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Sharp Asymptotics for Stochastic Dynamics with Parallel Updating Rule with self-interaction

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Abstract

In this paper we study metastability for a stochastic dynamics with a parallel updating rule in particular for a probabilistic cellular automata. The problem is addressed in the FreidlinWentzel regime, i.e., finite volume, small magnetic field, and in the limit when temperature tends to zero.

We are interested in how the system nucleates, i.e., in properties of the transition from the metastable state (the configuration with all minuses) to the stable state (the configuration with all pluses). In this paper we show that the nucleation time divided by its average converges to an exponential random variable and we express the proportionality constant for the average nucleation time in terms of parameters of the model. Our approach combines geometric and potential theoretic arguments.

A special feature of parallel dynamics is the existence of many fixed points and cyclic pairs of the zero temperature dynamics, in which the system can be trapped in its way to the stable phase. These cyclic points are corresponding to chessboard kind of configurations that under the parallel dynamics alternate between even and odd.

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1. Introduction

Metastable states are very common in nature and are typical of systems close to a first order phase transition. It is often observed that a system can persist for a long period of time in a phase which is not the one favored by the thermodynamic parameters; classical examples are the super-saturated vapor and the magnetic hysteresis. The rigorous description of this phenomenon in the framework of well defined mathematical models is relatively recent, dating back to the pioneering paper [CGOV], and has experienced substantial progress in the last decade. See [OV], [Bo], [Ho] for reviews and for a list of the most important papers on this subject.

A natural setup in which the phenomenon of metastability can be studied is that of Markov chains, or Markov processes, describing the time evolution of a statistical mechanical system. Think for instance to a stochastic lattice spin system. In this context powerful theories (see [BEGK], [MNOS], [OS]) have been developed with the aim to find answers valid with maximal generality and to reduce to a minimum the number of model dependent inputs necessary to describe the metastable behavior of the system.

Whatever approach is chosen, a key model dependent question is the computation of the minimal energy barrier, called communication energy, to be overcome by a path connecting the metastable to the stable state. Such a problem is in general quite complicated and becomes particularly difficult when the dynamics has a parallel character. Indeed, if simultaneous updates are allowed on the lattice, then no constraint on the structure of the trajectories in the configuration space is imposed. Therefore, to compute the communication energy, one must take into account all the possible transitions in the configuration space. The problem of the computation of the communication energy between the metastable (configuration with all minuses) to the stable state (configuration with all pluses) in a parallel dynamics setup has been addressed in [CN] in the probabilistic cellular automaton without self interaction. In particular, in [CN] the typical questions of metastability, that is the determination of the exit time and of the exit tube, have been answered for a reversible Probabilistic Cellular Automaton, in which each spin is coupled only with its nearest neighbors. In that paper it has been shown that, during the transition from the metastable minus state to the stable plus state, the system visits an intermediate chessboard-like phase. See [De] for introduction of the model probabilistic cellular automata and [CNP], [De], [GLD], [Ru], [St], [TVSMKP] for other results and applications.

In the present paper we study the reversible PCA in which each spin interacts both with itself and with its nearest neighbors; the metastable behavior of such a model has been investigated on heuristic and numerical grounds in [BCLS] and rigorously in [CNS]. The addition of the self-interaction changes completely the metastability scenario; in particular in [CNS] the authors show that the chessboard-like phase plays no role in the exit from the metastable phase. Another very interesting feature of this model is the presence of a large number of fixed points of the zero-temperature dynamics in which the system can be trapped.

We are interested in how the system nucleates, i.e., how it reaches the configuration with all pluses starting from all minuses. Our goal is to improve on earlier results by combining detailed results on the energy landscape for the dynamics and the potential theoretic approach to metastability that was developed [BEGK] and further exposed in [Bo]. Using the potential theoretic approach to metastability, developed in [BEGK] our main theorems sharpen earlier results obtained in [CNS]. We use the three model dependent ingredients proved in [CNS]: (1) the solution of the global variational problem for all the paths connecting the metastable and the stable state, i.e., the computation of the communication energy; (2) a sort of recurrence property stating that, starting from each configuration different from the metastable and the stable state, it is possible to reach a configuration at lower energy following a path with an energy cost strictly smaller than the communication energy; (3) determination of particular set of configurations, i.e., *critical configurations* for the transition from metastable to stable state and the neighbor of this set. This set of critical configurations is necessarily visited by the system during its excursion from the metastable to the stable state, it plays the role of the saddle configurations and represents the most important part of the typical tube of trajectories the system follows during the transition.

Our results are comparable with those derived for the Ising model in two and three dimensions evolving according to Glauber dynamics, respectively on finite box in [BM], and on volume growing exponentially fast with the inverse temperature in [BHS]. These works sharpened earlier results in [NS] in two dimensions and in [BC] in three dimensions.

Our results are comparable with those derived for lattice gas in two and three dimensions subject to kawasaki dynamics, respectively on finite box in [BHN], and on volume growing exponentially fast with the inverse temperature in [BHS] and in [GHNOS]. These works sharpened earlier results in [HOS] in two dimensions and in [HNOS] in three dimensions.

Kawasaki differs from Glauber in that it is a conservative dynamics: particles are conserved in the interior of the box. For critical comparison of Glauber and Kawasaki dynamics we refer to [MNOS] and [Ho]. Kawasaki and Glauber dynamics have very different properties, but they are both dynamics that allow only local changes, i.e, exchange of position of two particles for Kawasaki and change one spin per time for Glauber dynamics. Probabilistic Cellular automata are parallel dynamics where simultaneous updates are allowed on the lattice, then there is no continuity constraint on the cardinality of particles or plus spins can be used on the structure of the trajectories in the configuration space. Therefore, in the determination of the three model dependent inputs named above, one must take into account at any time all the possible transitions in the configuration space. For comparison Probabilistis Cellular Automata and serial dynamics in the context of metastability see [CNS3], [C], [CN], [CNS] and [CNS2].

The outline of the paper is as follows. In section 2 we define the model (in subsections 2.1 and 2.2), the stationary measure in 2.3 and the communication energy for probabilistic cellular automaton 2.4. Moreover, the end of section 2 in subsection 2.5 we recall earlier results. In section 3 we state our main theorems. In section 4 we proof of Lemma 2.1. In section 5 we review the results on Potential Theoretic approach that we will use in section 6. In section 6 we prove our main theorem 3.1.

2. Probabilistic Cellular Automata

In this section we introduce the basic notation, define the model of reversible Probabilistic Cellular Automata which will be studied in the sequel, and state the main known results.

2.1. The lattice and The configuration space

The spatial structure is modeled by the two–dimensional finite square $\Lambda \subset \mathbb{Z}^2$ of side length L with periodic boundary condition namely, by the torus Λ. We consider Λ endowed with the lattice distance $\mathrm{d}(x,y) := |x-y|$.

We say that $x, y \in \Lambda$ are *nearest neighbors* iff $\mathrm{d}(x,y) = 1$. For $X \subset \Lambda$ we let $\partial X := \{x \in X^{\mathrm{c}} : \mathrm{d}(x,X) = 1\}$ be the *external boundary* of X and $\overline{X} := X \cup \partial X$ be the *closure* of X. Two sets $X, Y \subset \Lambda$ are said to be *not interacting* if and only if $\mathrm{d}(X,Y) \geq 3$.

Let $x \in \Lambda$; for ℓ_1, ℓ_2 strictly positive reals we let $Q_{\ell_1,\ell_2}(x) := \{y \in \Lambda : x_1 \leq y_1 \leq x_1 + (\ell_1 - 1),\ x_2 \leq y_2 \leq x_2 + (\ell_2 - 1)\}$ be the rectangle of side lengths $[\ell_1]$ and $[\ell_2]$ with x the site with smallest coordinates. For $X \subset \Lambda$ and $\ell > 0$ we set $B_\ell(X) := \{y \in \Lambda : \mathrm{d}(X,y) \leq \ell\}$; if $\ell = 1$ we shall write $B(X)$ for $B_1(X)$, note that $B(X) = \overline{X}$. If $x \in \Lambda$ we write $B_\ell(x)$ for $B_\ell(\{x\})$, note that $B_\ell(x)$ is the ball of radius $[\ell]$ centered at x. Finally, we remark that $B(x)$ is the five site cross centered at $x \in \Lambda$.

The single spin state space is given by the finite set $\mathcal{S}_0 := \{-1, +1\}$ which we consider endowed with the discrete topology; the associated Borel σ–algebra is denoted by $\mathcal{F}_0$. The configuration space in $X \subset \Lambda$ is defined as $\mathcal{S}_X := \mathcal{S}_0^X$ and considered equipped with the product topology and the corresponding Borel σ algebra $\mathcal{F}_X$. The model and the related quantities that will be introduced later on will all depend on Λ, but since Λ is fixed it will be dropped from the notation; in this spirit we let $\mathcal{S}_\Lambda =: \mathcal{S}$ and $\mathcal{F}_\Lambda =: \mathcal{F}$.

Given $Y \subset X \subset \Lambda$ and $\sigma := \{\sigma_x \in \mathcal{S}_{\{x\}},\ x \in X\} \in \mathcal{S}_X$, we denote by σ_Y the *restriction* of σ to Y namely, $\sigma_Y := \{\sigma_x,\ x \in Y\}$.

2.2. The model: Probabilistic Cellular Automaton

Let $\beta > 0$ and $h \in \mathbb{R}$ such that $|h| < 1$ and $2/h$ is not integer, we consider the Markov chain on $\mathcal{S}$ with *transition matrix*

$$p(\sigma, \eta) := \prod_{x \in \Lambda} p_{x,\sigma}(\eta(x)) \quad \forall \sigma, \eta \in \mathcal{S} \tag{2.1}$$

where, for each $x \in \Lambda$ and $\sigma \in \mathcal{S}$, $p_{x,\sigma}(\cdot)$ is the probability measure on $\mathcal{S}_{\{x\}}$ defined as follows

$$p_{x,\sigma}(s) := \frac{1}{1 + \exp\{-2\beta s (S_\sigma(x) + h)\}} = \frac{1}{2}\left[1 + s \tanh \beta \left(S_\sigma(x) + h\right)\right] \tag{2.2}$$

with $s \in \{-1, +1\}$ and

$$S_\sigma(x) := \sum_{y \in V(x)} \sigma(y) \tag{2.3}$$

where $V(x)$ is a suitable neighborhood of the site x. Note that for x and s fixed $p_{x,\cdot}(s) \in \mathcal{F}_{V(x)}$ namely, the probability $p_{x,\sigma}(s)$ for the spin at site x to be equal to s depends only

on the values of the spins of σ inside the the neighborhood $V(x)$ of x. The normalization condition $p_{x,\sigma}(s) + p_{x,\sigma}(-s) = 1$ is trivially satisfied.
We study the metastability behavior of PCA model, appearing in (2.3) as

$$V(x) \quad = \quad V_1(x) := B(x) \tag{2.4}$$

The choice (2.4) corresponds to the model studied in [CNS], where also the self–interaction of spin in x is taken into account.

Such a Markov chain on the finite space $\mathcal{S}$ is an example of *reversible probabilistic cellular automata* (PCA). Let $n \in \mathbb{N}$ be the discrete time variable and $\sigma_n \in \mathcal{S}$ denote the state of the chain at time n, the configuration at time $n+1$ is chosen according to the law $p(\sigma_n, \cdot)$, see (2.1), hence all the spins are updated simultaneously and independently at any time. Finally, given $\sigma \in \mathcal{S}$ we consider the chain with initial configuration $\sigma_0 = \sigma$, we denote with $\mathbb{P}_\sigma$ the probability measure on the space of trajectories, by $\mathbb{E}_\sigma$ the corresponding expectation value, and by

$$\tau_A^\sigma := \inf\{t > 0 : \, \sigma_t \in A\} \tag{2.5}$$

the *first hitting time on* $A \subset \mathcal{S}$; we shall drop the initial configuration from the notation (2.5) whenever it is equal to $-\underline{1}$, we shall write τ_A for $\tau_A^{-\underline{1}}$, namely.

2.3. The stationary measure

It is straightforward, that the PCA (2.1) is reversible with respect to the finite volume Gibbs measure $\mu(\sigma) := \exp\{-\beta H(\sigma)\}/Z$ with $Z := \sum_{\eta \in \mathcal{S}} \exp\{-\beta H(\eta)\}$ and

$$H(\sigma) := H_{\beta,h}(\sigma) := -h \sum_{x \in \Lambda} \sigma(x) - \frac{1}{\beta} \sum_{x \in \Lambda} \log\cosh\left[\beta\left(S_\sigma(x) + h\right)\right] \tag{2.6}$$

in other words the *detailed balance* condition

$$p(\sigma, \eta)\, e^{-H(\sigma)} = p(\eta, \sigma)\, e^{-H(\eta)} \tag{2.7}$$

is satisfied for any $\sigma, \eta \in \mathcal{S}$. Hence, the measure μ is stationary for the PCA (2.1); to understand its most important features it is useful to study the related Hamiltonian. Since the Hamiltonian has the form (2.6) we shall often refer to $1/\beta$ as to the *temperature* and to h as to the *magnetic field*.
The definition of ground states is not completely trivial in our model, indeed the Hamiltonian H depends on β. The ground states are those configurations on which the Gibbs measure μ is concentrated when the limit $\beta \to \infty$ is considered, so they can be defined as the minima of the *energy*

$$E(\sigma) := \lim_{\beta \to \infty} H(\sigma) = -h \sum_{x \in \Lambda} \sigma(x) - \sum_{x \in \Lambda} |S_\sigma(x) + h| \tag{2.8}$$

uniformly in $\sigma \in \mathcal{S}$. Let $\mathcal{X} \subset \mathcal{S}$ if the energy E is constant on $\mathcal{X}$ namely, if all the configurations in $\mathcal{X}$ have the same energy, we shall misuse the notation by writing $E(\mathcal{X})$ for $E(\sigma)$ with $\sigma \in \mathcal{X}$.

2.4. Communication Energy

In our model the energy difference between two configurations σ and η is not sufficient to say if the system prefers to jump from σ to η or vice versa. Indeed, there are pair of configurations such that the system sees an energetic barrier in both directions. For this reason we associate a sort of *communication height* $\mathcal{H}(\sigma,\eta)$ to each pair of configurations $\sigma,\eta \in \mathcal{S}$. More precisely we extend the Hamiltonian (2.6) to $H : \mathcal{S} \cup \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ so that

$$\mathcal{H}(\sigma,\eta) := H(\sigma) - \log p(\sigma,\eta) \tag{2.9}$$

We consider the *communication energy*

$$\mathcal{E}(\sigma,\eta) := \lim_{\beta\to\infty} \mathcal{H}(\sigma,\eta) \geq \max\{E(\sigma), E(\eta)\} \tag{2.10}$$

and the *transition rate*

$$\Delta(\sigma,\eta) := \mathcal{E}(\sigma,\eta) - E(\sigma) = \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} 2|S_\sigma(x)+h| \geq 0 \tag{2.11}$$

where in the last equality we have used the definition (2.8) of $E(\sigma)$, (2.9), (2.1), and (2.2). We state and prove a simple Lemma that relates the communication height $\mathcal{H}(\sigma,\eta)$ to the communication energy $\mathcal{E}(\sigma,\eta)$.

Lemma 2.1 *For any $\sigma,\eta \in \mathcal{S}$ we have:*

$$\mathcal{H}(\sigma,\eta) = \mathcal{E}(\sigma,\eta) + \frac{|\Lambda|\,\log 2}{\beta} \tag{2.12}$$

We give now some useful definitions. A finite sequence of configurations $\omega = \{\omega_1,\ldots,\omega_n\}$ is called a *path* with starting configuration ω_1 and ending configuration ω_n; we denote the length of a path $|\omega| := n$. Given two paths ω and ω' such that $\omega_{|\omega|} = \omega'_1$ we let $\omega + \omega' := \{\omega_1,\ldots,\omega_{|\omega|},\omega'_2,\ldots,\omega'_{|\omega'|}\}$; note that $|\omega+\omega'| = |\omega| + |\omega'| - 1$. Given a path ω we define the *height* along ω as

$$\Phi_\omega := \begin{cases} E(\omega_1) & \text{if } |\omega| = 1 \\ \max_{i=1,\ldots,|\omega|-1} \mathcal{E}(\omega_i,\omega_{i+1}) & \text{otherwise} \end{cases} \tag{2.13}$$

Given two configurations $\sigma,\eta \in \mathcal{S}$, we denote by $\Theta(\sigma,\eta)$ the set of all the paths ω starting from σ and ending in η. The minimax between σ and η is defined as

$$\Phi(\sigma,\eta) := \min_{\omega\in\Theta(\sigma,\eta)} \Phi_\omega \tag{2.14}$$

2.5. Nucleation time and critical configurations

We pose now the problem of metastability, we state the known related theorem for the exit time for the model in (2.1) with $0 < h < 1$. Suppose that the system is prepared in the state $\sigma_0 = -\underline{1}$, where $-\underline{1}$ is the configuration with all the sites with spin -1, and let the dynamics evolve according with (2.1), we want to study the transition towards the phase $+\underline{1}$, the configuration with all spin $+1$.

It was shown in [CNS] that the minus one state is metastable in the sense that the system spends time exponentially big in configurations close to $-\underline{1}$ before visiting $+\underline{1}$, the configuration with all positive spins.

Let us consider the *critical length* λ defined as follows:

$$\lambda := \left[\frac{2}{h}\right] + 1 \tag{2.15}$$

Let us give the following definitions:

Definition 2.2 (Saddles) *1. We denote by $\mathcal{P}$ the set of the* protocritical droplets *consisting in the configurations with all the spins equal to -1 excepted those in a rectangle of side $\lambda - 1$ and λ in a neighboring site adjacent to one of the longest side.*

2. we denote with $\mathcal{C}$ the set of the critical droplet *consisting in the configurations with all the spins equal to -1 excepted those in a rectangle of side $\lambda - 1$ and λ in a pair of neighboring sites adjacent to one of the longest side.*

Figure 1: Protocritical and critical droplets

We restate a theorem from [CNS], giving a recurrence property on the set $\{-\underline{1}, +\underline{1}\}$, and estimate in probability of the first hitting time in the stable state.

Theorem 2.3 [CNS] *Recall Γ_1 has been defined in (3.1) and suppose $h > 0$ is chosen small enough; then*

1. for any $\sigma \in \mathcal{S} \setminus \{-\underline{1}\}$ we have

$$\Phi(\sigma, +\underline{1}) - E(\sigma) \leq \Gamma_0 < \Gamma_1 \tag{2.16}$$

2. we have

$$\Phi(-\underline{1}, +\underline{1}) - E(-\underline{1}) = \Gamma_1 \tag{2.17}$$

3. for each path $\omega = \{\omega_1, \ldots, \omega_n\} \in \Theta(-\underline{1}, +\underline{1})$ such that $\Phi_\omega = \Gamma_1$, then $\mathcal{E}(\omega_{i-1}, \omega_i) = \Gamma_1$ iff $\omega_{i-1} \in \mathcal{P}$, $\omega_i \in \mathcal{C}$ and $\omega_i = \omega_{i-1}^x$ for a suitable $x \in \Lambda$. .

We shall show that the first hitting time $\tau_{+\underline{1}}$, to $+\underline{1}$ is an exponential random variable with mean exponentially large in β. Let us consider the *critical length* λ defined as follows:

$$\lambda := \left[\frac{2}{h}\right] + 1 \tag{2.18}$$

3. Main results sharp description of nucleation

We shall show that the first hitting time $\tau_{+\underline{1}}$, to $+\underline{1}$ is an exponential random variable with mean exponentially large in β.

In order to state the main theorem for the PCA, we define the following *activation energy* of the configurations that trigger the nucleation:

$$\Gamma_1 \;:=\; -4h\lambda^2 + 4(4+h)\lambda + 2 - 6h \tag{3.1}$$

corresponding to the choice of V in 2.4.

Theorem 3.1 *Consider the probabilistic cellular automaton (2.1), for $h > 0$ small enough, we have:*

$$\mathbb{E}_{-\underline{1}}(\tau_{+\underline{1}}) = \frac{1}{K_i} e^{\beta\Gamma_i}[1 + o(1)] \qquad \beta \to \infty. \tag{3.2}$$

and

$$K_1 = 2^{-|\Lambda|}\, 8|\Lambda|(\lambda - 1); \tag{3.3}$$

Theorem 3.2 *Consider the probabilistic cellular automaton (2.1), for $h > 0$ small enough, we have:*

$$\mathbb{P}_{-\underline{1}}(\tau_{+\underline{1}} > t\mathbb{E}_{-\underline{1}}\tau_{+\underline{1}}) = [1 + o(1)]e^{-t[1+o(1)]} \;\; t \geq 0. \quad \beta \to \infty. \tag{3.4}$$

4. Proof of Lemma 2.1

By using (2.9), (2.1), (2.2), and (2.11) we get

$$\begin{aligned}
&\mathcal{H}(\sigma,\eta) - H(\sigma) - \beta\left[\mathcal{E}(\sigma,\eta) - E(\sigma)\right] \\
&\quad = \sum_{x\in\Lambda} \log(1 + e^{-2\beta\eta(x)[S_\sigma(x)+h]}) + \beta \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} 2\eta(x)[S_\sigma(x)+h] \\
&\quad = \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)>0}} \log(1 + e^{-2\beta\eta(x)[S_\sigma(x)+h]}) + \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} \log(1 + e^{-2\beta\eta(x)[S_\sigma(x)+h]}) \\
&\qquad + \beta \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} 2\eta(x)[S_\sigma(x)+h] \\
&\quad = \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)>0}} \log(1 + e^{-2\beta\eta(x)[S_\sigma(x)+h]}) + \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} \log(e^{+2\beta\eta(x)[S_\sigma(x)+h]} + 1) \\
&\quad = \sum_{x\in\Lambda} \log(1 + e^{-2\beta|S_\sigma(x)+h|})
\end{aligned} \tag{4.1}$$

Moreover, since

$$\begin{aligned}
H(\sigma) &:= -\sum_{x\in\Lambda}\log(\cosh(\beta(S_\sigma(x)+h))) - \beta\, h\sum_x \sigma(x)\\
&= -\sum_{x\in\Lambda}\log(\cosh(\beta|S_\sigma(x)+h|)) - \beta\, h\sum_x \sigma(x)\\
E(\sigma) &:= -h\sum_x \sigma(x) - \sum_x |S_\sigma(x)+h|
\end{aligned}$$

we have for the energy

$$\begin{aligned}
H(\sigma)-\beta\, E(\sigma) &= -\sum_{x\in\Lambda}\log(\cosh(\beta|S_\sigma(x)+h|)) + \beta\sum_x |S_\sigma(x)+h|\\
&= -\sum_x \left(\log(\cosh(\beta|S_\sigma(x)+h|)) - \log\exp(\beta|S_\sigma(x)+h|)\right)\\
&= -\sum_x \left(\log\frac{e^{\beta|S_\sigma(x)+h|}+e^{-\beta|S_\sigma(x)+h|}}{2e^{\beta|S_\sigma(x)+h|}}\right)\\
&= -\sum_x \left(\log\frac{1+e^{-2\beta|S_\sigma(x)+h|}}{2}\right) \qquad (4.2)
\end{aligned}$$

Adding ... (4.1) and (4.2) we get (2.12). □

5. Review on Potential Theoretic approach

The proof of the theorems (3.1) and (3.2) are based on the *potential–theoretic approach* to metastability developed in Bovier, Eckhoff, Gayrard and Klein [BEGK]. In this approach a key role is played by the notion of *capacity* between two sets of configurations, which we are going to recall.

5.1. Capacity

We define the *Dirichlet form* in the following way: for $h:\mathcal{S}\to\mathbb{R}$

$$\mathfrak{E}(h) = \frac{1}{2}\sum_{\sigma,\eta\in\mathcal{S}}\mu(\sigma)p(\sigma,\eta)[h(\sigma)-h(\eta)]^2 = \frac{1}{2}\sum_{\sigma,\eta\in\mathcal{S}}\frac{1}{Z}\,e^{-\beta\mathcal{H}(\sigma,\eta)}[h(\sigma)-h(\eta)]^2 \qquad (5.1)$$

where Z is the partition function.

From the definition of the communication energy (2.9), we have

$$\mu(\sigma)p(\sigma,\eta) = \mu(\sigma)\,e^{-\beta(\mathcal{H}(\sigma,\eta)-H(\sigma))} = \frac{1}{Z}\,e^{-\beta\mathcal{H}(\sigma,\eta)}.$$

Given two non-empty disjoint sets $\mathcal{A},\mathcal{B}$ the *capacity* of the pair $\mathcal{A},\mathcal{B}$ is defined by

$$\mathrm{CAP}(\mathcal{A},\mathcal{B}) := \min_{\substack{h:\mathcal{S}\to[0,1] \\ h|_{\mathcal{A}}=1, h|_{\mathcal{B}}=0}} \mathfrak{E}(h) \tag{5.2}$$

and from this definition it follows that the capacity is a *symmetric* function of the sets $\mathcal{A}$ and $\mathcal{B}$.
The right hand side of (5.2) has a unique minimizer $h^*_{\mathcal{A},\mathcal{B}}$ called *equilibrium potential* of the pair $\mathcal{A},\mathcal{B}$ given by

$$h^*_{\mathcal{A},\mathcal{B}}(\eta) = \mathbb{P}_\eta(\tau_{\mathcal{A}} < \tau_{\mathcal{B}}), \tag{5.3}$$

for any $\eta \notin \mathcal{A} \cup \mathcal{B}$.

The strength of this variational representation comes from the *monotonicity* of the Dirichlet form in the variable $p(\sigma,\eta)$. In fact, the Dirichlet form $\mathfrak{E}(h)$ is a monotone increasing function of the transition probabilities $p(\sigma,\eta)$ for $\sigma \neq \eta$, while it is independent of the value $p(\sigma,\sigma)$.
It is useful to recall that, due to 5.2, the Dirichlet for is monotone in terms of the $\mu(\sigma)p(\sigma,\eta)$, i.e., the following theorem holds:

Theorem 5.1 *Assume that $\mathfrak{E}$ and $\widetilde{\mathfrak{E}}$ are Dirichlet form associated to two Markov chains P and $\widetilde{P}$ with state space $\mathcal{S}$ and reversible with respect the measure μ. Assume that the transition probabilities p and $\widetilde{p}$ are given, for $x \neq y$, by*

$$p(x,y) = \frac{g(x,y)}{\mu(x)}, \tag{5.4}$$

$$\widetilde{p}(x,y) = \frac{\widetilde{g}(x,y)}{\mu(x)} \tag{5.5}$$

where $g(x,y) = g(y,x)$, $\widetilde{g}(x,y) = \widetilde{g}(y,x)$ and, for all $x \neq y$

$$\widetilde{g}(x,y) \leq g(x,y) \tag{5.6}$$

Then for any non intersecting sets $\mathcal{A}$, $\mathcal{B} \subset \mathcal{S}$,

$$\mathrm{cap}(\mathcal{A},\mathcal{B}) \geq \widetilde{\mathrm{CAP}}(\mathcal{A},\mathcal{B}) \tag{5.7}$$

Obviously the proof is evident, since $\mathfrak{E}(h) \geq \widetilde{\mathfrak{E}}(h)$ for any h. We remark that for the model studied, we have $g(x,y) = e^{-\beta\mathcal{H}(x,y)}$.

Thus, we will use Theorem 5.1 by simply setting some of the transition probabilities $p(x,y)$ equal to zero. Indeed if enough of these are zero, we obtain a chain where everything can be computed easily. In order to get a good lower bound, the trick will be to guess which transitions can be switched off without altering the capacities too much, and still to simplify enough to be able to compute it.

5.2. Relation between capacity and the mean hitting time

We want now to recall a theorem from [BEGK] that links mean hitting times and capacities. Indeed, if we define the set of metastable configuration $\mathcal{M}$:

Definition 5.2 *Consider a family of Markov chains, indexed by β, on a finite state space $\mathcal{S}$. A set $\mathcal{M} \subseteq \mathcal{S}$ is called* metastable *if*

$$\lim_{\beta\to\infty} \frac{\max_{\eta\notin\mathcal{M}} \mu(\eta)[\mathrm{CAP}(\eta,\mathcal{M})]^{-1}}{\min_{\eta\in\mathcal{M}} \mu(\eta)[\mathrm{CAP}(\eta,\mathcal{M}\setminus\eta)]^{-1}} = 0 \tag{5.8}$$

and the valley $A(\sigma)$ for a configuration $\sigma \in \mathcal{M}$:

Definition 5.3

$$A(\sigma) := \{\zeta \in \mathcal{S} : \mathbb{P}_\zeta(\tau_\sigma = \tau_\mathcal{M}) = \sup_{\eta\in\mathcal{M}} \mathbb{P}_\zeta(\tau_\eta = \tau_\mathcal{M})\} \tag{5.9}$$

With these definitions, we are able to write the mean hitting times in terms of the capacities, that satisfies a manageable variational principle. In fact, the following theorems holds:

Theorem 5.4 *Let $\sigma \in \mathcal{M}$ and $J \subseteq \mathcal{M} \setminus x$ be such that for all $m \notin J \cup x$ either $\mu(m) \ll \mu(\sigma)$ or $\mathrm{CAP}(m,J) \gg \mathrm{CAP}(m,\sigma)$. Then*

$$\mathbb{E}_\sigma \tau_J = \frac{\mu(A(\sigma))}{\mathrm{CAP}(\sigma,J)}(1+o(1)) \tag{5.10}$$

for β large enough.

Hence the strategy for proving the main theorems will be to identify the metastable sets (i.e., $-\underline{1}$, $+\underline{1}$), and to give sharp estimates for the capacities among the elements of this set, via a suitable application of the variational principle for the Dirichlet form.

6. Proof of Theorem 3.1

In order to proof the theorem, we first show that the metastable set $\mathcal{M}$ is the set $\{-\underline{1}, +\underline{1}\}$, then we give sharp estimates on $\mathrm{CAP}(-\underline{1}, +\underline{1})$ and we will conclude the proof using Theorem 5.4.

6.1. Metastable set $\mathcal{M} = \{-\underline{1}, +\underline{1}\}$

Recall definition 2.2 and theorem 2.3. Moreover we recall an elementary estimate given in [BHN]:

Lemma 6.1 [BHN] *For every non–empty disjoint sets $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$, there exist constants $0 < C_1 \leq C_2 < \infty$ (depending on $\mathcal{A}, \mathcal{B}$) such that for all β:*

$$C_1 \leq e^{\beta\Phi(\mathcal{A},\mathcal{B})} \, Z \, \mathrm{CAP}(\mathcal{A}, \mathcal{B}) \leq C_2. \tag{6.1}$$

Thus $\mathcal{M}$ is a metastable set in the sense that the following holds:

Theorem 6.2

$$\mathcal{M} = \{-\underline{1}, +\underline{1}\} \tag{6.2}$$

Proof
From (2.16) and (6.1), it follows that, for any $\eta \notin \{-\underline{1}, +\underline{1}\}$

$$\mu(\eta)[\mathrm{CAP}(\eta, \mathcal{M})]^{-1} \leq \frac{Z}{C_1} \, \mu(\eta) \, e^{\beta \, \phi(\eta, +\underline{1})} \leq \frac{e^{-H(\sigma)}}{C_1} \, e^{\beta \, \phi(\eta, \mathcal{M})} \leq C_3 \, e^{\beta(\Gamma_1 - \delta)} \tag{6.3}$$

for some $\delta > 0$, where in the last inequality we used (4.2).
If $\eta \in \{-\underline{1}, +\underline{1}\}$, we have:

$$\mu(\eta)[\mathrm{CAP}(-\underline{1}, +\underline{1})]^{-1} \geq C_4 e^{\beta \Gamma_1}. \tag{6.4}$$

Indeed, if $\eta = -\underline{1}$ equation (6.4) follows from (6.1), (2.17) and (4.2), while if $\eta = +\underline{1}$ we can write:

$$\frac{\mu(+\underline{1})}{\mathrm{CAP}(-\underline{1}, +\underline{1})} \geq Z \frac{\mu(+\underline{1})}{C_2 \, e^{-\beta \, \phi(-\underline{1}, +\underline{1})}} = \frac{e^{\beta\phi(-\underline{1}, +\underline{1}) - H(+\underline{1})}}{C_2} > C_5 \, e^{\beta \Gamma_1}$$

Hence, from (6.4) and (6.3), we get

$$\frac{\max_{\eta \notin \mathcal{M}} \mu(\eta)[\mathrm{CAP}(\eta, \mathcal{M})]^{-1}}{\min_{\eta \in \mathcal{M}} \mu(\eta)[\mathrm{CAP}(\eta, \mathcal{M} \setminus \eta)]^{-1}} \leq C_6 \, e^{-\beta \, \delta}$$

that concludes the proof. □

6.2. Sharp estimates for $\mathrm{CAP}(-\underline{1}, +\underline{1})$

We want to recall now the definition given in [CNS] of the *generalized basin of attraction* $\mathcal{G} \subset \mathcal{S}$ of $-\underline{1}$. In this set, containing $-\underline{1}$, but not $+\underline{1}$, one is able to evaluate the communication energy for all the possible direct jumps from the interior to the exterior of such a set $\mathcal{G}$. In the case of parallel dynamics, the lacking of continuity increases the difficulty of the computation of the communication energy between the metastable and the stable state and makes the definition of $\mathcal{G}$ more complicated. To define the set $\mathcal{G}$ one introduces two mappings $A, B : \mathcal{S} \to \mathcal{S}$.

The map A flips the first, in lexicographic order, unstable plus spin of σ to which corresponds a decrease of the energy. Under iteratively application of the map A: $\bar{A}$, the effect of the map A is that the number of pluses decreases, but only unstable pluses are flipped. Let $\sigma \in \mathcal{S}$ the configuration $B\sigma$ is the single step bootstrap percolation, i.e., it changes the value in $x \in \Lambda$ if the $\sigma(x) = -1$ and $\sum_{y \in V(x)} \sigma(x) \geq -1$ In other words B flips all the minus unstable spins and, among the stable minus spins, only those with two neighboring minuses.

Note that configurations σ such that $\bar{B}\bar{A}\sigma$ contains plus stripes winding around the torus Λ do not belong to $\mathcal{G}$.

The iteration of the map A and then of B converge to a fixed point in a finite time, and the composed mapping has the remarkable property of having the support consisting in well separated rectangles or stripes winding around the torus. We can now define the set $\mathcal{G}$. Denoting by $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ the collection of pairwise not interacting rectangles (or stripes) belonging to the support of the fixed point of the composed mapping, we define $\mathcal{G}$ as the set of all configurations such that the fixed point is either $-\underline{1}$ or $\min\{\ell_{i,1}, \ell_{i,2}\} \leq \lambda - 1$ and $\max\{\ell_{i,1}, \ell_{i,2}\} \leq L - 2$ for any $i = 1, \ldots, n$. Note that the set $\mathcal{G}$ contains the *subcritical configurations* and that configurations σ such that $\bar{B}\bar{A}\sigma$ contains plus stripes winding around the torus Λ do not belong to $\mathcal{G}$.
Following [CNS] we can evaluate the minmax between $\mathcal{G}$ and $\mathcal{G}^c$. Indeed

Proposition 6.3 [CNS]
With the definitions above, for $h > 0$ small enough and $L = L(h)$ large enough, we have

1. $-\underline{1} \in \mathcal{G}$, $+\underline{1} \in \mathcal{S} \setminus \mathcal{G}$, *and* $\mathcal{C} \subset \mathcal{S} \setminus \mathcal{G}$;
2. *for each* $\eta \in \mathcal{G}$ *and* $\zeta \in \mathcal{S} \setminus \mathcal{G}$ *we have* $E(\eta, \zeta) \geq E(-\underline{1}) + \Gamma_1$;
3. *for each* $\eta \in \mathcal{G}$ *and* $\zeta \in \mathcal{S} \setminus \mathcal{G}$ *we have* $E(\eta, \zeta) = E(-\underline{1}) + \Gamma_1$ *if and only if* $\zeta \in \mathcal{C}$, $\eta \in \mathcal{P}$ *and* $\zeta = \eta^x$ *for a suitable* $x \in \Lambda$.

We want to introduce the cycle $\mathcal{A}_{-\underline{1}}$ that is contained in $\mathcal{G}$:

$$\mathcal{A}_{-\underline{1}} := \{\eta \in \mathcal{G} : \exists \omega = \{\omega_0 = \eta, ..., \omega_n = -\underline{1}\} : \omega \subset \mathcal{G} \quad \text{and} \quad \Phi_\omega < \Gamma_1 + E(-\underline{1})\} \tag{6.5}$$

$\mathcal{A}_{-\underline{1}}$ is a subset of $\mathcal{G}$, such that the communication energies towards $-\underline{1}$ is strictly smaller than $\Gamma_1 + E(-\underline{1})$. Note that $\mathcal{P} \subset \mathcal{A}_{-\underline{1}}$.
Analogously we define $\mathcal{A}_{+\underline{1}}$:

$$\mathcal{A}_{+\underline{1}} := \{\eta \in \mathcal{G}^c : \exists \omega = \{\omega_0 = \eta, ..., \omega_n = +1\} : \omega \subset \mathcal{G} \quad \text{and} \quad \Phi_\omega < \Gamma_1 + E(-\underline{1})\} \tag{6.6}$$

Notice that $\mathcal{C} \subset \mathcal{A}_{+\underline{1}}$. We remark that from Proposition 6.3, for any $\eta \in \mathcal{A}_{-\underline{1}}$ and for any $\zeta \in \mathcal{A}_{+\underline{1}}$, $E(\eta, \zeta) \geq \Gamma_1 + E(-\underline{1})$ and $E(\eta, \zeta) = \Gamma_1 + E(-\underline{1})$ iff $\eta \in \mathcal{P}$ and $\zeta \in \mathcal{C}$.

Now we have all the ingredients to prove the sharp estimates for the capacity between $-\underline{1}$ and $+\underline{1}$.

Theorem 6.4 *With the notation of theorem 3.1*

$$\mathrm{CAP}(-\underline{1}, +\underline{1}) = K_1\, \mu(-\underline{1}) e^{-\beta\Gamma_1}(1 + o(1)) \tag{6.7}$$

in the limit $\beta \to \infty$

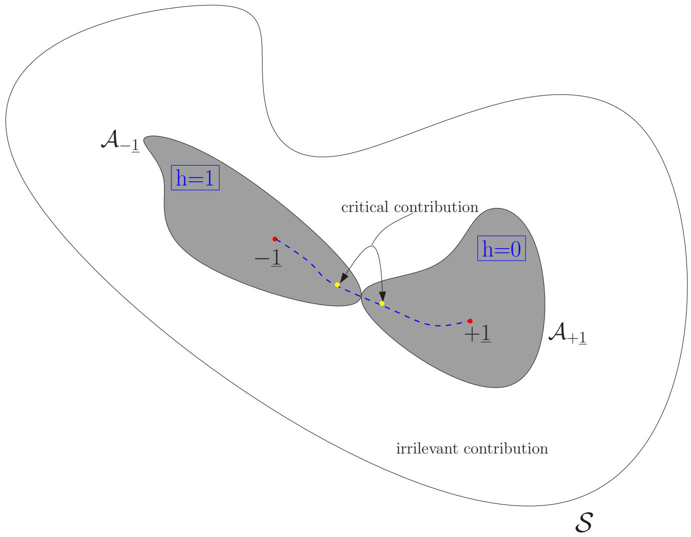

Figure 2: Upper bound

Proof.
In order to prove the theorem we give upper and lower bounds for the capacity $\mathrm{CAP}(-\underline{1}, +\underline{1})$.

i) Upper bound
We use the general strategy to prove an upper bound by guessing some a–priori properties of the minimizer $h^\star$, solution of the variational problem (5.2).
Let us consider the two basin of attraction $\mathcal{A}_{-\underline{1}}$ and $\mathcal{A}_{+\underline{1}}$, then we give an upper bound h^u for the equilibrium potential $h^\star$ choosing a test function in the following way:

$$h^u(\sigma) := \begin{cases} 1 & \sigma \in \mathcal{A}_{-\underline{1}}, \\ 0 & \sigma \in \mathcal{A}^c_{-\underline{1}} \end{cases} \tag{6.8}$$

Thus

$$\begin{aligned} \mathfrak{E}(h^u) &= \frac{1}{Z} \sum_{\substack{\sigma\in\mathcal{A}_{-\underline{1}},\\ \eta\in\mathcal{A}_{+\underline{1}}}} e^{-H(\sigma,\eta)} + \frac{1}{Z} \sum_{\substack{\sigma\in\mathcal{A}_{-\underline{1}},\\ \eta\in(\mathcal{A}_{+\underline{1}}\cup\mathcal{A}_{-\underline{1}})^c}} e^{-H(\sigma,\eta)} \\ &= \frac{1}{Z\,2^{|\Lambda|}} \sum_{\substack{\sigma\in\mathcal{A}_{-\underline{1}},\\ \eta\in\mathcal{A}_{+\underline{1}}}} e^{-\beta E(\sigma,\eta)} + \frac{1}{Z\,2^{|\Lambda|}} \sum_{\substack{\sigma\in\mathcal{A}_{-\underline{1}},\\ \eta\in(\mathcal{A}_{+\underline{1}}\cup\mathcal{A}_{-\underline{1}})^c}} e^{-\beta\,\mathcal{E}(\sigma,\eta)} \end{aligned} \tag{6.9}$$

where in the second step we used (2.12). We denote with N_1 the cardinality of the set of the saddles where the minmax Γ_1 is attained. Using

$$\frac{2^{|\Lambda|}\,\mathfrak{E}(h^u)}{\mu(-\underline{1})} \leq \sum \exp\{-\beta[\mathcal{E}(\sigma,\eta) - E(-\underline{1})]\} \tag{6.10}$$

we get contribution of terms $\mathcal{E}(\sigma,\eta)=\Gamma_1$, while all terms $\mathcal{E}(\sigma,\eta)>\Gamma_1$ give a lower contribution. Using this remark the l.h.s of (6.9), is bounded as follows:

$$\frac{2^{|\Lambda|}\,\mathfrak{E}(h^u)}{\mu(-\underline{1})} \le N_1\, e^{-\beta\Gamma_1} + |\mathcal{S}|\, e^{-\beta(\Gamma_1+\delta)}, \tag{6.11}$$

where $\delta>0$.
Indeed the minmax $\Phi(\mathcal{A}_{-\underline{1}},\mathcal{A}_{+\underline{1}})=\Gamma_1+E(-\underline{1})$ and by Proposition 6.3, it is attained only in the transitions between configurations $\eta\in\mathcal{P}$ and $\zeta\in\mathcal{C}$. Therefore, for all the other transitions we have $E(\eta,\zeta)>\Gamma_1+E(-\underline{1})$.

ii) Lower bound
In order to have a lower bound, let us estimate the equilibrium potential. We can prove the following Lemma:

Lemma 6.5 $\exists C>0,\exists\delta>0$ *such that for* β *large enough:*

$$\min_{\eta\in\mathcal{A}_{-\underline{1}}} h^\star(\eta) \;\ge\; 1-Ce^{-\delta\beta} \tag{6.12}$$

$$\max_{\eta\in\mathcal{A}_{+\underline{1}}} h^\star(\eta) \;\le\; Ce^{-\delta\beta} \tag{6.13}$$

Proof
Using a standard renewal argument, given $\eta\notin\{-\underline{1},+\underline{1}\}$:

$$\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}}) = \frac{\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}\cup\eta})}{1-\mathbb{P}_\eta(\tau_{-\underline{1}\cup+\underline{1}}>\tau_\eta)}; \tag{6.14}$$

$$\mathbb{P}_\eta(\tau_{-\underline{1}}<\tau_{+\underline{1}}) = \frac{\mathbb{P}_\eta(\tau_{-\underline{1}}<\tau_{+\underline{1}\cup\eta})}{1-\mathbb{P}_\eta(\tau_{-\underline{1}\cup+\underline{1}}>\tau_\eta)}. \tag{6.15}$$

Indeed if the process starting at point η wants to realize the event $\{\tau_{+\underline{1}}<\tau-\underline{1}\}$ it could do so either by going to $-\underline{1}$ immediately and without returning to η again, or it may return to η without going to $+\underline{1}$ or $-\underline{1}$. Clearly once the process return to η, we can use the strong Markov property. Thus

$$\begin{aligned}\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}}) &= \mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}\cup\eta})+\mathbb{P}_\eta(\tau_\eta<\tau_{+\underline{1}\cup-\underline{1}}\wedge\tau_{+\underline{1}}<\tau_{-\underline{1}})\\ &= \mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}\cup\eta})+\mathbb{P}_\eta(\tau_\eta<\tau_{+\underline{1}\cup-\underline{1}})\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}})\end{aligned}$$

and solving the equation for $\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}})$ we have the renewal equation.

Then $\forall\eta\in\mathcal{A}_{-\underline{1}}\setminus-\underline{1}$ we have:

$$h^\star(\eta)=1-\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}}\})=1-\frac{\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_{-\underline{1}\cup\eta})}{\mathbb{P}_\eta(\tau_{-\underline{1}\cup+\underline{1}}<\tau_\eta)}\ge 1-\frac{\mathbb{P}_\eta(\tau_{+\underline{1}}<\tau_\eta)}{\mathbb{P}_\eta(\tau_{-\underline{1}}<\tau_\eta)},$$

and for the last term we have the equality:

$$\frac{\mathbb{P}_\eta(\tau_{+\underline{1}} < \tau_\eta)}{\mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_\eta)} = \frac{\mathrm{CAP}(\eta, +\underline{1})}{\mathrm{CAP}(\eta, -\underline{1})}. \tag{6.16}$$

The upper bound for the numerator of (6.16) is easily obtained through the upper bound on $\mathrm{CAP}(-\underline{1}, +\underline{1})$. Indeed, $\mathrm{CAP}(\eta, +\underline{1}) \le \mathfrak{E}(h^u)$, with h^u the same as in (6.8). Hence, by (6.11) we have:

$$2^{|\Lambda|}\, Z\, \mathrm{CAP}(\eta, +\underline{1}) \le N_1\, e^{-\beta\,\Phi(-\underline{1},+\underline{1})} + |\mathcal{S}|\, e^{-\beta\,(\Phi(-\underline{1},+\underline{1})+\delta)}, \tag{6.17}$$

The lower bound on the denominator is obtained by reducing the state space to a single path from η to $-\underline{1}$, picking an *optimal* path $\omega = \{\omega_0, \omega_1, ..., \omega_n\}$ that realizes the minmax $\Phi(\eta, -\underline{1})$ and cutting all the transitions that are not in the path. On this new space we define a new process with transition probabilities $\widetilde{p}(\sigma, \eta)$ such that, for any $\sigma \ne \eta \in \omega$:

$$\widetilde{p}(\sigma, \eta) := \begin{cases} p(\sigma, \eta) & \text{whenever } \exists i :\ (\sigma, \eta) = (\omega_i, \omega_{i+1}) \\ 0 & \text{otherwise} \end{cases} \tag{6.18}$$

while $\widetilde{p}(\sigma, \sigma) = 1 - \sum_{\eta\ne\sigma} \widetilde{p}(\sigma, \eta)$. If we denote with $\mathfrak{E}^\omega(h)$ the Dirichlet form defined as in (5.1), with $\mathcal{S}$ replaced by ω and $p(\sigma, \eta)$ by $\widetilde{p}(\sigma, \eta)$, from Theorem 5.1 we have:

$$\mathrm{CAP}(\eta, -\underline{1}) \ge \min_{\substack{h:\omega\to[0,1] \\ h(\omega_0)=1, h(\omega_n)=0}} \mathfrak{E}^\omega(h) =: M \tag{6.19}$$

Due to the one–dimensional nature of the set ω, the variational problem in the right hand side can be solved explicitly by elementary computations. One finds that that the minimum equals

$$M = \left[\sum_{k=0}^{n-1} Z\, e^{H(\omega_k,\omega_{k+1})}\right]^{-1} \tag{6.20}$$

and it is uniquely attained at h given by

$$h(\omega_k) = M \sum_{l=0}^{k-1} Z\, e^{H(\omega_l,\omega_{l+1})} \qquad k = 0, 1, ..., n. \tag{6.21}$$

$$\begin{aligned} \mathrm{CAP}(\eta, -\underline{1}) &\ \ge\ M \ge \frac{1}{n\,Z} \max_k e^{-H(\omega_k,\omega_{k+1})} \\ &\ =\ 2^{-|\Lambda|}/n\, \frac{1}{Z} e^{-\beta\Phi(\eta,-\underline{1})} \end{aligned}$$

We know that if $\eta \in \mathcal{A}_{-\underline{1}}$ then $\Phi(\eta, -\underline{1}) < \Phi(\eta, +\underline{1})$. Indeed from the definition of the set $\mathcal{A}_{-\underline{1}}$:

$$\Phi(\eta, +\underline{1}) \ge \Gamma_1 + E(-\underline{1}) > \Phi(\eta, -\underline{1}). \tag{6.22}$$

For this reason

$$h^\star(\eta) \geq 1 - C(\eta)e^{-\beta(\Phi(-\underline{1},+\underline{1})-\Phi(\eta,-\underline{1}))} \geq 1 - C(\eta)e^{-\beta\delta}, \tag{6.23}$$

and we can take $C := \sup_{\eta\in\mathcal{A}_{-\underline{1}}\setminus-\underline{1}} C(\eta)$ and this concludes the proof of (6.12).
With a similar argument, $\forall\eta \in \mathcal{A}_{+\underline{1}} \setminus +\underline{1}$ we have:

$$\begin{aligned} h^\star(\eta) &= \mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_{+\underline{1}}\}) = \frac{\mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_{+\underline{1}\cup\eta})}{\mathbb{P}_\eta(\tau_{-\underline{1}\cup+\underline{1}} < \tau_\eta)} \leq \frac{\mathbb{P}_\eta(\tau_{-\underline{1}} < \tau_\eta)}{\mathbb{P}_\eta(\tau_{+\underline{1}} < \tau_\eta)} \\ &= \frac{\mathrm{CAP}(\eta,-\underline{1})}{\mathrm{CAP}(\eta,+\underline{1})} \leq C(\eta)e^{-\beta\delta}, \end{aligned} \tag{6.24}$$

which proves the second claim with $C = \max_{\eta\in\mathcal{A}_{+\underline{1}}\setminus+\underline{1}} C(\eta)$ □

Now we are able to give a lower bound for the capacity.
By Lemma (6.5) we have

$$\begin{aligned} \mathrm{CAP}(-\underline{1},+\underline{1}) = \mathfrak{E}(h^\star) &\geq \sum_{\substack{\sigma\in\mathbb{A}_{-\underline{1}} \\ \eta\in\mathcal{A}_{+\underline{1}}}} \mu(\sigma)p(\sigma,\eta)(h^\star(\sigma) - h^\star(\eta))^2 \\ &\geq 2^{-|\Lambda|}\, N_1\mu(-\underline{1})e^{-\beta\Gamma_1} + o(e^{-\beta\delta}). \end{aligned} \tag{6.25}$$

Now we want to evaluate the combinatorial factor N_1 of the sharp estimate.
We have to determinate all the possible ways to choose a protocritical droplet in the lattice with periodic boundary conditions. We know that the set $\mathcal{P}$ of such configurations contains all the rectangles $Q_{\lambda-1,\lambda}(x)$ with a single protuberance adjacent to one of the largest sides. Because of the translational invariance on the lattice, we can associate at each site x two rectangular droplets $Q_{\lambda-1,\lambda}(x)$ and $Q_{\lambda,\lambda-1}(x)$ such that their lower-left corner is in x. Considering the periodic boundary conditions, the number of such rectangles is

$$N_Q = 2|\Lambda|. \tag{6.26}$$

For every rectangle the possible transition between a protocritical droplet and a critical droplet are:

$$N_S = 2\,(2(\lambda - 2) + 2) \tag{6.27}$$

since there are two larger sides and given a single protuberance there are two ways two form a double protuberance if the spin is not in a corner and just one otherwise. Thus

$$N_1 = N_Q\, N_S = 8|\Lambda|(\lambda - 1)$$

and this completes the proof. □

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