EURANDOM PREPRINT SERIES
2010-052

The β Meixner model

Albert Ferreiro-Castilla and Wim Schoutens
ISSN 1389-2355

The β–Meixner model

Albert Ferreiro–Castilla[1] · **Wim Schoutens**[2]

December 17, 2010

Abstract.
We propose to approximate the Meixner model by a member of the β–family introduced in [Kuz10]. The advantage of such approximations are the *semi–explicit* formulas for the running extrema under the β–family processes which enables us to produce more efficient algorithms for certain exotic options.

Keywords: Lévy processes; Hitting probability; Barrier options.

2000 Mathematics Subject Classification 60G51, 97M30

---

1. Introduction

We propose to approximate the Meixner model by a member of the β–family introduced in [Kuz10]. The advantage of such approximations are the *semi–explicit* formulas for the running extrema under the β–family processes which enables us to produce more efficient algorithms for certain exotic options.

Therefore the aim of the present work is to rewrite the paper [SD10] for a new model which will be called β–M model from now on. We will calibrate the model to a vanilla surface by inverting a Fourier transform and compare such results with respect to the calibration with the Meixner process. Using the obtained parameters we will price digital down–and–out barrier options (DDOB) under the same underlying but using the *semi–explicit* formulas for the running minimum of the β–M model.

We will show that the approximation in [SD10] and the one described here are particular cases of the more general technique of approximating generalized hyperexponential Lévy processes by hyperexponential models - or hyperexponential jump–diffusion models -, which was used for the same objective in Jeannin and Pistorius [JP10].

2. The β–family and the Meixner process

From now on we will consider $X = \{X_t \mid t \geq 0\}$ to be a Lévy process with triplet (μ, σ, ν) and hence characterized by its Lévy exponent

$$\Psi_{X_1} = -i\mu z + \frac{\sigma^2}{2} z^2 - \int_{-\infty}^{\infty} (e^{izx} - 1 - izh(x))\nu(dx) \;, \tag{1}$$

where the cut–off function can be considered to be $h(x) \equiv x$ for the measures we will be looking at. Then the characteristic function for the Lévy process is

$$\varphi_{X_t}(z) = \mathbb{E}[e^{izX_t}] = e^{-t\Psi_{X_1}(z)} \;.$$

---

[1]Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra (Barcelona), Spain. e-mail: aferreiro@mat.uab.cat

[2]Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, P.O. Box 02400, 3000 Leuven, Belgium. email: wim@schoutens.be

2.1. **Meixner process.** The Meixner distribution, see [Sch03], is an infinitely divisible law and thus we can associate to it a Lévy process. The characteristic function of the Meixner distribution is

$$\varphi(u) = \left(\frac{\cos(b/2)}{\cosh((au - ib)/2)} \right)^{2d} ,$$

where $a > 0$, $-\pi < b < \pi$ and $d > 0$. It is a process with no Brownian part and thus its Lévy triplet is given by $(\mu, 0, \nu)$ where

$$\begin{aligned} \mu &= ad \tanh(b/2) - 2d \int_1^\infty \frac{\sinh(bx/a)}{\sinh(\pi x/a)} dx \\ \nu(x) &= d \frac{\exp(bx/a)}{x \sinh(\pi x/a)} . \end{aligned}$$

2.2. ***β*–family.** The a member of the $\boldsymbol{\beta}$–family is a Lévy process with triplet given by (μ, σ, ν) where

$$\nu(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} 1_{x>0} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} 1_{x<0} , \tag{2}$$

with $\alpha_i > 0$, $\beta_i > 0$, $c_i \geq 0$ and $\lambda_i \in (0, 3)$. Furthermore, the characteristic exponent satisfies

$$\Psi_{X_1} = -i\mu z + \frac{\sigma^2}{2} z^2 - [c_1 I(z; \alpha_1, \beta_1, \lambda_1) + c_2 I(-z; \alpha_2, \beta_2, \lambda_2)] , \tag{3}$$

where

$$\begin{aligned} I(z; \alpha, \beta, \lambda) &= \begin{cases} I_1(z; \alpha, \beta, \lambda); \lambda \in (0, 3) \setminus \{1, 2\}; \\ I_2(z; \alpha, \beta, \lambda); \lambda = 1; \\ I_3(z; \alpha, \beta, \lambda); \lambda = 2 , \end{cases} \\ I_1(z; \alpha, \beta, \lambda) &= \frac{1}{\beta} \mathbb{B} \left[\alpha - \frac{iz}{\beta}, 1\lambda \right] - \frac{1}{\beta} \mathbb{B}[\alpha, 1 - \lambda] \left(1 + \frac{iz}{\beta} [\psi(1 + \alpha - \lambda) - \psi(\alpha)] \right) \\ I_2(z; \alpha, \beta, \lambda) &= -\frac{1}{\beta} \left[\psi \left(\alpha - \frac{iz}{\beta} \right) - \psi(\alpha) \right] - \frac{iz}{\beta^2} \psi'(\alpha) \\ I_3(z; \alpha, \beta, \lambda) &= -\frac{1}{\beta} \left(1 - \alpha + \frac{iz}{\beta} \right) \left[\psi \left(\alpha - \frac{iz}{\beta} \right) - \psi(\alpha) \right] - \frac{iz(1 - \alpha)}{\beta^2} \psi'(\alpha) , \end{aligned}$$

and $\mathbb{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $\psi(x) = \frac{d}{du} \log(\Gamma(u))\big|_x$ the Digamma function.

3. Approximation

In [SD10] the authors approximate the density of the variance gamma (VG) process by a member of the $\boldsymbol{\beta}$–family. The VG process has triplet given by $(\mu, 0, \nu)$, where

$$\nu(x) = C \frac{e^{-Mx}}{x} 1_{x>0} + C \frac{e^{Gx}}{-x} 1_{x<0} ,$$

where $C \geq 0$ and $M, G > 0$. Therefore it seems reasonable to approximate the above Lévy measure by the measure

$$\nu(x) = c \frac{e^{-\alpha_1 x}}{1 - e^{-x}} 1_{x>0} + c \frac{e^{\alpha_2 x}}{1 - e^{x}} 1_{x<0} ,$$

which is the Lévy measure of (2) with parameters $c_1 = c_2 = c$, $\beta_1 = \beta_2 = 1$ and $\lambda_1 = \lambda_2 = 1$. The approximation is carried out under the asymptotic equality $1 - e^{-x} \approx x$ as $x \to 0$. In fact, the same sort of asymptotic behavior can be used to derive

$$\lim_{x \to 0} \frac{(1-e^{-x})^2}{x \sinh(x)} = 1 \ .$$

Hence the Lévy measure of the Meixner process can be approximated by a three parameter Lévy measure of a $\boldsymbol{\beta}$–process, which will be called $\boldsymbol{\beta}$–M process, as

$$\begin{aligned} \nu^M(x; a, b, d) &= d \frac{\exp(bx/a)}{x \sinh(\pi x/a)} \\ \nu^\beta(x; c, \alpha_1, \alpha_2) &= c \frac{e^{-\alpha_1 x}}{(1-e^{-x})^2} 1_{x>0} + c \frac{e^{\alpha_2 x}}{(1-e^{x})^2} 1_{x<0} \ , \end{aligned}$$

where $\nu^M(x; a, b, d)$ stands for the Lévy measure of the Meixner process and $\nu^\beta(x; c, \alpha_1, \alpha_2)$ for the Lévy measure of the $\boldsymbol{\beta}$–M process.

The asymptotic approximation works as long as $c = ad/\pi$. The values of α_1 and α_2 might not be related to a, b and d, this is a difference between our approximation and the one performed in [SD10] where all the parameters in the VG model had its counterpart in the $\boldsymbol{\beta}$–VG model. In this case tough, it makes sense that $\alpha_1 \approx (\pi - b)/a$ and $\alpha_2 \approx (\pi + b)/a$.

3.1. **The running extrema under the $\boldsymbol{\beta}$–M process.** The advantage of using a member of the $\boldsymbol{\beta}$–family as an approximation is that the Wiener–Hopf factors for the associated Lévy process are known in explicit form. According to [Kuz10], for a given $q > 0$, the Wiener–Hopf factors for a $\boldsymbol{\beta}$–process are

$$\begin{aligned} \Phi_q^-(z) &= \frac{1}{1 + \frac{iz}{\zeta_0^+}} \prod_{n \geq 1} \frac{1 + \frac{iz}{\beta_2(n-1+\alpha_2)}}{1 + \frac{iz}{\zeta_n}} \\ \Phi_q^+(z) &= \frac{1}{1 + \frac{iz}{\zeta_0^-}} \prod_{n \leq -1} \frac{1 + \frac{iz}{\beta_1(n+1-\alpha_1)}}{1 + \frac{iz}{\zeta_n}} \ , \end{aligned}$$

where ζ_n, ζ_0^+ and ζ_0^- are the zeros of $\Psi_{X_1}(i\zeta) + q = 0$ given in (3) which can be localized in the intervals

$$\begin{aligned} \zeta_0^- &\in (-\beta_1 \alpha_1, 0) \\ \zeta_0^+ &\in (0, \beta_2 \alpha_2) \\ \zeta_n &\in (\beta_2(\alpha_2 + n - 1), \beta_2(\alpha_2 + n)) \ , \quad n \geq 1 \\ \zeta_n &\in (\beta_1(-\alpha_1 + n), \beta_1(-\alpha_1 + n + 1)) \ , \quad n \leq -1 \ . \end{aligned}$$

Therefore one can also derive an expression for the running infimum as

$$\mathbb{P}[\inf_{0 \leq t \leq \tau(q)} X_t > x] = 1 - c_0^+ e^{\zeta_0^+ x} - \sum_{n \geq 1} c_n e^{\zeta_n x} \ , \tag{4}$$

where $\tau(q)$ is an exponential distributed random variable with parameter q and

$$c_0^+ = \prod_{n \geq 1} \frac{1 - \frac{\zeta_0^+}{\beta_2(n-1+\alpha_2)}}{1 - \frac{\zeta_0^+}{\zeta_n}} \ , \qquad c_k = \frac{1 - \frac{\zeta_k}{\beta_2(k-1+\alpha_2)}}{1 - \frac{\zeta_k}{\zeta_0^+}} \prod_{\substack{n \geq 1 \\ n \neq k}} \frac{1 - \frac{\zeta_k}{\beta_2(n-1+\alpha_2)}}{1 - \frac{\zeta_k}{\zeta_n}} \ . \tag{5}$$

One can recover the running infimum at a deterministic time by the inverse transform

$$\mathbb{P}[\inf_{0 \leq t \leq T} X_t > x] = \frac{1}{2\pi i} \int_{q \in \mathbb{C}} \frac{e^{qT}}{q} \mathbb{P}[\inf_{0 \leq t \leq \tau(q)} X_t > x] dq \ . \tag{6}$$

3.2. **Hyperexponential jump–diffusion framework.** The numerical implementation of the formulas (4) and (5) must be done by a truncation of the infinite sum and the infinite product. This means that essentially we are approximating the Wiener–Hopf factors of the process by a finite product. It turns out that this expressions for the Wiener–Hopf factors generate Hyperexponential jump–diffusion processes described in [Sau08]. In fact the idea comes from the possibility to approximate Generalized Hyperexponential (GHE) processes by Hyperexponential processes, see [AMP07] and [JP10].

GHE processes are Lévy processes with jumps given by $\nu(x) = k_+(x)1_{x>0}+k_-(-x)1_{x<0}$ where k_+ and k_- are completely monotone functions in $(0,\infty)$. It turns out that this Lévy measures can be written as

$$\nu(x) = 1_{x>0}\int_0^\infty e^{-ux}\nu_+(du) + 1_{x<0}\int_{-\infty}^0 e^{-|ux|}\nu_-(du)\ .$$

Heuristically, one can consider a finite Riemann sum of the above expression to end up with the approximation

$$\nu(x) \approx 1_{x>0}\sum_{i\in I}\omega_i e^{-\zeta_i x} + 1_{x<0}\sum_{j\in J}\omega_j e^{-|\zeta_j x|}\ ,$$

where I, J are finite partitions of $(0,\infty)$ and $(-\infty,0)$ respectively, and ω_i, ω_j are weights. For instance, one could choose $\zeta_i \in [t_i,t_{i+1}]$, $\zeta_j \in [t_{j+1},t_j]$, $\omega_i = \nu_+([t_i,t_{i+1}])$ and $\omega_j = \nu_-([t_{j+1},t_j])$. The process with such Lévy measure would be an Hyperexponential jump–diffusion.

The determination of the the intensity and the weights in the exponential approximation can vary. Jeannin and Pistorius [JP10] choose the number of exponentials and intensities beforehand and then fit the weights. Le Saux [Sau08] proposes a more systematic approach by approximating the Lévy exponent. In fact, the approximation made in [SC09] and the one here are particular choices of this procedure. To show that, consider Newton's generalized binomial theorem which sets the equality

$$(1-e^{-x})^{-n} = \sum_{k\geq 0}\begin{pmatrix} n+k-1 \\ k\end{pmatrix} e^{-kx} \qquad x\geq 0, n\in\mathbb{N}\ .$$

This means that in our approach we are approximating the jump part of the Meixner process by

$$\begin{aligned}\nu^\beta(x;c,\alpha_1,\alpha_2) &= c\frac{e^{-\alpha_1 x}}{(1-e^{-x})^2}1_{x>0} + c\frac{e^{\alpha_2 x}}{(1-e^{x})^2}1_{x<0}\\ &= 1_{x>0}\sum_{k\geq 0}c(k+1)e^{-(k+\alpha_1)x} + 1_{x<0}\sum_{k\geq 0}c(k+1)e^{(k+\alpha_2)x}\ .\end{aligned}$$

The same is valid for the approximation in [SD10]. When trying to numerical implement this approximation we will truncate the infinite sum representation and end up with the Hyperexponential jump–diffusion approximation.

4. Spot process

It is assumed that the underlying is modeled by an exponential Lévy process. That means that spot is of the form $S_t = S_0 e^{(r-q+w)t+X_t}$, where S_0 is the spot at time 0, r is the risk free rate, q is the dividend yield, ω is the mean correcting drift to ensure that the discounted prices are martingales and X_t is a Lévy process - here this will be either the Meixner or the $\boldsymbol{\beta}$–M process. A key function in the following will be the characteristic function of the $\log(S_t)$. This can be derived as

$$\varphi_{\log(S_t)}(u) = e^{iu(log(S_0)+(r-q+w)t)}\varphi_{X_t}(u) \tag{7}$$
$$= e^{iu(log(S_0)+(r-q+w)t)-t\Psi_{X_1}(u)}, \tag{8}$$

where $\omega = \Psi_{X_1}(-i) = -\varphi_{X_1}(-i)$.

5. Numerical results

The data set for the vanilla surface will be the one proposed in [Sch03, p. 6]. Since we already have a calibration of the Meixner model under this surface of call options (see [Sch03, p. 81]). For such data the risk free interest rate is $r = 1.20\%$, the dividend yield is $q = 1.90\%$ and $S_0 = 1124.47$. This data set was taken at the close of the market on 18/04/2002.

5.1. **Vanilla surface calibration.** One way of pricing call options is through the characteristic function of the process by the Carr and Madan formula. The price of a call option with strike K and maturity T is

$$\begin{aligned} C(K,T) &= e^{-rT}\mathbb{E}[\max((S_T - K), 0)] \\ &= \frac{e^{-rT}}{\pi}\int_0^\infty e^{-iuk}\rho(u)du \\ &\approx \frac{e^{-rT}}{\pi}\text{Real}\left(\text{FFT}\left[e^{iu_jb}\rho(u_j)\eta\left(\frac{3+(-1)^j-1_{\{j=1\}}}{3}\right)\right]_{j=1,\dots,n}\right), \end{aligned}$$

where $\alpha > 0$ is a damping factor, $u_j = \eta(j-1)$, $k = -b + \lambda(n-1) = \log(K)$, $\lambda\eta = 2\pi/N$ and

$$\rho(u) = \frac{e^{-rT}\varphi_{\log(S_T)}(u - i(\alpha+1))}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}.$$

In numerical implementation that follows we have set $\eta = 0.25$, $N = 4096$ and $\alpha = 1.5$. The minimization was done with respect to the root–mean–square–error

$$\text{RMSE} = \sqrt{\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}}.$$

The optimal parameters for the calibration of the Meixner model and the $\boldsymbol{\beta}$–M model are summarized in Fig. 1. On Fig. 2 and Fig. 3 one can see the performance of such optimal parameters. Essencially the two models fail and success on the same regions although the calibration of the $\boldsymbol{\beta}$–M model is better with respect to the RMSE error.

	$\boldsymbol{\beta}$–M model (c, α_1, α_2)	Meixner model (a, b, d)
Starting values	$(0.0438,\ 4.3835,\ 1.9255)$	$(0.3977,\ -1.4940,\ 0.3462)$
Optimal parameters	$(0.0538,\ 7.9017,\ 1.7344)$	$(0.4764,\ -1.4723,\ 0.2581)$
RMSE	3.1612	3.3506
CPU(s)	120.24	42.93

Figure 1: Calibration on the vanilla surface.

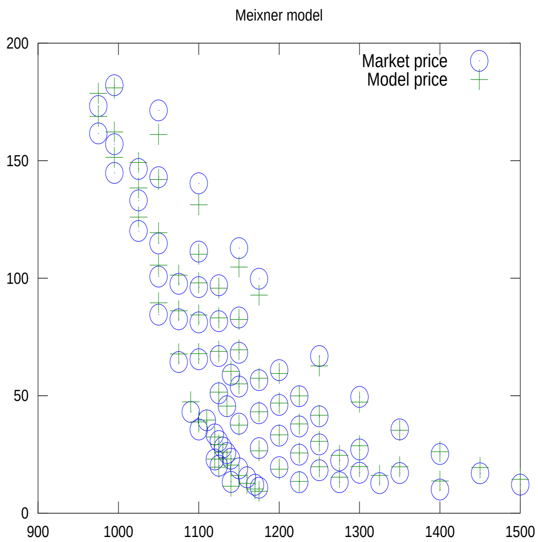

Figure 2: Meixner calibration on the vanilla surface.

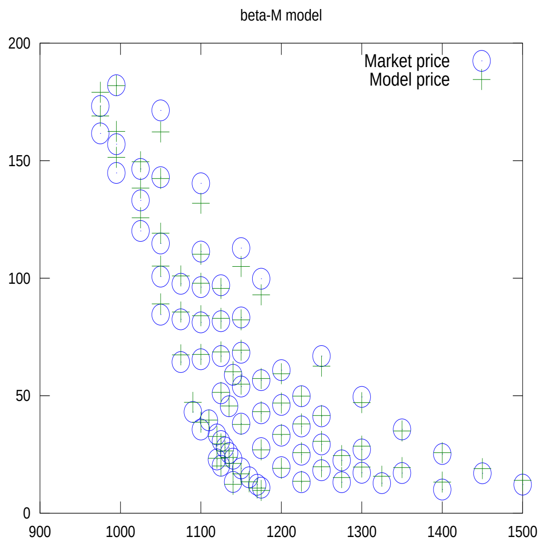

Figure 3: β–M calibration on the vanilla surface.

5.1.1. *Monte Carlo pricing.* We are going to check the DDOB pricing with the Wiener–Hopf factors by a comparison with a Monte Carlo method which in turn will be check with the vanilla surface with respect to the Carr Madan method.

A general setting for simulating Lévy processes with Monte Carlo technique is described in [Sch03, p. 102]. The idea is to approximate the big jumps of the process by a sum of Poisson process and the small ones by a Brownian motion or the mean. There is some discussion about this last step which can be found in [Sch03], but for our purposes we will approximate the small jumps by a Brownian motion. For a Lévy process with triplet (μ, σ, ν) we need to choose $\varepsilon \in (0, 1)$ and the partition

$$a_0 < a_1 < \cdots < a_k = -\varepsilon,\ \varepsilon = a_{k+1} < a_{k+2} < \cdots < a_{2k+1}\ ,$$

in such a way that $\nu((-\infty, a_0])$, $\nu([a_{2k+1}, \infty))$ and $\int_{-\varepsilon}^{\varepsilon} x^2\nu(dx)$ are all small enough. The approximation is then

$$X_t^{2k} = \mu t + \tilde{\sigma} W_t + \sum_{j=1}^{2k} c_j (N_t^j - \lambda_j t \underbrace{1_{|c_j|\leq 1}}_{\star})\ ,$$

where

$$\begin{aligned}
\tilde{\sigma}^2 &= \sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2 \nu(dx) \\
\lambda_j &= \begin{cases} \nu([a_{j-1}, a_j]);\ j = 1, \ldots, k \\ \nu([a_j, a_{j+1}]);\ j = k+1, \ldots, 2k+1 \end{cases} \\
c_j &= \begin{cases} -\sqrt{\frac{1}{\lambda_j}\int_{a_{j-1}}^{a_j} x^2\nu(dx)};\ j = 1, \ldots, k \\ \sqrt{\frac{1}{\lambda_j}\int_{a_j}^{a_{j+1}} x^2\nu(dx)};\ j = k+1, \ldots, 2k+1 \end{cases}\ ,
\end{aligned}$$

W is a Brownian motion and $\{N^j\}_j$ are independent Poisson process. Note that the indication function $(\star)$ might or might not be used depending on the cut–off function used in the Lévy–Khintchine formula for the original Lévy process. For instance it will not be needed for the simulation of the β–M model, but the Meixner triplet was computed assuming that the cut–off function of (1) was $h(x) = 1_{|x|<1}$, and thus it must also appear in the approximation.

For the numerical implementation we have set $k = 5000$ and done 100000 simulations. The partition was choose such that $a_{j-1} = -\alpha/j$ and $a_{2k+2-j} = \alpha/j$ for $j = 1, \ldots, k+1$ and $\alpha = 4.5$. The performance of the Monte Carlo method with respect to the Carr Madan is roughly the same, for the β–M model is almost negligible as you can see in Fig. 5, while in Fig. 4 we can appreciate some difference. In Fig. 6 we have depicted the coefficients λ_j with respect to c_j in both approximations.

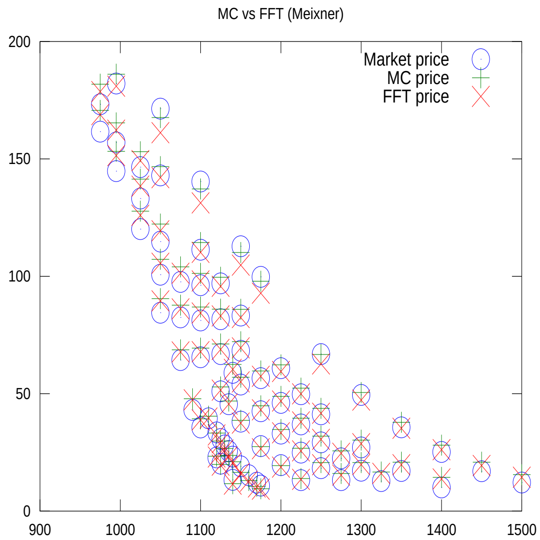

Figure 4: Meixner pricing using Monte Carlo.

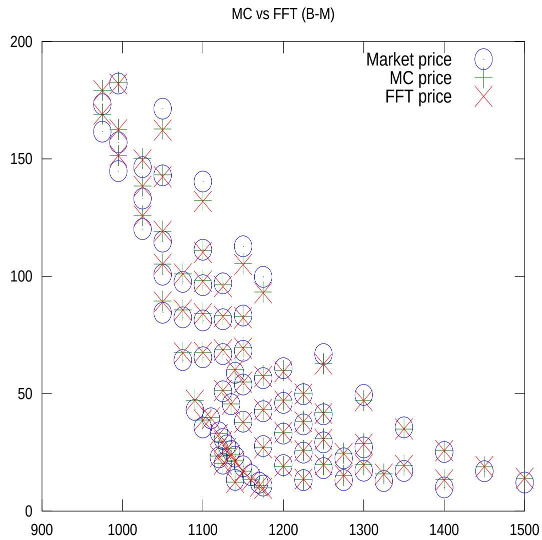

Figure 5: β–M pricing using Monte Carlo.

5.2. **DDOB pricing.** In this section we used the optimal parameters obtained in the previous sections to price DDOB options using the *semi–explicit* Wiener–Hopf factorization. The idea is to use also a Monte Carlo method to check the performance. The price of a DDOB option with barrier H and maturity T is

$$DDOB(H, T) = e^{-rT}\mathbb{P}[\inf_{0 \le t \le T} S_t > H] \, .$$

We price the exotic options under the range $T = \{1, 3, 5, 7, 10\}$ and $H = \{975, 995, 1025, 1050, 1075, 1090, 1100, 1110, 1120\}$.

For computing the coefficients c_0^+, ζ_0^+, c_n and ζ_n of equation (4) we have computed 100 roots of the equation $\Psi_{X_1}(i\zeta) + q = 0$ and used them to compute 75 coefficients c_n. Finally the integral (6) was discretized following a Gaver–Stehfest algorithm used in [SD10]. The figure Fig. 7 depicts the price using the Wiener–Hopf factors and the Monte Carlo method. The prices do not much a lot for large maturities or small barriers but this is just because we have used a step size of 0.1 in the Monte Carlo simulation - far too big for this purpose.

6. Further work

The next step in order to complete this project is to price credit default swaps (CDS) as it was done in [SD10]. One can also complete the study by comparing two methods to use the Wiener–Hopf factorization for the β–family. Here we have used equation (6) to compute the running extrema for a determinist time, but [KKPvS10] show an alternative method, this might not be more efficient for DDOB pricing but it seems to be more efficient on more complex exotic options.

Acknowledgement. This work was conducted during a short–stay visit by Albert Ferreiro–Castilla who would like to thank Eurandom.

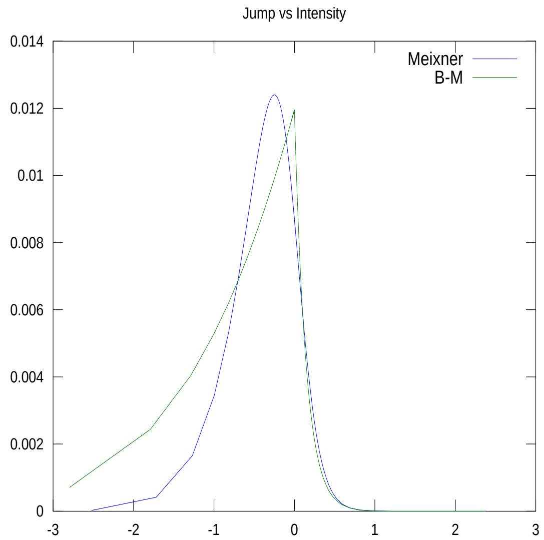

Figure 6: Jump with respect to intensity.

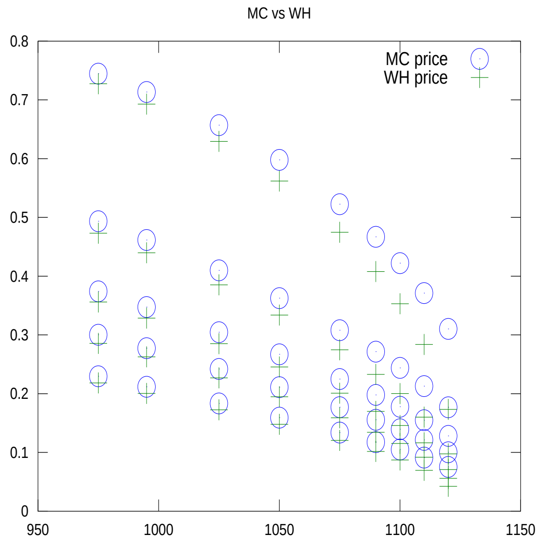

Figure 7: β–M pricing using Wiener–Hopf factors.

REFERENCES

[AMP07] S. Asmussen, D. Madan, and M. Pistorius. Pricing equity default swaps under an approximation to the CGMY Lévy models. *Journal of Computational Finance*, 11(1):79–93, 2007.

[JP10] Marc Jeannin and Martijn Pistorius. A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes. *Quant. Finance*, 10(6):629–644, 2010.

[KKPvS10] Alexey Kuznetsov, Andreas E. Kyprianou, Juan C. Pardo, and Kees van Schaik. A Wiener–Hopf Monte Carlo simulation technique for Lévy processes. *arXiv:0910.4743v2*, 2010.

[Kuz10] Alexey Kuznetsov. Wiener–Hopf factorization and distribution of the extrema for a family of Lévy processes. *The Annals of Applied Probability*, 20(5):1801–1830, 2010.

[Sau08] Nolwenn Le Saux. Approximating lévy processes by a hyperexponential jump–diffusion process with a view to option pricing. Master's thesis, Imperial College London, 2008.

[SC09] Wim Schoutens and Jessica Cariboni. *Lévy Processes in Credit risk*. John Weiley & Sons, England, 2009.

[Sch03] Wim Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives*. John Weiley & Sons, England, 2003.

[SD10] Wim Schoutens and Geert Van Damme. The β–variance gamma model. *Rev Deriv Res (to appear)*, 2010.