EURANDOM PREPRINT SERIES
2010-047

Parabolic Anderson model with a finite number of moving catalysts

F. Castell, O. Gün and G. Maillard
ISSN 1389-2355

Parabolic Anderson model with a finite number of moving catalysts

F. Castell, O. Gün and G. Maillard

Abstract We consider the parabolic Anderson model (PAM) which is given by the equation $\partial u/\partial t = \kappa\Delta u + \xi u$ with $u\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, and $\xi\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is a space-time random environment that drives the equation. The solution of this equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ.

In the present paper we focus on the case where ξ is a system of n independent simple random walks each with step rate $2d\rho$ and starting from the origin. We study the *annealed* Lyapunov exponents, i.e., the exponential growth rates of the successive moments of u w.r.t. ξ and show that these exponents, as a function of the diffusion constant κ and the rate constant ρ, behave differently depending on the dimension d. In particular, we give a description of the intermittent behavior of the system in terms of the annealed Lyapunov exponents, depicting how the total mass of u concentrates as $t \to \infty$. Our results are both a generalization and an extension of the work of Gärtner and Heydenreich [2], where only the case $n = 1$ was investigated.

F. Castell
CMI-LATP, Université de Provence, 39 rue F. Joliot-Curie, F-13453 Marseille Cedex 13, France, e-mail: castell@cmi.univ-mrs.fr

O. Gun
CMI-LATP, Université de Provence, 39 rue F. Joliot-Curie, F-13453 Marseille Cedex 13, France, e-mail: gun@cmi.univ-mrs.fr

G. Maillard
CMI-LATP, Université de Provence, 39 rue F. Joliot-Curie, F-13453 Marseille Cedex 13, France, e-mail: maillard@cmi.univ-mrs.fr, and EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

1 Introduction

1.1 Model

The parabolic Anderson model (PAM) is the partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \xi(x,t) u(x,t), \\ \quad u(x,0) = 1, \end{cases} \qquad x \in \mathbb{Z}^d,\, t \geq 0\,. \tag{1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, acting on u as

$$\Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ y \sim x}} [u(y,t) - u(x,t)]$$

($y \sim x$ meaning that y is nearest neighbor of x), and

$$\xi = (\xi_t)_{t \geq 0} \quad \text{with} \quad \xi_t = \{\xi(x,t) \colon\, x \in \mathbb{Z}^d\}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation.

One interpretation of (1) comes from population dynamics by considering a system of two types of particles A and B. A particles represent "catalysts", B particles represent "reactants" and the dynamics is subject to the following rules:

- A-particles evolve independently of B-particles according to a prescribed dynamics with $\xi(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to the number of A-particles present at the same location;
- the initial configuration of B-particles is that there is exactly one particle at each lattice site.

Then, under the above rules, $u(x,t)$ represents the average number of B-particles at site x at time t conditioned on the evolution of the A-particles.

It is possible to add that B-particles die at rate $\delta \in [0,\infty)$. This leads to the trivial transformation $u(x,t) \to u(x,t)e^{-\delta t}$. We will therefore always assume that $\delta = 0$. It is also possible to add a coupling constant $\gamma \in (0,\infty)$ in front of the ξ-term in (1), but this can be reduced to $\gamma = 1$ by a scaling argument.

In what follows, we focus on the case where

$$\xi(x,t) = \sum_{k=1}^{n} \delta_x\big(Y_k^\rho(t)\big) \tag{2}$$

with $\{Y_k^\rho \colon 1 \leq k \leq n\}$ a family of n independent simple random walks (ISRW), where for each $k \in \{1,\ldots,n\}$, $Y_k^\rho = (Y_k^\rho(t))_{t\geq 0}$ is a simple random walk with step rate $2d\rho$ starting from $Y_k^\rho(0) = 0$. We write $\mathbb{P}_0^{\otimes n}$ and $\mathbb{E}_0^{\otimes n}$ to denote respectively the law and the expectation of the family of n ISRW $\{Y_k^\rho : 1 \le k \le n\}$ where initially all of the walkers are located at 0.

Under this choice of catalysts, we will prove existence and derive both qualitative and quantitative properties of the annealed Lyapunov exponents (defined in Section 1.2). After that, we will discuss the intermittent behavior of the solution u of the PAM in terms of the Lyapunov exponents.

1.2 Lyapunov exponents and intermittency

Our focus will be on the *annealed* Lyapunov exponents that describes the exponential growth rate of the successive moments of the solution of (1).

By the Feynman-Kac formula, the solution of (1) reads

$$u(x,t) = \mathrm{E}_x\left(\exp\left[\int_0^t \xi\left(X^\kappa(s), t-s\right) ds\right]\right), \tag{3}$$

where $X^\kappa = (X^\kappa(t))_{t\ge 0}$ is the simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$ and E_x denotes expectation with respect to X^κ given $X^\kappa(0) = x$. Taking into account our choice of catalytic medium in (2) we define $\Lambda_p(t)$ as

$$\begin{aligned}\Lambda_p(t) &= \frac{1}{t}\log \mathbb{E}_0^{\otimes n}\left([u(x,t)]^p\right)^{1/p} \\ &= \frac{1}{pt}\log\left(\mathbb{E}_0^{\otimes n} \otimes \mathrm{E}_x^{\otimes p}\right)\left(\exp\left[\sum_{j=1}^p \sum_{k=1}^n \int_0^t \delta_0\left(X_j^\kappa(s) - Y_k^\rho(t-s)\right) ds\right]\right), \end{aligned} \tag{4}$$

where $\{X_j^\kappa,\ j = 1,\ldots,p\}$ is a family of p independent copies of X^κ and $\mathrm{E}_x^{\otimes p}$ stands for the expectation of this family with $X_j^\kappa(0) = x$ for all j.

If the last quantity admits a limit as $t \to \infty$ we define

$$\lambda_p := \lim_{t\to\infty} \Lambda_p(t). \tag{5}$$

λ_p is called the p-th (annealed) Lyapunov exponent of the solution u of the parabolic Anderson problem (1).

We will see in Theorem 1.1 that the limit in (5) exists and is independent of x. Hence, we suppress x in the notation. However, clearly, λ_p is a function of n, d, κ and ρ. In what follows, our main focus will be to analyze the dependence of λ_p on the parameters n, p, κ and ρ, therefore we will often write $\lambda_p^{(n)}(\kappa,\rho)$.

In particular, our main subject of interest will be to draw the qualitative picture of *intermittency* for these systems. First, note that by the moment inequality we have

$$\lambda_p^{(n)} \ge \lambda_{p-1}^{(n)}, \tag{6}$$

for all $p \in \mathbb{N} \setminus \{1\}$. The solution u of the system (1) is said to be *p-intermittent* if the above inequality is strict, namely,

$$\lambda_p^{(n)} > \lambda_{p-1}^{(n)}. \tag{7}$$

The solution u is *fully intermittent* if (7) holds for all $p \in \mathbb{N} \setminus \{1\}$.

Also note that, using Hölder's inequality, p-intermittency implies q-intermittency for all $q \geq p$ (see e.g. [2], Lemma 3.1). Thus, for any fixed $n \in \mathbb{N}$, p-intermittency in fact implies that

$$\lambda_q^{(n)} > \lambda_{q-1}^{(n)} \qquad \forall q \geq p\,,$$

and 2-intermittency means full intermittency.

Geometrically, intermittency corresponds to the solution being *asymptotically concentrated* on a thin set, which is expected to consist of "islands" located far from each other (see [8], Section 1 and references therein for more details).

1.3 Main results

Our first theorem states that the Lyapunov exponents exist and behave nicely as a function of κ and ρ. It will be proved in Section 2.

Theorem 1.1. *Let $d \geq 1$ and $n,p \in \mathbb{N}$.*
(i) For all $\kappa,\rho \in [0,\infty)$, the limit in (5) exists, is finite, and is independent of x if $(\kappa,\rho) \neq (0,0)$.
(ii) On $[0,\infty)^2$, $(\kappa,\rho) \mapsto \lambda_p^{(n)}(\kappa,\rho)$ is continuous, convex and non-increasing in both κ and ρ.

Let $G_d(x)$ be the Green function at lattice site x of simple random walk stepping at rate $2d$ and

$$\mu(\kappa) = \sup \mathrm{Sp}(\kappa\Delta + \delta_0) \tag{8}$$

be the supremum of the spectrum of the operator $\kappa\Delta + \delta_0$ in $l^2(\mathbb{Z}^d)$. It is well-known that (see e.g. [3], Lemma 1.3) $\mathrm{Sp}(\kappa\Delta + \delta_0) = [-4d\kappa, 0] \cup \{\mu(\kappa)\}$ with

$$\mu(\kappa) \begin{cases} = 0 & \text{if } \kappa \geq G_d(0), \\ > 0 & \text{if } \kappa < G_d(0). \end{cases} \tag{9}$$

Furthermore, $\kappa \mapsto \mu(\kappa)$ is continuous, non-increasing and convex on $[0,\infty)$, and strictly decreasing on $[0, G_d(0)]$. Our next two theorems, Theorem 1.2 and 1.3, give the limiting behavior of $\lambda_p^{(n)}$ as $\kappa \downarrow 0$, $\kappa \to \infty$ and $p \to \infty$, $n \to \infty$, respectively. They will be proved in Section 3.

Theorem 1.2. *Let $n,p \in \mathbb{N}$ and $\rho \in [0,\infty)$.*
(i) For all $d \geq 1$, $\lim_{\kappa\downarrow 0} \lambda_p^{(n)}(\kappa,\rho) = \lambda_p^{(n)}(0,\rho) = n\mu(\rho/p)$.
(ii) If $1 \leq d \leq 2$, then $\lambda_p^{(n)}(\kappa,\rho) > 0$ for all $\kappa \in [0,\infty)$. Moreover, $\kappa \mapsto \lambda_p^{(n)}(\kappa,\rho)$ is

strictly decreasing with $\lim_{\kappa\to\infty}\lambda_p^{(n)}(\kappa,\rho)=0$ *(see Fig. 1).*
(iii) If $d\geq 3$*, then* $\lambda_p^{(n)}(\kappa,\rho)=0$ *for* $\kappa\in[nG_d(0),\infty)$ *or* $\rho\in[pG_d(0),\infty)$ *(see Fig. 2).*

Theorem 1.3. *Let* $d\geq 1$ *and* $\kappa,\rho\in[0,\infty)$.
(i) For all $n\in\mathbb{N}$, $\lim_{p\to\infty}\lambda_p^{(n)}(\kappa,\rho)=n\mu(\kappa/n)$ *(see Fig. 1–2);*
(ii) For all $p>\rho/G_d(0)$, $\lim_{n\to\infty}\lambda_p^{(n)}(\kappa,\rho)=+\infty$*;*
(iii) For all $p\leq\rho/G_d(0)$ *and* $n\in\mathbb{N}$, $\lambda_p^{(n)}(\kappa,\rho)=0$.

Note that since $G_d(0)=\infty$ for dimensions 1 and 2 part (iii) of Theorem 1.3 is contained in part (iii) of Theorem 1.2.

By part (ii) of Theorem 1.1, $\lambda_p^{(n)}(\kappa,\rho)$ is non-increasing in κ. Hence, we can define $\{\kappa_p^{(n)}(\rho)\colon p\in\mathbb{N}\}$ as the non-decreasing sequence of critical κ's for which

$$\lambda_p^{(n)}(\kappa,\rho)\begin{cases}>0, & \text{for } \kappa\in\left[0,\kappa_p^{(n)}(\rho)\right),\\ =0, & \text{for } \kappa\in\left[\kappa_p^{(n)}(\rho),\infty\right),\end{cases}\qquad p\in\mathbb{N}. \tag{10}$$

As a consequence of Theorems 1.1 and 1.2 we have,

$$\begin{aligned}
&\kappa_p^{(n)}(\rho)=\infty && \text{if } 1\leq d\leq 2,\\
&0<\kappa_p^{(n)}(\rho)<\infty && \text{if } d\geq 3 \text{ and } p>\rho/G_d(0),\\
&\kappa_p^{(n)}(\rho)=0 && \text{if } d\geq 3 \text{ and } p\leq\rho/G_d(0).
\end{aligned} \tag{11}$$

Our fourth theorem gives bounds on $\kappa_p^{(n)}(\rho)$ which will be proved in Section 4. For this theorem we need to define the inverse of the function $\mu(\kappa)$. Note that by (8) and (9) we have $\mu(0)=1$ and $\mu(G_d(0))=0$. It is easy to see that $\mu(\kappa)$ restricted to the domain $[0,G_d(0)]$ is invertible with an inverse function $\mu^{-1}:[0,1]\to[0,G_d(0)]$. We extend μ^{-1} to $[0,\infty)$ by declaring $\mu^{-1}(t)=0$ for $t>1$.

Theorem 1.4. *Let* $n,p\in\mathbb{N}$.
(i) If $d\geq 3$, $\rho\in[0,\infty)\mapsto\kappa_p^{(n)}(\rho)$ *is a continuous, non-increasing and convex function such that*

$$\max\left(\frac{n}{4d}\,\mu(\rho/p),n\mu^{-1}(4d\rho/p)\right)\leq\kappa_p^{(n)}(\rho)\leq nG_d(0)\left(1-\frac{\rho}{pG_d(0)}\right)_+. \tag{12}$$

(ii) Assume that $d\geq 5$ *and let* $\alpha_d:=\frac{G_d(0)}{2d\|G_d\|_2^2}\in(0,\infty)$*. Then*

$$\kappa_p^{(n)}(\rho)\geq\left(nG_d(0)-\rho\,\frac{n}{p\alpha_d}\right)_+. \tag{13}$$

(iii) Assume that $d\geq 5$ *and* $p\in\mathbb{N}\setminus\{1\}$ *is such that* $\alpha_d>\frac{p-1}{p}$*. Then*

$$\kappa_{p-1}^{(n)}(\rho)<\kappa_p^{(n)}(\rho)\qquad\forall\rho\in(0,pG_d(0)). \tag{14}$$

Note that the condition $\alpha_d > \frac{p-1}{p}$ is always true if d is large enough by the following lemma whose proof is given in the appendix.

Lemma 1.1. *For all $d \geq 3$, $\alpha_d \leq 1$, and $\lim_{d\to\infty} \alpha_d = 1$.*

Our next theorem states some general intermittency results for all dimensions which will be proved in Section 5.

Theorem 1.5. *$d \geq 1$ and $n \in \mathbb{N}$.*
(i) If $\kappa \in [0, nG_d(0))$, then there exists $p \geq 2$ such that the system is p-intermittent.
(ii) If $\kappa \in [nG_d(0), \infty)$, then the system is not intermittent.

Note that since $G_d(0) = \infty$ for $d = 1,2$ the above Theorem implies that for dimensions 1 and 2 the system is always p-intermittent for some p.

Our last theorem describes several regimes in the intermittent behavior of the solution of the system (1). It will be proved in Section 5.

Theorem 1.6. *For all $n \in \mathbb{N}$, for any $p \in \mathbb{N} \setminus \{1\}$ given and for d large enough (s.t. $\alpha_d > (p-1)/p$), the system is*

- *2-intermittent for $\rho \in (0, 2G_d(0))$, and $\kappa \in [\kappa_1^{(n)}(\rho), \kappa_2^{(n)}(\rho))$;*
- *3-intermittent for $\rho \in [0, 3G_d(0))$, and $\kappa \in (\kappa_2^{(n)}(\rho), \kappa_3^{(n)}(\rho))$;*
- $\cdots$
- *p-intermittent for $\rho \in [0, pG_d(0))$, and $\kappa \in (\kappa_{p-1}^{(n)}(\rho), \kappa_p^{(n)}(\rho))$*

(see Fig. 3).

The intermittent behavior of the system is expected to be as follows.

Conjecture 1.1. In dimension $1 \leq d \leq 2$, the solution u is fully intermittent (see Fig. 1).

Conjecture 1.2. In dimension $d \geq 3$ the intermittency vanishes as κ increases (see Fig. 2). More precisely, if $d \geq 3$ there are three different regimes:

Regime A: for all $\kappa \in [0, \kappa_2^{(n)})$, the solution is full intermittent;
Regime B: for all $\kappa \in [\kappa_2^{(n)}, nG_d(0))$, there exists $p = p(\kappa) \geq 3$ such that the solution is q-intermittent for all $q \geq p$;
Regime C: for all $\kappa \in [nG_d(0), \infty)$, the solution is not p-intermittent for any $p \geq 2$

(see Fig. 2).

Conjecture 1.3. For all fixed $n \geq 1$ and d large enough the $\kappa_p^{(n)}$'s are distinct (see Fig. 2).

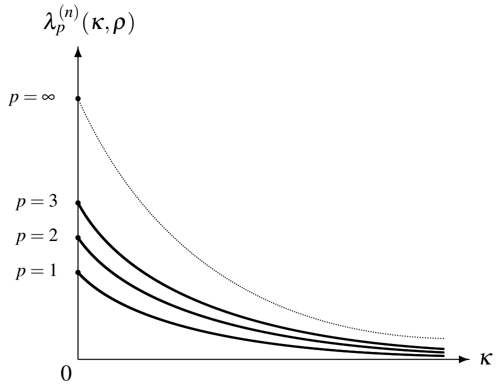

Fig. 1 Full intermittency when $1 \leq d \leq 2$. (Conjecture.)

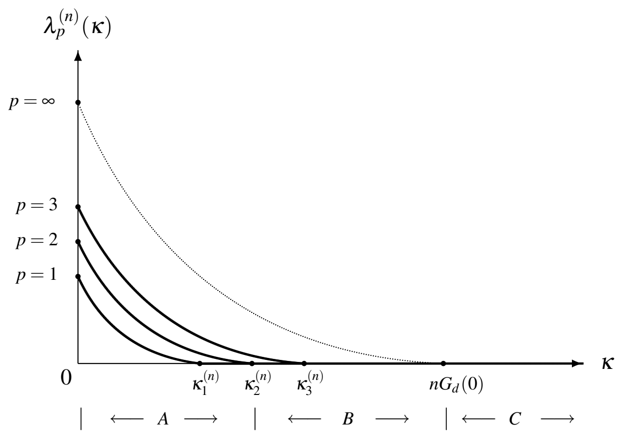

Fig. 2 Three intermittent regimes when $d \geq 3$ and $\rho < G_d(0)$. (Conjecture.)

1.4 Discussion

The behavior of the *annealed* Lyapunov exponents and more particularly the problem of intermittency for the PAM in a *space-time* random environment was subject to various studies. Carmona and Molchanov [1] obtained an essentially complete qualitative description of the annealed Lyapunov exponents and intermittency when ξ is white noise, i.e.,

$$\xi(x,t) = \frac{\partial}{\partial t} W(x,t),$$

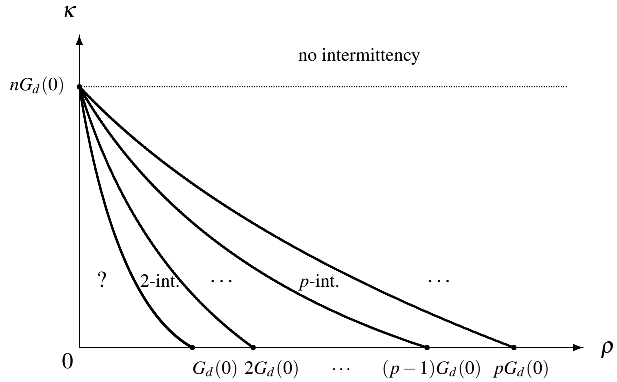

Fig. 3 Phase diagram of intermittency when d is large enough. The bold curves represent $\rho \in [0,\infty) \mapsto \kappa_q^{(n)}(\rho)$, $q = 1,\cdots,p$.

where $W = (W_t)_{t\geq 0}$ with $W_t = \{W(x,t)\colon\, x \in \mathbb{Z}^d\}$ is a white noise field. Further refinements on the behavior of the Lyapunov exponents were obtained in Greven and den Hollander [9]. In particular, it was shown that $\lambda_1 = 1/2$ for all $d \geq 1$ and $\lambda_p > 1/2$ for $p \in \mathbb{N}\setminus\{1\}$ in $d = 1,2$, while for $d \geq 3$ there exist $0 < \kappa_2 \leq \kappa_3 \leq \ldots$ satisfying

$$\lambda_p(\kappa) - \frac{1}{2}\begin{cases} > 0, \text{ for } \kappa \in \big[0,\kappa_p\big), \\ = 0, \text{ for } \kappa \in \big[\kappa_p,\infty\big), \end{cases} \qquad p \in \mathbb{N}\setminus\{1\}.$$

Upper and lower bounds on κ_p were derived, and the asymptotics of κ_p as $p \to \infty$ was computed. In addition, it was proved that for d large enough the κ_p's are distinct.

More recently various models where ξ is *non-Gaussian* were investigated. Kesten and Sidoravicius [10] and Gärtner and den Hollander [3], have considered the case where ξ is given by a Poisson field of independent simple random walks. In [10], the survival versus extinction of the system and in [3], the moment asymptotics were studied, in particular, their dependence on d, κ and the parameters controlling ξ. A partial picture of intermittency, depending on the parameters d and κ was obtained. The case where ξ is a single random walk –corresponding to our setting with $n = 1$– was studied by Gärtner and Heydenreich [2]. Analogous results to those contained in Theorems 1.1, 1.2 and 1.6 were obtained. In their situation $\lambda_1^{(1)} = \mu(\kappa+\rho)$ (recall (9)) and therefore $\kappa_1^{(1)} = G_d(0) - \rho$ corresponds to the critical value $\kappa_1^{(1)} = \inf\{\kappa \in [0,\infty)\colon\, \lambda_1^{(1)}(\kappa) = 0\}$. Because of this simplicity, a more complete picture of intermittency was obtained.

The investigation of annealed Lyapunov behavior and intermittency was extented to non-Gaussian and *space correlated* potentials first in Gärtner, den Hollander and Maillard, in [4] and [6], for the case where ξ is an exclusion process with symmetric random walk transition kernel, starting form a Bernoulli product measure and later in Gärtner, den Hollander and Maillard [7] for the case where ξ is a voter model starting either from Bernoulli product measure or from equilibrium (see Gärtner, den Hollander and Maillard [5], for an overview).

2 Proof of Theorem 1.1

Step 1: We first prove that if the limit in (5) exists, it does not depend on x as soon as $(\kappa,\rho)\neq(0,0)$. To this end, let us introduce some notations. For any $t>0$, we denote

$$Y_t \doteq (Y_1^{\rho}(t),\cdots,Y_n^{\rho}(t))\in\mathbb{Z}^{dn}\ ,\ X_t\doteq(X_1^{\kappa}(t),\cdots,X_p^{\kappa}(t))\in\mathbb{Z}^{dp}\,.$$

For $(x,y)\in\mathbb{Z}^{dp}\times\mathbb{Z}^{dn}$, $\mathbb{E}_{x,y}^{X,Y}$ denote the expectation under the law of $(X_t,Y_t)_{t\geq 0}$ starting from (x,y). The same notation is used for $x\in\mathbb{Z}^d$ and $y\in\mathbb{Z}^d$. In that case, it means that $X_0=(x,\cdots,x)$, $Y_0=(y,\cdots,y)$ and $\mathbb{E}_{x,y}^{X,Y}=\mathbb{E}_y^{\otimes n}\otimes \mathrm{E}_x^{\otimes p}$. Finally, for $x=(x_1,\cdots,x_p)\in\mathbb{Z}^{dp}$ and $y=(y_1,\cdots,y_n)\in\mathbb{Z}^{dn}$, set

$$I(x,y)=\sum_{j=1}^{p}\sum_{k=1}^{n}\delta_0(x_j-y_k)\,.$$

Then, by time reversal for Y in (4), for all $x\in\mathbb{Z}^d$ and $t>0$,

$$\mathbb{E}_0^{\otimes n}\left[u(x,t)^p\right]=\sum_{z\in\mathbb{Z}^{dn}}\mathbb{E}_{x,z}^{X,Y}\left[\exp\left(\int_0^t I(X_s,Y_s)\,ds\right)\delta_0(Y_t)\right]. \tag{15}$$

Using the Markov property at time 1 and the fact that $1\leq\exp\left(\int_0^1 I(X_s,Y_s)\,ds\right)$, we get for x_1 and x_2 any fixed points in $\mathbb{Z}^d$,

$$\begin{aligned}\mathbb{E}_0^{\otimes n}\left[u(x_1,t)^p\right] &\geq \sum_{z\in\mathbb{Z}^{dn}}\mathbb{E}_{x_1,z}^{X,Y}\left[\delta_{(x_2,\cdots,x_2)}(X_1)\delta_z(Y_1)\exp\left(\int_1^t I(X_s,Y_s)\,ds\right)\delta_0(Y_t)\right]\\ &=(p_1^{\kappa}(x_1,x_2))^p(p_1^{\rho}(0,0))^n\mathbb{E}_0^{\otimes n}\left([u(x_2,t-1)]^p\right),\end{aligned}$$

where p_t^{ν} is the transition kernel of a simple random walk on $\mathbb{Z}^d$ with step rate $2d\nu$. This proves the independence of λ_p w.r.t. x as soon as $\kappa>0$, since in this case for all $x_1,x_2\in\mathbb{Z}^d$, $p_1^{\kappa}(x_1,x_2)>0$.

For $\kappa=0$, since the X-particles do not move, we have

$$\mathbb{E}_0^{\otimes n}\left[u(x_1,t)^p\right]=\mathbb{E}_0\left[\exp\left(p\int_0^t\delta_{x_1}(Y_1^{\rho}(s))\,ds\right)\right]^n. \tag{16}$$

The same reasoning leads now to

$$\mathbb{E}_0^{\otimes n}\left[u(x_1,t)^p\right]\geq p_1^{\rho}(0,x_1-x_2)^n\mathbb{E}_0^{\otimes n}\left([u(x_2,t-1)]^p\right).$$

Step 2: Variational representation. From now on, we restrict our attention to the case $x = 0$. The aim of this step is to give a variational representation of $\lambda_p^{(n)}(\kappa,\rho)$. To this end, we introduce further notations. Let $(e_1,\cdots,e_d)$ be the canonical basis of $\mathbb{R}^d$. For $x = (x_1,\cdots,x_p) \in \mathbb{Z}^{dp}$, and $f : (x,y) \in \mathbb{Z}^{dp} \times \mathbb{Z}^{dn} \mapsto \mathbb{R}$, we set

$$\nabla_x f(x,y) = \big(\nabla_{x_1} f(x,y),\cdots,\nabla_{x_p} f(x,y)\big) \in \mathbb{R}^{dp},$$

where for $j \in \{1,\cdots,p\}$, and $i \in \{1,\cdots,d\}$,

$$\big\langle \nabla_{x_j} f(x,y), e_i \big\rangle = f(x_1,\cdots,x_j+e_i,\cdots,x_p,y) - f(x,y)\,.$$

The same notation is used for the y-coordinates, so that $\nabla_y f(x,y) \in \mathbb{R}^{dn}$. We also define

$$\begin{aligned}\Delta_x f(x,y) &= \sum_{j=1}^{p} \Delta_{x_j} f(x,y)\\ &= \sum_{j=1}^{p} \sum_{\substack{z_j\in\mathbb{Z}^d\\ z_j\sim x_j}} \big[f(x_1,\cdots,z_j,\cdots,x_p,y) - f(x_1,\cdots,x_j,\cdots,x_p,y)\big]\,.\end{aligned}$$

Proposition 2.1. *Let $d \geq 1$ and $n,p \in \mathbb{N}$. For all $\kappa,\rho \in [0,\infty)$,*

$$\begin{aligned}\lambda_p^{(n)}(\kappa,\rho) &= \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}_0^{\otimes n}\left[u(0,t)^p\right]\\ &= \frac{1}{p} \sup_{\substack{f\in l^2(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})\\ \|f\|_2=1}} \left\{ -\kappa \left\|\nabla_x f\right\|_2^2 - \rho \left\|\nabla_y f\right\|_2^2 + \sum_{(x,y)} I(x,y) f^2(x,y) \right\}.\end{aligned} \tag{17}$$

Proof. **Upper bound.** Let B_R^n be the ball in $\mathbb{Z}^{dn}$ of radius $R = t\log(t)$ centered at 0. Note that

$$\begin{aligned}\sum_{z\notin B_R^n} \mathbb{E}_{0,z}^{X,Y}\left[\exp\left(\int_0^t I(X_s,Y_s)\,ds\right)\delta_0(Y_t)\right] &\leq \exp(tnp)\mathbb{P}_0^Y\left(Y_t \notin B_R^n\right)\\ &\leq \exp(tnp)\exp\left(-C(n,d,\rho)\frac{R^2}{t}\right),\end{aligned}$$

for some constant $C(n,d,\rho) \in (0,\infty]$, and therefore

$$\lim_{t\to\infty} \frac{1}{t} \log \sum_{z\notin B_R^n} \mathbb{E}_{0,z}^{X,Y}\left[\exp\left(\int_0^t I(X_s,Y_s)\,ds\right)\delta_0(Y_t)\right] = -\infty.$$

Since in the same way,

$$\sum_{z\in B_R^n} \mathbb{E}_{0,z}^{X,Y}\left[\exp\left(\int_0^t I(X_s,Y_s)\,ds\right)\delta_0(Y_t)\ \mathbb{1}_{(B_R^p)^c}(X_t)\right]$$
$$\le \exp(npt)\,\mathbb{P}_0^Y(Y_t\in B_R^n)\,\mathbb{P}_0^X\left(X_t\notin B_R^p\right)$$
$$\le \exp(tnp)\exp\left(-C(p,d,\kappa)\frac{R^2}{t}\right),$$

we are thus led to study the existence of

$$\lim_{t\to\infty}\frac{1}{t}\log\sum_{z\in B_R^n}\mathbb{E}_{0,z}^{X,Y}\left[\exp\left(\int_0^t I(X_s,Y_s)\,ds\right)\delta_0(Y_t)\ \mathbb{1}_{B_R^p}(X_t)\right]$$
$$=\lim_{t\to\infty}\frac{1}{t}\log\left\langle f_1,e^{t\mathscr{L}}f_2\right\rangle,$$

where $f_1:(x,y)\in\mathbb{Z}^{dp}\times\mathbb{Z}^{dn}\mapsto\delta_0(x)\ \mathbb{1}_{B_R^n}(y)$, $f_2:(x,y)\in\mathbb{Z}^{dp}\times\mathbb{Z}^{dn}\mapsto\ \mathbb{1}_{B_R^p}(x)\delta_0(y)$, and $\mathscr{L}$ is the bounded self-adjoint operator in $l^2(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})$ defined by

$$\mathscr{L}f(x,y)=\kappa\Delta_x f(x,y)+\rho\Delta_y f(x,y)+I(x,y)f(x,y)\qquad\forall(x,y)\in\mathbb{Z}^{dp}\times\mathbb{Z}^{dn}.$$

Note that $\left\langle f_1,e^{t\mathscr{L}}f_2\right\rangle\le\|f_1\|_2\left\|e^{t\mathscr{L}}\right\|_{2,2}\|f_2\|_2=C(d,n,p)R^{d(n+p)/2}\left\|e^{t\mathscr{L}}\right\|_{2,2}$. Thus,

$$\lim_{t\to\infty}\frac{1}{t}\log\left\langle f_1,e^{t\mathscr{L}}f_2\right\rangle\le\|\mathscr{L}\|_{2,2}=\sup_{\substack{f\in l^2(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})\\ \|f\|_2=1}}\langle f,\mathscr{L}f\rangle,$$

which is the upper bound in (17).

Lower bound. By (15) with $x=0$, it follows that

$$\mathbb{E}_0^{\otimes n}[u(0,t)^p]\ge\mathbb{E}_{0,0}^{X,Y}\left[\exp\left(\int_0^t I(X_s,Y_s)\,ds\right)\delta_0(X_t)\delta_0(Y_t)\right]$$
$$=\left\langle\delta_0\otimes\delta_0,e^{t\mathscr{L}}(\delta_0\otimes\delta_0)\right\rangle=\left\|e^{\frac{t}{2}\mathscr{L}}(\delta_0\otimes\delta_0)\right\|_2^2$$
$$=\sum_{x\in\mathbb{Z}^{dp}}\sum_{y\in\mathbb{Z}^{dn}}\left(e^{\frac{t}{2}\mathscr{L}}(\delta_0\otimes\delta_0)(x,y)\right)^2.$$

Restricting the sum over $B_R^p\times B_R^n$, and applying Jensen's inequality, we get

$$
\begin{aligned}
&\mathbb{E}_0^{\otimes n}\left[u(0,t)^p\right] \\
&\quad \geq \sum_{x\in B_R^p}\sum_{y\in B_R^n}\left(e^{\frac{t}{2}\mathscr{L}}(\delta_0\otimes\delta_0)(x,y)\right)^2 \\
&\quad \geq \frac{1}{|B_R^n|}\frac{1}{|B_R^p|}\left(\sum_{x\in B_R^p}\sum_{y\in B_R^n} e^{\frac{t}{2}\mathscr{L}}(\delta_0\otimes\delta_0)(x,y)\right)^2 \\
&\quad = \frac{C(d,n,p)}{R^{d(n+p)}}\left(\sum_{x\in B_R^p}\sum_{y\in B_R^n}\mathbb{E}_{x,y}^{X,Y}\left[\exp\left(\int_0^{\frac{t}{2}} I(X_s,Y_s)\,ds\right)\delta_0(X_{\frac{t}{2}})\delta_0(Y_{\frac{t}{2}})\right]\right)^2 \\
&\quad = \frac{C(d,n,p)}{R^{d(n+p)}}\left(\mathbb{E}_{0,0}^{X,Y}\left[\exp\left(\int_0^{\frac{t}{2}} I(X_s,Y_s)\,ds\right)\ \mathbb{1}_{B_R^p}(X_{\frac{t}{2}})\ \mathbb{1}_{B_R^n}(Y_{\frac{t}{2}})\right]\right)^2 .
\end{aligned}
$$

Taking $R = t\log(t)$, we obtain that

$$
\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0^{\otimes n}\left[u(0,t)^p\right] \\
&\qquad \geq \liminf_{t\to\infty}\frac{2}{t}\log\mathbb{E}_{0,0}^{X,Y}\left[\exp\left(\int_0^{t/2} I(X_s,Y_s)\,ds\right)\ \mathbb{1}_{B_R^p}(X_{t/2})\ \mathbb{1}_{B_R^n}(Y_{t/2})\right] .
\end{aligned}
$$

As already noted, for $R = t\log(t)$,

$$
\begin{aligned}
&\mathbb{E}_{0,0}^{X,Y}\left[\exp\left(\int_0^{t/2} I(X_s,Y_s)\,ds\right)\ \mathbb{1}_{(B_R^p\times B_R^n)^c}(X_{t/2},Y_{t/2})\right] \\
&\qquad \leq \exp(tnp/2)\exp(-Ct\log(t)^2) ,
\end{aligned}
$$

and therefore, we get

$$
\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0^{\otimes n}\left[u(0,t)^p\right] \geq \liminf_{t\to\infty}\frac{2}{t}\log\mathbb{E}_{0,0}^{X,Y}\left[\exp\left(\int_0^{t/2} I(X_s,Y_s)\,ds\right)\right] .
$$

Now, the occupation measure $\frac{1}{t}\int_0^t \delta_{(X_s,Y_s)}\,ds$ satisfies a weak large deviations principle (LDP) in the space $\mathscr{M}_1(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})$ of probability measures on $\mathbb{Z}^{dp}\times\mathbb{Z}^{dn}$, endowed with the weak topology. The speed of this LDP is t and the rate function is given for all $\nu\in\mathscr{M}_1(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})$ by

$$
J(\nu) = \kappa\left\|\nabla_x\sqrt{\nu}\right\|_2^2 + \rho\left\|\nabla_y\sqrt{\nu}\right\|_2^2 .
$$

Since I is bounded, the lower bound in Varadhan's integral lemma yields

$$
\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0^{\otimes n}\left[u(0,t)^p\right] \geq \sup_{\nu\in\mathscr{M}_1(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})}\left\{\sum_{(x,y)} I(x,y)\nu(x,y) - J(\nu)\right\} .
$$

Setting $f(x,y) = \sqrt{\nu}(x,y)$ gives then the lower bound in (17). $\square$

Step 3: Properties of $\lambda_p^{(n)}$. Since $0 \le I(x,y) \le np$, we clearly have $0 \le \lambda_p^{(n)} \le n$. Using representation (17), the function $(\kappa,\rho) \mapsto \lambda_p^{(n)}(\kappa,\rho)$ is nonincreasing in κ and ρ, convex, and l.s.c. as a supremum of functions that are linear in κ and ρ. Since every finite convex function is also u.s.c., $\lambda_p^{(n)}$ is continuous.

3 Proof of Theorems 1.2–1.3

By symmetry, note that $\forall n, p \in \mathbb{N}$, $\forall \kappa, \rho \in [0,\infty)$,

$$\lambda_p^{(n)}(\kappa,\rho) = \frac{n}{p}\lambda_n^{(p)}(\rho,\kappa)\,. \tag{18}$$

3.1 Proof of Theorem 1.2

Proof of (i): We have already seen that $\lim_{\kappa\to 0}\lambda_p^{(n)}(\kappa,\rho) = \lambda_p^{(n)}(0,\rho)$ and that for $\kappa = 0$, the X particles do not move so that $\mathbb{E}_0^{\otimes n}[u(0,t)^p] = \mathbb{E}_0\left(\exp\left(pL_t^Y(0)\right)\right)^n$ (see (16)), where $L_t^Y(0)$ is the local time at 0 of a simple random walk in $\mathbb{Z}^d$ with rate $2d\rho$. Using the LDP for $L_t^Y(0)$, we obtain

$$\lambda_p^{(n)}(0,\rho) = \frac{n}{p}\sup_{\substack{f\in l^2(\mathbb{Z}^d)\\ \|f\|_2=1}}\langle f,(\rho\Delta + p\delta_0)f\rangle = n\mu(\rho/p)\,.$$

Proof of (ii): $\forall n, p \in \mathbb{N}$, $\forall \kappa, \rho \in [0,\infty)$,

$$\lambda_p^{(n)}(\kappa,\rho) \ge \lambda_1^{(n)}(\kappa,\rho) = n\lambda_n^{(1)}(\rho,\kappa) \ge n\lambda_1^{(1)}(\rho,\kappa) = n\mu(\kappa+\rho)\,,$$

where the last equality is proved in [2] and comes from the fact that $X_t^1 - Y_t^1$ is a simple random walk in $\mathbb{Z}^d$ with jump rate $2d(\kappa+\rho)$. Since $G_d(0) = \infty$ for $d = 1,2$, it follows from (9) that $\lambda_p^{(n)}(\kappa,\rho) > 0$ for $d = 1,2$.

Let us prove that $\lim_{\kappa\to\infty}\lambda_p^{(n)}(\kappa,\rho) = 0$. By monotonicity in ρ,

$$\lambda_p^{(n)}(\kappa,\rho) \le \lambda_p^{(n)}(\kappa,0) = n\mu(\kappa/n)\,, \tag{19}$$

so that the only thing to prove is that $\lim_{\kappa\to\infty}\mu(\kappa) = 0$. To this end, one can use the discrete Gagliardo-Nirenberg inequality: there exists a constant C such that for all $f : \mathbb{Z}^d \mapsto \mathbb{R}$,

$$\text{for } d=1\,,\ \ \|f\|_\infty^2 \le C\,\|f\|_2\,\|\nabla f\|_2\,;$$
$$\text{for } d=2\,,\ \ \|f\|_4^2 \le C\,\|f\|_2\,\|\nabla f\|_2\,.$$

From this, we get that for all $f \in l_2(\mathbb{Z}^d)$ with $\|f\|_2 = 1$,

$$-\kappa \|\nabla f\|_2^2 + f(0)^2 \le \begin{cases} -\kappa \|\nabla f\|_2^2 + \|f\|_\infty^2 & \text{for } d=1 \\ -\kappa \|\nabla f\|_2^2 + \|f\|_4^2 & \text{for } d=2 \end{cases}$$
$$\le -\kappa \|\nabla f\|_2^2 + C \|\nabla f\|_2 \,.$$

Taking the supremum over f yields

$$\mu(\kappa) \le \sup_{x \ge 0} \big(-\kappa x^2 + Cx\big) = \frac{C^2}{4\kappa}\,.$$

The strict monotonicity is now an easy consequence of the fact that $\kappa \mapsto \lambda_p^{(n)}(\kappa,\rho)$ is convex, positive, non increasing, and tends to 0 as $\kappa \to \infty$.

Proof of (iii): By (18) and (19), we get

$$\lambda_p^{(n)}(\kappa,\rho) \le n \min\left(\mu(\kappa/n), \mu(\rho/p)\right), \tag{20}$$

from which the claim follows.

3.2 Proof of Theorem 1.3

Proof of (i): Fix $\varepsilon > 0$. Let f approaching the supremum in the variational representation (17) of $\lambda_p^{(n)}(\kappa,0)$, so that

$$p\lambda_p^{(n)}(\kappa,0) - \varepsilon \le -\kappa \|\nabla_x f\|_2^2 + \sum_{x\in\mathbb{Z}^{dp}} \sum_{y\in\mathbb{Z}^{dn}} I(x,y) f^2(x,y)$$
$$\le p\lambda_p^{(n)}(\kappa,\rho) + \rho \sup_{\substack{f\in l^2(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn}) \\ \|f\|_2=1}} \left\|\nabla_y f\right\|_2^2 .$$

For $x \in \mathbb{Z}^{dp}$, set $f_x : y \in \mathbb{Z}^{dn} \mapsto f(x,y)$. Since the bottom of the spectrum of Δ in $l^2(\mathbb{Z}^{dn})$ is $-4dn$,

$$\sum_{y\in\mathbb{Z}^{dn}} \left\|\nabla_y f_x(y)\right\|_2^2 \le 4dn \sum_{y\in\mathbb{Z}^{dn}} f_x^2(y)\,,$$

for all $x \in \mathbb{Z}^{dp}$. Hence,

$$\sum_{x\in\mathbb{Z}^{dp}} \sum_{y\in\mathbb{Z}^{dn}} \left\|\nabla_y f_x(y)\right\|_2^2 \le 4dn \sum_{x\in\mathbb{Z}^{dp}} \sum_{y\in\mathbb{Z}^{dn}} f_x^2(y) = 4dn\,,$$

so that for all $\varepsilon > 0$,

$$p\lambda_p^{(n)}(\kappa,0)-\varepsilon \le p\lambda_p^{(n)}(\kappa,\rho)+4dn\rho\,.$$

Letting $\varepsilon \to 0$ yields,

$$\lambda_p^{(n)}(\kappa,0)-\frac{4dn\rho}{p} \le \lambda_p^{(n)}(\kappa,\rho) \le \lambda_p^{(n)}(\kappa,0)\,, \tag{21}$$

which, after letting $p \to \infty$, gives the claim.

Proof of (ii): By (18), $\lim_{n\to\infty} \lambda_p^{(n)}(\kappa,\rho) = \lim_{n\to\infty} \frac{n}{p}\lambda_n^{(p)}(\rho,\kappa)$ and by (i),

$$\lim_{n\to\infty} \lambda_n^{(p)}(\rho,\kappa) = \lambda_n^{(p)}(\rho,0) = p\mu(\rho/p) > 0\,, \text{ for } p > \rho/G_d(0)\,.$$

Hence, for $p > \rho/G_d(0)$, $\lim_{n\to\infty} \lambda_p^{(n)}(\kappa,\rho) = +\infty$.

Proof of (iii): This is a direct consequence of Theorem 1.2(iii).

4 Proof of Theorem 1.4

Proof of (i): We first prove that

$$\kappa_p^{(n)}(\rho) = \sup_{\substack{f\in l_2(\mathbb{Z}^{dp}\times\mathbb{Z}^{dn})\\ \|f\|_2=1}} \frac{\sum_{x,y} I(x,y)f^2(x,y) - \rho\left\|\nabla_y f\right\|_2^2}{\|\nabla_x f\|_2^2}\,. \tag{22}$$

Indeed, let us denote by S the supremum in the right-hand side of (22).

If $\kappa \ge \kappa_p^{(n)}(\rho)$, then $\lambda_p^{(n)}(\kappa,\rho) = 0$. Therefore, using (17), for all $f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$ such that $\|f\|_2 = 1$,

$$\sum_{x\in\mathbb{Z}^{dp}} \sum_{y\in\mathbb{Z}^{dn}} I(x,y)f^2(x,y) - \rho\left\|\nabla_y f\right\|_2^2 \le \kappa\|\nabla_x f\|_2^2\,,$$

so that $\kappa \ge S$. Hence $\kappa_p^{(n)}(\rho) \ge S$. On the opposite direction, we can assume that $S < \infty$. Then, by definition of S, for all $f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$ such that $\|f\|_2 = 1$,

$$\sum_{x\in\mathbb{Z}^{dp}} \sum_{y\in\mathbb{Z}^{dn}} I(x,y)f^2(x,y) - \rho\left\|\nabla_y f\right\|_2^2 \le S\|\nabla_x f\|_2^2\,.$$

Thus, for all $f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$ such that $\|f\|_2 = 1$, and all $\kappa \ge S$,

$$\sum_{x\in\mathbb{Z}^{dp}} \sum_{y\in\mathbb{Z}^{dn}} I(x,y)f^2(x,y) - \rho\left\|\nabla_y f\right\|_2^2 - \kappa\|\nabla_x f\|_2^2 \le (S-\kappa)\|\nabla_x f\|_2^2 \le 0\,.$$

Hence, for all $\kappa \geq S$, $\lambda_p^{(n)}(\kappa,\rho) = 0$, i.e., $\kappa \geq \kappa_p^{(n)}(\rho)$. Hence, $S \geq \kappa_p^{(n)}(\rho)$. This proves (22).

Since $\rho \mapsto \kappa_p^{(n)}(\rho)$ is a supremum of functions that are linear in ρ, it is l.s.c. and convex. It is also easily seen that $\rho \mapsto \kappa_p^{(n)}(\rho)$ is non increasing. The continuity follows then from the finiteness of $\kappa_p^{(n)}(\rho)$.

The lower bound in (12) is a direct consequence of (21). Indeed, since $\lambda_p^{(n)}(\kappa,0) = n\mu(\kappa/n)$, it follows from (21) that if $\mu(\kappa/n) > 4d\rho/p$, then $\kappa < \kappa_p^{(n)}(\rho)$. This yields the bound:

$$\kappa_p^{(n)}(\rho) \geq n\mu^{-1}(4d\rho/p)\,.$$

Using the symmetry relation (18), we also get from (21) that

$$\lambda_p^{(n)}(\kappa,\rho) \geq n\mu(\rho/p) - 4d\kappa\,.$$

This leads to $\kappa_p^{(n)}(\rho) \geq \frac{n}{4d}\mu(\rho/p)$. Hence, if $\rho/p < G_d(0)$, $\kappa_p^{(n)}(\rho) > 0$. We have already seen that $\kappa_p^{(n)}(\rho) = 0$ if $\rho/p \geq G_d(0)$. Since $\lambda_p^{(n)}(\kappa,0) = n\mu(\kappa/n)$, it follows that $\kappa_p^{(n)}(0) = nG_d(0)$. Using convexity, we have, for all $\rho \in [0, pG_d(0)]$,

$$\kappa_p^{(n)}(\rho) \leq \frac{\kappa_p^{(n)}(pG_d(0)) - \kappa_p^{(n)}(0)}{pG_d(0)}\rho + \kappa_p^{(n)}(0) = n\left(G_d(0) - \rho/p\right)\,.$$

Since $\kappa_p^{(n)}(\rho) = 0$ if $\rho/p \geq G_d(0)$, then the upper bound in (12) is proved.

Proof of (ii): To prove (13), let f_0 be the function

$$f_0(x,y) = \prod_{i=1}^{p} \frac{G_d(x_i)}{\|G_d\|_2} \prod_{j=1}^{n} \delta_0(y_j)\,.$$

Note that for $d \geq 5$, $\|G_d\|_2 < \infty$, so that f_0 is well-defined, and has l_2-norm equal to 1. From (22), we get

$$\kappa_p^{(n)}(\rho) \geq \frac{\sum_{x,y} I(x,y) f_0^2(x,y) - \rho \left\|\nabla_y f_0\right\|_2^2}{\|\nabla_x f_0\|_2^2}\,.$$

An easy computation then gives

$$\sum_{x,y} I(x,y) f_0^2(x,y) = np\frac{G_d^2(0)}{\|G_d\|_2^2}\,,$$

$$\left\|\nabla_y f_0\right\|_2^2 = n\left\|\nabla_{y_1}\delta_0\right\|_2^2 = 2dn\,,$$

and

$$\|\nabla_x f_0\|_2^2 = p\frac{\|\nabla_{x_1} G_d\|_2^2}{\|G_d\|_2^2} = p\frac{G_d(0)}{\|G_d\|_2^2}\,,$$

since $\|\nabla_{x_1} G_d\|_2^2 = \langle G_d, -\Delta G_d\rangle = \langle G_d, \delta_0\rangle = G_d(0)$. This gives (13).

Proof of (iii): The inequality (14) is clear if $\rho \in [(p-1)G_d(0), pG_d(0))$, since in this case, $\kappa_{p-1}^{(n)}(\rho) = 0 < \kappa_p^{(n)}(\rho)$. We assume therefore that $\rho \in (0,(p-1)G_d(0))$. From (12), we have $\kappa_{p-1}^{(n)}(\rho) \le nG_d(0) - \rho n/(p-1)$, whereas, from (13), $\kappa_p^{(n)}(\rho) \ge nG_d(0) - \rho n/(p\alpha_d)$. Hence $\kappa_{p-1}^{(n)}(\rho) < \kappa_p^{(n)}(\rho)$ as soon as $\alpha_d > \frac{p-1}{p}$. This gives the claim.

5 Proof of Theorems 1.5 and 1.6

Proof of Theorem 1.5(i): The function $p \mapsto \lambda_p^{(n)}(\kappa,\rho)$ increases from $\lambda_1^{(n)}(\kappa,\rho)$ to $n\mu(\kappa/n)$. Hence, there exists p such that $\lambda_p^{(n)}(\kappa,\rho) < \lambda_{p+1}^{(n)}(\kappa,\rho)$ as soon as $\lambda_1^{(n)}(\kappa,\rho) < n\mu(\kappa/n)$. But $n\mu(\kappa/n) = \lambda_1^{(n)}(\kappa,0)$. Hence, if $\lambda_1^{(n)}(\kappa,\rho) = n\mu(\kappa/n)$, the convex decreasing function $\rho \mapsto \lambda_1^{(n)}(\kappa,\rho)$ is constant. Being equal to 0 for $\rho \ge G_d(0)$, we get that $n\mu(\kappa/n) = 0$, which is not the case if $\kappa < nG_d(0)$.

Proof of Theorem 1.5(ii): if $\kappa \ge nG_d(0)$, then $\lambda_p^{(n)}(\kappa,\rho) = 0$, for all $p \ge 1$, and the system is not intermittent.

Proof of Theorem 1.6: For all $p \in \mathbb{N}\setminus\{1\}$ by Lemma 1.1 for d large enough we have $\alpha_d > \frac{p-1}{p}$. This implies that $\alpha_d > \frac{q-1}{q}$ for all $q \in \mathbb{N}\setminus\{1\}$ and $q \le p$. Hence, by Theorem 1.4(iii), for all $q \in \mathbb{N}\setminus\{1\}$ with $q \le p$ we have $\kappa_{q-1}^{(n)}(\rho) < \kappa_q^{(n)}(\rho)$, for all $\rho \in (0, pG_d(0))$. Hence, in the domain

$$\Big\{(\kappa,\rho)\colon \rho \in (0,qG_d(0))\,,\ \kappa_{q-1}^{(n)}(\rho) \le \kappa < \kappa_q^{(n)}(\rho)\Big\}$$

one has

$$\lambda_1^{(n)}(\kappa,\rho) = \cdots = \lambda_{q-1}^{(n)}(\kappa,\rho) = 0 < \lambda_q^{(n)}(\kappa,\rho)\,,$$

which proves the desired result.

Acknowledgements The research in this paper was supported by the ANR-project MEMEMO.

Appendix: Proof of lemma 1.1

For a function $f : \mathbb{Z}^d \mapsto \mathbb{R}$, let $\hat{f}$ denote the Fourier transform of f:

$$\hat{f}(\theta) = \sum_{x\in\mathbb{Z}^d} e^{i\langle\theta,x\rangle} f(x) \qquad \forall \theta \in [0,2\pi]^d\,.$$

Then, the inverse Fourier transform is given by

$$f(x) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} e^{-i\langle\theta,x\rangle} \hat{f}(\theta)\, d\theta\,,$$

and the Plancherel's formula reads

$$\sum_{x\in\mathbb{Z}^d} f^2(x) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} |\hat{f}(\theta)|^2\, d\theta\,.$$

Using the equation $\Delta G_d = -\delta_0$ we get that

$$\hat{G}_d(\theta) = \frac{1}{2\sum_{i=1}^d (1-\cos(\theta_i))}\,.$$

Hence,

$$\begin{aligned} G_d(0) &= \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{d\theta}{2\sum_{i=1}^d (1-\cos(\theta_i))} \\ &= \frac{1}{\pi^d} \int_{[0,\pi]^d} \frac{d\theta}{2\sum_{i=1}^d (1-\cos(\theta_i))} \\ &= \mathbb{E}\left[\frac{1}{2\sum_{i=1}^d (1-\cos(\Theta_i))}\right] \end{aligned}$$

where the random variables (Θ_i) are i.i.d. with uniform distribution on $[0,\pi]$. Moreover, by Plancherel's formula we have

$$\|G_d\|_2^2 = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{d\theta}{\left(2\sum_{i=1}^d (1-\cos(\theta_i))\right)^2} = \mathbb{E}\left[\frac{1}{\left(2\sum_{i=1}^d (1-\cos(\Theta_i))\right)^2}\right]\,.$$

Thus,

$$\alpha_d = \frac{G_d(0)}{2d\,\|G_d\|_2^2} = \frac{\mathbb{E}\left[\frac{1}{\bar{S}_d}\right]}{\mathbb{E}\left[\frac{1}{\bar{S}_d^2}\right]}\,,$$

where $\bar{S}_d = \frac{1}{d}\sum_{i=1}^d (1-\cos(\Theta_i))$. Applying Hölder's and Jensen's inequality, we get that

$$\alpha_d \le \frac{1}{\sqrt{\mathbb{E}\left[\frac{1}{\bar{S}_d^2}\right]}} \le \mathbb{E}(\bar{S}_d) = 1\,.$$

By the law of large numbers, $\bar{S}_d$ converges almost surely to $\mathbb{E}\,[1-\cos(\Theta)] = 1$ as d tends to infinity. We are now going to prove that $\bar{S}_d^{-2}$ is uniformly integrable by showing that $\forall p > 2$,

$$\sup_{d>2p} \mathbb{E}\left[\bar{S}_d^{-p}\right] < \infty. \tag{23}$$

Indeed, let $\varepsilon \in (0,\pi)$ be a small positive number to be fixed later. Let

$$\mathscr{I} = \{i \in \{1,\cdots,d\} \colon 0 \le \Theta_i \le \varepsilon\} \,.$$

$$\bar{S}_d \ge \frac{1}{d}\sum_{i\notin\mathscr{I}}(1-\cos(\varepsilon)) + \frac{c_\varepsilon}{d}\sum_{i\in\mathscr{I}}\Theta_i^2 \,,$$

where $c_\varepsilon = \inf_{0\le\theta\le\varepsilon}\frac{1-\cos(\theta)}{\theta^2} \to 1/2$ when $\varepsilon \to 0$. Therefore,

$$\mathbb{E}\left[\bar{S}_d^{-p}\right] \le d^p \sum_{k=0}^{d} \sum_{\substack{I\subset\{1,\cdots,d\}\\ |I|=k}} \mathbb{E}\left[\frac{\mathbb{1}_{\mathscr{I}=I}}{\left((1-\cos(\varepsilon))(d-k)+c_\varepsilon\sum_{i\in I}\Theta_i^2\right)^p}\right] .$$

Since the last expectation only depends on $|I|$, we get

$$\mathbb{E}\left[\bar{S}_d^{-p}\right] \le d^p \sum_{k=0}^{d} \binom{d}{k} a(k,\varepsilon,d)\,,$$

with

$$a(k,\varepsilon,d) := \frac{1}{\pi^d}\int_{\substack{0\le\theta_1,\cdots,\theta_k\le\varepsilon\\ \varepsilon\le\theta_{k+1},\cdots,\theta_d\le\pi}} \frac{d\theta_1\cdots d\theta_d}{\left((1-\cos(\varepsilon))(d-k)+c_\varepsilon(\theta_1^2+\cdots+\theta_k^2)\right)^p} .$$

Let ω_d denote the volume of the d-dimensional unit ball. For $k=d$,

$$\begin{aligned} a(d,\varepsilon,d) &= \frac{1}{\pi^d}\int_{0\le\theta_1,\cdots,\theta_d\le\varepsilon} \frac{d\theta_1\cdots d\theta_d}{c_\varepsilon^p\,\|\theta\|^{2p}} \\ &\le \frac{1}{c_\varepsilon^p\pi^d}\,\omega_d\int_0^{\sqrt{d}\varepsilon} r^{d-2p-1}dr \\ &= \left(\frac{\varepsilon}{\pi}\right)^d \frac{1}{(c_\varepsilon\varepsilon^2)^p} d^{\frac{d}{2}-p}\frac{\omega_d}{d-2p}\,, \end{aligned}$$

for $d>2p$.

Note that for large d, $\omega_d \simeq \frac{(2e\pi)^{d/2}}{\sqrt{\pi d}d^{d/2}}$. Therefore, as $d\to\infty$

$$d^p\binom{d}{d} a(d,\varepsilon,d) = O\left(d^{-3/2}(\varepsilon^2 2e/\pi)^{d/2}\right) .$$

If ε is chosen so that $\varepsilon^2 \le \pi/(2e)$, we obtain that $\lim_{d\to\infty} d^p\binom{d}{d}a(d,\varepsilon,d)=0$.

For $k \le d-1$,

$$a(k,\varepsilon,d) \le \frac{1}{(1-\cos(\varepsilon))^p}\frac{1}{(d-k)^p}\left(\frac{\varepsilon}{\pi}\right)^k\left(1-\frac{\varepsilon}{\pi}\right)^{d-k} ,$$

and $d^p \binom{d}{k} a(k,\varepsilon,d) \leq \frac{1}{(1-\cos(\varepsilon))^p}\mathbb{E}\left[\,\mathbb{1}_{N=k}(1-N/d)^{-p}\right]$, where N is a Binomial random variable with parameters d and ε/π. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\bar{S}_d^p}\right]\\
&\leq \frac{1}{(1-\cos(\varepsilon))^p}\mathbb{E}\left[\,\mathbb{1}_{N\leq d-1}(1-N/d)^{-p}\right] + O\left(d^{-3/2}\right)\\
&\leq \frac{d^p}{(1-\cos(\varepsilon))^p}\mathbb{P}\left[d\frac{2\varepsilon}{\pi} \leq N \leq d-1\right] + \frac{1}{(1-\cos(\varepsilon))^p(1-\frac{2\varepsilon}{\pi})^p} + O\left(d^{-3/2}\right).
\end{aligned}$$

Now, by the large deviations principle satisfied by N/d, there is an $i(\varepsilon) > 0$ such that $\mathbb{P}[N \geq d2\varepsilon/\pi] \leq \exp(-di(\varepsilon))$. This ends the proof of (23).

Using the uniform integrability (23), and the fact that $\bar{S}_d$ converges a.s. to 1, we obtain that $\mathbb{E}\left[\frac{1}{\bar{S}_d}\right]$ and $\mathbb{E}\left[\frac{1}{\bar{S}_d^2}\right]$ both converge to 1, when d goes to infinity.

References

1. Carmona R.A., Molchanov S.A.: Parabolic Anderson Problem and Intermittency. AMS Memoir 518, American Mathematical Society, Providence RI (1994)
2. Gärtner J., Heydenreich M.: Annealed asymptotics for the parabolic Anderson model with a moving catalyst. Stoch. Proc. Appl. **116**, 1511–1529 (2006)
3. Gärtner J., den Hollander F.: Intermittency in a catalytic random medium. Ann. Probab. **34**, 2219–2287 (2006)
4. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts: symmetric exclusion. Electronic J. Probab. **12**, 516–573 (2007)
5. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts. In: Blath J., Mörters P., Scheutzow M. (eds.) Trends in Stochastic Analysis, London Mathematical Society Lecture Note Series 353, pp. 235–248. Cambridge University Press, Cambridge (2009)
6. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts: three-dimensional simple symmetric exclusion. Electronic J. Probab. **72**, 2091–2129 (2009)
7. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts: voter model. Ann. Probab. **38**, 2066–2102 (2010)
8. Gärtner J., König W.: The parabolic Anderson model. In: Deuschel J.-D., Greven A. (eds.) Interacting Stochastic Systems, pp. 153-179, Springer, Berlin (2005)
9. Greven A., den Hollander F.: Phase transition for the long-time behavior of interacting diffusions. Ann. Probab. **35**, 1250–1306 (2007)
10. Kesten H., Sidoravicius V.: Branching random walk with catalysts. Electr. J. Prob. **8**, 1–51 (2003)