment to the issuer.

Loan Loss Allocation

If loan losses are allocated in reverse order of seniority, the notes outstanding principal amounts first have to be adjusted before any calculations of interest and principal due. The pro rata allocation method will have one additional change, the principal due to the Class A Notes and Class B Notes must now be based on the current outstanding balance of the notes after loss allocation.

Pari Passu

In the above waterfall Class A Notes interest payments are ranked senior to Class B Notes interest payments. Assume that the interest payments to Class A Notes and Class B Notes are paid *pari passu* instead. Then Class A Notes and Class B Notes have equal right to the available funds after level 1, and level 2 and 3 in the waterfall become effectively one level. Similarly, we can also assume that class A and class B principal due are allocated pro rata and paid pari passu.

For example, assume that principal due in month m to Class A Notes and Class B Notes is $P_{AD}(t_m) = 75$ and $P_{AD}(t_m) = 25$, respectively, and that the available amount after level 3 is $F_3(t_m) = 80$. In the original waterfall, Class A receives all its due principal and available amount after Class A principal is $F_4(t_m) = 5$. Class B receives in this case $P_{BP}(t_m) = 5$ and the shortfall is $P_{BS}(t_m) = 20$. If payments are done pari passu instead, Class A receives $P_{AP}(t_m) = 80 * 75/100 = 60$ and Class B $P_{BP}(t_m) = 80 * 25/100 = 20$, leading to a shortfall of $P_{AS}(t_m) = 20$ for Class A and $P_{BS}(t_m) = 5$ for Class B.

Chapter 3

Deterministic Models

3.1 Introduction

To be able to assess ABS deals one need to model the defaults and the prepayments in the underlying asset pool. The models discussed all refer to *static* pools.

Traditional models for these risks are the Logistic default model, the Conditional (or Constant) Default Rate model and the Conditional (Constant) Prepayment Rate model.

We focus on the time interval between the issue ($t = 0$) of the ABS notes and the weighted average maturity of the underlying assets (T).

The **default curve**, $P_d(t)$, refers to the default term structure, i.e., the cumulative default rate at time t (expressed as percentage of the initial outstanding principal amount of the asset pool or as the fraction of defaulted loans). By the **default distribution**, we mean the (probability) distribution of the cumulative default rate at time T.

The **prepayment curve**, $P_p(t)$, refers to the prepayment term structure, i.e., the cumulative prepayment rate at time t (expressed as percentage of the initial outstanding principal amount of the asset pool or as the fraction of prepaid loans). By the **prepayment distribution**, we mean the distribution of the cumulative prepayment rate at time T.

There are two approaches to choose between when modelling the defaults and prepayments: the top-down approach (portfolio level models) and the bottom-up approach (loan level models). In the top-down approach (portfolio level models) one model the cumulative default and prepayment rates of the portfolio. This is exactly what is done with the traditional models we shall present later in this chapter. In the bottom-up approach (loan level models) one models, to the contrary to the top-down approach, the individual loans default and prepayment behavior. Probably the most well-known loan level models are the factor or copula models, which are presented in the following chapter.

The choice of approach depends on several factors, such as, the number of assets in the reference pool and the homogeneity of the pool, see the discussion on the rating agencies methodologies in Chapter 5.

3.2 Default Modelling

3.2.1 Conditional Default Rate

The *Conditional* (or *Constant*) *Default Rate* (CDR) approach is the simplest way to use to introduce defaults in a cash flow model. The CDR is a sequence of (constant) annual default rates applied to the outstanding pool balance in the beginning of the time period, hence the model is conditional on the pool history and therefore called conditional. The CDR is an annual default rate that can be translated into a monthly rate by using the single-monthly mortality (SMM) rate:

$$SMM = 1 - (1 - CDR)^{1/12}.$$

The SMM rates and the corresponding cumulative default rates for three values of CDR $(2.5\%, 5\%, 7.5\%)$ are shown in Figure 3.1. The CDRs were applied to a pool of asset with no scheduled repayments or unscheduled prepayments, i.e., the reduction of the principal balance originates from defaults only.

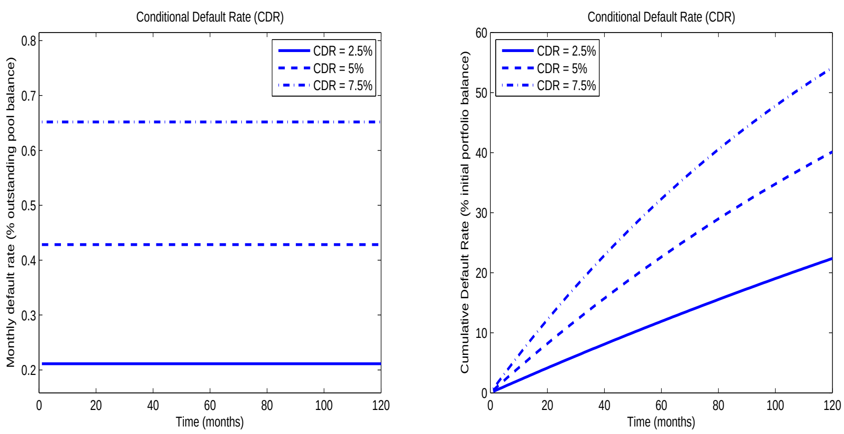

Figure 3.1: *Left panel: Single monthly mortality rate. Right panel: Cumulative default rates. The underlying pool contains non-amortising assets with no prepayments.*

An illustration of the CDR approach is given in Table 3.1 with SMM equal to 0.2%.

It is common to report historical defaults (defaulted principal amounts) realised in a pool in terms of CDRs, monthly or quarterly. To calculate the CDR for a specific month, one first calculates the monthly default rate as defaulted principal balance during the month divided by the outstanding principal balance in the beginning of the month less scheduled principal repayments during the month. This monthly default rate is then annualised

$$CDR = 1 - (1 - SMM)^{12}. \tag{3.1}$$

Month	Pool balance (beginning)	Defaulted principal	SMM (%)	Cumulative default rate (%)
1	100,000,000	200,000	0.20	0.2000
2	99,800,000	199,600	0.20	0.3996
3	99,600,400	199,201	0.20	0.5988
⋮	⋮	⋮	⋮	⋮
58	89,037,182	178,431	0.20	10.9628
59	88,859,108	178,074	0.20	11.1409
60	88,681,390	177,718	0.20	11.3186
61	88,504,027	177,363	0.20	11.4960
62	88,327,019	177,008	0.20	11.6730
⋮	⋮	⋮	⋮	⋮
119	78,801,487	157,919	0.20	21.1985
120	78,643,884	157,603	0.20	21.3561

Table 3.1: *Illustration of Conditional Default Rate approach. The single monthly mortality rate is fixed to* 0.2%*. No scheduled principal repayments or prepayments from the asset pool.*

Strengths and Weaknesses

The CDR models is simple, easy to use and it is straight forward to introduce stresses on the default rate. It is even possible to use the CDR approach to generate default scenarios, by using a probability distribution of the cumulative default rate. However, it is too simple, since it assumes that the default rate is constant over time.

3.2.2 The Default Vector Model

In the default vector approach, the total cumulative default rate is distributed over the life of the deal according to some rule. Hence, the *timing* of the defaults is modelled. Assume, for example, that 24% of the initial outstanding principal amount is assumed to default over the life of the deal, that is, the cumulative default rate is 24%. We could distribute these defaults uniformly over the life of the deal, say 120 months, resulting in assuming that 0.2% of the initial principal balance defaults each month. If the initial principal balance is euro 100 million and we assume 0.2% of the initial balance to default each month we have euro $200,000$ defaulting in every month. The first three months, five months in the middle and the last two months are shown in Table 3.2.

Note that this is not the same as the SMM given above in the Conditional Default Rate approach, which is the percentage of the outstanding principal balance in the *beginning* of the month that defaults. To illustrate the difference compare Table 3.1 (0.2% of the outstanding pool balance in the beginning of the month defaults) above with Table 3.2 (0.2% of the initial outstanding pool balance defaults each month). The SMM in Table 3.2 is calculated as the ratio of defaulted principal (200, 000) and the outstanding portfolio balance at the beginning of the month. Note that the SMM in Table 3.2 is increasing due to the fact that the outstanding portfolio balance is decreasing while the defaulted principal amount is fixed.

Month	Pool balance (beginning)	Defaulted principal	SMM (%)	Cumulative default rate (%)
1	100,000,000	200,000	0.2000	0.20
2	99,800,000	200,000	0.2004	0.40
3	99,600,000	200,000	0.2008	0.60
⋮	⋮	⋮	⋮	⋮
58	88,600,000	200,000	0.2257	11.60
59	88,400,000	200,000	0.2262	11.80
60	88,200,000	200,000	0.2268	12.00
61	88,000,000	200,000	0.2273	12.20
62	87,800,000	200,000	0.2278	12.40
⋮	⋮	⋮	⋮	⋮
119	76,400,000	200,000	0.2618	23.8
120	76,200,000	200,000	0.2625	24.0

Table 3.2: *Illustration of an uniformly distribution of the cumulative default rate (*24% *of the initial pool balance) over 120 months, that is, each month* 0.2% *of the initial pool balance is assumed to default. No scheduled principal repayments or prepayments from the asset pool.*

Of course many other default timing patterns are possible. Moody's methodology to rate granular portfolios is one such example, where default timing is based on historical data, see Section 5.2. S&P's apply this approach in its default stress scenarios in the cash flow analysis, see Section 5.3.

Strengths and Weaknesses

Easy to use and to introduce different default timing scenarios, for example, front-loaded or back-loaded. The approach can be used in combination with a scenario generator for the cumulative default rate.

3.2.3 The Logistic Model

The Logistic default model is used for modelling the default curve, that is, the cumulative default rate's evolution over time. Hence it can be viewed as an extension of the default vector approach where the default timing is given by a functional representation. In its most basic form, the Logistic default model has the following representation:

$$P_d(t) = \frac{a}{(1 + be^{-c(t-t_0)})},$$

where $a, b, c,\ t_0$ are positive constants and $t \in [0, T]$. Parameter a is the asymptotic cumulative default rate; b is a curve adjustment or offset factor; c is a time constant (spreading factor); and t_0 is the time point of maximum marginal credit loss. Note that the Logistic default curve has to be normalised such that it starts at zero (initially no defaults in the pool) and $P_d(T)$ equals the expected cumulative default rate.

From the default curve, which represents the cumulative default rate over time, we can find the marginal default curve, which describes the periodical default rate, by differentiating $P_d(t)$. Figure 1 shows a sample of default curves (left panel) and the corresponding marginal default curves (right panel) with time measured in months. Note that most of the default take place in the middle of the deal's life and that the marginal default curve is centered around month 60, which is due to our choice of t_0. More front-loaded or back-loaded default curves can be created by decreasing or increasing t_0.

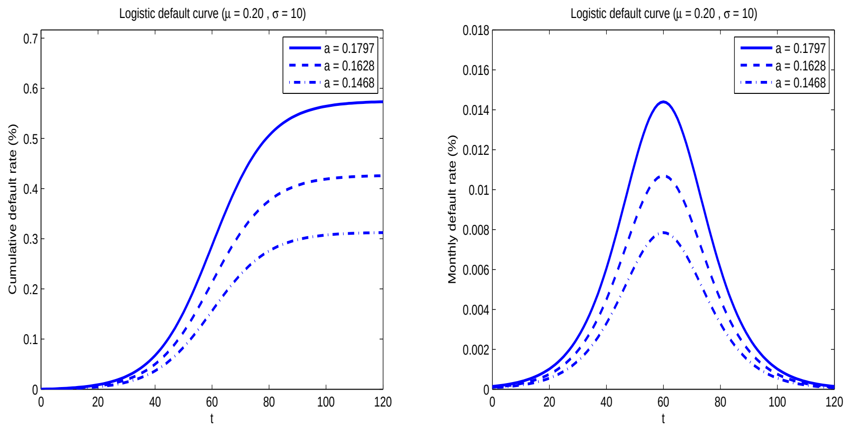

Figure 3.2: *Left panel: Sample of Logistic default curves (cumulative default rates). Right panel: Marginal default curves (monthly default rates). Parameter values: a is sampled from a log-normal distribution (with mean* 20% *and standard deviation* 10%*),* $b = 1$*,* $c = 0.1$ *and* $t_0 = 60$*.*

Table 3.3 illustrates the application of the Logistic default model to the same asset pool that was used in Table 3.2. The total cumulative default rate is 24% in both tables, however, the distribution of the defaulted principal is very different. For the Logistic model, the defaulted principal amount (as well as the SMM) is low in the beginning, very high in the middle and then decays in the second half of the time period. So the bulk of defaults occur in the middle of the deal's life. This is of course due to our choice of $t_0 = 60$. Something which is also evident in Figure 3.2.

Month	Pool balance (beginning)	Defaulted principal	SMM (%)	Cumulative default rate (%)
1	100,000,000	6,255	0.006255	0.006255
2	99,993,745	6,909	0.006909	0.013164
3	99,986,836	7,631	0.007632	0.020795
⋮	⋮	⋮	⋮	⋮
58	89,795,500	593,540	0.660991	10.204500
59	89,201,960	599,480	0.672048	10.798040
60	88,602,480	602,480	0.679981	11.397520
61	88,000,000	602,480	0.684636	12.000000
62	87,397,520	599,480	0.685923	12.602480
⋮	⋮	⋮	⋮	⋮
119	76,006,255	6,909	0.009089	23.993745
120	76,000,000	6,255	0.008230	24.000000

Table 3.3: *Illustration of an application of the Logistic default model. The cumulative default rate is assumed to be* 24% *of the initial pool balance. No scheduled principal repayments or prepayments from the asset pool. Parameter values:* $a = 0.2406$, $b = 1$, $c = 0.1$ *and* $t_0 = 60$.

The model can be extended in several ways. Seasoning could be taken into account in the model and the asymptotic cumulative default rate (a) can be divided into two factors, one systemic factor and one idiosyncratic factor (see Raynes and Ruthledge (2003)).

The Logistic default model thus has (at least) four parameters that have to be estimated from data (see, for example, Raynes and Ruthledge (2003) for a discussion on parameter estimation).

Introducing Randomness

The Logistic default model can easily be used to generate default scenarios. Assuming that we have a default distribution at hand, for example, the log-normal distribution, describing the distribution of the cumulative default rate at maturity T. We can then sample an expected cumulative default rates from the distribution and fit the 'a' parameter such that $P_d(T)$ equals the expected cumulative default rate. Keeping all the other parameters constant. Figure 3.3 shows a sample of Logistic default curves in the left panel, each curve has been generated from a cumulative default rate sampled from the log-normal distribution shown in the right panel.

Strengths and Weaknesses

The model is attractive because the default curve has an explicit analytic expression. With the four parameters (a, b, c, t_0) many different transformations of the basic shape is possible, giving

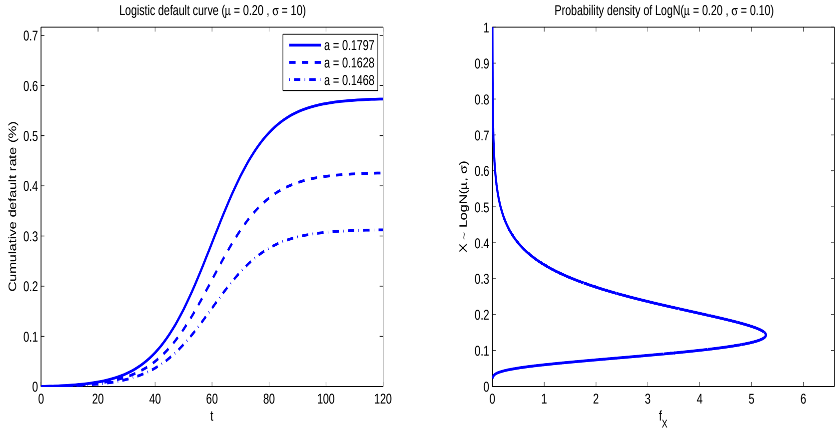

Figure 3.3: *Left panel: Sample of Logistic default curves (cumulative default rates). Parameter values: a is sampled from the log-normal distribution to the right, $b = 1$, $c = 0.1$ and $t_0 = 60$. Right panel: Log-normal default distribution with mean* 0.20 *and standard deviation* 0.10*.*

the user the possibility to create different default scenarios. The model is also easy to implement into a Monte Carlo scenario generator.

The evolutions of default rates under the Logistic default model has some important drawbacks: they are smooth, deterministic and static.

For the Logistic default model most defaults happen gradually and are a bit concentrated in the middle of the life-time of the pool. The change of the default rates are smooth. The model is, however, able of capturing dramatic changes of the monthly default rates.

Furthermore, the model is deterministic in the sense that once the expected cumulative default rate is fixed, there is no randomness in the model.

Finally, the defaults are modelled independently of prepayments.

3.3 Prepayment Modelling

3.3.1 Conditional Prepayment Rate

The *Conditional* (or *Constant*) *Prepayment Rate* (CPR) model is a top-down approach. It models the annual prepayment rate, which one applies to the outstanding pool balance that remains at the end of the previous month, hence the name conditional prepayment rate model. The CPR is an annual prepayment rate, the corresponding monthly prepayment rate is given by the single-monthly mortality rate (SMM) and the relation between the two is:

$$SMM = 1 - (1 - CPR)^{1/12}.$$

Strengths and Weaknesses

The strength of the CPR model lies in it simplicity. It allows the user to easily introduce stresses on the prepayment rate.

A drawback of the CPR model is that the prepayment rate is constant over the life of the deal, implying that the prepayments as measured in euro amounts are largest in the beginning of the deal's life and then decreases. A more reasonable assumption about the prepayment behavior of loans would be that prepayments ramp-up over an initial period, such that the prepayments are larger after the loans have seasoned.[1]

3.3.2 The PSA Benchmark

The *Public Securities Association* (PSA) benchmark for 30-year mortgages[2] is a model which tries to model the seasoning behaviour of prepayments by including a ramp-up over an initial period. It models a monthly series of annual prepayment rates: starting with a CPR of 0.2% for the first month after origination of the loans followed by a monthly increase of the CPR by an additional 0.2% per annum for the next 30 months when it reaches 6% per year, and after that staying fixed at a 6% CPR for the remaining years. That is, the marginal prepayment curve (monthly fraction of prepayments) is of the form:

$$\mathrm{CPR}(t) = \begin{cases} \frac{6\%}{30}t\,, & 0 \le t \le 30 \\ \\ 6\%\,, & 30 < t \le 360, \end{cases}$$

t=1,2,...,360 months. Remember that this is annual prepayment rates. The single-monthly prepayment rates are

$$\mathrm{SMM}(t) = 1 - (1 - \mathrm{CPR}(t))^{1/12}.$$

Speed-up or slow-down of the PSA benchmark is possible:

- 50 PSA means one-half the CPR of the PSA benchmark prepayment rate;
- 200 PSA means two times the CPR of the PSA benchmark prepayment rate.

Strengths and Weaknesses

The possibility to speed-up or slow-down the prepayment speed is giving the model some flexibility.

The PSA benchmark is a deterministic model, with no randomness in the prepayment curve's behaviour. And it assumes that the prepayment rate is changing smoothly over time, it is impossible to model dramatic changes in the prepayment rate of a short time interval, that is,

[1] Discussed in Fabozzi and Kothari (2008) page 33.

[2] The benchmark has been extended to other asset classes such as home equity loans and manufacturing housing, with adjustments to fit the stylized features of those assets, Fabozzi and Kothari (2008).

to introduce the possibility that the prepayment rate suddenly jumps. Finally, under the PSA benchmark the ramp-up of prepayments always takes place during the first 30 months and the rate is after that constant.

3.3.3 A Generalised CPR Model

A generalisation of the PSA benchmark is to model the monthly prepayment rates with the same functional form as the CPR above. That is, instead of assuming that CPR(t) has the functional form above, we assume now that SMM(t) can be described like that. The marginal prepayment curve (monthly fraction of prepayments) is described as follows:

$$p_p(t) = \begin{cases} a_p t \ , & 0 \leq t \leq t_{0p} \\ \\ a_p t_{0p} \ , & t_{0p} < t \leq \mathrm{T}, \end{cases}$$

where a_p is the single-monthly prepayment rate increase.

The prepayment curve, i.e., the cumulative prepayment rate, is found by calculating the area under the marginal prepayment curve:

$$P_p(t) = \begin{cases} \frac{a_p}{2} t^2 \ , & 0 \leq t \leq t_{0p} \\ \\ \frac{a_p}{2} t_{0p}^2 + a_p t_{0p}(t - t_{0p}) \ , & t_{0p} < t \leq \mathrm{T} \end{cases}$$

The model has two parameters:

- t_{0p}: the time where one switches to a constant CPR ($t_{0p} = 30$ months in PSA);
- $P_p(T)$: the cumulative prepayment rate at maturity. For example, $P_p(T) = 0.20$ means that 20% of the initial portfolio have prepaid at maturity T. Can be sampled from a prepayment distribution.

Once the parameters are set, one can calculate the rate increase per month

$$a_p = \frac{P_p(T)}{\frac{t_{0p}^2}{2} + t_{0p}(T - t_{0p})}.$$

Introducing Randomness

The generation of prepayment scenarios can easily be done with the generalised prepayment model introduced above. Assuming that we have a prepayment distribution at hand, for example, the log-normal distribution, describing the distribution of the cumulative prepayment rate at maturity T. We can then sample an expected cumulative prepayment rate from the distribution, and fit the a_p parameter such that $P_p(T)$ equals the expected cumulative prepayment rate. Figure 3.4 shows a sample of marginal prepayment curves and the corresponding cumulative prepayment curves.

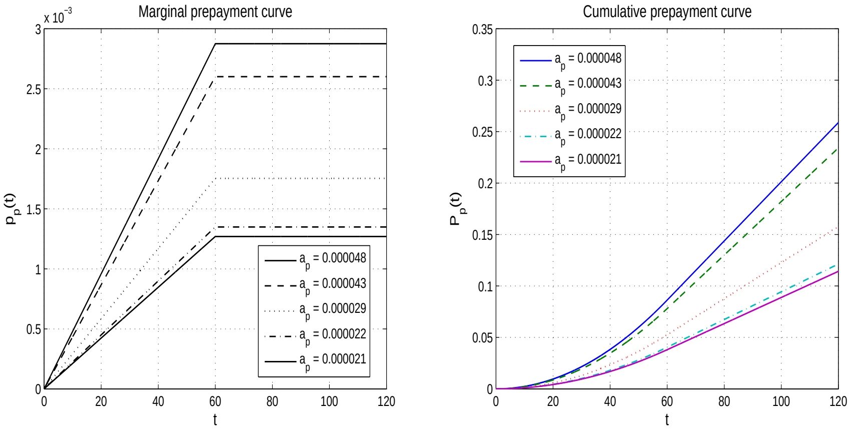

Figure 3.4: *Left panel: Sample of marginal prepayment curves (monthly fraction of prepayments) of the generalised CPR model. Right panel: The corresponding cumulative prepayment curves of the generalised CPR model. The prepayment distribution is assumed to be log-normal. The mean and standard deviation of the empirical prepayment distribution is* $\mu_p = 0.20$ *and* $\sigma_p = 0.10$.

Strengths and Weaknesses

The evolution of prepayment rates under the generalised CPR model is smooth and deterministic. The prepayment curve is smooth, no jumps are present, and it is completely determined once t_{0p} and $P_p(T)$ are chosen. Furthermore, after t_{0p} the model assumes that the prepayment rate is constant.

Chapter 4

Stochastic Models

4.1 Introduction

As was discussed in the previous chapter the traditional default and prepayment models has limited possibilities to capture the stochastic nature of the phenomena they are set to model. In the present chapter we propose a number of models that incorporate the stylized features of defaults and prepayments.

By modelling the evolution of defaults and prepayments with stochastic processes we can achieve three objectives:

- Stochastic timing of defaults and prepayments.
- Stochastic monthly default and prepayment rates.
- Correlation: between defaults; between prepayments; and between defaults and prepayments.

The models we present here can be divided into:

- Portfolio level models (top-down): Lévy Portfolio Models.
- Loan level models (bottom-up): One-factor models (Gaussian and Generic Lévy).

The family of stochastic processes know as Lévy processes is a powerful tool that has been used in financial modelling for quite some time now. In the recent years Lévy processes have been applied in the field of credit risk modelling and credit derivatives pricing, see Schoutens and Cariboni (2009).

4.2 Default Modelling

4.2.1 Lévy Portfolio Default Model

The Lévy portfolio default model models the cumulative default rate on portfolio level. The **default curve**, i.e., the fraction of loans that have defaulted at time t, is given by:

$$P_d(t) = 1 - \exp(-X_t),$$

where $X = \{X_t, t \geq 0\}$ is a stochastic process. Because we are modelling the cumulative default rate the default curve $P_d(t)$ must be non-decreasing over time (since we assume that a defaulted asset is not becoming cured). To achieve this we need to assume that $X = \{X_t, t \geq 0\}$ is non-decreasing over time, since then $\exp(-X_t)$ is non-decreasing. Furthermore, assuming that all assets in the pool are current ($\mathrm{P_d}(0) = 0$) at the time of issue ($t = 0$) we need $X_0 = 0$. Our choice of process comes from the family of stochastic processes called Lévy process, more precisely the single-sided Lévy processes. A single-sided Lévy process is non-decreasing and the increments are through jumps.

By using a stochastic process to "drive" the default curve, $\mathrm{P_d}(t)$ becomes a random variable, for all $t > 0$. In order to generate a default curve scenario, we must first draw a realization of the process $X = \{X_t, t \geq 0\}$. Moreover, $\mathrm{P_d}(0) = 0$, since we start the Lévy process at zero: $X_0 = 0$.

Example: Gamma Portfolio Default Model

As an example, let us consider a default curve based on a Gamma process $G = \{G_t, t \geq 0\}$ with shape parameter a and scale parameter b. The increment from time 0 to time t of the Gamma process, i.e., $G_t - G_0 = G_t$ (recall that $G_0 = 0$) is a Gamma random variable with distribution $\mathrm{Gamma}\,(at, b)$, for any $t > 0$. Consequently, the cumulative default rate at maturity follows the law $1 - \mathrm{e}^{-G_T}$, where $G_T \sim \mathrm{Gamma}\,(aT, b)$. Using this result, the parameters a and b can be found by matching the expected value and the variance of the cumulative default rate under the model to the mean and variance of the default distribution, that is, as the solution to the following system of equations:

$$\begin{aligned} \mathrm{E}\left[1 - \mathrm{e}^{-G_T}\right] &= \mu_d; \\ \mathrm{Var}\left(1 - \mathrm{e}^{-G_T}\right) &= \sigma_d^2, \end{aligned} \tag{4.1}$$

for predetermined values of the mean μ_d and standard deviation σ_d of the default distribution. Explicit expressions for the left hand sides of (4.1) can be found, by noting that the expected value and the variance can be written in terms of the characteristic function of the Gamma distribution.

A sample of Gamma portfolio default curves are shown in Figure 4.1 together with the corresponding default distribution. The mean and standard deviation of the default distribution is $\mu_d = 0.20$ and $\sigma_d = 0.10$, respectively, which implies that $X_T \sim Gamma(aT = 2.99, b =$

12.90). Note that the realisations of the Gamma default curve shown are very different. There is one path that very early has a large jump in the cumulative default rate (above 10% in month 2) and then evolves with a few smaller jumps and ends at about 25% in month 120. In contrast to this path we have a realisation that stays almost at zero until month 59 before jumping to just below 10% and then at month 100 makes a very large jump to around 30%. What is obvious from Figure 4.1 is that the Gamma portfolio default model gives a wide spectrum of default scenarios, from front-loaded default curves to back-loaded.

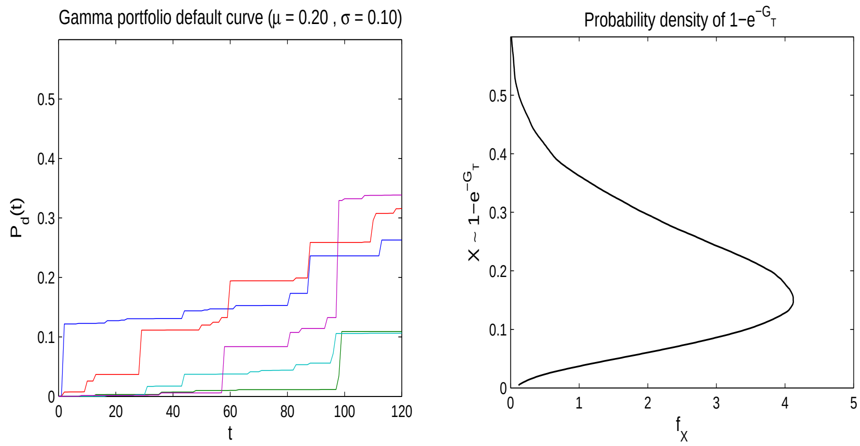

Figure 4.1: *Left panel: Sample of Lévy portfolio default curves. Right panel: corresponding default distribution. The mean and standard deviation of the empirical default distribution is* $\mu_d = 0.20$ *and* $\sigma_d = 0.10$*, respectively, which implies that* $X_T \sim Gamma(aT = 2.99, b = 12.90)$.

Note that the default distribution shown in Figure 4.1 is generated by the model. In contrast, the default distribution in Figure 3.3 is an assumption used to generate default curves, in this case a log-normal distribution.

Strengths and Weaknesses

The Lévy portfolio model is a stochastic portfolio-level approach to model the cumulative default rate. The model gives a wide range of default scenarios, from front-loaded default curves, where a majority of defaults takes place early, to back-loaded. The default curves are jump driven, increasing with random jump sizes.

4.2.2 Normal One-Factor Model

The **Normal one-factor** model (Vasicek (1987) and Li (1995)) models individual loan behaviors and introduce correlation between loans. The model is also used in pricing CDOs and other portfolio credit derivatives and is also called the Gaussian copula model. The link between the Normal one-factor model and the Gaussian copula was made by Li (2000). There is a link between the Normal one-factor model and the structural default model by Merton (1974), which assumes that an obligor defaulted by the maturity of its obligations if the value of the obligor's assets is below the value of its debt. In the Normal one-factor model we model the creditworthiness of an obligor through the use of a latent variable and records a default if the latent variable is below a barrier. The latent variable of an obligor is modelled as:

$$Z_n = \sqrt{\rho}X + \sqrt{1-\rho}X_n, \quad n = 1, 2, \ldots, N, \tag{4.2}$$

where X is the systemic factor and $X_n, n = 1, 2, \ldots, N$ are the idiosyncratic factors, all are standard normal random variables (mean 0, standard deviation 1), and ρ is the correlation between two assets:

$$\mathrm{Corr}(Z_m, Z_n) = \rho, \ m \neq n.$$

The nth loan defaulted by time t if

$$Z_n \leq K_n^d(t),$$

where $K_n^d(t)$ is a preset, time dependent *default barrier*.

If we assume that the pool consist of large number of homogeneous assets, we can use the representative line approach and model each individual asset as the "average" of the assets in the pool. By doing this, we only need to calculate one default barrier $K^d(t)$ and $K_n^d(t) = K^d(t)$ for all n. The default barrier can be chosen such that the default time is exponentially distributed:

$$P\left(Z_n \leq K^d(t)\right) = \Phi_{Z_n}\left(K^d(t)\right) = P\left(\tau < t\right) = 1 - \mathrm{e}^{-\lambda t},$$

where $\Phi_{Z_n}(\cdot)$ is the standard Normal cumulative distribution function. The λ parameter is set such that $P\left(Z_n \leq K^d(T)\right) = \mu_d$, where μ_d is the predetermined value for the mean of the default distribution. Note that $K^d(t)$ is non-decreasing in t, which implies that a defaulted loan stays defaulted and cannot be cured.

The correlation parameter ρ is set such that the standard deviation of the model match the standard deviation of the default distribution at time T, σ_d.

Given a sample $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$ of (correlated) standard Normal random variables, the default curve is then given by

$$\mathrm{P_d}(t; \mathbf{Z}) = \frac{\sharp\left\{Z_n \leq K^d(t); n = 1, 2, ..., N\right\}}{N}, \qquad t \geq 0. \tag{4.3}$$

In order to simulate default curves, one must thus first generate a sample of standard Normal random variables Z_n satisfying (4.2), and then, at each (discrete) time t, count the number of Z_i's that are less than or equal to the value of the default barrier $K^d(t)$ at that time.

The left panel of Figure 4.2 shows five default curves, generated by the Normal one-factor model (4.2) with $\rho \approx 0.121353$, such that the mean and standard deviation of the default distribution are 0.20 and 0.10. We have assumed in this realisation that all assets have the same default barrier. All curves start at zero and are fully stochastic, but unlike the Lévy portfolio model the Normal one-factor default model does not include any jump dynamics. The corresponding default distribution is again shown in the right panel.

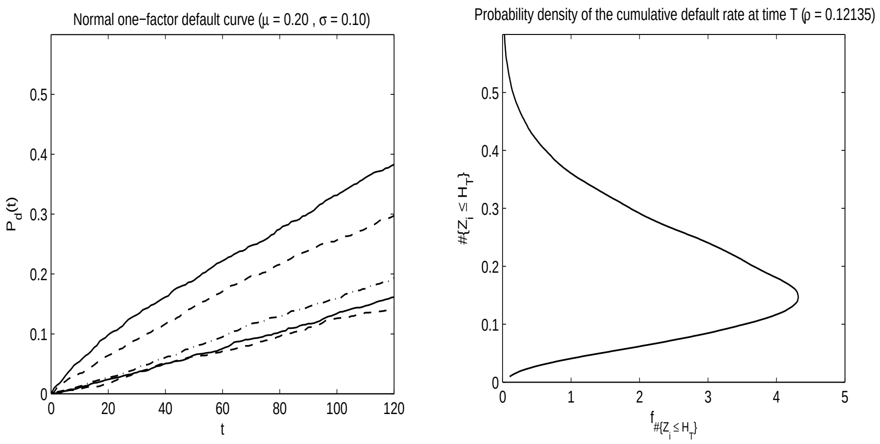

Figure 4.2: *Left panel: Sample of Normal one-factor default curves. Right panel: corresponding default distribution. The mean and standard deviation of the empirical default distribution is* $\mu_d = 0.20$ *and* $\sigma_d = 0.10$.

Just as for the Lévy portfolio default model we would like to point out that the default distribution is generated by the model, in contrast to the Logistic model. In Figure 4.2, an example of a default distribution is shown.

Examples of portfolio default rate (PDR) histograms for different default rate and correlation assumptions are given in Figure 4.3. As can be seen from the plots, changing the correlation assumption from 10% to 20%, keeping the default rate fixed will shift the mass of the distribution towards the lower end of the portfolio default rate range. However, at the same time the probability to have high PDRs increases. From the four plots in Figure 4.3 it is also possible to see the change of the PDR distribution when the correlation is kept fixed and the mean default rate assumption is changed.

Strengths and Weaknesses

The Normal one-factor model is a loan-level approach to modelling the cumulative portfolio default rate. In the loan-level approach one has the freedom to choose between assuming a homogeneous or a heterogeneous portfolio. For a large portfolio with quite homogeneous assets the representative line approach can be used, assuming that each of the assets in the portfolio behaves as the average asset. For a small heterogeneous portfolio it might be better to model the assets on an individual basis.

The Normal one-factor model can be used to model both the default and prepayment of an obligor, which will be evident in the section on prepayment modelling.

A known problem with the Normal one-factor model is that many joint defaults are very unlikely. The underlying reason is the too light tail-behavior of the standard normal distribution (a large number of joint defaults will be caused by a very large negative common factor X).

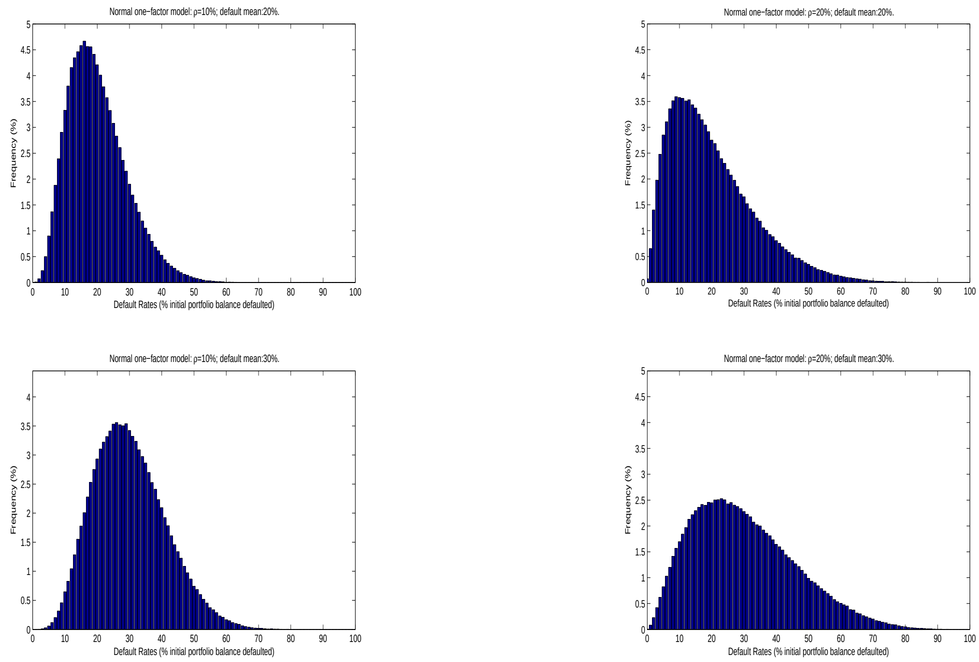

Figure 4.3: *Portfolio default rate (gross loss divided by initial outstanding pool balance) distributions versus correlation and default rate estimated by Monte Carlo simulations of a pool of* $2,000$ *loans using the Normal one-factor model. No prepayments. Bullet amortisation.*

Large Homogeneous Portfolio Approximation

The portfolio default rate (PDR) distribution can be found explicitly for the Normal one-factor model by assuming that the portfolio is homogeneous, which we already do, and consists of a large number of assets. Under these assumptions the distribution is given by

$$F_{PDR}(y) = P(PDR < y) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(y) - K^d(T)}{\sqrt{\rho}}\right), \quad 0 \le y \le 1, \tag{4.4}$$

where $K^d(T) = \Phi^{-1}(p(T))$.

The derivation of the distribution in (4.4) is described in Appendix A.

4.2.3 Generic One-Factor Lévy Model

To introduce heavier tails one can use **Generic one-factor Lévy** models (Albrecher et al (2006)) in which the latent variable of obligor i is of the form

$$Z_n = Y_\rho + Y_{1-\rho}^{(n)}, \quad n = 1, 2, \ldots, N, \tag{4.5}$$

where Y_t and $Y_t^{(n)}$ are Lévy processes with the same underlying distribution L with distribution function $H_1(x)$. Each Z_n has by stationary and independent increment property the same distribution L. If $E[Y_1^2] < \infty$, the correlation is again given by:

$$\mathrm{Corr}(Z_m, Z_n) = \rho, \quad m \neq n.$$

As for the Normal one-factor model, we again say that a borrower defaults at time t, if Z_n hits a predetermined barrier $K^d(t)$ at that time, where $K^d(t)$ satisfies

$$P\left(Z_n \leq K^d(t)\right) = 1 - \mathrm{e}^{-\lambda t}, \tag{4.6}$$

with λ determined as in the Normal one-factor model.

As an example we use the Shifted-Gamma model where $Y, Y_n,\ n = 1, 2, \ldots, N$ are independent and identically distributed shifted Gamma processes

$$Y = \{Y_t = t\mu - G_t : t \geq 0\},$$

where μ is a positive constant and G_t is a Gamma process with parameters a and b. Thus, the latent variable of obligor n is of the form:

$$Z_n = Y_\rho + Y_{1-\rho}^{(n)} = \mu - (G_\rho + G_{1-\rho}^{(n)}), \quad n = 1, 2, \ldots, N. \tag{4.7}$$

In order to simulate default curves, we first have to generate a sample of random variables $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$ satisfying (4.5) and then, at each (discrete) time t, count the number of Z_i's that are less than or equal to the value of the default barrier $K_d(t)$ at that time. Hence, the default curve is given by

$$\mathrm{P_d}(t; \mathbf{Z}) = \frac{\sharp\left\{Z_n \leq K^d(t); n = 1, 2, ..., N\right\}}{N}, \qquad t \geq 0. \tag{4.8}$$

The left panel of Figure 4.4 shows five default curves, generated by the Gamma one-factor model (4.7) with $(\mu, a, b) = (1, 1, 1)$, and $\rho \approx 0.095408$, such that the mean and standard deviation of the default distribution are 0.20 and 0.10. Again, all curves start at zero and are fully stochastic. The corresponding default distribution is shown in the right panel. Compared to the previous three default models, the default distribution generated by the Gamma one-factor model seems to be squeezed around μ_d and has a significantly larger kurtosis. Again we do not have to assume a given default distribution, the default distribution will be generated by the model.

It should also be mentioned that the latter default distribution has a rather heavy right tail (not shown in the graph), with a substantial probability mass at the 100 % default rate. This can be explained by looking at the right-hand side of equation (4.7). Since both terms between brackets are strictly positive and hence cannot compensate each other (unlike the Normal one-factor model), Z_i is bounded from above by μ. Hence, starting with a large systematic risk factor Y, things can only get worse, i.e. the term between the parentheses can only increase and therefore Z_i can only decrease, when adding the idiosyncratic risk factor Y_i. This implies that when we have a substantially large common factor, it is more likely that all borrowers will default, than with the Normal one-factor model.

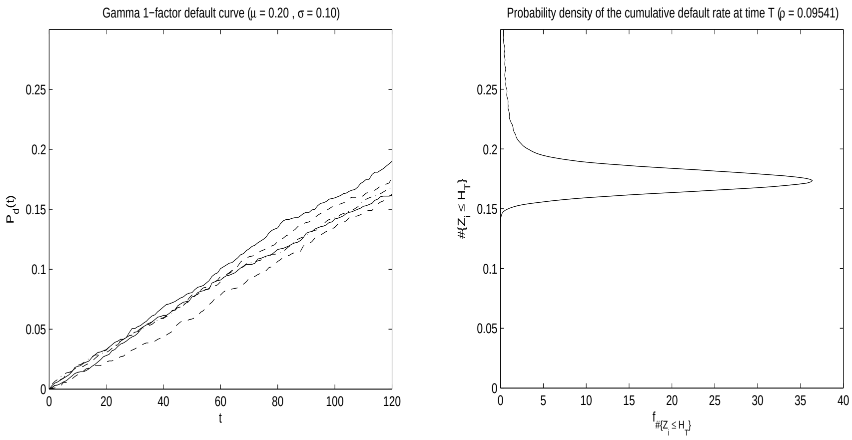

Figure 4.4: *Left panel: Sample of Gamma one-factor default curves. Right panel: corresponding default distribution. The mean and standard deviation of the empirical default distribution is* $\mu_d = 0.20$ *and* $\sigma_d = 0.10$.

Strengths and Weaknesses

The generic Lévy one-factor model is a loan-level model, just as the Normal one-factor model, but with the freedom to choose the underlying probability distribution from a large set of distributions. The distributions are more heavy tailed than the normal distribution, that is, give a higher probability to large positive or negative values. A higher probability that the common factor is a large negative number gives higher probability to have many defaults.

Large Homogeneous Portfolio Approximation

One can find the approximation of the portfolio default rate distribution for large homogeneous portfolios also under the generic one-factor Lévy models, see Albrecher et al (2006).

4.3 Prepayment Modelling

4.3.1 Lévy Portfolio Prepayment Model

The Lévy portfolio prepayment model is completely analogous to the Lévy portfolio default model described in Section 4.2.1.

4.3.2 Normal One-Factor Prepayment Model

The Normal one-factor prepayment model starts from the same underlying philosophy as its default equivalent. The idea is to model prepayment as an event that occurs if the credit worthiness of the obligor is above a certain level, the so called prepayment barrier, just as default was assumed to occur if the credit worthiness of the obligor was below a barrier, the so called default barrier.

The asset's latent variable is modelled by:

$$Z_n = \sqrt{\rho}X + \sqrt{1-\rho}X_n, \quad n = 1, 2, \ldots, N, \tag{4.9}$$

where X is the systemic factor and $X_n, n = 1, 2, \ldots, N$ are the idiosyncratic factors, all are standard normal random variables (mean 0, standard deviation 1), and ρ is the correlation between two assets: $\mathrm{Corr}(Z_m, Z_n) = \rho,\ m \neq n$.

The prepayment barrier $K_n^p(t)$ is chosen such that the probability of prepayment before time t equals $P_p(t)$ in the generalised CPR model:

$$P\left(Z_n \geq K_n^p(t)\right) = 1 - \Phi_{Z_n}\left(K_n^p(t)\right) = P_p(t).$$

Thus, $K_n^p(t) = \Phi^{-1}(1-P_p(t))$, where Φ^{-1} denotes the inverse of the standard Normal distribution function. Note that $K_n^p(t)$ is non-increasing in t, which implies that a prepaid loan does not reappear in the pool and, thus, that the prepayment curve is non-decreasing.

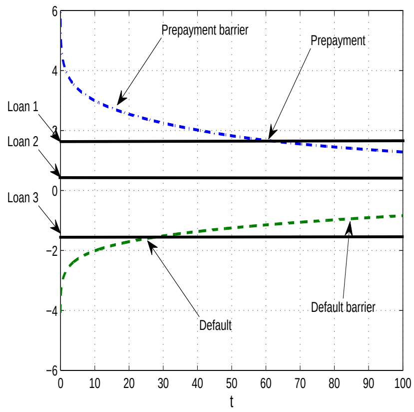

Figure 4.5: *Example of a default barrier and a prepayment barrier in a one-factor model.*

Figure 4.5 shows how a prepayment barrier and a default barrier can be combined in an one-factor model.

The prepayment curve is defined as:

$$\mathrm{P_p}(t; \mathbf{Z}) = \frac{\sharp\left\{Z_n \geq K^p(t); n = 1, 2, ..., N\right\}}{N}, \qquad t \geq 0. \tag{4.10}$$

Comparing the prepayment curves in Figure 4.6 with the curves generated by the generalised CPR model in the right panel of Figure 3.4, one can see that they are similar in shape due to the fact that the prepayment barrier is chosen such that the probability of prepayment of an individual obligor equals the cumulative prepayment rate given by the generalised CPR model. However, the prepayment curves generated by the Normal one-factor model are stochastic as can be seen from the non-linear behaviour of the curves.

Note that the prepayment distribution is generated by the model. This is in contrast with the prepayment distribution shown for the generalised CPR model in Figure 3.4 where we assumed the log-normal distribution.

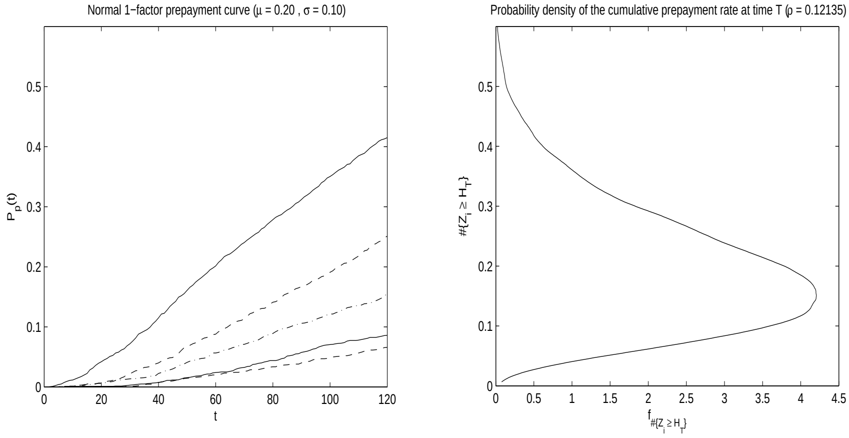

Figure 4.6: *Left panel: Sample of Normal one-factor prepayment curves. Right panel: Corresponding prepayment distribution. The mean and standard deviation of the empirical prepayment distribution is* $\mu_p = 0.20$ *and* $\sigma_p = 0.10$*.*

Strengths and Weaknesses

The evolution of the prepayment curve is stochastic, not deterministic. Furthermore, with the Normal one-factor model it is possible to model both default and prepayment of a single obligor at the same time.

See also comments on the Normal one-factor default model.

Chapter 5

Rating Agencies Methodologies

5.1 Introduction

To derive ratings of the asset backed securities (ABSs) in a securitisation structure the rating agencies assesses the risks in the deal and how well the structure mitigates these risks. The assessment is a combination of qualitative analysis and quantitative methodologies.

The present chapter gives an overview of two of the major rating agencies, Moody's and Standard & Poor's, quantitative methodologies to provide ratings to ABSs, in particular to ABSs backed by SME loans.

5.2 Moody's

In this section we focus on Moody's approach to rating SME transactions, although the basic methodologies is similar for other asset classes. As was already mentioned in Chapter 2, Moody's rating is an expected loss assessment, which incorporate the assessment of both the likelihood of default and the severity of loss, given default. The quantitative rating is based on the results from a quantitative model, which calculates the Expected Loss and the Expected Weighted Average Life of an ABS note. This quantitative rating is combined with a qualitative analysis, which includes an operational overview of the originator and the servicer and legal issues (transfer of assets and bankruptcy), to derive a final rating (Moody's (2001) and Moody's (2007b)).

The quantitative rating methodology used depends on the size and granularity of the underlying SME portfolio. For small or non-granular portfolios, such as CDO's, Moody's takes a bottom-up approach and use factor models (typically based on the Gaussian-copula approach, for example the Normal one-factor model presented previously) for the analysis. For granular portfolios Moody's adopt a default distribution (Lognormal or Normal Inverse) approach, to model the cumulative default rate at the deal maturity. The factor models are implemented in Moody's CDOROM^{TM} and Moody's STARFINDER^{TM} (see Moody's (2006b)); the granular approach in Moody's ABSROM^{TM} (see Moody's (2006a)).

In general, Moody's classifies SME portfolios with more than $1,000$ assets and no major concentrations as ABS SME.[1]

General information guidelines describing the data that Moody's would like to receive from the originator for SME securitisation transactions are given in Moody's (2007b).

5.2.1 Non-Granular Portfolios

For concentrated, heterogeneous pools the main tool for deriving a default distribution is Moody's CDOROM^{TM}. The portfolio default distribution will be directly derived from Monte Carlo simulations, which simulates the default of each individual asset based on a factor model as described previously in Section 4.2.2. The factor models used are typically based on one factor:

$$Z_n = \sqrt{\rho_c} X_c + \sqrt{1-\rho_c} X_n,$$

or, on two factors:

$$Z_n = \sqrt{\rho_c} X_c + \sqrt{\rho_i} X_i + \sqrt{1-\rho_c-\rho_i} X_n,$$

where X_c is the common global factor, X_i is an industry factor, X_n, $n = 1, 2, \ldots, N$ is the individual firm specific factor, and ρ_c and ρ_i is the global inter-industry and the sector specific intra-industry correlation assumptions, respectively, see Moody's (2007a).

The input parameters in the model are the probability of default of each individual asset and the asset correlation. The individual default probability is typically derived from either (i) public ratings, (ii) credit estimates or (iii) a mapping between the originator's internal rating system and Moody's rating scale. The correlation is derived from Moody's corporate correlation framework adopted in global CDOs. However, Moody's stresses the correlation parameters from 3% to 6% depending on the specific characteristics of the portfolio, to account for a higher geographical concentration and industrial clustering typically present in SME pools.[2]

In contrast to the approach for granular portfolios described below, the default timing is directly generated in the factor models since the default of each individual asset is simulated.

In CDOROM™ the recovery rates are stochastic and assumed to be distributed according to a Beta distribution applied to *each* defaulted asset.[3]

To derive a rating, the present value of the loss for the note and the note's weighted average life are calculated for each simulation run and the averages over all simulations are taken as estimates of the expected loss and the expected weighted average life. (A more detailed description is given in the section on granular portfolios.)

The rating of the note is found from Moody's Idealised Cumulative Expected Loss Table, which map the Expected Average Life and Expected Loss combination to a specific quantitative rating. An example of such a table is given in Moody's (2000b).

[1] To discriminate non-granular, granular and intermediate portfolios, Moody's calculates the Effective Number of Obligors, based on the Herfindahl index (see Moody's (2007d).

[2] Moody's (2007d), p. 4.

[3] Moody's (2007d), p. 8.

5.2.2 Granular Portfolios

For granular portfolios a default distribution for the total cumulative default rate (expressed as per cent of initial portfolio outstanding principal amount) over the life of the pool is assumed, typically a Normal Inverse[4] distribution (previously Moody's used the Lognormal distribution as standard, but this has changed (Moody's (2007d))). The default distribution is characterised by two parameters: the *mean* and the *standard deviation*, that has to be estimated. Moody's estimates these parameters from historical static cohort data provided by the originator. This data is typically given in a *Static cumulative default rate table* describing different cohorts (or vintages) of pools of loans and the cumulative default rate over a number of periods after origination. From this data estimates of the mean and standard deviation is derived. The basic methodology of how to extrapolate, clean and adjust for seasoning is described in Moody's (2005b). The parameter estimation based on historical cohort data is (almost) only applicable at the time the transaction is issued, because it is rarely that updated cohort data is made available at a later stage after the closing date. To handle this problem Moody's has developed a methodology to revise the default assumptions over the life of an ABS transaction (Moody's (2008)). The method takes as input transaction specific performance data, such as *delinquency rates*, *historical periodic default or loss rates* and *historical portfolio redemption rates*.

Based on the default distribution a set of *Default Scenarios* are derived and the scenario probability is given by the default distribution. The default scenarios are $0.00\%, 0.10\%, 0.20\%, \ldots$ and the scenario probability is the probability that the default rate falls between two consecutive default scenarios.[5] The Normal Inverse distribution and the 20% default scenario with its associated probability are illustrated in Figure 5.1.

To distribute the defaults over the life of the pool a *Default Timing* vector is defined. For each period, the corresponding element in the Default Timing vector is the percentage of the total cumulative default rate that will be applicable in that period. The Default Timing vector is used to calculate the outstanding amount of the defaulted loans per period in each default scenario:

$$\begin{aligned}\text{Defaulted Amount}(\text{period}\, i,\ \text{scenario}\, s) &= \text{Default Timing}\ (\text{period}\, i)\\ &\quad \times \text{Default Rate}(\text{scenario}\, s)\\ &\quad \times \text{Original Portfolio Amount}.\end{aligned}$$

The default timing is preferably derived from historical static cohort data on defaults (Moody's (2007d)).

For granular portfolios the recovery rate is assumed to be stochastic with a Normal distri-

[4] The Normal Inverse distribution is an approximation of the default distribution if the Normal one-factor model is used for a large homogeneous portfolio, see Appendix A and Moody's (2003) and Moody's (2007a).
[5] Moody's (2006a), p. 34.

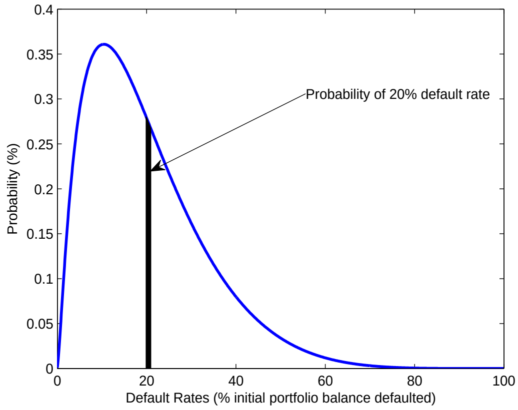

Figure 5.1: *Illustration of a Normal Inverse default distribution. The* 20% *Default Scenario and its associated probability is marked with a bar. The asset correlation was assumed to be* $\rho = 20\%$*, and the mean cumulative default rate* 20%*. The default barrier was estimated as described in the section on the Normal one-factor default model.*

bution and is applied on a portfolio basis.[6] Historical recovery data provided by the originator is used in order to determine the recovery rate. In ABSROM^{TM} a *Recovery Timing* vector is used to specify the timing of the recoveries.

For the prepayments Moody's assumes a fixed annual constant prepayment rate (CPR), which is estimated from the originator's historical data.

To come to a rating Moody′s ABSROM$^{\text{TM}}$ calculates the Expected Average Life (or weighted average life) and the Expected Loss of the note (see Moody's (2006a), p. 32-33). The Expected Average Life of the note is given by:

$$\sum_{s=\text{1st Default Scenario}}^{\text{Last Default Scenario}} \text{Weighted Average Life(scenario } s) \times \text{Probability(scenario } s),$$

where Weighted Average Life(scenario s) is:

$$\sum_{i=\text{1st Period}}^{\text{Legal Maturity Date}} \frac{\text{Outstanding Note Amount(Period } i, \text{scenario } s)}{\text{Original Note Amount} \times \text{Number of Periods per Annum}}.$$

The expected loss is calculated as the sum-product of the probability of each default scenario and the corresponding Relative Net Present Value-Loss. For each default scenario, the Relative Net Present Value-Loss for a note is calculated by discounting the cashflows (both interest and

[6] Moody's (2007d), p. 8. What is meant with "applied on a portfolio basis" is not clear. Using a Normal distribution for the recovery rate implies that the recovery rate can become negative. However, Moody's argues that by the Law of Large Numbers, if all LGDs are independent and identically distributed, the average LGD will be almost equal to its expected value for high default rates, which implies that the right tail of the loss distribution will not depend on the shape of the LGD distribution for each asset. See discussion in footnotes 30 and 31 in Moody's (2003), p. 18 and 19, respectively.

principal) received on that note with a discount rate which is equal to the rate of that note and by comparing it to the initial outstanding amount on the note (Moody's (2006a), p. 33):

$$\text{Relative NPV Loss(Scenario } s) = \frac{\text{Nominal Initial Amount} - \text{NPV Cashflow(Scenario } s)}{\text{Nominal Initial Amount}}.$$

The expected loss is then given by:

$$\text{Expected Loss} = \sum_{s=\text{1st Default Scenario}}^{\text{Last Default Scenario}} \text{Relative NPV Loss(Scenario } s) \cdot \text{Probability(Scenario } s).$$

For a fixed rate note the discount rate will be the promised coupon rate and for a floating rate note it will be the realised benchmark rate plus the note's margin.

The rating of the note is found from Moody's Idealised Cumulative Expected Loss Table, which map the Expected Average Life and Expected Loss combination to a specific quantitative rating. An example of such a table is given in Moody's (2000b).

V Scores and Parameter Sensitivity

Moody's has recently introduced two changes to the way structured finance ratings are presented: V Scores and Parameter Sensitivities. Moody's V Scores "provide a relative assessment of the quality of available credit information and the potential variability around various inputs to a rating determination.".[7] The Parameter Sensitivities "provide a quantitative/model-indicated calculation of the number of rating notches that a Moody's-rated structured finance security may vary if certain input parameters used in the initial rating process differed.".[8]

It is intended that the V Scores shall provide a ranking of transactions by the potential of rating changes due to uncertainty around the assumptions made during the rating process. V Scores are a qualitative assessment of the potential of rating changes due to, among others, data quality, historical performance, transaction complexity, and the transaction governance that underly the ratings.

To analyse the parameter sensitivity, typically, the two key input parameters that have the greatest impact within the sector will be stressed. For example, the mean portfolio default rate and the mean recovery rate can be assumed to vary between 12%, 14% and 16% and 30%, 40% and 50%, respectively. For each stressed scenario (i.e. each combination of default rate and recovery rate in our example) a new loss distribution is generated under which the notes are re-assessed.

5.3 Standard and Poor's

As mentioned before, the meaning of Standard and Poor's (S&P's) rating is the assessment of timely payment of interest and the ultimate payment of principal no later than the legal final

[7] Moody's (2009b), p. 1.
[8] Moody's (2009b), p. 1.

maturity date. It is only the credit quality of structure finance securities that is addressed, and the ratings framework is based on the likelihood of default and not on expected loss or loss given default (Standard and Poor's (2007b)).

S&P's employs a principle-based methodology for rating structured finance securities, outlined in Standard and Poor's (2007b). The core methodologies for analysing and rating securitisation transactions contains five key areas of analysis: credit quality of the securitised assets; payment structure and cash flow mechanics; legal and regulatory risks; operational and administrative risks; and counterparty risk. We will focus on the quantitative parts in the rating process here, namely credit quality of the securitised assets and payment structure and cash flow mechanics.

5.3.1 Credit Quality of Defaulted Assets

For most ABSs, RMBSs and CDOs backed by pools of loans, receivables or corporate debt the credit quality analysis focuses on determining under "worst-case" scenarios the portion of the original asset pool that will default and the portion of these defaulted assets that can be recovered. From this the potential ultimate loss on the debt issue can be derived (Standard and Poor's (2007b), p. 7).

S&P's has three main SME transaction categories (Standard and Poor's (2009a), p. 2):

- Granular SME transactions;
- Transactions with lumpy assets or high sector exposure; and
- Hybrid bespoke transactions.

SME transactions with highly granular characteristic with assets spread across different sectors and industries are categorised as granular transactions. Typically a granular transaction is a securitisation of a cross-section of a bank's SME loan portfolio. In the second category, the portfolio has a skewed risk profile due to an uneven and high exposure to a small number of obligors or economic sectors. Hybrid bespoke transactions are often created for the purpose of obtaining repo financing under central bank financing schemes and contains a mix of SME assets together with large corporate loans and residential and commercial mortgages.

Based on these categories, different analytical approaches and assumptions are applied to rate transactions backed by SME loans (Standard and Poor's (2009a), p. 3):

- The actuarial approach;
- Probability of default and stochastic modelling approach; and
- Secured real estate default analysis.

In the actuarial approach, base case portfolio default rates and recovery rates are derived using historical gross loss rates and recovery data. These default and recovery rates are then used to stress and simulate defaults and recoveries over time in different rating scenarios. Typically this approach is applied to granular SME transactions.

The second approach, probability of default and stochastic modelling, is based on S&P's CDO Evaluator® model. The model uses Monte Carlo simulation to assess the credit quality of an asset portfolio, taking as input the credit rating, notional exposure and maturity of each asset, as well as the correlation between each pair of assets. The output from the model is a probability distribution of potential portfolio default rates, which is the base for a set of *scenario default rates* (SDRs), one for each rating level. The SDR is the portfolio loss an ABS must be able to withstand without defaulting. The CDO Evaluator® is based on the *Gaussian copula* model by Li (2000).

The final approach is used for assets that are secured on real estate collateral and is a weighted average foreclosure frequency (WAFF) and weighted average loss severity (WALS) approach. To determine the likely default and loss on a loan underlying loan level characteristics, such as, loan-to-value (LTV) ratio, seasoning and regional concentrations are used.

The above described approaches are carried out together with a detailed cash flow analysis, which is described below.

CDO Evaluator® Model

As mentioned above the CDO Evaluator® model uses Monte Carlo simulations to assess the credit quality of the asset pool. The output of this assessment is a probability distribution of potential portfolio default rates. The CDO Evaluator® model is a bottom-up approach, where each individual asset is modelled. The modelling is based on the Gaussian copula model proposed by Li (2000). In fact, the Gaussian copula model is the Normal factor model "translated" into the language of copula functions. Hence, both Moody's and S&P's base their quantitative modelling of non-granular portfolios on the same mathematical model.

The CDO Evaluator® allows for both fixed and stochastic (beta distributed) recoveries, it is however not clear if stochastic recoveries are applicable for SMEs.[9]

From the probability distribution of default rates scenario default rates (SDRs) are derived. The SDR for a specific rating level is the largest portfolio default rate such that the probability of defaults in the portfolio exceeding the SDR is not greater than the probability of default for the given rating level and time horizon.

For example, assume that we want to find the SDR for the 'AA' rating level and a time horizon of 10 years. We lookup the probability of default associated with the 'AA' rating in a *credit curve* table. A credit curve table contains the probability of default for each rating level for a series of maturities. Let us say that the probability of default for a 10 year 'AA' rated tranche is 1.0%. We now have to find the largest portfolio default rate from the default rate distribution for which the likelihood of exceeding this value is less than or equal to 1.0%. This

[9] CDO Evaluator® Version 4.1 User Guide (version 1.36), Standard & Poor's Structured Finance Group, December 2008, p. 91.

is illustrated in Figure 5.2. The SDR equals 34% in this example and the likelihood that the defaults in the portfolio exceeds 34% is 0.96%.

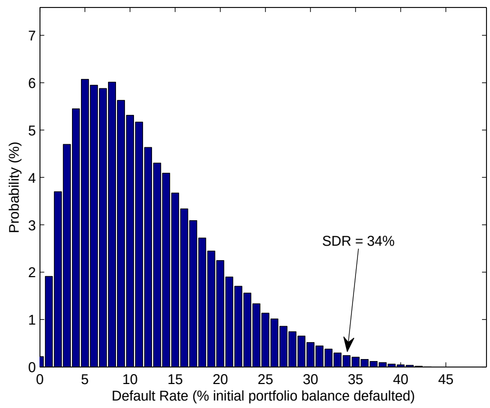

Figure 5.2: *Example of a probability distribution of portfolio default rates and the scenario default rate (34%) associated with a probability of default of 1.0%.*

5.3.2 Cash Flow Modelling

The cash flow analysis evaluates the availability of funds for timely payment of interest and ultimate payment of principal following the conditions of each rated class of notes and is used to determine the credit support levels for each rated class of notes. The cash flow analysis is done for each rated class of notes by stressing the cash flow from the asset pool. The severity of the stress scenarios applied to the cash flow depend on the desired rating. The cash flow analysis described here is the one used in combination with the CDO Evaluator® model and is based on the following reports: Standard and Poor's (2004b), Standard and Poor's (2006a), and Standard and Poor's (2006b).[10]

The stress tests are performed with respect to among other things:[11]

- Default timing;
- Delinquencies (if applicable);
- Recovery rates and timing;
- Interest rate hedging (including interest rate stresses);

[10] It is not clear if the cash flow analysis done in combination with the actuarial approach is the same from the documentation. In Standard and Poor's (2003) p. 11 a short description is made, but it does not clarify if the same set of stress scenarios for each rating level is used in combination with the actuarial approach as with the CDO Evaluator® model approach.

[11] Standard and Poor's (2004b) and Standard and Poor's (2006a). These reports are only discussing the cash flow modelling of stressed scenarios in combination with the CDO Evaluator® model.

- Prepayments (if applicable); and
- Senior fees.

We describe here only the default timing and the recovery timing stresses.

Default Timing Stresses

S&P's applies four standard default patterns and a few additional default patterns (saw-tooth patterns and expected case patterns) to stress the cash flow.[12] We will here describe the four standard patterns, given in Table 5.1, and refer the interested reader to Standard and Poor's (2004b) for a description of the additional patterns. Each pattern expresses the percentage of the cumulative default rate that occurs ever year once defaults starts. As can be seen from Table 5.1, all defaults are assumed to occur during four or five years once defaults starts. The annual default rates given in Table 5.1 can be distributed evenly across the four quarters of the year with defaults occurring on the last day of each quarter. This applies to all years except the first year of the transaction, in which the entire default amount is supposed to occur at the last day of the year, because S&P's assumes that some time elapse before defaults occur in a newly gathered portfolio. An exception to this is the case when the portfolio contains a large concentration of low credit quality assets.[13]

	Annual defaults (% of cumulative defaults)				
	Year 1	Year 2	Year 3	Year 4	Year 5
Pattern I	15	30	30	15	10
Pattern II	40	20	20	10	10
Pattern III	20	20	20	20	20
Pattern IV	25	25	25	25	-

Table 5.1: *Standard & Poor's standard default patterns. Annual defaults as a percentage of cumulative defaults. Source: "Update To General Cash Flow Analytics Criteria For CDO Securitizations", Standard and Poor's, October 17, 2006, p. 7.*

It is important to note that the default patterns are applied to the original par balance of the portfolio. As an example, assume that we apply Pattern I to a cumulative default rate of 20% and a pool with original balance 100. Then the original pool par balance experience defaults of $3\%(= 15\% \cdot 20\%)$, 6%, 6%, 3% and 2%, respectively, in the five years the pattern is covering, or, equivalently, 3, 6, 6, 3 and 2.

[12] S&P's uses this deterministic modelling approach with default patterns for application to cash flow CDO transactions. For synthetic CDO transactions S&P's uses the default timing patterns generated by the CDO Evaluator® model, see Standard and Poor's (2006b), p. 7.

[13] Standard and Poor's (2004b) p. 10.

These patterns are combined with default timing stresses, which means that the start of a specific pattern is delayed by a number of years. That is, the cash flow analysis is run for a specific pattern starting in year 1, and then for the same pattern starting in year 2, and so on. The starting times of the patterns are delayed to the point where the final default in the pattern occurs in the same year as the portfolio balance is expected to mature, which depends on the length of the reinvestment period and the weighted average life of the assets (given by the weighted average life covenant in the offering circular). These default timing stresses, that is, the delays, are different for different rating levels.[14] An example of the different starting years for different rating categories is given in Table 5.2.

Reinvestment	WAL	Tranche					
period	covenant*	'AAA'	'AA'	'A'	'BBB'	'BB'	'B'
5	4	1 to 5	1 to 5	1 to 4	1 to 3	1 to 2	1
5	6	1 to 7	1 to 7	1 to 6	1 to 5	1 to 4	1 to 3

Table 5.2: *Example of starting years for Standard & Poor's standard default patterns. *The WAL covenant at the end of the reinvestment period as stated in the offering circular. Source: "CDO Spotlight: General Cash Flow Analytics for CDO Securitizations", Standard and Poor's, August 25, 2004, p. 8.*

An illustration of how the default scenarios can look like when Pattern I in Table 5.1 is combined with the default timing stresses given in Table 5.2 for a 'AAA' or 'AA' rated tranche for a transaction with five years reinvestment period and a WAL covenant of four years is shown in Table 5.3. Note that in the table the annual defaults are shown as a percentage of the total cumulative defaults.

	Annual defaults (% of cumulative defaults)								
Scenario	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9
1	15	30	30	15	10	0	0	0	0
2	0	15	30	30	15	10	0	0	0
3	0	0	15	30	30	15	10	0	0
4	0	0	0	15	30	30	15	10	0
5	0	0	0	0	15	30	30	15	10

Table 5.3: *Example of default scenarios for analysing a 'AAA' or 'AA' tranche when the reinvestment period is 5 years and WAL covenant is a 4 years. Annual defaults as a percentage of cumulative defaults.*

[14] S&P's propose to change this in such a way that default timing stresses for rating level 'A' through 'B' are the same as for 'AAA' and 'AA', see Standard and Poor's (2009b).

Recoveries

Recovery rates are established on a transaction-by-transaction basis taking into account factors such as: the level of experience of the originator; transaction-specific investment guidelines and replenishment criteria; workout procedures and timing of expected recoveries; and location of the defaulted obligor. S&P's has established recovery ranges per country for each transaction, partly based on transaction-specific data available.[15]

Recoveries on defaulted loans are assumed to occur over a three-year workout period, with the recovery timing as given in Table 5.4. Note that the recoveries are realised in the end of the period.

	U.K.	France	Germany	Rest of Europe
End of year 1	75%	-	-	-
End of year 2	25%	50%	25%	50%
End of year 3	-	50%	75%	50%

Table 5.4: *Standard & Poor's recovery timing assumption as percentage of total recovery rate assumed in each year during workout period for SME loans. Source: "Credit Risk Tracker Strengthens Rating Analysis of CLOs of Eurpoean SME Loans", Standard and Poor's, June 10, 2004, p. 6.*

5.3.3 Achieving a Desired Rating

The stress scenarios used in the cash flow analysis aim to assess if the ABS under consideration can withstand the stresses associated with the sought rating level and therefore can receive the corresponding rating level.

For each stress scenario, the output from the cash flow analysis is the *break even default rate* (BDR) the portfolio can withstand and still generate adequate cash flow to meet contractual payments of interest and principal on the class of notes subject to the particular stress scenario. The break even default rate is found by first finding the minimum credit enhancement level given by the subordination structure, i.e., the note's attachment point, such that the note's overall credit performance is adequate for the targeted rating level. This minimum credit enhancement is then translated into a portfolio default rate, which is the so called *break even default rate*.

Thus, for each rated class of notes, the result of the cash flow analysis is a set of break even default rates (BDRs), one for each stress scenario. The desired rating of a class of notes is achieved by comparing the BDRs with the SDR for that rating level. Assume, for example, that we pick the minimum BDR for each rating level and compare it with the corresponding SDR. If the SDR for the 'AA' rating level is 34%, then a tranche can receive a 'AA' rating if the corresponding minimum BDR for that tranche is equal to or greater than 34%.

[15] Standard and Poor's (2004a), p. 5.

To pick out the BDR for each rated class that should be compared with the corresponding SDR, S&P's uses a percentile approach, which differentiate the application of BDRs across rating categories.[16] The break even percentiles by rating is shown in Table 5.5.

Tranche rating	Percentile
AAA	5th
AA	10th
A	35th
BBB	50th
BB	60th
B	70th

Table 5.5: *Break even percentiles by rating. Note that for all rating categories 'AA', 'A' and so until 'B' include rating subcategories, for example, 'AA' percentile also applies to 'AA+' and 'AA-'. Source: "Update To General Cash Flow Analytics Criteria For CDO Securitizations", Standard and Poor's, October 17, 2006, p. 3.*

In the example above, this would mean that the 10th percentile BDR should be equal to or greater than 34% if the tranche should receive a 'AA' rating.

5.4 Conclusions

The interpretation of a rating is different over the various rating agencies. Moody's rating is an assessment of the *expected loss* that a class of notes may experience, while S&P's rating is an assessment of the *probability of default* of the class of notes and addresses the timely payment of interest and the ultimate payment of principal.

Both Moody's and S&P's discriminate between granular and non-granular SME portfolios and applies different approaches to the two categories.

For non-granular SME portfolios both rating agencies use a loan-by-loan or bottom-up approach and model each individual asset in the pool. Moody's uses its CDOROM^{TM} tool, which uses Normal factor models (with dependence structures based on the Gaussian copula approach); S&P's is using its CDO Evaluator® model, which is based on the Gaussian copula approach. In both cases, thus, are the underlying mathematical tool to introduce dependence in the portfolios the Gaussian copula approach. Monte Carlo simulations are used to generate defaults in the asset pool and to derive a default distribution. The difference between the two methodologies lies in the use of the tool or model.

In Moody's methodology, the default scenario generated by *each* Monte Carlo simulation is fed into the cash flow model and the losses on the ABSs are derived. This is done for a large

[16] Earlier the minimum BDR produced by the cash flow analysis was compared to the scenario default rate (SDR) for each rating level, see Standard and Poor's (2006b).

number of simulations and an estimate of the expected loss on each ABS is derived. The cash flow analysis is thus an integrated part of the simulations. The expected losses are mapped to a rating for each ABS using Moody's loss rate tables.

In S&P's methodology, the Monte Carlo simulations generate a probability distribution of potential portfolio default rates that is used to derive a set of scenario default rates (SDRs), one for each rating level. Each SDR represents the maximum portfolio default rate that an ABS with the desired rating should be able to withstand without default. These SDRs are then used to create different stressed rating scenarios that are applied in a cash flow analysis, which assesses if the ABS under consideration can withstand the stresses associated with the targeted rating level and therefore can receive the corresponding rating level.

For granular SME portfolios, Moody's uses its ABSROM^{TM} tool, which uses a default rate distribution to generate default scenarios and the corresponding likelihood of each scenario. The default rate distribution's mean and standard deviation is estimated using historical data. Running a cash flow model with the different default scenarios, stressing the default timing, the expected loss on the notes are calculated. S&P's applies its *actuarial approach*, for granular SME portfolios, which is based on deriving base case default and recovery rates from historical data in order to stress defaults over the life of the transaction in different rating scenarios in a cash flow analysis.

Chapter 6

Model Risk and Parameter Sensitivity

6.1 Introduction

To derive ratings of ABSs is partly based on quantitative models for modelling defaults and prepayments generated in the asset pool. This introduces exposure to model and methodology risk, because there exists a vast amount of quantitative models and approaches to choose between, each producing different asset behaviour and portfolio loss distributions. The parameter values which are used as inputs to the quantitative models, such as, mean cumulative default and prepayment rates, recoveries, default probabilities, and asset correlations, are unknown quantities and are commonly estimated from historical data or based on subjective assumptions. This introduce parameter uncertainty into the assessment and it therefore becomes important to understand the ratings parameter sensitivity.

We observe in the present chapter the important impact the choices of model and parameter values have on ABS ratings by applying some of the default and prepayment models presented in Chapter 3 and 4 to a structure with two classes of notes backed by a pool of loans.

This chapter is mainly based on Jönsson et al (2009) and Jönsson and Schoutens (2010).

6.2 The ABS structure

Assets

The asset pool consists of 2,000 level-pay loans that pays principal and interest monthly. The interest rate is fixed. The pool is static, new loans are not added to the pool. The asset characteristics are shown in Table 6.1.

ASSETS		
Initial balance of the asset pool	V_0	\$30,000,000
Number of loans in the asset pool	N_0	2,000
Weighted Average Maturity of the assets	WAM	10 years
Weighted Average Coupon of the assets	WAC	12% p.a.
Payment frequency		monthly
Reserve target	r_T	5% (CB*)
Eligible reinvestment rate		3.92% p.a.
Loss-Given-Default	LGD	50%
Lag		5 months

Table 6.1: *Asset characteristics. * CB: Current Balance.*

Liabilities

This pool of assets backs two classes of notes: A (senior) and B (junior); both having fixed coupons. The notes are amortized *pro-rata* during the life of the deal. A reserve fund is used as an additional credit enhancement. The reserve fund target is 5% of outstanding balance of the pool. The characteristics of the notes are shown in Table 6.2.

LIABILITIES		
Initial balance of the senior note	A_0	\$24,000,000
Premium of the senior note	r_A	7% p.a.
Initial balance of the subordinated note	B_0	\$6,000,000
Premium of the subordinated note	r_B	9% p.a.
Servicing fee	r_{sf}	1% p.a.
Servicing fee shortfall rate	r_{sf-sh}	20% p.a.
Allocation method		Pro-rata; or Sequential

Table 6.2: *Characteristics of the Notes.*

6.3 Cashflow Modelling

The cashflow from the pool is modelled as described in the example in Section 2.2.1. The cash collections each month from the asset pool consists of interest payments and principal collections (scheduled repayments and unscheduled prepayments). These collections constitutes together with the principal balance of the reserve account *Available Funds* at the end of each month.

The Available Funds are distributed according to the *waterfall structure* in Table 6.3. The waterfall is a so called combined waterfall where the available funds at each payment date constitutes of both interest and principal collections. The items in the waterfall are calculated as described in Section 2.2.1.

Waterfall	
Level	Basic amortisation
1)	Servicing expenses
2)	Class A interest
3)	Class B interest
4)	Class A principal
5)	Class B principal
6)	Reserve account reimburs.
7)	Residual payments

Table 6.3: *The waterfall used in the analysis.*

6.4 Numerical Results I

To this ABS structure we applied some of the default models in and prepayment models Chapter 3 and Chapter 4 in different combinations analysing the rating, weighted average life and internal rate of return of the notes model dependence and also their sensitivity to changes in mean cumulative default rates and mean cumulative prepayment rates. We discuss in this chapter the model risk and parameter uncertainty present in ABS ratings related to default modelling and refer to Jönsson et al (2009) for the full study.

The ratings are based on cumulative expected loss, estimated by Monte Carlo simulations with one million scenarios. The losses on the notes are computed by calculating the notes internal rate of return (IRR) and comparing it to the promised yields. The difference between the yield and IRR is defined as the loss. The expected loss is given by adding the losses in each scenario and divide by the number of scenarios. For each scenario we also calculate the expected weighted average lives of the notes. Having calculated the expected loss and the expected weighted average life we can map these estimates to get a rating using Moody's idealized cumulative expected loss rates table.

The numerical results are based on four default models: Normal one-factor model, Logistic model, Lévy portfolio model, and Gamma one-factor model. The prepayments are modelled by the generalised CPR model and the mean prepayment rate is assumed to be 20%.

6.4.1 Model Risk

Model risk is omnipresent in the rating of the two notes in the case study. Table 6.4 shows the ratings of the Class A Notes and the Class B Notes. If we let the Normal one-factor model be our benchmark, we can measure the model risk by the number of notches the rating differs for the different default models within each column, that is, for a fixed mean default rate assumption. When mean default rate is 10%, we can observe that the rating output from the Gamma one-factor model differs from the Normal one-factor model by one notch. The other two models does not result in any rating differences. On the other hand increasing the mean default rate assumption to 20% and 40% we can observe discrepancies among all four models.

The rating of the Class B Notes is even more sensitive to model choice than the Class A Notes. Already for the 10% default rate mean assumption the rating differs by one or three notches between the models. For 20% mean default rate the rating difference is three to four notches and the difference is two to three notches at 40% mean default rate.

	Class A Notes			**Class B Notes**		
Default model	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$
Normal one-factor	Aaa (–)	Aaa (–)	Aa2 (–)	Aaa (–)	Aa1 (–)	Baa1 (–)
Logistic	Aaa (0)	Aa1 (1)	Aa3 (1)	Aa1 (1)	A1 (3)	Baa3 (2)
Lévy portfolio	Aaa (0)	Aaa (0)	A1 (2)	Aa1 (1)	A2 (4)	Baa3 (2)
Gamma one-factor	Aa1 (1)	Aa3 (3)	A2 (3)	Aa3 (3)	A2 (4)	Baa2 (1)

Table 6.4: *Ratings of the Class A Notes and Class B Notes with pro-rata allocation of principal. The numbers in parentheses are the rating changes (number of notches) compared to Normal one-factor model, assuming the same mean default rate (μ_d), i.e., column-wise comparison. Prepayment is modelled with the generalised CPR model. Mean cumulative prepayment rate* $\mu_p = 0.20$. *The rating is based on cumulative expected loss and expected weighted average life.*

To give a quantitative explanation of the rating differences reported in Table 6.4 we present in the expected loss and the expected weighted average life of the notes in Table 6.5 and Table 6.6, respectively. For a given default mean assumption is the expected weighted average life approximately the same for the A notes under all four default models. The expected loss varies quite a lot between the models for each default mean. Hence the differences in the ratings of the A notes are mainly caused by the differences in the expected loss. The same conclusion can be drawn for the B notes rating.

	Class A Notes			**Class B Notes**		
Default model	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$
Normal one-factor	0.00036114	0.034631	2.9626	0.033692	1.5642	57.936
Logistic	0.026746	0.3466	5.3712	0.93026	10.581	139.46
Lévy portfolio	0.0017992	0.16105	9.0857	1.4051	17.801	175.75
Gamma one-factor	1.4443	4.6682	18.431	6.288	20.736	85.662

Table 6.5: *Expected loss (in basis points) of the Class A Notes and Class B Notes with pro-rata allocation of principal. Prepayment is modelled with the generalised CPR model. Mean cumulative prepayment rate* $\mu_p = 0.20$.

	Class A Notes			Class B Notes		
Default model	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$
Normal one-factor	5.4775	5.2427	4.7309	5.4777	5.2502	4.9709
Logistic	5.4867	5.2742	4.8642	5.4901	5.3124	5.3358
Lévy portfolio	5.4799	5.2529	4.7895	5.4949	5.3525	5.4753
Gamma one-factor	5.4828	5.2599	4.7939	5.4955	5.3022	4.9739

Table 6.6: *Expected weighted average life (in years) of the Class A Notes and Class B Notes with pro-rata allocation of principal. Prepayment is modelled with the generalised CPR model. Mean cumulative prepayment rate* $\mu_p = 0.20$.

6.4.2 Parameter Sensitivity

We can use the same rating outputs as in Tabel 6.4 to analyse the rating outcomes sensitivity to changes in the mean default rate for each of the four default models. Table 6.7 shows the results of the rating when the mean cumulative default rate assumption changes $(10\%, 20\%, 40\%)$. From the results we may conclude that when increasing the average cumulative default rate the credit rating of the notes stays the same or is lowered for all default models. The rating of the Class A Notes changes with two notches when the Normal one-factor model is used, and with three to four notches for the other models. The rating of the senior notes is hence less uncertain if the Normal one-factor model is used than if any of the other models is used. The rating of the Class B Notes is much more uncertain and changes with seven notches for the Normal one-factor model and up to eight for the others, when the mean default rate is increased from 10% to 40%.

	Class A Notes			Class B Notes		
Default model	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$	$\mu_d = 10\%$	$\mu_d = 20\%$	$\mu_d = 40\%$
Normal one-factor	Aaa (–)	Aaa (0)	Aa2 (2)	Aaa (–)	Aa1 (1)	Baa1 (7)
Logistic	Aaa (–)	Aa1 (1)	Aa3 (3)	Aa1 (–)	A1 (3)	Baa3 (8)
Lévy portfolio	Aaa (–)	Aaa (0)	A1 (4)	Aa1 (–)	A2 (4)	Baa3 (8)
Gamma one-factor	Aa1 (–)	Aa3 (2)	A2 (4)	Aa3 (–)	A2 (2)	Baa2 (5)

Table 6.7: *Ratings of the Class A Notes and Class B Notes with pro-rata allocation of principal. The numbers in parentheses are the rating changes (number of notches) compared to* $\mu_d = 10\%$ *mean default rate, i.e., row-wise comparison. Prepayment is modelled with the generalised CPR model. Mean cumulative prepayment rate* $\mu_p = 0.20$. *The rating is based on cumulative expected loss and expected weighted average life.*

To give a quantitative explanation of the rating differences reported in Table 6.7 we again refer to Table 6.5 and Table 6.6. For each default model is the expected weighted average life of the A notes decreasing when the mean default rate assumption is increasing. This decrease appears due to the higher default rate forces the notes to be redeemed faster (and there exists enough available funds to redeem the A notes). Under all four default models the expected loss is increasing when the mean default rate is increasing. Hence the differences in ratings for a specific model are driven by both the increase in the expected loss and the decrease of the expected weighted average life.

The behaviour of expected weighted average life of the B notes is slightly different. For the two one-factor models the weighted average life is decreasing while for the other two models (Logistic and Lévy portfolio) the weighted average life is not monotonically decreasing. However, for these two models (Logistic and Lévy portfolio) the increase in the expected weighted average life is combined with a very high expected loss for $\mu_d = 40\%$, which forces the rating to be low. For the one-factor models the combination of shorter weighted average life and higher expected loss generates lower ratings.

6.5 Numerical Results II

The second numerical example we present is an illustration of the variability in the ratings due to changes in parameter values previously presented in Jönsson and Schoutens (2010). We focus here on the Normal one-factor model and the three parameters: mean default rate, asset correlation and recovery rate. The pool has the same characteristics as before, except that we now assume that all loans have bullet amortisation and that prepayment is not allowed. The study is based on Monte Carlo simulations with parameter assumptions as follows: cumulative default rates between 10% and 50%; correlations between 10% and 50%; and recovery rates between 10% and 90%.

Examples of portfolio default rate (PDR) distributions for different default rate and correlation assumptions are given in Figure 6.1. As can be seen from the plots, changing the correlation assumption from 10% to 20%, keeping the default rate fixed will shift the mass of the distribution towards the lower end of the portfolio default rate range. However, at the same time the probability to have high PDRs increases. From the four plots in Figure 6.1 it is also possible to see the change of the PDR distribution when the correlation is kept fixed and the mean default rate assumption is changed.

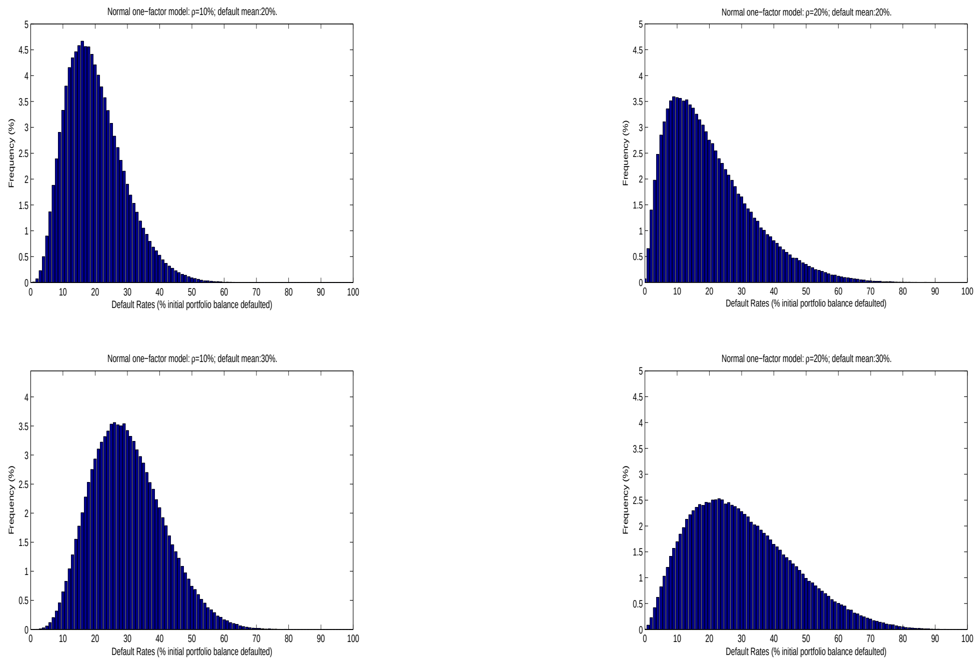

Figure 6.1: *Portfolio default rate (gross loss divided by initial outstanding pool balance) distributions versus correlation and default rate estimated by Monte Carlo simulations of a pool of* $2,000$ *loans using the Normal one-factor model. No prepayments. Bullet amortisation.*

6.5.1 Parameter Sensitivity

To illustrate the ratings sensitivity to the correlation and recovery rate assumptions we plot in Figure 6.2 the ratings of the Class A and B Notes for a range of correlations and recovery rates for two values of the mean default rate (20% and 30%). Each node on the grid corresponds to one rating output given one correlation and one recovery rate. For example, assuming 50% correlation and 10% recovery the rating is Baa3 and Ba2 for the A notes for 20% and 30% mean default rate, respectively.

The ratings sensitivity related to changes in correlation and recovery rate, keeping the mean default rate fixed (20% and 30%), is illustrated in Figure 6.3. We can first of all notice that the changes in the ratings when changing the recovery rate, while keeping the other parameters constant, are nonlinear. Secondly, the values of the mean default rate and the correlation influence the effect of changing the recovery rate.

Assume, for example, that the asset correlation is 10% and mean default rate is 20% (see the upper left graph in Figure 6.3). Under this assumption the rating for the Class A Notes differs one notch (from Aa1 to Aaa) when the recovery rate increases from 10% to 90%. Under the assumption that the mean default rate is 30% (the upper left graph in Figure 6.3) the rating changes from A2 to Aaa (five notches) for the same correlation assumption. Thus there is evidence of interaction effects in the ratings with respect to the parameters in the sense that the change in the rating due to a change of one of the parameters value depends on the values

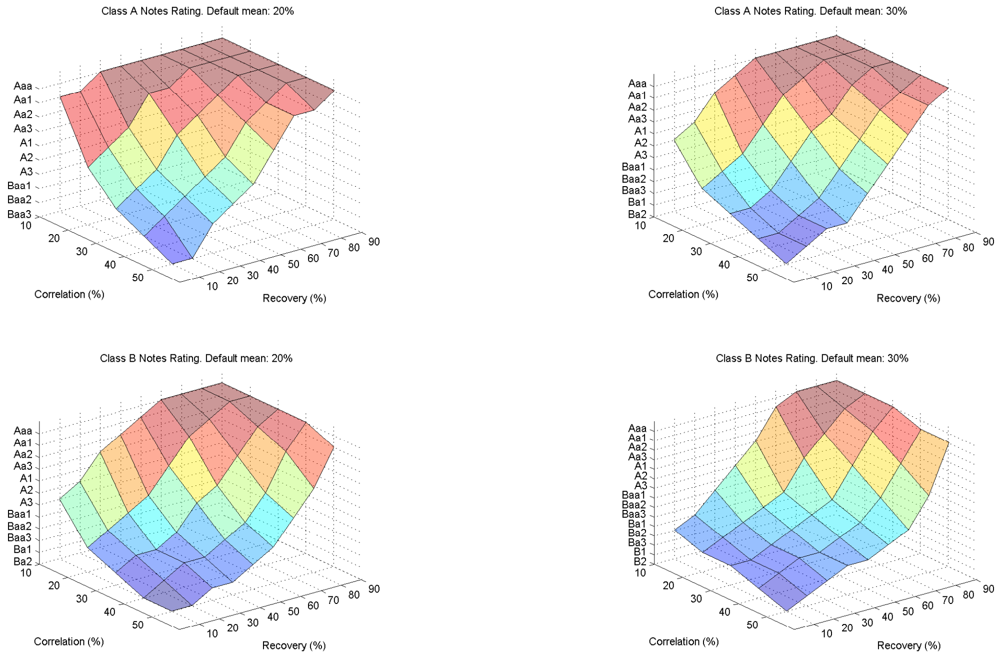

Figure 6.2: *Ratings versus correlation and recovery rate. Default rate:* 20% *(left) and* 30% *(right). No prepayments. Pro-rata allocation of principal. The rating is based on cumulative expected loss.*

of the other parameters.

The lower rating for higher default mean can be explained by looking at the PDR distribution in Figure 6.1, where we can observe that the distribution mean is shifted from 20% to 30% and the right tail of the distribution becomes heavier, i.e., it is more likely that we experience a high PDR. The increase in the rating as the recovery rate assumption is increasing is natural since we assume that more of a loan's principal is recovered after a default.

Finally, in Figure 6.3 it can also be seen that the rating is lowered if the correlation is increased, keeping the recovery rate and the mean default rate fixed. Again the variability in the rating is nonlinear and affected by the values of the values of the other two parameters. If, for example, the recovery rate is 10% and the default mean is 20% the change in the rating due to the value of the correlation parameter is five notches (from Ba2 to A3) for the B notes (see lower left graph in Figure 6.3). The change is eight notches (from Baa3 to Aa1) if the recovery rate is 50% and the default mean is kept at 20%. Note also that the rating changes are nonlinear for fixed recovery rate and default mean, for example, the change is three notches if the correlation is increased from 10% to 20% for 10% recovery rate and 20% default mean, while the change is only one notch if the correlation increases from 20% to 30%. The influence of the correlation is stronger for low recovery rates for the rating of the A note while the correlation has greatest impact for recovery rates in the medium or high values for the B note.

Increasing the correlation while keeping the other parameters fixed results in a fatter right hand tail of the distribution, increasing the likelihood of very high PDRs, as illustrated in Figure 6.1, which explains the lower ratings.

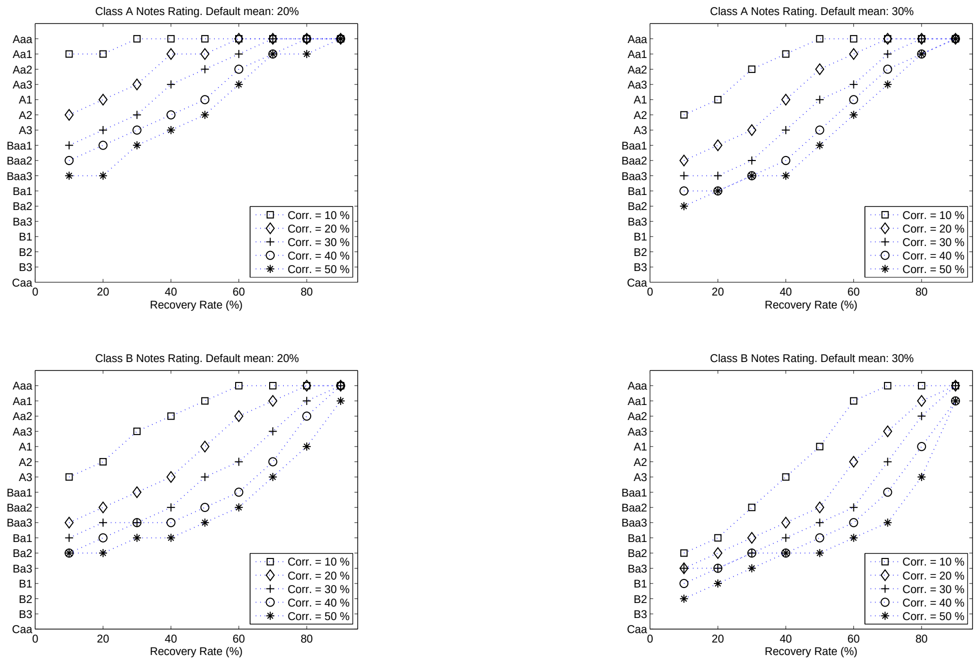

Figure 6.3: *Ratings versus correlation and recovery rate. Default rate:* 20% *(left) and* 30% *(right). No prepayments. Pro-rata allocation of principal. The rating is based on cumulative expected loss.*

In Figure 6.4 and 6.5, the rating output versus changes in mean default rate and recovery rate keeping the correlation fixed at 10% and 20% are presented. We can see in the figures that an increase in the default rate, keeping the recovery rate fixed, will lower the rating, which is what should be expected. We can again observe the ratings nonlinear dependence on the three parameters. For fixed default mean and correlation, the ratings change nonlinearly with respect to the recovery rate. For fixed recovery rate and correlation, the ratings change nonlinearly with respect to the default mean. The value of the default mean has greater impact on the ratings variability for low recovery rates than for high recovery rates for both classes of notes.

There are studies indicating that the recovery rate decreases when the default rate increases (see, for example, Altman et al (2005)), which implies that we could expect that the rating would be even further negatively affected due to an decrease in the recovery rate when the default rate increases.

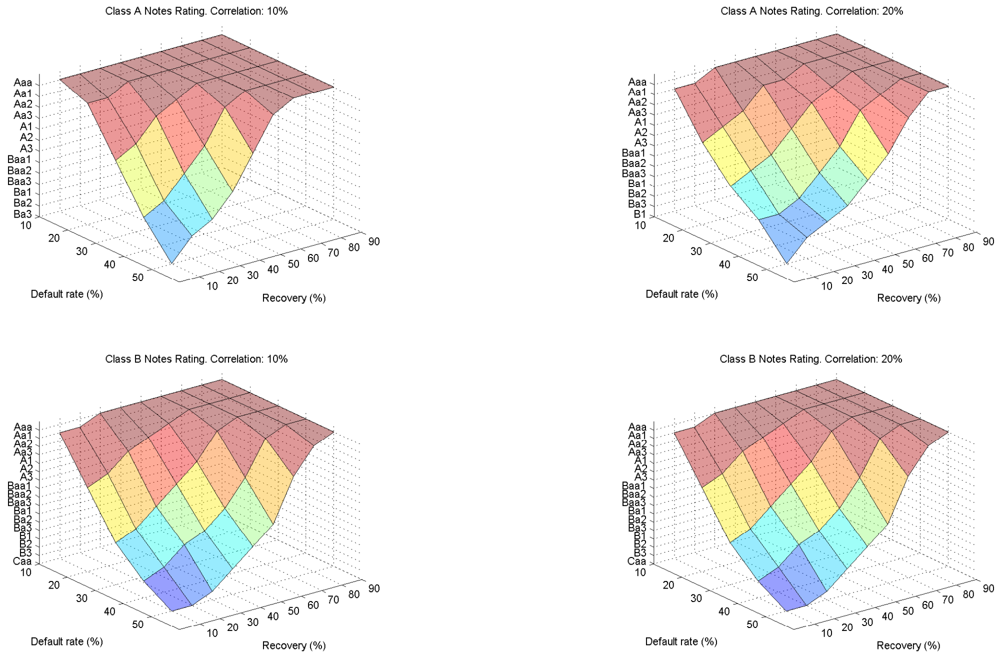

Figure 6.4: *Ratings versus default and recovery rate. Correlation:* 10% *(left) and* 20% *(right). No prepayments. Pro-rata allocation of principal. The rating is based on cumulative expected loss.*

6.6 Conclusions

In this chapter we highlighted the model risk and the influence of parameter uncertainty when rating ABSs. The model risk was assessed by comparing three different default models with a benchmark model, the Normal one-factor model. What could be observed for a low cumulative default rate assumption (10%) was that there was no or just one notch difference in rating for the senior notes and one to three notches difference for the junior notes. However, increasing the cumulative default rate to a high number (40%) the rating differed with as much as three notches for the senior notes and four notches for the junior notes. Thus, for high cumulative default rates the model risk becomes more significant.

The ratings uncertainty related to uncertainty in the cumulative default rate assumption was studied by analysing the number of notches the ratings changed for a given default model when the default rate increased. As could be expected, the ratings were very dependent on the cumulative default rate assumption and, of course, the uncertainty differed between the models. For the junior notes the rating differed with as much as seven to eight notches, when the cumulative default rate changes from 10% to 40%. For the senior notes the changes were one to four notches.

In a second analysis we analysed the variability in the ratings related to uncertainty in the mean default rate, asset correlation and recovery rate under the Normal one-factor model. Big

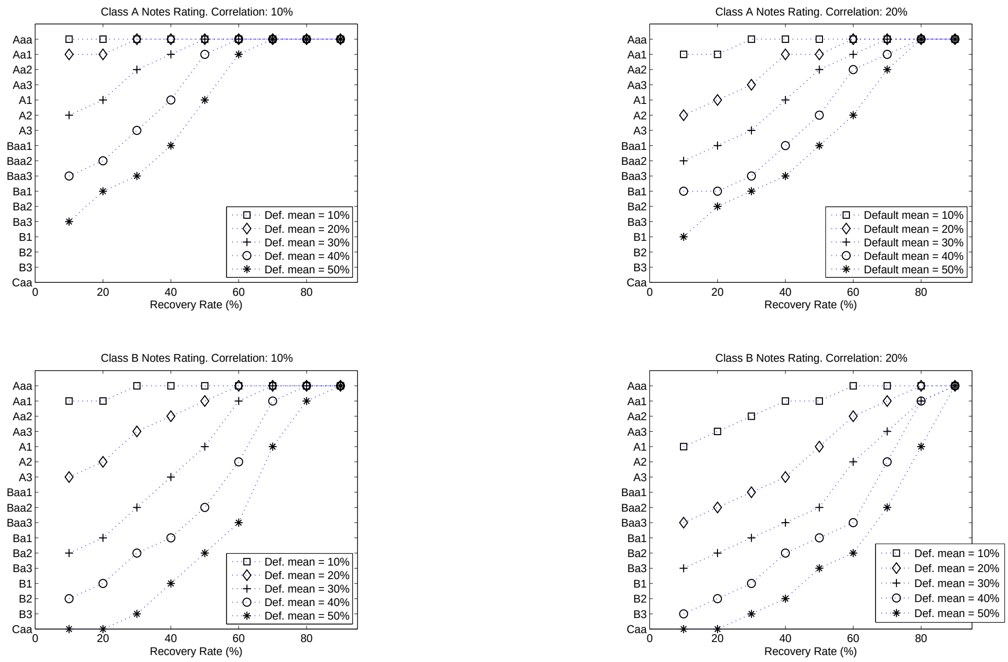

Figure 6.5: *Ratings versus default and recovery rate. Asset correlation:* 10% *(left) and* 20% *(right). No prepayments. Pro-rata allocation of principal. The rating is based on cumulative expected loss.*

variability in the ratings could be observed when the three parameters were allowed to take values within their ranges. It could also be observed that the respons in the ratings due to a change in one of the parameters depended on the values of the other two parameters. For example, the value of the default mean had greater impact on the ratings variability for low recovery rates than for high recovery rates for both classes of notes.

Chapter 7

Global Sensitivity Analysis for ABS

7.1 Introduction

In the previous chapter we looked at the ratings variability due to model choice and parameter values. In the present chapter we are going to investigate the ratings parameter sensitivity further by applying global sensitivity analysis techniques. We address the issue of identifying the main sources of uncertainties for structured finance ratings. Sensitivity analysis is a powerful methodology for analysing how uncertainty in the output can be allocated to different sources of uncertainty in the inputs, see for example Saltelli et al (2008) for an introduction to global sensitivity analysis.

Global sensitivity analysis is based on exploring the space of all possible combinations for the input parameters as effective as possible. We propose to work with a screening method called the elementary effect method. The aim of screening methods is to identify the subsets of influential and non-influential input factors using a small number of model evaluations.

This chapter is mainly based on results first presented in Campolongo et al (2010), which contains a much more extensive analysis than what is presented here.

7.2 The ABS Structure

We will assume a static and homogeneous collateral pool with the characteristics presented in Table 7.1.

This collateral pool is backing three classes of notes: A (senior), B (mezzanine), and C (junior). The details of the notes are given in Table 7.2 together with other structural characteristics. To this basic liability structure we have added a cash reserve account. The allocation of *principal due to be paid* to the notes is done sequentially. Note that this refers to the calculation of principal due to be paid. The actual amount of principal paid to the different notes depends on the available funds at the relevant level of the waterfall.

The waterfall of the structure is presented in Table 7.3. The waterfall is a so called combined waterfall where the available funds at each payment date constitutes of both interest and

Collateral	
Number of loans	2000
Initial principal amount	100,000,000
Weighted average maturity	5 years
Weighted average coupon (per annum)	9%
Amortisation	Level-Pay
Payment frequency	Monthly

Table 7.1: *Collateral characteristics.*

Liabilities			
Class of Notes	Initial Principal Amount	Interest Rate (per annum)	Credit enhancement (%)
A	80,000,000	1%	20%
B	14,000,000	2%	6%
C	6,000,000	4%	0%
General Features			
Final Maturity	10 years		
Payment frequency	Monthly		
Principal allocation	Sequential		
Shortfall rate (per annum)	Applicable note coupon		
Senior expenses			
Issuer fees	1% of Outstanding Pool Balance		
Servicer fees	1% of Outstanding Pool Balance		
Payment frequency	Monthly		
Shortfall rate (per annum)	20%		
Cash reserve			
Target amount	1% of Outstanding Pool Balance		
Minimum required amount	0% of Outstanding Pool Balance		

Table 7.2: *Liability and structural characteristics.*

principal collections.

7.3 Cashflow Modelling

The cashflow from the pool is modelled as described in Chapter 2. The cash collections each month from the asset pool consists of interest payments and principal collections (scheduled repayments). Note that there are no unscheduled prepayments in the pool. These collections

Waterfall	
Level	Basic amortisation
1)	Issuer expenses
2)	Servicer expenses
3)	Class A interest
4)	Class B interest
5)	Class A principal
6)	Class B principal
7)	Reserve account reimburs.
8)	Class C interest
9)	Class C principal
10)	Class C additional returns

Table 7.3: *The waterfall used in the analysis.*

constitutes together with the principal balance of the reserve account *Available Funds* at the end of each month.

7.4 Modelling Defaults

We model defaults in the asset pool by using the Logistic model that was treated in Section 3.2.3 and repeated here for convenience:

$$F(t) = \frac{a}{1 + be^{-c(t-t_0)}}, \ 0 \leq t \leq T, \tag{7.1}$$

where a, b, c, and t_0 are positive constants. Parameter a controls the right endpoint of the curve.

As explained in Section 3.2.3 the Logistic model can easily be combined with a Monte Carlo based scenario generator to generate default scenarios by sampling a value for a from a given default distribution. In this chapter the Normal Invers distribution will be used to describe the cumulative portfolio default rate (PDR) distribution at the maturity of the structure:

$$F_{PDR}(y) = P[PDR < y] = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(y) - K^d(T)}{\sqrt{\rho}}\right) \tag{7.2}$$

where $0\% \leq y \leq 100\%$ and $K^d(T) = \Phi^{-1}(p(T))$. The default distribution in (7.2) is a function of the obligor correlation, ρ, and the default probability, $p(T)$, which are unknown and unobservable. Instead of using these parameters as inputs it is common to fit the mean and standard deviation of the distribution to the mean and standard deviation estimated from historical data, see discussion in Appendix A.

7.4.1 Quasi-Monte Carlo Algorithm

To perform the sensitivity analysis we need to run our rating algorithm multiple of times with different parameter settings as will be explained in Section 7.5. For each run of the rating algorithm we are using Monte Carlo simulation to calculate the expected loss and the expected average life of the notes. To speed up the sensitivity analysis we are using Quasi-Monte Carlo simulations based on Sobol sequences.[1] (See Kucherenko (2008), Kucherenko et al (2010), Kucherenko (2007), and Kucherenko et al (2000) for more information on Sobol sequences and their applications.)

7.5 Sensitivity Analysis - Elementary Effects

A very efficient method within the screening methods in identifying important factors with few simulations is the elementary effects method (**EE method**). It is very simple, easy to implement and the results are clear to be interpreted. It was introduced in Morris (1991) and has been refined by Campolongo and co-workers in Campolongo et al (2007). Because of the ABS structure's complexity it is computationally expensive and EE method is very well suited for the sensitivity analysis of the ABS model's output.

The elementary effect (EE) of a specific input factor is the difference in the model output when this particular input factor is changed, while the rest of the input factors are kept constant. The method is thus based on one-at-a-time sensitivity analysis. However, in the EE method the one-at-a-time analysis is done many times for each input, each time under different settings of the other input factors, and the sensitivity measures are calculated from the empirical distribution of the elementary effects.

Let us assume that there are k uncertain input parameters $X_1, X_2, \ldots, X_k$ (assumed to be independent) in our model. Examples of input parameters are the mean and standard deviation of the default distribution.

To each input factor we assign a range and a distribution. For example, we could assume that X_1 is the mean of the default distribution and that it takes values in the range $[5\%, 30\%]$ uniformly, that is, each of the values in the range is equally likely to be chosen. We could of course use non-uniform distributions as well, for example, an empirical distribution.

These input parameters and their ranges create an input space of all possible combinations of values for the input parameters. To apply the EE method we map each of the ranges to the unit interval $[0, 1]$ such that the input space is completely described by a k-dimensional unit cube.

The original method by Morris provides two sensitivity measures for each input factor $i = 1, 2, \ldots k$:

[1] We have been using the 'sobolset' class (with the 'MatousekAffineOwen' scramble algorithm) and 'RandStream' class (with the 'mrg32k3a' generator algorithm) in MATLAB® for generating Sobol sequences and pseudo random numbers, respectively.

- μ_i used to detect input factors with an important overall influence on the output;
- σ_i used to detect input factors involved in interaction with other factors or whose effect is not linear.

In order to estimate the sensitivity measures, a number of elementary effects must be calculated for each input factor. Morris suggested an efficient design that builds r trajectories in order to compute r elementary effects. Each trajectory is composed by $(k+1)$ points in the input space such that each input factor changes value only once. A characteristic of this design is that the points on the same trajectory are not independent and in fact two consecutive points differ only in one component. Points belonging to different trajectories are independent since the starting points of the trajectories are independent.

Once a trajectory has been generated, the model is evaluated at each point of the trajectory and one elementary effect for each input factor can be computed. The EE of input factor i is either:

$$EE_i(\mathbf{X}^{(l)}) = \frac{Y\left(\mathbf{X}^{(l+1)}\right) - Y\left(\mathbf{X}^{(l)}\right)}{\Delta} \tag{7.3}$$

if the ith component of $\mathbf{X}^{(l)}$ has been increased by Δ or

$$EE_i(\mathbf{X}^{(l)}) = \frac{Y\left(\mathbf{X}^{(l)}\right) - Y\left(\mathbf{X}^{(l+1)}\right)}{\Delta} \tag{7.4}$$

if the ith component of $\mathbf{X}^{(l)}$ has been decreased by Δ, where $Y\left(\mathbf{X}^{(l)}\right)$ is the model output of interest calculated in the point $\mathbf{X}^{(l)}$ on the trajectory.

By randomly sampling r trajectories, r elementary effects can be estimated for each input. Usually the number of trajectories (r) depends on the number of factors, on the computational cost of the model, and on the number of levels (p) that each input can vary across. It has been proven that the best choice is to let p be an even integer and Δ to be equal to $\frac{p}{2(p-1)}$ (see Saltelli et al (2004) and Saltelli et al (2008)).

The sensitivity measures are defined as the mean and the standard deviation of the distribution for the elementary effects of each input:

$$\mu_i = \frac{\sum_{j=1}^{r} EE_i^j}{r} \tag{7.5}$$

and

$$\sigma_i = \sqrt{\frac{\sum_{j=1}^{r}(EE_i^j - \mu_i)^2}{r-1}}. \tag{7.6}$$

When considering elementary effects with opposite signs related to the ith factor, the effects may cancel each other out generating a low μ_i value. To overcome this problem Morris recommends to consider both μ_i and σ_i simultaneously in order to be able to draw some conclusions on the factor importance.

In Campolongo et al (2007), two improvements of the original EE method are proposed. Firstly, the sampling strategy to generate the trajectories is constructed such that optimized trajectories are generated. A large number of different trajectories (e.g. 1000) is constructed and then r of them are selected in order to get the maximum spread in the input space. (See Campolongo et al (2007) for the all details about the design that builds the r trajectories of $(k+1)$ points in the input space.) The second improvement is the introduction of a new sensitivity measure based on the absolute values of the elementary effects:

$$\mu_i^* = \frac{\sum_{j=1}^{r} |EE_i^j|}{r}. \tag{7.7}$$

This new sensitivity measure overcomes the cancelation effect mentioned earlier and can alone be used to assess the importance of each factor in the model.

Section 7.6 presents the results obtained by applying this methodology to the ABS model.

7.6 The SA Experiment

In order to apply the elementary effect method we first have to identify the outputs we want to study and which input factors that are controllable (i.e. know) and which are uncontrollable (i.e. unknown). We also have to identify suitable ranges for the uncontrollable input factors.

The sensitivity analysis (SA) is performed on the structure presented in Section 7.2 and the default model presented in Section 7.3. The fundamental output in our study is the rating of the ABSs. These ratings are derived from the **Expected Average Life** and the **Expected Loss** of the notes. Because of that, these two quantities are the outputs the SA should investigate in order to assess the influence of the unknown inputs in the ABS ratings.

Without loss of generality, the investor is assumed to be informed about the collateral pool's characteristics and the structural characteristics given in Table 7.1 and Table 7.2, respectively, and the waterfall in Table 7.3. These are treated as controllable input factors.

Assuming the default distribution of the pool to follow a Normal Inverse distribution and the default curve to be modelled by the Logistic model, the uncertain input factors in the SA are not related to the model choice but to the parameters of the cumulative default rate distribution, the default timing (the Logistic function) and the recoveries:

- the mean (μ_{cd}) and the standard deviation (σ_{cd}) of the Normal Inverse distribution;
- b, c, and t_0 of the Logistic Function;
- the recovery rate (RR) and the recovery lag (T_{RL}).

The input ranges are summarised in Table 7.4 and in the subsequent sections we will give some motivation to our choice of ranges.

Parameter	Range
μ_{cd}	$[5\%, 30\%]$
$Coeff.Variation\ (\frac{\sigma_{cd}}{\mu_{cd}})$	$[0.25, 1]$
b	$[0.5, 1.5]$
c	$[0.1, 0.5]$
t_0	$[\frac{T}{3}, \frac{2T}{3}]$
T_{RL}	$[6, 36]$
RR	$[5\%, 50\%]$

Table 7.4: *Ranges for the uncertain input factors.*

7.6.1 Ranges Associated with μ_{cd} and σ_{cd}

The mean and standard deviation of the default distribution are typically estimated using historical data provided by the originator of the assets (see Moody's (2005b) and Raynes and Ruthledge (2003)). In our SA we will assume that the mean cumulative default rate at maturity T (μ_{cd}) takes values in the interval $[5\%, 30\%]$. This is equivalent to assuming that the probability of default before T for a single asset in the pool ranges from 5% to 30%. (Recall that the mean of the Normal Inverse distribution is equal to the probability of default of an individual asset).

We make the range of the standard deviation (σ_{cd}) a function of μ_{cd} by using a range for the coefficient of variation, σ_{cd}/μ_{cd}. This gives us the opportunity to assume higher standard deviation (i.e. uncertainty) for high values of the default mean than for low values of the mean, which implies that we get higher correlation in the pool for high values of the mean than for low values, see Figure 7.1.

7.6.2 Ranges Associated with b, c, and t_0 in the Logistic Function

The parameters can be estimated from empirical loss curve by fitting the Logistic curve to a historical default curve (see Raynes and Ruthledge (2003)).

Because we want to cover a wide range of different default scenarios we have chosen the following parameter ranges:

- $c \in [0.1, 0.5]$;
- $t_0 \in [\frac{T}{3}, \frac{2T}{3}]$;
- $b \in [0.5, 1.5]$.

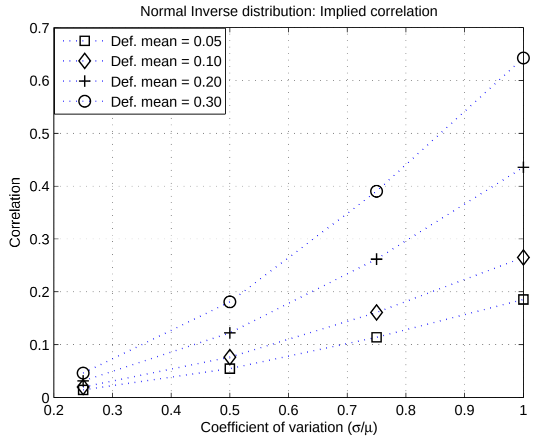

Figure 7.1: *Implied correlation versus coefficient of variation*

Inspecting the behavior of the Logistic functions in Figure 7.2 provides some insight to the possible scenarios generated with these parameter ranges and gives an intuitive understanding of the different parameters influence on the shape of the curve.

7.6.3 Ranges Associated with Recovery Rate and Recovery Lag

Recovery rates and recovery lags are very much dependent on the asset type in the underlying pool and the country where they are originated. For SME loans, for example, Standard and Poor's made the assumption that the recovery lag is between 12 months to 36 months depending on the country (see Standard and Poor's (2004a)). Moody's uses different recovery rate ranges for SME loans issued in, for example, Germany (25%−65%) and Spain (30%−50%), see Moody's (2009a).

The range associated with recovery lag T_{RL} has been fixed to be equal to $[6, 36]$ months and with the recovery rate to be equal to $[5\%, 50\%]$.

7.7 Numerical Results

The fundamental output in the ABS model is the rating that addresses the expected loss a note investor might suffer. With screening the input space of the ABS model using the elementary effect method we aim at answering the questions: Is the rating of the ABS reliable? Where is the uncertainty coming from, i.e. which input factors are more important in determining the uncertainty in the rating response? In this section we present results concerning the sensitivity measures μ^* and σ and refer to Campolongo et al (2010) for the full study.

Having estimated both the expected loss and the expected average life we can map these values into Moody's ratings using the Moody's Idealised Cumulative Expected Loss Table (see Moody's (2000b)).

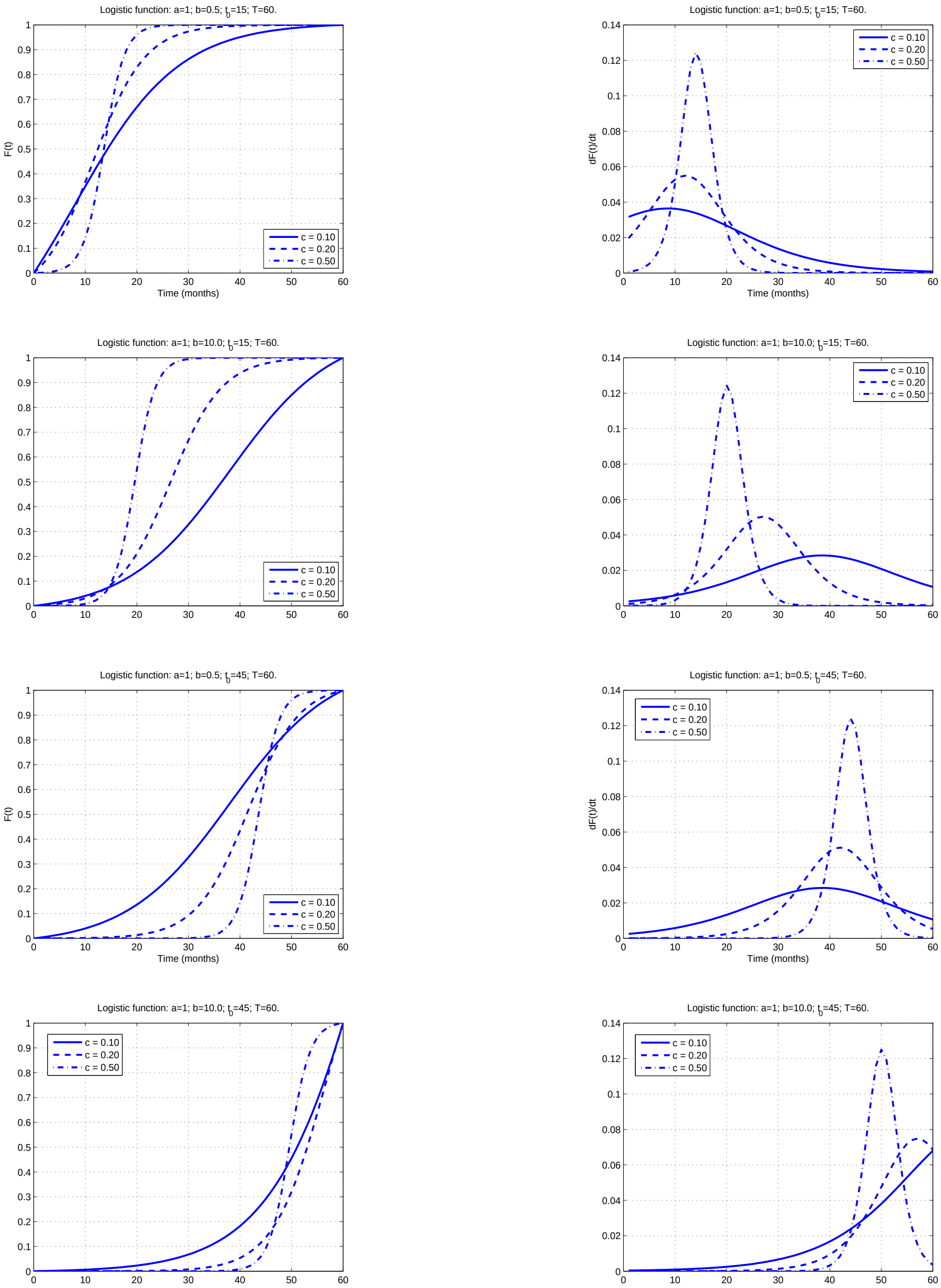

Figure 7.2: *The Logistic function and its derivative for different values of* b, c *and* t_0. *Parameter values:* $a = 1$ *and* $T = 60$.

We apply the elementary effect method improved by Campolongo et al (2007) with $r = 10$ optimized trajectories of $p = 4$ points. This choice has been demonstrated to produced valuable results in a general application of the sensitivity analysis.[2] Having $k = 7$ input parameters, a

[2] Saltelli et al (2004), p. 102.

total number of 80 ($N = r(k+1)$) model evaluations have to be performed. The ABS model runs 2^{14} times for each evaluation, in order to guarantee the convergence. Thus, we simulate 2^{14} scenarios to get the ratings for each parameter setting of the input factors.

7.7.1 Uncertainty Analysis

The empirical distribution of the ratings on each note can be used to obtain information on the uncertainty in the model (see Figure 7.3).

The A notes rating seems to be reliable and the performance is of high quality. We get good ratings with low degree of risk at 80% of the time and the best rating, Aaa, is obtained at 54% of time. The B notes ratings are not reliable because of the oscillation between 'Aaa' rating and 'Unrated'. The C notes are considered to be unrated at 46% of time so that the investor should be aware that the C notes are highly speculative and typically may suffer significant losses.

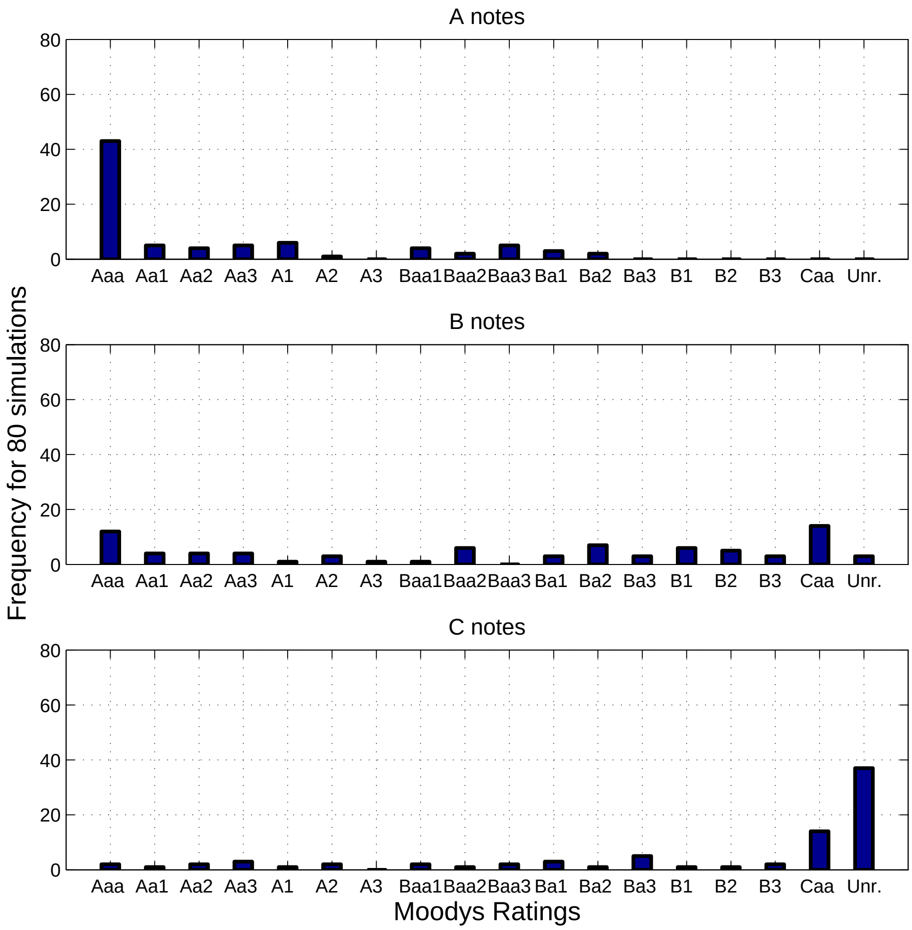

Figure 7.3: *Moody's Ratings empirical distribution obtained by* 80 *simulations*

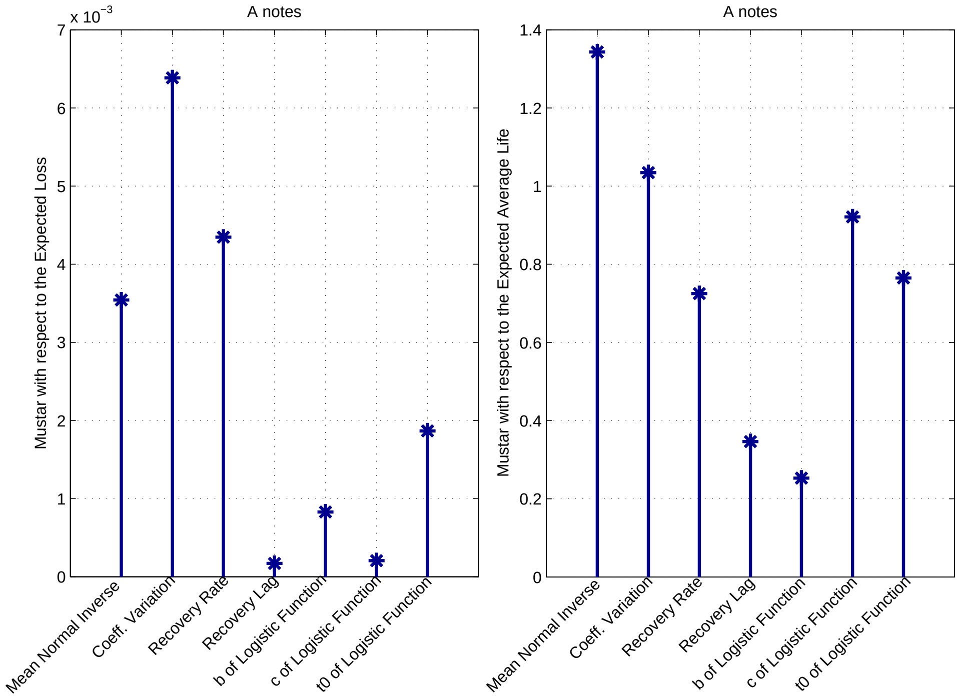

Figure 7.4: *Bar plots of the μ^* values for the A notes*

7.7.2 Sensitivity Measures μ^* and σ

In order to investigate where the uncertainty in the ratings comes from we focus on the expected loss and the expected average life. The SA provides two fundamental measures to rank each input factor in order of importance, the μ^* and σ for the elementary effects of each input. In Figure 7.4, Figure 7.5, and Figure 7.6 bar plots of μ^* are presented to visually depict the rank of the different input factors. The input factors nonlinear effects are illustrated in Figure 7.7, Figure 7.8, and Figure 7.9 with the help of scatter plots of (μ^*, σ).

At a first look at the bar plots it is clear that the least influential factors across all outputs are the recovery lag and the Logistic function's b parameter and hence they could be fixed without affecting the variance of the outputs of interest and therefore the ratings to a great extent.

Among the influential parameters the mean of the default distribution (μ_{cd}) is clearly the most influential input parameter over all for all three notes. It is characterized by high μ^* values for both the expected loss and the expected average life of all the notes, confirming the strong influence the mean default rate assumption has on the assessment of ABSs. The only exception from ranking the mean default rate as the most influential input factor is the expected loss of the A notes. Here the coefficient of variation is ranked the highest with the recovery rate as second and the mean default rate as third.

The input factors with the highest μ^* values for expected loss and expected average life, respectively are the same for the B notes with just small changes of the ranking. The same feature holds true also for the C notes. For these two classes of notes the mean default rate is the absolutely most influential input factor. For the A notes the ranking of the input factors differs more significantly between expected loss and expected average life. For the expected loss the two parameters of the default distribution together with the recovery rate are clearly the most influential. For the expected average life the Logistic function's c and t_0 should be counted as influential together with default distributions parameters and the recovery rate.

In the scatter plots in Figures 7.7 – 7.9 it is visible that the most influential factors also have the highest values of σ which indicate strong nonlinear effects and/or interactions with other factors.

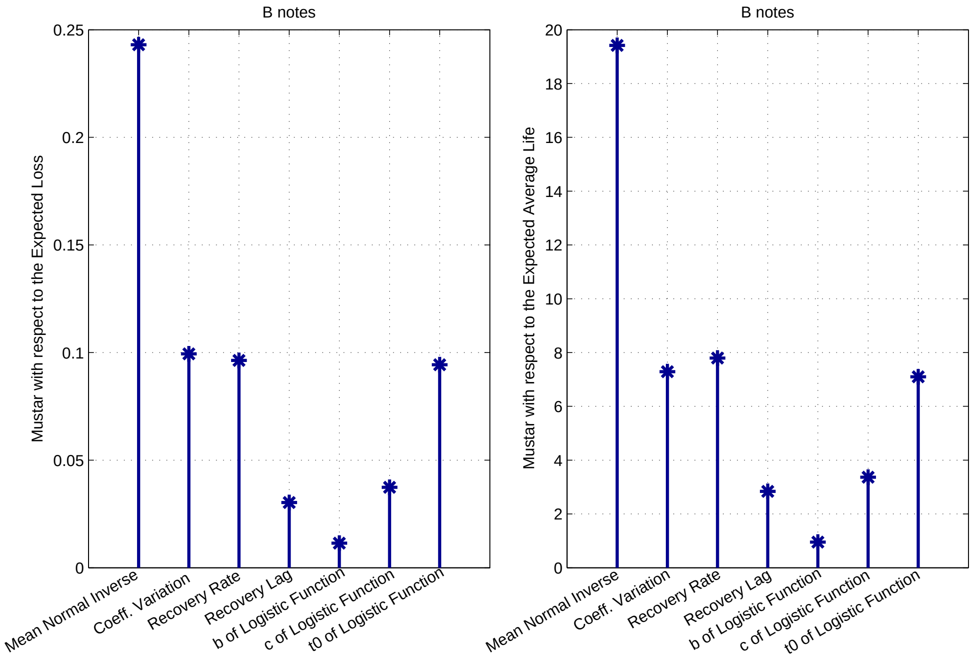

Figure 7.5: *Bar plots of the μ^* values for the B notes*

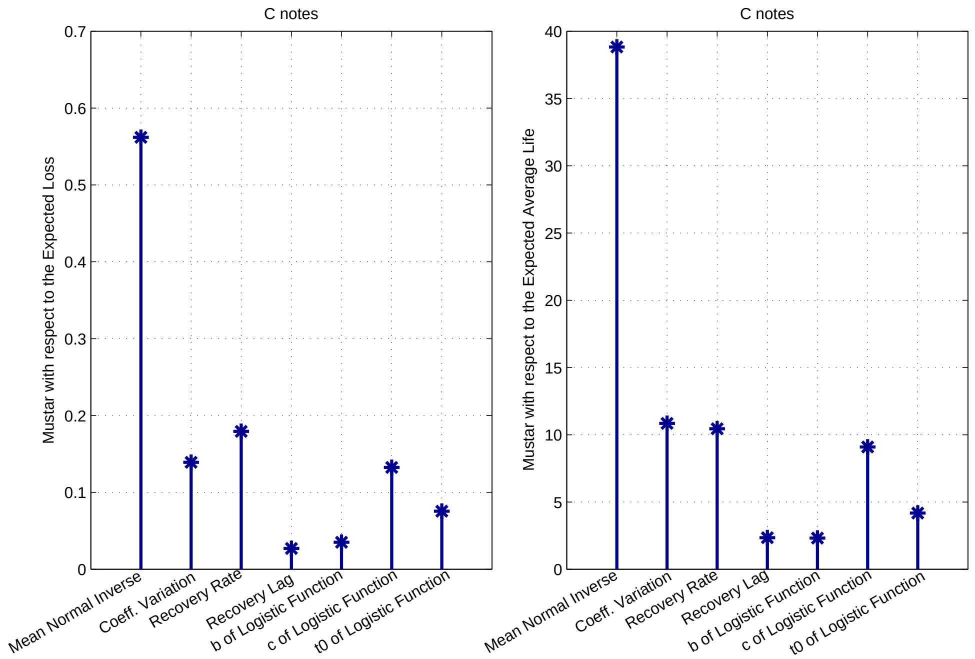

Figure 7.6: *Bar plots of the μ^* values for the C notes*

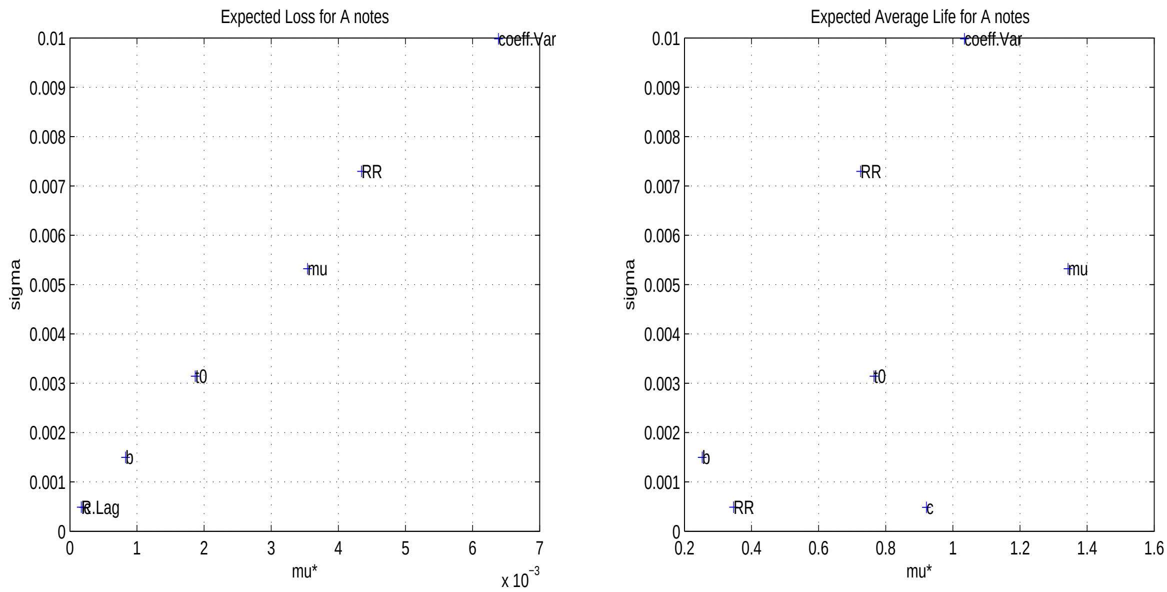

Figure 7.7: *Scatter plots of the sensitivity measures μ^* and σ for the Expected Loss and Expected Average Life in the A notes.*

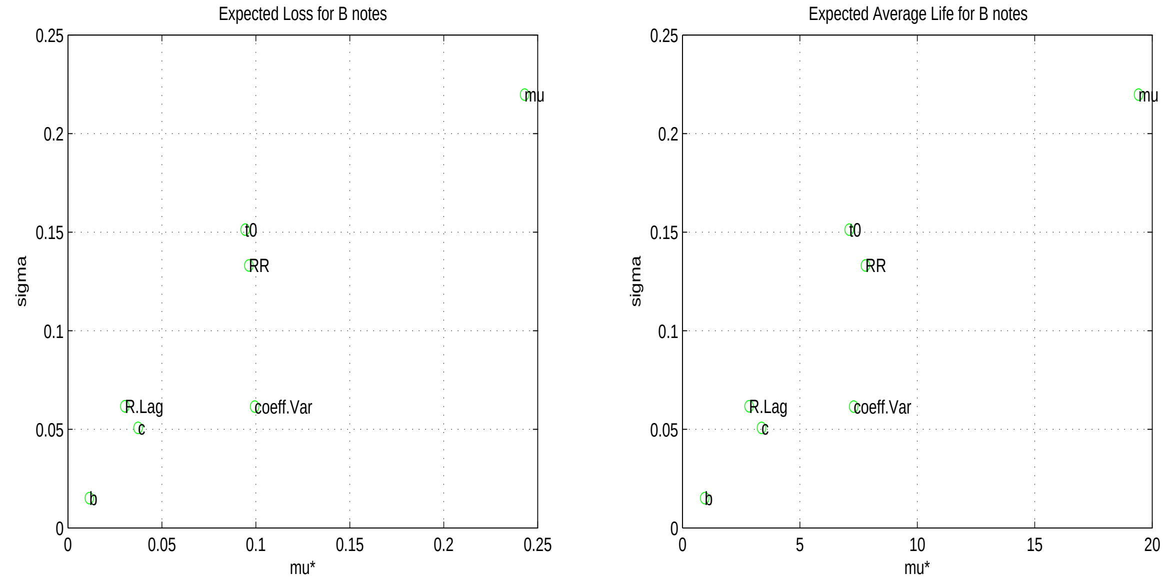

Figure 7.8: *Scatter plots of the sensitivity measures μ^* and σ for the Expected Loss and Expected Average Life in the B notes.*

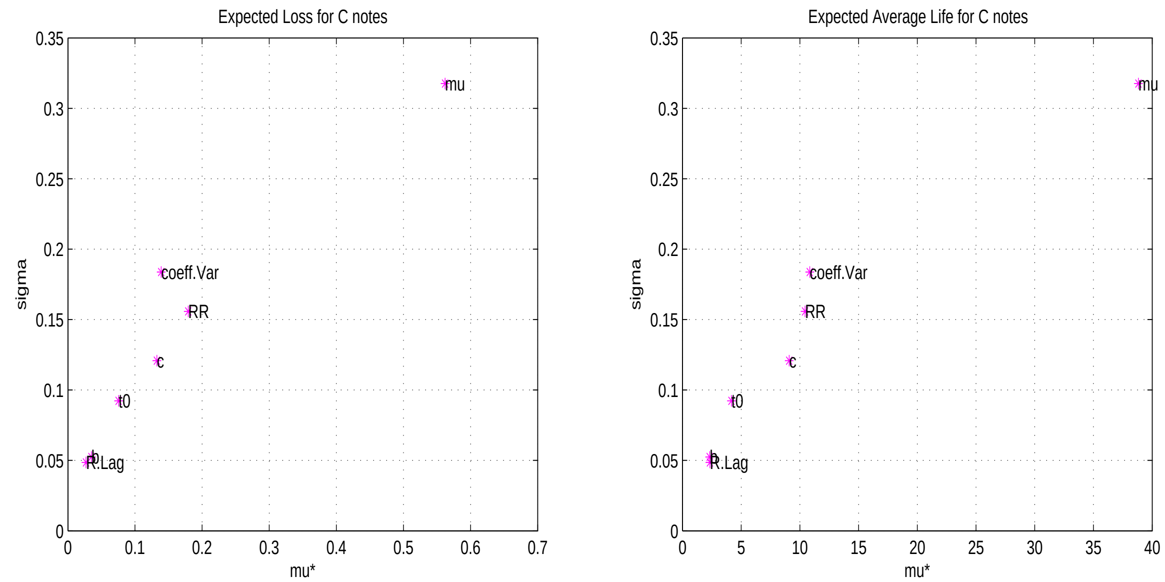

Figure 7.9: *Scatter plots of the sensitivity measures μ^* and σ for the Expected Loss and Expected Average Life in the C notes.*

7.8 Conclusions

In this chapter, we have shown how global sensitivity analysis can be used to analyse the main sources of uncertainty in the ratings of asset-backed securities (ABSs). Due to the fact that deriving ratings for ABSs is computationally expensive, the elementary effect method was chosen for the analysis. The elementary effect method was applied to a test example consisting of a large homogeneous pool of assets backing three classes of notes (senior, mezzanine and junior).

The result of the sensitivity analysis experiment has led to the conclusion that the least influential factors across all outputs are the recovery lag and the Logistic function's b parameter. Hence they could be fixed without affecting the variance of the outputs of interest and therefore the ratings to a great extent.

The mean of the default distribution (μ_{cd}) was found to be the most influential input parameter among all inputs for all three notes. It is characterized by high μ^* values for both the expected loss and the expected average life of all the notes, confirming the strong influence the mean default rate assumption has on the assessment of ABSs. The only exception from ranking the mean default rate as the most influential input factor could be found for the expected loss of the A notes. Here the coefficient of variation is ranked the highest with the recovery rate as second and the mean default rate as third.

It was further indicated that the most important input factors contributes to the rating uncertainty in a nonlinear way and/or with possible interactions between several of the factors.

Chapter 8

Summary

8.1 Introduction to Asset Backed Securities

Asset-Backed Securities (ABSs) are financial instrument backed by pools of assets. ABSs are created through a securitisation process, where assets are pooled together and the liabilities backed by these assets are tranched such that the ABSs have different seniority and risk-return profiles.

Due to the complex nature of securitisation deals there are many types of risks that have to be taken into account. The risks arise from the collateral pool, the structuring of the liabilities, the structural features of the deal and the counterparties in the deal. The main types of risks are *credit risk*, *prepayment risk*, *market risks*, *reinvestment risk*, *liquidity risk*, *counterparty risk*, *operational risk* and *legal risk*.

The quantitative analysis of an ABS is done through the modelling of the cashflows within the ABS deal. The modelling consists of two steps. The first step is to model the cash collections from the asset pool, which depends on the behaviour of the pooled assets. This can be done in two ways: with a top-down approach, modelling the aggregate pool behaviour; or with a bottom-up or loan-by-loan approach modelling each individual loan. It is in this step quantitative models and assumptions are needed. The second step is to model the distribution of the cash collections to the note holders, the issuer, the servicer and other transaction parties. This distribution of the cash collection, the so called *priority of payments* or *waterfall*, is described in detail in the *Offering Circular* or *Deal Prospectus*.

The cash collections from the asset pool consist of *interest collections* and *principal collections* (both scheduled repayments, unscheduled prepayments and recoveries). There are two parts of the modelling of the cash collections from the asset pool. Firstly, the modelling of performing assets, based on asset characteristics such as initial principal balance, amortisation scheme, interest rate and payment frequency and remaining term. Secondly, the modelling of the assets becoming delinquent, defaulted and prepaid, based on assumptions about the delinquency rates, default rates and prepayment rates together with recovery rates and recovery lags.

To be able to model cash collections from the asset pool it is needed to generate default and prepayment scenarios. We divide the default and prepayment models into two groups, deterministic (Chapter 3) and stochastic models (Chapter 4). The deterministic models are simple models with no built in randomness, i.e., as soon as the model parameters are set the evolution of the defaults and prepayments are know for all future times. The stochastic models are more advanced, based on stochastic processes and probability theory. By modelling the evolution of defaults with stochastic processes we can achieve three objectives: stochastic timing of defaults; stochastic monthly default rates; and correlation (between defaults, between prepayments and between defaults and prepayments).

The quantitative models and approaches used today are either deterministic, in the sense that the distribution of defaults or prepayments are certain as soon as the parameters of the models are fixed and the cumulative default rate and prepayment rate, respectively, are chosen, or they are stochastic and based on the Normal distribution. In the report a collection of default and prepayment models are presented, ranging from very simple deterministic models to advanced stochastic models. We have proposed a set of new stochastic models that are based on more flexible distributions than the Normal, which take into account more extreme events.

8.2 Rating Agencies Methodologies

Two of the major rating agencies, Moody's and Standard & Poor's (S&P's), methodologies for rating securitisation transactions has also been studied (Chapter 5). The focus in the study has been on their methodologies for rating SME (Small and Medium-sized Enterprizes) securitisation transactions. The two rating agencies have two different meanings of their ratings. Moody's rating is an assessment of the *expected loss* that a class of notes may experience during a certain time period, while S&P's rating is an assessment of the *probability of default* of the class of notes and addresses the likelihood of full and timely payment of interest and the ultimate payment of principal.

Both Moody's and S&P's discriminate between granular and non-granular SME portfolios and applies different approaches to the two categories.

For non-granular SME portfolios both rating agencies use a loan-by-loan or bottom-up approach and model each individual asset in the pool. Moody's uses its CDOROM^{TM} tool, which uses Normal factor models (with dependence structure based on the Gaussian copula approach); S&P's is using its CDO Evaluator® model, which is based on the Gaussian copula approach. In both cases, thus, are the underlying mathematical tool to introduce dependence in the portfolios the Gaussian copula approach. Monte Carlo simulations are used to generate defaults in the asset pool and to derive a default distribution. The difference between the two methodologies lies in the use of the tool or model.

In Moody's methodology, the default scenario generated by *each* Monte Carlo simulation is fed into the cash flow model and the losses on the ABSs are derived. This is done for a large number of simulations and an estimate of the expected loss on each ABS is derived. The cash flow analysis is thus an integrated part of the simulations. The expected losses are mapped to a rating for each ABS using Moody's loss rate tables.

In S&P's methodology, the Monte Carlo simulations generate a probability distribution of potential portfolio default rates that is used to derive a set of scenario default rates (SDRs), one for each rating level. Each SDR represents the maximum portfolio default rate that an ABS with the desired rating should be able to withstand without default. These SDRs are then used to create different stressed rating scenarios that are applied in a cash flow analysis, which assesses if the ABS under consideration can withstand the stresses associated with the targeted rating level and therefore can receive the corresponding rating level.

For granular SME portfolios, Moody's uses its ABSROM^{TM} tool, which uses a default rate distribution to generate default scenarios and the corresponding likelihood of each scenario. The default rate distribution's mean and standard deviation is estimated using historical data provided by the originator. Running a cash flow model with the different default scenarios, stressing the default timing, the expected loss on the notes are calculated. S&P's applies its *actuarial approach* for granular SME portfolios, which is based on deriving base case default and recovery rates from historical data in order to stress defaults over the life of the transaction in different rating scenarios in a cash flow analysis.

8.3 Model Risk and Parameter Sensitivity

The models influence on the ratings of structured finance transactions were studied on a transaction with two classes of notes (Chapter 6). The findings can be summarised by saying that model risk is omnipresent. The model risk was assessed by comparing three different default models with a benchmark model, the Normal one-factor model. What could be observed for a low cumulative default rate assumption (10%) was that there was no or just one notch difference in rating for the senior notes and one to three notches difference for the junior notes, between the models output. However, increasing the cumulative default rate to a high number (40%) the rating differed with as much as three notches for the senior notes and four notches for the junior notes. Thus, for high cumulative default rates the model risk becomes more significant.

The ratings sensitivity to the cumulative default rate assumption was also studied by analysing the number of notches the ratings changed for a given default model when the default rate increased. As could be expected, the ratings are very dependent on the cumulative default rate assumption. For the junior notes the rating differed with as much as seven to eight notches, when the cumulative default rate changes from 10% to 40%. For the senior notes the changes were one to four notches.

In a second analysis we analysed the variability in the ratings related to uncertainty in the mean default rate, asset correlation and recovery rate under the Normal one-factor model. Big variability in the ratings could be observed when the three parameters were allowed to take values within their ranges. It could also be observed that the respons in the ratings due to a change in one of the parameters depended on the values of the other two parameters. For example, the value of the default mean had greater impact on the ratings variability for low recovery rates than for high recovery rates for both classes of notes.

8.4 Global Sensitivity Analysis

To further investigate the ABS ratings parameter sensitivity we have proposed to apply global sensitivity analysis techniques (Chapter 7). Global sensitivity analysis is the study of how the uncertainty in a model's input affects the model's output and investigates the relative importance of each input in determining this uncertainty. In global sensitivity analysis the input parameters take values in ranges given by the analyst. These input ranges creates an input space and an aim with global sensitivity analysis is to explore this input space and analyse the model output's respons to different combinations of input parameter values. To identify which input parameters are the most influential and which are non-influential a screening method is suitable for ranking the input parameters in a computationally expensive problem as the rating of ABSs. We have chosen to work with the elementary effect (EE) method, which is best practise among the screening methods. The EE method explores the input space in an optimal way.

To illustrate the method we applied the EE method to an ABS structure with three classes of notes: A (senior), B (mezzanine) and C (junior), backed by a pool of homogeneous assets, which could default but not prepay. Prepayment was excluded to simplify the experiment. Seven input parameters were assumed to be uncertain in the experiment: the mean and coefficient of variation of the default distribution, three parameters controlling the default timing (modelled by the Logistic function), and the recovery rate and the recovery lag. The output studied were the expected loss (EL) and the expected weighted average life (EAL) of the notes. The result of the experiment indicated that the default distributions parameters and the recovery rate were the most influential inputs for both EL and EAL for all classes of notes. The recovery lag and one of the parameters of the default timing were found to be non-influential.

The identification of an input parameter as non-influential gives an opportunity to simplify the model by fix this parameter to a constant value within the parameter's range (so called factor fixing) without influencing the variability of the output of the model. On the other hand, the categorisation of an input parameter as influential indicates that it requires careful analysis of its range (so called factor prioritisation).

Appendix A

Large Homogeneous Portfolio Approximation

A.1 The Gaussian One-Factor Model and the LHP Approximation

In the Gaussian one-factor model an obligor is assumed to default if the value of its creditworthiness is below a pre-specified value. The creditworthiness of an obligor is modeled through a latent variable:

$$Z_n = \sqrt{\rho}X + \sqrt{1-\rho}X_n, \; n = 1, 2, ..., N, \tag{A.1}$$

where X is the systemic factor and X_n with $n = 1, 2, ..., N$ are the idiosyncratic factors; all assumed to be standard normal random variables with mean zero and unit variance and ρ is the correlation between two assets:

$$Corr(Z_m, Z_n) = \rho, \; m \neq n.$$

The nth loan defaulted by time t if

$$Z_n \leq K_n^d(t),$$

where $K_n^d(t)$ is the time dependent default barrier. Under the assumption of the homogenous pool, each asset behaves as the average of the assets in the pool and we can set $K_n^d(t) = K^d(t)$ for the all n. The default barrier can be chosen such that:

$$P(Z_n \leq K^d(T)) = p(T), \tag{A.2}$$

where $p(T)$ is the probability of default of a single obligor in the pool by maturity T. It implies $K^d(T) = \Phi^{-1}(p(T))$.

The cumulative portfolio default rate is given by:

$$PDR(T) = \sum_{n=1}^{N} \frac{D_n(T)}{N} \tag{A.3}$$

where $D_n(T)$ is the default indicator of asset n. The default indicator $D_n(T)$ equals one (with probability $p(T)$) if asset n defaulted by time T and zero otherwise.

The expected value of the portfolio default rate at time T is

$$\begin{aligned} E[PDR(T)] &= E[\tfrac{1}{N}\textstyle\sum_{n=1}^{N} D_n(T)] \\ &= \tfrac{1}{N}\textstyle\sum_{n=1}^{N} E[D_n(T)] \\ &= E[D_1(T)] \\ &= P(D_1(T) = 1) \\ &= P(Z_1 \leq K^d(T)) = p(T), \end{aligned} \tag{A.4}$$

where the third equality follows by the homogeneous portfolio assumption and the last equality holds by definition. Thus under the homogeneous portfolio assumption the portfolio default rate mean is equal to the individual loan's probability of default $p(T)$.

The default indicators in (A.3) are correlated and we can not use the Law of Large numbers to derive a limiting distribution. However, conditional on the common factor X, the default indicators are independent and we can apply the Law of Large Numbers.

Conditional on the common factor the portfolio default rate at time T is given by

$$PDR(T; X = x) = \sum_{n=1}^{N} \frac{D_n(T;\ X = x)}{N} \tag{A.5}$$

where $D_n(T;\ X = x)$ is the default indicator of asset n given the systematic factor X.

By the Law of Large Numbers, as N tends to infinity we get:

$$PDR(T; X = x) \underset{N\to\infty}{\to} E[P_{cd}|X = x] = \frac{1}{N}\sum_{n=1}^{N} p(x) = \frac{1}{N} N p(x) = p(T, x), \tag{A.6}$$

where $p(T, x)$ is the default probability for an individual asset given $X = x$:

$$\begin{aligned} p(T, x) &= P(Z_n \leq K^d(T)|X = x) \\ &= P(\sqrt{\rho}X - \sqrt{1-\rho}X_n \leq K^d(T)|X = x) \\ &= \Phi\left(\frac{K^d(T) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \end{aligned} \tag{A.7}$$

It follows that the distribution of $PDR(T;X)$ is[1]:

$$
\begin{aligned}
F_{PDR(T;X)}(y) &= P\left(PDR(T;X) < y\right) \\
&= P\left(p(X) < y\right) \\
&= P\left(\Phi\left(\frac{K^d(T) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) < y\right) \\
&= P\left(X > \frac{K^d(T) - \sqrt{1-\rho}\,\Phi^{-1}(y)}{\sqrt{\rho}}\right).
\end{aligned}
\tag{A.8}
$$

Using the symmetry of the normal distribution, we get:

$$
F^{LHP}_{PDR(T;X)}(y) = P(PDR(T;X) < y) = \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(y) - K^d(T)}{\sqrt{\rho}}\right) \tag{A.9}
$$

where $0\% \leq y \leq 100\%$ and $K^d(T) = \Phi^{-1}(p(T))$. Note that the right hand side of (A.9) is independent of the systemic factor X. The distribution in (A.9) is sometimes called the **Normal Inverse distribution**, see, for example, Moody's (2003).

Thus for a reasonably large homogeneous portfolio we can use the distribution in (A.9) as an approximation to the portfolio default rate distribution.

We illustrate in Figure A.1 the portfolio default rate distribution's dependence on the correlation parameter ρ, under the assumption that the default mean is 30%. As can be seen from the plots, under a low correlation assumption the PDR distribution will have a bell shaped form, but as the asset correlation increases the mass of the distribution is shifted towards the end points of the PDR interval, increasing the likelihood of zero or a very small fraction of the portfolio defaulting and the likelihood of the whole portfolio defaulting. This is natural since a very high correlation (close to one) means that the loans in the pool are likely to either survive together or default together. In general, it can be said that the PDR distribution becomes flatter and more mass is shifted towards the tails of the distribution when the default mean is increased.

A.2 Calibrating the Distribution

The default distribution in (A.9) is a function of the obligor correlation, ρ, and the default probability, $p(T)$, which are unknown and unobservable. Instead of using these parameters as inputs it is common to fit the mean and standard deviation of the distribution to the mean and standard deviation, respectively, estimated from historical data (see, for example, Moody's (2005b) and Raynes and Ruthledge (2003)). Let us denote by μ_{cd} and σ_{cd} the estimated mean and standard deviation, respectively.

The mean of the distribution is equal to the probability of default for a single obligor, $p(T)$, so $p(T) = \mu_{cd}$. As a result there is only one free parameter, the correlation ρ, left to adjust to fit the distribution's standard deviation to σ_{cd}, which can be done numerically by minimising $\sigma^2_{cd} - Var_\rho(PDR(T))$, where the subscript is used to show that the variance is a function of ρ.

[1] The above convergence is in probability, which implies convergence in distribution.

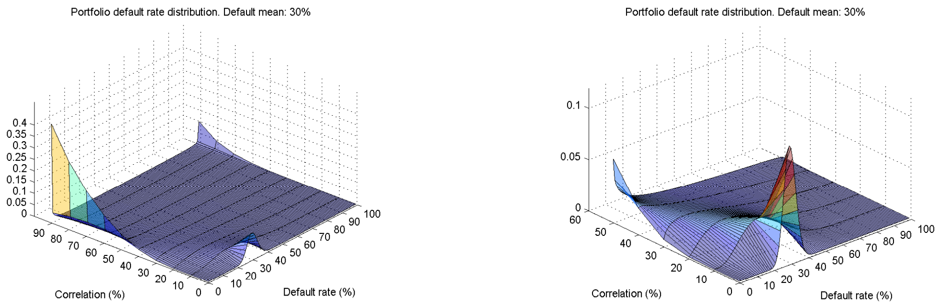

Figure A.1: *Portfolio default rate versus correlation. Large homogeneous portfolio approximation. Left panel: correlation between* 1% *and* 90%. *Right panel: correlation between* 1% *and* 50%. *Mean default rate:* 30%.

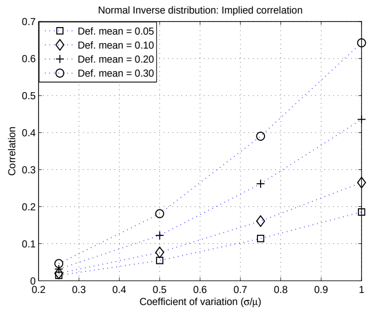

Figure A.2: *Implied correlation for different values of mean and coefficient of variation (standard deviation divided by mean) equal to 0.25, 0.5, 0.75, 1.0 and 1.25.*

Looking at the correlation values given in Figure A.2 and the density plots in Figure A.1 one can see that the corresponding default distributions will have very different shapes. Ranging from bell shaped curves to very heavy tailed ones with the mass almost completely concentrated at zero and one.

It is important to understand that the behavior of the correlation and the default probability shown in Figure A.2 should not be taken as a general rule. The graphs show the result of fitting the distribution to means and standard deviations in the distribution's "comfort zone", i.e, values that will give good fits. (The root mean squared error is of the order of magnitude of 10^{-11} for the shown results.) For combinations of the default mean and the coefficient of variation that result in an implied correlation equal to one the calibration will stop since it cannot improve the root mean squared error, which in these situations will be much larger than for the values shown in Figure A.2.

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