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in light and heavy-traffic**

K. Dębicki, K.M. Kosiński and M. Mandjes
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GAUSSIAN QUEUES
IN LIGHT AND HEAVY-TRAFFIC

K. DĘBICKI, K.M. KOSIŃSKI, AND M. MANDJES

Abstract. In this paper we investigate Gaussian queues in the light-traffic and in the heavy-traffic regime. The setting considered is that of a centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments and variance function $\sigma_X^2(\cdot)$, equipped with a deterministic drift $c > 0$, reflected at 0:

$$Q_X^{(c)}(t) = \sup_{-\infty < s \le t} (X(t) - X(s) - c(t - s)).$$

We study the resulting stationary workload process $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \ge 0\}$ in the limiting regimes $c \to 0$ (heavy traffic) and $c \to \infty$ (light traffic). The primary contribution is that we show for both limiting regimes that, under mild regularity conditions on the variance function, there exists a normalizing function $\delta(c)$ such that $Q_X^{(c)}(\delta(c)\cdot)/\sigma_X(\delta(c))$ converges to a non-trivial limit in $C[0, \infty)$.

1. Introduction

A substantial research effort has been devoted to the analysis of queues with Gaussian input, often also called *Gaussian queues* [9, 10, 11]. The interest in this model can be explained from the fact that the Gaussian input model is highly flexible in terms of incorporating a broad set of correlation structures, including those that exhibit *long-range dependence* (strong positive correlations on a broad range of time-scales). Furthermore, the Gaussian input model is an adequate approximation for various real-life systems. A key result in this area is [17], where it is shown that large aggregates of Internet sources converge to a fractional Brownian motion (being a specific Gaussian process).

The setting considered in this paper is that of a centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments and variance function $\sigma_X^2(\cdot)$, equipped with a deterministic drift $c > 0$, reflected at 0:

$$Q_X^{(c)}(t) = \sup_{-\infty < s \le t} (X(t) - X(s) - c(t - s)).$$

The resulting *stationary workload process* can be regarded as a *queue* [13]. The objective of the paper is to study $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \ge 0\}$ in the limiting regimes $c \downarrow 0$ (heavy traffic) and $c \to \infty$ (light traffic).

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Under mild conditions on the variance function $\sigma_X^2(\cdot)$, $Q_X^{(c)}$ is a properly defined, almost surely (a.s.) finite stochastic process. However, if $c \downarrow 0$, then $Q_X^{(c)}$ grows to infinity (in a distributional sense). The branch of queueing theory investigating *how fast* $Q_X^{(c)}$ grows to infinity (as $c \downarrow 0$) is commonly referred to as the domain of *heavy-traffic approximations*. In many situations this regime allows manageable expressions for performance metrics that are, under 'normal' load conditions, highly complex or even intractable, see for instance the seminal paper by Kingman [8] on the classical single-server queue. Since then, a similar approach has been followed in various other settings, see, e.g., [5, 12, 14, 16, 18] and many others.

Analogously, one can ask what happens in the *light-traffic* regime, i.e., $c \to \infty$; then evidently $Q_X^{(c)}$ decreases to zero. So far, hardly any attention has been paid to the light-traffic and heavy-traffic regimes for Gaussian queues. An exception is Dębicki and Mandjes [7], where the focus is on a special family of Gaussian processes, in a specific heavy-traffic setting. The primary contribution of the present paper concerns the analysis of $Q_X^{(c)}$ under both limiting regimes, for quite a broad class of Gaussian input processes X.

We now give a somewhat more detailed introduction to the material presented in this paper. To this end, first observe that if $X \equiv B_H$, where $B_H \equiv \{B_H(t) : t \in \mathbb{R}\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, then due to the self-similarity of B_H,

$$\left\{Q_{B_H}^{(c)}\left(c^{\frac{1}{H-1}}t\right) : t \geq 0\right\} \stackrel{d}{=} c^{\frac{H}{H-1}}\left\{Q_{B_H}^{(1)}(t) : t \geq 0\right\}.$$

The variance function of B_H satisfies $\sigma_{B_H}^2(t) = t^{2H}$, and this immediately implies the following statement.

Proposition 1. *For a fractional Brownian motion B_H with Hurst parameter $H \in (0, 1)$,*

$$\frac{Q_{B_H}^{(c)}\left(c^{\frac{1}{H-1}}\cdot\right)}{\sigma_{B_H}\left(c^{\frac{1}{H-1}}\right)} \xrightarrow{d} Q_{B_H}^{(1)}(\cdot)$$

in $C[0, \infty)$, both as $c \to 0$ and $c \to \infty$.

Our focus is on the space $C[0, \infty)$ of all continuous functions on $[0, \infty)$; it is the most natural in our setting, as we consider Gaussian processes X with almost surely continuous sample paths. In this paper we extend Proposition 1 to Gaussian processes X that are equipped with a variance function $\sigma_X^2(\cdot)$ that meets mild regularity assumptions.

For instance, it is well known that under the assumption that $\sigma_X(\cdot)$ varies regularly at infinity with parameter $\alpha \in (0, 1)$, for any function δ such that $\delta(c) \to \infty$ as $c \to 0$, there is convergence to fractional Brownian motion in the heavy-traffic regime:

$$(1) \qquad \frac{X(\delta(c)\cdot)}{\sigma_X(\delta(c))} \xrightarrow{d} B_\alpha(\cdot), \text{ as } c \to 0.$$

We shall show that an analogous statement holds in the light-traffic regime, that is, if $\sigma_X(\cdot)$ varies regularly at zero with parameter $\lambda \in (0, 1)$, then for any function δ such that

$\delta(c) \to 0$ as $c \to \infty$,

$$(2) \qquad \frac{X(\delta(c)\cdot)}{\sigma_X(\delta(c))} \stackrel{d}{\to} B_\lambda(\cdot), \text{ as } c \to \infty.$$

It will turn out that additional regularity conditions are required to make sure that both (1) and (2) apply in $C(\mathbb{R})$.

Crucially, our work shows that the statements (1) and (2), which relate to the input processes, carry over to the corresponding stationary buffer content processes $Q_X^{(c)}$. That is, that we identify a function $\delta(\cdot) \equiv \hat{\delta}(\cdot)$ such that

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma_X(\delta(c))} \stackrel{d}{\to} Q_{B_\alpha}^{(1)}(\cdot), \text{ as } c \to 0,$$

and a function $\delta(\cdot) \equiv \check{\delta}(\cdot)$ such that

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma_X(\delta(c))} \stackrel{d}{\to} Q_{B_\lambda}^{(1)}(\cdot), \text{ as } c \to \infty,$$

both in $C[0,\infty)$.

This paper is organized as follows. In Section 2 we introduce the notation and give some preliminaries. Subsection 3.1 presents the results for the heavy-traffic regime, whereas Subsection 3.2 covers the light-traffic regime. We give the proofs of the main theorems, (i.e., Theorem 1 and Theorem 2) in Section 4.

2. Preliminaries

In this paper we use the following notation. By id : $\mathbb{R} \to \mathbb{R}$ we shall denote the identity operator on $\mathbb{R}$, that is $\mathrm{id}(t) = t$ for every $t \in \mathbb{R}$. We write $f(x) \sim g(x)$ as $x \to x_0 \in [0,\infty]$ when $\lim_{x\to x_0} f(x)/g(x) = 1$. Let $\mathscr{RV}_\infty(\alpha)$ and $\mathscr{RV}_0(\lambda)$ denote the class of regularly varying functions at infinity with parameter α and at zero with parameter λ, respectively.

2.1. **Spaces of continuous functions.** For any $T > 0$, let $C[-T,T]$ be the space of all continuous functions f defined on $[-T,T]$ such that $f(0) = 0$. Equip $C[-T,T]$ with the topology of uniform convergence, i.e., the topology generated by the norm $\|f\|_{[-T,T]} := \sup_{t\in[-T,T]} |f(t)|$. Finally, let $C(\mathbb{R})$ be the space of all functions f such that $f_{|[-T,T]} \in C[-T,T]$ for all $T > 0$. Weak convergence in $C(\mathbb{R})$ is equivalent to weak convergence in $C[-T,T]$ for all $T > 0$. Convergence of a family of random elements $\{X^{(c)}\}$ in $C[-T,T]$ as $c \to c_0$ is implied by convergence of finite-dimensional distributions in conjunction with

$$(3) \qquad \lim_{\zeta\to 0} \limsup_{c\to c_0} \mathbb{P}\left(\sup_{\substack{|t-s|\le\zeta \\ s,t\in[-T,T]}} \left| X^{(c)}(t) - X^{(c)}(s) \right| \ge \eta \right) = 0,$$

for any $\eta > 0$, see [3, Thm. 7.5]. Note that these two conditions imply tightness of $\{X^{(c)}\}$ in $C[-T,T]$. For notational convenience, we leave out the requirement $s,t \in [-T,T]$ explicitly in the remainder. The above definitions extend in an obvious way to $C[0,T]$ and $C[0,\infty)$.

For $\gamma \geq 0$, let Ω^γ be the space of all continuous functions f such that $f(0) = 0$ and $\lim_{t\to\pm\infty} f(t)/(1 + |t|^\gamma) = 0$, and equip Ω^γ with the topology generated by the norm $\|f\|_{\Omega^\gamma} := \sup_{t\in\mathbb{R}} |f(t)|/(1 + |t|^\gamma)$ under which Ω^γ is a separable Banach space. Therefore, by Prokhorov's theorem [3, Section 5] weak convergence of random elements of Ω^γ is implied by convergence of finite-dimensional distributions and tightness.
The following property is known [6, Lemma 4].

Proposition 2. *Let a family of random elements $\{X^{(c)}\}$ on Ω^γ be given. Suppose that the image of $\{X^{(c)}\}$ under the projection mapping $p_T : \Omega^\gamma \to C[-T,T]$ is tight at c_0 in $C[-T,T]$ for all $T > 0$. Then $\{X^{(c)}\}$ is tight in Ω^γ if and only if for any $\eta > 0$,*

$$\lim_{T\to\infty} \limsup_{c\to c_0} \mathbb{P}\left(\sup_{|t|\geq T} \frac{|X^{(c)}(t)|}{1 + |t|^\gamma} \geq \eta\right) = 0. \tag{4}$$

2.2. **Fluid Queues.** Let $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ denote a stationary buffer content process for a fluid queue fed by a centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments, $X(0) = 0$ and continuous variance function $\sigma_X^2(\cdot) \equiv \sigma^2(\cdot)$. The system is drained with a constant rate $c > 0$, so that for any $t \geq 0$,

$$Q_X^{(c)}(t) = \sup_{-\infty<s\leq t} (X(t) - X(s) - c(t - s)).$$

Additionally, an equivalent representation for $Q_X^{(c)}(t)$ holds [15, p. 375]:

$$Q_X^{(c)}(t) = Q_X^{(c)}(0) + X(t) - ct + \max\left(0, \sup_{0<s<t}\left(-Q_X^{(c)}(0) - (X(s) - cs)\right)\right). \tag{5}$$

Throughout the paper we impose the following assumptions (or a subset of them):

C: $\lim_{t\to0} \sigma^2(t)|\log|t||^{1+\varepsilon} < \infty$, for some $\varepsilon > 0$;
RV1: $\sigma \in \mathscr{RV}_\infty(\alpha)$, for $\alpha \in (0,1)$;
RV2: $\sigma \in \mathscr{RV}_0(\lambda)$, for $\lambda \in (0,1)$.

We say that X satisfies **HT**, if it satisfies both **C** and **RV1**; it satisfies **LT** if it satisfies both **RV1** and **RV2**.

Remark 1. We make the following observations:

- **C** implies that X belongs to $C[0,\infty)$, see [1, Thm. 1.4].
- **RV1** implies that, when $t \to \pm\infty$, then $X(t)/t \to 0$ a.s., so that $Q_X^{(c)}$ is a properly defined stochastic process for any $c > 0$, see [6, Lemma 3].
- **RV2** implies **C**. Indeed, since $\sigma \in \mathscr{RV}_0(\lambda)$, then $t \mapsto \sigma(1/t)$ belongs to $\mathscr{RV}_\infty(-\lambda)$, thus $\sigma^2(1/t)t^\lambda \to 0$ as $t \to \infty$. Equivalently, $\sigma^2(t)t^{-\lambda} \to 0$ as $t \to 0$, implying the condition in **C**.

Due to the stationarity of increments, all finite-dimensional distributions of X are specified by the variance function, since we have

$$\mathbb{C}\mathrm{ov}(X(t), X(s)) = \frac{1}{2}\left(\sigma^2(s) + \sigma^2(t) - \sigma^2(|t - s|)\right). \tag{6}$$

Recall that by $B_H \equiv \{B_H(t) : t \in \mathbb{R}\}$ we denote fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a centered Gaussian process with stationary increments, continuous sample paths, $B_H(0) = 0$ and covariance function

$$\mathbb{C}\mathrm{ov}(B_H(t), B_H(s)) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right). \tag{7}$$

As mentioned in the introduction, if $c \to 0$, then $Q_X^{(c)} \to \infty$ a.s., which is called the *heavy-traffic regime.* On the other hand, if $c \to \infty$, then $Q_X^{(c)} \to 0$ a.s., which is called the *light-traffic regime.*

2.3. **Metric entropy.** For any $\mathbb{T} \subset \mathbb{R}$ define the *semimetric*

$$d(t,s) := \sqrt{\mathbb{E}|X(t) - X(s)|^2} = \sigma(|t-s|), \quad t, s \in \mathbb{T}.$$

We say that $S \subset \mathbb{T}$ is a ϑ-net in $\mathbb{T}$ with respect to the semimetric d, if for any $t \in \mathbb{T}$ there exists an $s \in S$ such that $d(t,s) \le \vartheta$. The metric entropy $\mathbb{H}_d(\mathbb{T}, \vartheta)$ is defined as $\log \mathbb{N}_d(\mathbb{T}, \vartheta)$, where $\mathbb{N}_d(\mathbb{T}, \vartheta)$ denotes the minimal number of points in a ϑ-net in $\mathbb{T}$ with respect to d. Later on we use the following proposition, see [2, Thm. 1.3.3] and [2, Corollary 1.3.4], respectively.

Proposition 3. *There exists a universal constant K such that for a d-compact set $\mathbb{T}$*

$$\mathbb{E}\left(\sup_{t\in\mathbb{T}} X(t)\right) \le K \int_0^{\mathrm{diam}(\mathbb{T})/2} \sqrt{\mathbb{H}_d(\mathbb{T}, \vartheta)}\, \mathrm{d}\vartheta.$$

and

$$\mathbb{E}\left(\sup_{\substack{(s,t)\in\mathbb{T}\times\mathbb{T}\\ d(s,t)<\zeta}} |X(t) - X(s)|\right) \le K \int_0^{\zeta} \sqrt{\mathbb{H}_d(\mathbb{T}, \vartheta)}\, \mathrm{d}\vartheta.$$

3. Main Results

In this section we list the result for the heavy-traffic and light-traffic regime, respectively. It is emphasized that these results are highly symmetric.

3.1. **Heavy-traffic Regime.** In the heavy-traffic regime we are interested in the analysis of $Q_X^{(c)}$ as $c \to 0$, under the assumption that X satisfies **HT**. The following statement follows from [6, Thms. 5 and 6].

Proposition 4. *If X satisfies* **HT**, *then for any function $\delta(c)$ such that $\delta(c) \to \infty$ as $c \to 0$,*

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot), \text{ as } c \to 0,$$

in $C(\mathbb{R})$ and Ω^γ for any $\gamma > \alpha$.

Now define the function $\hat{\delta}$ through $\hat{\delta}(c) := \inf\{x > 0 : x/\sigma(x) \geq 1/c\}$. The function $\hat{\delta}$ is chosen such that

$$\frac{c\hat{\delta}(c)}{\sigma(\hat{\delta}(c))} \sim 1, \text{ as } c \to 0. \tag{8}$$

From the definition of $\hat{\delta}$ it follows that $\hat{\delta} \in \mathscr{RV}_0(1/(\alpha-1))$. Combining the above with Proposition 4 leads to the following statement.

Corollary 1. *If X satisfies* **HT***, then*

$$\frac{X(\hat{\delta}(c)\cdot) - c\hat{\delta}(c)\,\mathrm{id}(\cdot)}{\sigma(\hat{\delta}(c))} \xrightarrow{d} B_\alpha(\cdot) - \mathrm{id}(\cdot) \text{ as } c \to 0,$$

in $C(\mathbb{R})$.

Now we are in the position to present the main result of this subsection.

Theorem 1. *If X satisfies* **HT***, then*

$$\frac{Q_X^{(c)}(\hat{\delta}(c)\cdot)}{\sigma(\hat{\delta}(c))} \xrightarrow{d} Q_{B_\alpha}^{(1)}(\cdot) \text{ as } c \to 0, \tag{9}$$

in $C[0,\infty)$.

We postpone the proof of Theorem 1 to Section 4.

Remark 2. Theorem 1 extends the findings of [7, Theorem 3.2] where, under the heavy-traffic regime, the weak convergence in $C[0,\infty)$ of $Q_X^{(c)}(\hat{\delta}(c)\cdot)/\sigma(\hat{\delta}(c))$ as $c \to 0$ was obtained for the class of input processes having differentiable sample paths a.s., i.e., of the form $X(t) = \int_0^t Z(s)\mathrm{d}s$, where $\{Z(s) : s \geq 0\}$ is a stationary centered Gaussian process whose variance function satisfies specific regularity conditions.

3.2. **Light-traffic Regime.** In the light-traffic regime we analyze the convergence of $Q_X^{(c)}$ as $c \to \infty$, under the assumption that X satisfies **LT**. We begin by stating the counterpart of Proposition 4.

Proposition 5. *If X satisfies* **RV2***, then for any function $\delta(c)$ such that $\delta(c) \to 0$ as $c \to \infty$,*

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot), \text{ as } c \to \infty,$$

in $C(\mathbb{R})$. If, moreover, X satisfies **LT***, then the convergence also holds in Ω^γ, for any $\gamma > \max\{\lambda, \alpha\}$.*

Now define the function $\check{\delta}$ through $\check{\delta}(c) := \sup\{x > 0 : x/\sigma(x) \geq 1/c\}$. The function $\check{\delta}$ is chosen such that

$$\frac{c\check{\delta}(c)}{\sigma(\check{\delta}(c))} \sim 1, \text{ as } c \to \infty. \tag{10}$$

From the definition of $\check{\delta}$ it follows that $\check{\delta} \in \mathscr{RV}_\infty(1/(\lambda-1))$. Like in the heavy-traffic case, combining the above with Proposition 5 leads to the counterpart of Corollary 1.

Corollary 2. *If X satisfies* **RV2***, then*

$$\frac{X(\check{\delta}(c)\cdot) - c\check{\delta}(c)\,\mathrm{id}(\cdot)}{\sigma(\check{\delta}(c))} \xrightarrow{d} B_\lambda(\cdot) - \mathrm{id}(\cdot) \text{ as } c \to \infty,$$

in $C(\mathbb{R})$.

The main result of this subsection is now stated as follows.

Theorem 2. *If X satisfies* **LT***, then*

$$\frac{Q_X^{(c)}(\check{\delta}(c)\cdot)}{\sigma(\check{\delta}(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(\cdot) \text{ as } c \to \infty, \tag{11}$$

in $C[0, \infty)$.

We postpone the proof of Proposition 5 and Theorem 2 to Section 4.

Remark 3. The assumption **LT** excludes the class of input processes of the structure $X(t) = \int_0^t Z(s)\mathrm{d}s$, with $\{Z(s) : s \geq 0\}$ being a centered stationary Gaussian process with continuous sample paths a.s. (since $\lambda = 1$ in this case). In [7, Theorem 4.1] it was shown that, for this class of Gaussian processes, $Q_X^{(c)}(0)/\sigma(\check{\delta}(c))$ does *not* converge weakly to $Q_{B_\lambda}^{(1)}(0)$ as $c \to \infty$.

Remark 4. Note that we are interested in values of $\hat{\delta}$ for small c and in values of $\check{\delta}$ for large c. Therefore we can put $\delta(c) = \hat{\delta}(c)$ for $c \leq c^*$ and $\delta(c) = \check{\delta}(c)$ for $c > c^*$, for any $c^* > 0$ and claim that under **RV1** and **RV2** (which we named **LT**) $Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))$ converges weakly both as $c \to 0$ and $c \to \infty$ to $Q_{B_\alpha}^{(1)}(\cdot)$ and $Q_{B_\lambda}^{(1)}(\cdot)$, respectively. In the sequel we will no longer use the notation $\hat{\delta}$ and $\check{\delta}$, and write δ instead.

4. Proofs

In this section we prove our results, but we start by presenting an auxiliary result.

Lemma 1. *If X satisfies* **LT***, then for any $\epsilon > 0$, there exist constants $C, a > 0$, such that for all $x \leq a$ and $t > 0$,*

$$\frac{\sigma(tx)}{\sigma(x)} \leq C \times \begin{cases} t^\ell & t \leq 1, \\ t^u & t > 1, \end{cases}$$

where $\ell := \min\{\lambda - \epsilon, \alpha + \epsilon\}$ and $u := \max\{\alpha + \epsilon, \lambda + \epsilon\}$.

Proof Take any $\epsilon > 0$, then because $\sigma \in \mathscr{RV}_0(\lambda)$, then there exists an $a \leq 1$ such that

$$\frac{\sigma(tx)}{\sigma(x)} \leq 2t^{\lambda-\epsilon}, \text{ for all } x \leq a \text{ and } tx \leq a. \tag{12}$$

Moreover, there exists a constant K_1 such that $\sigma(x) \geq K_1 x^{\lambda+\epsilon}$ for all $x \leq a$.
Because $\sigma \in \mathscr{RV}_\infty(\alpha)$, there exist constants $A, K_2 > 0$ such that $\sigma(x) \leq K_2 t^{\alpha+\epsilon}$ for all $x \geq A$. Because σ is continuous, we can in fact find a K_2 such that $\sigma(x) \leq K_2 t^{\alpha+\epsilon}$ for all $x \geq a$. Therefore

$$\frac{\sigma(tx)}{\sigma(x)} \leq \frac{K_2 (tx)^{\alpha+\epsilon}}{K_1 x^{\lambda+\epsilon}} =: K t^{\alpha+\epsilon} x^{\alpha-\lambda}, \text{ for all } x \leq a \text{ and } tx \geq a.$$

Note that, if $\alpha - \lambda \geq 0$, then we have

$$\frac{\sigma(tx)}{\sigma(x)} \leq K t^{\alpha+\epsilon}, \text{ for all } x \leq a \text{ and } tx \geq a. \tag{13}$$

If $\alpha - \lambda < 0$, then

$$\frac{\sigma(tx)}{\sigma(x)} \leq K a^{\alpha-\lambda} t^{\lambda+\epsilon}, \text{ for all } x \leq a \text{ and } tx \geq a. \tag{14}$$

Combining (12)-(14), we conclude that there exists a constant $C > 0$, such that

$$\frac{\sigma(tx)}{\sigma(x)} \leq C \max\left\{t^{\lambda-\epsilon}, t^{\alpha+\epsilon}, t^{\lambda+\epsilon}\right\}, \text{ for all } x \leq a \text{ and all } t > 0.$$

□

In what follows, we will use the following notation. Let

$$X^{(c)}(t) := \frac{X(\delta(c)t)}{\sigma(\delta(c))}$$

and denote the variance of $X^{(c)}$ by $(\sigma^{(c)})^2$, that is,

$$\sigma^{(c)}(t) := \frac{\sigma(\delta(c)t)}{\sigma(\delta(c))}.$$

Proof of Proposition 5 We begin by showing the convergence in $C(\mathbb{R})$. To this end, we need to show the convergence in $C[-T,T]$ for any fixed $T > 0$.

Convergence in $C[-T,T]$: From the fact that $\sigma \in \mathscr{RV}_0(\lambda)$, it is immediate that the finite-dimensional distributions of $X^{(c)}$ converge in distribution to B_λ as $c \to \infty$, cf. (6)-(7). Therefore, the weak convergence of $X^{(c)}$ in $C[-T,T]$ would be implied by (3); we therefore now establish (3).
By the UCT, see [4, Thm. 1.5.2], for any $t \in (0,\zeta]$, we have $\sigma^{(c)}(t) \leq 2\zeta^\lambda$. Thus, Proposition 3 yields that for some universal constant $K > 0$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{|s-t|\leq\zeta} \left|X^{(c)}(t) - X^{(c)}(s)\right| \geq \eta\right) &\leq \mathbb{P}\left(\sup_{\sigma^{(c)}(|s-t|)\leq 2\zeta^\lambda} \left|X^{(c)}(t) - X^{(c)}(s)\right| \geq \eta\right) \\
&\leq \frac{1}{\eta}\,\mathbb{E}\left(\sup_{\sigma^{(c)}(|s-t|)\leq 2\zeta^\lambda} \left|X^{(c)}(t) - X^{(c)}(s)\right|\right) \\
&\leq \frac{K}{\eta}\int_0^{2\zeta^\lambda} \sqrt{\mathbb{H}^{(c)}([-T,T],\vartheta)}\,\mathrm{d}\vartheta,
\end{aligned}$$

where $\mathbb{H}^{(c)}([-T,T],\cdot)$ is the metric entropy induced by $\sigma^{(c)}$.
By Potter's bound [4, Thm. 1.5.6] for any $\epsilon, \zeta > 0$, $\epsilon < \lambda$ and $t \in (0,\zeta]$ and sufficiently large c (corresponding to small $\delta(c)$), we have $\sigma^{(c)}(t) \leq 2t^{\lambda-\epsilon}$. Hence

$$\mathbb{H}^{(c)}([-T,T],\vartheta) \leq \mathbb{H}_{\tilde{d}}\left([-T,T], \frac{\vartheta}{2}\right),$$

where $\tilde{d}$ is a semimetric such that $\tilde{d}(s,t) = |t-s|^{\lambda-\epsilon}$. The inverse of $x \mapsto x^{\lambda-\epsilon}$ is given by $x \mapsto x^{1/(\lambda-\epsilon)}$, so that

$$\mathbb{H}_{\tilde{d}}([-T,T],\vartheta) \leq \log\left(\frac{T}{\vartheta^{1/(\lambda-\epsilon)}}+1\right) \leq C\log\left(\frac{1}{\vartheta}\right),$$

for some constant $C>0$ and $\vartheta>0$ small. It follows that

$$\begin{aligned}\int_0^{2\zeta^\lambda}\sqrt{\mathbb{H}^{(c)}([-T,T],\vartheta)}\,\mathrm{d}\vartheta \;&\leq\; \mathbb{H}_{\tilde{d}}\left([-T,T],\frac{\vartheta}{2}\right)\\ &\leq\; \sqrt{C}\int_0^{2\zeta^\lambda}\sqrt{\log\left(\frac{2}{\vartheta}\right)}\,\mathrm{d}\vartheta = 2\sqrt{C}\int_{\zeta^{-\lambda}}^\infty \frac{\sqrt{\log\vartheta}}{\vartheta^2}\,\mathrm{d}\vartheta.\end{aligned}$$

Summarizing, we have

$$\limsup_{c\to\infty}\mathbb{P}\left(\sup_{|s-t|\leq\zeta}\left|X^{(c)}(t)-X^{(c)}(s)\right|\geq\eta\right) \leq \frac{2K\sqrt{C}}{\eta}\int_{\zeta^{-\lambda}}^\infty\frac{\sqrt{\log\vartheta}}{\vartheta^2}\,\mathrm{d}\vartheta;$$

we obtain (3) by letting $\zeta\to 0$.

Convergence in Ω^γ: To show the convergence in Ω^γ, we need to verify (4). Observe that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\geq e^k}\frac{|X^{(c)}(t)|}{1+t^\gamma}\geq\eta\right) &\leq \frac{1}{\eta}\sum_{j=k}^\infty\frac{\mathbb{E}\sup_{t\in[e^j,e^{j+1}]}|X^{(c)}(t)|}{1+e^{j\gamma}}\\ &\leq \frac{1}{\eta}\sum_{j=k}^\infty\frac{\mathbb{E}|X^{(c)}(e^j)|}{1+e^{j\gamma}} + \frac{2}{\eta}\sum_{j=k}^\infty\frac{\mathbb{E}\sup_{t\in[e^j,e^{j+1}]}X^{(c)}(t)}{1+e^{j\gamma}}\\ &=: I_1(k)+I_2(k).\end{aligned}$$

$I_1(k)$ and $I_2(k)$ are dealt with separately. According to Lemma 1, for large c (that is, small $\delta(c)$), we have

$$\sigma^{(c)}(t) \leq C\times\begin{cases} t^\ell & t\leq 1,\\ t^u & t>1,\end{cases}$$

where ℓ and u can be chosen such that $\ell,u<\gamma$. Therefore,

$$I_1(k)\leq\frac{1}{\eta}\sum_{j=k}^\infty\frac{\sigma^{(c)}(e^j)}{1+e^{j\gamma}}\leq\frac{C}{\eta}\sum_{j=k}^\infty\frac{e^{ju}}{1+e^{j\gamma}},$$

and the resulting upper bound tends to zero as $k\to\infty$.
Now focus on $I_2(k)$. For some universal constant $K>0$ and because of the stationarity of the increments of X, Proposition 3 yields that $I_2(k)$ is majorized by

$$\frac{2K}{\eta}\sum_{j=k}^\infty\frac{\int_0^\infty\sqrt{\mathbb{H}^{(c)}([e^j,e^{j+1}],\vartheta)}\,\mathrm{d}\vartheta}{1+e^{j\gamma}} = \frac{2K}{\eta}\sum_{j=k}^\infty\frac{\int_0^\infty\sqrt{\mathbb{H}^{(c)}([0,e^j(e-1)],\vartheta)}\,\mathrm{d}\vartheta}{1+e^{j\gamma}}.$$

We observe that, for some constants $C_1, C_2 > 0$ (that is, not depending on j),

$$\begin{aligned}\int_0^1 \sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)}\, \mathrm{d}\vartheta &\le \int_0^1 \sqrt{\log\left(\frac{e^j(e-1)}{2\vartheta^{1/\ell}} + 1\right)}\, \mathrm{d}\vartheta \\ &\le \int_0^1 \sqrt{C_1 + j + \frac{1}{\ell}\log\left(\frac{1}{\vartheta}\right)}\, \mathrm{d}\vartheta \\ &= \ell e^{\ell(C_1+j)} \int_{C_1+j}^{\infty} \sqrt{\vartheta} e^{-\ell\vartheta}\, \mathrm{d}\vartheta \\ &\le \ell e^{\ell(C_1+j)} \int_0^{\infty} \sqrt{\vartheta} e^{-\ell\vartheta}\, \mathrm{d}\vartheta = C_2 e^{\ell j}.\end{aligned}$$

Recall that $\ell < \gamma$, so that

$$\lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{\int_0^1 \sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)}\, \mathrm{d}\vartheta}{1 + e^{j\gamma}} \le \frac{2K}{\eta} \lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{C_2 e^{\ell j}}{1 + e^{j\gamma}} = 0.$$

So it remains to show the analogous statement for the integration interval $[1, \infty)$. Using a similar argumentation as the one above, one can show that

$$\int_1^{\infty} \sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)}\, \mathrm{d}\vartheta \le C_3 e^{uj},$$

for some constant $C_3 > 0$, from which the claim is readily obtained. □

Since the proof of Theorem 1 is analogous to the proof of Theorem 2, we choose to focus on the heavy-traffic case only.

Proof of Theorem 1 The proof consists of the usual three steps: convergence of the one-dimensional distributions, the finite-dimensional distributions, and a tightness argument.

Step 1: Convergence of one-dimensional distributions. In this step we show that, for a fixed $t \ge 0$, that

$$\frac{Q_X^{(c)}(t)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\alpha}^{(1)}(t).$$

Since $Q_X^{(c)}$ is stationary, it is enough to show the above convergence for $t = 0$ only. Observe that, due to the time-reversibility property of Gaussian processes,

$$Q_X^{(c)}(0) \stackrel{d}{=} \sup_{t\ge 0} \left(X(t) - ct\right) = \sup_{t\ge 0} \left(X(\delta(c)t) - c\delta(c)t\right).$$

Upon combining Corollary 1 with the continuous mapping theorem, for each $T > 0$,

$$\frac{\sup_{t\in[0,T]} (X(\delta(c)t) - c\delta(c)t)}{\sigma(\delta(c))} \xrightarrow{d} \sup_{t\in[0,T]} (B_\alpha(t) - t),$$

as $c \to 0$. Thus is suffices to show that

$$\lim_{T\to\infty} \limsup_{c\to 0} \mathbb{P}\left(\sup_{t\ge T} \left(\frac{X(\delta(c)t) - c\delta(c)t}{\sigma(\delta(c))}\right) \ge \eta\right) = 0, \tag{15}$$

for any $\eta > 0$. Recall the definition of $X^{(c)}$ and observe that for sufficiently small c,

$$\mathbb{P}\left(\sup_{t\geq T}\left(\frac{X(\delta(c)t)-c\delta(c)t}{\sigma(\delta(c))}\right)\geq\eta\right)\leq\mathbb{P}\left(\sup_{t\geq T}\frac{|X^{(c)}(t)|}{\eta+2t}\geq 1\right),$$

where we used (8). Proposition 4 implies that the family $\{X^{(c)}\}$ is tight in Ω^γ, for some $\gamma \leq 1$. Now (15) follows from Proposition 2.

Step 2: Convergence of finite-dimensional distributions. The argumentation of this step is analogous to Step 1. First note that for any $t_i \geq 0$, $\eta_i > 0$ and $s_i < t_i$, where $i = 1, \ldots, n$, for any $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\frac{Q_X^{(c)}(\delta(c)t_i)}{\sigma(\delta(c))}>\eta_i,\, i=1,\ldots,n\right)\\
&\quad=\mathbb{P}\left(\sup_{s\leq\delta(c)t_i}\left(\frac{X(\delta(c)t_i)-X(s)-c(\delta(c)t_i-s)}{\sigma(\delta(c))}\right)>\eta_i,\, i=1,\ldots,n\right)\\
&\quad\leq\mathbb{P}\left(\sup_{s\in[s_i,t_i]}\left(\frac{X(\delta(c)t_i)-X(\delta(c)s)-c\delta(c)(t_i-s)}{\sigma(\delta(c))}\right)>\eta_i, i=1,\ldots,n\right)\\
&\qquad+\sum_{i=1}^{n}\mathbb{P}\left(\sup_{s\leq s_i}\left(\frac{X(\delta(c)t_i)-X(\delta(c)s)-c\delta(c)(t_i-s)}{\sigma(\delta(c))}\right)>\eta_i\right).
\end{aligned}$$

Now the same procedure can be followed as in Step 1.

Step 3: Tightness in $C[0,T]$. In this step, for any $T > 0$, we show the tightness of $\{Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))\}$ in $C[0,T]$. Given that we have established Step 2 already, it is readily seen that we are left with proving (3), with $c \to 0$ and $s, t \in [0,T]$; the remainder of the proof is devoted to settling this claim.

Stationarity of $Q_X^{(c)}$ implies that $\{Q_X^{(c)}(\delta(c)t) - Q_X^{(c)}(\delta(c)s) : t \geq s\}$ is distributed as

$$\{Q_X^{(c)}(\delta(c)(t-s)) - Q_X^{(c)}(0) : t \geq s\},$$

so that it suffices to prove (3) for $s = 0$ only. Furthermore, cf. (5),

$$\sup_{0<t<\zeta}\left|Q_X^{(c)}(\delta(c)t)-Q_X^{(c)}(0)\right|\leq 2\sup_{0<t<\zeta}\left|X^{(c)}(\delta(c)t)-c\delta(c)t\right|.$$

From Corollary 1 it follows that

$$\sup_{0<t<\zeta}\frac{\left|X^{(c)}(\delta(c)t)-c\delta(c)t\right|}{\sigma(\delta(c))}\xrightarrow{d}\sup_{0<t<\zeta}|B_\alpha(t)-t|.$$

Now notice that

$$\mathbb{P}\left(\sup_{0<t\leq\zeta}|B_\alpha(t)-t|\geq\frac{\eta}{2}\right)\leq 2\mathbb{P}\left(\sup_{0<t\leq\zeta}(B_\alpha(t)-t)\geq\frac{\eta}{2}\right),$$

which for $\zeta < \eta/4$ can be bounded from above by

$$2\mathbb{P}\left(\sup_{0<t\leq\zeta}B_\alpha(t)\geq\frac{\eta}{4}\right)=2\mathbb{P}\left(\sup_{0<t\leq 1}B_\alpha(t)\geq\frac{\eta}{4}\zeta^{-\alpha}\right).$$

Now it is straightforward to conclude that the last expression tends to zero as $\zeta \to 0$. $\square$

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Instytut Matematyczny, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.
E-mail address: `Krzysztof.Debicki@math.uni.wroc.pl`

Korteweg-de Vries Institute for Mathematics, University of Amsterdam, the Netherlands; Eurandom, Eindhoven University of Technology
E-mail address: `K.M.Kosinski@uva.nl`

Korteweg-de Vries Institute for Mathematics, University of Amsterdam, the Netherlands; Eurandom, Eindhoven University of Technology, the Netherlands; CWI, Amsterdam, the Netherlands
E-mail address: `M.R.H.Mandjes@uva.nl`