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Abstract

In this paper we prove a law of large numbers for a general class of $\mathbb{Z}^d$-valued random walks in dynamic random environments, including examples that are *non-elliptic*. We assume that the random environment has a certain space-time mixing property, which we call *conditional cone-mixing*, and that the random walk has a tendency to stay inside wide enough space-time cones. The proof is based on a generalization of the regeneration scheme developed by Comets and Zeitouni [5] for static random environments, which was recently adapted by Avena, den Hollander and Redig [1] to dynamic random environments. We exhibit a number of one-dimensional examples to which our law of large numbers applies. For some of these examples the sign of the speed can be determined.

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1 Introduction

1.1 Background

Random walk in random environment (RWRE) has been an active area of research for more than three decades. Informally, RWRE's are random walks in discrete or continuous space-time whose transition kernels or transition rates are not fixed but are random themselves, constituting a random environment. Typically, the law of the random environment is taken to be translation invariant. Once a realization of the random environment is fixed, we say that the law of the random walk is *quenched*. Under the quenched law, the random walk is Markovian but not translation invariant. It is also interesting to consider the quenched law averaged over the law of the random environment, which is called the *annealed law*. Under the annealed law, the random walk is not Markovian but translation invariant. For an overview on RWRE, we refer the reader to Zeitouni [11, 12], Sznitman [9, 10], and references therein.

In the past decade, several models have been considered in which the random environment itself evolves in time. These are referred to as *random walk in dynamic random environment* (RWDRE). By viewing time as an additional spatial dimension, RWDRE can be seen as a special case of RWRE, and as such it inherits the difficulties present in RWRE in dimensions two or higher. However, RWDRE is harder than RWRE because it is an interpolation between RWRE and homogeneous random walk, which arise as limits when the dynamics is slow, respectively, fast. For a list of mathematical papers dealing with RWDRE, we refer the reader to Avena, den Hollander and Redig [2]. Most of the literature on RWDRE is restricted to situations in which the space-time correlations of the random environment are either absent or rapidly decaying.

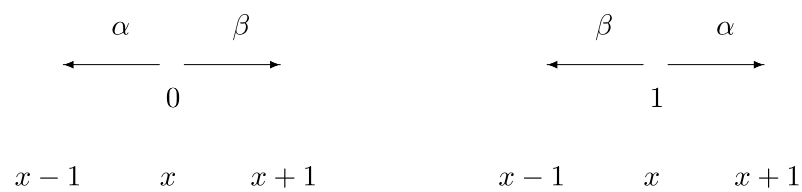

Figure 1: Jump rates of the random walk on top of a hole (= 0), respectively, a particle (= 1).

One paper in which a milder space-time mixing property is considered is Avena, den Hollander and Redig [1], where a law of large numbers (LLN) is derived for a class of one-dimensional RWDRE's in which the role of the random environment is taken by an *interacting particle system* (IPS) with configuration space

$$\Omega := \{0,1\}^{\mathbb{Z}}. \tag{1.1}$$

The transition rates of the random walk are as in Fig. 1: on a *hole* (i.e., on a 0) the random walk has rate α to jump one unit to the left and rate β to jump one unit to the right, while on a *particle* (i.e., on a 1) the rates are reversed (w.l.o.g. it may be assumed that $0 < \beta < \alpha < \infty$, so that the random walk has a drift to the left on holes and a drift to the right on particles). Hereafter, we will refer to this model as the (α, β)-model. The random walk starts at 0 and a LLN is proved under the assumption that the IPS satisfies a space-time mixing property called *cone-mixing*, which means that the states inside a space-time cone are almost independent of the states in a space plane far below this cone. The proof uses a regeneration scheme originally developed by Comets and Zeitouni [5] for RWRE and adapted to deal with RWDRE. This proof can be easily extended to $\mathbb{Z}^d$, $d \geq 2$, with the appropriate corresponding notion of cone-mixing.

1.2 Elliptic vs. non-elliptic

The original motivation for the present paper was to study the (α, β)-model in the limit as $\alpha \to \infty$ and $\beta \downarrow 0$. In this limit, which we will refer to as the $(\infty, 0)$-model, the walk almost is a deterministic functional of the IPS and therefore is non-elliptic. The challenge was to find a way to deal with the *lack of ellipticity.* As we will see in Section 3, our set-up will be rather general and will include the (α, β)-model, the $(\infty, 0)$-model, as well as various other models. Two papers that deal with non-elliptic (actually, deterministic) RW(D)RE's are Madras [7] and Matic (arXiv:0911.1809v2), where a recurrence vs. transience criterion, respectively, a large deviation principle are derived.

In the RW(D)RE literature, ellipticity assumptions play an important role. RW(D)RE on $\mathbb{Z}^d$, $d \geq 1$, is called *elliptic* when, almost surely w.r.t. the random environment, all the rates are *finite* and there is a basis $\{e_i\}_{1 \leq i \leq d}$ of $\mathbb{Z}^d$ such that the rate to go from x to $x + e_i$ is *positive* for $1 \leq i \leq d$. RW(D)RE is called *uniformly elliptic* when, almost surely w.r.t. the random environment, these rates are *bounded away from infinity*, respectively, *bounded away from zero.* In [5] and [1], uniform ellipticity is crucial in order to take advantage of the cone-mixing property. More precisely, it is crucial that the rates are uniformly elliptic in a direction in which the walk is transient. By this we mean that there is a non-zero vector e and a deterministic time T such that the quenched probability for the random walk to displace itself by e during time T is uniformly positive for almost all realizations of the random environment. The (α, β)-model is uniformly elliptic for e pointing in the time direction, since the total jump rate is $\alpha + \beta$ at every site. For the $(\infty, 0)$-model, however, there is no such e. In fact, there are many interesting models where the probability to move to any fixed space-time position is zero inside a set of environments of positive probability, and for all of these models the approach in [1] fails.

In the present paper, to deal with the possible lack of ellipticity we require a different space-time mixing property for the dynamic random environment, which we call *conditional cone-mixing.* Moreover, as in [5] and [1], we require the random walk to have a tendency to stay inside space-time cones. Under these assumptions, we are able to set up a regeneration scheme and prove a LLN. Our result includes the LLN for the (α, β)-model in [1], the $(\infty, 0)$-model for at least two subclasses of IPS's that we will exhibit, as well as models that are intermediate, in the sense that they are neither uniformly elliptic in any direction, nor are as environment-dependent as the $(\infty, 0)$-model.

1.3 Outline

The rest of the paper is organized as follows. In Section 2 we discuss, still informally, the $(\infty, 0)$-model and the regeneration strategy. This section serves as a motivation for the formal definition in Section 3 of the class of models we are after, which is based on three *structural assumptions.* Section 4 contains the statement of our LLN under four *hypotheses*, and a description of two classes of one-dimensional IPS's that satisfy these hypotheses for the $(\infty, 0)$-model. Section 5 contains preparation material, given in a more general context, that is used in the proof of the LLN given in Section 6. In Section 7 we verify the hypotheses for the two classes of IPS's described in Section 4. We also obtain a criterion to determine the sign of the speed in the LLN, via a comparison with independent spin-flip systems. Finally, in Section 8, we discuss how to adapt the proofs in Section 7 to other models, namely, generalizations of the (α, β)-model and the $(\infty, 0)$-model, and mixtures thereof. We also give an example where our hypotheses fail.

The examples in our paper are all one-dimensional, even though our LLN is valid in $\mathbb{Z}^d$, $d \geq 1$.

2 Motivation

2.1 The $(\infty, 0)$-model

Let

$$\xi := (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t := \big(\xi_t(x)\big)_{x\in\mathbb{Z}} \tag{2.1}$$

be a one-dimensional IPS on Ω with bounded and translation-invariant transition rates. We will interpret the states $\xi_t(x)$ by saying that at time t site x contains either a *hole* ($\xi_t(x) = 0$) or a *particle* ($\xi_t(x) = 1$). Typical examples to have in mind are independent spin-flips and simple exclusion.

Suppose that, given a realization of ξ, we run the (α, β)-model with $0 < \beta \ll 1 \ll \alpha < \infty$. Then the behavior of the random walk is as follows. Suppose that $\xi_0(0) = 1$ and that the walk starts at 0. The walk rapidly moves to the first hole on its right, typically before any of the particles it encounters manages to flip to a hole. When it arrives at the hole, the walk starts to rapidly jump back and forth between the hole and the particle to the left of the hole: we say that it sits in a *trap*. If $\xi_0(0) = 0$ instead, then the walk rapidly moves to the first particle on its left, where it starts to rapidly jump back and forth in a trap. In both cases, before moving away from the trap, the walk typically waits until one or both of the sites in the trap flip. If only one site flips, then the walk typically moves in the direction of the flip until it hits a next trap, etc. If both sites flip simultaneously, then the probability for the walk to sit at either of these sites is close to $\frac{1}{2}$, and hence it leaves the trap in a direction that is close to being determined by an independent fair coin.

The limiting dynamics when $\alpha \to \infty$ and $\beta \downarrow 0$ can be obtained from the above description by removing the words "rapidly, "typically" and "close to". Except for the extra Bernoulli($\frac{1}{2}$) random variables needed to decide in which direction to go to when both sites in a trap flip simultaneously, the walk up to time t is a deterministic functional of $(\xi_s)_{0\leq s\leq t}$. In particular, if we take ξ to be a spin-flip system with only single-site flips, then apart from the first jump the walk is completely deterministic. Since the walk spends all of its time in traps where it jumps back and forth between a hole and a particle, we may imagine that it lives on the edges of $\mathbb{Z}$. We implement this observation by associating with each edge its left-most site, i.e., we say that the walk is at x when we actually mean that it is jumping back and forth between x and $x + 1$.

Let

$$W := (W_t)_{t\geq 0} \tag{2.2}$$

denote the random walk path, which is càdlàg and, by the observations made above, is of the form

$$W_t := F_t\big((\xi_s)_{0\leq s\leq t}, Y\big), \qquad t \geq 0, \tag{2.3}$$

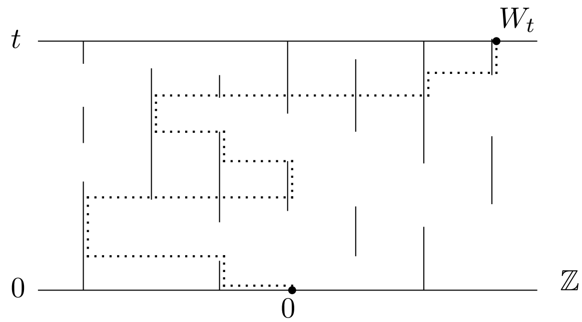

Figure 2: The vertical lines represent presence of particles. The dotted line is the path of the $(\infty, 0)$-walk.

where F_t is a measurable function taking values in $\mathbb{Z}$, and Y is a sequence of i.i.d. Bernoulli($\frac{1}{2}$) random variables independent of ξ. Note that W also has the following three properties:

(1) For any fixed time s, given $(\xi_u)_{0 \le u \le s}$ and $(W_u)_{0 \le u \le s}$, $(W_{s+t} - W_s)_{t \ge 0}$ is equal in distribution to $(W_t)_{t \ge 0}$ when ξ is started from $\bar{\xi}_s = (\bar{\xi}_s(x))_{x \in \mathbb{Z}}$, where $\bar{\xi}_s(x) = \xi_s(x + W_s)$. In particular, $(\xi_t, W_t)_{t \ge 0}$ is Markovian.

(2) Given that W stays inside a space-time cone until time t, $(W_s)_{0 \le s \le t}$ is a functional only of Y and of the states in ξ up to time t inside this cone.

(3) Each jump of the path follows the same mechanism as the first jump, i.e.,

$$P_\eta\big(W_t - W_{t-} = x \mid (\xi_s)_{0 \le s \le t},\, (W_s)_{0 \le s < t}\big) = P_{\theta_{W_{t-}} \xi_t}(W_0 = x). \tag{2.4}$$

For the $(\infty, 0)$-model the path is always in a trap, and so the r.h.s. of (2.4) is a.s. equal to $\delta_0(x)$. However, in the sequel we will have occasion to also consider discrete-time models for which the r.h.s. may be different.

The reason for emphasizing these properties will become clear in Section 3.

2.2 Regeneration

The cone-mixing property that is assumed in [1] to prove the LLN for the (α, β)-model can be loosely described as the requirement that all the states of the IPS inside a space-time cone opening upwards depend weakly on the states inside a space plane far below the tip (see Fig. 3). Let us give a rough idea of how this property can lead to *regeneration*. Consider the event that the walk stands still for a long time. Since the jump times of the walk are independent of the IPS, so is this event. During this pause, the environment around the walk is allowed to mix, which by the cone-mixing property means that by the end of the pause all the states inside a cone with a tip at the space-time position of the walk are almost independent of the past of the walk. If thereafter the walk stays confined to the cone, then its future increments will be almost independent of its past, and so we get an approximate regeneration. Since in the (α, β)-model there is a uniformly positive probability for the walk to stay inside a space-time cone with a large enough inclination, we see that the idea of regeneration can indeed be made to work.

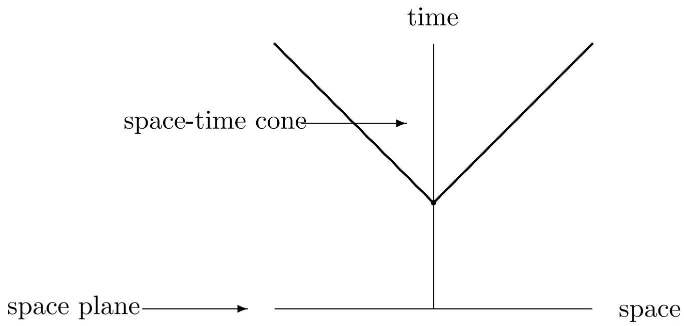

Figure 3: Cone-mixing property: asymptotic independence of states inside a space-time cone from states inside a space plane.

For the actual proof of the LLN in [1], cone-mixing must be more carefully defined. For technical reasons, there must be some uniformity in the decay of correlations between events in the space-time cone and in the space plane. This uniformity holds, for instance, for any spin-flip system in the $M < \epsilon$ regime (Liggett [6], Section I.3), but not for the exclusion process or the supercritical contact process. Therefore the approach outlined above works for the first IPS, but not for the other two.

There are three properties of the (α, β)-model that make the above heuristics plausible. First, to be able to apply the cone-mixing property relative to the space-time position of the walk, it is important that the pair (IPS,walk) is Markovian and that the law of the environment as seen from the walk at any time is comparable to the initial law. Second, there is a uniformly positive probability for the walk to stand still for a long time and afterwards stay inside a space-time cone. Third, once the walk stays inside a space-time cone, its increments depend on the IPS only through the states inside that cone. Let us compare these observations with what happens in the $(\infty, 0)$-model. Property (1) from Section 2.1 gives us the Markov property, while property (2) gives us the measurability inside cones. As we will see, property (3) implies absolute continuity of the law of the environment as seen from the walk at any positive time with respect to its counterpart at time zero. Therefore, as long as we can make sure that the walk has a tendency to stay inside space-time cones (which is reasonable when we are looking for a LLN), the main difference is that the event of standing still for a long time is not independent of the environment, but rather is a *deterministic* functional of the environment. Consequently, it is not at all clear whether cone-mixing is enough to allow for regeneration. On the other hand, the event of standing still is local, since it only depends on the states of the two neighboring sites of the trap where the walk is pausing. For most IPS's, the observation of a local event will not affect the weak dependence between states that are far away in space-time. Hence, if such IPS's are cone-mixing, then states inside a space-time cone remain almost independent of the initial configuration even when we condition on seeing a trap for a long time.

Thus, under suitable assumptions, the event "standing still for a long time" is a candidate to induce regeneration. In the (α, β)-model this event does not depend on the environment whereas in the $(\infty, 0)$-model it is a deterministic functional of the environment. If we put the (α, β)-model in the form (2.3) by taking for Y two independent Poisson processes with rates α and β, then we can restate the previous sentence by saying that in the (α, β)-model the regeneration-inducing event depends only on Y, while in the $(\infty, 0)$-model it depends only on ξ. We may therefore imagine that, also for other models that can be put in the form (2.3) and that share properties (1)–(3), it will be possible to find more general regeneration-inducing events that depend on both ξ and Y in a non-trivial manner. This motivates our setup in Section 3.

3 Model setting

So far we have been discussing RWDRE driven by an IPS. However, there are convenient constructions of IPS's on richer state spaces (like graphical representations) that can facilitate the construction of the regeneration-inducing events mentioned in Section 2.2. We will therefore allow for more general Markov processes to represent the dynamic random environment ξ. Notation is set up in Section 3.1. Section 3.2 contains the three structural assumptions that define the class of models we are after.

3.1 Notation

Let E be a separable metric space and $\xi := (\xi_t)_{t\geq 0}$ a Markov process with state space $E^{\mathbb{Z}^d}$ where $d \in \mathbb{N}$. Let $Y := (Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables independent of ξ. For $I \subset [0,\infty)$, abbreviate $\xi_I := (\xi_u)_{u\in I}$, and analogously for Y. The joint law of ξ and Y when $\xi_0 = \eta \in E^{\mathbb{Z}^d}$ will be denoted by $\mathbb{P}_\eta$.

For $t \geq 0$ and $x \in \mathbb{Z}^d$, let θ_t and θ_x denote the time-shift and space-shift operators given by

$$\theta_t(\xi, Y) := \big((\xi_{t+s})_{s\geq 0}, (Y_{\lfloor t\rfloor+n})_{n\in\mathbb{N}}\big), \qquad \theta_x(\xi, Y) := \big((\theta_x\xi_t)_{t\geq 0}, (Y_n)_{n\in\mathbb{N}}\big), \tag{3.1}$$

where $\theta_x\xi_t(y) = \xi_t(x + y)$. In the sequel, whether θ is a time-shift or a space-shift operator will always be clear from the index.

We assume that ξ is translation-invariant, i.e., $\theta_x\xi$ under $\mathbb{P}_\eta$ has the same distribution as ξ under $\mathbb{P}_{\theta_x\eta}$. We also assume the existence of a (not necessarily unique) translation-invariant equilibrium distribution μ for ξ, and write $\mathbb{P}_\mu(\cdot) := \int \mu(d\eta)\,\mathbb{P}_\eta(\cdot)$ to denote the joint law of ξ and Y when ξ_0 is drawn from μ.

For $n \in \mathbb{N}$, let $\mathscr{Y}_n := \sigma\{Y_k\colon\, 1 \leq k \leq n\}$ be the σ-algebra generated by $(Y_k)_{1\leq k\leq n}$. For $m > 0$ and $R \in \mathbb{N}$, define the m-cone, respectively, the R-enlarged m-cone by

$$\begin{aligned} C(m) &:= \big\{(x,t) \in \mathbb{Z}^d \times [0,\infty)\colon\, \|x\| \leq mt\big\},\\ C_R(m) &:= \big\{(x,t) \in \mathbb{Z}^d \times [0,\infty)\colon\, \exists\,(y,t) \in C(m) \text{ with } \|x-y\| \leq R\big\}, \end{aligned} \tag{3.2}$$

where $\|\cdot\|$ is the L^1 norm. Let $\mathscr{C}_t(m)$ and $\mathscr{C}_{R,t}(m)$ be the σ-algebras generated by the states of ξ up to time t inside $C(m)$ and $C_R(m)$, respectively.

3.2 Structural assumptions

In what follows we make *three structural assumptions*:

(A1) **(Additivity)**
$W = (W_t)_{t\geq 0}$ is a random translation of a càdlàg random walk that starts at 0 and is a functional of ξ and Y. More precisely, let $(F_t)_{t\in[0,1]}$ be a family of $\mathbb{Z}^d$-valued measurable functions. Define a random process Z by putting

$$\begin{aligned} Z_0 &:= 0\\ Z_{n+t} - Z_n &:= F_t(\theta_{Z_n}\xi_{(n,n+t]}, Y_{n+1}), \qquad n \in \mathbb{N}_0,\ t \in (0,1]. \end{aligned} \tag{3.3}$$

Then Z has càdlàg paths and

$$W_t - W_0 \quad = \quad \theta_{W_0} Z_t, \tag{3.4}$$

where W_0 is a $\mathbb{Z}^d$-valued random variable that depends on ξ only through ξ_0, i.e.,

$$\mathbb{P}_\mu(W_0 = x \mid \xi) = \mathbb{P}_\mu(W_0 = x \mid \xi_0) \;\text{ a.s. } \forall\, x \in \mathbb{Z}^d. \tag{3.5}$$

(A2) **(Locality)**
There exists an $R \in \mathbb{N}$ such that, for all $m > 0$, Z is measurable w.r.t. $\mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty$ on the event $\{Z_t \in C(m)\ \forall\, t \geq 0\}$.

(A3) **(Homogeneity of jumps)**
For all $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$,

$$\mathbb{P}_\mu\left(W_n - W_{n-} = x \mid \xi_{[0,n]}, W_{[0,n)}\right) = \mathbb{P}_{\theta_{W_{n-}}\xi_n}\left(W_0 = x\right) \qquad \mathbb{P}_\mu\text{-a.s.} \tag{3.6}$$

These are the analogues of properties (1)–(3) of the $(\infty, 0)$-model mentioned in Section 2.1.

Let us denote by $\bar{\xi} := (\bar{\xi}_t)_{t\geq 0}$ the *environment process* associated to W, i.e., $\bar{\xi}_t := \theta_{W_t}\xi_t$, and let $\bar{\mu}_t$ denote the law of $\bar{\xi}_t$ under $\mathbb{P}_\mu$. We abbreviate $\bar{\mu} := \bar{\mu}_0$. Note that $\bar{\mu} = \mu$ when $\mathbb{P}_\mu(W_0 = 0) = 1$. From (3.4) we see that $(W_t - W_0)_{t\geq 0}$ under $\mathbb{P}_\mu$ has the same distribution as Z under $\mathbb{P}_{\bar{\mu}}$.

4 Main results

Theorems 4.1 and 4.2 below are the main results of our paper. Theorem 4.1 in Section 4.1 is our LLN. Theorem 4.2 in Section 4.2 verifies the hypotheses in this LLN for the $(\infty, 0)$-model for two classes of one-dimensional IPS's. For these classes some more information is available, namely, convergence in L^p, $p \geq 1$, and a criterion to determine the sign of the speed.

4.1 Law of large numbers

In order to develop a regeneration scheme for a random walk subject to assumptions (A1)–(A3) based on the heuristics discussed in Section 2.2, we must have suitable regeneration-inducing events. In the *four hypotheses* stated below, these regeneration-inducing events appear as a sequence of events $(\Gamma_L)_{L\in\mathbb{N}}$ such that $\Gamma_L \in \mathscr{C}_{R,L}(m) \vee \mathscr{Y}_L$ for all $L \in \mathbb{N}$ and some $m > 0$.

(H1) **(Determinacy)**
On Γ_L, $Z_t = 0$ for all $t \in [0, L]$ $\mathbb{P}_{\bar{\mu}}$-a.s.

(H2) **(Non-degeneracy)**
For L large enough, there exists a $\gamma_L > 0$ such that $\mathbb{P}_\eta(\Gamma_L) \geq \gamma_L$ for $\bar{\mu}$-a.e. η.

(H3) **(Cone constraints)**
Let $\mathcal{S} := \inf\{t \geq 0\colon\ Z_t \notin C(m)\}$ denote the first exit time of $C(m)$. Then there exist $a \in (1, \infty)$, $\kappa_L \in (0, 1]$ and $\psi_L > 0$ such that, for L large enough and $\bar{\mu}$-a.e. η,

$$\begin{aligned} &(1) \quad \mathbb{P}_\eta(\theta_L \mathcal{S} = \infty \mid \Gamma_L) \geq \kappa_L, \\ &(2) \quad \mathbb{E}_\eta\left[1_{\{\theta_L\mathcal{S}<\infty\}}(\theta_L\mathcal{S})^a \mid \Gamma_L\right] \leq \psi_L^a. \end{aligned} \tag{4.1}$$

(H4) **(Conditional cone-mixing)**
There exists a sequence of numbers $(\phi_L)_{L\in\mathbb{N}}$ in $[0,\infty)$ satisfying $\lim_{L\to\infty}\phi_L = 0$ such that, for L large enough and for $\bar{\mu}$-a.e. η,

$$|\mathbb{E}_\eta\left(\theta_L f \mid \Gamma_L\right) - \mathbb{E}_{\bar{\mu}}(\theta_L f \mid \Gamma_L)| \le \phi_L \|f\|_\infty \qquad \forall\, f \in \mathscr{C}_{R,\infty}(m), f \ge 0. \tag{4.2}$$

We are now ready to state our LLN.

Theorem 4.1. *Under assumptions* (A1)–(A3) *and hypotheses* (H1)–(H4), *there exists a* $w \in \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} t^{-1} W_t = w \qquad \mathbb{P}_\mu - a.s. \tag{4.3}$$

Remark 1: Hypothesis (H4) above without the conditioning on Γ_L in (4.2) is the same as the cone-mixing condition used in Avena, den Hollander and Redig [1]. There, $W_0 = 0$ $\mathbb{P}_\mu$-a.s., so that $\bar{\mu} = \mu$.

Remark 2: Theorem 4.1 provides no information about the value of w, not even its sign when $d = 1$. Understanding the dependence of w on model parameters is in general a highly non-trivial problem.

4.2 Examples

We next describe two classes of one-dimensional IPS's for which the $(\infty,0)$-model satisfies hypotheses (H1)–(H4). Further details will be given in Section 7. In both classes, ξ is a spin-flip system in $\Omega = \{0,1\}^{\mathbb{Z}}$ with bounded and translation-invariant single-site flip rates. We may assume that the flip rates at the origin are of the form

$$c(\eta) = \begin{cases} c_0 + \lambda_0 p_0(\eta) & \text{if } \eta(0) = 1, \\ c_1 + \lambda_1 p_1(\eta) & \text{if } \eta(0) = 0, \end{cases} \qquad \eta \in \Omega, \tag{4.4}$$

for some $c_i, \lambda_i \ge 0$ and $p_i \colon \Omega \to [0,1]$, $i = 0,1$.

Example 1: $c(\cdot)$ is in the $M < \epsilon$ regime (see Liggett [6], Section I.3).

Example 2: $p(\cdot)$ has range 1 and $(\lambda_0 + \lambda_1)/(c_0 + c_1) < \lambda_c$, where λ_c is the critical infection rate of the one-dimensional nearest-neighbor contact process.

Theorem 4.2. *Consider the* $(\infty,0)$*-model. Suppose that* ξ *is a spin-flip system with flip rates given by* (4.4). *Then for Examples* 1 *and* 2 *there exist a version of* ξ *and events* $\Gamma_L \in \mathscr{C}_{R,L}(m) \vee \mathscr{Y}_L$, $L \in \mathbb{N}$, *satisfying hypotheses* (H1)–(H4). *Furthermore, the convergence in Theorem* 4.1 *holds also in* L^p *for all* $p \ge 1$, *and*

$$\begin{aligned} w &\ge \tfrac{c_0+\lambda_0}{c_1+c_0+\lambda_0}(c_1 - c_0 - \lambda_0) \qquad \textit{if } c_1 > c_0 + \lambda_0, \\ w &\le -\tfrac{c_1+\lambda_1}{c_0+c_1+\lambda_1}(c_0 - c_1 - \lambda_1) \quad \textit{if } c_0 > c_1 + \lambda_1. \end{aligned} \tag{4.5}$$

For independent spin-flip systems (i.e., when $\lambda_0 = \lambda_1 = 0$), we are able to show that w is positive, zero or negative when the density $c_1/(c_0 + c_1)$ is, respectively, larger than, equal to or smaller than $\frac{1}{2}$. Criterion (4.5) for other ξ is obtained by comparison with independent spin-flip systems.

We expect hypotheses (H1)–(H4) to hold for a very large class of IPS's and walks. For each choice of IPS and walk, the *verification* of hypotheses (H1)–(H4) constitutes a *separate problem.* Typically, (H1)–(H2) are immediate, (H3) requires some work, while (H4) is hard.

Additional models will be discussed in Section 8. We will consider generalizations of the (α, β)-model and the $(\infty, 0)$-model, namely, *pattern* models and *internal noise* models, as well as mixtures of them. The verification of (H1)–(H4) will be largely similar to the two models discussed above and will therefore not be carried out in detail.

This concludes the motivation and the statement of our main results. The remainder of the paper will be devoted to the proofs of Theorems 4.1 and 4.2, with the exception of Section 8, which contains additional examples and remarks.

5 Preparation

The aim of this section is to prove two propositions (Propositions 5.2 and 5.4 below) that will be needed in Section 6 to prove the LLN. In Section 5.1 we deal with approximate laws of large numbers for general discrete- or continuous-time random walks in $\mathbb{R}^d$. In Section 5.2 we specialize to additive functionals of a Markov chain whose law at any time is absolutely continuous with respect to its initial law.

5.1 Approximate law of large numbers

This section contains two fundamental facts that are the basis of our proof of the LLN. They deal with the notion of an approximate law of large numbers.

Definition 5.1. *Let $W = (W_t)_{t \geq 0}$ be a random process in $\mathbb{R}^d$ with $t \in \mathbb{N}$ or $t \in [0, \infty)$. For $\varepsilon \geq 0$ and $v \in \mathbb{R}^d$, we say that W has an ε-approximate asymptotic velocity v, written $W \in AV(\varepsilon, v)$, if*

$$\limsup_{t \to \infty} \left\| \frac{W_t}{t} - v \right\| \leq \varepsilon \qquad a.s. \tag{5.1}$$

We take $\| \cdot \|$ to be the L_1-norm. A simple observation is that W a.s. has an asymptotic velocity if and only if for every $\varepsilon > 0$ there exists a $v_\varepsilon \in \mathbb{R}^d$ such that $W \in AV(\varepsilon, v_\varepsilon)$. In this case $\lim_{\varepsilon \downarrow 0} v_\varepsilon$ exists and is equal to the asymptotic velocity v.

5.1.1 First key proposition: skeleton approximate velocity

The following proposition gives conditions under which an approximate velocity for the process observed along a random sequence of times implies an approximate velocity for the full process.

Proposition 5.2. *Let W be as in Definition 5.1. Let $(\tau_k)_{k \in \mathbb{N}_0}$ be an increasing sequence of random times in $[0, \infty)$ (or $\mathbb{N}_0$) with $\lim_{k \to \infty} \tau_k = \infty$ a.s., and let $X_k := (W_{\tau_k}, \tau_k) \in \mathbb{R}^{d+1}$, $k \in \mathbb{N}_0$. Suppose that the following hold:*
(i) There exists an $m > 0$ such that

$$\limsup_{k \to \infty} \sup_{s \in (\tau_k, \tau_{k+1}]} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| \leq m \quad a.s. \tag{5.2}$$

(ii) There exist $v \in \mathbb{R}^d$, $u > 0$ and $\varepsilon \geq 0$ such that $X \in AV(\varepsilon, (v, u))$.
Then $W \in AV((3m + 1)\varepsilon/u, v/u)$.

Proof. First, let us check that (i) implies

$$\limsup_{t\to\infty} \frac{\|W_t\|}{t} \le m \quad \text{a.s.} \tag{5.3}$$

Suppose that

$$\limsup_{k\to\infty} \sup_{s>\tau_k} \left\| \frac{W_s - W_{\tau_k}}{s-\tau_k} \right\| \le m \quad \text{a.s.} \tag{5.4}$$

Since, for every k and $t > \tau_k$,

$$\left\| \frac{W_t}{t} \right\| \le \frac{\|W_{\tau_k}\|}{t} + \left\| \frac{W_t - W_{\tau_k}}{t-\tau_k} \right\| \left|1 - \frac{\tau_k}{t}\right| \le \frac{\|W_{\tau_k}\|}{t} + \sup_{s>\tau_k} \left\| \frac{W_s - W_{\tau_k}}{s-\tau_k} \right\| \left|1 - \frac{\tau_k}{t}\right|, \tag{5.5}$$

(5.3) follows from (5.4) by letting $t \to \infty$ followed by $k \to \infty$.

To check (5.4), define, for $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$,

$$m(k,l) := \sup_{s\in(\tau_k,\tau_{k+l}]} \left\| \frac{W_s - W_{\tau_k}}{s-\tau_k} \right\| \quad \text{and} \quad m(k,\infty) := \sup_{s>\tau_k} \left\| \frac{W_s - W_{\tau_k}}{s-\tau_k} \right\| = \lim_{l\to\infty} m(k,l). \tag{5.6}$$

Using the fact that $(x_1+x_2)/(y_1+y_2) \le (x_1/y_1) \vee (x_2/y_2)$ for all $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 > 0$, we can prove by induction that

$$m(k,l) \le \max\{m(k,1), \dots, m(k+l-1,1)\}, \qquad l \in \mathbb{N}. \tag{5.7}$$

Fix $\varepsilon > 0$. By (i), a.s. there exists a k_ε such that $m(k,1) \le m + \varepsilon$ for $k > k_\varepsilon$. By (5.7), the same is true for $m(k,l)$ for all $l \in \mathbb{N}$, and therefore also for $m(k,\infty)$. Since ε is arbitrary, (5.4) follows.

Let us now proceed with the proof of the proposition. Assumption (ii) implies that, a.s.,

$$\limsup_{k\to\infty} \left\| \frac{W_{\tau_k}}{k} - v \right\| \le \varepsilon \quad \text{and} \quad \limsup_{k\to\infty} \left| \frac{\tau_k}{k} - u \right| \le \varepsilon. \tag{5.8}$$

Assume w.l.o.g. that $\tau_0 = 0$. For $t \ge 0$, let k_t be the (random) non-negative integer such that

$$\tau_{k_t} \le t < \tau_{k_t+1}. \tag{5.9}$$

Then, from (5.8) and (5.9) we deduce that

$$\limsup_{t\to\infty} \left| \frac{t}{k_t} - u \right| \le \varepsilon \quad \text{and so} \quad \limsup_{t\to\infty} \left| \frac{t}{k_t} - \frac{\tau_{k_t}}{k_t} \right| \le 2\varepsilon. \tag{5.10}$$

Observe that, since $\tau_1 < \infty$ a.s., $k_t > 0$ a.s. for large enough t. For such t we may write

$$\begin{aligned} \left\| \frac{uW_t}{t} - v \right\| &\le \frac{\|W_t\|}{t} \left| u - \frac{t}{k_t} \right| + \left\| \frac{W_t - W_{\tau_{k_t}}}{k_t} \right\| + \left\| \frac{W_{\tau_{k_t}}}{k_t} - v \right\| \\ &\le \frac{\|W_t\|}{t} \left| u - \frac{t}{k_t} \right| + \sup_{s\in(\tau_{k_t},\tau_{k_t+1}]} \left\| \frac{W_s - W_{\tau_{k_t}}}{s - \tau_{k_t}} \right\| \left| \frac{t-\tau_{k_t}}{k_t} \right| + \left\| \frac{W_{\tau_{k_t}}}{k_t} - v \right\|, \end{aligned} \tag{5.11}$$

from which we obtain the conclusion by taking the limsup as $t \to \infty$ in (5.11), using (i), (5.3), (5.8) and (5.10), and then dividing by u. □

5.1.2 Conditions for the skeleton to have an approximate velocity

The following lemma states sufficient conditions for a discrete-time process to have an approximate velocity. It will be used in the proof of Proposition 5.4 below.

Lemma 5.3. *Let $X = (X_k)_{k\in\mathbb{N}_0}$ be a sequence of random vectors in $\mathbb{R}^d$ with joint law P. Suppose that there exist a probability measure Q on $\mathbb{R}^d$ and numbers $\phi \in [0,1)$, $a > 1$, $K > 0$ with $\int_{\mathbb{R}^d} \|x\|^a \, Q(dx) \le K^a$, such that, P-a.s. for all $k \in \mathbb{N}_0$,*
(i) $|P(X_{k+1} - X_k \in A \mid X_0, \dots, X_k) - Q(A)| \le \phi$ for all A measurable;
(ii) $E[\|X_{k+1} - X_k\|^a | X_0, \dots, X_k] \le K^a$.
Then

$$\limsup_{n\to\infty} \left\| \frac{X_n}{n} - v \right\| \le 2K\phi^{(a-1)/a} \qquad P\text{-a.s.}, \tag{5.12}$$

where $v = \int_{\mathbb{R}^d} x \, Q(dx)$. In other words, $X \in AV(2K\phi^{(a-1)/a}, v)$.

Proof. The proof is an adaptation of the proof of Lemma 3.13 in [5]; we include it here for completeness. With regular conditional probabilities, we can, using (i), couple P and $Q^{\otimes\mathbb{N}_0}$ according to a standard splitting representation (see e.g. Berbee [3]). More precisely, on an enlarged probability space we can construct random variables

$$(\Delta_k, \tilde{X}_k, \hat{X}_k)_{k\in\mathbb{N}_0} \tag{5.13}$$

such that

(1) $(\Delta_k)_{k\in\mathbb{N}_0}$ is an i.i.d. sequence of Bernoulli(ε) random variables.

(2) $(\tilde{X}_k)_{k\in\mathbb{N}_0}$ is an i.i.d. sequence of random vectors with law Q.

(3) $(\Delta_l)_{l\ge k}$ is independent of $(\tilde{X}_l, \hat{X}_l)_{0\le l<k}$ and of $\hat{X}_k$.

(4) $((1-\Delta_k)\tilde{X}_k + \Delta_k\hat{X}_k)_{k\in\mathbb{N}_0}$ is equal in distribution to $(X_k)_{k\in\mathbb{N}_0}$.

(5) With $\mathcal{G}_k = \sigma\{\Delta_l, \tilde{X}_l, \hat{X}_l \colon\ 0 \le l \le k\}$, for any Borel function $f \ge 0$, $E[f(\tilde{X}_k) \mid \mathcal{G}_{k-1}]$ is measurable w.r.t. $\sigma\{X_l \colon\ 0 \le l \le k-1\}$.

Using (4), we may write

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \frac{1}{n}\sum_{k=1}^{n} \tilde{X}_k - \frac{1}{n}\sum_{k=1}^{n} \Delta_k\tilde{X}_k + \frac{1}{n}\sum_{k=1}^{n} \Delta_k\hat{X}_k, \tag{5.14}$$

where the equality holds in distribution. As $n \to \infty$, the first term converges a.s. to v by the LLN for i.i.d. random variables. By Hölder's inequality, the norm of the second term is at most

$$\left(\frac{1}{n}\sum_{k=1}^{n} \Delta_k\right)^{(a-1)/a} \left(\frac{1}{n}\sum_{k=1}^{n} \|\tilde{X}_k\|^a\right)^{1/a}, \tag{5.15}$$

which, by (1) and (2), converges a.s. as $n \to \infty$ to

$$\varepsilon^{(a-1)/a} \left(\int_{\mathbb{R}^d} \|x\|^a \, Q(dx)\right)^{1/a} \le K\varepsilon^{(a-1)/a}. \tag{5.16}$$

For the third term, put $X_k^* = E[\hat{X}_k \mid \mathcal{G}_{k-1}]$. Fix $y \in \mathbb{R}^d$ and put

$$M_n^y = \sum_{k=1}^n \frac{\Delta_k}{k} \langle \hat{X}_k - X_k^*, y \rangle. \tag{5.17}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Then $(M_n^y)_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$-martingale whose quadratic variation is

$$\langle M^y \rangle_n = \sum_{k=1}^n \frac{\Delta_k}{k^2} \langle \hat{X}_k - X_k^*, y \rangle^2. \tag{5.18}$$

By the Burkholder-Gundy inequality, we have

$$\begin{aligned} E\left[\sup_{n\in\mathbb{N}} |M_n^y|^{a\wedge 2}\right] &\le C\,E\left[\langle M^y \rangle_\infty^{(a\wedge 2)/2}\right] \\ &\le C\,E\left[\sum_{k=1}^\infty \frac{\Delta_k}{k^{a\wedge 2}} \left|\langle \hat{X}_k - X_k^*, y \rangle\right|^{a\wedge 2}\right] \le C\,\|y\|^{a\wedge 2} K^{a\wedge 2}, \end{aligned} \tag{5.19}$$

where C denotes a generic constant. This implies that M_n^y is uniformly integrable for every y and therefore converges a.s. as $n \to \infty$. Hence Kronecker's lemma gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \Delta_k \langle \hat{X}_k - X_k^*, y \rangle = 0 \ \text{ a.s.} \tag{5.20}$$

Since y is arbitrary, this in turn implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \Delta_k (\hat{X}_k - X_k^*) = 0 \ \text{ a.s.} \tag{5.21}$$

On the other hand, since $\|\Delta_k \hat{X}_k\| \le \|X_k\|$, we have by (1), (3) and (5) that

$$\varepsilon E[\|\hat{X}_k\|^a \mid \mathcal{G}_{k-1}] = E[\Delta_k \|\hat{X}_k\|^a \mid \mathcal{G}_{k-1}] \le E[\|X_k\|^a \mid \mathcal{G}_{k-1}] \le K^a, \tag{5.22}$$

where the last inequality uses condition (ii). Combining (5.22) with Jensen's inequality, we obtain

$$\|X_k^*\| \le E\big[\|\hat{X}_k\|^a \mid \mathcal{G}_{k-1}\big]^{1/a} \le \frac{K}{\varepsilon^{1/a}}, \tag{5.23}$$

so that

$$\left\| \frac{1}{n} \sum_{k=1}^n \Delta_k X_k^* \right\| \le \frac{K}{\varepsilon^{1/a}} \left(\frac{1}{n} \sum_{k=1}^n \Delta_k \right). \tag{5.24}$$

Since the right-hand side converges a.s. to $K\varepsilon^{(a-1)/a}$ as $n \to \infty$, the proof is finished. □

5.2 Additive functionals of a discrete-time Markov chain

5.2.1 Notation

Let $(\eta_n)_{n\in\mathbb{N}_0}$ be a Markov process in the canonical space equipped with the time-shift operators $(\theta_n)_{n\in\mathbb{N}_0}$. Put $\mathscr{F}_n := \sigma\{\eta_i \colon 0 \le i \le n\}$ and let P_η denote the law of $(\eta_n)_{n\in\mathbb{N}_0}$ when $\eta_0 = \eta$. Fix an initial measure ν and suppose that

$$\nu_n \ll \nu, \tag{5.25}$$

where ν_n is the law of η_n under P_ν, where $P_\nu(\cdot) := \int \nu(d\eta) P_\eta(\cdot)$.

Let F be a $\mathbb{R}^d$-valued measurable function and put $Z_0 = 0$, $Z_n := \sum_{k=0}^{n-1} F(\eta_k)$, $n \in \mathbb{N}$. Then

$$Z_{n+k} - Z_k = \theta_k Z_n, \qquad k, n \in \mathbb{N}_0. \tag{5.26}$$

We are interested in finding regeneration times $(\tau_k)_{k\in\mathbb{N}_0}$ such that $(Z_{\tau_k}, \tau_k)_{k\in\mathbb{N}_0}$ satisfies the hypotheses of Lemma 5.3. In the Markovian setting it makes sense to look for τ_k of the form

$$\tau_0 = 0, \qquad \tau_{k+1} = \tau_k + \theta_{\tau_k}\tau, \quad k \in \mathbb{N}_0, \tag{5.27}$$

where τ is a random time.

Condition (i) of Lemma 5.3 is a "decoupling condition". It states that the law of an increment of X depends weakly on the previous increments. Such a condition can be implemented by the occurrence of a "decoupling event" under which the increments of $(Z_{\tau_k}, \tau_k)_{k\in\mathbb{N}_0}$ lose dependence. In this setting, τ is a time at which the decoupling event is observed.

5.2.2 Second key proposition: approximate regeneration times

Proposition 5.4 below is a consequence of Lemma 5.3 and is the main result of this section. It will be used together with Proposition 5.2 to prove the LLN in Section 6. It gives a way to construct τ when the decoupling event can be detected by "probing the future" with a stopping time.

For a random variable $\mathcal{T}$ taking values in $\mathbb{N}_0$, we define the *image* of $\mathcal{T}$ by $\mathcal{I}_\mathcal{T} := \{n \in \mathbb{N}_0 \colon P_\nu(\mathcal{T} = n) > 0\}$, and its closure under addition by $\bar{\mathcal{I}}_\mathcal{T} := \{n \in \mathbb{N}_0 \colon \exists\, l \in \mathbb{N}, i_1, \ldots, i_l \in \mathcal{I}_\mathcal{T} \colon n = i_1 + \cdots + i_l\}$.

Proposition 5.4. *Let $\mathcal{T}$ be a stopping time for the filtration $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$ taking values in $\mathbb{N} \cup \{\infty\}$. Put $D := \{\mathcal{T} = \infty\}$. Suppose that the following properties hold P_ν-a.s.:*
(i) For every $n \in \bar{\mathcal{I}}_\mathcal{T}$ there exists a $D_n \in \mathscr{F}_n$ such that

$$D \cap \theta_n D = D_n \cap \theta_n D. \tag{5.28}$$

(ii) There exist numbers $\rho \in (0, 1]$, $a > 1$, $C > 0$, $m > 0$ and $\phi \in [0, 1)$ such that

$$\begin{array}{ll} (a) & P_{\eta_0}(\mathcal{T} = \infty) \geq \rho, \\ (b) & E_{\eta_0}[\mathcal{T}^a, \mathcal{T} < \infty] \leq C^a, \\ (c) & \text{On } D,\ \|Z_t\| \leq mt \text{ for all } t \in \mathbb{N}_0, \\ (d) & \big|P_{\eta_0}\big((Z_n, \theta_n\mathcal{T})_{n\in\mathbb{N}_0} \in A \mid D\big) - P_\nu\big((Z_n, \theta_n\mathcal{T})_{n\in\mathbb{N}_0} \in A \mid D\big)\big| \leq \phi. \end{array} \tag{5.29}$$

Then there exists a random time $\tau \in \mathscr{F}_\infty$ taking values in $\mathbb{N}$ such that $X \in AV(\varepsilon, (v, u))$, where $(v, u) = E_\nu[(Z_\tau, \tau) \mid D]$, $u > 0$ and $\varepsilon = 12(m + 1)u\phi^{(a-1)/a}$. Here, $X_k := (Z_{\tau_k}, \tau_k)$ with τ_k as in (5.27)*.*

5.2.3 Two further propositions

In order to prove Proposition 5.4, we will need two further propositions (Propositions 5.5 and 5.6 below).

Proposition 5.5. *Let τ be a random time measurable w.r.t. $\mathscr{F}_\infty$ taking values in $\mathbb{N}$. Put τ_k as in (5.27) and $X_k := (Z_{\tau_k}, \tau_k)$. Suppose that there exists an event $D \in \mathscr{F}_\infty$ such that the following hold P_ν-a.s.:*
(i) For $n \in \mathcal{I}_\tau$, there exist events H_n and $D_n \in \mathscr{F}_n$ such that

$$\begin{array}{ll} (a) & \{\tau = n\} = H_n \cap \theta_n D, \\ (b) & D \cap \theta_n D = D_n \cap \theta_n D. \end{array} \tag{5.30}$$

(ii) There exist $\phi \in [0,1)$, $K > 0$ and $a > 1$ such that that, on $\{P_{\eta_0}(D) > 0\}$,

$$\begin{array}{ll} (a) & E_{\eta_0}[\|X_1\|^a | D] \le K^a, \\ (b) & |P_{\eta_0}(X_1 \in A | D) - P_\nu(X_1 \in A|D)| \le \phi \quad \forall\, A \text{ measurable.} \end{array} \tag{5.31}$$

Then $X \in AV\big(\varepsilon, (v,u)\big)$, where $\varepsilon = 2K\phi^{(a-1)/a}$ and $(v,u) := E_\nu[X_1|D]$.

Proof. Let $\mathscr{F}_{\tau_k}$ be the σ-algebra of the events before time τ_k, i.e., all events $B \in \mathscr{F}_\infty$ such that for all $n \in \mathbb{N}_0$ there exist $B_n \in \mathscr{F}_n$ such that $B \cap \{\tau_k = n\} = B_n \cap \{\tau_k = n\}$. We will prove that, P_ν-a.s., for all $k \in \mathbb{N}_0$,

$$|P_\nu(\theta_{\tau_k} X_1 \in A | \mathscr{F}_{\tau_k}) - P_\nu(X_1 \in A|D)| \le \phi \quad \forall\, A \text{ measurable} \tag{5.32}$$

and

$$E_\nu\left[\|\theta_{\tau_k} X_1\|^a | \mathscr{F}_{\tau_k}\right] \le K^a. \tag{5.33}$$

By putting $Q(\cdot) := P_\nu(X_1 \in \cdot | D)$ and noting that $X_{k+1} - X_k = \theta_{\tau_k} X_1$ and $X_j \in \mathscr{F}_{\tau_k}$ for all $0 \le j \le k$, this will imply that the conditions of Lemma 5.3 are all satisfied.

To prove (5.32–5.33), we note that, by (i), we can verify by induction that (i)(a) holds also for τ_k, i.e., for all $n \in \mathcal{I}_{\tau_k}$ there exist $H_{k,n} \in \mathscr{F}_n$ such that

$$\{\tau_k = n\} = H_{k,n} \cap \theta_n D. \tag{5.34}$$

For $B \in \mathscr{F}_{\tau_k}$ and a measurable nonnegative function f, we may write

$$\begin{aligned} E_\nu\left[1_B \theta_{\tau_k} f(X_1)\right] &= \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B \cap \{\tau_k = n\}} \theta_n f(X_1)\right] = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B_n \cap H_{k,n}} \theta_n\big(1_D f(X_1)\big)\right] \\ &= \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B_n \cap H_{k,n}} P_{\eta_n}(D) E_{\eta_n}\left[f(X_1)|D\right]\right]. \end{aligned} \tag{5.35}$$

To obtain (5.32), choose $f(x) = \|x\|^a$ and conclude by using (ii)(a) together with (5.25). To obtain (5.33), choose $f = 1_A$, subtract $P_\nu(B) E_\nu\left[f(X_1)|D\right]$ from (5.35) using that, by (i)(a), $P_\nu(B) = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B_n \cap H_{k,n}} P_{\eta_n}(D)\right]$ and conclude by using (ii)(b) together with (5.25). $\square$

Observe that, by (i)(a), (5.25) and the assumption that $\tau < \infty$, we must have $P_\nu(D) > 0$.

Proposition 5.6. *Let $\mathcal{T}$ be a stopping time as in Proposition 5.4 and suppose that conditions (ii)(a) and (ii)(b) of that proposition are satisfied. Define a sequence of stopping times $(T_k)_{k \in \mathbb{N}_0}$ as follows. Put $T_0 = 0$ and, for $k \in \mathbb{N}_0$,*

$$T_{k+1} := \begin{cases} \infty & \text{if } T_k = \infty \\ T_k + \theta_{T_k} \mathcal{T} & \text{otherwise.} \end{cases} \tag{5.36}$$

Put

$$N := \inf\{k \in \mathbb{N}_0 \colon\ T_k < \infty \text{ and } T_{k+1} = \infty\}. \tag{5.37}$$

Then $N < \infty$ a.s. and there exists a constant $\chi = \chi(a, \rho) > 0$ such that, P_ν-a.s.,

$$E_{\eta_0}\left[T_N^a\right] \le (\chi C)^a. \tag{5.38}$$

Furthermore, $\mathcal{I}_{T_N} \subset \bar{\mathcal{I}}_{\mathcal{T}}$.

Proof. First, let us check that

$$P_{\eta_0}(N \ge n) \le (1-\rho)^n. \tag{5.39}$$

Indeed, $N \ge n$ if and only if $T_n < \infty$, so that, for $k \in \mathbb{N}_0$,

$$P_{\eta_0}(T_{k+1} < \infty) = E_{\eta_0}\left[1_{\{T_k<\infty\}} P_{\eta_{T_k}}(\mathcal{T} < \infty)\right] \le (1-\rho) P_{\eta_0}(T_k < \infty), \tag{5.40}$$

where we use (ii)(a) and the fact that (5.25) implies that the law of η_{T_k} is also absolutely continuous w.r.t. ν. Clearly, (5.39) follows from (5.40) by induction. In particular, $N < \infty$ a.s.

From (5.36) we see that, for $0 \le k \le n$ and on $\{T_k < \infty\}$,

$$T_n = T_k + \theta_{T_k} T_{n-k}. \tag{5.41}$$

Using (ii)(a) and (b), with the help of (5.25) again, we can a.s. estimate, for $0 \le k < n$,

$$\begin{aligned}
E_{\eta_0}\left[1_{\{T_n<\infty\}} \left|T_{k+1} - T_k\right|^a\right] &= E_{\eta_0}\left[1_{\{T_{k+1}<\infty\}} \left|T_{k+1} - T_k\right|^a P_{\eta_{T_{k+1}}}(T_{n-k-1} < \infty)\right] \\
&\le (1-\rho)^{n-k-1} E_{\eta_0}\left[1_{\{T_k<\infty, \theta_{T_k}\mathcal{T}<\infty\}} \theta_{T_k}\mathcal{T}^a\right] \\
&= (1-\rho)^{n-k-1} E_{\eta_0}\left[1_{\{T_k<\infty\}} E_{\eta_{T_k}}\left[1_{\{\mathcal{T}<\infty\}}\mathcal{T}^a\right]\right] \\
&\le (1-\rho)^{n-k-1} C^a P_{\eta_0}(T_k < \infty) \\
&\le (1-\rho)^{n-1} C^a.
\end{aligned} \tag{5.42}$$

Now write, using (5.36),

$$T_N = \sum_{k=0}^{N-1} T_{k+1} - T_k. \tag{5.43}$$

By Jensen's inequality,

$$T_N^a \le N^{a-1} \sum_{k=0}^{N-1} \left|T_{k+1} - T_k\right|^a \tag{5.44}$$

so that, by (5.42),

$$E_{\eta_0}\left[T_N^a\right] \le \sum_{n=1}^{\infty} n^{a-1} \sum_{k=0}^{n-1} E_{\eta_0}\left[1_{\{N=n\}} \left|T_{k+1} - T_k\right|^a\right] \le C^a \sum_{n=1}^{\infty} n^a (1-\rho)^{n-1} \quad \text{a.s.} \tag{5.45}$$

and (5.38) follows by taking $\chi(a,\rho) = \left(\sum_{n=1}^{\infty} n^a (1-\rho)^{n-1}\right)^{1/a}$.

As for the claim that $\mathcal{I}_{T_N} \subset \bar{\mathcal{I}}_{\mathcal{T}}$, write

$$\{T_N = n\} = \sum_{k=0}^{\infty} \{T_k = n, N = k\} \tag{5.46}$$

to see that $\mathcal{I}_{T_N} \subset \bigcup_{k=0}^{\infty} \mathcal{I}_{T_k}$. Using (5.36), we can verify by induction that, for each k, $\mathcal{I}_{T_k} \subset \{n \in \mathbb{N}_0 \colon \exists\, i_1, \ldots, i_k \in \mathcal{I}_{\mathcal{T}} \colon\ n = i_1 + \cdots + i_k\} \subset \bar{\mathcal{I}}_{\mathcal{T}}$, and the claim follows. □

5.2.4 Proof of Proposition 5.4

We can now combine Propositions 5.5 and 5.6 to prove Proposition 5.4.

Proof. In the following we will refer to the hypotheses/results of Proposition 5.5 with the prefix P. For example, P(i)(a) denotes hypothesis (i)(a) in that proposition. The hypotheses in Proposition 5.4 will be referred to without a prefix. Since the hypotheses of Proposition 5.6 are a subset of those of Proposition 5.4, the conclusions of the former are valid.

We will show that, if $\tau := t_0 + \theta_{t_0} T_N$ for a suitable t_0, then τ satisfies the hypotheses of Proposition 5.5 for a suitable K. There are two cases. If $\mathcal{I}_{\mathcal{T}} = \emptyset$, then $T_N \equiv 0$. Choosing $t_0 = 1$, we basically fall in the context of Lemma 5.3. P(i)(a) and P(i)(b) are trivial, (ii)(c) implies that P(ii)(a) holds with $K = (m+1)$, while P(ii)(b) follows immediately from (ii)(d). Therefore, we may suppose that $\mathcal{I}_{\mathcal{T}} \neq \emptyset$ and put $\iota := \min \mathcal{I}_{\mathcal{T}} \geq 1$. Let $\hat{C} := 1 \vee (\chi C)$ and $t_0 := \iota \lceil \hat{C} \rho^{-1/a} \rceil$. We will show that τ satisfies the hypotheses of Proposition 5.5 with $K = 6\iota(m+1)\hat{C}\rho^{-1/a}$.

<u>P(i)(a)</u>: First we show that this property is true for T_N. Indeed,

$$\{T_N = n\} = \sum_{k \in \mathbb{N}_0} \{N = k, T_k = n\} = \sum_{k \in \mathbb{N}_0} \{T_k = n, \theta_n \mathcal{T} = \infty\} \tag{5.47}$$

$$= \theta_n D \cap \left(\bigcup_{k \in \mathbb{N}_0} \{T_k = n\} \right), \tag{5.48}$$

and $\hat{H}_n := \bigcup_{k \in \mathbb{N}_0} \{T_k = n\} \in \mathscr{F}_n$ since the T_k's are all stopping times. Now we observe that $\{\tau = n\} = \theta_{t_0}\{T_N = n - t_0\}$, so we can take $H_n := \emptyset$ if $n < t_0$ and $H_n := \theta_{t_0} \hat{H}_{n-t_0}$ otherwise.

<u>P(i)(b)</u>: By (i), it suffices to show that $\mathcal{I}_\tau \subset \bar{\mathcal{I}}_{\mathcal{T}}$. Since $t_0 \in \bar{\mathcal{I}}_{\mathcal{T}}$ (as an integer multiple of ι), this follows from the definition of τ and the last conclusion of Proposition 5.6.

<u>P(ii)(a)</u>: By (ii)(c), $\|X_1\|^a = (\|Z_\tau\| + \tau)^a \leq ((m+1)\tau)^a$ on D. Therefore, we just need to show that

$$E_{\eta_0}[\tau^a | D] \leq (6\iota\hat{C})^a/\rho. \tag{5.49}$$

Now, $\tau^a \leq 2^{a-1}(t_0^a + \theta_{t_0} T_N^a)$ and, by Proposition 5.6 and (5.25),

$$E_{\eta_0}[\theta_{t_0} T_N^a] = E_{\eta_0}\left[E_{\eta_{t_0}}[T_N^a]\right] \leq \hat{C}^a. \tag{5.50}$$

Using (ii)(a), we obtain

$$E_{\eta_0}[\theta_{t_0} T_N^a | D] \leq \hat{C}^a/\rho. \tag{5.51}$$

Since $t_0 \leq 2\iota\hat{C}\rho^{-1/a}$ and $\iota \geq 1$, (5.49) follows.

<u>P(ii)(b)</u>: Let $S = (S_n)_{n \in \mathbb{N}_0}$ with $S_n := \theta_n \mathcal{T}$. By (ii)(d), it suffices to show that $X_1 = (Z_\tau, \tau) \in \sigma(Z, S)$. Since $Z_\tau = \sum_{n=0}^\infty 1_{\{\tau = n\}} Z_n \in \sigma(Z, \tau)$, it suffices to show that $\tau \in \sigma(S)$. From the definition of the T_k's, we verify by induction that each T_k is measurable in $\sigma(S)$. Since $N \in \sigma((T_k)_{k \in \mathbb{N}_0})$, both N and T_N are also in $\sigma(S)$. Therefore, $\tau \in \sigma(\theta_{t_0} S) \subset \sigma(S)$.

With all hypotheses verified, Proposition 5.5 implies that $X \in AV(\hat{\varepsilon}, (v, u))$, where $(v, u) = E_\nu[X_1 | D]$ and $\hat{\varepsilon} = 2K\phi^{(a-1)/a}$. To conclude, observe that $u = E_\nu[\tau | D] \geq t_0 \geq \iota\hat{C}\rho^{-1/a} > 0$, so that $K = 6(m+1)\iota\hat{C}\rho^{-1/a} \leq 6(m+1)u$. Therefore, $\hat{\varepsilon} \leq \varepsilon$ and the proposition follows. In the case $\mathcal{I}_{\mathcal{T}} = \{0\}$, we conclude similarly since $u = 1$ and $K = (m+1)$. □

6 Proof of Theorem 4.1

In this section we show how to put the model defined in Section 3 in the context of Section 5, and we prove the LLN using Propositions 5.2 and 5.4.

6.1 Two further lemmas

Before we start, we first derive two lemmas (Lemmas 6.1 and 6.2 below) that will be needed in Section 6.2. The first lemma relates the laws of the environment as seen from W_n and from W_0. The second lemma is an extension of the conditional cone-mixing property for functions that depend also on Y.

Lemma 6.1. *$\bar{\mu}_n \ll \bar{\mu}$ for all $n \in \mathbb{N}$.*

Proof. For $t \geq 0$, let $\bar{\mu}_{t-}$ denote the law of $\theta_{W_{t-}}\xi_t$ under $\mathbb{P}_\mu$. First we will show that $\bar{\mu}_{t-} \ll \mu$. This is a consequence of the fact that μ is translation-invariant equilibrium, and remains true if we replace W_{t-} by any random variable taking values in $\mathbb{Z}^d$. Indeed, if $\mu(A) = 0$ then $\mathbb{P}_\mu(\theta_x \xi_t \in A) = 0$ for every $x \in \mathbb{Z}^d$, so

$$\bar{\mu}_{t-}(A) = \mathbb{P}_\mu(\theta_{W_{t-}}\xi_t \in A) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}_\mu(W_{t-} = x, \theta_x \xi_t \in A) = 0. \tag{6.1}$$

Now take $n \in \mathbb{N}$ and let $g_n := \frac{d\bar{\mu}_{n-}}{d\mu}$. For any measurable $f \geq 0$,

$$\begin{aligned}
\mathbb{E}_\mu\left[f(\theta_{W_n}\xi_n)\right] &= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[1_{\{W_n - W_{n-} = x\}} f(\theta_x \theta_{W_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[P_{\theta_{W_{n-}}\xi_n}(W_0 = x) f(\theta_x \theta_{W_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[g_n(\xi_0) P_{\xi_0}(W_0 = x) f(\theta_x \xi_0)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[g_n(\xi_0) 1_{\{W_0 = x\}} f(\theta_x \xi_0)\right] \\
&= \mathbb{E}_\mu\left[g_n(\xi_0) f(\theta_{W_0}\xi_0)\right]
\end{aligned} \tag{6.2}$$

where, for the second equality, we use (A3). □

Lemma 6.2. *For L large enough and for all nonnegative $f \in \mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty$,*

$$\left|\mathbb{E}_\eta\left[\theta_L f \mid \Gamma_L\right] - \mathbb{E}_{\bar{\mu}}[\theta_L f \mid \Gamma_L]\right| \leq \phi_L \|f\|_\infty \qquad \bar{\mu} - a.s. \tag{6.3}$$

Proof. Put $f_y(\eta) = f(\eta, y)$ and abbreviate $Y^{(L)} = (Y_k)_{k>L}$. Then $\theta_L f = \theta_L f_{Y^{(L)}}$. Since Γ_L depends on Y only through $(Y_k)_{k \leq L}$, we have

$$\mathbb{E}_\eta[\theta_L f \, 1_{\Gamma_L} \mid Y^{(L)}] = \mathbb{E}_\eta\left[\theta_L f_{(\cdot)} \, 1_{\Gamma_L}\right] \circ (Y^{(L)}), \tag{6.4}$$

and the claim follows from (H4) applied to f_y. □

6.2 Proof of Theorem 4.1

Proof. Fix an $L \in \mathbb{N}$ large enough and define

$$\begin{aligned} \eta_n &:= \left(\theta_{Z_n}\xi_{(n,n+1]}, Y_{n+1}\right),\; n \in \mathbb{N}_0 \\ \mathcal{T}_L &:= \begin{cases} L & \text{on } \Gamma_L^c, \\ L + \theta_L\lceil \mathcal{S}\rceil & \text{on } \Gamma_L. \end{cases} \end{aligned} \tag{6.5}$$

Observe that $\lceil \mathcal{S}\rceil$ is still a stopping time, and that, by (3.3), Z_n is an additive functional of the Markov chain $(\eta_n)_{n\in\mathbb{N}_0}$. To avoid confusion, we will denote the time-shift operator for η_n by $\bar\theta$, which is given by $\bar\theta_n = \theta_{Z_n}\theta_n$.

Next, we verify (5.25) and the hypotheses of Proposition 5.4 for Z_n and $\mathcal{T}_L$ under $\mathbb{P}_{\bar\mu}$. These hypotheses will be referred to with the prefix P. The notation here is consistent in the sense that parameters in Section 3 are named according to their role in Section 5; the presence/absence of a subscript L indicates whether the parameter depends on L or not.

Observe that the law of $(\eta_n)_{n\in\mathbb{N}}$ given η_0 is the same as the law of $(\eta_n)_{n\in\mathbb{N}_0}$ under $\mathbb{P}_{\theta_{Z_1}\xi_1}$. Therefore, by Lemma 6.1, in order to prove results P_ν-a.s., it suffices to prove them under $\mathbb{P}_\eta$ for $\bar\mu$-a.e. η.

<u>(5.25)</u>: Since $(Y_n)_{n\in\mathbb{N}}$ is i.i.d. and Y_{n+1} is independent of (Z_n, ξ), we just need to worry about the first coordinate of η_n. Put $\varphi_n := \frac{d\bar\mu_n}{d\bar\mu}$ (which exists by Lemma 6.1). For $f \geq 0$ measurable, we may write

$$\begin{aligned} \mathbb{E}_{\bar\mu}\left[f\left(\theta_{Z_n}\xi_{(n,n+1]}\right)\right] &= \mathbb{E}_\mu\left[\mathbb{E}_{\bar\xi_n}\left[f\left(\xi_{(0,1]}\right)\right]\right] = \mathbb{E}_{\bar\mu}\left[\varphi_n(\xi_0)\mathbb{E}_{\xi_0}\left[f\left(\xi_{(0,1]}\right)\right]\right] \\ &= \mathbb{E}_{\bar\mu}\left[\varphi_n(\xi_0) f\left(\xi_{(0,1]}\right)\right], \end{aligned} \tag{6.6}$$

so (5.25) follows.

<u>P(i)</u>: We will find D_n for $n \geq L$. This is enough, since both $\mathcal{I}_{\mathcal{T}_L}$ and $\bar{\mathcal{I}}_{\mathcal{T}_L}$ are subsets of $[L,\infty)\cap\mathbb{N}$. We may write

$$\begin{aligned} \{\mathcal{T}_L = \infty\} &= \Gamma_L \cap \{|Z_{t+L} - Z_L| \leq mt \;\forall\, t \geq 0\}, \\ \bar\theta_n\{\mathcal{T}_L = \infty\} &= \theta_{Z_n}\theta_n\Gamma_L \cap \{|Z_{t+n+L} - Z_{n+L}| \leq mt \;\forall\, t \geq 0\}. \end{aligned} \tag{6.7}$$

Furthermore, by (H1), on $\bar\theta_n\Gamma_L$, $Z_{t+n} = Z_n$ for $t \in [0, L]$. Therefore when we intersect the two above events, we get

$$D \cap \bar\theta_n D = \Gamma_L \cap \{|Z_t - Z_L| \leq mt \;\forall\, t \in [L, n]\} \cap \bar\theta_n D, \tag{6.8}$$

i.e., the hypothesis holds with $D_n := \Gamma_L \cap \{|Z_t - Z_L| \leq mt \;\forall\, t \in [L,n]\} \in \mathscr{F}_n$ for $n \geq L$.

<u>P(ii)(a)</u>: Since $\{\mathcal{T}_L = \infty\} = \{\theta_L\mathcal{S} = \infty\} \cap \Gamma_L$, we get from (H2) and (H3)(1) that, for for $\bar\mu$-a.e.η,

$$\mathbb{P}_\eta\left(\mathcal{T}_L = \infty\right) = \mathbb{P}_\eta\left(\theta_L\mathcal{S} = \infty \mid \Gamma_L\right)\mathbb{P}_\eta(\Gamma_L) \geq \kappa_L\gamma_L > 0, \tag{6.9}$$

so that we can take $\rho_L := \kappa_L\gamma_L$.

<u>P(*ii*)(*b*)</u>: By the definition of $\mathcal{T}_L$, we have

$$\begin{aligned} \mathcal{T}_L^a 1_{\{\mathcal{T}_L<\infty\}} &= L^a 1_{\Gamma_L^c} + \left(L + \theta_L\lceil\mathcal{S}\rceil\right)^a 1_{\Gamma_L\cap\{\theta_L\lceil\mathcal{S}\rceil<\infty\}} \\ &\leq L^a 1_{\Gamma_L^c} + \left(L + 1 + \theta_L\mathcal{S}\right)^a 1_{\Gamma_L\cap\{\theta_L\mathcal{S}<\infty\}} \\ &\leq 2^{a-1}(L+1)^a + 2^{a-1}\theta_L\left(\mathcal{S}^a 1_{\{\mathcal{S}<\infty\}}\right)1_{\Gamma_L}. \end{aligned} \tag{6.10}$$

Therefore, by (H3)(2), we get, for $\bar{\mu}$-a.e. η,

$$\mathbb{E}_\eta\left[\mathcal{T}_L^a 1_{\{\mathcal{T}_L<\infty\}}\right] \le 2^a((L+1)^a + (1\vee\psi_L)^a) \le [2(L+1+1\vee\psi_L)]^a\,, \tag{6.11}$$

so that we can take $C_L := 2(L+1+1\vee\psi_L)$.

P(ii)(c): This follows from (H1) and the definition of $\mathcal{S}$.

P(ii)(d): First note that $\bar{\theta}_n\mathcal{T}_L \in \sigma(Z)$ for all $n\in\mathbb{N}_0$. Since $\{\mathcal{T}_L=\infty\} = \Gamma_L \cap \theta_L\{\mathcal{S}=\infty\}$, we have $Z \in \theta_L\mathscr{C}_{R,\infty}(m)\vee\mathscr{Y}_\infty$ on $\{\mathcal{T}_L=\infty\}$ by (H1) and (A3), so this claim follows from Lemma 6.2.

Thus, for large enough L, we can conclude by Proposition 5.4 that there exists a sequence of times $(\tau_k)_{k\in\mathbb{N}_0}$ with $\lim_{k\to\infty}\tau_k=\infty$ a.s. such that $(Z_{\tau_k},\tau_k)_{k\in\mathbb{N}_0}\in AV(\varepsilon_L,(v_L,u_L))$, where

$$\begin{aligned} v_L &= \mathbb{E}_{\bar{\mu}}[Z_{\tau_1}|D],\\ u_L &= \mathbb{E}_{\bar{\mu}}[\tau_1|D]>0,\\ \varepsilon_L &= 12(m+1)u_L\phi_L^{(a-1)/a}. \end{aligned} \tag{6.12}$$

From (6.12) and (ii)(c), Proposition 5.2 implies that $Z\in AV(\delta_L,w_L)$, where

$$\begin{aligned} w_L &= v_L/u_L,\\ \delta_L &= (3m+1)12(m+1)\phi_L^{(a-1)/a}. \end{aligned} \tag{6.13}$$

By (H4), $\lim_{L\to\infty}\delta_L=0$. As was observed after Definition 5.1, this implies that $w := \lim_{L\to\infty} w_L$ exists and that $\lim_{t\to\infty} t^{-1}Z_t = w$ $\mathbb{P}_{\bar{\mu}}$-a.s., which, by (3.4), implies the same for W_t, $\mathbb{P}_\mu$-a.s. □

We have at this point finished the proof of our LLN. In the following sections, we will look at examples that satisfy (H1)–(H4). Section 7 is devoted to the $(\infty,0)$-model for two classes of one-dimensional spin-flip systems. In Section 8 we discuss three additional models where the hypotheses are satisfied, and one where they are not.

7 Proof of Theorem 4.2

We begin with a proper definition of the $(\infty,0)$-model in Section 7.1, where we identify the functions F_t of Section 2 and check assumptions (A1)–(A3). In Section 7.2, we first concern ourselves with finding events Γ_L satisfying (H1) and (H2) in suitable versions of spin-flip systems with bounded rates, and then show that (H3) holds. We also derive uniform integrability properties of $t^{-1}W_t$ which are the key to showing convergence in L^p once we have the LLN. In Sections 7.3 and 7.4, we specialize to particular constructions in order to prove (H4), which is the hardest of the four hypotheses. Section 7.5 is devoted to proving a criterion for positive or negative speed.

7.1 Definition of the model

Assume that ξ is a càdlàg process with state space $E := \{0,1\}^{\mathbb{Z}}$. We will define the walk W in several steps.

7.1.1 Identification of F_t

First, let $Tr^+ = Tr^+(\eta)$ and $Tr^- = Tr^-(\eta)$ denote the locations of the closest traps to the right and to the left of the origin in the configuration $\eta \in E$, i.e.,

$$\begin{array}{rcl} Tr^+(\eta) & := & \inf\{x \in \mathbb{N}_0\colon\ \eta(x) = 1, \eta(x+1) = 0\}, \\ Tr^-(\eta) & := & \sup\{x \in -\mathbb{N}_0\colon\ \eta(x) = 1, \eta(x+1) = 0\}, \end{array} \tag{7.1}$$

with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. For $i, j \in \{0, 1\}$, abbreviate $\langle i, j\rangle := \{\eta \in E\colon\ \eta(0) = i, \eta(1) = j\}$. Let $\bar{E} := \langle 1, 0\rangle$, i.e., the set of all the configurations with a trap at the origin.

Next, we define the functional J that gives us the jumps in W. For $b \in \{0, 1\}$ and $\eta \in E$, let

$$J(\eta, b) := T^+(\eta)\left(1_{\langle 1,1\rangle} + b1_{\langle 0,1\rangle}\right) + T^-(\eta)\left(1_{\langle 0,0\rangle} + (1-b)1_{\langle 0,1\rangle}\right), \tag{7.2}$$

i.e., J is equal to either the left or the right trap, depending on the configuration around the origin. In the case where the configuration is an inverted trap ($\langle 0, 1\rangle$), the direction of the jump is decided by the value of b. Observe that $J = Tr^+ = Tr^- = 0$ when $\eta \in \bar{E}$, independently of the value of b.

Suppose that $\xi_0 \in \bar{E}$, and let $(b_k)_{k\in\mathbb{N}}$ be a sequence of numbers in $\{0, 1\}$. We define $(F_t)_{t\geq 0}$ as a function of ξ and this sequence as follows. Put $X_0 = \tau_0 := 0$ and, recursively for $k \geq 0$,

$$\begin{array}{rcl} \tau_{k+1} & := & \inf\{t > \tau_k\colon\ (\eta_t(X_k), \eta_t(X_k + 1)) \neq (1, 0)\}, \\ X_{k+1} & := & X_k + J(\theta_{X_k}\xi_{\tau_k}, b_{k+1}). \end{array} \tag{7.3}$$

Since ξ is càdlàg, we have $\tau_{k+1} - \tau_k > 0$ for all $k \in \mathbb{N}_0$. We define $(F_t)_{t\geq 0}$ as the path that jumps $X_{k+1} - X_k$ at time τ_{k+1} and is constant between jumps, i.e., for $t < \lim_{k\to\infty} \tau_k$,

$$F_t := \sum_{k=0}^{\infty} 1_{\{\tau_k \leq t < \tau_{k+1}\}} X_k. \tag{7.4}$$

The above definition makes sense as long as the jumps are finite. When this fails, we can declare $\pm\infty$ to be absorbing states.

When ξ is an IPS such that the total flip rate for each site is uniformly bounded, then $\lim_{k\to\infty} \tau_k = \infty$ a.s. This is true, for example, for any IPS with translation-invariant rates satisfying the existence conditions in Liggett [6], Chapter I. If, additionally, there are a.s. minimum and maximum densities under the equilibrium measure, then $J < \infty$ a.s. and, by induction, $X_k < \infty$ a.s. for every k as well, since, as we saw in the proof of Lemma 6.1, the law of the environment as seen from an integer-valued random variable is absolutely continuous w.r.t. the equilibrium. Therefore, in this case F_t is defined and is finite for all t.

7.1.2 Check of (A1)–(A3)

When, as in the case of the last paragraph, F_t is defined and finite for all t, we may define W_t so as to satisfy (A1) in the following way. Let $(b_{n,k})_{n,k\in\mathbb{N}_0}$ be a double-indexed sequence of i.i.d. Bernoulli($\frac{1}{2}$) random variables. Put $W_{0-} := 0$ and, recursively for $n \geq 0$ and $t \in [0, 1)$,

$$\begin{array}{rcl} W_n - W_{n-} & := & J\big(\theta_{W_{n-}}\xi_n, b_{n,0}\big), \\ W_{t+n} - W_n & := & F_t\big(\theta_{W_n}\xi_{[n,n+t]}, (b_{n+1,k})_{k\geq 1}\big). \end{array} \tag{7.5}$$

From (7.5), assumption (A1) follows with $(Y_n)_{n\in\mathbb{N}} := ((b_{n,k})_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$, and with $Z = W$ if $\xi_0 \in \bar{E}$, while Z is defined arbitrarily otherwise. This is enough because $\bar{\mu}(\bar{E}) = 1$. Assumption (A2) is a consequence of the fact that the value of J can be found by exploring contiguous pairs of sites, one pair at a time, starting around the origin. Therefore (A2) holds with $R = 1$. Assumption (A3) also follows from (7.5).

7.1.3 Monotonicity

The following monotonicity property will be helpful in checking (H3). In order to state it, we first endow both E and $D([0,\infty), E)$ with the usual partial ordering, i.e., for $\eta_1, \eta_2 \in E$, $\eta_1 \leq \eta_2$ means that $\eta_1(x) \leq \eta_2(x)$ for all $x \in \mathbb{Z}$, while, for $\xi^{(1)}, \xi^{(2)} \in D([0,\infty), E)$, $\xi^{(1)} \leq \xi^{(2)}$ means that $\xi^{(1)}_t \leq \xi^{(2)}_t$ for all $t \geq 0$.

Lemma 7.1. *Fix a realization of $(b_{n,k})_{n,k\in\mathbb{N}_0}$. If $\xi^{(1)} \leq \xi^{(2)}$, then $W_t(\xi^{(1)}, (b_{n,k})_{n,k\in\mathbb{N}_0}) \leq W_t(\xi^{(2)}, (b_{n,k})_{n,k\in\mathbb{N}_0})$ for all t for which both are defined.*

Proof. This is a straightforward consequence of the definition. To see why, we need only understand what happens when the two walks separate; when this happens, the second walk is always to the right of the first. □

7.2 Spin-flip systems with bounded flip rates

7.2.1 Dynamical random environment

From now on we will take ξ to be a single-site spin-flip system with translation-invariant and bounded flip rates. We may assume that the rates at the origin are of the form

$$c(\eta) = \begin{cases} c_0 + \lambda_0 p_0(\eta) & \text{when} \quad \eta(0) = 1, \\ c_1 + \lambda_1 p_1(\eta) & \text{when} \quad \eta(0) = 0, \end{cases} \tag{7.6}$$

where $c_i, \lambda_i > 0$ and $p_i \in [0,1]$. We assume the existence conditions of Liggett [6], Chapter I, which in our setting amounts to the additional requirement that c has bounded triple norm.

From (7.6), we see that the IPS is stochastically dominated by the system ξ^+ with rates

$$c^+(\eta) = \begin{cases} c_0 & \text{when} \quad \eta(0) = 1, \\ c_1 + \lambda_1 & \text{when} \quad \eta(0) = 0, \end{cases} \tag{7.7}$$

while it stochastically dominates the system ξ^- with rates

$$c^-(\eta) = \begin{cases} c_0 + \lambda_0 & \text{when} \quad \eta(0) = 1, \\ c_1 & \text{when} \quad \eta(0) = 0. \end{cases} \tag{7.8}$$

These are the rates of two independent spin-flip systems with respective densities $\rho^+ := (c_1 + \lambda_1)/(c_0 + c_1 + \lambda_1)$ and $\rho^- := c_1/(c_0 + \lambda_0 + c_1)$. Consequently, any equilibrium for ξ is stochastically dominated by, respectively, dominates a Bernoulli product measure with density ρ^+, respectively, ρ^-. Thus, the walk is defined and is finite for all times by the remarks made in the Section 7.1.

We will take as the dynamic random environment the triple $\Xi := (\xi^-, \xi, \xi^+)$ starting from initial configurations η^-, η, η^+ satisfying $\eta^- \leq \eta \leq \eta^+$, and coupled together via the basic (or

Vasershtein) coupling, which implements the stochastic ordering as an a.s. partial ordering. More precisely, Ξ is the IPS with state space E^3 whose rates are translation invariant and at the origin are given schematically by (the configuration of the middle coordinate is η),

$$\begin{aligned}
(000) \quad &\to \quad \begin{cases} (111) & c_1, \\ (011) & c(\eta)-c_1, \\ (001) & c_1+\lambda_1-c(\eta), \end{cases} \\
(001) \quad &\to \quad \begin{cases} (111) & c_1, \\ (011) & c(\eta)-c_1, \\ (000) & c_0, \end{cases} \\
(011) \quad &\to \quad \begin{cases} (111) & c_1, \\ (000) & c_0, \\ (001) & c(\eta)-c_0, \end{cases} \\
(111) \quad &\to \quad \begin{cases} (000) & c_0, \\ (001) & c(\eta)-c_0, \\ (011) & c_0+\lambda_0-c(\eta). \end{cases}
\end{aligned} \tag{7.9}$$

7.2.2 Definition of Γ_L and verification of (H1)–(H3)

Using Ξ, we can define the events Γ_L by

$$\Gamma_L := \left\{\xi_t^{\pm}(x) = \xi_0^{\pm}(x)\ \forall\, t\in[0,L],\ x=0,1\right\}. \tag{7.10}$$

Since $\xi^- \le \xi \le \xi^+$, this event implies that also $\xi_t(x) = \xi_0(x)$ for all $t \in [0,L]$ and $x = 0,1$. Therefore, when $\xi_0 \in \bar{E}$, Γ_L implies that there is a trap at the origin between times 0 and L. Since $\bar{\mu}$ is concentrated on $\bar{E}$, (H1) holds. The probability of Γ_L is positive and depends on the initial configuration only through the states at 0 and 1, so (H2) is also satisfied.

In order to verify (H3), we will take advantage of the stochastic domination in Ξ to reduce to the case of independent spin-flips. This will also allow us to deduce convergence in L^p, $p \ge 1$.

Lemma 7.2. *Let ξ be an independent spin-flip system. Let $\rho \in [0,1)$, and let ν be the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$ with density ρ, except at the sites 0 and 1, where the states are a.s. 1. Then*

- *(a) The process $(t^{-1}W_t^+)_{t\ge1}$ is bounded in L^p for all $p>1$.*
- *(b) Let $S := \sup\{t>0\colon W_t > mt\}$. There exist positive constants $m = m(\rho)$, $K_1 = K_1(\rho)$ and $K_2 = K_2(\rho)$ such that*

$$\mathbb{P}_\nu(S>t) \le K_1 e^{-K_2 t} \ \text{ for all } t>0. \tag{7.11}$$

Before proving this lemma, let us see how it leads to (H3). We will show that, for any $a>0$, there exists a constant $K \ge 0$ such that, for all $L \ge 1$ and $\eta \in \bar{E}$,

$$\mathbb{E}_\eta\left[\theta_L\left(\mathcal{S}^a 1_{\{\mathcal{S}<\infty\}}\right) \mid \Gamma_L\right] \le K. \tag{7.12}$$

To start, we note that

$$\mathcal{S} := \min(\mathcal{S}_+, \mathcal{S}_-), \tag{7.13}$$

where

$$\mathcal{S}_+ := \inf\{t \geq 0\colon\ W_t > mt\} \quad \text{and} \quad \mathcal{S}_- := \inf\{t \geq 0\colon\ W_t < -mt\}. \tag{7.14}$$

Let us focus on $\mathcal{S}_+$. Denote by $\underline{1}$ the configuration with 1's everywhere except at site 1, so that $\underline{1} \in \bar{E}$. Also, for $\eta \in E$, denote by $(\eta)_{1,1}$ the configuration obtained from η by setting the state at sites 0 and 1 equal to 1. Noting that $\mathcal{S}_+ 1_{\{\mathcal{S}_+ < \infty\}} \leq S$ and S is monotone in the initial configuration, and using stochastic domination, we may write

$$\begin{aligned} \mathbb{E}_\eta\left[\theta_L\left(\mathcal{S}_+^a 1_{\{\mathcal{S}_+<\infty\}}\right), \Gamma_L\right] &\leq \mathbb{E}_\eta\left[\Gamma_L, \mathbb{E}_{\xi_L}\left[S^a\right]\right] \leq \mathbb{E}_\eta\left[\Gamma_L, \mathbb{E}_{(\xi_L^+)_{1,1}}\left[S^a\right]\right] \\ &= \mathbb{P}_\eta(\Gamma_L)\,\mathbb{E}_\eta\left[\mathbb{E}_{(\xi_L^+)_{1,1}}\left[S^a\right]\right] \leq \mathbb{P}_\eta(\Gamma_L)\,\mathbb{E}_{\underline{1}}\left[\mathbb{E}_{(\xi_L^+)_{1,1}}\left[S^a\right]\right]. \end{aligned} \tag{7.15}$$

Under $\mathbb{P}_{\underline{1}}$, the distribution of ξ_L^+ outside $\{0,1\}$ is product Bernoulli(ρ_L) with $\rho_L = \rho_e + e^{-\lambda L}(1-\rho_e)$, where λ is the total flip rate and ρ_e is the equilibrium density of the independent spin-flip system. Denote by ν_1 the measure that is product Bernoulli(ρ_1) outside $\{0,1\}$ and is concentrated at 1 on these two sites. Since $\rho_L \leq \rho_1$ for $L \geq 1$, it follows from the above observations and Lemma 7.2 that

$$\mathbb{E}_\eta\left[\theta_L\left(\mathcal{S}_+^a 1_{\{\mathcal{S}_+<\infty\}}\right) \mid \Gamma_L\right] \leq \mathbb{E}_{\nu_1}\left[S^a\right] < \infty. \tag{7.16}$$

We can similarly control $\mathcal{S}_-$ by noting that Lemma 7.2 implies the symmetric result for $\rho \in (0,1]$ and $S' := \sup\{t > 0\colon\ W_t < -mt\}$ (by exchanging the role of particles and holes). Therefore, in order to verify (H3), all that is left to do is to prove Lemma 7.2.

We now give the proof of Lemma 7.2.

Proof. First, suppose that ρ is the equilibrium density for the system, and let λ be its total flip rate. For a path ξ, define $G_0 = U_0 := 0$ and, recursively for $k \geq 0$,

$$\begin{aligned} G_{k+1} &:= G_k + Tr^+(\theta_{G_k}\xi_{U_k}), \\ U_{k+1} &:= \inf\{t > U_k\colon\ \xi_t(G_{k+1}+1) = 1\}. \end{aligned} \tag{7.17}$$

Put

$$H_t := \sum_{k=0}^{\infty} 1_{\{U_k \leq t < U_{k+1}\}} G_{k+1}. \tag{7.18}$$

Then $H = (H_t)_{t\geq 0}$ is the process that waits to the left of a hole until it flips to a particle, and then jumps to the right to the site just before the next hole. Therefore, $W_t \leq H_t$ by construction and, since $H_t \geq 0$, also $W_t^+ \leq H_t$. Since ξ is an independent spin-flip system that (apart from two sites) starts from equilibrium, the increments $G_{k+1} - G_k$ are i.i.d. Geometric($1-\rho$), and $U_{k+1} - U_k$ are i.i.d. Exponential($\lambda\rho$) and independent from $(G_k)_{k\in\mathbb{N}_0}$. From this we see, using Jensen's inequality, that $(t^{-1}H_t)_{t\geq 1}$ is bounded in L^p for all $p > 1$, which proves (a), and that $\lim_{t\to\infty} t^{-1}H_t = (\lambda\rho(1-\rho))^{-1}$ a.s. Moreover, since the G_k have exponential moments and the jump times U_k have a minimal rate, H satisfies a large deviation estimate of the type

$$\mathbb{P}_\nu(\exists\ s > t \text{ such that } H_s > ms) \leq K_1 e^{-K_2 t} \ \text{ for all } t > 0, \tag{7.19}$$

where $m := 2]\lambda\rho(1-\rho)]^{-1}$. The claim follows from (7.19), since $\{S > t\} \subset \{\exists\, s > t \text{ such that } H_s > ms\}$.

Next, consider the general case. Let ρ_e be the equilibrium density of the independent spin-flip system. Using stochastic domination, we may suppose that $\rho \geq \rho_e$. If $\rho > \rho_e$, then let ξ' be the independent spin-flip system with rate $\lambda\rho$ to flip from hole to particle and rate $\lambda(1-\rho)$ to flip from particle to hole. This system stochastically dominates the original system and has ρ as its equilibrium density, and so we fall back to the previous case. □

7.2.3 Uniform integrability

The following corollary implies that, for systems given by (7.6), $(t^{-1}|W_t|^p)_{t\geq 1}$ is uniformly integrable for any $p \geq 1$, so that, whenever we have a LLN, the convergence holds also in L^p.

Corollary 7.3. *Let ξ be a spin-flip system with rates as in* (7.6)*, starting from equilibrium. Then $(t^{-1}W_t)_{t\geq 1}$ is bounded in L^p for all $p > 1$.*

Proof. This is a straightforward consequence of Lemma 7.2(a) since, by monotonicity, W_t^+ in ξ is smaller than its counterpart in ξ^+. Moreover, since $-W$ is also a $(\infty, 0)$-walk in the same class of environments, this reasoning is valid for W_t^- as well. □

We still need to verify (H4). This will be done in Sections 7.3 and 7.4 below.

7.3 Example 1: $M < \epsilon$

We recall the definition of M and ϵ for a translation-invariant spin-flip system:

$$
\begin{aligned}
M &:= \sum_{x\neq 0} \sup_{\eta} |c(\eta^x) - c(\eta)| , && (7.20)\\
\epsilon &:= \inf_{\eta} \left\{ c(\eta) + c(\eta^0) \right\} , && (7.21)
\end{aligned}
$$

where η^x is the configuration obtained from η by flipping the x-coordinate.

7.3.1 Mixing for ξ

If ξ is in the $M < \epsilon$ regime, then there is exponential decay of space-time correlations (see Liggett [6], Section I.3). In fact, if ξ, ξ' are two copies starting from initial configurations η, η' and coupled according to the Vasershtein coupling, then, as was shown in Maes and Shlosman [8], the following estimate holds uniformly in $x \in \mathbb{Z}$ and in the initial configurations:

$$
\mathbb{P}_{\eta,\eta'}\left(\xi_t(x) \neq {\xi'}_t(x)\right) \leq e^{-(\epsilon - M)t}. \tag{7.22}
$$

Since the system has uniformly bounded flip rates, it follows that there exist constants $K_1, K_2 > 0$, independent of $x \in \mathbb{Z}$ and of the initial configurations, such that

$$
\mathbb{P}_{\eta,\eta'}\left(\exists s > t \text{ s.t. } \xi_s(x) \neq {\xi'}_s(x)\right) \leq K_1 e^{-K_2 t}. \tag{7.23}
$$

For $A \subset \mathbb{Z}\times\mathbb{R}_+$ measurable, let Discr(A) be the event in which there is a discrepancy between ξ and ξ' in A, i.e., $\mathrm{Discr}(A) := \{\exists (x,t) \in A \colon\ \xi_t(x) \neq \xi'_t(x)\}$. Recall the definition of $C_R(m)$ and $C_{R,t}(m)$ from Section 3.1. From (7.23) we deduce that, for any fixed $m > 0$ and $R \in \mathbb{N}$, there exist (possibly different) constants $K_1, K_2 > 0$ such that

$$
\mathbb{P}_{\eta,\eta'}(\mathrm{Discr}(C_R(m) \setminus C_{R,t}(m))) \leq K_1 e^{-K_2 t}. \tag{7.24}
$$

7.3.2 Mixing for Ξ

Bounds of the same type as (7.22)–(7.24) hold for $\xi^{\pm}$, since $M = 0$ and $\epsilon > 0$ for independent spin-flips. Therefore, in order to have such bounds for the triple Ξ, we need only couple a pair Ξ, Ξ' in such a way that each coordinate is coupled with its primed counterpart by the Vasershtein coupling. This can be accomplished by the following set of rates at the origin (the configurations of the middle coordinates, ξ and ξ', are η and η'; the configurations of $\xi^{\pm}$ and $\xi'^{\pm}$ outside the origin play no role):

$$
\begin{aligned}
(000)(000) \quad &\to \quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) \wedge c(\eta') - c_1, \\ (011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\ (001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\ (001)(001) & c_1 + \lambda_1 - c(\eta) \vee c(\eta'), \end{cases} \\
(001)(001) \quad &\to \quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) \wedge c(\eta') - c_1, \\ (011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\ (001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\ (000)(000) & c_0, \end{cases} \\
(000)(001) \quad &\to \quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) \wedge c(\eta') - c_1, \\ (011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\ (001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\ (001)(001) & c_1 + \lambda_1 - c(\eta) \vee c(\eta'), \\ (000)(000) & c_0, \end{cases} \\
(000)(011) \quad &\to \quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) - c_1, \\ (001)(011) & c_1 + \lambda_1 - c(\eta), \\ (000)(000) & c_0, \\ (000)(001) & c(\eta') - c_0, \end{cases} \\
(000)(111) \quad &\to \quad \begin{cases} (111)(111) & c_1, \\ (011)(111) & c(\eta) - c_1, \\ (001)(111) & c_1 + \lambda_1 - c(\eta), \\ (000)(000) & c_0, \\ (000)(001) & c(\eta') - c_0, \\ (000)(011) & c_0 + \lambda_0 - c(\eta'). \end{cases}
\end{aligned}
\tag{7.25}
$$

The other transitions, starting from

$$
(111)(111), \quad (011)(011), \quad (111)(011), \quad (111)(001), \quad (111)(000), \tag{7.26}
$$

can be obtained from the ones above by symmetry, exchanging the roles of $\xi^{\pm}$ and of particles and holes. Redefining $\mathrm{Discr}(A) := \{\exists\ (x,t) \in A\colon\ \Xi_t(x) \neq \Xi'_t(x)\}$, by the previous observation we see that (7.24) still holds, with possibly different constants. As a consequence, we get the following lemma.

Lemma 7.4. *Define $d(\eta,\eta') := \sum_{x\in\mathbb{Z}} 1_{\{\eta(x)\neq\eta'(x)\}} 2^{-|x|-1}$. For any $m > 0$ and $R \in \mathbb{N}$,*

$$
\lim_{d(\Xi_0,\Xi'_0)\to 0} \mathbb{P}_{\Xi_0,\Xi'_0}\big(\mathrm{Discr}(C_R(m))\big) = 0. \tag{7.27}
$$

Proof. For any $t > 0$, we may split $\mathrm{Discr}(C_R(m)) = \mathrm{Discr}(C_{R,t}(m)) \cup \mathrm{Discr}(C_R(m) \setminus C_{R,t}(m))$, so that

$$\mathbb{P}_{\eta,\eta'}\big(\mathrm{Discr}(C_R(m))\big) \leq \mathbb{P}_{\eta,\eta'}\big(\mathrm{Discr}(C_{R,t}(m)) + \mathbb{P}_{\eta,\eta'}\big(\mathrm{Discr}(C_R(m) \setminus C_{R,t}(m))\big). \tag{7.28}$$

Fix $\varepsilon > 0$. By (7.24), for t large enough the second term in (7.28) is smaller than ε uniformly in η, η'. For this fixed t, the first term goes to zero as $d(\eta, \eta') \to 0$, since $C_{R,t}(m)$ is contained in a finite space-time box and the coupling in (7.25) is Feller with uniformly bounded total flip rates per site. (Note that the metric d generates the product topology, under which the configuration space is compact.) Therefore $\limsup_{d(\eta,\eta')\to 0} \mathbb{P}_{\eta,\eta'}\left(\mathrm{Discr}(C_R(m))\right) \leq \varepsilon$. Since ε is arbitrary, (7.27) follows. □

7.3.3 Conditional mixing

Next, we define an auxiliary process $\bar{\Xi}$ that, for each L, has the law of Ξ conditioned on Γ_L up to time L. We restrict to initial configurations $\eta \in \bar{E}$. In this case, $\bar{\Xi}$ is a process on $\big(\{0,1\}^{\mathbb{Z}\setminus\{0,1\}}\big)^3$ with rates that are equal to those of Ξ, evaluated with a trap at the origin. More precisely, for $\bar{\eta} \in \{0,1\}^{\mathbb{Z}\setminus\{0,1\}}$, denote by $(\bar{\eta})_{1,0}$ the configuration in $\{0,1\}^{\mathbb{Z}}$ that is equal to $\bar{\eta}$ in $\mathbb{Z} \setminus \{0,1\}$ and has a trap at the origin. Then set $\bar{C}_x(\bar{\eta}) := C_x((\bar{\eta})_{1,0})$, where $\bar{C}_x$ is the rate of $\bar{\Xi}$ and C_x is the rates Ξ at site $x \in \mathbb{Z}$. Observe that the latter depend only on the middle configuration η, and not on $\eta^{\pm}$. These rates give the correct law for $\bar{\Xi}$ because Ξ conditioned on Γ_L is Markovian up to time L. Indeed, the probability of Γ_L does not depend on η (for $\eta \in \bar{E}$) and, for $s < L$, $\Gamma_L = \Gamma_s \cap \theta_s \Gamma_{L-s}$. Thus, the rates follow by uniqueness. Observe that they are no longer translation-invariant.

Two copies of the process $\bar{\Xi}$ can be coupled similiarly as Ξ by using rates analogous to (7.25). Since each coordinate of $\bar{\Xi}$ has similar properties as the corresponding coordinate in Ξ (i.e., $\bar{\xi}^{\pm}$ are independent spin-flip systems and $\bar{\xi}$ is the in $M < \epsilon$ regime), it satisfies an estimate of the type

$$\bar{\mathbb{P}}_{\eta,\eta'}\left(\mathrm{Discr}([-t,t] \times \{t\})\right) \leq K_1 e^{-K_2 t} \tag{7.29}$$

for appropriate constants $K_1, K_2 > 0$. From this estimate we see that $d(\bar{\Xi}_t, \bar{\Xi}'_t) \to 0$ in probability as $t \to \infty$, uniformly in the initial configurations. By Lemma 7.4, this is also true for $\mathbb{P}_{(\bar{\Xi}_t)_{1,0},(\bar{\Xi}'_t)_{1,0}}(\mathrm{Discr}(C_R(m)))$. Since the latter is bounded, the convergence holds in L_1 as well.

7.3.4 Proof of (H4)

Let f be a bounded function, measurable in $\mathscr{C}_{R,\infty}(m)$, and estimate

$$\begin{aligned}\big|\mathbb{E}_\eta\left[\theta_L f \mid \Gamma_L\right] - \mathbb{E}_{\eta'}\left[\theta_L f \mid \Gamma_L\right]\big| &\leq 2\|f\|_\infty \mathbb{P}_{\eta,\eta'}\big(\theta_L \mathrm{Discr}(C_R(m)) \mid \Gamma_L\big) \\ &\leq 2\|f\|_\infty \sup_{\eta,\eta'} \bar{\mathbb{E}}_{\eta,\eta'}\Big[\mathbb{P}_{(\bar{\Xi}_L)_{1,0},(\bar{\Xi}'_L)_{1,0}}\left(\mathrm{Discr}(C_R(m))\right)\Big],\end{aligned} \tag{7.30}$$

where $\bar{\mathbb{E}}$ denotes expectation under the (coupled) law of $\bar{\Xi}$. Therefore (H4) follows with

$$\phi(L) := 2 \sup_{\eta,\eta'} \bar{\mathbb{E}}_{\eta,\eta'}\Big[\mathbb{P}_{(\bar{\Xi}_L)_{1,0},(\bar{\Xi}'_L)_{1,0}}\left(\mathrm{Discr}(C_R(m))\right)\Big], \tag{7.31}$$

which converges to zero as $L \to \infty$ by the previous discussion.

7.4 Example 2: subcritical dependence spread

In this section, we suppose that the rates $c(\eta)$ have a finite range of dependence, say r. In this case, the system can be constructed via a graphical representation as follows.

7.4.1 Graphical representation

For $x \in \mathbb{Z}$, let $I_t^j(x)$ and $\Lambda_t^j(x)$ be independent Poisson processes with rates c_j and λ_j respectively, where $j = 0, 1$. At each event of $I_t^j(x)$, put a j-cross on the corresponding space-time point. At each event of $\Lambda^j(x)$, put two j-arrows pointing at x, one from each side, extending over the whole range of dependence. Start with an arbitrary initial configuration $\xi_0 \in \{0,1\}^{\mathbb{Z}}$. Then obtain the subsequent states $\xi_t(x)$ from ξ_0 and the Poisson processes by, at each j-cross, choosing the next state at site x to be j and, at at each j-arrow pair, choosing the next state to be j if an independent Bernoulli($p_j(\theta_x \eta_s)$) trial succeeds, where s is the time of the j-arrow event. This construction is well defined since, because of the finite range, for each fixed time $t \geq 0$ it can a.s. be performed locally.

Any collection of processes with the same range and with rates of the form (7.6) with c_i, λ_i fixed ($i = 0, 1$) can be coupled together via this representation by fixing additionally for each site x a sequence $(U_n(x))_{n\in\mathbb{N}}$ of independent Uniform[0, 1] random variables to evaluate the Bernoulli trials at j-arrow events. In particular, $\xi^{\pm}$ can be coupled together with ξ in the graphical representation by noting that, for ξ^-, $p_0 \equiv 1$ and $p_1 \equiv 0$ and the opposite is true for ξ^+. For example, ξ^- is the process obtained by ignoring all 1-arrows and using all 0-arrows. This gives the same coupling as the one given by the rates (7.9). In particular, we see that in this setting the events Γ_L are given by (when $\xi_0 \in \bar{E}$)

$$\Gamma_L := \left\{ I_L^0(0) = \Lambda_L^0(0) = I_L^1(1) = \Lambda_L^1(1) = 0 \right\}. \tag{7.32}$$

7.4.2 Coupling with a contact process

We will couple Ξ with a contact process $\zeta = (\zeta_t)_{t\geq 0}$ in the following way. We keep all Poisson events and start with a configuration $\eta_0 \in \{i, h\}^{\mathbb{Z}}$, where i stands for "infected" and h for "healthy". We then interpret every cross as a recovery, and every arrow pair as infection transmission from any infected site within a neighborhood of radius r to the site the arrows point to. This gives rise to a 'threshold contact process' (TCP), i.e., a process with transitions at a site x given by

$$\begin{array}{lll} i \to h & \text{with rate} & c_0 + c_1, \\ h \to i & \text{with rate} & (\lambda_0 + \lambda_1) 1_{\{\exists \text{ infected site within range } r \text{ of } x\}}. \end{array} \tag{7.33}$$

In the graphical representation for ξ, we can interpret crosses as moments of memory loss and arrows as propagation of influence from the neighbors. Therefore, looking at the pair $(\Xi_t(x), \zeta_t(x))$, we can interpret the second coordinate being healthy as the first coordinate being independent of the initial configuration.

Proposition 7.5. *Let $\underline{i}$ represent the configuration with all sites infected, and let Ξ_0, $\Xi_0' \in E^3$. Couple Ξ, Ξ' and ζ by fixing a realization of all crosses, arrows and uniform random variables, where Ξ and Ξ' are obtained from the respective initial configurations and ζ is started from $\underline{i}$. Then a.s. $\Xi_t(x) = \Xi_t'(x)$ for all $t > 0$ and $x \in \mathbb{Z}$ such that $\zeta_t(x) = h$.*

Proof. Fix $t > 0$ and $x \in \mathbb{Z}$. With all Poisson and Uniform random variables fixed, an algorithm to find the state at (x,t), simultaneously for any collection of systems of type (7.6) with fixed c_i, λ_i, from their respective initial configurations runs as follows. Find the first Poisson event before t at site x. If it is a j-cross, then the state is j. If it is a j-arrow, then to decide the state we must evaluate p_j and, therefore, we must first take note of the states at this time at each site within range r of x, including x itself. In order to do so, we restart the algorithm for each of these sites. This process ends when time 0 or a cross is reached along every possible path from (x,t) to $\mathbb{Z} \times \{0\}$ that uses arrows (transversed in the direction opposite to which they point) and vertical lines. In particular, if along each of these paths time 0 is never reached, then the state at (x,t) does not change when we change the initial configuration. On the other hand, 0 is not reached if and only if every path ends in a cross, which is exactly the description of the event $\{\zeta_t(x) = h\}$. □

7.4.3 Subcritical regime

The process $(\zeta_t)_{t \geq 0}$ is stochastically dominated by a standard (linear) contact process (LCP) with the same range and rates. Therefore, if the LCP is subcritical, i.e., if $\lambda := (\lambda_0 + \lambda_1)/(c_0 + c_1) < \lambda_c$ where λ_c is the critical parameter for the corresponding LCP, then the TCP will die out also. When $r = 1$, we have the following lemma, which follows from results Liggett [6], Chapter VI.

Lemma 7.6. *Let B_t be the cluster of healthy sites around the origin at time t. If $\lambda < \lambda_c$, then there exist positive constants K_1, K_2, K_3, K_4 such that*

$$\mathbb{P}_{\underline{i}}\big(\exists\, s > t\colon\ |B_s| < K_1 e^{K_2 s}\big) \leq K_3 e^{-K_4 t}. \tag{7.34}$$

Lemma 7.6 says that the infection disappears exponentially fast around the origin. The proof in Liggett [6], Chapter VI, relies on the nearest-neighbour nature of the interaction, but using the techniques in Bezuidenhout and Grimmett [4] it should be possible to obtain the same result for any r. From here on we will take $r = 1$, but the rest of the argument will follow for any r as soon as a result like Lemma 7.6 is shown to be true for subcritical LCP with range r.

Pick a cone C with any inclination and tip at time t. Because of Lemma 7.6, if t is large, then with high probability all sites inside C are healthy. Therefore, inside a set of high probability, the states of ξ in the cone are equal to a random variable that is independent of the initial configuration, which implies the cone-mixing property.

7.4.4 Proof of (H4)

In order to prove the *conditional* cone-mixing property, we couple the conditioned process with a conditioned contact process as follows. First, let

$$\tilde{\Gamma}_L := \Big\{ I^j_L(i) = \Lambda^j_L(i) = 0\colon\ j, i \in \{0,1\} \Big\}. \tag{7.35}$$

Proposition 7.7. *Let $\hat{\imath}$ represent the configuration with all sites infected except for $\{0,1\}$, which are healthy. Let Ξ_0, $\Xi'_0 \in \bar{E}^3$. Couple Ξ, Ξ' conditioned on Γ_L and ζ conditioned on $\tilde{\Gamma}_L$ by fixing a realization of all crosses, arrows and uniform random variables as in Proposition 7.5 and starting, respectively, from Ξ_0, Ξ'_0 and $\hat{\imath}$, but, for Ξ and Ξ', remove the Poisson events that characterize Γ_L and, for ζ, remove all Poisson events up to time L at sites 0 and 1, which characterizes $\tilde{\Gamma}_L$. Then a.s. $\Xi_t(x) = \Xi'_t(x)$ for all $t > 0$ and $x \in \mathbb{Z}$ such that $\zeta_t(x) = h$.*

Proof. On Γ_L, the states at sites 0 and 1 are fixed for time $[0, L]$. Therefore, in order to determine the state at (x, t), we need not extend paths that touch $\{0, 1\} \times [0, L]$: when every path from (x, t) either ends in a cross or touches $\{0, 1\} \times [0, L]$, the state at (x, t) does not change when the initial configuration is changed in $\mathbb{Z} \setminus \{0, 1\}$. But this is precisely the characterization of $\{\eta_t(x) = h\}$ on $\tilde{\Gamma}_L$ when started from $\hat{i}$. $\square$

The proof of (H4) is finished by noting that $(\eta_t)_{t\geq 0}$ starting from $\hat{i}$ and conditioned on $\tilde{\Gamma}_L$ is stochastically dominated by $(\eta_t)_{t\geq 0}$ starting from $\underline{i}$. Therefore, by Lemma 7.6, the "dependence infection" still dies out exponentially fast, and we conclude as for the unconditioned cone-mixing.

7.5 The sign of the speed

For independent spin flips, we are able to characterize with the help of a coupling argument the regimes in which the speed is positive, zero or negative. By the stochastic domination described in Section 7.2, this gives us a criterion for positive (or negative) speed in the two classes addressed in Sections 7.3 and 7.4 above.

7.5.1 Lipschitz property of the speed for independent spin-flip systems

Let ξ be an independent spin-flip system with rates d_0 and d_1 to flip to holes and particles, respectively. Since it fits both classes of IPS considered in Sections 7.3 and 7.4, by Theorem 4.1 there exists a $w(d_0, d_1) \in \mathbb{R}$ that is the a.s. speed of the $(\infty, 0)$-walk in this environment. This speed has the following Lipschitz property.

Lemma 7.8. *Let $d_0, d_1, \delta > 0$. Then*

$$w(d_0, d_1 + \delta) - w(d_0, d_1) \geq \frac{d_0 \delta}{d_0 + d_1 + \delta}. \tag{7.36}$$

Proof. An independent spin flip system ξ with rates d_0, d_1 can be constructed via a graphical representation by taking, for each site $x \in \mathbb{Z}$, two Poisson processes $N^i(x)$ with rates d_i, $i = 0, 1$, with each event of N^i representing a flip to state i. For a fixed $\delta > 0$, a second system ξ^δ with rates d_0 and $d_1 + \delta$ can be coupled to ξ by starting from a configuration $\xi_0^\delta \geq \xi_0$ and adding to each site another Poisson process $N^\delta(x)$ with rate δ, whose events also represent flips to particles, but only for ξ^δ. Let us denote by W and W^δ the walks in these respective environments. Since under the coupling $\xi \leq \xi^\delta$, we have by monotonicity $W_t \leq W_t^\delta$ for all $t \geq 0$. Define a third walk, W^*, that is allowed to use one and only one event of N^δ. More precisely, W^* will see the same environment as W up to the first time S when its path encounters an event of N^δ, and the configuration around W_{S-}^* is then taken to be the same as the one around W_S, except for an additional particle at $W_{S-}^* + 1$. At further times, W^* sees no more N^δ events. By construction, we have $W_t \leq W_t^* \leq W_t^\delta$. We aim prove that

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0 + d_1} \delta t, \tag{7.37}$$

where μ and μ^δ are the equilibria of the respective systems. From this the conclusion will follow after dividing by t and letting $t \to \infty$.

Let $\eta_1 := \theta_{W_S}\xi_S \in \bar{E}$ and $\eta_2 := (\eta_1)^1$ be the configurations around W_S and W^*_{S-}, respectively. Then

$$\begin{aligned}\mathbb{E}_{\mu^\delta}\left[W^\delta_t\right] - \mathbb{E}_\mu\left[W_t\right] &\geq \mathbb{E}_\mu\left[W^*_t - W_t, S \leq t\right] \geq \mathbb{E}_\mu\left[W^*_t - W_t, S \leq t, \eta_1(2) = 0\right] \\ &= \mathbb{E}_\mu\left[\mathbb{E}_{\eta_1,\eta_2}\left[W^2_{t-S} - W^1_{t-S}\right], \eta_1(2) = 0, S \leq t\right], \end{aligned} \tag{7.38}$$

where W^i, $i = 1, 2$ are copies of W starting from η_i and coupled via the graphical representation. Note that, for any $s > 0$, when starting from η_1, η_2, the difference $W^2_s - W^1_s$ can only decrease when we flip all states to the left of W^1_0 to particles and all states to the right of W^2_0 to holes. But, in the event that $\eta_1(2) = 0$, after doing these operations, we find that W^2 has the same distribution as $W^1 + 1$, so that

$$\mathbb{E}_{\mu^\delta}\left[W^\delta_t\right] - \mathbb{E}_\mu\left[W_t\right] \geq \mathbb{P}_\mu\left(\eta_1(2) = 0, S \leq t\right). \tag{7.39}$$

Next, consider the event $\eta_1(2) = 0$. There are two possible situations: either at time S the site $W_S + 2$ was not yet visited, in which case the state there is still distributed as a Bernoulli($d_1/(d_0 + d_1)$), or it was visited before S. In the latter case, let s be the time of the last visit to this site before S. By geometrical constraints, at time s only a hole could have been observed at this site, so the probability that its state at time S is a hole is larger than at equilibrium, which is $d_0/(d_0 + d_1)$. In other words,

$$\mathbb{P}_\mu\left(\eta_1(2) = 0 \mid S, W_{[0,S]}\right) \geq \frac{d_0}{d_0 + d_1}, \tag{7.40}$$

which, together with (7.39) and the fact that S is distributed as Geometric(δ), gives us

$$\mathbb{E}_{\mu^\delta}\left[W^\delta_t\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0 + d_1}\left(1 - e^{\delta t}\right). \tag{7.41}$$

Since δ is arbitrary, we may repeat the argument for systems with rates $d_1 + (k/n)\delta$, $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$, to obtain

$$\begin{aligned}\mathbb{E}_{\mu^\delta}\left[W^\delta_t\right] - \mathbb{E}_\mu\left[W_t\right] &\geq \sum_{k=0}^{n-1} \frac{d_0}{d_0 + d_1 + (k/n)\delta}\left(1 - e^{(\delta/n)t}\right) \\ &\geq \frac{d_0}{d_0 + d_1 + \delta} n\left(1 - e^{\delta t/n}\right), \end{aligned} \tag{7.42}$$

and we get (7.37) by letting $n \to \infty$. □

7.5.2 Sign of the speed

If $d_0 = d_1$, then $w = 0$, since by symmetry $W_t = -W_t$ in distribution. Hence we can summarize:

Proposition 7.9. *For an independent spin-flip system with rates d_0 and d_1,*

$$\begin{aligned} &w \geq \tfrac{d_0}{d_0+d_1}\left(d_1 - d_0\right) && \text{if } d_1 > d_0, \\ &w = 0 && \text{if } d_1 = d_0, \\ &w \leq -\tfrac{d_1}{d_0+d_1}\left(d_0 - d_1\right) && \text{if } d_1 < d_0. \end{aligned} \tag{7.43}$$

Applying this result to the systems $\xi^{\pm}$ of Section 7.2, we obtain the following.

Proposition 7.10. *Suppose that there exists a speed w for the $(\infty,0)$-model in a spin-flip system with rates given by (7.6). Then,*

$$\begin{aligned} v &\geq \tfrac{c_0+\lambda_0}{c_1+c_0+\lambda_0}\,(c_1-c_0-\lambda_0) \qquad \text{if } c_1 > c_0+\lambda_0, \\ v &\leq -\tfrac{c_1+\lambda_1}{c_0+c_1+\lambda_1}\,(c_0-c_1-\lambda_1) \quad \text{if } c_0 > c_1+\lambda_1. \end{aligned} \tag{7.44}$$

This concludes the proof of Theorem 4.2 and the discussion of our two classes of IPS's for the $(\infty,0)$-model. In Section 8 we give additional examples and discuss the limitations of our setting.

8 Other examples

We discuss three types of examples: generalizations of the (α,β)-model and the $(\infty,0)$-model, and mixtures thereof.

1. Pattern models. Take $\aleph$ to be a finite sequence of 0's and 1's, which we call a *pattern*, and let R be the length of this sequence. Take the environment ξ to be of the same type used to define the $(\infty,0)$-walk. Let $q:\{0,1\}^R\setminus\{\aleph\}\to[0,1]$. The pattern walk is defined similarly as the $(\infty,0)$ walk, with the trap being substituted by the pattern, and a Bernoulli(q) random variable being used to decide whether the walk jumps to the right or to the left. More precisely, let $\vartheta=(\eta_0(0),\dots,\eta_0(R-1))$. If $\vartheta=\aleph$, then we set $W_0=0$, otherwise we sample b_0 as an independent Bernoulli($q(\vartheta)$) trial. If $b_0=1$, then W_0 is set to be the starting position of the first occurrence of $\aleph$ in η_0 to the right of the origin, while if $b_0=0$, then the first occurrance of $\aleph$ to the left of the origin is taken instead. Then the walk waits at this position until the configuration of one of the R states to its right changes, at which time the procedure to find the jump is repeated with the environment as seen from W_0. Subsequent jumps are obtained analogously. The $(\infty,0)$-model is a pattern model with $\aleph:=(1,0)$, $q(1,1):=1$, $q(0,0):=0$ and $q(0,1):=1/2$.

For the spin-flip systems given by (7.6), the pattern walk is defined and is finite for all times, no matter what $\aleph$ is, the reasoning being exactly the same as for the $(\infty,0)$-walk. Also, it may be analogously defined so as to satisfy assumptions (A1)–(A3). When we define the events Γ_L as

$$\Gamma_L := \left\{\xi^{\pm}_s(j)=\xi^{\pm}_0(j)\ \forall\ s\in[0,L] \text{ and } j\in\{0,...,R-1\}\right\}, \tag{8.1}$$

we may indeed, by completely analogous arguments, reobtain all the results of Section 7, so that hypotheses (H1)–(H4) hold and, therefore, the LLN as well.

2. Internal noise models. For $x\in\mathbb{Z}\setminus\{0\}$ and $\eta\in E$, let $\pi_x(\eta)$ be functions with a finite range of dependence R. These are the rates to jump x from the position of the walk. Let $\pi_x:=\sup_\eta \pi_x(\eta)$ and suppose that, for some $u>0$,

$$\sum_{x\in\mathbb{Z}\setminus\{0\}} e^{u|x|}\pi_x < \infty. \tag{8.2}$$

This implies that also

$$\Pi := \sum_{x\in\mathbb{Z}\setminus\{0\}} \pi_x < \infty. \tag{8.3}$$

The walk starts at the origin, and waits an independent Exponential(Π) time τ until it jumps to x with probability $\pi_x(\xi_\tau)/\Pi$. These probabilities do not necessarily sum up to one, so the walk may well stay at the origin. The subsequent jumps are obtained analogously, with ξ_τ substituted by the environment around the walk at the time of the attempted jump. It is clear that (A1)–(A3) hold. The walk has a bounded probability of standing still independently of the environment, and its jumps have an exponential tail. We choose

$$\Gamma_L := \{\tau > L\}. \tag{8.4}$$

By defining an auxiliary walk $(H_t)_{t\geq 0}$ that also tries to jump at time τ, but only to sites $x > 0$ with probability π_x/Π, we see that $W_t \leq H_t$ and that H_t has properties analogous to the process defined in the proof of Lemma 7.2. Therefore, (H1)–(H3) are always satisfied for this model. Since Γ_L is independent of ξ, (H4) is the (unconditional) cone-mixing property. Observe that $W_0 = 0$, so that $\bar{\mu} = \mu$. Therefore the LLN for this model holds in both examples discussed in Section 7, and also for the IPS's for which cone-mixing was shown in Avena, den Hollander and Redig [1]. The (α, β)-model is an internal noise model with $R = 0$ (the rates depend only on the state of the site where the walker is) and $\pi_x(\eta) = 0$, except for $x = \pm 1$, for which $\pi_1(1) = \alpha = \pi_{-1}(0)$ and $\pi_1(0) = \beta = \pi_{-1}(1)$.

3. Mixtures of pattern and internal noise. Let $\mathcal{I}$ be a finite set, and suppose that for each $i \in \mathcal{I}$ we are given a model that is either pattern or internal noise. Let $(I_t)_{t\geq 0}$ be an irreducible continuous-time Markov chain on $\mathcal{I}$, starting from equilibrium. Then the mixture is the model for which the dynamics associated to i is applied in the time intervals when I_t is in state i. A choice of Γ_L is given by

$$\Gamma_L = \sum_{i\in\mathcal{I}} \{I_0 = i\} \cap \Gamma_L^i, \tag{8.5}$$

where Γ_L^i is the corresponding event for the model associated to i. It is easily checked that the mixed model satisfies (A1)–(A3) and that, whenever (H1)–(H4) are satisfied for each i, they are also satisfied for the mixed model.

An open example. Let us close by giving an example of a model that does not satisfy the hypotheses of our LLN (in dynamic random environments given by spin-flip systems). When $\xi(0) = j$, let C^j be the cluster of j's around the origin. Define jump rates for the walk as follows:

$$\begin{aligned} \pi_1(\eta) &= \begin{cases} |C^1| & \text{if} \quad \eta(0) = 1, \\ |C^0|^{-1} & \text{if} \quad \eta(0) = 0, \end{cases} \\ \pi_{-1}(\eta) &= \begin{cases} |C^0| & \text{if} \quad \eta(0) = 0, \\ |C^1|^{-1} & \text{if} \quad \eta(0) = 1. \end{cases} \end{aligned} \tag{8.6}$$

Even though this walk seems to be a natural example (and is quite close to our setup), it does not satisfy (A3). Nor does it satisfy (H2) for any reasonable choice of Γ_L, which is actually the hardest obstacle. The problem is that, while we are able to transport a.s.-properties of the equilibrium measure to the environment measure (i.e., from the point of view of the walk) using absolute continuity, we cannot control the distortion in events of positive measure. Indeed, even if Γ_L has a high probability at time zero, there is no a priori guarantee that it will have high probability frequently enough at later times under the environment measure. And even if we are able to find enough time points where this happens, correlations between the event $\{W_t \in C(m) \ \forall\, t \geq 0\}$ and translations of Γ_L start to play a role. Because of such complications, the proof for the LLN breaks down and the challenge remains.

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