**Parabolic Anderson model with voter catalysts:
maximality of the annealed asymptotics**

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Abstract We consider the parabolic Anderson model $\partial u / \partial t = \kappa \Delta u + \gamma \xi u$ with $u \colon \mathbb{Z}^d \times \mathbb{R}^+ \to \mathbb{R}^+$, where $\kappa \in \mathbb{R}^+$ is the diffusion constant, Δ is the discrete Laplacian, $\gamma \in \mathbb{R}^+$ is the coupling constant, and $\xi \colon \mathbb{Z}^d \times \mathbb{R}^+ \to \{0,1\}$ is the voter model starting from Bernoulli product measure ν_ρ with density $\rho \in (0,1)$. The solution of this equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ.

In Gärtner, den Hollander and Maillard [6] the behavior of the *annealed* Lyapunov exponents, i.e., the exponential growth rates of the successive moments of u w.r.t. ξ, was investigated. It was shown that these exponents exhibit an interesting dependence on the dimension and on the diffusion constant.

In the present paper we address some questions left open in [6] by considering specifically when the Lyapunov exponents are the a priori maximal value in terms of strong transience of the Markov process underlying the voter model.

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1 Introduction

1.1 Model

The parabolic Anderson model (PAM) is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \gamma\xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d, t \geq 0, \qquad (1)$$

with u a $\mathbb{R}^+$-valued field, $\kappa \in \mathbb{R}^+$ a diffusion constant, Δ the discrete Laplacian, acting on u as

$$\Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ y \sim x}} [u(y,t) - u(x,t)]$$

($y \sim x$ meaning that y is a nearest neighbor of x), $\gamma \in \mathbb{R}^+$ a coupling constant and

$$\xi = (\xi_t)_{t \geq 0} \quad \text{with} \quad \xi_t = \{\xi_t(x) := \xi(x,t) \colon x \in \mathbb{Z}^d\}$$

the *Voter Model* (VM) taking values in $\{0,1\}^{\mathbb{Z}^d \times \mathbb{R}^+}$. As initial condition, we choose

$$u(x,0) = 1, \qquad x \in \mathbb{Z}^d. \qquad (2)$$

One can interpret (1) in terms of population dynamics. Consider a system of two types of particles, A "catalyst" and B "reactant", subject to:

- A-particles evolve autonomously according to the voter dynamics;
- B-particles perform independent random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location;
- the initial configuration of B-particles is one particle everywhere.

Then $u(x,t)$ can be interpreted as the average number of B-particles at site x at time t conditioned on the evolution of the A-particles.

1.2 Voter Model

The VM is the Markov process on $\{0,1\}^{\mathbb{Z}^d}$ with generator L acting on cylindrical functions f as

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} p(x,y) \sum_{\substack{y \in \mathbb{Z}^d \\ y \sim x}} \Big(f(\eta^{x,y}) - f(\eta)\Big),$$

where $p \colon \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ is the transition kernel of an irreducible random walk and $\eta^{x,y}$ is the configuration

$$\begin{cases} \eta^{x,y}(z) = \eta(z) & \forall z \neq x, \\ \eta^{x,y}(x) = \eta(y). \end{cases}$$

In words, $\xi(x,t) = 1$ and $\xi(x,t) = 0$ mean the presence and the absence of a particle at site x at time t, respectively. Under the VM dynamics, the presence and absence of particles are imposed according to the random walk transition kernel $p(\cdot,\cdot)$.

The VM was introduced independently by Clifford and Sudbury [3] and by Holley and Liggett [7], where the basic results concerning equilibria were shown. Let $(S_t)_{t \geq 0}$ be the Markov semigroup associated with L, $p^{(s)}(x,y) = (1/2)[p(x,y) + p(y,x)]$, $x,y \in \mathbb{Z}^d$, be the *symmetrized* transition kernel associated with $p(\cdot,\cdot)$, and μ_ρ the *equilibrium measure* with density $\rho \in (0,1)$. When $p^{(s)}(\cdot,\cdot)$ is *recurrent* all equilibria are *trivial*, i.e., of the form $\mu_\rho = (1-\rho)\delta_0 + \rho\delta_1$, while when $p^{(s)}(\cdot,\cdot)$ is *transient* there are also *non-trivial* equilibria, i.e., ergodic measures μ_ρ, different from the previous one, which are the unique shift-invariant and ergodic equilibrium with density $\rho \in (0,1)$.

For both cases we have

$$\mu S_t \rightarrow \mu_\rho \qquad \text{weakly as } t \rightarrow \infty \tag{3}$$

for any starting measure μ that is stationary and ergodic with density ρ (see Liggett [10], Corollary V.1.13). This is in particular the case for our choice $\mu := \nu_\rho$, the Bernoulli product measure with density $\rho \in (0,1)$.

1.3 Lyapunov exponents

Our focus of interest will be on the p-th *annealed* Lyapunov exponent, defined by

$$\lambda_p = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho} \left([u(0,t)]^p\right)^{1/p}, \qquad p \in \mathbb{N}, \tag{4}$$

which represents the exponential growth rate of the p-th moment of the solution of the PAM (1), where $\mathbb{E}_{\nu_\rho}$ denotes the expectation w.r.t. the ξ-process starting from Bernoulli product measure ν_ρ with density $\rho \in (0,1)$. Note that λ_p depends on the parameters κ, d, γ and ρ with the two latter being fixed from now. If the above limit exists, then, by Hölder's inequality, $\kappa \mapsto \lambda_p(\kappa)$ satisfies

$$\lambda_p(\kappa) \in [\rho\gamma, \gamma] \qquad \forall \kappa \in [0,\infty).$$

The behavior of the annealed Lyapunov exponents with VM catalysts has already been investigated by Gärtner, den Hollander and Maillard [6], where it was shown that:

- the Lyapunov exponents defined in (4) exist and do not depend on the choice of the starting measure $\nu_\rho S_T$, $T \in [0,\infty]$, where $\nu_\rho S_\infty := \mu_\rho$ denotes the equilibrium measure of density ρ (recall (3));

- the function $\kappa \mapsto \lambda_p(\kappa)$ is globally Lipschitz outside any neighborhood of 0 and satisfies $\lambda_p(\kappa) > \rho\gamma$ for all $\kappa \in [0,\infty)$;
- the Lyapunov exponents satisfy the following dichotomy (see Figure 1):
  - when $1 \le d \le 4$, if $p(\cdot,\cdot)$ has zero mean and finite variance, then $\lambda_p(\kappa) = \gamma$ for all $\kappa \in [0,\infty)$;
  - when $d \ge 5$,
    · $\lim_{\kappa \to 0} \lambda_p(\kappa) = \lambda_p(0)$;
    · $\lim_{\kappa \to \infty} \lambda_p(\kappa) = \rho\gamma$;
    · if $p(\cdot,\cdot)$ has zero mean and finite variance, then $p \mapsto \lambda_p(\kappa)$ is strictly increasing for $\kappa \ll 1$.

The following questions were left open (see [6], Section 1.8):

(Q1) Does $\lambda_p < \gamma$ when $d \ge 5$ if $p(\cdot,\cdot)$ has zero mean and finite variance?

(Q2) Is there a full dichotomy in the behavior of the Lyapunov exponents? Namely, $\lambda_p < \gamma$ if and only if $p^{(s)}(\cdot,\cdot)$ is strongly transient, i.e.,

$$\int_0^\infty t\, p_t^{(s)}(0,0)\, dt < \infty.$$

Since any transition kernel $p(\cdot,\cdot)$ in $d \ge 5$ satisfies $\int_0^\infty t\, p_t(0,0)dt < \infty$, a positive answer to (Q2) will also ensure a positive one to (Q1) in the particular case when $p(\cdot,\cdot)$ is symmetric. Theorems 1.2–1.4 in Section 1.4 give answers to question (Q2), depending on the symmetry of $p(\cdot,\cdot)$. A positive answer to (Q1), given in Theorem 1.1, can also be deduced from our proof of Theorem 1.2.
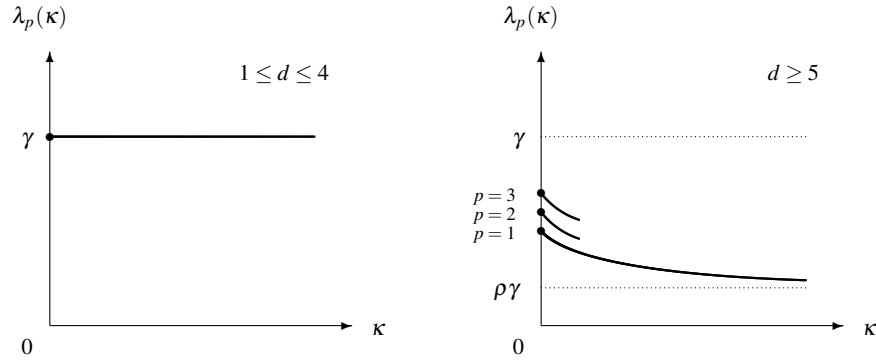

Fig. 1 Dichotomy of the behavior of $\kappa \mapsto \lambda_p(\kappa)$ when $p(\cdot,\cdot)$ has zero mean and finite variance.

By the Feynman-Kac formula, the solution of (1–2) reads

$$u(x,t) = \mathbb{E}_x \left(\exp\left[\gamma \int_0^t \xi\left(X^\kappa(s), t - s\right) ds \right] \right),$$

where $X^\kappa = (X^\kappa(t))_{t \geq 0}$ is a simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$ and $\mathbb{E}_x$ denotes the expectation with respect to X^κ given $X^\kappa(0) = x$. This leads to the following representation of the Lyapunov exponents

$$\lambda_p = \lim_{t \to \infty} \Lambda_p(t)$$

with

$$\Lambda_p(t) = \frac{1}{pt} \log \left(\mathbb{E}_{\nu_\rho} \otimes \mathbb{E}_0^{\otimes p} \right) \left(\exp \left[\gamma \int_0^t \sum_{j=1}^p \xi \left(X_j^\kappa(s), t-s \right) ds \right] \right),$$

where X_j^κ, $j = 1, \ldots, p$, are p independent copies of X^κ. In the above expression, the ξ and X^κ processes are evolving in time reversed directions. It is nevertheless possible to let them run in the same time evolution by using the following arguments. Let $\widetilde{\Lambda}_p(t)$ denote the ξ-*time-reversal* analogue of $\Lambda_p(t)$ defined by

$$\widetilde{\Lambda}_p(t) = \frac{1}{pt} \log \left(\mathbb{E}_{\nu_\rho} \otimes \mathbb{E}_0^{\otimes p} \right) \left(\exp \left[\gamma \int_0^t \sum_{j=1}^p \xi \left(X_j^\kappa(s), s \right) ds \right] \right)$$

and denote $\underline{\Lambda}_p(t) =$

$$\frac{1}{pt} \log \max_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\nu_\rho} \otimes \mathbb{E}_0^{\otimes p} \right) \left(\exp \left[\gamma \int_0^t \sum_{j=1}^p \xi \left(X_j^\kappa(s), t-s \right) ds \right] \prod_{j=1}^p \delta_x \left(X_j^\kappa(t) \right) \right)$$

$$= \frac{1}{pt} \log \max_{x \in \mathbb{Z}^d} \left(\mathbb{E}_{\nu_\rho} \otimes \mathbb{E}_0^{\otimes p} \right) \left(\exp \left[\gamma \int_0^t \sum_{j=1}^p \xi \left(X_j^\kappa(s), s \right) ds \right] \prod_{j=1}^p \delta_x \left(X_j^\kappa(t) \right) \right),$$

where in the last line we reverse the time of the ξ-process by using that ν_ρ is shift-invariant and X_j^κ, $j = 1, \ldots, p$, are time-reversible. As noted in [4], Section 2.1, $\lim_{t \to \infty} [\Lambda_p(t) - \underline{\Lambda}_p(t)] = 0$ and, using the same argument, $\lim_{t \to \infty} [\widetilde{\Lambda}_p(t) - \underline{\Lambda}_p(t)] = 0$, after which we can conclude that

$$\lambda_p(\kappa) = \lim_{t \to \infty} \frac{1}{pt} \log \left(\mathbb{E}_{\nu_\rho} \otimes \mathbb{E}_0^{\otimes p} \right) \left(\exp \left[\gamma \int_0^t \sum_{j=1}^p \xi \left(X_j^\kappa(s), s \right) ds \right] \right).$$

1.4 Main results

In what follows we give answers to questions (Q1) and (Q2) addressed in [6] concerning when the Lyapunov exponents are trivial, i.e., equal to their a priori maximal value γ.

Our first theorem gives a positive answer to (Q1). It will be proved in Section 2 as a consequence of the proof of Theorem 1.2.

Theorem 1.1. *If $d \geq 5$ and $p(\cdot,\cdot)$ has zero mean and finite variance, then $\lambda_p(\kappa) < \gamma$ for all $p \geq 1$ and $\kappa \in [0,\infty)$.*

Our two next theorems state that the full dichotomy in (Q2) holds in the case when $p(\cdot,\cdot)$ is symmetric (see Fig. 2). They will be proved in Section 2 and 3, respectively.

Theorem 1.2. *If $p(\cdot,\cdot)$ is symmetric and strongly transient, then $\lambda_p(\kappa) < \gamma$ for all $p \geq 1$ and $\kappa \in [0,\infty)$.*

Theorem 1.3. *If $p(\cdot,\cdot)$ is symmetric and not strongly transient, then $\lambda_p(\kappa) = \gamma$ for all $p \geq 1$ and $\kappa \in [0,\infty)$.*
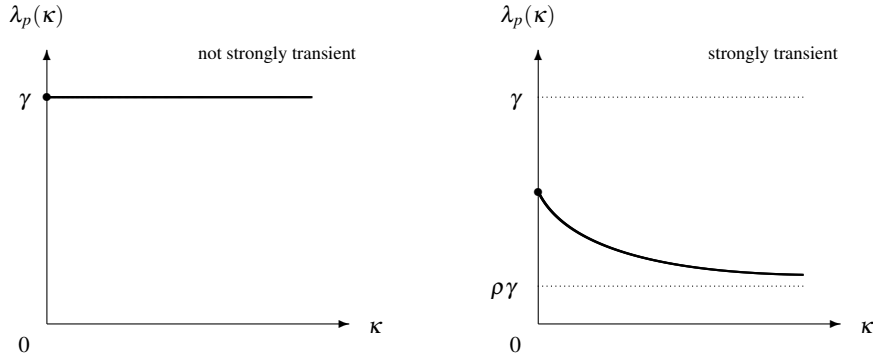

Fig. 2 Full dichotomy of the behavior of $\kappa \mapsto \lambda_p(\kappa)$ when $p(\cdot,\cdot)$ is symmetric.

A similar full dichotomy also holds for the case where ξ is symmetric exclusion process in equilibrium, between recurrent and transient $p(\cdot,\cdot)$ (see [5]).

Our fourth theorem shows that this full dichotomy only holds for symmetric transition kernels $p(\cdot,\cdot)$, ensuring that the assertion in (Q2) is not true in its full generality.

Theorem 1.4. *There exists $p(\cdot,\cdot)$ not symmetric with $p^{(s)}(\cdot,\cdot)$ not strongly transient such that $\lambda_p(\kappa) < \gamma$ for all $p \geq 1$ and $\kappa \in [0,\infty)$.*

In the strongly transient regime, the following problems remain open:

(a) $\lim_{\kappa \to 0} \lambda_p(\kappa) = \lambda_p(0)$;
(b) $\lim_{\kappa \to \infty} \lambda_p(\kappa) = \rho\gamma$;
(c) $p \mapsto \lambda_p(\kappa)$ is strictly increasing for $\kappa \ll 1$;
(d) $\kappa \mapsto \lambda_p(\kappa)$ is convex on $[0,\infty)$.

In [6], (a) and (b) were established when $d \geq 5$, and (c) when $d \geq 5$ and $p(\cdot, \cdot)$ has zero mean and finite variance. Their extension to the case when $p(\cdot, \cdot)$ is strongly transient remains open.

In what follows, we use generic notation $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation whatever the corresponding process is (even for joint processes) and denote $\xi_s(x) := \xi(s, x)$.

2 Proof of Theorems 1.1 and 1.2

We first give the proof of Theorem 1.2. In this section, the random walk X on ξ is symmetric. At the end of the section we will explain how to derive the proof of Theorem 1.1. Recall that in this part we suppose that the coalescing random walks X of the Voter Model ξ are symmetric.

We have to show that $\lambda_p(\kappa) < \gamma$ for all $\kappa \in [0, \infty)$. In what follows we assume without loss of generality that $p = 1$, the extension to arbitrary $p \geq 1$ being straightforward. Our approach is to pick a *bad environment set* B_E associated to the ξ-process and a *bad random walk set* B_W associated to the random walk X^κ so that, for all $n \in \mathbb{N}$,

$$\mathbb{E}\left(\exp\left[\gamma \int_0^n \xi_s(X^\kappa(s)) \, ds \right] \right) \tag{5}$$

$$\leq \left(\mathbb{P}(B_E) + \mathbb{P}(B_W) \right) e^{\gamma n} + \mathbb{E}\left(\mathbb{1}_{B_E^c \cap B_W^c} \exp\left[\gamma \int_0^n \xi_s(X^\kappa(s)) \, ds \right] \right)$$

with, for some $0 < \delta < 1$,

$$\mathbb{P}(B_E) \leq e^{-\delta n}, \qquad \mathbb{P}(B_W) \leq e^{-\delta n}, \tag{6}$$

and,

$$\int_0^n \xi_s(X^\kappa(s)) \, ds \leq n(1 - \delta) \qquad \text{on } B_E^c \cap B_W^c. \tag{7}$$

Since, combining (5–7), we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}\left(\exp\left[\gamma \int_0^n \xi_s(X^\kappa(s)) \, ds \right] \right) < \gamma,$$

it is enough to prove (6) and (7).

The proof of (6) is given in Sections 2.1–2.3 below, and (7) will be obvious from our definitions of B_E and B_W.

2.1 Coarse-graining and skeletons

Write $\mathbb{Z}_e^d = 2\mathbb{Z}^d$ and $\mathbb{Z}_o^d = 2\mathbb{Z}^d + 1$, where $1 = (1,\ldots,1) \in \mathbb{Z}^d$. We are going to use a *coarse-graining representation* defined by a *space-time block partition* B_y^j and a *random walk skeleton* $(y_i)_{i \geq 0}$. To that aim, for a fixed M, consider

$$B_y^j = \prod_{k=1}^d \left[(y_k - 1)M, (y_k + 1)M \right) \times \left[jM, (j+1)M \right) \subset \mathbb{Z}^d \times \mathbb{R}_+,$$

where $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$y \in \begin{cases} \mathbb{Z}_e^d & \text{when } j \text{ is even,} \\ \mathbb{Z}_o^d & \text{when } j \text{ is odd.} \end{cases}$$

Without loss of generality we can consider random walks trajectories on interval $[0,n]$ with $n \in \mathbb{N}$ multiple of M. Define the *M-skeleton set* set by

$$\Xi = \left\{ \left(y_0, \ldots, y_{\frac{n}{M}} \right) \in (\mathbb{Z}^d)^{\frac{n}{M}+1} : y_{2k} \in \mathbb{Z}_e^d, \ y_{2k+1} \in \mathbb{Z}_o^d \ \forall k \in \mathbb{N}_0 \right\}$$

and the *M-skeleton set associated to a random walk X* by

$$\Xi(X) = \left\{ \left(y_0, \ldots, y_{\frac{n}{M}} \right) \in \Xi : X(kM) \in B_{y_k}^k \ \forall k \in \{0, \ldots, n/M\} \right\}.$$

In what follows, we will consider the *M*-skeleton $\Xi(X^\kappa)$, but, as X^κ starts from $0 \in \mathbb{Z}^d$, the first point of our *M*-skeleton will always be $y_0 := 0 \in \mathbb{Z}^d$ (see Fig. 3).
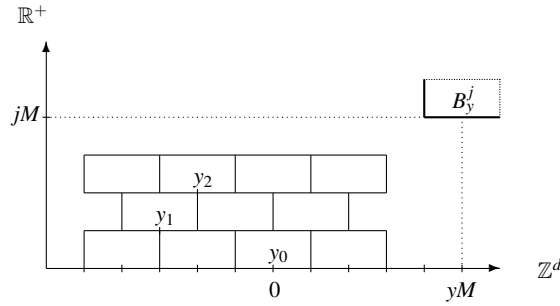

Fig. 3 Illustration of a *M*-skeleton $(y_0, y_1, y_2, \ldots) \in \Xi$ and coarse-grained B_y^j blocks.

In the next lemma we prove that the number of *M*-skeletons not oscillating too much is at most exponential in n/M. For that, define

$$\Xi_A = \left\{ (y_0,\ldots,y_{\frac{n}{M}}) \in \Xi : \sum_{j=1}^{n/M} (\|y_j - y_{j-1}\|_\infty - 1) \leq \frac{n}{Md} \right\}, \tag{8}$$

where $\|\cdot\|_\infty$ is the standard l_∞ norm, the set of all M-skeletons that are *appropriate*.

Lemma 2.1. *There exists some universal constant $K \in (1,\infty)$ such that, for any $n, M \in \mathbb{N}$,*

$$|\Xi_A| \leq K^{n/M}.$$

Proof. For any fixed $y_1 \in \mathbb{Z}^d$ and $N \in \mathbb{N}_0$, let

$$I(N) = \left| \left\{ y_2 \in \mathbb{Z}^d : \|y_1 - y_2\|_\infty - 1 = N \right\} \right|$$

be the number of elements of $\mathbb{Z}^d$ on the boundary of the cube of size $2N+3$ centered at y_1. For any $N \in \mathbb{N}$, we have $I(N) = (2N+3)^d - (2N+1)^d$ and $I(0) = 3^d - 1$, therefore, for any $N \in \mathbb{N}_0$,

$$I(N) \leq 3^d (N+1)^d. \tag{9}$$

Define, for any $N, k \in \mathbb{N}$,

$$I(N,k) = \left| \left\{ (y_j)_{0 \leq j \leq k} \in (\mathbb{Z}^d)^{k+1} : \sum_{j=1}^{k} (\|y_j - y_{j-1}\|_\infty - 1) = N \right\} \right| \tag{10}$$

the number of sequences in $(\mathbb{Z}^d)^{k+1}$ having size N. By (9) and (10), we have

$$
\begin{aligned}
I(N,k) &= \sum_{\substack{(N_1,\ldots,N_k): \\ \sum_{i=1}^{k} N_i = N}} \left(\prod_{i=1}^{k} I(N_i) \right) \\
&\leq 3^{dk} \sum_{\substack{(N_1,\ldots,N_k): \\ \sum_{i=1}^{k} N_i = N}} \left(\prod_{i=1}^{k} (N_i + 1) \right)^d \\
&\leq 3^{dk} \binom{k+N+1}{N} 2^{dN},
\end{aligned}
\tag{11}
$$

where, in the last line, we used that

$$\max_{\substack{(N_1,\ldots,N_k): \\ \sum_{i=1}^{k} N_i = N}} \prod_{i=1}^{k} (N_i + 1) = 2^N.$$

Using (11) and the fact that $\Xi \subset \mathbb{Z}^d$, we obtain

$$|\Xi_A| \leq \sum_{N=0}^{\lfloor \frac{n}{Md} \rfloor} I\left(N, \frac{n}{M}\right) \leq 3^{\frac{n}{M}} \sum_{N=0}^{\lfloor \frac{n}{Md} \rfloor} \binom{\frac{n}{M}+N+1}{N} 2^{dN}$$

$$\leq 3^{\frac{n}{M}} \sum_{N=0}^{\lfloor \frac{n}{Md} \rfloor} \binom{\frac{2n}{M}}{N} 2^{dN} \leq 3^{\frac{n}{M}} \sum_{N=0}^{\frac{2n}{M}} \binom{\frac{2n}{M}}{N} 2^{dN}$$

$$= 3^{\frac{n}{M}} (2^d+1)^{\frac{2n}{M}},$$

which ends the proof of the lemma. □

2.2 The bad environment set B_E

This section is devoted to the proof of the leftmost part of (6) for suitable set B_E defined below.

We say that an environment ξ is *good* w.r.t. an M-skeleton $(y_0, \cdots, y_{\frac{n}{M}})$ if we have

$$\left| \left\{ 0 \leq j < \frac{n}{M} : \exists (x, jM) \in B_{y_j}^j \text{ s.t. } \xi_s(x) = 0 \ \forall s \in [jM, jM+1] \right\} \right| \geq \frac{n}{4M}.$$

Since we want the environment to be good w.r.t. all appropriate M-skeleton, we define the *bad environment set* as

$$B_E = \{ \exists \text{ an } M\text{-skeleton} \in \Xi_A \text{ s.t. } \xi \text{ is not good w.r.t. it} \}.$$

In the next lemma we prove that for any fixed M-skeleton, the probability that ξ is not good w.r.t. it is at most exponentially small in n/M.

Lemma 2.2. *Take $(y_i)_{0 \leq i \leq \frac{n}{M}} \in \Xi$ an M-skeleton. For M big enough, we have*

$$\mathbb{P}\left(\xi \text{ is not good w.r.t. } (y_i)_{0 \leq i \leq \frac{n}{M}} \right) \leq (4K)^{-n/M},$$

where K is the universal constant defined in Lemma 2.1.

Therefore, combining Lemmas 2.1 and 2.2, we get

$$\mathbb{P}(B_E) \leq 4^{-n/M}$$

for M big enough, from which we obtain the leftmost part of (6).

Before proving Lemma 2.2, we first give an auxiliary lemma. In order to study the evolution of the VM, we consider, as usual, the dual process, namely, a coalescing random system that evolves backward in time. To that aim, define $(X^{x,t}(s))_{0 \leq s \leq t}$ to be the random walk starting from 0 at time t, (i.e., $X^{x,t}(0) = 0$). From the graphical representation of the VM, we can write $\xi_t(x) = \xi_0(X^{x,t}(t)), x \in \mathbb{Z}^d$, and therefore the the VM process can be expressed in terms of its initial configuration and a system of coalescing random walks. Two random walks $X^{x,s}$ and $X^{x',s'}$ with $s' < s$ meet if

there exists $u \leq s'$ such that $X^{x',s'}(s'-u) = X^{x,s}(s-u)$. It is therefore the same to say that $X^{x,s}$ and $X^{x',s'}$ with $s' < s$ meet (in some appropriate time interval) if there exists $t \geq 0$ in this interval (by letting $t = s' - u$) such that

$$X^{x',s'}(t) = X^{x,s}(t+s-s').$$

For convenience we will adopt this notation in the rest of the section.

Lemma 2.3. *Take two independent random walks $X^{x,s}$ and $X^{x',s'}$ with $s' < s$. Then the probability they ever meet is bounded above by*

$$\int_{s-s'}^{\infty} p_t(0,0)\,dt.$$

Proof. Consider the random variable

$$W = \int_0^{\infty} \mathbf{1}\left\{X^{x,s}(t) = X^{x',s'}(t+(s-s'))\right\} dt.$$

By symmetry, its expectation satisfies

$$\begin{aligned}
\mathbb{E}(W) &= \int_0^{\infty} \mathbb{P}\left(X^{0,0}(2t+s-s') = x-x'\right) dt \\
&\leq \int_0^{\infty} \mathbb{P}\left(X^{0,0}(2t+s-s') = 0\right) dt \\
&= \frac{1}{2}\int_{s-s'}^{\infty} p_t(0,0)\,dt.
\end{aligned}$$

Moreover, we have

$$\mathbb{E}(W \mid W > 0) = \int_0^{\infty} p_{2t}(0,0)\,dt \geq \frac{1}{2}$$

and then, since $\mathbb{E}(W) = \mathbb{E}(W \mid W > 0)\,\mathbb{P}(W > 0)$, it follows that

$$\mathbb{P}(W > 0) \leq 2\mathbb{E}(W) = \int_{s-s'}^{\infty} p_t(0,0)\,dt.$$

$\square$

We are now ready to prove the Lemma 2.2.

Proof. Recall that $\xi_s(x) = \xi_0(X^{x,s}(s))$, where ξ_0 is distributed by a product Bernoulli law with density $\rho \in (0,1)$. We first consider any M-skeleton $(y_0,\ldots,y_{n/M}) \in \Xi$ (even not appropriate). For each $0 \leq j < n/M$, we choose R sites $(x_1^j, jM),\ldots,$ $(x_R^j, jM) \in B_{y_j}^j$ such that

$$\mathbb{P}\left(\exists 0 \leq k,k' \leq R,\ k \neq k': X^{x_k^j,jM}(s) = X^{x_{k'}^j,jM}(s) \text{ for some } s \in [0, jM]\right) \leq \varepsilon \quad (12)$$

for $\varepsilon \ll 1$ to be specified later (see Fig. 4). Remark that we first fix ε and R and then we choose M large enough so we can find these R sites. As we are in the strongly transient regime, we know that these points exist. If two such random walks hit each other, then we freeze all the random walks issuing from the corresponding block j.
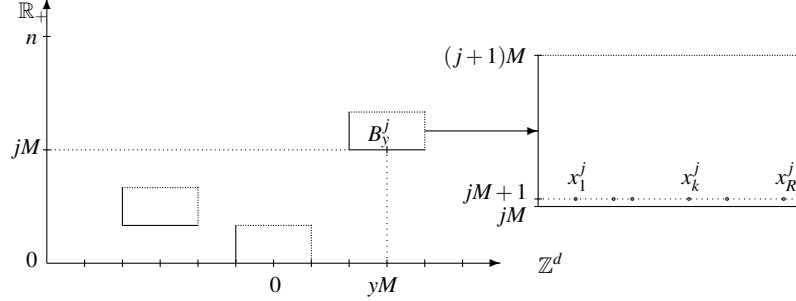

Fig. 4 Illustration of sites (x_k^j, jM), $k \in \{1, \ldots, R\}$, for a fixed M-skeleton.

For any $1 \leq j < \frac{n}{M}$, $1 \leq k \leq R$ for some $R > 0$, we have

$$\mathbb{E}\left(\sum_{j'=j+1}^{\frac{n}{M}-1} \sum_{k'=1}^{R} 1\left\{ X^{x_{k'}^{j'}, j'M}(s+(j'-j)M) = X^{x_k^j, jM}(s) \text{ for some } s \in [0, jM] \right\} \right)$$

$$\leq R \sum_{j'=j+1}^{\frac{n}{M}-1} \int_{(j'-j)M}^{\infty} p_t(0,0) \, dt$$

$$\leq R \int_{M}^{\infty} \frac{t}{M} p_t(0,0) \, dt,$$

and therefore, summing over $1 \leq k \leq R$, we get

$$\mathbb{E}\left(\sum_{j'=j+1}^{\frac{n}{M}-1} \sum_{k,k'=1}^{R} 1\left\{ X^{x_{k'}^{j'}, j'M}(s+(j'-j)M) = X^{x_k^j, jM}(s) \text{ for some } s \in [0, jM] \right\} \right)$$

$$\leq \frac{R^2}{M} \int_{M}^{\infty} t \, p_t(0,0) \, dt \leq \varepsilon^2, \tag{13}$$

for M sufficiently large. Again, remark that we first fix ε and R, then we choose M large enough. For each j, we now define the filtration

$$\mathscr{F}_t^j = \sigma\left(X^{x_k^j, jM}(s) \colon 0 \leq s \leq t, \ 1 \leq k \leq R \right)$$

and the sub-martingale $Z^j(t) :=$

$$\mathbb{E}\left(\sum_{j'=j+1}^{\frac{n}{M}-1} \sum_{k,k'=1}^{R} 1\left\{ X^{x_{k'}^{j'},j'M}(s+(j'-j)) = X^{x_k^j,jM}(s) \text{ for some } s \in [0,t] \right\} \ \Bigg| \ \mathscr{F}_t^j \right),$$

with the stopping time

$$\tau^j = \inf\left\{ t \geq 0 \colon Z^j(t) > \varepsilon \right\}.$$

We freeze every random walk issuing from block j at time $jM \wedge \tau^j$. Using (13) and the Doob's inequality, we can see that

$$\mathbb{P}(\tau^j < \infty) \leq \frac{R^2}{M\varepsilon} \int_M^{\infty} t\, p_t(0,0)\, dt \leq \varepsilon. \tag{14}$$

Since, Z^j is a continuous sub-martingale except at jump times of one of the random walks $X^{x_k^j,jM}$ and when a jump occurs, the increment is at most

$$R \sum_{j'=j+1}^{\frac{n}{M}-1} p_{(j'-j)M}(0,0) \leq \varepsilon^2$$

if M is big enough. Therefore, for all $0 \leq t \leq \tau^j$, we get

$$Z^j(t) < \varepsilon + \varepsilon^2 \leq 2\varepsilon \qquad \mathbb{P}\text{-a.s.}. \tag{15}$$

Now we say that j is *good* if

- $\tau^j > jM$;
- the R random walks $X^{x_1^j,jM}, \cdots, X^{x_R^j,jM}$ do not meet;
- the random walks $X^{x_k^j,jM}$ do not hit any point $x_{k'}^{j'}$ during interval $[(j-j')M - 1, (j-j')M]$ for $j' < j$;
- the random walks $X^{x_k^j,jM}$ do not meet $X^{x_{k'}^{j'},j'M}$ for $j' < j$.

By (14), we know that the probability that the first condition does not occur is smaller than ε. By definition of the sites $x_1^j,\ldots,x_R^j$, we know that the probability that random walks issuing from the same block j at sites x_k^j, $k = 1,\ldots,R$, hit each other is smaller than ε (recall (12)). Moreover, the probability that the third condition does not occur is bounded from above by

$$R \sum_{j'=1}^{j-1} \int_{(j-j')M-1}^{(j-j')M} p_t(0,0)\, dt$$

which is as small as we want for M large because we are in a transient case. We still have to compute the probability that the fourth condition does not occur. Furthermore, we can see that two random walks issuing from the same block evolve independently until they meet, provided they do not meet a previous random walk. from the above consideration, we get

$$\mathbb{P}\left(j \text{ is not good} \mid \mathscr{G}^{j-1}\right) \leq 3\varepsilon + \sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathscr{G}^{j'}\right),$$

where

$$\mathscr{G}^j = \sigma\left(X^{x_k^{j'}, j'M}(s) \colon 1 \leq k \leq R, \, 1 \leq j' \leq j, \, 0 \leq s \leq j'M \wedge \tau^{j'}\right).$$

Here, we recall that $X^{x_k^j, jM}$ meets $X^{x_{k'}^{j'}, j'M}$ (with $j' < j$) if we have

$$X^{x_k^j, jM}(s + (j - j')M) = X^{x_{k'}^{j'}, j'M}(s) \text{ for some } s \in \left[0, \tau^{j'} \wedge j'M\right].$$

By (15), we have, for all j' fixed,

$$\sum_{j=j'+1}^{\frac{n}{M}-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathscr{G}^{j'}\right) \leq 2\varepsilon.$$

Summing over all $1 \leq j' \leq n/M - 2$, we get

$$\sum_{j'=1}^{\frac{n}{M}-2} \sum_{j=j'+1}^{\frac{n}{M}-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathscr{G}^{j'}\right) \leq \frac{n}{M} 2\varepsilon,$$

and then, interchanging the sums, we arrive at

$$\sum_{j=2}^{\frac{n}{M}-1} \sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathscr{G}^{j'}\right) \leq \frac{n}{M} 2\varepsilon.$$

Thus, there are at most $\lfloor \frac{n}{2M} \rfloor$ random positions j with the property

$$\sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathscr{G}^{j'}\right) \geq 4\varepsilon,$$

and so at least $\lceil \frac{n}{2M} \rceil - 2$ random positions j have the property

$$\sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathscr{G}^{j'}\right) < 4\varepsilon.$$

For these random positions j, we then have

$$\mathbb{P}\left(j \text{ is good} \mid \mathscr{G}^{j-1}\right) \geq 1 - 7\varepsilon.$$

Using an elementary coupling, we have at least $\frac{n}{3M}$ positions that are good with probability bounded by

$$\mathbb{P}\left(Y \geq \frac{n}{3M}\right) \geq 1 - e^{-c(\varepsilon)n/M}$$

for

$$Y \sim B\left(\left\lceil \frac{n}{2M} \right\rceil - 2, 1 - 7\varepsilon\right)$$

and $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Therefore, outside of a small probability $e^{-c(\varepsilon)n/M}$, for at least $\frac{n}{3M}$ positions j, we have that the random walks $X^{x_k^j, jM}$ are disjoint and so the values $\xi_0(X^{x_k^j, jM}(s))$ are independent until time $s \leq jM$. Then, using the fact that $(\xi_0(x))_{x \in \mathbb{Z}^d}$ are i.i.d. Bernoulli product with parameter ρ, we have that the number of positions j so that there exists $(x, jM) \in B_{y_j}^j$ with $\xi_s(x) = 0$ and $s \in [jM, jM + 1]$ is at least $\frac{n}{4M}$ outside the probability

$$\mathbb{P}\left(Y' \geq \frac{n}{12M}\right) \leq \left(\frac{1}{4K}\right)^{n/M}$$

with $Y' \sim B\left(\frac{n}{3M}, \left((1 - e^{-1}(1 - \rho))^R\right)\right)$, where ε and R are chosen small and large enough, respectively. $\square$

The proof of the leftmost part of (6) is now completed.

2.3 The bad random walk set B_W

This section is devoted to the proof of the rightmost part of (6).

We are now interested in the random walk X^κ. We are going to prove that $(X^\kappa(s))_{0 \leq s \leq n}$ has an appropriate M-skeleton and touches enough zeros outside a probability event exponentially small in n (see Lemmas 2.4 and 2.5 below, respectively). To define the bad random set B_W announced in (3.4), we are going to define $B_W = B_{W_1} \cup B_{W_2}$, where the bad sets B_{W_1} and B_{W_2} correspond, respectively, to random walks trajectories X^κ which do not have appropriate M-skeleton and do not touch enough sites occupied by a zero configuration of the VM. To be more precise, define

$$B_{W_1} = \left\{(\Xi(X^\kappa) \notin \Xi_A\right\}.$$

In the next lemma, we prove that the probability of B_{W_1} is exponentially small in n.

Lemma 2.4. *Take* $(X^\kappa(s))_{0 \leq s \leq n}$ *and* $\Xi(X^\kappa) = (y_0, \cdots, y_{n/M})$ *the associated M-skeleton. Then, there exists a constant K' not depending on n such that*

$$\mathbb{P}(B_{W_1}) \leq e^{-K'n}.$$

Proof. In order to have the random walk moving from one block of the skeleton to a nonadjacent one, the random walk has to make at least M steps in the same direction. Keeping that in mind, define

$$Y_j(s) = X^\kappa(jM+s) - X^\kappa(jM)$$

and let

$$\tau_1^j = \inf\{s\colon \|Y_j(s)\|_\infty \geq M\}, \qquad \tau_i^j = \inf\left\{s > \tau_{i-1}^j\colon \|Y_j(s) - Y_j(\tau_{i-1}^j)\|_\infty \geq M\right\}.$$

Next, define

$$W_i^j = 1_{\{\tau_j^i < M\}}$$

and use an elementary coupling to have

$$\mathbb{P}(W_1^j = 1) \leq e^{-c/M} \quad \text{and} \quad \mathbb{P}(W_i^j = 1 | W_{i-1}^j = 1) \leq e^{-c(i)/M} \leq e^{-c/M}$$

for some constants c and $c(i)$ which verify $c(i) \geq c$. Therefore, we have that the number of jumps for the jth block is bounded above by the number of W_i^j equals to 1. Using a coupling we can see that this is bounded above by a geometric law with parameter $e^{-c/M}$. Now if we consider all the blocks, by elementary properties of geometric random variables, we have

$$\mathbb{P}(B_{W_1}) \leq \mathbb{P}\left(Y \geq \frac{n}{Md}\right) \leq e^{-c'n}$$

for $Y \sim B\left(\frac{n}{M}\left(1 + \frac{1}{d}\right), e^{-c/M}\right)$, some constant $c' > 0$, M being large and the proof is done. $\square$

Lemma 2.4 proves the first part of the rightmost part of (6), namely the part concerned with bad set B_{W_1}. Now we look at the number of times X^κ stays on a site where the VM has zero value. For that, define

$$\tau_{i+1} = \inf\left\{t > \tau_i + 1\colon \exists x \in \mathbb{Z}^d \text{ s.t. } \|x - X^\kappa(t)\|_\infty \leq 2M, \xi_s(x) = 0 \;\forall s \in [t, t+1]\right\}$$

with $\tau_0 = 0$ and

$$k(M) = e^{-1/2} \inf_{\|x\|_\infty \leq 2M} \mathbb{P}\left(X^\kappa(1/2) = x \mid X^\kappa(0) = 0\right)$$

(remark that $k(M)$ does not depend on n and is strictly positive). Finally, we define

$$B_{W_2} = \left\{(X^\kappa, \xi)\colon \tau_{\lfloor \frac{n}{2M}\rfloor} \leq n - 1 \text{ and } \int_0^n \xi_s(X^\kappa(s))\,ds \geq n\left(1 - \frac{k(M)}{8M}\right)\right\}$$

as being the bad set corresponding to random walks trajectories X^κ which do not touch enough sites occupied by a zero configuration of the VM. In the next lemma, we prove that such a set has an exponentially small probability in n.

Lemma 2.5. *There exists a constant $\delta > 0$ not depending on n, such that, for M big enough we have*

$$\mathbb{P}(B_{W_2}) \leq e^{-n\delta}.$$

Proof. Take any realization of ξ and for each time τ_i, define a random variable Y_i which take value 1 if X^κ reaches a site with value zero at time $\tau_i + \frac{1}{2}$ and stays at that point until time $\tau_i + 1$, and takes value 0 otherwise. Remark that after having fixed n, we can choose the state of ξ_t, $t > n$, as we want, for example, full of zeros. Continue until $\tau_{\lfloor n/(2M) \rfloor}$ which is finite if ξ is well chosen after time n. Using the strong Markov property, for every $k_i \in \{0,1\}$, we see that

$$\mathbb{P}(Y_i = 1 | Y_j = k_j, j < i) = \mathbb{P}(Y_i = 1 | Y_{i-1} = k_{i-1}) \geq k(M).$$

Then, it follows that $Y := \sum_{i=1}^{\lfloor n/(2M) \rfloor} Y_i$ is stochastically greater than Y' the binomial random variable $B\left(\frac{n}{2M}, k(M)\right)$. Moreover, if $\tau_{\lfloor n/(2M) \rfloor} \leq n - 1$, we have

$$\int_0^n \xi_s(X(s)) \, ds \leq n - \frac{1}{2} Y.$$

Hence, we get

$$\mathbb{P}(B_{W_2}) \leq \mathbb{P}\left(\tau_{\lfloor \frac{n}{2M} \rfloor} \leq n-1 \text{ and } n - \frac{1}{2} Y \geq n - \frac{nk(M)}{8M} \right)$$

$$\leq \mathbb{P}\left(n - \frac{Y}{2} \geq n - \frac{nk(M)}{8M} \right)$$

$$= \mathbb{P}\left(Y \leq \frac{nk(M)}{4M} \right)$$

$$\leq e^{-cn}$$

for n sufficiently large and c, a positive constant not depending on n. This result being shown for any realization of ξ (up to time n), this ends the proof. □

Lemma 2.5 proves the second part of the rightmost part of (6), namely the part concerned with bat set B_{W_2}. To complete the proof of (7), it suffices to use the definition of B_E and $B_W = B_{W_1} \cup B_{W_2}$, and to remark that if B_E and B_{W_1} do not occur, then the first condition of B_{W_2}, namely $\tau_{\lfloor n/2M \rfloor} \leq n - 1$, is satisfied and therefore the second must be violated.

2.4 Proof of Theorem 1.1

The proof of Theorem 1.1 can be deduced from the proof of Theorem 1.2. Without assuming that $p(\cdot, \cdot)$ is symmetric, it is enough to see that

- $\int_0^\infty t p_t(0,0) \, dt < \infty$, by local CLT, and
- there is enough symmetry because there exists some $C > 0$ such that $p_t(x,0) \leq C p_t(0,0)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,\infty)$. Therefore, Lemma 2.2 can still be applied.

¿From these two observations, the proof of Theorem 1.1 goes through the same lines as the one of Theorem 1.2.

3 Proof of Theorem 1.3

In this section we consider the Lyapunov exponents when the random walk kernel associated to the voter model noise is symmetric and also not strongly transient, that is

$$\int_0^\infty t\, p_t(0,0)\, dt = \infty .$$

We want to show that when $p(\cdot,\cdot)$ is symmetric and not strongly transient, then have that

$$\lambda_p(\kappa) \equiv \gamma \qquad \forall \kappa \in (0,\infty),\ \forall p \geq 1 .$$

Since the result is easily seen for recurrent random walks, we can and will assume in the following that

$$\int_0^\infty p_t(0,0)\, dt < \infty .$$

Given the reasoning of [5], Section 3.1 and [6], Section 5.1 this result will follow from Proposition 3.1, below. Consider, in the graphical representation associated to the VM ξ,

$\chi(t) :=$ number of distinct coalescing random walks produced on $\{0\} \times [0,t]$

(This quantity is discussed in Bramson, Cox and Griffeath [1]).

Proposition 3.1. *Assume that* $p(\cdot,\cdot)$, *is symmetric and not strongly transient, then for any* $\varepsilon > 0$, *we have that*

$$\lim_{t \to \infty} \mathbb{P}\big(\chi(t) \leq \varepsilon t\big) = 1 .$$

Before proving Proposition 3.1, we will first give the proof of Theorem 1.3.

Proof. It is immediate from the graphical representation of the VM and Proposition 3.1 that for all $\delta > 0$ and $M < \infty$,

$$\mathbb{P}\Big(\xi(x,s) = 1\ \forall \|x\|_\infty \leq M,\ \forall s \in [0,t]\Big) \geq e^{-\delta t}$$

for all t sufficiently large. Thus, just as in [6], Section 5.1, we have for all $p \geq 1$,

$$
\begin{aligned}
\mathbb{E}&([u(0,t)]^p) \\
&\geq e^{\gamma p t}\,\mathbb{P}\Big(\|X^\kappa(s)\|_\infty < M\ \forall s \in [0,t]\Big)^p \mathbb{P}\Big(\xi(x,s) = 1\ \forall \|x\|_\infty < M,\ \forall s \in [0,t]\Big) \\
&\geq e^{t(\gamma p - \delta - c(M)p)}
\end{aligned}
$$

for $c(M) \to 0$ as $M \to \infty$. From this, it is immediate that

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}([u(0,t)]^p)^{1/p} = \gamma.$$

$\square$

To prove Proposition 3.1, we consider the following system of coalescing random walks

$$I = \{X^t : t \in \mathscr{P}\}$$

for $\mathscr{P}$ a two sided, rate one Poisson process and X^t a random walk defined on $s \in [t, \infty)$, starting at 0 at time t (we could equally well consider a system of random walks indexed by $h\mathbb{Z}$ for some constant h). The coalescence is such that for $t < t' \in \mathscr{P}$, X^t, $X^{t'}$ evolve independently until $T^{t,t'} = \inf\{s > t' : X^t(s) = X^{t'}(s)\}$, and then, for $s \geq T^{t,t'}$, $X^t(s) = X^{t'}(s)$.

We will be interested in the density or number of distinct random walks at certain times. To aid this line we will adopt via a labelling procedure for the random walks, whereby effectively when two random walks meet for the first time, one of them (chosen at random) dies; in this optic the number of distinct random walks will be the number still alive. Our labeling scheme involves defining for each $t \in \mathscr{P}$, the label process l_s^t for $s \geq t-$ (it will be helpful to be able to define $l_{t-}^t = t$, though since at time t there may well be other random walks present at the origin, it will not necessarily be the case that $l_t^t = t$). These processes will be defined by the following properties:

- if for $t \neq t' \in \mathscr{P}, X^t(s) \neq X^{t'}(s)$, then $l_s^t \neq l_s^{t'}$;
- if $t_1, t_2, \ldots, t_r$ are elements of $\mathscr{P}$, then at $s \geq \max\{t_1, \ldots, t_r\}$, if $X^{t_1}(s) = X^{t_2}(s) = \cdots = X^{t_r}(s)$, then $l_s^{t_1} = l_s^{t_2} = \cdots = l_s^{t_r} = u$ for some $u \in \mathscr{P}$ with $X^{t_1}(s) = X^u(s)$;
- if for $t \neq t' \in \mathscr{P}$, X^t meets $X^{t'}$ for the first time at s, then independently of past and future random walks or labeling decisions $l_s^t = l_s^{t'} = l_{s-}^{t'}$ with probability $\frac{1}{2}$ and with equal probability $l_s^t = l_s^{t'} = l_{s-}^t$;
- the process l_s^t can only change at moments where X^t meets a distinct random walk for the first time.

For $t \in \mathscr{P}$, $s > t$, we say that t is *alive at time s*, if $l_s^t = t$; it *dies* at time s if $l_{s-}^t = t, l_s^t \neq t$. We say X^t, X^u *coalesce* at time s if this is the first time at which the two labels are equal. The following are easily seen:

- the events $A_s^t = \{l_s^t = t\}$ for $t \in \mathscr{P}$ are decreasing in s;
- A_s^t depends only on the random motions of the coalescing random walks and on the labeling choices involving X^t;
- for $s > 0$, the number of independent random walks $X^t(s)$, $t \in \mathscr{P} \cap [-n, 0]$ is simply equal to the number of distinct labels $l_s^t, t \in \mathscr{P} \cap [-n, 0]$.

Let

$$c_0 = \lim_{s \to \infty} \mathbb{P}^t(A_s^t) \in [0, 1], \tag{16}$$

according to palm measure, $\mathbb{P}^t$, for $t \in \mathscr{P}$. We obtain easily:

Proposition 3.2.

$$\lim_{s \to \infty} \frac{1}{s} \left| \left\{ \text{distinct random walks } X^t(0) : t \in \mathscr{P} \cap [-s, 0) \right\} \right| = c_0 \qquad \text{a.s.}$$

Proof. Using the definition of c_0 in (16) and ergodicity of the system we see that the limit is greater than c_0. Then, Lemma 3.1 gives the result.

Lemma 3.1. *For c_0 as defined in (16), for each $\varepsilon > 0$, there exists $R < \infty$ so that if we consider the finite system of coalescing random walks $(X^t)_{t \in (-R, 0] \cap \mathscr{P}}$, then with probability at least $1 - \varepsilon$ at time R there are less than $(c_0 + \varepsilon)R$ distinct random walks labels.*

Proof. By definition of c_0, for all $\varepsilon > 0$ there exists a T_0 so that

$$\mathbb{P}^0 \left(\text{label 0 is alive at time } s \right) < c_0 + \frac{\varepsilon}{100} \qquad \forall s \geq T_0.$$

Now pick R_1 so that

$$\mathbb{P} \left(\|X^0(s)\|_\infty \leq R_1 \ \forall s \in (0, T_0) \right) \geq 1 - \frac{\varepsilon}{100}.$$

Therefore,

$$\mathbb{P}^0 \left(\text{label 0 is not alive at time } s \geq T_0 \text{ and } \|X^0(s)\|_\infty \leq R_1 \ \forall s \in (0, T_0) \right)$$
$$\geq 1 - c_0 - \frac{2\varepsilon}{100}.$$

We then pick T_1 so that

$$\mathbb{P} \left(\exists t \in \mathscr{P} \cap [-T_1, T_1]^c : \|X^t(s)\|_\infty \leq R_1 \text{ for some } s \in (0, T_0) \right) < \frac{\varepsilon}{100}.$$

Thus

$$\mathbb{P}^0 \left(\exists t \in \mathscr{P} \cap [-T_1, T_1] \setminus \{0\} : l_{R_1}^0 = t \right) \geq 1 - c_0 - \frac{\varepsilon}{30}.$$

¿From the translation invariant property of the system and ergodicity if

$$\lambda_s := \left| \left\{ t \in [-s, s] \cap \mathscr{P} : X^t \text{ loses its label to a random walk } X^{t'} \text{ with } |t - t'| \leq T_1 \right\} \right|,$$

then

$$\liminf_{s \to \infty} \frac{\lambda_s}{2s} \geq 1 - c_0 - \frac{\varepsilon}{30} \qquad \text{a.s.}$$

The result now follows easily. □

Proposition 3.1 will be proven by showing:

Proposition 3.3. *If $p(\cdot, \cdot)$ is symmetric and not strongly transient, then $c_0 = 0$.*

To show Proposition 3.1, this is sufficient to note that by changing time scale we have that Proposition 3.3 implies that the limiting density for a Poisson system of random walks of any rate must be zero. But it is elementary that if we consider the number of distinct random walks contributing to $\chi(t)$, that are not part of the collection of random walks generated by the same Harris system but starting at points of an independent Poisson process of rate M for large M, we see that it is small compared to t with high probability as M becomes large.

The argument for Proposition 3.3 is low level and intuitive. We argue by contradiction and suppose that $c_0 > 0$. From this we can deduce, loosely speaking, that after a certain time either a random walk has lost its original label, or it will keep it forever. We then introduce coupling on these random walks so that we may regard these random walks as essentially independent random walks starting at 0 (at different times). We then introduce convenient comparison systems so that we can analyze subsequent coalescences. We will use automatically, without reference, the following "obvious" result:

Lemma 3.2. *Consider two collections of coalescing random walks $\{Y^i\}$ and $\{(Y')^i\}$ for i in some index set. If the coalescence rule is weaker for the $\{(Y')^i\}$ system, in that if two walks $(Y')^i$ and $(Y')^j$ are permitted to coalesce at time t, then so are Y^i and Y^j, then there is a coupling of the two systems so that the weaker contains the stronger.*

We now fix $\varepsilon > 0$ so that $\varepsilon \ll c_0$ (by hypothesis $c_0 > 0$). We choose R according to Lemma 3.1 and divide up time into intervals $I_j = [jR, (j+1)R)$. We consider the coalescing system where random walks X^t, $X^{t'}$, $t, t' \in \mathscr{P}$, can only "coalesce" (or destroy a label t or t') if t, t' are in the same I_j interval. Thus we have a system of random walks that is invariant to time shifts by integer multiples of R. We now introduce a system of random walks Y^t, $t \in V := \cup_j \{[jR, (j+1)R) \cap jR + \frac{1}{c_0}\mathbb{Z}\}$. The random walks Y^t, $t \in [jR, (j+1)R)$ are not permitted to coalesce up until time $(j+1)R$ (at least) and will evolve independently of the system $(X^t)_{t \in \mathscr{P}}$ until time $(j+1)R$. We will match up the points in $V \cap I_j$, with those in $\mathscr{P} \cap I_j \cap K$ in a maximal measurable way for $K = \{t \in \mathscr{P} : \text{label } t \text{ survives to time } (j+1)R\}$.

Lemma 3.3. *Unmatched points in $\cup_j \mathscr{P} \cap I_j \cap K$ and in V have density less than 2ε for R fixed sufficiently large.*

Remark: the system is not translation invariant with respect to all shifts but it possesses enough invariance for us to speak of densities.

We similarly have

Lemma 3.4. *Unmatched Y particles have density less than 2ε for R fixed sufficiently large.*

It is elementary that two random walks X, Z can be coupled so that for t sufficiently large $X(t) = Z(t)$. For given $\varepsilon > 0$ we choose M_0 and then M_1 so that

$$\mathbb{P}\left(\sup_{t \leq R} \|X(t)\|_\infty \geq M_0\right) < \frac{\varepsilon}{10} \tag{17}$$

and

$$\sup_{|z| \leq 2M_0} \mathbb{P}\Big(X_0, Z_z \text{ not coupled by time } M_1\Big) < \frac{\varepsilon}{10}, \tag{18}$$

where X_0 and Z_z denote that the random walks X and Z start at point 0 and point z respectively.

We then on interval I_j couple systems Y and X by letting married pairs Y^t, $X^{t'}$, $t \in V$, $t' \in \mathscr{P} \cap K$, evolve independently of other Y, X random walks so that they couple by time $M_1 + (j+1)R$ with probability at least $1 - \frac{3\varepsilon}{10}$.

Thus we have two types of random walk labels, l^t, for the X system which are equal to t at time $t + R + M_1$: those for which the associated random walk was paired with a Y random walk and such that the random walks have coupled by time $t + R + M_1$ said to be *coupled* and the others, said to be *decoupled*. Similarly for the points in V associated to Y random walks. We note that the foregoing implies that the density of uncoupled labels is bounded by $2\varepsilon + \frac{3\varepsilon}{10} \leq 3\varepsilon$. The point is that modulo this small density, we have an identification of the coalescing X random walks and the Y random walks.

We now try to show that enough Y particles will coalesce in a subsequent time interval to imply that there will be a significant decrease in surviving labels for the X system. To do this we must bear in mind that, essentially, it will be sufficient to show a decrease in the density of Y random walk labels definitely greater than ε. Secondly, as already noted, we will adopt a coalescence scheme that is a little complicated namely $Y^i, Y^{i'}$ in V can only "coalesce" at time $t \geq \max\{i, i'\}$ if

- $(t - i', t - i)$ are in some time set to be specified;
- for $i < i'$, $\frac{t-i'}{i'-i} \in \left(\frac{9}{10}, \frac{11}{10}\right)$.

We now begin to specify our coalescence rules for the random walk system $\{Y^i\}_{i \in V}$. The objective here will be to facilitate the necessary calculations. A first objective is to have coalescence of $Y^i, Y^{i'}$ at times $t > \max\{i, i'\}$ so that $p_t(0,0)$ is well behaved around $t - i$, $t - i'$. It follows from symmetry of the random walk that $t \mapsto p_t(0,0)$ is decreasing. The problem we address is that it is not immediate how to achieve bounds in the opposite direction. This is the purpose of the next result.

Lemma 3.5. *Consider positive $\{a_n\}_{n \geq 0}$ so that $\sum_{n=0}^{\infty} a_n = \infty$. For all $r \in \mathbb{Z}_+$, there exists a subsequence $\{a_{n_i}\}_{i \geq 0}$ so that*
(i) $\sum a_{n_i} = \infty$;
(ii) $a_{n_i} > \frac{1}{2} a_{n_i - r}, \forall i \geq 0$.

Proof. If $r > 1$ we may consider the r subsequences $\{a_{ri+j}\}_{i \geq 0}$ for $j \in \{0, 1, \cdots, r-1\}$. At least one of these must satisfy $\sum a_{ri+j} = \infty$ so, without loss of generality, we take $r = 1$.

Now we classify i as good or bad according to whether $a_i > a_{i-1}/2$ or not. This decomposes $\mathbb{Z}$ into intervals of bad sites, alternating with intervals of good sites. By geometric bounds, the sum of bad sites is bounded by the sum of the good a_i for which i is the right end point of a good interval. Thus we have

$$\sum_{i \text{ good}} a_i = \infty,$$

from which the result is immediate.   □

Corollary 3.1. *For our symmetric kernel $p_t(0,0)$ we can find $n_i \uparrow \infty$ so that*

(i) $\sum_i \int_{2^{n_i}}^{2^{n_i+1}} t p_t(0,0) dt = \infty$;

(ii) $p_{2^{n_i}-1}(0,0) \leq 2^{12} p_{2^{n_i}+3}(0,0)$.

Proof. In Lemma 3.5 take

$$a_n = \int_{2^{n+3}}^{2^{n+4}} t p_t(0,0) dt$$

and take $r = 5$. Then by the monotonicity of $t \to p_t(0,0)$, we have

$$2^{2n_i+7} p_{2^{n_i}+3}(0,0) \geq a_{n_i} \geq \frac{1}{2} a_{n_i-5} \geq \frac{1}{2} 2^{2n_i-4} p_{2^{n_i}-1}(0,0).$$

   □

We fix such a sequence $\{n_j\}_{j \geq 1}$ once and for all.

   We assume, as we may, that $n_j < n_{j+1} - 4$ for all $j \geq 1$ and also assume again, as we may, that

$$\int_{2^{n_j}}^{2^{n_j+1}} t p_t(0,0) dt < \frac{\varepsilon}{100}$$

for all $j \geq 1$. We are now ready to consider our coalescence rules. We choose $\varepsilon \ll \alpha \ll 1$ (we will fully specify α later on but we feel it more natural to defer the technical relations). We then choose k_0 so that $2^{n_{k_0}} > R + M_1$ with R as in Lemma 3.1 and Lemma 3.3 and M_1 as in (18), and

$$k_1 := \inf\left\{ k > k_0 : \sum_{j=k_0}^{k} \int_{2^{n_j}}^{2^{n_j+1}} t p_t(0,0) dt > \alpha \right\}. \tag{19}$$

We have coalescence between Y^i and $Y^{i'}$, for $i < i'$ only at $t \in [i' + 2^{n_j}, i' + 2^{n_j+1}]$, $j \in [k_0, k_1]$ if

(a) $\frac{t-i'}{i'-i} \in (9/10, 11/10)$;

(b)   the interval of $t \in [i' + 2^{n_j}, i' + 2^{n_j+1}]$ satisfying (a) is of length at least 2.

We say (i, i') and (i', i) are in j and write $(i, i') \in j$ if the above relations hold. To show that sufficient coalescence occurs, we essentially use Bonferroni inequalities.

   To aid our argument we introduce a family of independent (non coalescing) random walks $\{(Z^i(s))_{s \geq i}\}_{i \in V}$ such that for each $i \in V$, $Y^i(s) = Z^i(s)$ for $s \geq i$ such that $l_s^i = i$. In the following we will deal with random walks Y^0, Z^0, but lack of total translation invariance notwithstanding, it will be easy to see that all bounds obtained for these random walks remain valid for more general random walks Y^i, Z^i. For a given random walk Y^0, say, the probability that Y^0 is killed by Y^i (with i possible

in the sense of the above rules) is in principle a complicated event given the whole system of coalescing random walks. Certainly the event

$$\{Z^0 \text{ meets } Z^i \text{ in appropriate time interval after first having met } Z^k\}$$

is easier to deal with than the corresponding Y event. ¿From this point on we will shorten our phraseology by taking "Z^i hits Z^k" to mean that Z^i meets Z^k at a time t satisfying the conditions (a) and (b) above with respect to i, k.

For $(Z^0(s))_{s \geq 0}$ and $(Z^i(s))_{s \geq i}$ independent random walks each beginning at 0, we first estimate

$$\sum_i \mathbb{P}(Z^0 \text{ hits } Z^i).$$

This of course decomposes as

$$\sum_j \sum_{(0,i) \in j} \mathbb{P}(Z^0 \text{ hits } Z^i).$$

We fix j and consider $i > 0$ so that $(0, i) \in j$ (the case $i < 0$ is similar). That is the interval of times s with

$$(s-i)/i \in (9/10, 11/10), \ s \in [i + 2^{n_j}, i + 2^{n_j+1}]$$

is at least 2 in length: we note that for each $i \in \left(\frac{5}{4} 2^{n_j}, \frac{7}{4} 2^{n_j}\right)$ the relevant interval, $\frac{19}{10} i \leq s \leq \frac{21}{10} i$ is an interval of length greater than $\frac{5}{4} 2^{n_j} \frac{1}{5} = \frac{2^{n_j}}{4}$.

Lemma 3.6. *There exists $c_2 \in (0, \infty)$ so that for any interval I of length at least 1 contained in $(1, \infty)$,*

$$\frac{1}{c_2} \int_I p_t(0,0) \, dt \leq \mathbb{P}\big(X^0(t) = 0 \text{ for some } t \in I\big) \leq c_2 \int_I p_t(0,0) \, dt$$

Proof. Consider random variable $W = \int_a^{b+1} 1_{\{X(s)=0\}} \, ds$ for $I = [a, b]$. Then

$$\mathbb{E}(W) = \int_a^{b+1} \mathbb{P}(X(s) = 0) \, ds = \int_a^{b+1} p_s(0,0) \, ds \leq 2 \int_I p_s(0,0) \, ds,$$

by monotonicity of $p_s(0,0)$ and the fact that $b - a \geq 1$. But for $\tau := \inf\{s \in I : X(s) = 0\}$ we have $\mathbb{E}(W | \mathscr{F}_\tau) \geq e^{-1}$ on $\{\tau < \infty\}$ so

$$\mathbb{P}(\tau < \infty) = \mathbb{P}\big(X^0(t) = 0 \text{ for some } t \in I\big) \leq \mathbb{E}(W)e \leq 2e \int_I p_s(0,0) \, ds.$$

Equally for $W' = \int_a^b 1_{\{X(s)=0\}} \, ds$, we have

$$\mathbb{E}(W' | \mathscr{F}_\tau) \leq \gamma = \int_0^\infty p_s(0,0) \, ds \quad \text{on } \{\tau < \infty\}$$

and so

$$\mathbb{P}(\tau < \infty) \geq \frac{\mathbb{E}(W')}{\gamma} = \frac{1}{\gamma}\int_a^b p_s(0,0)\,ds.$$

$\square$

Proposition 3.4. *For some universal $c_3 \in (0,\infty)$,*

$$c_3^{-1} 2^{2n_j} p_{2^{n_j}}(0,0) \leq \sum_{(0,i)\in j} \mathbb{P}\big(Z^0 \text{ hits } Z^i\big) \leq c_3 2^{2n_j} p_{2^{n_j}}(0,0).$$

Proof. We consider first the upper bound. There are less than 2^{n_j} relevant i. For such an i,

$$\mathbb{P}\big(Z^0 \text{ hits } Z^i\big) \leq \mathbb{P}\big(X^0(t) \text{ hits } 0 \text{ for some } t \in \big[a+2^{n_j}, a+3\cdot 2^{n_j}\big]\big)$$

for some $a \geq 0$ and, by monotonicity of $t \to p_t(0,0)$ and using Lemma 3.6, this is bounded by

$$c_2 \int_{2^{n_j}}^{3\cdot 2^{n_j}} p_s(0,0)\,ds \leq c_3 2^{n_j} p_{2^{n_j}}(0,0)$$

for some $c_3 > 0$. On the other side the number of $i \in \big(\frac{5}{4}2^{n_j}, \frac{7}{4}2^{n_j}\big)$ is greater than $c_1 \frac{1}{3} 2^{n_j}$ if R was fixed sufficiently large and for each such i, $\big(\frac{9}{10}i, \frac{11}{10}i\big) \subset [2^{n_j}, 2^{n_j+1}]$. Moreover, we have

$$\mathbb{P}\big(Z^0 \text{ hits } Z^i\big) \geq \frac{1}{c_2}\int_{\frac{28}{10}i}^{\frac{32}{10}i} p_s(0,0)\,ds \geq \frac{1}{c_2}\frac{4}{10}i p_{2^{n_j+3}}(0,0) \geq \frac{1}{c_2} 2^{n_j-1} p_{2^{n_j+3}}(0,0)$$

$$\geq \frac{2^{-13}}{c_2} 2^{n_j} p_{2^{n_j-1}}(0,0) \geq c_3^{-1} 2^{n_j} p_{2^{n_j}}(0,0),$$

because of Lemma 3.6, Corollary 3.1 (by our choice of j), monotonicity of $t \to p_t(0,0)$ and possibly after increasing c_3. $\square$

Thus, using that $j \in [k_0, k_1]$ (recall 19), we have a universal c_4 such that

$$c_4 \alpha \geq \sum_j \sum_{(0,i)\in j} \mathbb{P}\big(Z^0 \text{ hits } Z^i\big) \geq \frac{\alpha}{c_4}.$$

There are two issues to address

(a)   to show that
$$\mathbb{P}\big(\exists j, \exists i \text{ so that } (0,i) \in j, \ Z^0 \text{ hits } Z^i\big)$$

   is of the order α;

(b)   to show that (a) holds with Z^0, Z^i replaced by our coalescing random walks Y^0, Y^i.

In fact both parts are resolved by the same calculation.

   We consider the probability that random walk Z^0 is involved in a "3-way" collision with Z^i and $Z^{i'}$ either due to Z^0 hitting Z^i in the appropriate time interval and

then hitting $Z^{i'}$, or Z^0 hitting Z^i and, subsequently Z^i hitting $Z^{i'}$. The first case is important to bound so that one can use simple Bonferroni bounds to get a lower bound on $\mathbb{P}(\exists i \text{ so that } Z^i \text{ hits } Z^0)$. The second is to take account of the fact that we are interested in the future coalescence of a given random walk Y^0. As already noted, we can couple the systems in the usual way so that for all t, $\{\cup_i Y_t^i\} \subseteq \{\cup_i Z_t^i\}$. The problem is that if for some i, Z^i hits Z^0 due to coalescence this need not imply that Y^i hits Y^0: if the Y^i particles coalesced with a $Y^{i'}$ before Z^i hits Z^0. Fortunately this event is contained in the union of events above over i, i'.

Proposition 3.5. *There exists universal constant K so that for all i and i' with* $(0, i') \in j'$

$$\mathbb{P}\big(Z^0 \text{ hits } Z^i \text{ and then } Z^{i'}\big) \le K 2^{n_{j'}} p_{2^{n_{j'}}}(0, 0) \mathbb{P}\big(Z^0 \text{ hits } Z^i\big).$$

Proof. There are several cases to consider: $i < 0 < i'$, $i < i' < 0$, $i' < i < 0$, $i' < 0 < i$, $0 < i < i'$ and $0 < i' < i$. All are essentially the same so we consider explicitly $0 < i < i'$. We leave the reader to verify that the other cases are analogous. We choose j, j' so that $(0, i) \in j$ and $(0, i') \in j'$ (so necessarilly $j' \ge j$). We condition on T_j, $Z^i(T_j)(= Z^0(T_j))$, for

$$T_j := \inf\left\{s \in \left(\frac{19i}{10}, \frac{21i}{10}\right) \cap \left[i + 2^{n_j}, i + 2^{n_j+1}\right] : Z^i(s) = Z^0(s)\right\} < \infty.$$

With $x = Z^0(T_j)$ we have

$$\mathbb{P}\left(\exists s' \ge T_j \in \left(\frac{19i'}{10}, \frac{21i'}{10}\right) \cap \left[i' + 2^{n_{j'}}, i' + 2^{n_{j'}+1}\right] : Z^{i'}(s') = Z^0(s') \ \middle| \ G^{0,i}\right)$$
$$= \mathbb{P}\big(Z^0(t) = x \text{ for some } t \in I_j\big),$$

where I_j is the image of the interval

$$\left[(i' + 2^{n_{j'}}) \vee T_j \vee \frac{19i'}{10}, i' + 2^{n_{j'}+1} \wedge \frac{21i'}{10}\right],$$

by the function $t \mapsto 2t - T_j - i'$, for $G^{0,i} = \sigma(Z^0(s), Z^i(s) : s \le T_j)$. By elementary algebra this is less than

$$\mathbb{P}\left(Z^0(t) = x \text{ for } t \in \left(\frac{9i'}{10}, \frac{16i'}{5}\right)\right),$$

but by arguing as in Lemma 3.6, this is bounded by

$$c_2 \int_{\frac{9i'}{10}}^{\frac{16i'}{5}} p_s(0, x)\, ds \le c_2 \int_{\frac{9}{10}i'}^{\frac{16}{5}i'} p_s(0, 0)\, ds \le c_2 \frac{23}{10} i' p_{2^{n_{j'}-1}}(0, 0)$$

$$\le c_2 \frac{23}{9} 2^{13} 2^{n_{j'}} p_{2^{n_{j'}+3}}(0, 0) \le c' 2^{n_{j'}} p_{2^{n_{j'}}}(0, 0)$$

for some universal constant c', where we use symmetry and monotonicity of $p_s(\cdot,\cdot)$ and Corollary 3.1, by the choice of our $n_{j'}$. So given that

$$\mathbb{P}(T_i < \infty) = \mathbb{P}(Z^0 \text{ hits } Z^i),$$

the desired bound is achieved. $\square$

Corollary 3.2. *For α sufficiently small*

$$\mathbb{P}(\exists i\colon Z^0 \text{ hits } Z^i) \geq \frac{\alpha}{2}.$$

Proof. By Bonferroni, the desired probability is superior to

$$\sum_i \mathbb{P}(Z^0 \text{ hits } Z^i) - \sum_{i,i'} \mathbb{P}(Z^0 \text{ hits } Z^i \text{ and then } Z^{i'}) \geq \alpha - Kc_3^2\alpha^2 \geq \frac{\alpha}{2}$$

if $\alpha \leq 1/(2Kc_3^2)$. $\square$

We similarly show

Proposition 3.6. *There exists universal constant K so that for all i' and i with $(0,i) \in j$,*
$$\mathbb{P}(Z^0 \text{ hits } Z^i \text{ after } Z^i \text{ hits } Z^{i'}) \leq K2^{n_j}p_{2^{n_j}}(0,0)\mathbb{P}(Z^i \text{ hits } Z^{i'}).$$

This gives as a corollary

Corollary 3.3. *For the coalescing system $\{Y^i\}_{i \in V}$ provided α is sufficiently small,*

$$\mathbb{P}(Y^i \text{ dies after time } R) \geq \frac{\alpha}{5}.$$

Proof. We have of course from the labeling scheme

$$\begin{aligned}
\mathbb{P}&(Y^i \text{ dies after time } R)\\
&\geq \frac{1}{2}\mathbb{P}(Y^i \text{ hits } Y^{i'} \text{ in appropriate time interval for some } i')\\
&\geq \frac{1}{2}\mathbb{P}(Z^i \text{ hits } Z^{i'} \text{ in appropriate time interval for some } i')\\
&\quad -\frac{1}{2}\mathbb{P}(Z^i \text{ hits } Z^{i'} \text{ in appropriate time interval for some } i' \text{ so that}\\
&\qquad\qquad Z^{i'} \text{ hits some } Z^{i''} \text{ previously})\\
&\geq \frac{\alpha}{4} - Kc_3^2\alpha^2 \geq \frac{\alpha}{5}
\end{aligned}$$

for $\alpha \leq 1/(20Kc_3^2)$. $\square$

We can now complete the proof of Proposition 3.3 and hence that of Proposition 3.1. If we have $c_0 > 0$, then we can find $0 < \varepsilon < \alpha/200$ and α so small that the

relevant results above hold, in particular Corollary 3.3. Thus the density of Y's is reduced by at least $\alpha/5$. But by our choice of ε and Lemma 3.3, the density of X's is reduced by at least $\alpha/5 - 6\varepsilon \geq \alpha/6 \geq 3\varepsilon$ which is a contradiction with Proposition 3.2, because it would entail the density falling strictly below c_0.

4 Proof of Theorem 1.4

In what follows we assume, as in Section 2, that $p = 1$, the extension to arbitrary $p \geq 1$ being straightforward.

We begin by specifying the random walk $(X(t))_{t \geq 0}$ on $\mathbb{Z}^4$ defined by

$$X(t) = S(t) + e_1 N(t)$$

with $(S(t))_{t \geq 0}$ denoting a simple random walk on $\mathbb{Z}^4$, $(N(t))_{t \geq 0}$ a rate 1 Poisson process and $e_1 = (1,0,0,0)$ the first unit vector in $\mathbb{Z}^4$.

Thus our random walk $(X(t))_{t \geq 0}$ is highly transient but its symmetrization is a mean zero random walk and by the local central limit theorem, we have

$$\int_0^\infty t p_t^{(s)}(0,0)\, dt = \infty,$$

where $p_t(\cdot, \cdot)$ is the semigroup associated to $(X(t))_{t \geq 0}$.

It remains to show that $\lambda_p(\kappa) < \gamma$ for all $\kappa \in [0, \infty)$. Our approach is modeled on the proof of the first part of Theorem 1.2. We wish again to pick *bad environment set* B_E associated to the ξ-process and *bad random walk set* B_W associated to the random walk X^κ so that

$$\mathbb{E}\left(\exp\left[\gamma \int_0^n \xi_s(X^\kappa(s))\, ds \right] \right) \tag{20}$$

$$\leq \left(\mathbb{P}(B_E) + \mathbb{P}(B_W) \right) e^{\gamma n} + \mathbb{E}\left(\mathbb{1}_{B_E^c \cap B_W^c} \exp\left[\gamma \int_0^n \xi_s(X^\kappa(s))\, ds \right] \right) \tag{21}$$

with, for some $0 < \delta < 1$,

$$\mathbb{P}(B_E) \leq e^{-\delta n}, \qquad \mathbb{P}(B_W) \leq e^{-\delta n}, \tag{22}$$

and, automatically from the definition of B_E and B_W,

$$\int_0^n \xi_s(X^\kappa(s))\, ds \leq n(1 - \delta) \qquad \text{on } B_E^c \cap B_W^c \tag{23}$$

(as in the proof of Theorem 1.2). Since, combining (20–23), we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}\left(\exp\left[\gamma \int_0^n \xi_s(X^\kappa(s))\, ds \right] \right) < \gamma,$$

it is enough to prove (22). All of this has been done in the proof of Theorem 1.2 in a different situation. The major difference is that we need to modify the collection of skeletons used.

Lemma 4.1. *Let $X^\kappa(\cdot)$ be a speed κ simple random walk in four dimensions. Fix $M \in \mathbb{N} \setminus \{1\}$. There exists $c > 0$ so that for M large and all n, outside of an e^{-cn} probability event, there exists $0 \le i_1 < i_2 < \cdots < i_{n/2M} \le n$ so that*

$$X^\kappa_{(1)}(i_j M + kM) - X^\kappa_{(1)}(i_j M) > -\frac{kM}{2}, \qquad j \in \{1, \cdots, n/(2M)\}, k \ge 0, \quad (24)$$

where $(X^\kappa_{(1)}(t))_{t \ge 0}$ denotes the first coordinate of $(X^\kappa(t))_{t \ge 0}$.

Proof. Define
$$\sigma_1 = \inf\{kM > 0 \colon X^\kappa_{(1)}(kM) \le -kM/2\}$$

and recursively
$$\sigma_{i+1} = \inf\{kM > \sigma_i \colon X^\kappa_{(1)}(kM) - X^\kappa_{(1)}(\sigma_i) \le -(kM - \sigma_i)/2\}.$$

Since the event
$$\{rM \le \sigma_1 < \infty\} \subset \cup_{k=r}^\infty \{X^\kappa_{(1)}(kM) \le -kM/2\},$$

we have easily that, for all r,
$$\mathbb{P}(rM \le \sigma_1 < \infty) \le e^{-rMc}$$

for $c > 0$ not depending on n or M. If we now define
$$\tau_1 = \inf\{kM > 0 \colon X^\kappa_{(1)}(jM) - X^\kappa_{(1)}(kM) > -(j-k)M/2 \ \forall j > k\}$$

and recursively
$$\tau_{i+1} = \inf\{kM > \tau_i \colon X^\kappa_{(1)}(jM) - X^\kappa_{(1)}(kM) > -(j-k)M/2 \ \forall j > k\},$$

it is easily seen that
$$\mathbb{P}(\tau_1 \ge rM) \le \mathbb{P}(\exists 1 \le k \le r \colon rM \le \sigma_k < \infty)$$
$$\le \sum_{k=1}^{r-1} \sum_{0 < x_1 < \cdots < x_k < r} \mathbb{P}(\sigma_i = x_i M \ \forall i \le k, rM \le \sigma_{k+1} < \infty)$$
$$\le \sum_{k=1}^{r-1} \sum_{0 < x_1 < \cdots < x_k < r} e^{-rMc} \le e^{-rMc} 2^r$$

which is less than $e^{-rMc/2}$ if M is fixed sufficiently large. We have

- $(\tau_{i+1} - \tau_i)_{i \ge 1}$ are i.i.d. (this follows from Kuczek's argument (see [8])).

- Provided M has been fixed sufficiently large for each integer $r \geq 1$, $\mathbb{P}(\tau_{i+1} - \tau_i \geq rM) \leq e^{-rMc/4}$. This follows from the fact that random variable $\tau_{i+1} - \tau_i$ is simply the random variable τ_1 conditioned on an event of probability at least $1/2$ (provided M was fixed large).

Thus by elementary properties of geometric random variables we have

$$\mathbb{P}(\tau_{n/2M} > n) \leq \mathbb{P}\left(Y \geq \frac{n}{2M}\right) \leq e^{-cn}$$

for $Y \sim B\left(\frac{n}{M}, e^{-cM/4}\right)$, $c > 0$ and M large. This completes the proof of the lemma.
$\square$

Given that the path of the random walk satisfies the condition of this lemma, we call the (not uniquely defined) points $i_1, i_2, \cdots$ *regular* points.

Given this result, we consider the M-skeleton induced by the values $X^{\kappa}(jM)$, $0 \leq j \leq n/M\}$, discretized via spatial cubes of length $M/8$ (rather than $2M$ as in the proof of Theorem 1.2). It is to be noted that if $(X^{\kappa}(t))_{0 \leq t \leq n}$ satisfies the claim for Lemma 4.1 and $y_0 := 0, y_1, y_2, \cdots, y_{n/M}$ with $y_k \in \mathbb{Z}^4$, $0 \leq k \leq n/M$, is its M-skeleton, namely,

$$X^{\kappa}(kM) \in C_{y_k} := \prod_{j=1}^{4} \left[y_k^{(j)}\frac{M}{8}, (y_k^{(j)}+1)\frac{M}{8}\right), \qquad 0 \leq k \leq n/M, \qquad (25)$$

where $y_k^{(j)}$ denotes the j-th coordinate of y_k (we suppose without loss of generality that M is a multiple of 8). Then, by (24) and (25), we must have

$$y_{i_j}^{(1)} - 4k \leq y_{i_j+k}^{(1)} + 1.$$

In particular, we must have

$$y_{i_{j'}}^{(1)} - 4(i_j - i_{j'}) \leq y_{i_j}^{(1)} + 1 \qquad \forall i_{j'} < i_j. \qquad (26)$$

In the following we modify the definition of appropriate skeletons by adding in the requirement that the skeleton must possess at least $n/2M$ indices $i_1, i_2, \ldots, i_{n/2M}$ with the corresponding y_{i_j} satisfying (26). We note that the resizing of the cubes makes the notion of acceptability a little more stringent but does not change the essentials.

Remark first that Lemma 2.5 is still valid in our new setting. Lemma 4.1 immediately gives that with this new definition, Lemma 2.4 remains true. Of course since this definition is more restrictive we have

$$|\Xi_A| \leq K^{n/M}$$

for K as in Lemma 2.1.

In fact in our program all that remains to do, that is in any substantive way different from the proof of Theorem 1.2, is to give a bound on the probability of B_E for

appropriate B_E. This is the content of the lemma below (analogous to Lemma 2.2). Given this lemma, we can then proceed exactly as with the proof of Theorem 1.2.

Lemma 4.2. *For any skeleton* $(y_k)_{0 \leq k \leq n/M}$ *in* Ξ_A*, the probability that* ξ *is not good for* $(y_k)_{0 \leq k \leq n/M}$*, i.e.,*

$$\nexists \frac{n}{4M} \text{ indices } 1 \leq j \leq \frac{n}{M} : \xi_s(z) = 0 \ \forall s \in [jM, jM+1] \text{ for some } z \in C_{y_j},$$

is less than $(4K)^{-n/M}$*.*

Proof. We note that proving the analogous result for Theorem 1.2, we did not need our skeleton to be in Ξ_A, the proof worked over any skeleton. For us however it is vital that our skeleton satisfies (26).

We consider a skeleton in Ξ_A. Let the first $n/2M$ regular points of our skeleton be $i_1, i_2, \cdots i_{n/2M}$. For each $1 \leq i_j \leq n/2M$, we choose R points

$$x_1^{i_j}, \ldots, x_R^{i_j} \in C_{y_{i_j}}$$

so spread out that for random walks $(X(t))_{t \geq 0}$ as in (4) beginning at the points $x_k^{i_j}$, $k = 1, \cdots, R$, the chance that two of them meet is less than $0 < \varepsilon \ll 1$.

Now, consider $i_{j'} < i_j$ and the probability that a random walk starting at $(x_k^{i_j}, i_j M)$ meets a random walk starting at $(x_{k'}^{i_{j'}}, i_{j'}M)$ satisfies the following lemma.

Lemma 4.3. *For* $i_{j'} < i_j$*, there exits* $K > 0$ *such that*

$$\mathbb{P}\left(X^{x_k^{i_j}, i_j M} \ \text{meets} \ X^{x_{k'}^{i_{j'}}, i_{j'}M}\right) \leq \frac{K}{M^2 (i_j - i_{j'})^2}.$$

Proof. The important point is that since our skeleton is in Ξ_A,

$$\left(x_{k'}^{i_{j'}}\right)^{(1)} \leq \left(x_k^{i_j}\right)^{(1)} + (i_j - i_{j'})\frac{M}{2} + \frac{M}{4}, \tag{27}$$

and so we have

$$\mathbb{P}\left(X^{x_k^{i_j}, i_j M} \ \text{meets} \ X^{x_{k'}^{i_{j'}}, i_{j'}M}\right)$$

$$\leq \mathbb{P}\left(\left(X^{x_k^{i_j}, i_j M}\right)^{(1)} ((i_j - i_{j'})M) \geq \left(x_k^{i_j}\right)^{(1)} + (i_j - i_{j'})\frac{3M}{4},\right.$$

$$\left. X^{x_k^{i_j}, i_j M} \ \text{meets} \ X^{x_{k'}^{i_{j'}}, i_{j'}M}\right)$$

$$+ \mathbb{P}\left(\left(X^{x_k^{i_j}, i_j M}\right)^{(1)} ((i_j - i_{j'})M) \leq \left(x_k^{i_j}\right)^{(1)} + (i_j - i_{j'})\frac{3M}{4}\right).$$

By standard large deviations bounds,

$$\mathbb{P}\left(\left(X^{x_k^{i_j},i_jM}\right)^{(1)}\left((i_j-i_{j'})M\right)\le \left(x_k^{i_j}\right)^{(1)}+(i_j-i_{j'})\frac{3M}{4}\right)\le e^{-CM(i_j-i_{j'})},$$

for some universal $C \in (0,\infty)$. For the other term, we have the following lemma.

Lemma 4.4. *For* 2 *independent processes* $X = (X(t))_{t\ge0}$ *and* $Y = (Y(t))_{t\ge0}$ *with* $X(0) = x \in \mathbb{Z}^4$ *and* $Y(0) = y \in \mathbb{Z}^4$, *the probability that* X *ever meets* Y *is bounded by* $K/\|x-y\|_\infty^2$.

Proof. $X - Y$ is not exactly a simple random walk, but it is a symmetric random walk and so local CLT gives appropriate random walks bounds (see, e.g., [9]). □

From this and inequality (27), we have

$$\mathbb{P}\left(X^{x_k^{i_j},i_jM}\text{ meets }X^{x_{k'}^{i_{j'}},i_{j'}M}\left|\left(X^{x_k^{i_j},i_jM}\right)^{(1)}\left((i_j-i_{j'})M\right)\ge\left(x_k^{i_j}\right)^{(1)}+(i_j-i_{j'})\frac{3M}{4}\right.\right)$$
$$\le\frac{K}{M^2(i_j-i_{j'})^2}\,.$$

□

Thus, for any R large but fixed, we can choose M so that for all skeleton in Ξ_A and each $i_{j'}$, we have

$$\sum_{i_j<i_{j'}}\sum_{k,k'}\mathbb{P}\left(X^{x_k^{i_j},i_jM}\text{ meets }X^{x_{k'}^{i_{j'}},i_{j'}M}\right)\le\frac{R^2K}{M^2}\sum_{r=1}^{+\infty}\frac{1}{r^2}$$
$$\le\frac{R^2K'}{M^2}<\varepsilon^2$$

with M chosen sufficiently large, which is analogous to (13). ¿From this point on, the rest follows as for the proof of Lemma 2.2. □

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