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Abstract

We consider a stochastic fluid EOQ-type model with demand rates that operate in a two-state random environment. This environment alternates between exponentially distributed periods of high demand and generally distributed periods of low demand. The inventory level starts at some level q, and decreases linearly at rate a during the periods of high demand, and at rate $b < a$ at periods of low demand. The inventory level is refilled to level q when level 0 is hit or when an expiration date is reached, whichever comes first.

We determine the steady-state distribution of the inventory level, as well as other quantities of interest like the distribution of the time between successive refills. Finally, for a given cost/revenue structure, we determine the long-run average profit, and we consider the problem of choosing q such that the profit is optimized.

1 Introduction

We consider a stochastic fluid EOQ-type model with demand rates that operate in a two-state random environment. This environment alternates between a *good* state (a period of high demand rate) and a *bad* state (a period of low demand rate) according to a continuous-time semi-Markov chain. We assume that the net demand rate equals a during the good state and b during the bad state. We assume that $a > b$, since the sales during good periods are higher than those during bad ones (but the latter assumption is not necessary for the analysis). The high demand and the low demand periods follow each other according to an alternating renewal process as follows: the good periods are independent and exponentially distributed random variables with parameter λ and the bad periods are *i.i.d.* random variables with distribution G, mean $1/\mu$.

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In the present paper we discuss a fluid inventory model with *perishable* items which have a fixed expiration date. A plant that produces icecream may serve as a motivating example. When the weather is good, then the demand for icecream is high; otherwise it is considerably lower. As a first approximation, one might represent the weather condition as a two-state random environment. More generally, the demand rates for many goods go up and down between different levels due to fashion or other recurring external effects. Models including a multistate Markovian or semi-Markovian environment can be suitable for such situations, with the two-state case presented in this paper serving as a first approximation.

We denote by $\mathbf{X} = \{X(t) : t \geq 0\}$ the content level process of the inventory where $\mathbf{X}$ is a regenerative process that operates under the alternating good and bad states. A regeneration cycle is the time interval between two successive epochs in which the process switches to a good state while the inventory is refilled to level q. At the beginning of the cycle $\mathbf{X}$ decreases linearly at rate a and during the cycle it decreases linearly either at rate a or at rate b, depending on the state of the environment. By shifting the origin to the beginning of the cycle we have $X(0) = q$, where level q is the decision variable of the problem.

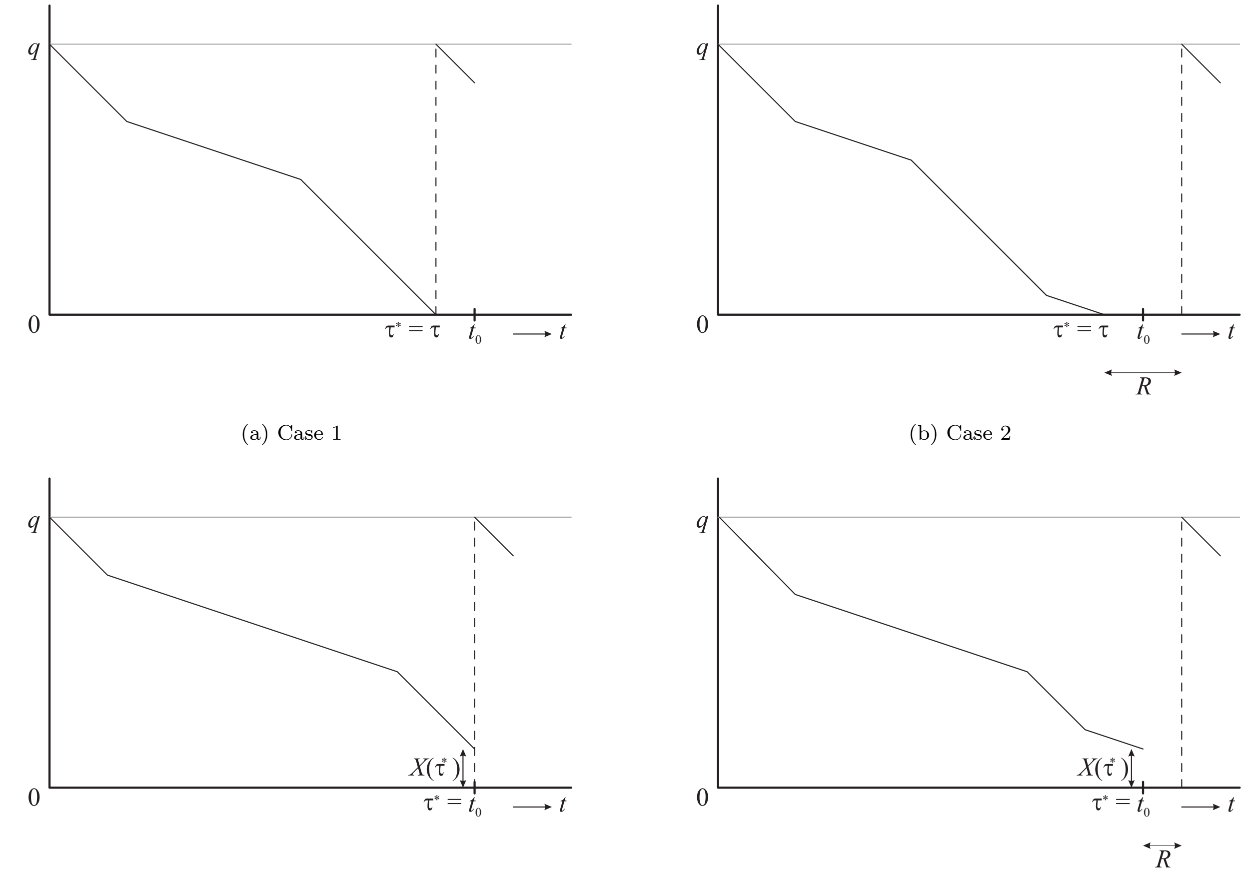
The dynamics of the model is such that at the origin the content level is refilled with fresh perishable items up to level q (the items of the same batch have common shelf life). We define the stopping times $\tau = \inf\{t > 0 : X(t) = 0\}$ and $\tau^* = \min\{\tau, t_0\}$, where t_0 indicates the expiration date of the items. As time progresses the stored items age together and perish together (if they perish).

There are four possible cases for the termination of a cycle; two cases may occur under the event $\{X(\tau^*) = 0\}$ and two cases may occur under the event $\{X(\tau^*) > 0\}$.

Case 1: $\{X(\tau^*) = 0\}$ (which means that $\tau^* < t_0$) and the state of the environment at τ^* is good (high demand period). In this case the cycle length is τ^*, and an order of size q fresh items is placed (in this study we assume that the lead time is negligible).

Case 2: $\{X(\tau^*) = 0\}$ and the state of the environment at τ^* is bad (low demand period).

The controller will wait until the end of the low demand period and only then an order for



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

Figure 1: The four possible cases for the termination of a cycle

q fresh items is placed. In the latter case the cycle length is τ^*+R where R is the residual time to the end of the last low demand period (note that τ^* and R are not necessarily independent; however, they are independent in case that the low demand periods are exponential).

Case 3: $\{X(\tau^*) > 0\}$ (which means that $\tau^* = t_0$) and the state of the environment at τ^* is good (high demand period).

The amount $X(\tau^*)$ is discarded (due to perishability) at a loss, and an order of size q fresh items is placed. Again, in this case the cycle length is τ^*.

Case 4: $\{X(\tau^*) > 0\}$ and the state of the environment at τ^* is bad (low demand period).

The controller will wait until the end of the bad period and only then an order for q fresh items is placed. In the latter case, the cycle length is $\tau^* + R$. Note that the above control policy guarantees that the cycle always starts with high demand rate, namely, whenever the environment is good.

The controller's objective is to maximize the long-run average profit by considering the following optimization problem. He wishes to select the optimal $q := q^*$ so as to properly balance revenues and costs. Revenues are earned by selling units; π is the net profit of selling one unit (sale minus purchase price). The costs concern setup cost K, which is incurred each time an order is placed (so it the setup cost per cycle), the proportional holding cost c_h per unit time and per unit of stored items, the cost c_d for one discarded unit (outdating) and the cost c_s for one unit of unsatisfied demand. The long-run average holding cost is $c_h EX(\infty)$, where $X(\infty)$ is a random variable having the steady-state distribution of $\mathbf{X}$. It should be noted that $X(t) \to X(\infty)$ in distribution. Thus, by the limit theorem for regenerative processes (see [1] p. 170) and the fact that $0 \leq X(t) \leq q$ it follows that $EX(t) \to EX(\infty)$. Accordingly, with C denoting the length of a regenerative cycle, the profit objective function is:

$$P(q) = \frac{\pi q - K}{EC} - c_d \frac{EX(\tau^*)\mathbf{1}_{\{\tau^*=t_0\}}}{EC} - c_s \frac{ER}{EC} - c_h EX(\infty), \tag{1}$$

where the functionals in (1) are all functions of q, and where R has an atom at zero.

Literature review

The literature about Perishable Inventory Systems (PIS) is quite rich. Over the last three decades, five comprehensive reviews ([6, 8, 12, 13, 17]) have been published. They reveal a strong emphasis on the design of algorithms for optimization and/or control (when to place an order and how much to order). Only a small minority of the papers focuses on the stochastic analysis of PIS that operate under certain heuristic control policies; see [8]. Our study belongs to the latter category. First, we introduce the performance analysis of a certain stochastic fluid EOQ model (the case of compound Poisson demand is introduced in [2], but without the randomness of the environment). Then, a specific objective function is introduced and the functionals obtained lay the groundwork for some numerical and sensitivity analysis.

The survey [8] classifies continuous review models into three categories: without fixed ordering cost or lead times, without fixed ordering cost having positive lead time, and with fixed ordering cost (typically with zero lead time). The first category was originated by Graves [7] who assumed that items are continuously produced and perish after a deterministic time, and that demand follows a compound Poisson process with either a single-unit or an exponential demand at each arrival. The second category was originated by Pal [14] who investigated the performance of an $(S-1, S)$ control policy. The third category, originated by Weiss [20], is of relevance to our model; [9], [10], [11], [5] and [16] made significant contributions to models in this category. Lian, Liu and Neuts [10] consider discrete demand for items and time to perishability that is either fixed-and-known or that follows a Phase-Type distribution.

There are two types of perishable inventory models. The first family of models assumes that the quality of the items is slowly decreasing over time. For an early review of work of this family of models see Nahmias [12] and [13]. A generic work by Rajan and Steinberg [18] considers perishable items with order set-up cost. They assume that the quantity of good items deteriorates at an exponential rate, and solve the problem of finding the joint optimal ordering and pricing strategy. The second family is of models with obsolescences ([19]). Here, the items might perish at each period with some probability which is typically increasing over time ([15], [3]).

A large variety of inventory models is presented in detail in the monograph [22]. The stochastic models are based on point processes for the demand arrivals in random environments (in the book called “world-driven”). The fluid systems in [22] are deterministic EOQ models with the classical extensions such as planned backorders, limited capacity, quantity discounts, and imperfect quality. In the deterministic setting, time-varying demands are considered also, but without multiple order quantities.

To the best of the authors’ knowledge, the stochastic EOQ model with the added features of perishability and a non-Markovian environment expounded in this article is new. The work that is most closely related to this study is [4]. In fact, the present study is a generalization of [4] with two important added features: (i) The items stored are subject to perishability with a pre-determined expiration date, viz., the constant t_0 (not a decision variable). (ii) The lengths of the bad periods of the random environment are not necessarily exponential.

In the next section we model the length of the intermittent high and low demand periods as an alternating renewal process. Such a model can be described as a generalized integrated telegrapher process (see Zacks [21]), with a particle moving with velocities a and b intermittently. The distribution of $X(t)$ can be derived as in Zacks [21] (see the appendix).

Organization of the paper

Section 2 contains a detailed model description. Section 3 presents a compound Poisson representation of the problem. Section 4 presents the distributions of related stopping times. In Section 5 we consider cycle lengths and related quantities, which are necessary for the optimization problem. Combining the various results from that section allows us to determine the expected profit per time unit. In Section 6 we present explicit formulae of cost functionals for the case of exponential length of low demand period. Numerical examples are given, in which we explore the effect of several parameters on the cost function.

2 Formulation of the Model

We consider an inventory system of perishable items, whose shelf life is t_0 time units. At the beginning of a cycle, a quantity q is placed in stock. All items left in stock after t_0 are discarded (at cost of c_d per unit). The demand for items follows alternating periods of high demand (HD) and low demand (LD). The lengths of successive HD periods are *i.i.d.* (independent, identically distributed) random variables $H_1, H_2, \ldots$ with common distribution F. The lengths of successive LD periods are *i.i.d.* random variables $L_1, L_2, \ldots$ with common distribution G. Furthermore, these two sequences are also independent.

As in a fluid model, demand is assumed to be continuous. The quantity demanded in an LD period during one time unit is β_L, and that during an HD period is β_H, where $0 < \beta_L < \beta_H < \infty$. We assume that $\beta_L t_0 < q < \beta_H t_0$.

A cycle starts during an HD period. If a cycle ends during an LD period, we extend it until an HD period starts. More precisely, let $Q(t)$ be the total amount demanded up to time t in a given cycle, where $Q(0) = 0$. Since a cycle starts during an HD period, $\beta_L t_0 < Q(t_0) \leq q$. All sample paths of $Q(t)$ satisfy $\beta_L t < Q(t) \leq \beta_H t$ for all t in a cycle. Define the stopping times (which were already informally defined in Section 1)

$$\tau = \inf\{t > 0 : Q(t) = q\}, \tag{2}$$

and

$$\tau^* = \min\{t_0, \tau\}. \tag{3}$$

Notice that $\dfrac{q}{\beta_H} \leq \tau^* \leq t_0$. In Figure 1 we present four possible sample paths of the demand process. These four sample paths correspond respectively to the case that the cycle ends at t_0 in an HD period, ends at t_0 in an LD period, ends at τ when q is reached in an HD period, and ends at τ when q is reached in an LD period.

Notice that when $\tau^* = t_0$, then generally $Q(t_0) < q$, and $q - Q(t_0)$ items are discarded.

The cycle length is

$$\begin{aligned} C &= \tau^*, \quad \text{if } \ \tau^* \in \ \text{an HD period} \\ &= \tau^* + R, \quad \text{else.} \end{aligned} \tag{4}$$

Here R is the length of the residual LD period following τ^*. If $C > \tau^*$ then there is a penalty $c_s R$ for shortage of items.

If the distribution of the HD-periods is exponential, i.e., $F(u) = 1 - e^{-\lambda u}$, then the first HD-period of a cycle is $\exp(\lambda)$ distributed not only when it is following an LD-period but also when it is the continuation of the HD-period which ended at τ^*. Hence the inventory cycles are renewal cycles in this case. In the remainder of the paper we indeed assume that F is $\exp(\lambda)$, whereas the distribution G of the LD-periods is general.

3 A compound Poisson representation

Let $\{N(t), t \geq 0\}$ be an ordinary Poisson process with intensity λ. It represents the renewal process of HD-periods. Let $X_i = \beta_L L_i$ represent the quantities demanded during LD-periods. Notice that $X_1, X_2, \ldots$ are *i.i.d.* with c.d.f. $G_X(x) = G(x/\beta_L)$ and p.d.f. $g_X(x) = \dfrac{1}{\beta_L} g(x/\beta_L)$, where g is the density of G. Consider the compound Poisson process

$$Y(t) = \sum_{n=0}^{N(t)} X_n, \quad t \geq 0, \tag{5}$$

where $X_0 = 0$. $N(t)$ is the number of completed HD-periods during the time interval $(0, t)$. If $N(t) = 0$, the first HD-period is still on at time t. For $N(t) \geq 1$, $Y(t)$ is the total demand during the LD-periods following the HD-periods in $(0, t)$.

During t time units of HD-periods the quantity demanded is $\beta_H t$. The quantity left for demands during LD-periods is $q - \beta_H t$. Also, with t time units on HD-periods, all the remaining $t_0 - t$ time periods are for LD-periods, so the maximum possible LD-demand then is $\beta_L(t_0 - t)$. Accordingly, let

$$B(t) = \min(q - \beta_H t, \beta_L(t_0 - t)), \tag{6}$$

and define the stopping time

$$\tilde{\tau} = \inf(t > 0 : Y(t) \geq B(t)). \tag{7}$$

Below we shall see that $\tilde{\tau}$ is linearly related to τ^*; obtaining the distribution of $\tilde{\tau}$ will immediately give us the distribution of τ^*.

In Figure 2 we show a possible realization of $\tilde{\tau}$. The time at which the two lines $y = q - \beta_H t$ and $y = \beta_L(t_0 - t)$ intersect is denoted by $t^* = \dfrac{q - \beta_L t_0}{\beta_H - \beta_L}$. Notice that $\tilde{\tau} \leq \dfrac{q}{\beta_H}$. Also, if $\tilde{\tau} < t^*$ then $Y(\tilde{\tau}) \geq \beta_L(t_0 - \tilde{\tau})$. The total time required to attain this value is $\tilde{\tau} + Y(\tilde{\tau})/\beta_L = t_0$. Thus, all values of $\tilde{\tau} < t^*$ correspond to $\tau^* = t_0$ (Cases 3 and 4), whereas all values of $\tilde{\tau}$ in $(t^*, q/\beta_H]$ correspond to $\dfrac{q}{\beta_H} \leq \tau^* < t_0$. Furthermore, the total number of units demanded at $\tilde{\tau}$, when $0 < \tilde{\tau} \leq t^*$, is

$$Q(\tilde{\tau}) = \beta_L t_0 + (\beta_H - \beta_L)\tilde{\tau}, \quad \tilde{\tau} \leq t^*. \tag{8}$$

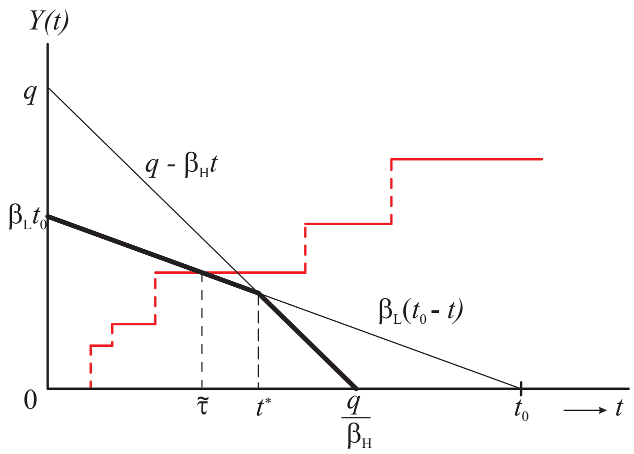


Figure 2: $Y(t)$ and $B(t)$ (bold) in Case 3

Notice that $Q(\tilde{\tau}) \leq q$. Thus, if $\tilde{\tau} \leq t^*$ the number of discarded units is

$$D(\tilde{\tau}) = (\beta_H - \beta_L)(t^* - \tilde{\tau}), \quad \tilde{\tau} \leq t^*. \tag{9}$$

If $\tilde{\tau} > t^*$ then $D(\tilde{\tau}) = 0$. The relationship between $\tilde{\tau}$ and τ^* is given by

$$\tau^* = \frac{q}{\beta_L} - \left(\frac{\beta_H}{\beta_L} - 1\right)\tilde{\tau}. \tag{10}$$

4 The distribution of $\tilde{\tau}$ and τ^*

In this section we successively determine the distribution of $\tilde{\tau}$, the distribution of τ^* (using (10)), and the joint density of $\tilde{\tau}$ and R. Let $p(n;\mu)$ and $P(n;\mu)$ denote the p.d.f. and c.d.f. of the Poisson distribution with mean μ. The c.d.f. of $Y(t)$ is

$$H(y;t) = \sum_{n=0}^{\infty} p(n;\lambda t) G^{(n)}\left(\frac{y}{\beta_L}\right), \tag{11}$$

where $G^{(n)}(\cdot)$ is the n-fold convolution of $G(\cdot)$, and $G^{(0)}(\cdot) \equiv 1$. $H(y;t)$ has an atom $e^{-\lambda t}$ at $y = 0$. On $(0,\infty)$ it is absolutely continuous, with density

$$h(y;t) = \frac{1}{\beta_L} \sum_{n=1}^{\infty} p(n;\lambda t) g^{(n)}\left(\frac{y}{\beta_L}\right). \tag{12}$$

It follows from the definition of $\tilde{\tau}$ (see (7)) that $P(\tilde{\tau} > t) = H(B(t);t)$. Since $B(t)$ is a decreasing function,

$$\begin{aligned} P(\tilde{\tau} > t) &= H(\beta_L(t_0 - t);t) \quad \text{if} \ \ 0 < t < t^*, \\ &= H(q - \beta_H t;t) \quad \text{if} \ \ t^* \leq t \leq \frac{q}{\beta_H}. \end{aligned} \tag{13}$$

Recall that $\{\tau^* = t_0\} = \{0 < \tilde{\tau} \leq t^*\}$. Thus

$$P\{\tau^* = t_0\} = 1 - H(B(t^*);t^*) = 1 - \sum_{n=0}^{\infty} p(n;\lambda t^*) G^{(n)}(t_0 - t^*). \tag{14}$$

The density of $\tilde{\tau}$ over $\left(0, \dfrac{q}{\beta_H}\right)$ is $\psi_{\tilde{\tau}}(t) = -\dfrac{d}{dt} P(\tilde{\tau} > t)$. The density $\psi_{\tilde{\tau}}(t)$ can be written as $\psi_{\tilde{\tau}}^{(1)}(t) + \psi_{\tilde{\tau}}^{(2)}(t)$, where

$$\psi_{\tilde{\tau}}^{(1)}(t) = \lambda \mathrm{e}^{-\lambda t} + \lambda \sum_{n=1}^{\infty} (p(n;\lambda t) - p(n-1;\lambda t)) G^{(n)}(\frac{B(t)}{\beta_L}), \tag{15}$$

$$\psi_{\tilde{\tau}}^{(2)}(t) = \beta_L h(B(t);t) \quad if \quad 0 \leq t \leq t^*, \tag{16}$$

$$\psi_{\tilde{\tau}}^{(2)}(t) = \beta_H h(B(t);t) \quad if \quad t^* \leq t \leq \frac{q}{\beta_H}. \tag{17}$$

$\psi_{\tilde{\tau}}^{(1)}(t)$ corresponds to the case that $\tilde{\tau} \in LD$, and $\psi_{\tilde{\tau}}^{(2)}(t)$ to the case that $\tilde{\tau} \in HD$. Thus,

$$P(\tilde{\tau} \in LD) = \frac{\int_0^{q/\beta_H} \psi_{\tilde{\tau}}^{(1)}(t)dt}{\int_0^{q/\beta_H} \psi_{\tilde{\tau}}(t)dt}. \tag{18}$$

From (10) the c.d.f. of τ^* now follows. Recall that all values of $\tilde{\tau} < t^*$ correspond to $\tau^* = t_0$, whereas all values of $\tilde{\tau}$ in $(t^*, q/\beta_H]$ correspond to $\dfrac{q}{\beta_H} \leq \tau^* < t_0$. For $\dfrac{q}{\beta_H} < t < t_0$, with $\Psi_{\tau^*}(\cdot)$ denoting the distribution of τ^*:

$$\begin{aligned} \Psi_{\tau^*}(t) &= P\left(\frac{q}{\beta_L} - \left(\frac{\beta_H}{\beta_L} - 1\right)\tilde{\tau} \leq t\right) \\ &= P\left\{\tilde{\tau} \geq \frac{q - \beta_L t}{\beta_H - \beta_L}\right) = H\left(\frac{\beta_L(\beta_H t - q)}{\beta_H - \beta_L}; \frac{q - \beta_L t}{\beta_H - \beta_L}\right). \end{aligned} \tag{19}$$

The joint density $f_{\tilde{\tau},R}(t,r)$ of $\tilde{\tau}$ and R, the residual LD-period length after stopping. Distinguishing the two cases where shortages occur, we have, for $t^* < t < \dfrac{q}{\beta_H}$, $r > 0$:

$$f_{\tilde{\tau},R}(t,r) = \frac{1}{\beta_L} \lambda e^{-\lambda t} g\left(\frac{q - \beta_H t + \beta_L r}{\beta_L}\right) + \frac{\lambda}{\beta_L} \int_0^{q - \beta_H t} h(y;t) g\left(\frac{q - \beta_H t - y + \beta_L r}{\beta_L}\right) dy, \tag{20}$$

and for $0 < t < t^*$,

$$f_{\tilde{\tau},R}(t,r) = \frac{1}{\beta_L}\lambda e^{-\lambda t} g\left(\frac{\beta_L(t_0-t)+\beta_L r}{\beta_L}\right) + \frac{\lambda}{\beta_L}\int_0^{\beta_L(t_0-t)} h(y;t) g\left(\frac{\beta_L(t_0-t)-y+\beta_L r}{\beta_L}\right) dy. \tag{21}$$

From (21) we get

$$\int_0^\infty f_{\tilde{\tau},R}(t,r)dr = \psi_{\tilde{\tau}}^{(1)}(t). \tag{22}$$

Indeed, for $y \geq 0$,

$$\int_0^\infty g(\frac{B(t)-y}{\beta_L}+r)dr = 1 - G(\frac{B(t)-y}{\beta_L}). \tag{23}$$

Moreover,

$$\begin{aligned}
& \frac{\lambda}{\beta_L}\int_0^{B(t)} h(y;t)(1-G(\frac{B(t)-y}{\beta_L}))dy \\
= & \frac{\lambda}{\beta_L}\int_0^{B(t)} \sum_{n=1}^\infty p(n;\lambda t) g^{(n)}(\frac{y}{\beta_L})(1-G(\frac{B(t)-y}{\beta_L}))dy \\
= & \lambda \sum_{n=1}^\infty p(n;\lambda t)\int_0^{B(t)/\beta_L} g^{(n)}(x)(1-G(\frac{B(t)}{\beta_L}-x))dx \\
= & \lambda \sum_{n=1}^\infty p(n;\lambda t)(G^{(n)}(\frac{B(t)}{\beta_L}) - G^{(n+1)}(\frac{B(t)}{\beta_L})).
\end{aligned} \tag{24}$$

Thus,

$$\psi_{\tilde{\tau}}^{(1)}(t) = \int_0^\infty f_{\tilde{\tau},R}(t,r)dr = \lambda \mathrm{e}^{-\lambda t} + \lambda \sum_{n=1}^\infty (p(n;\lambda t) - p(n-1;\lambda t))G^{(n)}(\frac{B(t)}{\beta_L}). \tag{25}$$

Finally,

$$\begin{aligned}
ER &= \int_0^{q/\beta_H}\int_0^\infty r f_{\tilde{\tau},R}(t,r)drdt \\
&= \lambda \int_0^{q/\beta_H} \mathrm{e}^{-\lambda t}\int_{B(t)/\beta_L}^\infty \bar{G}(x)dxdt \\
&+ \lambda \int_0^{q/\beta_H}\int_0^{B(t)/\beta_L} h(y;t)\int_{\frac{B(t)}{\beta_L}-y}^\infty \bar{G}(x)dxdydt,
\end{aligned} \tag{26}$$

where $\bar{G}(x) = 1 - G(x)$.

5 The expected profit per cycle

Assume that the net profit of selling one item (the difference of sale and purchase price) is π, and that the set-up costs per cycle are K. Then the total net profit $P(q)$ per cycle, as a function of q, is the difference of the net profit of selling all items that were produced in a cycle, and the various costs: set-up costs K per cycle, costs for shortage (c_s per time unit), costs for discarding material (c_d per unit of material), and holding cost (c_h per unit material per time unit):

$$P(q) = \frac{\pi q - K}{EC} - c_d \frac{EX(\tau^*)\mathbf{1}_{\{\tau^*=t_0\}}}{EC} - c_s \frac{ER}{EC} - c_h EX(\infty). \tag{27}$$

We wish to choose q such that this profit is maximized. We now show how the terms in (27) can be obtained from the results of Section 4.

1. Determination of EC. In view of (4), we have to determine $E\tau^*$ and ER. $E\tau^*$ immediately follows from (14) and (19). We further have, with $f_{\tilde{\tau},R}(t,r)$ being given by (19):
$$ER = E[R\mathbf{1}_{R>0}] = \int_{t=0}^{q/\beta_H} \int_{r=0}^{\infty} r f_{\tilde{\tau},R}(t,r)dtdr. \tag{28}$$

2. Determination of $E[X(\tau^*)\mathbf{1}_{\{\tau^*=t_0\}}]$. Recall that $\{\tau^* = t_0\} = \{0 < \tilde{\tau} \leq t^*\}$. For such values of $\tilde{\tau} = t$, $Y(t) = \beta_L(t_0 - t)$. Hence, when $\tilde{\tau} = t$ and $\tau^* = t_0$, the inventory level decreases in $[0, t_0]$ at rate β_H during a total period of length t and at rate β_L during a total period of length $t_0 - t$. Consequently, when $\tilde{\tau} = t$ and $\tau^* = t_0$, we have $X(t_0) = q - \beta_H t - \beta_L(t_0 - t)$. So
$$E[X(\tau^*)\mathbf{1}_{\{\tau^*=t_0\}}] = \int_{t=0}^{t^*} (q - \beta_H t - \beta_L(t_0 - t))\psi_{\tilde{\tau}}(t)dt. \tag{29}$$

3. Determination of $EX(\infty)$. Notice that $X(t) = q - Q(t)$, so $EX(\infty) = q - EQ(\infty)$. We make the following observation. Consider the cycle for as long as $X(t) > 0$. If $Y(t) = y$, then there were LD periods during y/β_L, as well as HD periods during $t - y/\beta_L$. Hence if $Y(t) = y$ then $X(t) = q - Q(t) = q - y - \beta_H \left(t - \frac{y}{\beta_L}\right)$. Finally, remembering that

$h(y;t)$ is the density of $Y(t)$:

$$\begin{aligned} EX(\infty) &= \int_{t=0}^{t^*} \psi_{\tilde{\tau}}(t) \int_{y=0}^{\beta_L(t_0-t)} \left(q - \beta_H \left(t - \frac{y}{\beta_L}\right) - y\right) h(y;t)dydt \\ &+ \int_{t=t^*}^{q/\beta_H} \psi_{\tilde{t}}(t) \int_{y=0}^{q-\beta_H t} \left(q - \beta_H \left(t - \frac{y}{\beta_L}\right) - y\right) h(y;t)dydt. \end{aligned} \tag{30}$$

6 Cost functionals for exponential demand

In the present section we develop the formulae for the required cost functionals, in the special case of $G(y) = 1 - e^{-\mu y}$.

The c.d.f. and p.d.f. of $Y(t)$ are (cf. (11) and (12))

$$H(y;t) = \sum_{j=0}^{\infty} P(j;\lambda t) p\left(j; \frac{\mu y}{\beta_L}\right), \tag{31}$$

and

$$h(y;t) = \frac{\mu}{\beta_L} \sum_{j=0}^{\infty} p(j+1;\lambda t) p\left(j; \frac{\mu y}{\beta_L}\right). \tag{32}$$

Moreover, cf. (14),

$$P(\tau^* = t_0) = \sum_{n=1}^{\infty} p(n;\lambda t^*) P(n-1;(t_0 - t^*)\mu). \tag{33}$$

The density of $\tilde{\tau}$, viz. $\psi_{\tilde{\tau}}(t)$, becomes (cf. (15)-(17)): For $0 < t \leq t^*$,

$$\begin{aligned} \psi_{\tilde{\tau}}(t) &= \lambda \sum_{n=1}^{\infty} (p(n-1;\lambda t) - p(n;\lambda t)) P(n-1;(t_0 - t)\mu) \\ &+ \mu \sum_{n=1}^{\infty} p(n;\lambda t) p(n-1);(t_0 - t)\mu). \end{aligned} \tag{34}$$

And for $t^* < t < \dfrac{q}{\beta_H}$,

$$\begin{aligned} \psi_{\tilde{\tau}}(t) &= \lambda \sum_{n=1}^{\infty} (p(n-1;\lambda t) - p(n;\lambda t)) P(n-1;(q - \beta_H t)\mu/\beta_L) \\ &+ \frac{\beta_H}{\beta_L} \mu \sum_{n=1}^{\infty} p(n;\lambda t) p(n-1;(q - \beta_H t)\mu/\beta_L). \end{aligned} \tag{35}$$

The density $\psi_{\tilde{\tau}}(t)$, given in (34) and (35), is composed of two parts. One part corresponding to a jump over the boundary $\psi_{\tilde{\tau}}^{(1)}(t)$ (LD period) and one part, $\psi_{\tilde{\tau}}^{(2)}(t)$, where

$Y(\tilde{\tau}) = B(\tilde{\tau})$ (HD period). According to (34) and (35),

$$\psi_{\tilde{\tau}}^{(1)}(t) = \lambda \sum_{n=1}^{\infty} (p(n-1;\lambda t) - p(n;\lambda t)) P(n-1;\mu B(t)/\beta_L) \tag{36}$$

and

$$\psi_{\tilde{\tau}}^{(2)}(t) = \mu \frac{|B'(t)|}{\beta_L} \sum_{n=1}^{\infty} p(n;\lambda t) p(n-1;\mu B(t)/\beta_L), \tag{37}$$

where $B'(t) = \frac{d}{dt}B(t)$. When G is $\exp(\mu)$, R is independent of $\tilde{\tau}$ and exponential, and the joint density becomes:

$$f_{\tilde{\tau},R}(t,r) = \psi_{\tilde{\tau}}(t)\mu e^{-\mu r}, \quad 0 < t \leq q/\beta_H. \tag{38}$$

Finally, the c.d.f. of τ^* is (cf. (19))

$$\Psi_{\tau^*}(t) = \sum_{j=0}^{\infty} p\left(j;\mu\frac{\beta_H t - q}{\beta_H - \beta_L}\right) P\left(j;\lambda\frac{q-\beta_L t}{\beta_H - \beta_L}\right). \tag{39}$$

We are now ready to evaluate the cost function (27). Starting with the expected value of τ^*, we have

$$E\{\tau^*\} = \frac{q}{\beta_H} e^{-\lambda q/\beta_H} + t_0 P\{\tau^* = t_0\} + \int_{q/\beta_H}^{t_0} t\psi_{\tau^*}(t)dt, \tag{40}$$

where $\psi_{\tau^*}(t)$ is the density of the c.d.f. (19), and where the first two terms correspond to the atoms of τ^* at q/β_H and at t_0. Now,

$$\begin{aligned} \int_{q/\beta_H}^{t_0} t\psi_{\tau^*}(t)dt &= \int_{q/\beta_H}^{t_0} \left(\int_0^t dy\right) \psi_{\tau^*}(t)dt \\ &= q/\beta_H(\Psi_{\tau^*}(t_0) - \Psi_{\tau^*}(q/\beta_H)) \\ &+ \int_{q/\beta_H}^{t_0} \int_y^{t_0} \psi_{\tau^*}(t)dtdy. \end{aligned} \tag{41}$$

Accordingly (notice that $\Psi_{\tau^*}(t_0) = 1 - P\{\tau^* = t_0\}$ and $\Psi_{\tau^*}(q/\beta_H) = e^{-\lambda q/\beta_H}$),

$$\begin{aligned} E\{\tau^*\} &= \frac{q}{\beta_H} e^{-\lambda q/\beta_H} + t_0 P\{\tau^* = t_0\} \\ &+ \frac{q}{\beta_H}(1 - P\{\tau^* = t_0\}) - \frac{q}{\beta_H} e^{-\lambda q/\beta_H} \\ &+ (t_0 - \frac{q}{\beta_H})(1 - P\{\tau^* = t_0\}) \\ &- \int_{q/\beta_H}^{t_0} H\left(\frac{\beta_L(\beta_H y - q)}{\beta_H - \beta_L}; \frac{q - \beta_L y}{\beta_H - \beta_L}\right) dy \\ &= t_0 - \int_{q/\beta_H}^{t_0} H\left(\frac{\beta_L(\beta_H y - q)}{\beta_H - \beta_L}; \frac{q - \beta_L y}{\beta_H - \beta_L}\right) dy. \end{aligned} \tag{42}$$

In the present special exponential case, $P\{\tau^* = t_0\}$ is given by (33). Thus, using (31),

$$\begin{aligned} E\{\tau^*\} = t_0 - \left(t_0 - \frac{q}{\beta_H}\right) \cdot \sum_{j=0}^{\infty} \int_0^1 P\left(j; \frac{\lambda}{\beta_H - \beta_L}\left(q\left(1 - \frac{\beta_L}{\beta_H}\right) - \beta_L\left(t_0 - \frac{q}{\beta_H}\right)u\right)\right) \cdot \\ \cdot\ p\left(j; \frac{\mu}{\beta_H - \beta_L}(t_0\beta_H - q)\, u\right) du. \end{aligned} \tag{43}$$

Recall that $C = \tau^* + I\{\tau^* \in \text{LD}\}R$. In the present case, since G is exponential, $R \sim \exp(\mu)$ independently of τ^*. Thus,

$$E\{C\} = E\{\tau^*\} + \frac{1}{\mu} P\{\tau^* \in \text{LD}\}. \tag{44}$$

Moreover

$$P\{\tau^* \in \text{LD}\} = \frac{\int_0^{t_0} \psi_{\tilde{\tau}}^{(1)}(t)dt}{\int_0^{t_0} \psi_{\tilde{\tau}}(t)dt}. \tag{45}$$

Notice that

$$\int_0^{t_0} \psi_{\tilde{\tau}}(t)dt = 1 - e^{-\lambda q/\beta_H}. \tag{46}$$

Moreover,

$$\begin{aligned} \int_0^{t_0} \psi_{\tilde{\tau}}^{(1)}(t)dt = \lambda \sum_{n=1}^{\infty} \int_0^{t^*} (p(n-1; \lambda t) - p(n; \lambda t)) \cdot P(n-1; (t_0 - t)\mu)dt \\ + \lambda \sum_{n=1}^{\infty} \int_{t^*}^{q/\beta_H} (p(n-1; \lambda t) - p(n; \lambda t)) \cdot P(n-1; (q - \beta_H t)\mu/\beta_L)dt. \end{aligned} \tag{47}$$

For the first integral in (47) we make the transformation $u = t/t^*$, and for the second integral we set $u = (t - t^*)/(q/\beta_H - t^*)$. We then obtain

$$\int_0^{t_0} \psi_{\tilde{\tau}}^{(1)}(t)dt = \lambda t^* \sum_{n=1}^{\infty} \int_0^1 (p(n-1; \lambda t^* u) - p(n; \lambda t^* u)) \cdot P(n-1; \mu(t_0 - t^* u))du$$
$$+ \lambda \left(\frac{q}{\beta_H} - t^*\right) \sum_{n=1}^{\infty} \int_0^1 \left(p\left(n-1; \lambda\left(t^*(1-u) + \frac{q}{\beta_H}u\right)\right) - p\left(n; \lambda\left(t^*(1-u) + \frac{q}{\beta_H}u\right)\right)\right)$$
$$\cdot \ P\left(n-1; \frac{\mu}{\beta_L}(q - \beta_H t^*)(1-u)\right) du. \tag{48}$$

The expected discarded quantity is, according to (9):

$$\begin{aligned} E\{D(\tilde{\tau})I(\tilde{\tau} \le t^*)\} &= (\beta_H - \beta_L)t^* E\{I(\tilde{\tau} \le t^*)\} - (\beta_H - \beta_L)E\{\tilde{\tau} I(\tilde{\tau} \le t^*)\} \\ &= (\beta_H - \beta_L)t^* P\{\tau^* = t_0\} - (\beta_H - \beta_L)\int_0^{t^*} t\psi_{\tilde{\tau}}(t)dt. \end{aligned} \tag{49}$$

Furthermore, according to (34),

$$\begin{aligned} \int_0^{t^*} t\psi_{\tilde{\tau}}(t)dt = \lambda \sum_{n=1}^{\infty} \int_0^{t^*} t(p(n-1; \lambda t) - p(n; \lambda t))P(n-1; \mu(t_0 - t))dt \\ + \mu \sum_{n=1}^{\infty} \int_0^{t^*} tp(n; \lambda t)p(n-1; \mu(t_0 - t))dt. \end{aligned} \tag{50}$$

As before, let $u = t/t^*$, then

$$\begin{aligned} \int_0^{t^*} t\psi_{\tilde{\tau}}(t)dt = \lambda t^{*2} \sum_{n=1}^{\infty} \int_0^1 u(p(n-1; \lambda t^* u) - p(n; \lambda t^* u)) \ \cdot \\ \cdot \ P(n-1; \mu(t_0 - t^* u))du \\ + \mu t^{*2} \sum_{n=1}^{\infty} \int_0^1 up(n; \lambda t^* u)p(n-1; \mu(t_0 - t^* u))du. \end{aligned} \tag{51}$$

Introduce the function, for $a > 0$,

$$M(a; t) = \int_0^a \left(q - \beta_H t + \left(\frac{\beta_H}{\beta_L} - 1\right) y\right) h(y; t)dy, \tag{52}$$

where $h(y; t)$ is the density of $Y(t)$ given by (32). We have

$$M(a; t) = (q - \beta_H t) \cdot H(a; t) + \left(\frac{\beta_H}{\beta_L} - 1\right) \int_0^a yh(y; t)dy. \tag{53}$$

Moreover,

$$\begin{aligned} \int_0^a yh(y;t)dy &= \int_0^a \int_x^a h(y;t)dydx \\ &= \int_0^a (H(a;t) - H(x;t))dx \\ &= aH(a,t) - \int_0^a H(x;t)dx. \end{aligned} \tag{54}$$

According to (31),

$$\begin{aligned} \int_0^a H(x;t)dx &= \sum_{j=0}^{\infty} P(j;\lambda t) \int_0^a p\left(j;\frac{\mu}{\beta_L}x\right) dx \\ &= \frac{\beta_L}{\mu} \sum_{j=0}^{\infty} P(j;\lambda t) \left(1 - P\left(j;\frac{\mu}{\beta_L}a\right)\right). \end{aligned} \tag{55}$$

Thus, for $t < q/\beta_H$,

$$\begin{aligned} M(a;t) = &\sum_{j=0}^{\infty} P(j;\lambda t)p\left(j;\frac{\mu}{\beta_L}a\right)\left(q - \beta_H t + \left(\frac{\beta_H}{\beta_L} - 1\right)a\right) \\ &- \left(\frac{\beta_H}{\beta_L} - 1\right)\frac{\beta_L}{\mu}\sum_{j=0}^{\infty} P(j;\lambda t)\left(1 - P\left(j;\mu\frac{a}{\beta_L}\right)\right). \end{aligned} \tag{56}$$

With the aid of (30) and (56) we get

$$E\{X(\infty)\} = \int_0^{t^*} M(\beta_L(t_0 - t);t)\psi_{\tilde{\tau}}(t)dt + \int_{t^*}^{q/\beta_H} M(q - \beta_H t;t)\psi_{\tilde{\tau}}(t)dt. \tag{57}$$

Using the expressions derived above, in Table 6.1 we explore the effect of several parameters on the cost function. We start with a collection of base values for the parameters, and then vary one parameter at a time.

Table 6.1: The effect of several parameters on the cost function

Base values: $\pi = 0.5, K = 10, c_d = 10, c_s = 20, c_h = 10, \lambda = 0.1, \mu = 0.2, \beta_L = 10,$ $\beta_H = 30, t_0 = 24, q = 300.$

NA = Not Available; the parameters do not satisfy the constraints.

parameter	$P(q)$	*parameter*	P(q)
$\beta_L = 5$	0.8621	$K = 6$	3.4776
$\beta_L = 10$	3.1281	$K = 8$	3.3028
$\beta_L = 15$	NA	$K = 10$	3.1281
$\beta_H = 20$	1.4208	$K = 12$	2.9533
$\beta_H = 30$	3.1281	$K = 14$	2.7785
$\beta_H = 40$	4.3455	$\pi = 0.48$	2.6038
$q = 250$	1.4849	$\pi = 0.49$	2.8659
$q = 300$	3.1281	$\pi = 0.50$	3.1281
$q = 350$	3.7141	$\pi = 0.51$	3.3902
$q = 400$	3.1642	$\pi = 0.52$	3.6524
$t_0 = 20$	1.7607		
$t_0 = 24$	3.1281		
$t_0 = 30$	NA		

In Table 6.2 we use the same base values as in the previous table, but we vary q from 200 to 400. Zooming in, it turns out that the optimal value of q is 352 (bold item in the table).

Table 6.2: Optimal choice of q

$\pi = 0.5, K = 10, c_d = 10, c_s = 20, c_h = 10, \lambda = 0.1, \mu = 0.2, \beta_L = 10, \beta_H = 30, t_0 = 24.$

q	$P(q)$	q	$P(q)$
200	NA	349	3.7132
225	NA	350	3.7141
250	1.4849	351	3.7145
275	2.4361	**352**	**3.7146**
300	3.1281	353	3.7141
325	3.5571	354	3.7132
330	3.6106	355	3.7118
335	3.6532	360	3.6981
340	3.6846	375	3.5873
345	3.7050	400	3.1643

Appendix: An alternative derivation of the distribution of the stopping time τ^*

In this appendix we present a different method for deriving the distribution of τ^*. This method may also be useful in studying variants of the model of this paper.

Let $W(t)$ denote the total time in the interval $(0, t)$ in which the system is in HD periods. Obviously, $0 \le W(t) \le t$. The total demand up to time t is then

$$Q(t) = \beta_H W(t) + \beta_L(t - W(t)) = \beta_L t + (\beta_H - \beta_L)W(t). \tag{58}$$

The distribution of $W(t)$ can be found in the following manner. Let

$$N_H(w) = \max\{n \ge 0 : \sum_{i=0}^{n} H_i \le w\}, \tag{59}$$

where $H_0 = 0$ and where H_i is the length of the ith HD period. Since $H_i \sim \exp(\lambda)$, $\{N_H(w); w \ge 0\}$ is a Poisson process with intensity λ. Construct the compound Poisson process (with $L_0 = 0$ and L_n denoting the length of the nth LD period)

$$\tilde{Y}(w) = \sum_{n=0}^{N_H(w)} L_n. \tag{60}$$

Notice that $P\{\tilde{Y}(w) = 0\} = \mathrm{e}^{-\lambda w}$, and that $\tilde{Y}(w) = Y(w)/\beta_L$, cf. (5). Also, the c.d.f. of $\tilde{Y}(w)$ is

$$\tilde{H}(y; w) = H(\beta_L y; w) = \sum_{n=0}^{\infty} p(n; \lambda w) G^{(n)}(y). \tag{61}$$

Realize that $W(t)$ is the stopping time

$$W(t) = \inf\{w > 0 : \tilde{Y}(w) \geq t - w\}. \tag{62}$$

Thus, since $\tilde{Y}(w)$ is increasing,

$$P\{W(t) > w\} = P\{\tilde{Y}(w) < t - w\} = H(\beta_L(t - w); w), \quad \text{for } \ 0 < w < t. \tag{63}$$

Also, $P\{W(t) = t\} = \mathrm{e}^{-\lambda t}$.

Since $\tau = \inf\{t > 0 : Q(t) = q\}$,

$$\begin{aligned} P\{\tau \leq t\} = 1 - P\{\tau > t\} &= 1 - P\{Q(t) < q\} = 1 - P\{\beta_L t + (\beta_H - \beta_L) W(t) < q\} \\ &= 1 - P\{W(t) < \frac{q - \beta_L t}{\beta_H - \beta_L}\} = H(\beta_L \frac{\beta_H t - q}{\beta_H - \beta_L}; \frac{q - \beta_L t}{\beta_H - \beta_L}) \end{aligned} \tag{64}$$

Finally, since $\tau^* = \min(\tau, t_0)$, we have (cf. (14)),

$$P\{\tau^* = t_0\} = 1 - H(\beta_L \frac{\beta_H t_0 - q}{\beta_H - \beta_L}; \frac{q - \beta_L t_0}{\beta_H - \beta_L}), \tag{65}$$

and, for $\frac{q}{\beta_H} < t < t_0$, cf. (19),

$$P\{\tau^* \leq t\} = H(\beta_L \frac{\beta_H t - q}{\beta_H - \beta_L}; \frac{q - \beta_L t}{\beta_H - \beta_L}) = \sum_{n=0}^{\infty} p(n; \lambda \frac{q - \beta_L t}{\beta_H - \beta_L}) G^{(n)}(\frac{\beta_H t - q}{\beta_H - \beta_L}). \tag{66}$$

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