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ABSTRACT. This paper considers a Lévy-driven queue (i.e., a Lévy process reflected at 0), and focuses on the distribution of $M(t)$, that is, the minimal value attained in an interval of length $t$ (where it is assumed that the queue is in stationarity at the beginning of the interval). The first contribution is an explicit characterization of this distribution, in terms of Laplace transforms, for spectrally one-sided Lévy processes (i.e., either only positive jumps or only negative jumps). The second contribution concerns the asymptotics of $\mathbb{P}(M(T_u) > u)$ (for different classes of functions $T_u$ and $u$ large); here we have to distinguish between heavy-tailed and light-tailed scenarios.

## 1. INTRODUCTION

The class of processes with stationary and independent increments, known as *Lévy processes*, form a key object in applied probability. A substantial body of literature is devoted to Lévy processes that are reflected at 0, sometimes also referred to as *Lévy-driven queues*, and are regarded as a valuable generalization of the classical M/G/1 queues; also the important special case of reflected Brownian motion is covered.

These reflected Lévy processes are defined as follows. Let $X \equiv \{X(t) : t \in \mathbb{R}\}$ be a Lévy process with (without loss of generality) zero drift: $\mathbb{E}X(1) = 0$. Then define the queueing process (or: workload process, storage process) $Q \equiv \{Q(t) : t \geq 0\}$ through

$$Q(t) := \sup_{s \leq t} \left(X(t) - X(s) - c(t-s)\right),$$

where it is assumed that the workload is in *equilibrium* (stationarity) at time 0, i.e., $Q(0) =_{\rm d} Q_e$. We refer to this process $Q$ as the reflection of the Lévy process $Y = \{Y(t) : t \in \mathbb{R}\}$ at 0, where $Y(t) := X(t) - ct$. In the sequel we normalize time such that $c = 1$.

When considering the steady state $Q_e$ of the reflected process introduced above, the literature can be roughly divided into two categories. (A) In the first place there are results on the full distribution of $Q_e$, in terms of the corresponding Laplace transform. Particularly for the case of *one-sided jumps*, these transforms are fairly explicit. If $X$ is such that it has only positive jumps, $X \in \mathscr{S}^+$ (which is often referred to as the *spectrally positive case*), then a generalization of the classical Pollaczek-Khintchine formula was derived [20], while in the case of only negative jumps, $X \in \mathscr{S}^-$ (*spectrally negative*), $Q_e$ was seen to be exponentially distributed. In the Lévy processes literature [5, 16], this type of results can be found under the denominator *fluctuation theory*. We recall that there are powerful tools available for numerical inversion of Laplace transforms [1, 13]. (B) In the second place there are results that describe the *asymptotics* of $\mathbb{P}(Q_e > u)$ for $u$ large. Then one has to distinguish between results in which the upper tail of the Lévy increments is light on one hand, sometimes referred to as the Cramér case, and results that correspond to the heavy-tailed regime on the other hand; see for instance [15] and references therein.

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In the present short communication, we consider a related problem: we analyze how long the process consecutively spends above a given level. More formally, we consider the distribution of $M(t) := \inf_{s\in[0,t]} Q(s)$, i.e., the minimum value attained by the workload process in a window of length $t$, where it is assumed that the queue is in stationarity at the beginning of the interval. This problem has various applications: one could for instance think of the analysis of persistent overload in an element of a communication network or a node in a supply chain; see e.g. [17]. A related study on the situation of infinitely-divisible self-similar input is [2].

Our results correspond to both branches (A) and (B) mentioned above: in Section 2 we present results on the Laplace transform of $M(t)$, relying on known results for Lévy fluctuation theory; we also consider the special case of Brownian motion. Section 3 identifies the asymptotics of $\mathbb{P}\left(M(T_u) > u\right)$ for different classes of functions $T_u$ and $u$ large; as expected, we need to distinguish between heavy-tailed and light-tailed input.

Recall that

$$Y(t) = X(t) - t, \quad Q(t) = \sup_{s\le t}\left(Y(t) - Y(s)\right), \quad M(t) = \inf_{s\in[0,t]} Q(s);$$

we will also extensively use the following notation:

$$K(t) := \inf_{s\in[0,t]} Y(s)$$

so that $M(t) = Q(0) + K(t)$. Notice that due to the independent increments property of $X$, the random variables $Q(0)$ and $K(t)$ are independent, and hence $M(t) =_{\rm d} Q_e + K(t)$.

## 2. Transforms for the spectrally one-sided case

In this section we evaluate the double transform, with $x \ge 0$, $y > 0$,

$$\mathscr{L}(x,y) := \int_0^\infty \int_0^\infty e^{-xu}e^{-yt}{\rm d}\mathbb{P}\left(M(t)\le u\right){\rm d}t = \int_0^\infty \mathbb{E}e^{-xM(t)}e^{-yt}{\rm d}t.$$

As indicated in the introduction, we do so for Lévy processes with one-sided jumps. We separately treat the spectrally-positive and spectrally-negative case.

Let us start with computations that are valid for any Lévy process $X$ as introduced in Section 1. Integration by parts yields

$$\mathscr{L}(x,y) = \int_0^\infty e^{-yt}\left(1 - x\int_0^\infty e^{-xu}\mathbb{P}\left(M(t) > u\right){\rm d}u\right){\rm d}t = \frac{1}{y} - x\mathscr{K}(x,y), \tag{1}$$

where

$$\begin{aligned}\mathscr{K}(x,y) &:= \int_0^\infty\int_0^\infty e^{-xu}e^{-yt}\mathbb{P}\left(M(t) > u\right){\rm d}u{\rm d}t\\ &= \int_0^\infty\int_0^\infty e^{-xu}e^{-yt}\mathbb{P}\left(Q(0) + K(t) > u\right){\rm d}u{\rm d}t\\ &= \int_0^\infty\int_0^\infty e^{-xu}e^{-yt}\int_u^\infty \mathbb{P}\left(K(t) > u - z\right){\rm d}\mathbb{P}\left(Q(0)\le z\right){\rm d}u{\rm d}t.\end{aligned}$$

Our goal is to evaluate the 'double transforms' $\mathscr{K}(x,y)$ and $\mathscr{L}(x,y)$ that uniquely determine the distribution of $M(t)$.

Let $R_z := \inf\{t\ge 0 : -Y(t) > z\}$ denote the first passage time of $-Y$ over level $z > 0$; note that $Y$ has a negative drift, and therefore $R_z$ is finite almost surely. As the event $\{K(t) > -z\}$ coincides with $\{R_z > t\}$, we obtain, after interchanging the order of integration,

$$\mathscr{K}(x,y) = \int_0^\infty\int_0^z e^{-xu}\left(\int_0^\infty e^{-yt}\mathbb{P}\left(R_{z-u} > t\right){\rm d}t\right){\rm d}u{\rm d}\mathbb{P}\left(Q(0)\le z\right).$$

### 2.1. **Spectrally positive case.**

**Theorem 1.** *Let $X \in \mathscr{S}^+$. Denote $\vartheta(s) = \log \mathbb{E}e^{-sY(1)}$. Then with $x \geq 0$, $y > 0$,*

$$\mathscr{L}(x,y) = \frac{x}{\vartheta(x)y} + \frac{x\vartheta^{-1}(y)}{(x-\vartheta^{-1}(y))y^2} - \frac{x^2}{(x-\vartheta^{-1}(y))y\vartheta(x)}.$$

*Proof.* It is well-known that $\mathbb{E}e^{-yR_z} = e^{-z\vartheta^{-1}(y)}$. Noting that

$$\int_0^\infty e^{-yt}\mathbb{P}\left(R_{z-u} > t\right)\mathrm{d}t = \frac{1}{y}\left(1 - \mathbb{E}e^{-yR_{z-u}}\right) = \frac{1}{y}\left(1 - e^{-(z-u)\vartheta^{-1}(y)}\right),$$

it follows that

$$\mathscr{K}(x,y) = \int_0^\infty \left(\left(\frac{1-e^{-xz}}{xy}\right) - \left(\frac{e^{-z\vartheta^{-1}(y)} - e^{-xz}}{(x-\vartheta^{-1}(y))y}\right)\right)\mathrm{d}\mathbb{P}\left(Q(0) \leq z\right).$$

Recalling that $Q(0) =_{\mathrm{d}} Q_e$ and using 'Pollaczek-Khintchine' we know that $\mathbb{E}e^{\alpha Q_e} = \alpha\vartheta'(0)/\vartheta(\alpha)$. Therefore the claim follows from (1) and the fact that

$$\mathscr{K}(x,y) = \frac{1}{xy}\left(1 - \frac{x}{\vartheta(x)}\right) - \frac{1}{(x-\vartheta^{-1}(y))y}\left(\frac{\vartheta^{-1}(y)}{y} - \frac{x}{\vartheta(x)}\right).$$

□

### 2.2. **Spectrally negative case.**

**Theorem 2.** *Let $X \in \mathscr{S}^-$. Denote by $\eta^{-1}(x) = \sup\{s \geq 0 : \eta(s) = x\}$ the right-inverse of $\eta(s) = \log \mathbb{E}e^{sY(1)}$. Then with $x \geq 0$, $y > 0$,*

$$\mathscr{L}(x,y) = \frac{1}{y}\left(1 - \frac{x}{\eta^{-1}(0)+x}\frac{\eta^{-1}(y)}{\eta^{-1}(0)+\eta^{-1}(y)}\right).$$

*Proof.* Notice that in this case $Q(0) =_{\mathrm{d}} Q_e$ is exponentially distributed with parameter $\alpha := \eta^{-1}(0) > 0$. Thus,

$$\mathscr{K}(x,y) = \int_0^\infty \int_0^z e^{-xu}.\frac{1}{y}\left(1 - \mathbb{E}e^{-yR_{z-u}}\right)\alpha e^{-\alpha z}\mathrm{d}u\mathrm{d}z = \frac{1}{(\alpha+x)y}\left(1 - \int_0^\infty \mathbb{E}e^{-yR_z}\alpha e^{-\alpha z}\mathrm{d}z\right).$$

The second factorization identity [16], states that

$$\int_0^\infty \mathbb{E}e^{-yR_z}e^{-\alpha z}\mathrm{d}z = \frac{\kappa(y,\alpha) - \kappa(y,0)}{\alpha\kappa(y,\alpha)},$$

where in this spectrally negative case $\kappa(y,x) = \eta^{-1}(y) + x$. Now the claim follows from (1) and the fact that

$$\mathscr{K}(x,y) = \frac{1}{(\alpha+x)y}\frac{\kappa(y,0)}{\kappa(y,\alpha)}.$$

□

### 2.3. **Brownian motion.**

**Theorem 3.** *Let $X$ be a standard Brownian motion $B \equiv \{B(t) : t \in \mathbb{R}\}$. Then, for each $t > 0$,*

$$\mathbb{P}\left(M(t) > u\right) = \exp(-2u)\left(2(1+t)\Psi(\sqrt{t}) - \sqrt{\frac{2t}{\pi}}\exp\left(-\frac{t}{2}\right)\right),$$

*where $\Psi(x) = \mathbb{P}\left(\mathcal{N} > x\right)$ for a standard normal random variable $\mathcal{N}$.*

*Proof.* Because $B \in \mathscr{S}^-$, $Q(0) =_{\rm d} Q_e$ has an exponential distribution with mean $1/2$. Thus,

$$\begin{aligned}\mathbb{P}\left(M(t) > u\right) &= \mathbb{P}\left(Q(0) + \inf_{s\in[0,t]}(B(s)-s) > u\right)\\ &= \int_u^\infty \mathbb{P}\left(\inf_{s\in[0,t]}(B(s)-s) > u-x\right) 2\exp(-2x)\mathrm{d}x\\ &= 2\exp(-2u)\int_0^\infty \mathbb{P}\left(\sup_{s\in[0,t]}(B(s)+s) < y\right)\exp(-2y)\mathrm{d}y\\ &= \exp(-2u)\mathbb{E}\exp\left(-2\sup_{s\in[0,t]}(B(s)+s)\right)\end{aligned}$$

and the claim follows after some elementary computations (see also [7, Eqn. (1.1.3)] or [4]). □

## 3. Asymptotics

In this section we consider the asymptotics of $\mathbb{P}\left(M(T_u) > u\right)$ for a variety of functions $T_u$ and $u$ large. As usual, heavy-tailed and light-tailed scenarios need to be addressed separately.

### 3.1. **Heavy-tailed case.**

In this section we shall work with the following assumption about the Lévy process $X$:

**Assumption 1.** *For $\alpha > 1$, let $X(1) \in \mathscr{RV}(-\alpha)$ – the class of distributions with a complementary distribution function that is regularly varying at $\infty$ with index $-\alpha$. Moreover, if $\alpha \in (1,2)$, then in addition*

$$\lim_{x\to\infty}\frac{\mathbb{P}\left(X<-x\right)}{\mathbb{P}\left(X>x\right)} = \rho \in [0,\infty).$$

We start with the following general proposition.

**Proposition 1.** *For a Lévy process $X$ such that $\mathbb{E}X(1) = 0$, as $u \to \infty$,*

$$K(u)/u \to -1 \quad \textit{almost surely.}$$

*Proof.* Observe that, for any $\varepsilon > 0$ and any fixed $T \le u$,

$$\begin{aligned}\mathbb{P}\left(\left|\frac{K(u)}{u}+1\right| > \varepsilon\right) &\le \mathbb{P}\left(\frac{Y(u)}{u} > -1+\varepsilon\right)\\ &\quad + \mathbb{P}\left(\inf_{t\in[0,T]} Y(t) < \inf_{t\in(T,u]} Y(t)\right) + \mathbb{P}\left(\frac{1}{u}\inf_{t\in[0,T]} Y(t) < -1-\varepsilon\right)\\ &\le \varepsilon,\end{aligned}$$

which can be realized due to the fact that $Y(u)/u \to -1$ and $\inf_{t\in[0,T]} Y(t)/u \to 0$, almost surely. □

In the sequel we say that $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$.

**Proposition 2.** *Assume that the Lévy process $X$ satisfies Assumption 1.*
*(i) If $f(n) \ge n$, then*

$$\mathbb{P}\left(X(n) > f(n)\right) \sim n\mathbb{P}\left(X(1) > f(n)\right),$$

*as $n \to \infty$, $n \in \mathbb{N}$.*
*(ii) As $u \to \infty$,*

$$\mathbb{P}\left(Q_e > u\right) \sim \frac{u}{\alpha-1}\mathbb{P}\left(X(1) > u\right).$$

*Proof. Ad (i).* These asymptotics can be found in, e.g., [10] for $\alpha \geq 2$ and [8, 9] for $\alpha \in (1,2)$; see also [14] for a recent treatment. *Ad (ii).* See, e.g., [3, 15]. □

We now state the main result of this subsection: the exact asymptotics of $\mathbb{P}(M(T_u) > u)$.

**Theorem 4.** *Assume that the Lévy process $X$ satisfies Assumption 1. Then*

$$\mathbb{P}(M(T_u) > u) \sim \mathbb{P}(Q_e > u + T_u) + T_u \mathbb{P}(X(1) > u + T_u), \quad \textit{as} \quad u \to \infty. \tag{2}$$

The asymptotics in Theorem 4 can be made more explicit. Part (ii) of Proposition 2 immediately leads to the following corollary.

**Corollary 1.** *Assume that the Lévy process $X$ satisfies Assumption 1. Then*

$$\mathbb{P}(M(T_u) > u) \sim \begin{cases} \frac{1}{\alpha-1} u \mathbb{P}(X(1) > u) & \textit{when} \quad T_u = o(u), \\ \frac{A+\alpha}{\alpha-1}(A+1)^{-\alpha} T_u \mathbb{P}(X(1) > T_u) & \textit{when} \quad u \sim A T_u, \\ \frac{\alpha}{\alpha-1} T_u \mathbb{P}(X(1) > T_u) & \textit{when} \quad u = o(T_u), \end{cases}$$

*as $u \to \infty$.*

**Proof of Theorem 4.** The proof consists of an upper bound and a lower bound. We use the notation $T_u^- := \lfloor T_u \rfloor$ and $T_u^+ := \lceil T_u \rceil$.

*Upper bound.* To prove an (asymptotically) tight upper bound for $\mathbb{P}(M(T_u) > u)$, first we observe that for any $\varepsilon > 0$, using that $Q(0) =_{\mathrm{d}} Q_e$ is independent of $\{X(t) : t \geq 0\}$,

$$\begin{aligned} \mathbb{P}(M(T_u) > u) &\leq \mathbb{P}\left(M(T_u^-) > u\right) \leq \mathbb{P}\left(Q_e + X(T_u^-) \geq u + T_u^-\right) \\ &\leq \mathbb{P}\left(Q_e > (1-\varepsilon)(u + T_u^-)\right) + \mathbb{P}\left(X(T_u^-) > (1-\varepsilon)(u + T_u^-)\right) \\ &\quad + \mathbb{P}\left(Q_e > \varepsilon(u + T_u^-)\right) \mathbb{P}\left(X(T_u^-) > \varepsilon(u + T_u^-)\right) \\ &=: \pi_1^+(u) + \pi_2^+(u) + \pi_3^+(u). \end{aligned}$$

Using (i) of Proposition 2 and the strong law of large numbers for $X$, it is easy to show that $\pi_3^+(u) = o(\pi_1^+(u))$ for a fixed $\varepsilon$. It is standard now to show that

$$\lim_{\varepsilon \to 0} \limsup_{u \to \infty} \frac{\pi_1^+(u)}{\mathbb{P}(Q_e > u + T_u)} = 1.$$

Moreover,

$$\lim_{\varepsilon \to 0} \limsup_{u \to \infty} \frac{\pi_2^+(u)}{T_u \mathbb{P}(X(1) > u + T_u)} = 1,$$

due to item (i) in Proposition 2. This establishes the upper bound.

*Lower bound.* As for the lower bound observe that

$$\begin{aligned} \mathbb{P}(M(T_u) > u) &\geq \mathbb{P}\left(M(T_u^+) > u\right) \geq \mathbb{P}\left(Q_e + K(T_u^+) > u, X(T_u^+) - T_u^+ - K(T_u^+) < \varepsilon T_u^+\right) \\ &\geq \mathbb{P}\left(Q_e + X(T_u^+) > u + (1+\varepsilon) T_u^+\right) \mathbb{P}\left(X(T_u^+) - T_u^+ - K(T_u^+) < \varepsilon T_u^+\right) \\ &=: \pi_1^-(u) \pi_2^-(u). \end{aligned}$$

By Proposition 1, $\pi_2^-(u) \to 1$ as $u \to \infty$. Also,

$$\begin{aligned} \pi_1^-(u) &\geq \mathbb{P}\left(Q_e + X(T_u^+) - \varepsilon T_u^+/2 > u + (1+\varepsilon/2) T_u^+, X(T_u^+) > -\varepsilon T_u^+/2\right) \\ &\geq \mathbb{P}\left(\{Q_e > u + (1+\varepsilon/2) T_u^+, X(T_u^+) > -\varepsilon T_u^+/2\} \cup \{X(T_u^+) > u + (1+\varepsilon/2) T_u^+\}\right) \\ &= \mathbb{P}\left(Q_e > u + (1+\varepsilon/2) T_u^+\right) \mathbb{P}\left(X(T_u^+) > -\varepsilon T_u^+/2\right) + \mathbb{P}\left(X(T_u^+)) > u + (1+\varepsilon/2) T_u^+\right) \\ &\quad - \mathbb{P}\left(Q_e > u + (1+\varepsilon/2) T_u^+\right) \mathbb{P}\left(X(T_u^+) > u + (1+\varepsilon/2) T_u^+\right) \\ &=: \pi_3^-(u) \pi_4^-(u) + \pi_5^-(u) - \pi_6^-(u) \pi_7^-(u), \end{aligned}$$

where we again used that $Q(0) =_{\rm d} Q_e$ and $\{X(t) : t \geq 0\}$ are independent. By the strong law of large numbers, $\pi_4^-(u) \to 1$ as $u \to \infty$. Moreover, it is easy to show that $\pi_6^-(u)\pi_7^-(u) = o(\pi_3^-(u))$. Now the lower bound follows by noting that

$$\lim_{\varepsilon \downarrow 0} \liminf_{u \to \infty} \frac{\pi_3^-(u)}{\mathbb{P}\left(Q_e > u + T_u\right)} = 1,$$

and that (i) of Proposition 2 yields

$$\lim_{\varepsilon \downarrow 0} \liminf_{u \to \infty} \frac{\pi_5^-(u)}{T_u \, \mathbb{P}\left(X(1) > u + T_u\right)} = 1.$$

This completes the proof. □

3.1.1. *Stable Lévy processes.* Following the notation from [19], let $S_\alpha(\sigma, \beta, \mu)$ be a stable law with index $\alpha \in (0, 2)$, scale parameter $\sigma > 0$, skewness parameter $\beta \in [-1, 1]$ and drift $\mu \in \mathbb{R}$. We call $X$ an $(\alpha, \beta)$-stable Lévy process if $X$ is a Levy process and $X(1)$ has the same distribution as $S_\alpha(1, \beta, 0)$.
Let

$$B(\alpha, \beta) := \frac{\Gamma(1+\alpha)}{\pi} \sqrt{1 + \beta^2 \tan^2\left(\frac{\pi\alpha}{2}\right)} \sin\left(\frac{\pi\alpha}{2} + \arctan\left(\beta \tan\left(\frac{\pi\alpha}{2}\right)\right)\right),$$

and let $X$ be an $(\alpha, \beta)$-stable Lévy process with $\alpha \in (1, 2)$ and $\beta \in (-1, 1]$. Then,

$$\mathbb{P}\left(X(1) > u\right) \sim \frac{B(\alpha, \beta)}{\alpha} u^{-\alpha},$$

see, e.g., [18, Prop. 2.1]. Now Theorem 4 can be rephrased as follows.

**Corollary 2.** *For an $(\alpha, \beta)$-stable Lévy process $X$ with $\alpha \in (1, 2)$ and $\beta \in (-1, 1]$,*

$$\mathbb{P}\left(M(T_u) > u\right) \sim \begin{cases} \frac{1}{\alpha-1} \frac{B(\alpha,\beta)}{\alpha} u^{1-\alpha} & \text{when} \quad T_u = o(u), \\ \frac{A+\alpha}{\alpha-1} (A+1)^{-\alpha} \frac{B(\alpha,\beta)}{\alpha} {T_u}^{1-\alpha} & \text{when} \quad u \sim A T_u, \\ \frac{\alpha}{\alpha-1} \frac{B(\alpha,\beta)}{\alpha} {T_u}^{1-\alpha} & \text{when} \quad u = o(T_u), \end{cases}$$

*as $u \to \infty$.*

3.2. **Light-tailed case.** In this subsection, we consider the light-tailed situation, also frequently referred to as the Cramér case. Throughout, with $\phi(\vartheta) := \log \mathbb{E} \exp(\vartheta X(1))$ denoting the *cumulant function*, we impose the following assumption.

**Assumption 2.** *Let*

$$\beta^\star := \sup\{\beta : \mathbb{E} e^{\beta X(1)} < \infty\}$$

*Assume that $\beta^\star > 0$ and there exists $\vartheta^\star \in (0, \beta^\star)$, such that $\phi(\vartheta^\star) = \vartheta^\star$. Moreover, assume that* 0 *is* regular *for $X$, that is, $\mathbb{P}\left(\inf\{t > 0 : X(t) > 0\} = 0\right) = 1$.*

For $r \geq 0$, define

$$I(r) := \sup_{\vartheta > 0} \left(\vartheta r - \phi(\vartheta)\right).$$

**Proposition 3.** *Under Assumption 2, the following statements hold.*
*(i) As $u \to \infty$,*

$$\log \mathbb{P}\left(Q_e > u\right) \sim -\vartheta^\star u.$$

*(ii) For all $u > 0$,*

$$\mathbb{P}\left(Q_e > u\right) \leq e^{-\vartheta^\star u}.$$

*(iii) The function $I$ obeys*

$$I(1) < \infty \quad \textit{and} \quad I'(1) \leq \vartheta^\star.$$

*(iv) For any $\varepsilon > 0$,*

$$\liminf_{u\to\infty} \frac{1}{u} \log \mathbb{P}\left(K(u) > -\varepsilon u\right) \geq -I(1)$$

*Proof.* For (i) and (ii), we refer to [6]. For (iii), notice that $I(1) = \sup_{\vartheta>0}(\vartheta - \psi(\vartheta))$ is attained for $\vartheta \in (0, \vartheta^\star)$; therefore also $I'(1) \leq \vartheta^\star$. As for (iv), observe that

$$\mathbb{P}\left(K(u) > -\varepsilon u\right) = \mathbb{P}\left(\frac{Y(u\,\cdot)}{u} \in A_\varepsilon\right),$$

where

$$A_\varepsilon := \{f \in D[0,1] : f(t) > -\varepsilon, \forall t \in [0,1]\}$$

and $D[0,1]$ is the space of càdlàg functions on $[0,1]$. Using sample-path large deviations results for Lévy processes, see [11, Theorems 5.1 and 5.2], we now obtain that

$$\liminf_{u\to\infty} \frac{1}{u} \log \mathbb{P}\left(K(u) > -\varepsilon u\right) \geq -\inf\{\psi(f) : f \in A_\varepsilon \cap C[0,1]\},$$

where $\psi(f) := \int_0^1 I(f'(t) + 1)\,\mathrm{d}t$. Now observe that the path $f^\star \equiv 0$ is in $A_\varepsilon$. The stated follows by realizing that $\psi(f^\star) = I(1)$. □

Now we can proceed with the main result of this subsection.

**Theorem 5.** *Assume that the Lévy process $X$ satisfies Assumption 2. Then*

$$\log \mathbb{P}\left(M(T_u) > u\right) \sim -u\vartheta^\star - T_u I(1), \quad \textit{as} \quad u \to \infty.$$

The asymptotics in Theorem 5 can trivially be made more explicit by comparing both exponential decay rates. The intuition behind the following corollary is that, in large deviations language, the most likely path corresponding to the rare event under study first builds up from an empty system to level $u$ (at time 0), and then remains at level $u$ for the nest $T_u$ time units; both parts of the path result in both contributions to the decay rate (i.e., $-u\vartheta^\star$ and $-T_u I(1)$). Then, depending on whether $T_u$ is small or large with respect to $u$, one of these two contributions dominates.

**Corollary 3.** *Assume that the Lévy process $X$ satisfies Assumption 2. Then*

$$\mathbb{P}\left(M(T_u) > u\right) \sim \begin{cases} -u\vartheta^\star & \textit{when} \quad T_u = o(u), \\ -T_u(A\vartheta^\star + I(1)) & \textit{when} \quad u \sim AT_u, \\ -T_u I(1) & \textit{when} \quad u = o(T_u), \end{cases}$$

*as $u \to \infty$.*

**Proof of Theorem 5.** The proof again consists of two bounds.

*Lower bound.* Observe that the probability of interest is, for any $\varepsilon > 0$, bounded from below by

$$\mathbb{P}\left(Q_e > u + \varepsilon T_u\right) \mathbb{P}\left(K(T_u) > -\varepsilon T_u\right).$$

Now the lower bound follows by combining parts (i) and (iv) of Proposition 3, and then sending $\varepsilon \downarrow 0$.

*Upper bound.* Observe that

$$K(t) \leq \left(X(t) - t\right) \mathbb{I}(t),$$

where $\mathbb{I}(t)$ denotes the indicator function $1_{\{X(t)/t\in(0,1)\}}$. Thus,

$$\begin{aligned}\mathbb{P}\left(M(T_u)>u\right) &\leq \mathbb{P}\left(Q_e+(X(T_u)-T_u)\mathbb{I}(T_u)>u\right)\\ &= \int_{\mathbb{R}}\mathbb{P}\left(Q_e>u-xT_u+T_u\mathbb{I}()\right)\mathrm{d}\mathbb{P}\left(\frac{X(T_u)}{T_u}\mathbb{I}(T_u)\leq x\right)\\ &= \int_0^1 \mathbb{P}\left(Q_e>u-xT_u+T_u\right)\mathrm{d}\mathbb{P}\left(\frac{X(T_u)}{T_u}\mathbb{I}(T_u)\leq x\right)\\ &\leq e^{-\vartheta^\star u}\int_0^1 e^{-\vartheta^\star T_u(1-x)}\mathrm{d}\mathbb{P}\left(\frac{X(T_u)}{T_u}\mathbb{I}(T_u)\leq x\right),\end{aligned}$$

where the last inequality follows from part (ii) of Proposition 3. The sequence $\{X(u)\mathbb{I}(u)/u\}$ satisfies the large deviations principle on $((0,1),\mathcal{B}(0,1))$ with rate $u$ and rate function $I(\cdot)$. Thus, Varadhan's Lemma [12, Theorem. 4.3.1] implies

$$\lim_{u\to\infty}\frac{1}{T_u}\log\int_0^1 e^{-\vartheta^\star T_u(1-x)}\mathrm{d}\mathbb{P}\left(\frac{X(T_u)}{T_u}\mathbb{I}(T_u)\leq x\right)=-\inf_{x\in(0,1)}\left(\vartheta^\star(1-x)+I(x)\right)=I(1),$$

where the last equality is due to part (iii) of Proposition 3 and convexity of $I(\cdot)$. □

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