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**Tenor Specific Pricing**

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# Tenor Specific Pricing

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## Abstract

Observing that pure discount curves are now based on a variety of tenors giving rise to tenor specific zero coupon bond prices, the question is raised on how to construct tenor specific prices for all financial contracts. Noting that in conic finance one has the law of two prices, bid and ask, that are nonlinear functions of the random variables being priced, we model dynamically consistent sequences of such prices using the theory of nonlinear expectations. The latter theory is closely connected to solutions of backward stochastic difference equations. The drivers for these stochastic difference equations are here constructed using concave distortions that implement risk charges for local tenor specific risks. It is then observed that tenor specific prices given by the mid quotes of bid and ask converge to the risk neutral price as the tenor is decreased and liquidity increased when risk charges are scaled by the tenor. Square root tenor scaling can halt the convergence to risk neutral pricing, preserving bid ask spreads in the limit. The greater liquidity of lower tenors may lead to an increase or decrease in prices depending on whether the lower liquidity of a higher tenor has a mid quote above or below the risk neutral value. Generally for contracts with a large upside and a bounded downside the prices fall with liquidity while the opposite is the case for contracts subject to a large downside and a bounded upside.

## 1 Introduction

The aftermath of the financial crisis of 2008 has brought with it the existence of tenor specific yield curves. The existence of these curves was first brought to our attention by Mercurio (2010a, 2010b). We now have explicit constructions of discount curves or zero coupon bond prices at different tenors. Somewhat more precisely, there is the *OIS* curve, along with the one, three, six and twelve month curves. Differences reflect the fact, for example, that the first three consecutive one month forward rates when compounded fall short of the first three month forward rate. In the past these differences also existed but the gaps were small and possibly well within bid ask bounds. Following the crisis of 2008 the differences have become quite substantial, leading financial institutions to explicitly construct tenor specific discount curves.

The simplest of all contracts is the pure discount bond and if its price is tenor specific then the same probably holds for other more complicated claims like stocks, and derivatives on underlying stock prices. The question we address in this paper are the theoretical foundations for tenor specific pricing, what possibly do these prices mean, and can we develop procedures for the explicit theoretical computation of tenor specific prices on all contracts.

It is clear that with multiple prices for pure discount bonds among other assets, the law of one price is abandoned, or preserved depending on how we see the tenor specific price. Some argue that longer tenors embody higher credit risk and hence are not the same cash flow (Morini (2008)). The higher rates on longer tenors being the compensation for the additional credit exposure.

However, shorter tenors are in some sense more liquid and the higher price could just reflect the value of the additional liquidity. The possibility that both credit and liquidity considerations may be simultaneously involved is recognized for example in Mercurio (2009). In this paper we develop a liquidity based model for tenor specific yield curves that can then be applied to other assets as well. Other approaches to multiple yield curves include for example Kijima, Tanaka, and Wong (2008).

Our approach to liquidity modeling builds on the two price model of markets introduced in Cherny and Madan (2010). The two price model of markets takes the market to be an abstract counterparty for all financial transactions by economic agents. As a counterparty, the market takes the otherside of all transactions and this typically involves holding the opposite risk position to the maturity of the contingent claim or holding it for the length of the period in a static one period model. Unlike economic agents who optimize their objectives, the market is a passive counterparty that accepts certain cash flows. The market will take a nonnegative cash flow and more generally is modeled as accepting a convex cone of cash flows that contains the nonnegative cash flows. The underlying probability under which the market evaluates the possibilities is given by a single market selected risk neutral measure. The set of acceptable cash flows are then those that have a positive expectation with respect to a collection of test measures that are equivalent to the selected base risk neutral measure (Artzner, Delbaen, Eber and Heath (1999), Carr Geman and Madan (2001), Jaschke and Kuchler (2001)). We may denote this class of test measures  $\mathcal{M}$  with elements  $Q \in \mathcal{M}$ . The base risk neutral measure is denoted  $Q^0$  and the measures  $Q \in \mathcal{M}$  are equivalent to  $Q^0$ .

It is shown in Cherny and Madan (2010) that the ask price  $a(X)$ , respectively bid price  $b(X)$  for a potentially hedged cash flow  $X$  are then given by the supremum, respectively infimum, over all  $Q \in \mathcal{M}$  of the expectation under  $Q$  of  $X$ , or

$$\begin{aligned} a(X) &= \sup_{Q \in \mathcal{M}} E^Q[X] \\ b(X) &= \inf_{Q \in \mathcal{M}} E^Q[X]. \end{aligned}$$

It is further argued in Carr, Madan and Vicente Alvarez (2010) that one may take the mid quote as a candidate for a two way price in such two price markets. Typically this mid quote price is not equal to the risk neutral expectation and is above or below the risk neutral expectation depending on the nature of the cash flow. For cash flows with a large upside, like out-of-the-money options, the ask price pulls the mid quote above the risk neutral expectation, while the opposite holds true for risky loans with a bounded upside and a large downside risk exposure. The difference between the mid quote and the risk neutral expectation is taken as profit in Carr, Madan and Vicente Alvarez (2010). The underlying model in Cherny and Madan (2010) and Carr, Madan and Vicente Alvarez (2010) is a static one period model.

This paper generalizes these methods to a dynamic model operating at mul-

multiple tenors. For financial transactions it is realized by the market that whenever an economic agent enters a financial transaction they may hold it to maturity or to some horizon or they may seek to reverse the transaction before this time. The market then asks the economic agents as to what are the frequencies at which they wish to consider a possible reversal. The economic agents then ask the market, what are the tenor specific exit prices at different tenors expressed as functions of the underlying risks whose future values are currently unknown. What we provide in this paper are model specific procedures determining tenor specific exit prices on a variety of contracts.

We apply the recently developed theory of nonlinear expectations (Shige Peng (2004), Rosazza Gianin (2006)) to construct dynamically consistent sequences of bid and ask prices on multiple tenors. Our tenor specific exit price schedules are then given by the mid quotes of these sequences. The shorter the tenor, the more liquid the pricing and this leads us to the construction of liquidity contingent pricing. The limiting prices may or may not be risk neutral depending on the scaling employed in risk charges.

The outline of the rest of the paper is as follows. Section 2 presents evidence on tenor specific yield curves or pure discount bond prices post crisis. Section 3 presents a theoretical determination of tenor specific pricing of financial claims via nonlinear expectations. Section 4 introduces drivers for nonlinear expectations based on concave distortions. Section 5 presents the computations in a simple binomial context. Section 6 develops tenor specific discount curves when the underlying spot rate process satisfies the Cox, Ingersoll and Ross square root process. Section 7 reports on tenor specific stock prices when the underlying risk is geometric Brownian motion or the variance gamma process. Section 8 takes up tenor specific option pricing. Section 9 reports on the pricing of stocks and options on stocks under square root tenor scaling. Section 10 concludes.

## 2 Tenor Specific Yield Curves

Most banks post the crisis of 2008 construct pure discount curves using as base instruments fixed income contracts like certificates of deposit, forward rate agreements, futures contracts, and swaps to build discount curves at a variety of tenors, with the most popular ones being the *OIS* curve for the daily tenor, followed by tenors of 1, 3, 6 and 12 months. By way of an example we present in Figure (1) the gap in basis points between the pure discount price of maturity  $t$  on a tenor above *OIS* and the *OIS* price on December 15 2010. The price gap is almost 200 basis points near a ten year maturity.

From this data one may also construct the spread between forward rates on the higher tenors and the *OIS* forward rate. Figure (2) presents a graph of these spreads at various maturities. The spread in the forward rates reach up to 70 basis points.

We have to ask ourselves what these prices are and what is their basis. A possibility is that the differences are credit related, but the instruments employed are quite varied with multiple counterparties and it is unclear that the biases

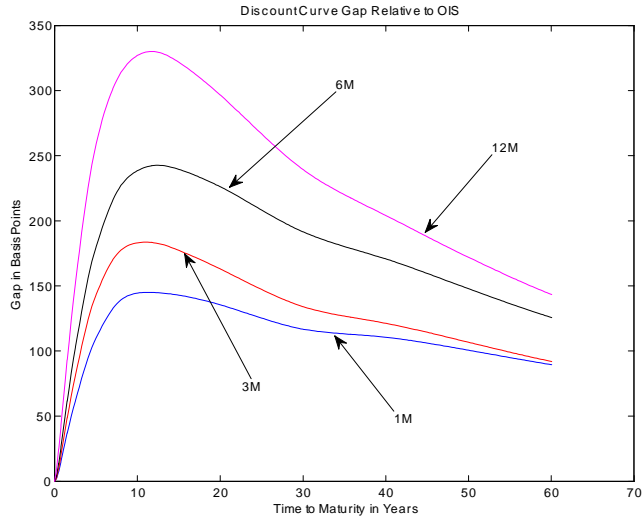


Figure 1: Zero coupon bond prices at tenors of one, three, six and twelve months less the OIS price in basis points.

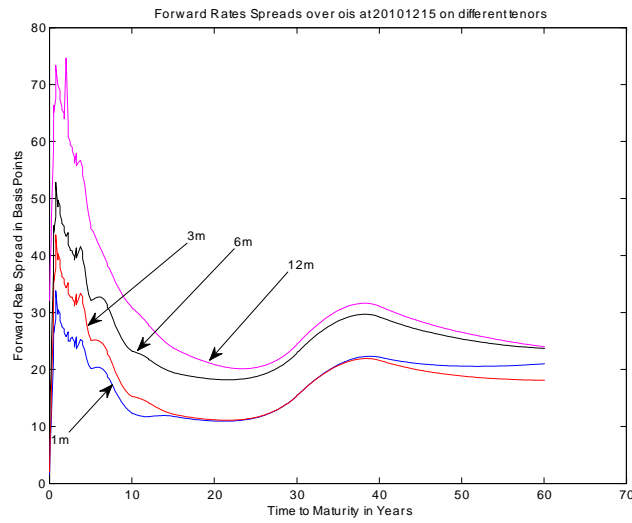


Figure 2: Forward Rate spreads at various tenors over OIS forward rates.

built in are purely credit related. For example, Eberlein, Madan and Schoutens (2011) show using a joint model of credit and liquidity that the Lehman default was a liquidity event for the remaining banks and not a credit issue. Certainly lower tenors represent a greater liquidity so might the difference be to some extent due to this enhanced liquidity. How does liquidity expressed via a lower trading tenor theoretically affect prices. These are the questions we now address.

### 3 Theoretical Tenor Specific Pricing

This section develops the theory for tenor specific pricing in general. However, to focus attention we begin with the simplest security of the pure discount bond. All economic agents must trade with the market and in line with the principles of conic finance the market serves as the passive counterparty for all financial transactions. The market is aware of a single risk neutral instantaneous spot rate process  $r = (r(t), t \geq 0)$  at which funds may be transferred by the market through time. Suppose for simplicity that the underlying process for  $r$  is a one dimensional Markov process.

Consider in this context the desire by an economic agent to buy from the market a unit face pure discount bond of unit maturity. If the market fixes the ask price at  $a$ , the market holds the random present value cash flow of

$$X(0, 1) = a - e^{-\int_0^1 r(u)du}.$$

The economic agent could hold the bond for unit time and then collect the unit face value. If the market prices this contract to acceptability using a convex set of test measures  $\mathcal{M}$  then the ask price is given by

$$a = \sup_{Q \in \mathcal{M}} E^Q \left[ e^{-\int_0^1 r(u)du} \right]$$

while the bid price is

$$b = \inf_{Q \in \mathcal{M}} E^Q \left[ e^{-\int_0^1 r(u)du} \right],$$

and the mid quote or the reference two way price is the average of the bid and ask prices.

Suppose now the economic agent wishes to have from the only market he or she must trade with, the opportunity to unwind this position at some earlier date and he or she wishes to see the terms at which this unwind may be possible. Essentially the economic agent asks the market for a schedule of bid and ask prices as functions of the prevailing spot rate at a frequency of  $h = 1/N$ . For  $N = 4$  we have a quarterly schedule while  $N = 12$  yields a monthly schedule.

The market then has to first determine the bid and ask prices at time  $1 - h$ . At this time the present value of the risk  $X^a, X^b$  for an ask respectively bid price is

$$\begin{aligned} X^a(1 - h, 1) &= a - e^{-\int_{1-h}^1 r(u)du} \\ X^b(1 - h, 1) &= e^{-\int_{1-h}^1 r(u)du} - b \end{aligned}$$

If the market uses the same cone of acceptability then these ask and bid prices are

$$\begin{aligned} a_{1-h}(r(1-h)) &= \sup_{Q \in \mathcal{M}} E^Q \left[ e^{-\int_{1-h}^1 r(u) du} \right] \\ b_{1-h}(r(1-h)) &= \inf_{Q \in \mathcal{M}} E^Q \left[ e^{-\int_{1-h}^1 r(u) du} \right]. \end{aligned}$$

In principle this schedule may be computed. We now consider the determination of the schedule at the next time step of  $1-2h$ .

Now the market is ready to sell for  $a(1-h)$  at time  $1-h$  and we ask what price is the market willing to sell for at time  $1-2h$ . If it sells for  $a$  at time  $1-2h$  we have the present value cash flow at time  $1-h$  of earning the interest and buying back at  $1-h$  for the ask price of  $a_{1-h}(r(1-h))$ .

$$X^a(1-2h, 1-h) = a - a_{1-h}(r(1-h))e^{-\int_{1-2h}^{1-h} r(u) du}$$

The corresponding bid cash flow is

$$X^b(1-2h, 1-h) = b_{1-h}(r(1-h))e^{-\int_{1-2h}^{1-h} r(u) du} - b$$

It follows from making these risks acceptable that

$$\begin{aligned} a_{1-2h}(r(1-2h)) &= \sup_{Q \in \mathcal{M}} E^Q \left[ a_{1-h}(r(1-h))e^{-\int_{1-2h}^{1-h} r(u) du} \right] \\ b_{1-2h}(r(1-2h)) &= \inf_{Q \in \mathcal{M}} E^Q \left[ b_{1-h}(r(1-h))e^{-\int_{1-2h}^{1-h} r(u) du} \right] \end{aligned}$$

We thus get the ask and bid recursions on tenor  $h$  of

$$\begin{aligned} a_h(t-h) &= \sup_{Q \in \mathcal{M}} \left( E^Q \left[ e^{-\int_{t-h}^t r(u) du} a_h(t) \right] \right) \\ b_h(t-h) &= \inf_{Q \in \mathcal{M}} \left( E^Q \left[ e^{-\int_{t-h}^t r(u) du} b_h(t) \right] \right) \end{aligned}$$

The tenor specific discount curve is then given by the time zero mid quotes computed on each tenor  $h$  as

$$m_h(T) = \frac{a_h(0) + b_h(0)}{2}.$$

The spreads between different tenors arise in these computations from liquidity considerations embedded in the cones of acceptable risks. They are not credit related as we do not have any defaults but just a reluctance to take exposures. Observe that if we go back to the law of one price with a base risk neutral measure  $Q^0$  we may rewrite the recursion as

$$b_h(t-h) = E^{Q^0} \left[ e^{-\int_{t-h}^t r(u) du} b_h(t) \right] + \inf_{Q \in \mathcal{M}} \left( E^Q \left[ e^{-\int_{t-h}^t r(u) du} b_h(t) \right] - E^{Q^0} \left[ e^{-\int_{t-h}^t r(u) du} b_h(t) \right] \right)$$



where we have the one step ahead expectation plus a risk charge based on the deviation. These risk charges are for exposure to unhedgeable risk and the motivations are related to those considered for example by Bernardo and Ledoit (2000), Cochrane and Saa-Requejo (2000), Černý and Hodges (2000), Carr, Geman and Madan (2001) and Jaschke and Kuchler (2001).

This risk charge is for exposure to deviations and could in principle be the same for two different tenors. However, the charge is for a risk exposure over an interval of length  $h$  and should be levied as a rate per unit time with the charge for  $h$  units of time being proportional to  $h$ . Hence the recursion employed for the tenor  $h$  is

$$\begin{aligned} b_h(t-h) &= E^{Q^0} \left[ e^{-\int_{t-h}^t r(u)du} b_h(t) \right] + h \inf_{Q \in \mathcal{M}} \left( E^Q \left[ e^{-\int_{t-h}^t r(u)du} b_h(t) \right] - E^{Q^0} \left[ e^{-\int_{t-h}^t r(u)du} b_h(t) \right] \right) \\ a_h(t-h) &= E^{Q^0} \left[ e^{-\int_{t-h}^t r(u)du} a_h(t) \right] + h \sup_{Q \in \mathcal{M}} \left( E^Q \left[ e^{-\int_{t-h}^t r(u)du} a_h(t) \right] - E^{Q^0} \left[ e^{-\int_{t-h}^t r(u)du} a_h(t) \right] \right) \end{aligned}$$

Section 9 briefly investigates scaling by the square root of the tenor. One may observe from binomial model computations in Section 5.2 below that the risk charge at a time step of  $1/n$  is proportional to the square root of  $1/n$ . Since this is summed over  $n$  terms in a portion of unit time, the resulting total risk charges would go to infinity with no time scaling. Scaling by  $1/n$  sends the limiting risk charge to zero with convergence of prices to risk neutral values. Square root scaling is motivated and reported on in section 9.

The resulting bid and ask price sequences are dynamically consistent nonlinear expectations operators associated with the solution of backward stochastic difference equations. We have presented them here without reference to this underlying framework. To establish this connection we first briefly review these concepts and the connection between them as they have been established in the literature.

In the context of a discrete time finite state Markov chain with states  $e_i$  identified with the unit vectors of  $\mathbb{R}^M$  for some large integer  $M$ , Cohen and Elliott (2010) have defined dynamically consistent translation invariant nonlinear expectation operators  $\mathcal{E}(\cdot|\mathcal{F}_t)$ . The operators are defined on the family of subsets  $\{\mathbb{Q}_t \subset L^2(\mathcal{F}_T)\}$ . For completeness we recall here this definition of an  $\mathcal{F}_t$ -consistent nonlinear expectation for  $\{\mathbb{Q}_t\}$ . This  $\mathcal{F}_t$ -consistent nonlinear expectation for  $\{\mathbb{Q}_t\}$  is a system of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t), \quad 0 \leq t \leq T$$

satisfying the following properties:

1. For  $Q, Q' \in \mathbb{Q}_t$ , if  $Q \geq Q'$   $\mathbb{P}$ -a.s. componentwise, then

$$\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(Q'|\mathcal{F}_t)$$

$\mathbb{P}$ -a.s. componentwise, with for each  $i$ ,

$$e_i \mathcal{E}(Q|\mathcal{F}_t) = e_i \mathcal{E}(Q'|\mathcal{F}_t)$$

only if  $e_i Q = e_i Q'$   $\mathbb{P}$ -a.s.

2.  $\mathcal{E}(Q|\mathcal{F}_t) = Q$   $\mathbb{P}$ -a.s. for any  $\mathcal{F}_t$ -measurable  $Q$ .

3.  $\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$   $\mathbb{P}$ -a.s. for any  $s \leq t$

4. For any  $A \in \mathcal{F}_t$ ,  $\mathbf{1}_A \mathcal{E}(Q|\mathcal{F}_t) = \mathcal{E}(\mathbf{1}_A Q|\mathcal{F}_t)$   $\mathbb{P}$ -a.s.

Furthermore the system of operators is dynamically translation invariant if for any  $Q \in L^2(\mathcal{F}_T)$  and any  $q \in L^2(\mathcal{F}_t)$ ,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

Such dynamically consistent translation invariant nonlinear expectations may be constructed from solutions of Backward Stochastic Difference and or Differential Equations (Cohen and Elliott (2010), El Karoui and Huang (1997)). These are equations to be solved simultaneously for processes  $Y, Z$  where  $Y_t$  is the nonlinear expectation and the pair  $(Y, Z)$  satisfy

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \leq u < T} Z_u M_{u+1} = Q$$

for a suitably chosen adapted map  $F : \Omega \times \{0, \dots, T\} \times \mathbb{R}^K \times \mathbb{R}^{K \times N} \rightarrow \mathbb{R}^K$  called the driver and for  $Q$  an  $\mathbb{R}^K$  valued  $\mathcal{F}_T$  measurable terminal random variable. We shall work in this paper generally with the case  $K = 1$ . For all  $t$ ,  $(Y_t, Z_t)$  are  $\mathcal{F}_t$  measurable. Furthermore for a translation invariant nonlinear expectation the driver  $F$  must be independent of  $Y$  and must satisfy the normalisation condition  $F(\omega, t, Y_t, 0) = 0$ .

The drivers of the backward stochastic difference equations are the risk charges and for our ask and bid price sequences at tenor  $h$  we have drivers  $F^a, F^b$  where

$$\begin{aligned} F^a(\omega, u, Y_u, Z_u) &= h \sup_{Q \in \mathcal{M}} E^Q [Z_u M_{u+1}] \\ F^b(\omega, u, Y_u, Z_u) &= h \inf_{Q \in \mathcal{M}} E^Q [Z_u M_{u+1}], \end{aligned}$$

and the drivers are independent of  $Y$ . The process  $Z_t$  represents the residual risk in terms of a set of spanning martingale differences  $M_{u+1}$  and in our applications we solve for the nonlinear expectations  $Y_t$  without in general identifying either  $Z_t$  or the set of spanning martingale differences. We define risk charges directly for the risk defined for example as the zero mean random variable

$$e^{-\int_{t-h}^t r(u) du} a_h(t) - E^{Q^0} \left[ e^{-\int_{t-h}^t r(u) du} a_h(t) \right].$$

Leaving aside pure discount bonds we may consider for example a one year call option written on a forward or futures price  $S(t)$  with zero risk neutral drift, unit maturity, strike  $K$  and payoff

$$(S(1) - K)^+.$$

Dynamically consistent forward bid and ask price sequences on the tenor  $h$  may be constructed as nonlinear expectations starting with

$$a(1) = b(1) = (S(1) - K)^+.$$

Thereafter we apply the recursions

$$\begin{aligned} a_{t-h}(S(t-h)) &= E^{Q^0} [a_t(S(t))] + h \sup_{Q \in \mathcal{M}} \left( E^Q \left[ a_t(S(t)) - E^{Q^0} [a_t(S(t))] \right] \right) \\ b_{t-h}(S(t-h)) &= E^{Q^0} [b_t(S(t))] + h \inf_{Q \in \mathcal{M}} \left( E^Q \left[ b_t(S(t)) - E^{Q^0} [b_t(S(t))] \right] \right) \end{aligned}$$

Similar recursions apply to put options and other functions of the terminal stock price. For path dependent claims of an underlying Markov process with payoff on tenor  $h$  of

$$F((S(jh), 0 \leq j \leq J)) = V_j^a = V_j^b$$

We determine the ask and bid value of the remaining uncertainty  $V_j^a(S(jh)), V_j^b(S(jh))$  by the recursions

$$\begin{aligned} V_j^a(S(jh)) &= E^{Q^0} [F(S(kh), 0 \leq k \leq j+1) - F(S(kh), 0 \leq k \leq j) + V_{j+1}^a(S((j+1)h))] \\ &+ h \sup_{Q \in \mathcal{M}} E^Q \left[ \begin{array}{l} F(S(kh), 0 \leq k \leq j+1) - F(S(kh), 0 \leq k \leq j) + V_{j+1}^a(S((j+1)h)) \\ -E^{Q^0} [F(S(kh), 0 \leq k \leq j+1) - F(S(kh), 0 \leq k \leq j) + V_{j+1}^a(S((j+1)h))] \end{array} \right] \end{aligned}$$

The ask value of the claim is then

$$F(S(kh), 0 \leq k \leq j) + V_j^a(S(jh)).$$

Similarly for the bid we have

$$\begin{aligned} V_j^b(S(jh)) &= E^{Q^0} [F(S(kh), 0 \leq k \leq j+1) - F(S(kh), 0 \leq k \leq j) + V_{j+1}^b(S((j+1)h))] \\ &+ h \inf_{Q \in \mathcal{M}} E^Q \left[ \begin{array}{l} F(S(kh), 0 \leq k \leq j+1) - F(S(kh), 0 \leq k \leq j) + V_{j+1}^b(S((j+1)h)) \\ -E^{Q^0} [F(S(kh), 0 \leq k \leq j+1) - F(S(kh), 0 \leq k \leq j) + V_{j+1}^b(S((j+1)h))] \end{array} \right] \end{aligned}$$

and the bid value of the claim is

$$F(S(kh), 0 \leq k \leq j) + V_j^b(S(jh)).$$

Tenor specific values may be constructed for a vast array of financial claims using the procedures developed for nonlinear expectations after the selection of an appropriate driver. The lower the tenor or the greater the frequency of quotations the more the liquidity that is being offered to economic agents. One might enquire into the nature of the limiting price associated with various drivers. These interesting questions are left for a future research effort. For results in a diffusion context in this direction we refer to Stadje (2010). For the moment we investigate the resulting tenor specific prices for bonds, stocks and options on stocks in a variety of contexts for a specific set of drivers based on distortions.

## 4 Drivers for nonlinear expectations based on distortions

The driver for a translation invariant nonlinear expectation is basically a positive risk charge for the ask price and a positive risk shave for a bid price applied to a zero mean risk exposure to be held over an interim. We are then given as input the risk exposure ideally spanned by some martingale differences as  $Z_u M_{u+1}$  or alternatively a zero mean random variable  $X$  with a distribution function  $F(x)$ . Cherny and Madan (2010) have constructed in the context of a static model law invariant bid and ask prices based on concave distortions. The bid and prices for a local exposure are then defined in terms of a concave distribution function  $\Psi(u)$  defined on the unit interval as

$$\begin{aligned} b &= \int_{-\infty}^{\infty} x d\Psi(F(x)) \\ a &= - \int_{-\infty}^{\infty} x d\Psi(1 - F(-x)). \end{aligned}$$

It is shown in Cherny and Madan (2010) that the set  $\mathcal{M}$  of test measures seen as measure changes on the unit interval applied to  $G(u) = F^{-1}(u)$  are all densities  $Z(u)$  with respect to Lebesgue measure for which the antiderivative  $H' = Z$  is distortion bounded, or  $H \leq \Psi$ .

We consider in the rest of the paper drivers based on the distortion *minmaxvar*. In this case

$$\begin{aligned} F^b(Z_u M_{u+1}) &= \int_{-\infty}^{\infty} x d\Psi^\gamma(\Theta(x)) \\ F^a(Z_u M_{u+1}) &= - \int_{-\infty}^{\infty} x d\Psi^\gamma(1 - \Theta(-x)) \\ \Theta(x) &= \Pr(Z_u M_{u+1} \leq x). \end{aligned}$$

The distortion  $\Psi^\gamma(u)$  is given by

$$\Psi^\gamma(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

Importantly it was shown in Carr, Madan and Vicente Alvarez (2010) that for such distortions in general the mid quote lies above the risk neutral expectation if a claim has large exposures at quantiles above the median and low exposures below the median. The opposite is the case for large exposures at the lower quantiles and low ones above. The quantile exposure is measured by the sensitivity or derivative of the inverse of the distribution function. The following sections take up numerical evaluations of tenor specific pricing of discount bonds, stocks and options.

## 5 Tenor Specific Binomial Trees

This section illustrates the computation of nonlinear expectations using distortion based drivers on binomial trees. The first example illustrates how mid quotes constructed from the bid and ask prices of conic finance differ from risk neutral valuations and the use of such mid quotes as candidates for the prices of economies satisfying the law of one price will often display arbitrage opportunities that are not really there. The second example constructs tenor specific prices of tenors one and two on a two period tree. Two subsections cover the two examples.

### 5.1 Mid Quote Arbitrage

Consider a one period two state binomial tree with up and down factors of  $u = 1.1$  and  $d = 0.8$  respectively. For an initial stock price of 100 with zero rates and dividends and the distortion *minmaxvar* at the stress level of 0.5 the bid and ask prices of an at the money call are 3.74 and 8.84 respectively. The corresponding bid and ask put prices are 2.31, 12.52. The risk neutral up probability is

$$p = \frac{1-d}{u-d} = .67.$$

The risk neutral call and put prices are with  $S_u = S_0u$ ,  $S_d = S_0d$

$$\begin{aligned} rC &= p(S_u - S_0) = 6.7 \\ rP &= (1-p)(S_0 - S_d) = 6.7 \end{aligned}$$

The corresponding bid and ask prices for the call are

$$\begin{aligned} bC &= rC + (-rC * \Psi^\gamma(1-p) + (S_u - S_0 - rC) * (1 - \Psi^\gamma(1-p))) = 3.74 \\ aC &= rC + (-rC * (1 - \Psi^\gamma(p)) + (S_u - S_0 - rC) * \Psi^\gamma(p)) = 8.85 \end{aligned}$$

while the put prices are

$$\begin{aligned} bP &= rP + (-rP * \Psi^\gamma(p) + (S_0 - S_d - rP) * (1 - \Psi^\gamma(p))) = 2.31 \\ aP &= rP + (-rP * (1 - \Psi^\gamma(1-p)) + (S_0 - S_d - rP) * \Psi^\gamma(1-p)) = 12.52. \end{aligned}$$

The mid quote for the call and the put are

$$\begin{aligned} mC &= 6.29 \\ mP &= 7.41 \end{aligned}$$

and the value of the stock using these option midquotes is

$$mC - mP + 100 = 98.88$$

reflecting an arbitrage, but none exists. The cost of getting the stock via options is

$$aC - bP + 100 = 8.85 - 2.31 + 100 = 106.54$$

while the revenue from selling the stock via options is

$$bC - aP + 1 = 3.74 - 12.52 + 100 = 91.22.$$

The use of midquotes as prices for the law of one price may reflect illusory arbitrages that are absent at unseen risk neutral valuations with negative cash flows when spreads are taken into account.

## 5.2 Nonlinear Expectations on Two Period Binomial Trees with Multiple Tenors

We consider here a two period tree with an initial stock price of 100 and up and down factors of 1.1, 0.95 respectively. The risk neutral probability  $p$  of an up move is then  $1/3$ . There are three states at the end of the tree and the bid and ask at the money call prices on a single tenor of two periods with cash flows  $c_0 = 0 < c_1 = 4.5 < c_2 = 21$  and probabilities  $p_0 = (1 - p)^2 = .4444$ ,  $p_1 = 2p(1 - p) = .4444$ ,  $p_2 = p^2 = .1111$  are given by the equations

$$\begin{aligned} bC2 &= c_0 \Psi^\gamma(p_0) + c_1 (\Psi^\gamma(p_0 + p_1) - \Psi^\gamma(p_0)) + c_2 (1 - \Psi^\gamma(p_0 + p_1)) = 1.5569 \\ aC2 &= c_2 \Psi^\gamma(p_2) + c_1 (\Psi^\gamma(p_2 + p_1) - \Psi^\gamma(p_2)) + c_0 (1 - \Psi^\gamma(p_2 + p_1)) = 9.0451. \end{aligned}$$

The mid quote is 5.3010. For the roll over of bid and ask prices in two steps of one period we show the results on a tree. The bid price at time  $1/2$  in the upstate is

$$\begin{aligned} bC1u &= rC1u + .5 \left( (S_0ud - 100)^+ - rC1u \right) \Psi^\gamma(1 - p) + ((S_0u^2 - 100)^+ - rC1u) (1 - \Psi^\gamma(1 - p)) \\ aC1u &= rC1u + .5 \left( (S_0ud - 100)^+ - rC1u \right) (1 - \Psi^\gamma(p)) + ((S_0u^2 - 100)^+ - rC1u) \Psi^\gamma(p) \\ rC1u &= (1 - p) * (S_0ud - 100)^+ + p * (S_0u^2 - 100)^+. \end{aligned}$$

Similarly, one may compute  $bC1d$ ,  $aC1d$ . Finally the equations for the ask price at the root of the tree are

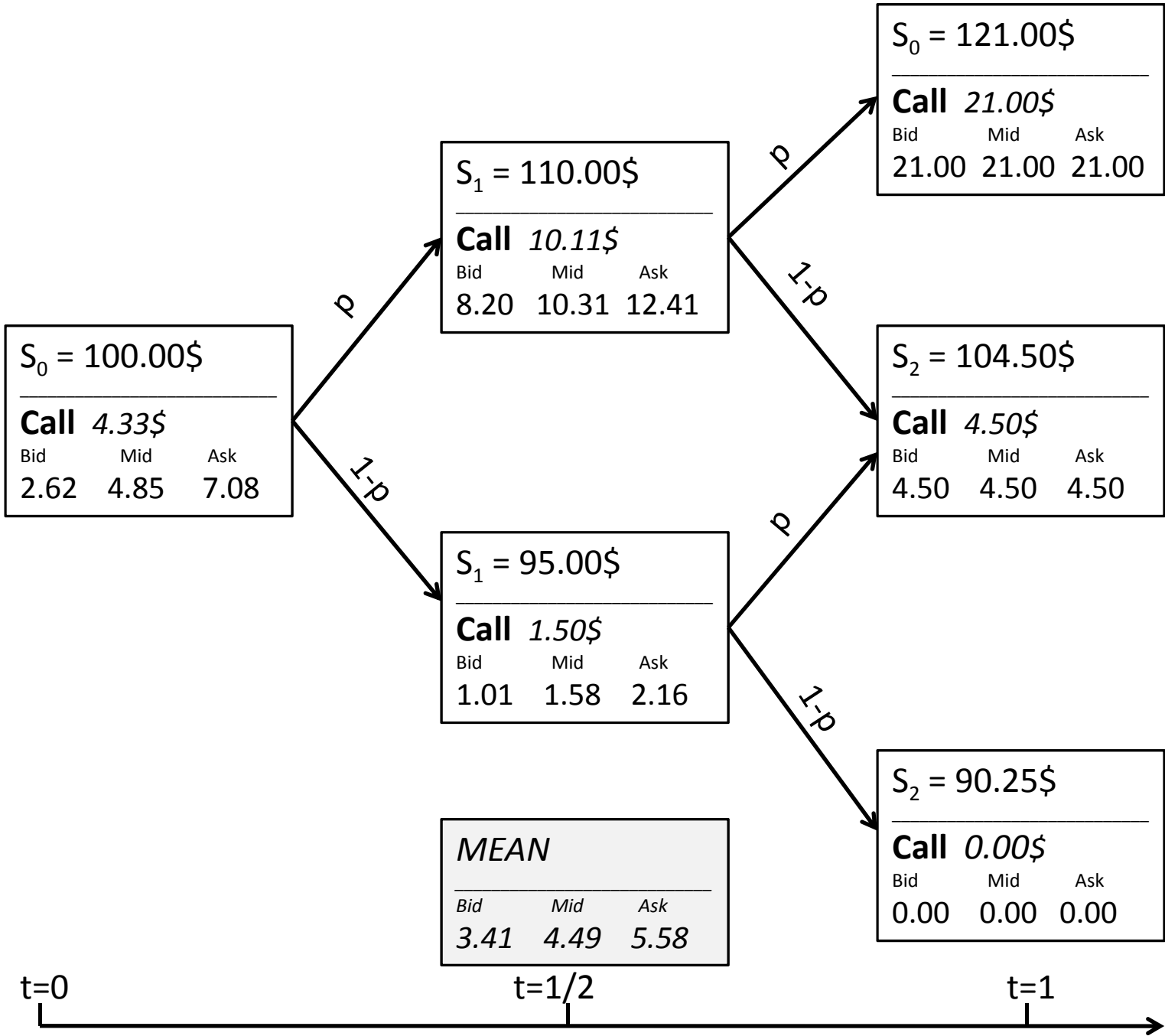
$$\begin{aligned} bC1 &= maC1 + .5 ((aC1d - maC1) \Psi^\gamma(1 - p) + (aC1u - maC1) (1 - \Psi^\gamma(1 - p))) \\ aC1 &= maC1 + .5 ((aC1d - maC1) (1 - \Psi^\gamma(p)) + (aC1u - maC1) \Psi^\gamma(p)) \\ maC1 &= p * aC1u + (1 - p) * aC1d. \end{aligned}$$

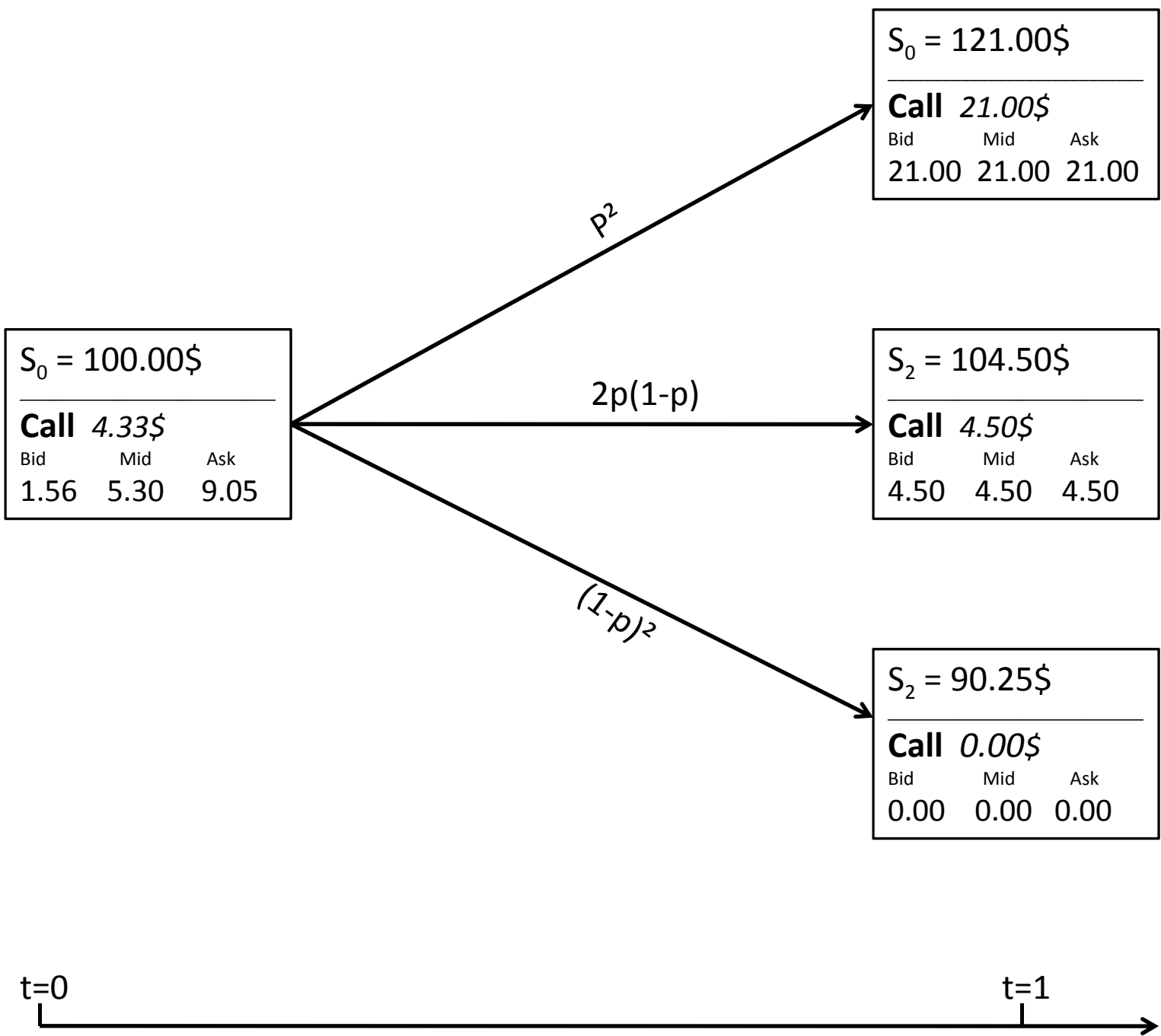
The full trees for a single step of two periods and two half steps are presented in a Figure with bid, ask and mid prices computed at each node with the minmaxvar distortion at stress 0.5.

## 6 Tenor Specific Discount Curves for the CIR spot rate model

The construction of tenor specific discount curves require access to the probability law of random variables of the form

$$X^a(t, t + h) = e^{-\int_t^{t+h} r(u) du} a_{t+h}(r(t + h)).$$







Hence one needs access to the joint law of the forward spot rate and the integral over the interim. This is available for the Cox, Ingersoll, Ross (1985) spot rate process defined by the stochastic differential equation

$$dr = \kappa(\theta - r)dt + \lambda\sqrt{r}dW$$

where  $\kappa$  is the rate of mean reversion,  $\theta$  is the long term equilibrium interest rate and  $\lambda$  is the spot rate volatility parameter. The Laplace transform of the forward spot rate given the current rate is available in closed form and an application of an inverse Laplace transform along the lines of Abate and Whitt (1995) gives us access to the distribution function. The forward spot rate may then be simulated by the inverse uniform method.

The Laplace transform of the integral given the initial spot rate and the final spot rate is also available in closed form (Pitman and Yor (1982)) and once again an inverse Laplace transform allows us to draw from the density of the integral given the rates at the two ends. In this way we may simulate readings on  $X^a(t, t+h)$  and  $X^b(t, t+h)$ . This simulation method has been suggested in the literature by Glasserman (2003), Broadie and Kaya (2006) and Chan and Joshi (2010). Working backwards from a one year maturity for the first step we just need the law of the integral. Thereafter we first simulate  $r(t+h)$  we then interpolate from stored values of bid and ask prices at the later time step the value for  $a_{t+h}(r(t+h)), b_{t+h}(r(t+h))$ . Then we draw from the distribution of the integral given the rates at the two ends to do the discounting and construct a single reading on  $X^a$  or  $X^b$ . We are then in a position to perform the recursion at different tenors back to time zero.

For a sample of parameter values to work with we employ the joint characteristic function for the rate and its integral

$$\phi_r(u, v) = E \left[ \exp \left( iur(t) + iv \int_0^t r(s)ds \right) \right]$$

and determine the risk neutral pure discount bond prices as

$$P(0, t) = \phi_r(0, i).$$

This model for bond prices was fitted to the *OIS* discount curve for data on December 15 2010 presented in Section 2. The estimated parameters were

$$\begin{aligned} \kappa &= 0.3712 \\ \theta &= 0.0477 \\ \lambda &= 0.0599 \\ r_0 &= 0.00004. \end{aligned}$$

A graph of the actual and fitted bond price curves is presented in Figure (3)

The recursions for bid and ask prices were performed using the *minmaxvar* distortion at a stress level of 0.75, for all the local risk charges. Figure (4) presents discount five year bond prices at time 0 for 3, 6 and 12 month tenors

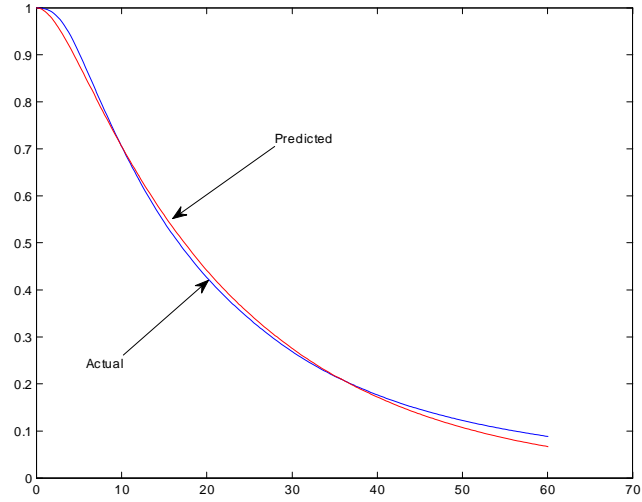


Figure 3: Actual and CIR model predicted OIS discount bond prices for maturities up to 60 years.

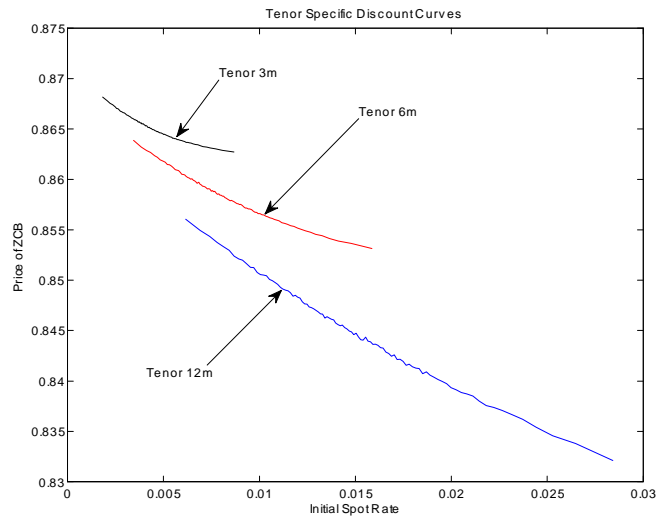


Figure 4: Tenor Specific Discount curves generated from mid quotes of dynamic sequences of bid and ask prices constructed at the tenors of 3, 6, and 12 months in black, red and blue respectively.

as a function of the initial spot rate that we let vary to levels reached at the first time point of 3, 6 and 12 months.

An increase in the price of pure discount bonds associated with the shorter tenor is observed in the model in line with market data for such tenor specific discount curves. The theory for tenor specific pricing presented in this paper is capable of generating tenor specific discount curves of the form observed in markets post the crisis of 2008.

## 7 Tenor Specific Forward Stock Prices

This section considers tenor specific pricing for an underlying risk neutral process that is forward price martingale. Our first example is that of geometric Brownian motion. The risk neutral process here is

$$S(t) = S(0) \exp\left(\sigma W(t) - \frac{\sigma^2 t}{2}\right)$$

for a Brownian motion  $W(t)$  and we take the initial stock price  $S(0)$  to be 100. Economic agents trading with the market do not have access to this risk neutral process that represents the underlying risk priced by the market.

Consider first the forward prices for delivery of stock in one year for a variety of volatilities and quoting tenors. We take for the two way price of the market, the mid quote constructed using the distortion *minmaxvar* at the stress level 0.75. Table 1 presents the resulting midquotes.

TABLE 1  
Mid Quotes under GBM

	Volatility		
Tenor	.2	.3	.4
12m	101.8420	103.7851	106.9213
6m	101.1343	102.4458	104.5791
3m	100.4782	101.4591	102.5282
1m	100.1019	100.0572	100.2699

It is clear that for the geometric Brownian motion model the single time step mid quote is above the risk neutral value and furthermore as one enhances liquidity by decreasing the tenor the prices fall towards the risk neutral value. The positive skewness of the lognormal distribution lifts the supremum and results in a mid quote above the risk neutral value. This effect is dampened for the shorter tenors. Actual risk neutral stock price distributions have a considerable left skewness as reflected in the implied volatility smiles. It is therefore instructive to investigate mid quotes in tenor specific pricing for models that fit the smile. For this we turn next to the variance gamma model.

For a set of stylized parameter values we fix  $\sigma$  the volatility of the Brownian motion at .2 as a control on volatility. We then take some moderate and high values for skewness and excess kurtosis via setting  $\theta$  at  $-.3$ ,  $-.6$  and setting  $\nu$

at .5,1.5. For these four settings we report in Table 2 on the midquotes for a quarterly tenor on a one year forward quote.

TABLE 2  
Mid Quotes under VG

$\sigma$	$\nu$	$\theta$	midquote
.2	.5	-.3	97.7905
.2	.5	-.6	97.0130
.2	1.5	-.3	96.0998
.2	1.5	-.6	95.2685

It may be observed that in all these cases the mid quote is below the risk neutral value. Preliminary numerical investigations confirm that as we increase liquidity we do get a convergence to the risk neutral value and hence it appears that an increase in liquidity raises the two way price quote on stocks for two price markets.

## 8 Tenor Specific Option Prices

This section reports on the mid quote and the risk neutral value of out of the money options and loan type contracts for an underlying geometric Brownian motion with a 30% volatility and the four VG processes considered in section 6. The out of the money options are a put struck at 80 and a call struck at 120 with an annual maturity. The loan or risky debt type contract pays the minimum of 1.25 times the stock price and a 100 dollars. Loss is then taken for stock prices below 80. In each case the risk neutral value and the mid quote are reported at each of two tenors, quarterly and monthly. The results are in tables 3 and 4, one for the quarterly tenor and the other for the monthly tenor. The loan mid quotes are below risk neutral values and rise as the tenor comes down. The opposite is the case with out-of-the-money options reflecting the expected convergence to risk neutral values.

TABLE 3  
Tenor Specific Options, Tenor 3m

Model	Risky Debt		80 Put		120 Call	
	RNV	MidQuote	RNV	MidQuote	RNV	MidQuote
GBM .3	95.7828	94.1758	3.4018	4.6827	5.1379	7.7697
VG(.2,.5,-.3)	95.8767	93.1988	3.3304	5.4839	2.4325	3.4840
VG(.2,.5,-.6)	91.2	87.4621	7.0161	10.0061	6.95	7.9183
VG(.2,1.5,-.3)	93.4180	89.6225	5.3079	8.3473	2.4641	3.2026
VG(.2,1.5,-.6)	87.3013	82.3885	10.2863	14.2151	8.9318	9.0814

TABLE 4

Tenor Specific Options, Tenor 1m

Model	Risky Debt		80 Put		120 Call	
	RNV	MidQuote	RNV	MidQuote	RNV	MidQuote
GBM .3	95.8017	95.1889	3.3705	3.8599	4.8799	5.8331
VG(.2,.5,-.3)	96.0179	94.4431	3.1921	4.4529	2.4178	2.7529
VG(.2,.5,-.6)	91.4011	88.9371	6.8378	8.8070	6.8789	6.7116
VG(.2,1.5,-.3)	93.6408	91.3086	5.1132	6.9834	2.4269	2.6336
VG(.2,1.5,-.6)	87.3896	84.0681	10.0263	12.6801	8.8675	8.3056

## 9 Square root tenor scaling

The random variable for a tenor  $h$  may be considered to have a variance broadly proportional to  $h$ . One may therefore contemplate annualizing by scaling it by  $1/\sqrt{h}$  before computing a risk charge that is then scaled by  $h$ . The net effect is to scale the risk charge by  $\sqrt{h}$  in place of  $h$ . We are grateful to Mitja Stajdich for making this observation and pointing us to his result that in the diffusion case such square root scaling preserves spreads in the limit, thereby halting the convergence to risk neutral values as the tenor is decreased. We report in this section the equivalents of Tables one through four with such a square root tenor scaling in place of tenor scaling.

TABLE 5

Mid Quotes under GBM and root tenor scaling

Tenor	Volatility		
	.2	.3	.4
12m	101.8420	103.7851	106.9213
6m	101.6839	104.2970	107.9990
3m	101.9386	104.4102	107.3796
1m	101.5409	103.6056	106.2558

The 40% volatility contract with a tenor of 4 days had a mid quote of 105.0538. The square root scaling does reduce the speed of convergence and maintains a spread in the limit. We next report on quarterly scaling on *VG* with square root tenor scaling.

TABLE 6

Mid Quotes under VG under root tenor scaling

$\sigma$	$\nu$	$\theta$	midquote
.2	.5	-.3	97.1448
.2	.5	-.6	96.0687
.2	1.5	-.3	94.0838
.2	1.5	-.6	92.7382

Once again we see that with square root scaling the distance to risk neutral is enhanced. Finally we report on option pricing under root tenor scaling.

TABLE 7

Tenor Specific Options, Tenor 3m

Model	Risky Debt		80 Put		120 Call	
	RNV	MidQuote	RNV	MidQuote	RNV	MidQuote
GBM .3	95.6883	91.5419	3.4529	6.7707	5.0577	12.0317
VG(.2,.5,-.3)	95.9270	90.8465	3.2644	7.3303	2.4817	4.8811
VG(.2,.5,-.6)	91.3507	84.6722	6.8638	12.2119	6.8526	9.5767
VG(.2,1.5,-.3)	93.5032	86.8443	5.2722	10.6001	2.4456	4.1219
VG(.2,1.5,-.6)	87.1819	78.9192	10.1248	16.7625	8.9112	10.0043

TABLE 8

Tenor Specific Options, Tenor 1m

Model	Risky Debt		80 Put		120 Call	
	RNV	MidQuote	RNV	MidQuote	RNV	MidQuote
GBM .3	95.8005	91.9145	3.3495	6.4607	4.9204	11.2801
VG(.2,.5,-.3)	96.0447	90.1850	3.2029	7.8863	2.4073	4.5261
VG(.2,.5,-.6)	91.3790	83.3989	6.8634	13.2570	6.8914	8.2685
VG(.2,1.5,-.3)	93.5995	86.0582	5.0495	11.0829	2.6226	3.7702
VG(.2,1.5,-.6)	87.4155	77.6397	10.0508	17.8745	8.9499	8.7296

Additionally we recomputed the tenor specific discount curves of Section 6 for a square root scaling to find the 3 month tenor above the 12 month but now by a smaller amount. It is anticipated that scalings higher than square root will also lead to a convergence to risk neutral but now at a slower rate. Assuming a sufficiently fast scaling it is reasonable to conjecture that the limit is generally the risk neutral price.

## 10 Conclusion

Fixed income markets now construct pure discount curves based on a variety of tenors for rolling over funds between time points. This gives rise to tenor specific prices for zero coupon bonds and raises the issue of the possibility of tenor specific pricing for all financial contracts. It is recognized that the law of two prices, bid and ask, as constructed in theory of conic finance set out in Cherny and Madan (2010), yields prices that are nonlinear functions of the random variables being priced. Dynamically consistent sequences of such prices are then related to the theory of nonlinear expectations and its connections with solutions to backward stochastic difference equations. The drivers for the stochastic difference equations are related to concave distortions that implement risk charges for the local risk specific to the tenor.

This theory is applied at a variety of tenors to generate such tenor specific bid and ask prices for discount bonds, stocks, and options on stocks. It is

observed that such tenor specific prices given by the mid quotes of bid and ask converge to the risk neutral price as the tenor is decreased. The convergence to the risk neutral may be halted by adjusting the scaling of risk charges to square root tenor scaling for example. The greater liquidity of lower tenors may lead to an increase or decrease in prices depending on whether the lower liquidity of a higher tenor has a mid quote above or below the risk neutral value. Generally for contracts with a large upside and a bounded downside the prices fall with liquidity while the opposite is the case for contracts subject to a large downside and a bounded upside.

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