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Abstract

We study a multi-location inventory problem with a so-called quick response warehouse. In case of a stock-out at a local warehouse, the demand might be satisfied by a stock transfer from the quick response warehouse. We derive the optimal policy for when to accept and when to reject such a demand at the quick response warehouse. We also derive conditions under which it is always optimal to accept these demands. Furthermore, we conduct a numerical study and consider model variations.

Keywords: inventory, quick response warehouse, lateral transshipment, optimal policy structure.

1 Introduction

In this paper we study a multi-location inventory model, with the special feature of a so-called *Quick Response* (QR) warehouse. When a local warehouse is out-of-stock, a part can be transshipped from this QR warehouse. In this way the demand is satisfied much more quickly compared to an emergency shipment from outside the network.

A relevant application of this is found in spare parts inventory models, where ready-for-use parts are kept on stock for the critical component of advanced technical systems. Examples of these include the key manufacturing machines in production lines, trucks for a transportation company,

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and expensive medical equipment in a hospital. Upon break-down of a system, it demands a spare part. During this time, the system is down at very high costs because of loss of production/revenue. So, in order to reduce down time, it is important that demand is quickly satisfied, and a quick response warehouse is a good option for doing so. Axsäter et al. [2] (and also Howard et al. [8]) describe the setting at Volvo Parts Corporation, a global spare parts service provider, which makes use of QR warehouses (referred to as 'support warehouses'). Rijk [18] studies the stock control of Océ, a company in printing and document management. They use quick response stocks for storing parts that need to be within a short time range of the customers, but for which it is not possible or efficient to store these in the car stocks of the maintenance engineers.

Another application of the model with a QR warehouse is the combination of physical stores with an on-line shop, e.g. for books or fashion. The on-line shop keeps some items in inventory as well, located centrally. The demands of customers visiting the physical stores are satisfied immediately when there is stock on-hand, but their demands can be routed to the inventory of the on-line shop in case of a stock-out at the store. Hence, this on-line shop, which also has its own demand stream, acts as a QR warehouse.

Although a relevant problem, to the best of our knowledge, no results are known about the optimal use of a QR warehouse. For that, we study the policy for when the QR warehouse should accept and when it should reject a demand originating at a local warehouse. We formulate the problem as a Markov decision problem (MDP) and use event-based dynamic programming (cf. [9, 10]) to derive the optimal policy structure of the QR warehouse. We furthermore derive simple, sufficient conditions under which it is always optimal to accept a demand.

In inventory models, shipments of stock between warehouses of the same echelon are usually referred to as lateral transshipments (LTs), see [16] for an overview. However, LTs are typically possible between multiple (sets of) warehouses and often in multiple ways. We mention [1, 11, 14] as work where LTs are limited in specific directions. None of these previous works, however, derive the optimal policy structure.

Our model is also related to overflow models in telecommunication, more precisely, to call center models (see [6] for an overview). Among others, Örmeci [15] and Chevalier et al. [3] derive optimal admission and routing policies for calling customers (i.e. demands). In these models no costs are incorporated for the routing and loosing of customers. These costs, however, turn out to play an important role in the optimal policy.

Furthermore, there is a link between our model and stock rationing models, in which multiple types of customers (demand classes) demand a part at a single stock point. The customers of each of these demand classes arrive according to a Poisson process, and have different penalty costs when their demand is not satisfied. The optimal policy for when to accept and when to reject demands is a critical level policy (see e.g. Ha [7]). Such a policy prescribes a stock level (the critical level) for each demand class from which on their demands are satisfied. Focusing on the overflow demands of the local warehouses, our model is related to this framework. However, these demand streams are *not* Poisson processes and, as a consequence, the critical level policy fails to be optimal in our setting.

Each of the overflow demand streams, however, can be described as a Markov modulated Poisson process (MMPP, see [5] for an overview). The optimal policy then depends on the states of each of these processes, as we find the optimal policy to do. Hence, we generalize the result of [7] for Poisson processes to this MMPP. Namely, our model includes Poisson demand processes as a special case, when setting the basestock levels at the local warehouses equal zero. As there is only one state in this case, the optimal policy reduces to the state *in*dependent critical level policy.

Finally, we mention the link of our model with dual supplier problems (see Minner [13] for an overview). In these problems a warehouse has the option of using a second supplier that can deliver emergency shipments at an extra cost. However, this emergency supplier is exogenous, while we include the QR warehouse in our model. Hence, the stock level of the QR warehouse is decisive for whether to apply a QR shipment or not.

The outline of this paper is as follows. We start by describing the model and introducing the notation in Section 2. We formulate the problem as an MDP and introduce the value function. In Section 3 we show that the value function satisfies certain structural results. From this the optimal policy at the QR warehouse is derived, as well as the simplifying conditions. Section 4 shows numerical results on how much cost savings are achieved by executing the optimal policy. Three model extensions are discussed in Section 5. In Section 6 we conclude and present options for further research. All proofs are in the Appendix.

2 Model and Notation

2.1 Problem Description

We consider the following multi-location inventory model. We have J local warehouses, with index $j = 1, \dots, J$, and a quick response (QR) warehouse with index $j = 0$, keeping on stock a single stock-keeping-unit. Warehouse j follows a base stock policy with base stock level S_j, $j = 0, 1, \dots, J$. We assume one-for-one replenishments, where the replenishment lead times are exponentially distributed with mean $1/\mu_j$, $j = 0, 1, \dots, J$. To avoid trivialities, we assume $S_0 \geq$

1. Furthermore, we assume replenishments from a central warehouse with infinite stock, but equivalently these can come from an external supplier outside the network. Here, one can also interpret a replenishment as a production to stock, or a repair procedure of a repairable. The local warehouses and the QR warehouse each face a demand stream, which is Poisson with rate λ_j, $j = 0, 1, \ldots, J$. We assume the interarrival and replenishment times to be all mutually independent.

Each warehouse satisfies demands from a group of machines assigned to it. When *local* warehouse j is out-of-stock, the demand can be fulfilled by a stock transfer from the QR warehouse, at costs P_j^{QR}. This is referred to as an overflow demand of warehouse j. In this case a part from the QR warehouse is directly assigned to this demand, and shipped to the local warehouse. Hence, the demand and part are instantaneously coupled, where P_j^{QR} includes the possible extra down time costs (e.g. because of loss of production) of a machine during the extra time required for the quick response procedure. We assume this procedure to be much faster than waiting for a regular replenishment. When the demand is not fulfilled from the QR warehouse, it has to be fulfilled by an emergency procedure, at penalty costs P_j^{EP}, e.g. by a shipment from the central warehouse or an external supplier. This is equivalent to considering the demand to be lost (for this set of warehouses). As a model extension we consider in Section 5.3 backlogging at the local warehouses.

Demands that occur at the QR warehouse itself, are either satisfied directly, or fulfilled by an emergency procedure at penalty costs P_0^{EP}. To avoid trivialities, we assume that $0 \leq P_j^{QR} \leq P_j^{EP}$ for all j, and define $\Delta P_j = P_j^{EP} - P_j^{QR}$. For ease of notation, we define $P_0^{QR} = 0$ and hence $\Delta P_0 = P_0^{EP}$. This inventory model is graphically presented in Figure 1. The question is when the QR warehouse should accept, i.e. satisfy, an (overflow) demand, and when it is better to reject it.

2.2 Markov Decision Process Formulation

We model the problem as a Markov decision process (MDP, cf. [17]). Let x_j be the stock level of location j, and let $x = (x_0, x_1, \ldots, x_J)$ be the vector consisting of all stock levels. So x is the state of the system, on the state space $\mathcal{S}$ consisting of all possible combinations of stock levels. We have two types of events that can occur: demands and replenishments. At rate λ_j a demand arrives at location j. For the local warehouses, the demand is fulfilled directly from stock if $x_j > 0$. Otherwise the demand is routed to the QR warehouse, which may accept it (if $x_0 > 0$ at costs P_j^{QR}) or may reject it (at costs P_j^{EP}). The demands that arise directly at the QR warehouse, may be accepted (if $x_0 > 0$, no costs) or may be rejected (at costs P_0^{EP}). The replenishment rate at warehouse j is $(S_j - x_j)\mu_j$, where $S_j - x_j$ is the number of outstanding orders. We apply uniformization (cf. [12]) by adding fictitious transitions, to let the replenishment event occur at rate $S_j\,\mu_j$.

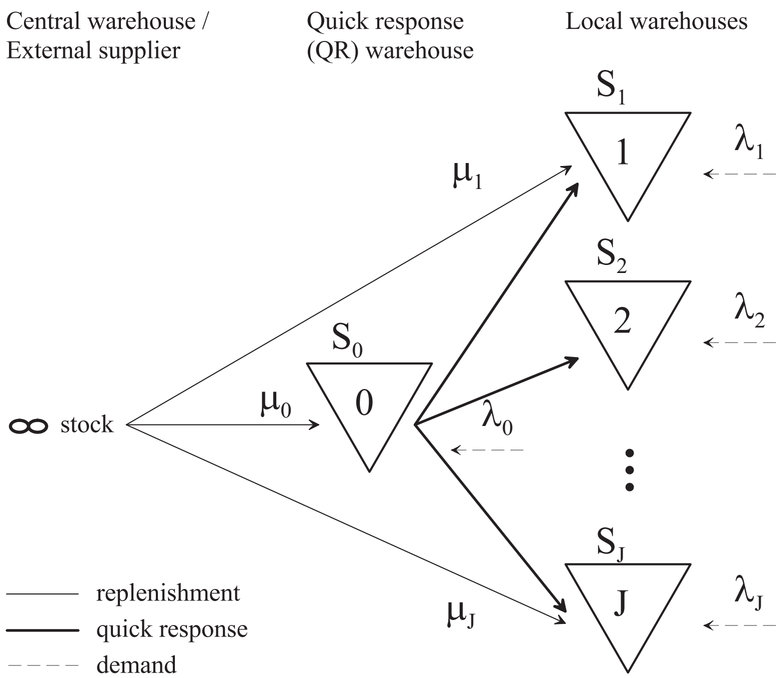

Figure 1: Multi-location inventory model with a Quick Response warehouse.

Let $V_n : \mathcal{S} \mapsto \mathbb{R}$ be the *value function*, the minimum cost function when there are n events (demands or replenishments) left. It is given by:

$$V_{n+1}(x) = \frac{1}{\nu} \left(\sum_{j=0}^{J} h_j(x_j) + \sum_{j=0}^{J} \mu_j G_j V_n(x) + \sum_{j=1}^{J} \lambda_j H_j V_n(x) + \lambda_0 H_{QR} V_n(x) \right), \text{ for } x \in \mathcal{S}, n \geq 0, \tag{1}$$

starting with $V_0 \equiv 0$, where $\nu = \sum_{j=0}^{J} S_j \, \mu_j + \sum_{j=0}^{J} \lambda_j$ is the uniformization rate. The operators G_j (replenishments at j), H_j (demands at local warehouse j), and H_{QR} (demands at QR warehouse) are defined below. The holding costs, denoted by $h_j(x_j)$, represent the costs for keeping x_j parts in stock at location j during one time unit. We assume that $h_0(\cdot)$ is convex.

The operator H_j models the demands at local warehouse $j = 1, \ldots, J$, and is defined by:

$$H_j f(x) = \begin{cases} f(x - e_j) & \text{if } x_j > 0; \\ \min\{P_j^{QR} + f(x - e_0), & \\ \qquad P_j^{EP} + f(x)\} & \text{if } x_j = 0, x_0 > 0; \\ P_j^{EP} + f(x) & \text{otherwise.} \end{cases}$$

Here e_j is the unit vector of length $J + 1$ with a 1 at position j ($j = 0, 1, \ldots, J$). When $x_j > 0$ a part is taken from stock, hence the stock level decreases by one. When $x_j = 0$ (and $x_0 > 0$), H_j selects the costs–minimizing action of taking a part from the QR warehouse, or loosing the demand. If both the local and the QR warehouse are out-of-stock, the only option is that the demand is lost.

The operator H_{QR} models the demands at QR warehouse, and is defined by:

$$H_{QR}f(x) = \begin{cases} \min\{f(x-e_0), P_0^{EP} + f(x)\} & \text{if } x_0 > 0; \\ P_0^{EP} + f(x) & \text{if } x_0 = 0. \end{cases}$$

When $x_0 > 0$ the minimizing action is chosen over accepting or rejecting. Otherwise, when $x_0 = 0$, the only option is to reject the demand.

The operator G_j models the replenishments at warehouse $j = 0, 1, \ldots, J$, and is defined by:

$$G_j f(x) = \begin{cases} (S_j - x_j)f(x+e_j) + x_j f(x) & \text{if } x_j < S_j; \\ S_j f(x) & \text{if } x_j = S_j. \end{cases}$$

Note that both the second term in the first line, as well as the second line, represent fictitious transitions, hence assuring that the total rate at which G_j occurs is $S_j\,\mu_j$, independently of the stock level x_j. We have taken the replenishment rate to be linear in the number of outstanding orders. In Section 5.2 we consider state-dependent replenishment rates as a model extension.

3 Structural Results

In this section we prove our main result: the structure of the optimal policy of the QR warehouse. For this, we first introduce the properties convexity and supermodularity. Each of the operators in the value function preserves these properties, hence the value function satisfies them. From this the optimal policy structure is derived, as well as conditions under which it is optimal to always accept a demand.

3.1 Structural Properties

Consider the following properties of a function f, defined for all x such that the states appearing in the right–hand and left–hand side of the inequalities exist in the state space $\mathcal{S}$:

$$\text{Conv}(x_i): \quad f(x) + f(x + 2\,e_i) \geq 2\,f(x + \,e_i),$$

$$\text{Supermod}(x_i, x_j): \quad f(x) + f(x + e_i + e_j) \geq f(x + e_i) + f(x + e_j) \;\; \text{for } i \neq j.$$

Conv(x_i) stands for convexity of f in x_i. This means that the difference $f(x) - f(x + e_i)$ is decreasing in x_i. Supermod(x_i, x_j) stands for supermodularity of f in the pair (x_i, x_j). By definition it is symmetric in x_i and x_j.

The next lemma shows that the operators H_j, H_{QR}, and G_j preserve these properties. We use the following notation, cf. [10]: for an operator X we denote by $X\colon\ P_1,\ldots,P_N\ \to\ P_1$ that when a function f satisfies properties $P_1,\ldots,P_N$, then Xf satisfies property P_1. The proofs of these and all other lemmas and theorems are given in the Appendix.

Lemma 1. ***a)*** *For all $j = 1,\ldots,J$:*

$$H_j : Conv(x_0) \ \to \ Conv(x_0),$$
$$H_j : Supermod(x_0, x_k),\ Conv(x_0) \ \to \ Supermod(x_0, x_k),\ \text{for all } k = 1,\ldots,J.$$

b)
$$H_{QR} : Conv(x_0) \ \to \ Conv(x_0),$$
$$H_{QR} : Supermod(x_0, x_j),\ Conv(x_0) \ \to \ Supermod(x_0, x_j),\ \text{for } j = 1,\ldots,J.$$

c) *For all $j = 0, 1,\ldots,J$:*
$$G_j : Conv(x_0) \ \to \ Conv(x_0),$$
$$G_j : Supermod(x_0, x_k) \ \to \ Supermod(x_0, x_k),\ \text{for all } k = 1,\ldots,J.$$

These results contribute to the literature, like [10], as they can be used in other models as well. Using induction, this lemma directly leads to the following result.

Theorem 2. *For all $n \geq 0$, V_n is $Conv(x_0)$ and $Supermod(x_0, x_j)$ for all $j = 1,\ldots,J$, when V_0 satisfies these properties.*

3.2 Structure of Optimal Policy

The following theorem describes the structure of the optimal policy at the QR warehouse. For this, we denote by $x^{(0,j)}$, $j = 1,\ldots,J$, the vector x without the component x_0 and with $x_j = 0$. That is, $x^{(0,j)} := (x_1,\ldots,x_{j-1}, 0, x_{j+1},\ldots,x_J)$. Furthermore, define $x^{(0,0)} := (x_1,\ldots,x_J)$.

Theorem 3. *The optimal policy at the QR warehouse is a state-dependent threshold policy. That is, for all $j = 0, 1,\ldots,J$ there exists a switching curve $T_j(x^{(0,j)})$ that characterizes the optimal decision for a demand at the QR warehouse (i.e., $j = 0$), or an overflow demand from local warehouse j when $x_j = 0$ (for $j = 1,\ldots,J$):*

- *if $x_0 > T_j(x^{(0,j)})$: accept;*
- *if $x_0 \leq T_j(x^{(0,j)})$: reject.*

$T_j(x^{(0,j)})$ is decreasing in each component of $x^{(0,j)}$.

The proof makes use of the fact that the value function is $\text{Conv}(x_0)$ and $\text{Supermod}(x_0, x_j)$. Figure 2 shows the optimal policy structure for demand at local warehouses when $J = 2$. It shows that a demand is more likely to be accepted when the QR stock level is high, and/or when the stock levels at the other local warehouses are high. This is in line with the fact that $T_j(x^{(0,j)})$ is decreasing.

Let $j_1, j_2 \in \{1, \ldots, J\}$ with $\Delta P_{j_1} \geq \Delta P_{j_2}$, i.e. the cost difference ΔP_{j_1} at local warehouse j_1 is larger than or equal to the cost difference ΔP_{j_2} at local warehouse j_2. Then their optimal actions for applying a quick response are ordered accordingly. More precisely, let $a_j^*(x)$ denote the optimal action at local warehouse j in case of a demand, where we encode $a_j^*(x) = 1$ for a quick response (i.e. accepting the demand) and $a_j^*(x) = 0$ for an emergency procedure (reject), then the following holds.

Proposition 1. *Let $j_1, j_2 \in \{1, \ldots, J\}$ with $\Delta P_{j_1} \geq \Delta P_{j_2}$. Then $a_{j_1}^*(x) \geq a_{j_2}^*(x)$ for all x with $x_{j_1} = x_{j_2} = 0$.*

When the basestock levels at all local warehouses equal zero, the overflow demand stream from each of the local warehouses is a Poisson process with rate λ_j. By Theorem 3 the switching curve T_j is a function of $x^{(0,j)}$, however, there is only *one* such vector, namely the all zero vector. Hence, in this case, the switching curve T_j reduces to a constant, say $C_j \in \{0, 1, \ldots, S_0\}$, for all j. An (overflow) demand is satisfied when $x_0 > C_j$, and is rejected otherwise. This state *in*dependent threshold policy is known as a *critical level policy*, where the C_j's are called the critical levels. It is known to be optimal in this setting for Poisson demand streams, cf. [7]. So, we have proven this result as a special case of our model. Moreover, Proposition 1 shows that the critical levels are ordered based on the ΔP_j, that is, if $\Delta P_1 \geq \Delta P_2 \geq \ldots \geq \Delta P_J$, then $C_1 \geq C_2 \geq \ldots \geq C_J$, as in [7]. This model, where inventory at a single warehouse is allocated to multiple customers classes with different costs factors ΔP_1, is known as a stock rationing problem.

An overflow demand stream from local warehouse j is a special case of a Markov modulated Poisson process (MMPP [5]): one with S_j states, demand rate λ_j at the QR warehouse when in state $x_j = 0$ (and zero otherwise), and transition probabilities following from the replenishments and local demands. Hence, Theorem 3 generalizes the optimal policy for a stock rationing problem with this form of MMPP demand streams, showing that a state dependent threshold policy is optimal in this case.

3.3 Conditions

The following theorem provides a condition under which a simpler policy is always optimal. It turns out that only the holding cost of the last part at the QR warehouse is important, hence define $\Delta h_0 = h_0(1) - h_0(0)$. Furthermore, write $(x)^+ = \max\{x, 0\}$.

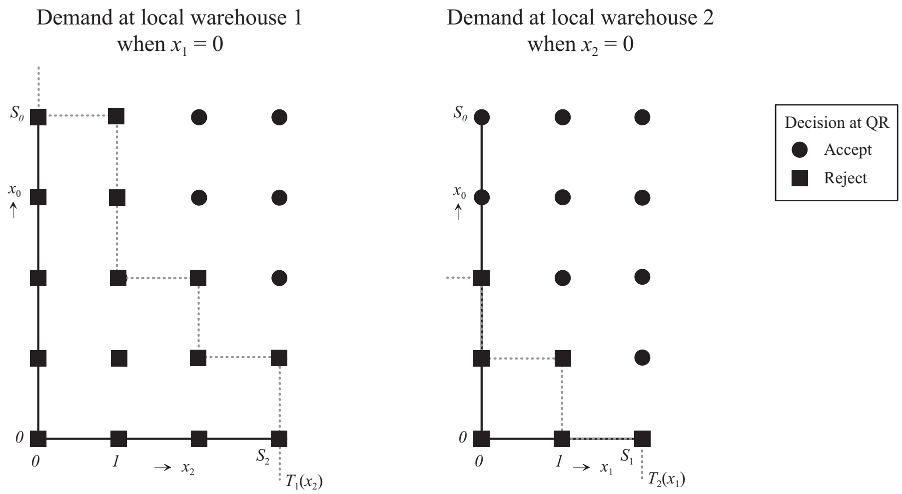

Figure 2: Optimal policy structure when $J = 2$ for demands at the local warehouses $j = 1, 2$ when $x_j = 0$.

Theorem 4. *It is optimal to always accept a demand from local warehouse j at the QR warehouse (for $j = 1, \ldots, J$, when $x_0 > 0$ and $x_j = 0$), or a demand at the QR warehouse (for $j = 0$, when $x_0 > 0$) if:*

$$\sum_{k=0}^{J} \lambda_k \Big(\Delta P_k - \Delta P_j\Big)^+ \;\leq\; \mu_0 \Delta P_j + \Delta h_0. \tag{2}$$

Recall that $\Delta P_k = P_k^{EP} - P_k^{QR}$, where by definition, $\Delta P_0 = P_0^{EP}$. Furthermore, note that the larger μ_0 is, the easier the condition is satisfied, which is intuitively correct. Basically, these conditions give a trade-off between the cost parameters. It follows directly from the theorem, that it is optimal to always satisfy *any* demand at the QR warehouse, if (2) holds for all $j = 0, 1, \ldots, J$.

4 Numerical Results

In a numerical study we show how much is to be gained by executing the optimal policy, compared to two simpler policies. For that, we consider two examples, one excluding and one including a direct demand stream at the QR warehouse. In both cases we vary the arrival rates and cost parameters. We then compare the average costs per time unit of executing three possible policies: the optimal policy, a naive policy always satisfying all demands, and a state-*in*dependent threshold policy with optimal thresholds, the so-called optimal critical level policy.

In a critical level policy, for each warehouse a critical level is prescribed, say C_j for warehouse $j = 0, 1, \ldots, J$ where $C_j \in \{1, \ldots, S_j\}$. Only when the inventory level at the QR warehouse is

$\frac{P_i^{QR}}{P_i^{EP}}$	λ_1: 1.5	2.2	2.9
0.1	+ 2.34%	+ 4.93%	+ 7.79%
0.5	+ 0.74%	+ 1.63%	+ 2.66%
0.9	+ 0.10%	+ 0.23%	+ 0.39%

(a) Example 1: Optimal policy vs. naive policy

$\frac{P_i^{QR}}{P_i^{EP}}$	λ_1: 1.5	2.2	2.9
0.1	+ 2.34%	+ 1.72%	+ 2.35%
0.5	+ 0.74%	+ 0.57%	+ 0.91%
0.9	+ 0.10%	+ 0.08%	+ 0.13%

(b) Example 1: Optimal policy vs. optimal critical level policy

$\frac{P_i^{QR}}{P_i^{EP}}$	λ_1: 0.7	1.2	1.7
0.1	+ 0.11%	+ 1.62%	+ 4.29%
0.5	+ 0.39%	+ 0.58%	+ 0.78%
0.9	+ 6.04%	+ 4.59%	+ 3.16%

(c) Example 2: Optimal policy vs. naive policy

$\frac{P_i^{QR}}{P_i^{EP}}$	λ_1: 0.7	1.2	1.7
0.1	+ 0.11%	+ 1.62%	+ 4.29%
0.5	+ 0.01%	+ 0.02%	+ 0.06%
0.9	+ 0.02%	+ 0.01%	+ 0.01%

(d) Example 2: Optimal policy vs. optimal critical level policy

Table 1: Improvements of optimal policy.

above this level, an (overflow) demand from warehouse j is satisfied. The critical level is a fixed constant which does not depend on the state of the system. By an exhaustive search we optimize the vector $(C_0, C_1, \ldots, C_J)$. Note that at least one critical level will equal zero, as otherwise at least one part of the stock at the QR warehouse remains untouched in any case.

We consider two examples, both with three local warehouses and all base stock levels equal to 3. All replenishment rates equal $\mu_i = 1$ and all holdings costs are 0. The emergency costs are $P_0^{EP} = 10$, $P_1^{EP} = 50$, $P_2^{EP} = 20$, and $P_3^{EP} = 10$. We specify the quick response costs by setting the ratio P_i^{QR}/P_i^{EP}, taking values in $\{0.1, 0.5, 0.9\}$. In Example 1, the QR warehouse is facing *no* direct demand stream: $\lambda_0 = 0$. Furthermore, we vary λ_1 and $\lambda_2 = \lambda_3 = 2.9$. In Example 2, λ_0 is positive: $\lambda_0 = \lambda_2 = \lambda_3 = 1.7$, and again we vary λ_1.

We calculate the relative extra costs per time unit that executing the naive and optimal critical level policy impose compared to the optimal policy. The results are given in Table 1, showing that there might be an almost 8% difference compared to the optimal policy. Cost saving can be reached especially when λ_1 is high.

5 Model Extensions

In this section we study three model extensions. Firstly we study the situation that with some probability p_j a demand might be fulfilled by a quick response, followed we the study of stock level dependent replenishment rates, and backlogging at the local warehouses.

5.1 Quick response with probability p_j

Suppose that a customer who's demand at local warehouse j is not satisfied due to a stock-out, is only interested in a quick response with probability $p_j \in [0,1]$ (i.i.d.). This generalization of the model may be relevant for a setting with physical stores and an on-line shop (see Section 1). For this, we have to adjust the operator H_j into, say, $H_j^{(p_j)}$:

$$H_j^{(p_j)} f(x) = \begin{cases} f(x-e_j) & \text{if } x_j > 0; \\ p_j \min\{P_j^{QR} + f(x-e_0),\ P_j^{EP} + f(x)\} & \\ \qquad +(1-p_j)\Big(P_j^{EP} + f(x)\Big) & \text{if } x_j = 0, x_0 > 0; \\ P_j^{EP} + f(x) & \text{otherwise.} \end{cases}$$

We can write $H_j^{(p_j)} = p_j H_j + (1-p_j)\tilde{H}_j$ where

$$\tilde{H}_j f(x) = \begin{cases} f(x-e_j) & \text{if } x_j > 0; \\ P_j^{EP} + f(x) & \text{otherwise.} \end{cases}$$

So, $\tilde{H}_j$ models at demand a warehouse j that is either directly satisfied when $x_j > 0$, and otherwise fulfilled via an emergency procedure, i.e., without having the option of a quick response. Now in order to prove that $H_j^{(p_j)}$ preserves Conv(x_0) and Supermod(x_0, x_k) for all k, we only have to prove that $\tilde{H}_j$ does so, with is rather straightforward. Hence, analogously to Lemma 1a), the following results hold.

Lemma 5. ***a)*** *For all $j = 1, \dots, J$:*

$$\begin{aligned} \tilde{H}_j :& \textit{Conv}(x_0) \ \rightarrow \ \textit{Conv}(x_0), \\ \tilde{H}_j :& \textit{Supermod}(x_0, x_k) \ \rightarrow \ \textit{Supermod}(x_0, x_k), \ \textit{for all } k = 1, \dots, J. \end{aligned}$$

b)

$$\begin{aligned} H_j^{(p_j)} :& \textit{Conv}(x_0) \ \rightarrow \ \textit{Conv}(x_0), \\ H_j^{(p_j)} :& \textit{Supermod}(x_0, x_j),\ \textit{Conv}(x_0) \ \rightarrow \ \textit{Supermod}(x_0, x_j), \ \textit{for } j = 1, \dots, J. \end{aligned}$$

Hence, part b) is a direct consequence of part a). From this lemma, it follows that Theorems 2 and 3 remain valid, as does Theorem 4 with (2) replaced by:

$$\sum_{k=0}^{J} \lambda_k p_k \Big(\Delta P_k - \Delta P_j\Big)^+ \ \le \ \mu_0 \Delta P_j + \Delta h_0.$$

Note that the only difference is the extra term p_k. This gives a weaker condition, i.e. it is more easily satisfied, which is intuitively correct.

5.2 Stock Level Dependent Replenishment Rates

When the stock level at warehouse j ($j = 0, 1, \ldots, J$) equals x_j, there are $y_j := S_j - x_j$ outstanding orders, where $0 \le y_j \le S_j$. In the preceding we assumed that the replenishment rate at this warehouse is $y_j\,\mu_j$. We now investigate the case of stock level dependent replenishment rates. That is, the replenishment rate is given by $\phi_j : \{0, 1, \ldots, S_j\} \mapsto \mathbb{R}^+$, where $\phi_j(0) = 0$ and furthermore $\phi_j(y_j)$ is assumed to be a concave, increasing function of y_j. Hence, its maximum is attained in S_j, so $\max_{y_j \in \{0,1,\ldots,S_j\}} \phi_j(y_j) = \phi_j(S_j) =: \overline{\phi}_j$, assuming $\overline{\phi}_j < \infty$.

The replenishment operator, say $\tilde{G}_j$, now is given by:

$$\tilde{G}_j f(x) = \begin{cases} \phi_j(y_j) f(x + e_j) + (\overline{\phi}_j - \phi_j(y_j)) f(x) & \text{if } x_j < S_j \text{ (if } y_j > 0) \\ \overline{\phi}_j f(x) & \text{if } x_j = S_j \text{ (if } y_j = 0). \end{cases}$$

for $j = 0, 1, \ldots, J$. The rate out of each state because of $\tilde{G}_j$ is equal to $\overline{\phi}_j$. Analogously to Lemma 1c), the following results hold.

Lemma 6. *For all $j = 0, 1, \ldots, J$:*

$$\begin{aligned} \tilde{G}_j :& \mathit{Decr}(x_0) \;\to\; \mathit{Decr}(x_0), \\ \tilde{G}_j :& \mathit{Conv}(x_0), \mathit{Decr}(x_0) \;\to\; \mathit{Conv}(x_0), \\ \tilde{G}_j :& \mathit{Supermod}(x_0, x_k) \;\to\; \mathit{Supermod}(x_0, x_k) \textit{ for all } k = 1, \ldots, J. \end{aligned}$$

Here Decr(x_0) stands for (non-strict) decreasingness in x_0, that is: $f(x) \ge f(x + e_0)$.

Example 1. *An example of a stock level dependent replenishment rate where $\phi(.)$ is increasing and concave, is a multi-server model with T servers. Each server processes a replenishment at rate μ. So, the replenishment rate is linear in y, namely $y\mu$, with maximum rate $T\mu$:*

$$\phi(y) = \begin{cases} y\mu & \textit{if } 0 \le y < T, \\ T\mu & \textit{if } T \le y \le S. \end{cases}$$

Special cases are $T = 1$ (single server) and $T = S$ (ample repair capacity, as in the current model). This might also be an appropriate model when T machines are producing (i.e. replenishing) to stock, with exponentially distributed production lead times.

As we need Decr(x_0) in order for $\tilde{G}_j$ to preserve convexity, we cannot include holding costs anymore in the QR warehouse (as these are increasing in x_0). So, the new value function, say $\tilde{V}_n$, becomes:

$$\tilde{V}_{n+1}(x) = \sum_{j=1}^{J} h_j(x_j) + \frac{1}{\tilde{\nu}} \left(\sum_{j=0}^{J} \tilde{G}_j \tilde{V}_n(x) + \sum_{i=1}^{J} \lambda_j H_j \tilde{V}_n(x) \right), \text{ for } x \in \mathcal{S}, n \ge 0,$$

starting again with e.g. $\tilde{V}_0 \equiv 0$, where now $\tilde{\nu} = \sum_{j=0}^{J} \overline{\phi}_j + \sum_{j=1}^{J} \lambda_j$ is the uniformization rate.

As a consequence of Lemma 6, we have, like in Theorem 2, that $\tilde{V}_n$ is Decr(x_0), Conv(x_0) and Supermod(x_0, x_j) for all $j = 1, \ldots, J$, when $\tilde{V}_0$ satisfies these properties. Hence, Theorem 3 remains to hold. Theorem 4 remains valid when (2) is replaced by:

$$\sum_{k=0}^{J} \lambda_k \Big(\Delta P_k - \Delta P_j \Big)^+ \;\leq\; \Big(\mu_0(S_0) - \mu_0(S_0 - 1) \Big) \Delta P_j + \Delta h_0. \tag{3}$$

Note that instead of μ_0 we now have $\mu_0(S_0) - \mu_0(S_0 - 1)$.

5.3 Backlogging at Local Warehouses

In Ching [4] an approximate evaluation is given for a model that is almost identical to our model as described in Section 2. He allows backlogging at the local warehouse, up to a (finite) maximum B_j. Only when this maximum is reached, a demand from a local warehouse flows over to the QR warehouse. Ching assumes that such a demand is always satisfied at the QR warehouse.

Instead of the stock level x_j we now focus on the stock level *plus* the maximum number of outstanding backorders B_j, that is: $x_j^{(b)} := x_j + B_j$. Taking the vector $(x_0, x_1^{(b)}, \ldots, x_J^{(b)})$ as the state of the system, we are now back at the original model, however, with a stock level dependent replenishment rate (cf. Section 5.2) at each of the local warehouses. The replenishment rate is given by:

$$\phi_j(x_j^{(b)}) = \begin{cases} (S_j - x_j^{(b)} + B_j)\mu_j & \text{if } x_j^{(b)} > B_j, \\ S_j \mu_j & \text{if } x_j^{(b)} \leq B_j. \end{cases}$$

Hence, the results of Section 5.2 apply to this model as well. As a consequence, when (3) holds for all j, always accepting any (overflow) demand at the QR warehouse, the policy assumed in [4], is optimal in this setting. Even if we charge backlog costs per outstanding backorder per time unit (adding the term $\sum_{j=1}^{J} b_j(\max(0, -x_j))$ where $b_j(\cdot)$ is a non-increasing function with $b_j(0) = 0$), the given structural results and conditions remain valid.

6 Conclusion

We presented a multi-location inventory model with a QR warehouse. Using the structure of the value function, we characterized the optimal policy at this QR warehouse for when to satisfy an (overflow) demand as a state-dependent threshold policy. We furthermore derived conditions under which it is always optimal to satisfy a demand. As model extensions, we considered demands for a quick response with some probability p_j, state-dependent replenishment rates and backlogging at the local warehouses. We furthermore conducted a numerical study showing by how much the performance of the system deteriorates when a state-*in*dependent threshold policy is executed.

It would be interesting for further research to study how well the sufficient condition (2) covers the parameter settings under which all overflow demands from warehouse j are accepted at the QR warehouse under the optimal policy. Moreover, an interesting question is whether the same structural results of the optimal policy hold for more general arrival processes at the QR warehouse. When the overflow demand streams at the QR warehouse are Poisson processes, the optimal policy is known to be state independent threshold policy. We generalized this by letting the demand processes be the overflow streams of the local warehouses, hence being a special form of Markov modulated Poisson processes. The question is whether this can be generalized even further, to more general MMPPs or Markov arrival processes.

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A Proofs

A.1 Proof of Lemma 1

Proof. **a)** For all $j = 1, \ldots, J$ the following holds.
• $H_j : \text{Conv}(x_0) \;\to\; \text{Conv}(x_0)$.
Assume that f is $\text{Conv}(x_0)$, then we show that $H_j f$ is $\text{Conv}(x_0)$ as well. For $x_j > 0$ we have:

$$H_j f(x) + H_j f(x + 2\,e_0) = f(x - e_j) + f(x + 2\,e_0 - e_j) \geq 2\,f(x + e_0 - e_j) = 2\,H_j f(x + e_0),$$

as f is $\text{Conv}(x_0)$. For $x_j = 0,\ x_0 > 0$ we have:

$$\begin{aligned} H_j f(x) + H_j f(x + 2\,e_0) = \min\Big\{ & f(x - e_0) + P_j^{QR} + f(x + e_0) + P_j^{QR}, f(x - e_0) + P_j^{QR} + f(x + 2\,e_0) + P_j^{EP}, \\ & f(x) + P_j^{EP} + f(x + e_0) + P_j^{QR}, f(x) + P_j^{EP} + f(x + 2\,e_0) + P_j^{EP} \Big\}, \end{aligned}$$

which has to be greater than or equal to $2\,H_j f(x+e_0) = 2\,\min\{f(x)+P_j^{QR}, f(x+e_0)+P_j^{EP}\}$. For the third term in the minimization this trivially holds, for the first and fourth term it directly follows as f is $\text{Conv}(x_0)$, and for the second term we have to use this twice:

$$f(x-e_0)+P_j^{QR}+f(x+2\,e_0)+P_j^{EP} \geq f(x-e_0)+P_j^{QR}+2\,f(x+e_0)+P_j^{EP}-f(x) \geq f(x)+P_j^{QR}+f(x+e_0)+P_j^{EP}.$$

For $x_j = 0,\ x_0 = 0$ analogously:

$$\begin{aligned} H_j f(x) + H_j f(x + 2\,e_0) &= \min\Big\{ f(x) + P_j^{EP} + f(x + e_0) + P_j^{QR}, f(x) + P_j^{EP} + f(x + 2\,e_0) + P_j^{EP} \Big\} \\ &\geq 2\,\min\{f(x) + P_j^{QR}, f(x + e_0) + P_j^{EP}\} = 2\,H_j f(x + e_0). \end{aligned}$$

• $H_j : \text{Supermod}(x_0, x_j),\ \text{Conv}(x_0) \;\to\; \text{Supermod}(x_0, x_j)$.
Assume that f is $\text{Supermod}(x_0, x_j)$ and $\text{Conv}(x_0)$, then we show that $H_j f$ is $\text{Supermod}(x_0, x_j)$. For $x_j > 0$:

$$\begin{aligned} H_j f(x) + H_j f(x + e_0 + e_j) &= f(x - e_j) + f(x + e_0) \\ &\geq f(x + e_0 - e_j) + f(x) = H_j f(x + e_0) + H_j f(x + e_j), \end{aligned}$$

as f is $\text{Supermod}(x_0, x_j)$. For $x_j = 0,\ x_0 > 0$ we have:

$$H_j f(x) + H_j f(x + e_0 + e_j) = \min\left\{ f(x - e_0) + P_j^{QR} + f(x + e_0), f(x) + P_j^{EP} + f(x + e_0) \right\},$$

which has to be greater than or equal to $H_j f(x+e_0) + H_j f(x+e_j) = \min\left\{ f(x) + P_j^{QR}, f(x + e_0) + P_j^{EP} \right\} + f(x)$. For the second term in the minimization this trivially hold, for the first term we use that f is $\text{Conv}(x_0)$:

$$f(x - e_0) + P_j^{QR} + f(x + e_0) \geq 2\,f(x) + P_j^{QR}.$$

For $x_j = 0,\ x_0 = 0$ analogously:

$$\begin{aligned} H_j f(x) + H_j f(x + e_0 + e_j) &= f(x) + P_j^{EP} + f(x + e_0) \\ &\geq \min\left\{ f(x) + P_j^{QR}, f(x + e_0) + P_j^{EP} \right\} + f(x) = H_j f(x + e_0) + H_j f(x + e_j). \end{aligned}$$

• H_j : Supermod(x_0, x_k), Conv(x_0) $\rightarrow$ Supermod(x_0, x_k) for all $k \neq j$.
Assume that f is Supermod(x_0, x_k) and Conv(x_0), then we show that $H_j f$ is Supermod(x_0, x_k). For $x_j > 0$:

$$\begin{aligned} H_j f(x) + H_j f(x + e_0 + e_k) &= f(x - e_j) + f(x + e_0 + e_k - e_j) \\ &\geq f(x + e_0 - e_j) + f(x + e_k - e_j) = H_j f(x + e_0) + H_j f(x + e_k). \end{aligned}$$

as f is Supermod(x_0, x_k). For $x_j = 0,\ x_0 > 0$ we have:

$$\begin{aligned} H_j f(x) + H_j f(x + e_0 + e_k) = \min\Big\{ & f(x - e_0) + P_j^{QR} + f(x + e_k) + P_j^{QR}, f(x - e_0) + P_j^{QR} \\ & + f(x + e_0 + e_k) + P_j^{EP}, f(x) + P_j^{EP} + f(x + e_k) + P_j^{QR}, f(x) + P_j^{EP} + f(x + e_0 + e_k) + P_j^{EP} \Big\}, \end{aligned}$$

which has to be greater than or equal to $H_j f(x + e_0) + H_j f(x + e_k) = \min\{f(x) + P_j^{QR}, f(x + e_0) + P_j^{EP}\} + \min\{f(x - e_0 + e_k) + P_j^{QR}, f(x + e_k) + P_j^{EP}\}$. For the third term in the minimization this trivially holds, and for the first and fourth term we use that f is Supermod(x_0, x_k). For the second term we first use this, followed by using that f is Conv(x_0):

$$\begin{aligned} f(x - e_0) + P_j^{QR} + f(x + e_0 + e_k) + P_j^{EP} &\geq f(x) + f(x - e_0 + e_k) + P_j^{QR} + f(x + e_0 + e_k) + P_j^{EP} - f(x + e_k) \\ &\geq f(x) + P_j^{QR} + f(x + e_k) + P_j^{EP}. \end{aligned}$$

For $x_j = 0,\ x_0 = 0$ analogously:

$$\begin{aligned} H_j f(x) + H_j f(x + e_0 + e_k) &= \min\Big\{ f(x) + P_j^{EP} + f(x + e_k) + P_j^{QR}, f(x) + P_j^{EP} + f(x + e_0 + e_k) + P_j^{EP} \Big\} \\ &\geq \min\{f(x) + P_j^{QR}, f(x + e_0) + P_j^{EP}\} + \min\{f(x - e_0 + e_k) + P_j^{QR}, f(x + e_k) + P_j^{EP}\} \\ &= H_j f(x + e_0) + H_j f(x + e_k). \end{aligned}$$

b) • H_{QR} : Conv(x_0) $\rightarrow$ Conv(x_0).
Assume that f is Conv(x_0), then we show that $H_{QR} f$ is Conv(x_0) as well. For $x_0 > 0$:

$$\begin{aligned} H_{QR} f(x) + H_{QR} f(x + 2\,e_0) = \min\Big\{ & f(x - e_0) + f(x + e_0), f(x - e_0) + f(x + 2\,e_0) + P_0^{EP}, \\ & f(x) + P_0^{EP} + f(x + e_0), f(x) + P_0^{EP} + f(x + 2\,e_0) + P_0^{EP} \Big\}. \end{aligned}$$

which has to be greater than or equal to $2\,H_{QR} f(x + e_0) = 2\ \min\{f(x), f(x + e_0) + P_0^{EP}\}$. For the third term in the minimization this trivially holds, for the first and fourth term we use that f is Conv(x_0), and for the second term we have to use this twice:

$$f(x - e_0) + f(x + 2\,e_0) + P_0^{EP} \geq f(x - e_0) + 2\,f(x + e_0) - f(x) + P_0^{EP} \geq f(x) + f(x + e_0) + P_0^{EP}.$$

For $x_0 = 0$ analogously:

$$\begin{aligned} H_{QR} f(x) + H_{QR} f(x + 2\,e_0) &= \min\Big\{ f(x) + P_0^{EP} + f(x + e_0), f(x) + P_0^{EP} + f(x + 2\,e_0) + P_0^{EP} \Big\} \\ &\geq 2\ \min\{f(x), f(x + e_0) + P_0^{EP}\} = 2\,H_{QR} f(x + e_0). \end{aligned}$$

• H_{QR} : Supermod(x_0, x_j), Conv(x_0) $\rightarrow$ Supermod(x_0, x_j), for $j = 1, \ldots, J$.
Assume that f is Supermod(x_0, x_j) and Conv(x_0), then we show that $H_{QR} f$ is Supermod(x_0, x_j). For $x_0 > 0$:

$$\begin{aligned} H_{QR} f(x) + H_{QR} f(x + e_0 + e_j) = \min\Big\{ & f(x - e_0) + f(x + e_j), f(x - e_0) + f(x + e_0 + e_j) + P_0^{EP}, \\ & f(x) + P_0^{EP} + f(x + e_j), f(x) + P_0^{EP} + f(x + e_0 + e_j) + P_0^{EP} \Big\}, \end{aligned}$$

which has to be greater than or equal to $H_{QR} f(x + e_0) + H_{QR} f(x + e_j) = \min\{f(x), f(x + e_0) + P_0^{EP}\} + \min\{f(x + e_j - e_0), f(x + e_j) + P_0^{EP}\}$. For the third term in the minimization this trivially holds, and for the first and fourth term we use that f is Supermod(x_0, x_j). For the second term we first use this, followed by using that f is Conv(x_0):

$$\begin{aligned} & f(x - e_0) + f(x + e_0 + e_j) + P_0^{EP} \\ & \quad \geq f(x) + f(x - e_0 + e_j) + f(x + e_0 + e_j) + P_0^{EP} - f(x + e_j) \geq f(x) + f(x + e_j) + P_0^{EP}. \end{aligned}$$

For $x_0 = 0$ analogously:

$$\begin{aligned} H_{QR} f(x) + H_{QR} f(x + e_0 + e_j) &= \min\Big\{ f(x) + P_0^{EP} + f(x + e_j), f(x) + P_0^{EP} + f(x + e_0 + e_j) + P_0^{EP} \Big\} \\ & \min\{f(x), f(x + e_0) + P_0^{EP}\} + \min\{f(x + e_j - e_0), f(x + e_j) + P_0^{EP}\} \\ &= H_{QR} f(x + e_0) + H_{QR} f(x + e_j). \end{aligned}$$

c) We prove the following, for all $j = 0, 1, \ldots, J$:

$$
\begin{aligned}
&1)\ G_j : \text{Conv}(x_j) \ \rightarrow \ \text{Conv}(x_j),\\
&2)\ G_j : \text{Conv}(x_k) \ \rightarrow \ \text{Conv}(x_k) \text{ for all } k \neq j,\\
&3)\ G_0 : \text{Supermod}(x_0, x_j) \ \rightarrow \ \text{Supermod}(x_0, x_j) \text{ for } j \neq 0,\\
&4)\ G_j : \text{Supermod}(x_0, x_j) \ \rightarrow \ \text{Supermod}(x_0, x_j) \text{ for } j \neq 0,\\
&5)\ G_j : \text{Supermod}(x_0, x_k) \ \rightarrow \ \text{Supermod}(x_0, x_k) \text{ for } j \neq 0 \text{ and all } k \neq 0,\ j.
\end{aligned}
$$

From this the result of the lemma directly follows. For that, we note that 1) and 2) imply that $G_j : \text{Conv}(x_0) \rightarrow \text{Conv}(x_0)$ for $j = 0, 1, \ldots, J$.

• 1) $G_j : \text{Conv}(x_j) \ \rightarrow \ \text{Conv}(x_j)$.
Assume that f is $\text{Conv}(x_j)$, then we show that $G_j f$ is $\text{Conv}(x_j)$ as well. For $x_j + 2 < S_j$:

$$
\begin{aligned}
G_j f(x) + G_j f(x + 2\, e_j) &= (S_j - x_j) f(x + e_j) + x_j f(x) + (S_j - x_j - 2) f(x + 3\, e_j) + (x_j + 2) f(x + 2\, e_j)\\
&= (S_j - x_j - 2)\Big[f(x + e_j) + f(x + 3\, e_j)\Big] + x_j \Big[f(x) + f(x + 2\, e_j)\Big] + 2\, f(x + e_j) + 2\, f(x + 2\, e_j)\\
&\geq 2\,(S_j - x_j - 2) f(x + 2\, e_j) + 2\, x_j f(x + e_j) + 2\, f(x + e_j) + 2\, f(x + 2\, e_j)\\
&= 2\,(S_j - x_j - 1) f(x + 2\, e_j) + 2\,(x_j + 1) f(x + e_j) = 2\, G_j f(x + e_j),
\end{aligned}
$$

where the inequality holds by applying that f is $\text{Conv}(x_j)$ on the parts between brackets. For $x_j + 2 = S_j$ analogously:

$$
\begin{aligned}
G_j f(x) + G_j f(x + 2\, e_j) &= 2\, f(x + e_j) + (S_j - 2) f(x) + S_j f(x + 2\, e_j)\\
&= 2\, f(x + e_j) + (S_j - 2)[f(x) + f(x + 2\, e_j)] + 2\, f(x + 2\, e_j)\\
&\geq 2\, f(x + e_j) + 2\,(S_j - 2) f(x + e_j) + 2\, f(x + 2\, e_j)\\
&= 2\, f(x + 2\, e_j) + 2\,(S_j - 1) f(x + e_j) = 2\, G_j f(x + e_j).
\end{aligned}
$$

• 2) $G_j : \text{Conv}(x_k) \ \rightarrow \ \text{Conv}(x_k)$ for all $k \neq j$.
Assume that f is $\text{Conv}(x_k)$, then we show that $G_j f$ for $k \neq j$ is $\text{Conv}(x_k)$ as well. For $x_j < S_j$:

$$
\begin{aligned}
G_j f(x) + G_j f(x + 2\, e_k) &= (S_j - x_j) f(x + e_j) + x_j f(x) + (S_j - x_j) f(x + e_j + 2\, e_k) + x_j f(x + 2\, e_k)\\
&\geq 2\,(S_j - x_j) f(x + e_j + e_k) + 2\, x_j f(x + e_k) = 2\, G_j f(x + e_k),
\end{aligned}
$$

and for $x_j = S_j$:

$$
G_j f(x) + G_j f(x + 2\, e_k) = S_j f(x) + S_j f(x + 2\, e_k) \geq 2\, S_j f(x + e_k) = 2\, G_j f(x + e_k).
$$

• 3) $G_0 : \text{Supermod}(x_0, x_j) \ \rightarrow \ \text{Supermod}(x_0, x_j)$ for $j \neq 0$.
Assume that f is $\text{Supermod}(x_0, x_j)$ for $j \neq 0$, then we show that $G_0 f$ is $\text{Supermod}(x_0, x_j)$ as well (for $j \neq 0$). For $x_0 + 1 < S_0$:

$$
\begin{aligned}
&G_0 f(x) + G_0 f(x + e_0 + e_j) = (S_0 - x_0) f(x + e_0) + x_0 f(x) + (S_0 - x_0 - 1) f(x + 2\, e_0 + e_j) + (x_0 + 1) f(x + e_0 + e_j)\\
&= (S_0 - x_0 - 1)\Big[f(x + e_0) + f(x + 2\, e_0 + e_j)\Big] + f(x + e_0) + x_0 \Big[f(x) + f(x + e_0 + e_j)\Big] + f(x + e_0 + e_j)\\
&\geq (S_0 - x_0 - 1)\Big[f(x + e_0 + e_j) + f(x + 2\, e_0)\Big] + f(x + e_0) + x_0 \Big[f(x + e_j) + f(x + e_0)\Big] + f(x + e_0 + e_j)\\
&= (S_0 - x_0) f(x + e_0 + e_j) + x_0 f(x + e_j) + (S_0 - x_0 - 1) f(x + 2\, e_0) + (x_0 + 1) f(x + e_0) = G_0 f(x + e_j) + G_j f(x + e_0),
\end{aligned}
$$

and for $x_0 + 1 = S_0$:

$$
\begin{aligned}
G_0 f(x) + G_0 f(x + e_0 + e_j) &= f(x + e_0) + (S_0 - 1) f(x) + S_0\, f(x + e_0 + e_j)\\
&= (S_0 - 1)\Big[f(x) + f(x + e_0 + e_j)\Big] + f(x + e_0) + f(x + e_0 + e_j)\\
&\geq (S_0 - 1)\Big[f(x + e_j) + f(x + e_0)\Big] + f(x + e_0) + f(x + e_0 + e_j)\\
&= f(x + e_0 + e_j) + (S_0 - 1) f(x + e_j) + S_0\, f(x + e_0) = G_0 f(x + e_j) + G_j f(x + e_0).
\end{aligned}
$$

• 4) $G_j : \text{Supermod}(x_0, x_j) \ \rightarrow \ \text{Supermod}(x_0, x_j)$ for $j \neq 0$.
This follows directly from 3) by symmetry of $\text{Supermod}(x_0, x_j)$ in x_0 and x_j. Hence, there is no need to distinguish between G_0 and G_j, and the statement follows.

• 5) G_j : Supermod$(x_0, x_k) \rightarrow$ Supermod(x_0, x_k) for $j \neq 0$ and all $k \neq 0,\ j$.
Assume that f is Supermod(x_0, x_k), then we show that $G_j f$ is Supermod(x_0, x_k) as well (for $j \neq 0$). For $x_j < S_j$:

$$G_j f(x) + G_j f(x+e_0+e_k) = (S_j - x_j) f(x+e_j) + x_j f(x) + (S_j - x_j) f(x+e_0+e_j+e_k) + x_j f(x+e_0+e_k)$$
$$\geq (S_j - x_j) f(x+e_0+e_j) + x_j f(x+e_0) + (S_j - x_j) f(x+e_k+e_j) + x_j f(x+e_k) = G_j f(x+e_0) + G_j f(x+e_k),$$

and for $x_j = S_j$:

$$G_j f(x) + G_j f(x+e_0+e_k) = S_j f(x) + S_j f(x+e_0+e_k) \geq S_j f(x+e_0) + S_j f(x+e_k)$$
$$= G_j f(x+e_0) + G_j f(x+e_k).$$

□

A.2 Proof of Theorem 3

Proof. Consider a demand directly at the QR warehouse ($j = 0$), or an overflow demand from local warehouse $j \in \{1, \ldots, J\}$ when $x_j = 0$. There are two options for such a demand: accepting it (if $x_0 > 0$) or rejecting it at the QR warehouse. Let

$$w_j(u, x) := \begin{cases} P_j^{QR} + V_n(x - e_0) & \text{if } u = 1 \text{ (accept)}, \\ P_j^{EP} + V_n(x) & \text{if } u = 0 \text{ (reject)}. \end{cases}$$

Then $H_j V_n(x) = \min_{u \in \{0,1\}} w_j(u, x)$ for x such that $x_j = 0$ and $x_0 > 0$. Also, as $P_0^{QR} = 0$ by definition, $H_{QR} V_n(x) = \min_{u \in \{0,1\}} w_0(u, x)$ for x such that $x_0 > 0$.

Let $\Delta_{x_k} w_j(u, x) := w_j(u, x + e_k) - w_j(u, x)$ for all x with $x_0 > 0$ and $x_k < S_k$. Then, for $x_0 > 0$:

$$\Delta_{x_0} w_j(1, x) - \Delta_{x_0} w_j(0, x) = w_j(1, x+e_0) - w_j(1, x) - w_j(0, x+e_0) + w_j(0, x)$$
$$= V_n(x) - V_n(x - e_0) - V_n(x + e_0) + V_n(x) \leq 0,$$

as V_n is Conv(x_0). Furthermore, for $x_0 > 0,\ k \neq j$:

$$\Delta_{x_k} w_j(1, x) - \Delta_{x_k} w_j(0, x) = w_j(1, x+e_k) - w_j(1, x) - w_j(0, x+e_k) + w_j(0, x)$$
$$= V_n(x - e_0 + e_k) - V_n(x - e_0) - V_n(x + e_k) + V_n(x) \leq 0,$$

as V_n is Supermod(x_0, x_k).

This implies that, for every $n \geq 0$, there exists a switching curve, say T_j^n, which is a function of $x^{(0,j)}$, such that the optimal decision at the QR warehouse is to accept the demand if $x_0 > T_j^n(x^{(0,j)})$, and to reject it if $x_0 \leq T_j^n(x^{(0,j)})$. As overflow demands from local warehouse j can only occur when $x_j = 0$, the switching curve does not depend on x_j. Moreover, it follows that T_j^n is decreasing in each of its components.

Hence, if f_{n+1} is the minimizing policy in (1), then f_{n+1} is a state dependent threshold policy described by the switching curves T_j^{n+1}, $j = 0, 1, \ldots, J$. Note that the transition probability matrix of every stationary policy is unichain (since every state can access $(S_0, S_1, \ldots, S_J)$) and aperiodic (since the transition probability from state $(S_0, S_1, \ldots, S_J)$ to itself is positive). Then, by [17, Theorem 8.5.4], the long run average costs under the stationary policy f_{n+1} converge to the minimal long run average costs as n tends to infinity. Since there are only finitely many stationary threshold policies, this implies that there exists an optimal stationary policy that is a state dependent threshold type policy. □

A.3 Proof of Proposition 1

Proof. We show that when $a_{j_2}^*(x) = 1$ then $a_{j_1}^*(x) = 1$ as well, for all states x such that $x_{j_1} = x_{j_2} = 0$. Suppose that $a_{j_2}^*(x) = 1$, i.e. applying a quick response at warehouse j_2 in case of a stock out is optimal, then

$$f(x - e_0) + P_{j_2}^{QR} \leq f(x) + P_{j_2}^{EP}.$$

Hence

$$f(x - e_0) \leq f(x) + \Delta P_{j_2} \leq f(x) + \Delta P_{j_1},$$

where the second inequality holds by the condition $\Delta P_{j_1} \geq \Delta P_{j_2}$. Now

$$f(x - e_0) + P_{j_1}^{QR} \leq f(x) + P_{j_1}^{EP},$$

and so $a_{j_1}^*(x) = 1$. □

A.4 Proof of Theorem 4

Proof. We prove the theorem 1) for local warehouse j, $j = 1, \dots, J$, and 2) for the QR warehouse.

1) We prove that when (2) is satisfied, the following holds:

$$V_n(x - e_0) + P_j^{QR} \leq V_n(x) + P_j^{EP}, \text{ for all } x \text{ with } x_j = 0 \text{ and } x_0 = 1.$$

Hence, a demand from local warehouse j is always accepted at the QR warehouse in this case. It follows from the structural results of Theorem 3 that this action is optimal as well for $x_0 = 2, \dots, S_0$ (and $x_j = 0$).

Write $x_{(a,b)}$ for x with $x_0 = a$ and $x_j = b$:

$$x_{(a,b)} := (a, x_1, \dots, x_{j-1}, b, x_{j+1}, \dots, x_J).$$

We prove that when (2) holds, then $V_n(x_{(0,0)}) - V_n(x_{(1,0)}) \leq P_j^{EP} - P_j^{QR} = \Delta P_j$, for all $n \geq 0$, where the entries other than x_0 and x_j are equal for $x_{(0,0)}$ and $x_{(1,0)}$. For this, we use induction on V_n and consider each of the operators separately. For $V_0 \equiv 0$ the inequality clearly holds. Assume that it holds for a certain n and denote this V_n by f. That is, the induction hypothesis (*i.h.*) is given by:

$$f(x_{(0,0)}) - f(x_{(1,0)}) \leq \Delta P_j. \qquad (i.h.)$$

We apply each of the operators separately to $f(x_{(0,0)}) - f(x_{(1,0)})$. All inequalities hold by (*i.h.*) unless stated otherwise.

- H_{QR}:

$$H_{QR} f(x_{(0,0)}) - H_{QR} f(x_{(1,0)}) = \max\{P_0^{EP}, f(x_{(0,0)}) - f(x_{(1,0)})\} \leq \max\{\Delta P_0, \Delta P_j\},$$

as by definition $\Delta P_0 = P_0^{EP}$.

- H_j:

$$H_j f(x_{(0,0)}) - H_j f(x_{(1,0)}) = \max\{P_j^{EP} - P_j^{QR}, f(x_{(0,0)}) - f(x_{(1,0)})\} \leq \Delta P_j.$$

- H_k (for $k \neq j$): if $x_k > 0$:

$$H_k f(x_{(0,0)}) - H_k f(x_{(1,0)}) = f(x_{(0,0)} - e_k) - f(x_{(1,0)} - e_k) \leq \Delta P_j,$$

and if $x_k = 0$:

$$H_k f(x_{(0,0)}) - H_k f(x_{(1,0)}) = \max\{P_k^{EP} - P_k^{QR}, f(x_{(0,0)}) - f(x_{(1,0)})\} \leq \max\{\Delta P_k, \Delta P_j\}.$$

- G_j:

$$G_j f(x_{(0,0)}) - G_j f(x_{(1,0)}) = S_j[f(x_{(0,1)}) - f(x_{(1,1)})] \leq S_j[f(x_{(0,0)}) - f(x_{(1,0)})] \leq S_j \Delta P_j,$$

where the first inequality holds as f (i.e. V_n) is Supermod(x_0, x_j) (cf. Theorem 4).

- G_0:

$$\begin{aligned} G_0 f(x_{(0,0)}) - G_0 f(x_{(1,0)}) &= S_0 f(x_{(1,0)}) - (S_0 - 1) f(x_{(2,0)}) - f(x_{(1,0)}) \\ &= (S_0 - 1)[f(x_{(1,0)}) - f(x_{(2,0)})] \leq (S_0 - 1)[f(x_{(0,0)}) - f(x_{(1,0)})] \leq (S_0 - 1)\Delta P_j, \end{aligned}$$

where the first inequality holds as f (i.e. V_n) is Conv(x_0) (cf. Theorem 4).

- G_k (for $k \neq 0,\ j$):

$$\begin{aligned} G_k f(x_{(0,0)}) - G_k f(x_{(1,0)}) &= (S_k - x_k) f(x_{(0,0)} + e_k) + x_k f(x_{(0,0)}) - (S_k - x_k) f(x_{(1,0)} + e_k) - x_k f(x_{(1,0)}) \\ &= (S_k - x_k)[f(x_{(0,0)} + e_k) - f(x_{(1,0)} + e_k)] + x_k[f(x_{(0,0)}) - f(x_{(1,0)})] \\ &\leq (S_k - x_k)\Delta P_j + x_k \Delta P_j = S_k \Delta P_j. \end{aligned}$$

We now combining these results. Recall that $\Delta h_0 = h_0(1) - h_0(0)$ and note that trivially $\Delta P_j \leq \max\{\Delta P_k, \Delta P_j\}$:

$$\begin{aligned} \nu \left(V_{n+1}(x_{(0,0)}) - V_{n+1}(x_{(1,0)})\right) &= -\Delta h_0 + \lambda_0 \left(H_{QR} f(x_{(0,0)}) - H_{QR} f(x_{(1,0)})\right) \\ &\quad + \sum_{k=1}^{J} \lambda_k \left(H_k f(x_{(0,0)}) - H_k f(x_{(1,0)})\right) + \sum_{k=0}^{J} \mu_k \left(G_k f(x_{(0,0)}) - G_k f(x_{(1,0)})\right) \\ &\leq -\Delta h_0 + \sum_{k=0}^{J} \lambda_k \max\{\Delta P_k, \Delta P_j\} + \Big(\sum_{k=0}^{J} \mu_k S_k - \mu_0\Big)\Delta P_j \\ &= -\Delta h_0 + \sum_{k=0}^{J} \lambda_k \max\{\Delta P_k - \Delta P_j, 0\} + \Big(\nu - \mu_0\Big)\Delta P_j \\ &\leq \mu_0 \Delta P_j + \Big(\nu - \mu_0\Big)\Delta P_j = \nu \Delta P_j, \end{aligned}$$

where the last inequality holds by (2). Hence, we have proven that the induction step holds.

2) We show that when (2) is satisfied, the following holds:

$$V_n(x-e_0) \le V_n(x) + P_0^{EP}, \text{ for all } x \text{ with } x_0 = 1.$$

Hence, a demand at the QR from warehouse j is always accepted in this case. It follows from the structural results of Theorem 3 that this action is optimal as well for $x_0 = 2, \dots, S_0$.

Write $x_{(a)}$ for x with $x_0 = a$, that is $x_{(a)} := (a, x_1, \dots, x_J)$. We prove that when (2) holds, then $V_n(x_{(0)}) - V_n(x_{(1)}) \le P_0^{EP}$, for all $n \ge 0$, where the entries other than x_0 are equal for $x_{(0)}$ and $x_{(1)}$. For this, we use induction on V_n and consider each of the operators separately. For $V_0 \equiv 0$ the inequality clearly holds. Assume that it holds for a certain n and denote this V_n by f. That is, the induction hypothesis (*i.h.*) is given by:

$$f(x_{(0)}) - f(x_{(1)}) \le P_0^{EP}. \qquad (i.h.)$$

We apply each of the operators separately to $f(x_{(0)}) - f(x_{(1)})$. All inequalities hold by (*i.h.*) unless stated otherwise.

• H_{QR}:

$$H_{QR}f(x_{(0)}) - H_{QR}f(x_{(1)}) = \max\{P_0^{EP}, f(x_{(0)}) - f(x_{(1)})\} \le P_0^{EP}.$$

• H_k: if $x_k > 0$:

$$H_k f(x_{(0)}) - H_k f(x_{(1)}) = f(x_{(0)} - e_k) - f(x_{(1)} - e_k) \le P_0^{EP},$$

and if $x_k = 0$:

$$H_k f(x_{(0)}) - H_k f(x_{(1)}) = \max\{P_k^{EP} - P_k^{QR}, f(x_{(0)}) - f(x_{(1)})\} \le \max\{\Delta P_k, P_0^{EP}\}.$$

• G_0:

$$\begin{aligned} G_0 f(x_{(0)}) - G_0 f(x_{(1)}) &= S_0 f(x_{(1)}) - (S_0 - 1) f(x_{(2)}) - f(x_{(1)}) \\ &= S_0[f(x_{(1)}) - f(x_{(2)})] \le (S_0 - 1)[f(x_{(0)}) - f(x_{(1)})] \le (S_0 - 1) P_0^{EP}, \end{aligned}$$

where the first inequality holds as f (i.e. V_n) is Conv(x_0) (cf. Theorem 4).
• G_k (for $k \ne 0$):

$$\begin{aligned} G_k f(x_{(0)}) - G_k f(x_{(1)}) &= (S_k - x_k) f(x_{(0)} + e_k) + x_k f(x_{(0)}) - (S_k - x_k) f(x_{(1)} + e_k) - x_k f(x_{(1)}) \\ &= (S_k - x_k)[f(x_{(0)} + e_k) - f(x_{(1)} + e_k)] + x_k[f(x_{(0)}) - f(x_{(1)})] \le (S_k - x_k) P_0^{EP} + x_k P_0^{EP} = S_k P_0^{EP}. \end{aligned}$$

Combining these results yields (recall that by definition $\Delta P_0 = P_0^{EP}$):

$$\begin{aligned} \nu\left(V_{n+1}(x_{(0)}) - V_{n+1}(x_{(1)})\right) &= -\Delta h_0 + \lambda_0 \left(H_{QR}f(x_{(0)}) - H_{QR}f(x_{(1)})\right) \\ &\quad + \sum_{k=1}^{J} \lambda_k \left(H_k f(x_{(0)}) - H_k f(x_{(1)})\right) + \sum_{k=0}^{J} \mu_k \left(G_k f(x_{(0)}) - G_k f(x_{(1)})\right) \\ &\le -\Delta h_0 + \sum_{k=0}^{J} \lambda_k \max\{\Delta P_k, P_0^{EP}\} + P_0^{EP}\Big(\sum_{k=0}^{J} \mu_k S_k - \mu_0\Big) \\ &= -\Delta h_0 + \sum_{k=0}^{J} \lambda_k \max\{\Delta P_k - P_0^{EP}, 0\} + P_0^{EP}\Big(\nu - \mu_0\Big) \\ &\le \mu_0 P_0^{EP} + P_0^{EP}\Big(\nu - \mu_0\Big) = \nu P_0^{EP}, \end{aligned}$$

where the last inequality holds by (2), applied for $j = 0$. Hence, we have proven that the induction step holds. □

A.5 Proof of Lemma 5

Proof. **a)** For all $j = 1, \dots, J$ the following holds.
• $\tilde{H}_j$: Conv(x_0) $\to$ Conv(x_0).
Assume that f is Conv(x_0), then we show that $\tilde{H}_j f$ is Conv(x_0) as well. For $x_j > 0$ we have:

$$\tilde{H}_j f(x) + \tilde{H}_j f(x + 2\,e_0) = f(x - e_j) + f(x + 2\,e_0 - e_j) \ge 2\,f(x + e_0 - e_j) = 2\,\tilde{H}_j f(x + e_0),$$

as f is Conv(x_0), and for $x_j = 0$ we have:

$$\tilde{H}_j f(x) + \tilde{H}_j f(x + 2\,e_0) = f(x) + P_j^{EP} + f(x + 2\,e_0) + P_j^{QR} \geq 2\,f(x + e_0) + P_j^{EP} = 2\,\tilde{H}_j f(x + e_0),$$

as f is Conv(x_0)

• $\tilde{H}_j$: Supermod$(x_0, x_j) \;\to\;$ Supermod(x_0, x_j).
Assume that f is Supermod(x_0, x_j), then we show that $\tilde{H}_j f$ is Supermod(x_0, x_j). For $x_j > 0$:

$$\tilde{H}_j f(x) + \tilde{H}_j f(x + e_0 + e_j) = f(x - e_j) + f(x + e_0) \geq f(x + e_0 - e_j) + f(x) = \tilde{H}_j f(x + e_0) + \tilde{H}_j f(x + e_j),$$

as f is Supermod(x_0, x_j), and for $x_j = 0$ we have:

$$\tilde{H}_j f(x) + \tilde{H}_j f(x + e_0 + e_j) = f(x) + P_j^{EP} + f(x + e_0) = \tilde{H}_j f(x + e_0) + \tilde{H}_j f(x + e_j).$$

• $\tilde{H}_j$: Supermod$(x_0, x_k) \;\to\;$ Supermod(x_0, x_k) for all $k \neq j$.
Assume that f is Supermod(x_0, x_k), then we show that $\tilde{H}_j f$ is Supermod(x_0, x_k). For $x_j > 0$:

$$\begin{aligned}\tilde{H}_j f(x) + \tilde{H}_j f(x + e_0 + e_k) &= f(x - e_j) + f(x + e_0 + e_k - e_j) \\ &\geq f(x + e_0 - e_j) + f(x + e_k - e_j) = \tilde{H}_j f(x + e_0) + \tilde{H}_j f(x + e_k),\end{aligned}$$

as f is Supermod(x_0, x_k), and for $x_j = 0$ we have:

$$\begin{aligned}\tilde{H}_j f(x) + \tilde{H}_j f(x + e_0 + e_k) &= f(x) + P_j^{EP} + f(x + e_0 + e_k) + P_j^{EP} \\ &\geq f(x + e_0) + P_j^{EP} + f(x + e_k) + P_j^{EP} = \tilde{H}_j f(x + e_0) + \tilde{H}_j f(x + e_k).\end{aligned}$$

b) Direct consequence of part a) combined with Lemma 1, part a) and the identity $H_j^{(p_j)} = p_j H_j + (1 - p_j)\tilde{H}_j$.

□

A.6 Proof of Lemma 6

Proof. • $\tilde{G}_j$: Decr$(x_0) \;\to\;$ Decr(x_0).
Assume that f is Decr(x_0), then we show that $\tilde{G}_j f$ is Decr(x_0) as well. The cases $j \in \{1, \dots, J\}$ are trivial, so we only show $j = 0$. For $x_0 + 1 < S_0$ we have:

$$\begin{aligned}&\tilde{G}_0 f(x) - \tilde{G}_0 f(x + e_0) \\ &= \phi_0(y_0) f(x + e_0) + \Big(\overline{\phi}_0 - \phi_0(y_0)\Big) f(x) - \phi_0(y_0 - 1) f(x + 2\,e_0) - \Big(\overline{\phi}_0 - \phi_0(y_0 - 1)\Big) f(x + e_0) \\ &= \phi_0(y_0 - 1)\Big(f(x + e_0) - f(x + 2\,e_0)\Big) + \Big(\overline{\phi}_0 - \phi_0(y_0)\Big)\Big(f(x) - f(x + e_0)\Big) \;\geq 0,\end{aligned}$$

and for $x_0 + 1 = S_0$:

$$\tilde{G}_0 f(x) - \tilde{G}_0 f(x + e_0) = \phi_0(y_0) f(x + e_0) + \Big(\overline{\phi}_0 - \phi_0(y_0)\Big) f(x) - \overline{\phi}_0 f(x + e_0) = \Big(\overline{\phi}_0 - \phi_0(y_0)\Big)\Big(f(x) - f(x + e_0)\Big) \geq 0.$$

• $\tilde{G}_j$: Conv(x_0), Decr$(x_0) \;\to\;$ Conv(x_0).
For $j \neq 0$ trivially $\tilde{G}_j$: Conv$(x_0) \;\to\;$ Conv(x_0). Hence we only show the proof for $j = 0$. Assume that f is Conv(x_0) and Decr(x_0), then we show that $\tilde{G}_0 f$ is Conv(x_0). Write $\overline{\phi}_j(y_j) = \overline{\phi}_j - \phi_j(y_j)$. For $x_0 + 2 < S_0$ we have

$$\begin{aligned}&\tilde{G}_0 f(x) + \tilde{G}_0 f(x + 2\,e_0) - 2\,\tilde{G}_0 f(x + e_0) \\ &= \phi_0(y_0) f(x + e_0) + \overline{\phi}_0(y_0) f(x) + \phi_0(y_0 - 2) f(x + 3\,e_0) \\ &\qquad + \overline{\phi}_0(y_0 - 2) f(x + 2\,e_0) - 2\,\phi_0(y_0 - 1) f(x + 2\,e_0) - 2\,\overline{\phi}_0(y_0 - 1) f(x + e_0) \\ &= \phi_0(y_0 - 2)\Big(f(x + 3\,e_0) + f(x + e_0) - 2\,f(x + 2\,e_0)\Big) + 2\Big(\overline{\phi}_0 - \overline{\phi}_0(y_0 - 2)\Big) f(x + 2\,e_0) \\ &\qquad - \Big(\overline{\phi}_0 - \overline{\phi}_0(y_0 - 2)\Big) f(x + e_0) + \Big(\overline{\phi}_0 - \overline{\phi}_0(y_0)\Big) f(x + e_0) - 2\Big(\overline{\phi}_0 - \overline{\phi}_0(y_0 - 1)\Big) f(x + 2\,e_0) \\ &\qquad + \overline{\phi}_0(y_0) f(x) + 2\,\overline{\phi}_0(y_0 - 2) f(x + 2\,e_0) - 2\,\overline{\phi}_0(y_0 - 1) f(x + e_0) \\ &\geq -\overline{\phi}_0(y_0 - 2) f(x + 2\,e_0) + \overline{\phi}_0(y_0 - 2) f(x + e_0) - \phi_0(y_0) f(x + e_0) + 2\,\phi_0(y_0 - 1) f(x + 2\,e_0) \\ &\qquad + \overline{\phi}_0(y_0) f(x) - 2\,\overline{\phi}_0(y_0 - 1) f(x + e_0) \\ &= \overline{\phi}_0(y_0 - 2)\Big(f(x + e_0) - f(x + 2\,e_0)\Big) - 2\,\overline{\phi}_0(y_0 - 1)\Big(f(x + e_0) - f(x + 2\,e_0)\Big) + \overline{\phi}_0(y_0)\Big(f(x) - f(x + e_0)\Big) \\ &\geq \Big(\overline{\phi}_0(y_0 - 2) - 2\,\overline{\phi}_0(y_0 - 1) + \overline{\phi}_0(y_0)\Big)\Big(f(x + e_0) - f(x + 2\,e_0)\Big) \geq 0,\end{aligned}$$

where the first inequality hold as f is Conv(x_0) (hence the term $f(x+3\,e_0)+f(x+e_0)-2\,f(x+2\,e_0)$ is positive), the second inequality holds again as f is Conv(x_0) (hence $f(x)-f(x+e_0)\geq f(x+e_0)-f(x+2\,e_0)$), and the last inequality holds as f is Decr(x_0) and $\overline{\phi}_0(\cdot)$ is convex (which holds as $\phi_0(\cdot)$ is concave).

For $x_0+2=S_0$ we have:

$$\begin{aligned}
&\tilde{G}_0 f(x)+\tilde{G}_0 f(x+2\,e_0)-\tilde{G}_0 f(x+e_0)\\
&\quad=\phi_0(2)f(x+e_0)+\overline{\phi}_0(2)f(x)+\overline{\phi}_0(0)f(x+2\,e_0)-2\,\phi_0(1)f(x+2\,e_0)-2\,\overline{\phi}_0(1)f(x+e_0)\\
&\quad=\Big(\overline{\phi}_0-\overline{\phi}_0(2)\Big)f(x+e_0)+\overline{\phi}_0(2)f(x)+\overline{\phi}_0(0)f(x+2\,e_0)-2\Big(\overline{\phi}_0-\overline{\phi}_0(1)\Big)f(x+2\,e_0)-2\,\overline{\phi}_0(1)f(x+e_0)\\
&\quad=\overline{\phi}_0(2)\Big(f(x)-f(x+e_0)\Big)-2\,\overline{\phi}_0(1)\Big(f(x+e_0)-f(x+2\,e_0)\Big)+\overline{\phi}_0(0)\Big(f(x+e_0)-f(x+2\,e_0)\Big)\\
&\quad\geq\Big(\overline{\phi}_0(2)-2\,\overline{\phi}_0(1)+\overline{\phi}_0(0)\Big)\Big(f(x+e_0)-f(x+2\,e_0)\Big)\geq 0.
\end{aligned}$$

the first inequality holds as f is Conv(x_0) (hence $f(x)-f(x+e_0)\geq f(x+e_0)-f(x+2\,e_0)$) and the last inequality holds as f is Decr(x_0) and $\overline{\phi}_0(\cdot)$ is convex. Note that we have used that $\overline{\phi}_0(0)=\overline{\phi}_0-\phi_0(0)=\overline{\phi}_0$.

• $\tilde{G}_j$: Supermod(x_0,x_j) $\rightarrow$ Supermod(x_0,x_j), for$j=1,\dots,J$.
Assume that f is Conv(x_0) and Supermod(x_0,x_j) for $j\neq 0$, then we show that $\tilde{G}_k f$ is Supermod(x_0,x_j) as well. We consider $\tilde{G}_0$ (then $\tilde{G}_j$ follows by symmetry) and $\tilde{G}_k$ for $k\neq j$ separately. For $\tilde{G}_0$ with $x_0+1<S_0$ we have:

$$\begin{aligned}
&\tilde{G}_0 f(x)+\tilde{G}_0 f(x+e_0+e_j)-\tilde{G}_0 f(x+e_0)-\tilde{G}_0 f(x+e_j)\\
&=\phi_0(y_0)f(x+e_0)+(\overline{\phi}_0-\phi_0(y_0))f(x)+\phi_0(y_0-1)f(x+2\,e_0+e_j)+(\overline{\phi}_0-\phi_0(y_0-1))f(x+e_0+e_j)\\
&\quad-\phi_0(y_0-1)f(x+2\,e_0)-(\overline{\phi}_0-\phi_0(y_0-1))f(x+e_0)-\phi_0(y_0)f(x+e_0+e_j)-(\overline{\phi}_0-\phi_0(y_0))f(x+e_j)\\
&=\phi_0(y_0-1)\,(f(x+2\,e_0+e_j)+f(x+e_0)-f(x+2\,e_0)-f(x+e_0+e_j))\\
&\quad+(\overline{\phi}_0-\phi_0(y_0))\,(f(x)+f(x+e_0+e_j)-f(x+e_0)-f(x+e_j))\geq 0,
\end{aligned}$$

as f is Supermod(x_0,x_j), and for $x_0+1=S_0$ we have:

$$\begin{aligned}
\tilde{G}_0 f(x)+\tilde{G}_0 f(x+e_0+e_j)-\tilde{G}_0 f(x+e_0)-\tilde{G}_0 f(x+e_j)&=\phi_0(y_0)f(x+e_0)+(\overline{\phi}_0-\phi_0(y_0))f(x)\\
&\quad+\overline{\phi}_0 f(x+e_0+e_j)-\overline{\phi}_0 f(x+e_0)-\phi_0(y_0)f(x+e_0+e_j)-(\overline{\phi}_0-\phi_0(y_0))f(x+e_j)\\
&=(\overline{\phi}_0-\phi_0(y_0))\,(f(x)+f(x+e_0+e_j)-f(x+e_0)-f(x+e_j))\geq 0,
\end{aligned}$$

as f is Supermod(x_0,x_j). For $\tilde{G}_k$ $(k\neq 0,j)$ with $x_k<S_k$ we have:

$$\begin{aligned}
&\tilde{G}_k f(x)+\tilde{G}_k f(x+e_0+e_j)-\tilde{G}_k f(x+e_0)-\tilde{G}_k f(x+e_j)\\
&=\phi_k(y_k)f(x+e_k)+(\overline{\phi}_k-\phi_k(y_k))f(x)+\phi_k(y_k)f(x+e_0+e_j+e_k)+(\overline{\phi}_k-\phi_k(y_k))f(x+e_0+e_j)\\
&\quad-\phi_k(y_k)f(x+e_0+e_k)-(\overline{\phi}_k-\phi_k(y_k))f(x+e_0)-\phi_k(y_k)f(x+e_j+e_k)-(\overline{\phi}_k-\phi_k(y_k))f(x+e_j)\\
&=\phi_k(y_k)\Big(f(x+e_k)+f(x+e_0+e_j+e_k)-f(x+e_0+e_k)-f(x+e_j+e_k)\Big)\\
&\quad+(\overline{\phi}_k-\phi_k(y_k))\Big(f(x)+f(x+e_0+e_j)-f(x+e_0)-f(x+e_j)\Big)\ \geq 0,
\end{aligned}$$

as f is Supermod(x_0,x_j), and for $x_k=S_k$ we have:

$$\begin{aligned}
&\tilde{G}_k f(x)+\tilde{G}_k f(x+e_0+e_j)-\tilde{G}_k f(x+e_0)-\tilde{G}_k f(x+e_j)\\
&\quad=\overline{\phi}_k\Big(f(x)+f(x+e_0+e_j)-f(x+e_0)-f(x+e_j)\Big)\geq 0,
\end{aligned}$$

as f is Supermod(x_0,x_j). □