The uniform measure on a Galton-Watson tree without the XlogX condition

Elie Aïdékon
ISSN 1389-2355
The uniform measure on a Galton–Watson tree without the XlogX condition

Elie Aïdékon

Summary. We consider a Galton–Watson tree with offspring distribution $\nu$ of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the $n$-th generation and taking the limit $n \to \infty$. In the case $E[\nu \ln(\nu)] < \infty$, this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to $\ln(m)$ ([2], [9]). When $E[\nu \ln(\nu)] = \infty$, we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres [10].

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d’enfants $\nu$ a une moyenne finie. La mesure uniforme sur la frontière de l’arbre s’obtient en chargeant chaque sommet de la $n$-ième génération avec une masse 1, puis en prenant la limite $n \to \infty$. Dans le cas $E[\nu \ln(\nu)] < \infty$, cette mesure a été très étudiée, et l’on sait que la dimension de Hausdorff de la mesure est égale à $\ln(m)$ ([2], [9]). Lorsque $E[\nu \ln(\nu)] = \infty$, nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres [10].

Keywords: Galton–Watson tree, Hausdorff dimension.

AMS subject classifications: 60J80, 28A78.

1 Introduction

Let $T$ be a Galton–Watson tree of root $e$, associated to the offspring distribution $q := (q_k, k \geq 0)$. We denote by GW the distribution of $T$ on the space of rooted trees, and $\nu$ a generic random variable on $\mathbb{N}$ with distribution $q$. We suppose that $q_0 = 0$ and $m := \sum_{k \geq 0} kq_k \in (1, \infty)$:

1Department of Mathematics and Computer science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. email: elie.aidekon@gmail.com
the tree has no leaf (hence survives forever) and is not degenerate. For any vertex \( u \), we write \(|u|\) for the height of vertex \( u \) (\(|e| = 0\)), \( \nu(u) \) for the number of children of \( u \), and \( Z_n \) is the population at height \( n \). We define \( S(\mathcal{T}) \) as the set of all infinite self-avoiding paths of \( \mathcal{T} \) starting from the root and we define a metric on \( S(\mathcal{T}) \) by \( d(r, r') := e^{-|r \wedge r'|} \) where \( r \wedge r' \) is the highest vertex belonging to \( r \) and \( r' \). The space \( S(\mathcal{T}) \) is called boundary of the tree, and elements of \( S(\mathcal{T}) \) are called rays.

When \( E[\nu \ln(\nu)] < \infty \), it is well-known that the martingale \( m^{-n}Z_n \) converges in \( L^1 \) and almost surely to a positive limit ([4]). Seneta [13] and Heyde [3] proved that in the general case (i.e allowing \( E[\nu \ln(\nu)] \) to be infinite), there exist constants \( (c_n)_{n \geq 0} \) such that

(a) \( W_\infty := \lim_{n \to \infty} \frac{Z_n}{c_n} \) exists a.s.
(b) \( W_\infty > 0 \) a.s.
(c) \( \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = m \).

In particular, for each vertex \( u \in \mathcal{T} \), if \( Z_k(u) \) stands for the number of descendants \( v \) of \( u \) such that \(|v| = |u| + k\), we can define

\[ W_\infty(u) := \lim_{k \to \infty} \frac{Z_k(u)}{c_k} \]

and we notice that \( m^{-n} \sum_{|u| = n} W_\infty(u) = W_\infty(e) \).

**Definition.** The uniform measure (also called branching measure) is the unique Borel measure on \( S(\mathcal{T}) \) such that

\[ \text{UNIF}(\{r \in S(\mathcal{T}), r_n = u\}) := \frac{m^{-n}W_\infty(u)}{W_\infty(e)} \]

for any integer \( n \) and any vertex \( u \) of height \( n \).

We observe that, for any vertex \( u \) of height \( n \),

\[ \text{UNIF}(\{r \in S(\mathcal{T}), r_n = u\}) = \lim_{k \to \infty} \frac{Z_k(u)}{Z_{n+k}}. \]

Therefore the uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of \( \text{UNIF} \), defined by

\[ \dim(\text{UNIF}) := \min\{\dim(E), \text{UNIF}(E) = 1\} \]
where the minimum is taken over all subsets $E \subset S(T)$ and $\dim(E)$ is the Hausdorff dimension of set $E$. The case $E[\nu \ln(\nu)] < \infty$ has been well studied. In [2] and [9], it is shown that $\dim(\text{UNIF}) = \ln(m)$ almost surely. A description of the multifractal spectrum is available in [5],[9],[12],[14]. The case $E[\nu \ln(\nu)] = \infty$ presented as Question 3.1 in [10] was left open. This case is proved to display an extreme behaviour.

**Theorem 1.1.** If $E[\nu \ln(\nu)] = \infty$, then $\dim(\text{UNIF}) = 0$ for GW-a.e tree $T$.

The drop in the dimension comes from bursts of offspring at some places of the tree $T$. Namely, for UNIF-a.e. ray $r$, the number of children of $r_n$ will be greater than $(m - o(1))^n$ for infinitely many $n$. To prove it, we work with a particular measure $Q$, under which the distribution of the numbers of children of a uniformly chosen ray is more tractable. Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3.

## 2 A spine decomposition

For $k \geq 1$ and $s \in (0,1)$, we call $\phi_k(s)$ the probability generating function of $Z_k$

$$\phi_k(s) := E[s^{Z_k}].$$

We denote by $\phi_k^{-1}(s)$ the inverse map on $(0,1)$ and we let $s \in (0,1)$. Then $M_n := \phi_n^{-1}(s)^Z_n$ defines a martingale and converges in $L^1$ to some $M_\infty > 0$ a.s ([3]). Therefore we can take in (a)

$$c_n := -\frac{1}{\ln(\phi_n^{-1}(s))}$$

which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(e)}$. For any vertex $u$ at generation $n$, we define similarly

$$M_\infty(u) := \frac{1}{\phi_n^{-1}(s)}e^{-m^{-n}W_\infty(u)} = e^{1/c_n}e^{-m^{-n}W_\infty(u)}$$

which is the limit of the martingale $M_k(u) := e^{1/c_n}e^{-Z_k(u)/c_n+k}$. In [6], Lynch introduces the so-called derivative martingale

$$\partial M_n := e^{1/c_n}Z_n \frac{\phi_n'}{\phi_n(\phi_n^{-1}(s))}M_n$$
and shows that the derivative martingale also converges almost surely and in $L^1$ ($\partial M_n$ is in fact bounded). Moreover the limit $\partial M_\infty$ is positive almost surely. We deduce that the ratio $\frac{\phi'_n(\phi_n^{-1}(s))}{c_n}$ converges to some positive constant. In particular, it follows from (c) that

$$\lim_{n \to \infty} \frac{\phi'_{n+1}(\phi_{n+1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} = m. \tag{2.1}$$

We are interested in the probability measure $Q$ on the space of rooted trees defined by

$$\frac{dQ}{dGW} := \partial M_\infty.$$

Let us describe this change of measure. We call a marked tree a couple $(T, r)$ where $T$ is a rooted tree and $r$ a ray of the tree $T$. Let $(\mathbb{T}, \xi)$ be a random variable in the space of all marked trees (equipped with some probability $P(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level $k$ and on the location of the ray at level $k$, (which we denote respectively by $T_k$ and $\xi_k$),

- the number of children of the vertices at generation $k$ are independent
- the vertex $\xi_k$ has a number $\nu(\xi_k)$ of children such that for any $\ell$

$$P(\nu(\xi_k) = \ell) = \tilde{q}_\ell := \frac{q_\ell \ell}{1/c_k+1} e^{-\ell/c_k+1} \frac{\phi'_k(\phi_k^{-1}(s))}{\phi'_k(\phi_k^{-1}(s))} \tag{2.2}$$

- the number of children of a vertex $u \neq \xi_k$ at generation $k$ verifies for any $\ell$

$$P(\nu(u) = \ell) = \tilde{q}_\ell := q_\ell e^{1/c_k} \exp \left(-\frac{\ell}{c_k+1}\right) \tag{2.3}$$

- the vertex $\xi_{k+1}$ is chosen uniformly among the children of $\xi_k$

As often in the literature, we will call the ray $\xi$ the spine. We refer to [8], [7] for motivation on spine decompositions. In our case, we can see $\mathbb{T}$ as a Galton-Watson tree in varying environment and with immigration. The fact that (2.2) and (2.3) define probabilities come from the equations (remember that by definition $e^{-1/c_k} = \phi_k^{-1}(s)$)

$$E \left[ (\phi_k^{-1}(s))^{\nu} \right] = \phi_k^{-1}(s),$$

$$E \left[ \nu(\phi_k^{-1}(s))^{\nu-1} \right] = \frac{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}{\phi'_k(\phi_k^{-1}(s))}.$$
We mention that in [8], a similar decomposition was presented using the martingale \( Z_n \). In this case, the offspring distribution of the spine is the size-biased distribution \((\frac{m}{n})_{\ell \geq 0}\) whereas the other particles generate offspring according to the original distribution \( q \). In particular, the offspring distributions do not depend on the generation. When \( E[\nu \ln(\nu)] < \infty \), the process, which is a Galton–Watson process with immigration, has a distribution equivalent to GW. It is no longer true when \( E[\nu \ln(\nu)] = \infty \), in which case the spine can give birth to a super-exponential number of children.

**Proposition 2.1.** Under \( Q \), the tree \( T \) has the distribution of \( T \). Besides, for \( P \)-almost every tree \( T \), the distribution of \( \xi \) conditionally on \( T \) is the uniform measure \( \text{UNIF} \).

**Proof.** For any tree \( T \), we define \( T_n \) the tree \( T \) obtained by keeping only the \( n \)-first generations. Let \( T \) be a tree. We will prove by induction that, for any integer \( n \) and any vertex \( u \) at generation \( n \),

\[
P(T_n = T_n, \xi_n = u) = \frac{\partial M_n}{Z_n} GW(T_n = T_n).
\]

For \( n = 0 \), it is straightforward since \( T_0 \) and \( T_0 \) are reduced to the root. We suppose that this is true for \( n - 1 \), and we prove it for \( n \). Let \( \overset{\leftarrow}{u} \) denote the parent of \( u \), and, for any vertex \( v \) at height \( n - 1 \), let \( k(v) \) denote the number of children of \( v \) in the tree \( T \). We have

\[
P(T_n = T_n, \xi_n = u | T_{n-1} = T_{n-1}, \xi_{n-1} = \overset{\leftarrow}{u}) = \frac{1}{k(\overset{\leftarrow}{u})} \prod_{v|v=n-1} \tilde{q}_k(v)
\]

\[
= \frac{e^{1/c_n} \phi'_n(\phi^{-1}_n(s))}{e^{1/c_n} \phi'_n(\phi^{-1}_n(s))} \frac{e^{Z_{n-1}/c_n}}{e^{Z_n/c_n}} \prod_{|v|=n-1} q_k(v)
\]

\[
= \frac{e^{1/c_n} \phi'_n(\phi^{-1}_n(s))}{e^{1/c_n} \phi'_n(\phi^{-1}_n(s))} \frac{e^{Z_{n-1}/c_n}}{e^{Z_n/c_n}} GW(T_n = T_n | T_{n-1} = T_{n-1}).
\]

We use the induction assumption to get

\[
P(T_n = T_n, \xi_n = u) = e^{1/c_n} \frac{1}{\phi'_n(\phi^{-1}_n(s))} e^{-\frac{Z_n}{c_n}} GW(T_n = T_n)
\]

which proves (2.4). Summing over the \( n \)-th generation of \( T \) gives

\[
P(T_n = T_n) = \partial M_n GW(T_n = T_n) = Q(T = T_n).
\]
This computation also shows that $P(\xi_n = u | T_n) = 1/Z_n$ which implies that $\xi$ is uniformly distributed on the boundary $S(T)$. □

**Remark A.** For $u$ a vertex of $T$ at generation $n$, call $T(u)$ the subtree rooted at $u$. A similar computation shows that if $u \notin \xi$, then the distribution $P_u$ of $T(u)$ (conditionally on $T_n$ and on $\xi_n$) verifies

$$
\frac{dP_u}{dGW} = M_\infty(u).
$$

### 3 Proof of Theorem 1.1

The following proposition shows that in the tree $T$, there exist infinitely many times when the ball $\{r \in S(T) : r_n = \xi_n\}$ has a 'big' weight.

**Proposition 3.1.** Suppose that $E[\nu \ln(\nu)] = \infty$. Then we have P-a.s.

$$
\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m).
$$

**Proof.** Let $1 < a < b < m$ and $n \geq 0$. We get from (2.2)

$$
P(\nu(\xi_n) \in (a^n, b^n)) = \frac{\phi'_n(\phi_n^{-1}(s))}{\phi'_{n+1}(\phi_{n+1}^{-1}(s))} E[\nu e^{-(\nu-1)/c_n+1}, \nu \in (a^n, b^n)]
$$

$$
\geq \frac{\phi'_n(\phi_n^{-1}(s))}{\phi'_{n+1}(\phi_{n+1}^{-1}(s))} e^{-b^n/c_n} E[\nu, \nu \in (a^n, b^n)].
$$

From (c) and (2.1), we deduce that for $n$ large enough, we have

$$
P(\nu(\xi_n) \in (a^n, b^n)) \geq \frac{1}{2m} E[\nu, \nu \in (a^n, b^n)].
$$

Therefore, under the condition $E[\nu \ln(\nu)] = \infty$, we have

$$
\sum_{n \geq 0} P(\nu(\xi_n) \in (a^n, b^n)) = \infty.
$$

(3.1)

Let $H(\xi_n) := \{u \in T : u \text{ child of } \xi_n, u \neq \xi_{n+1}\}$. By Remark A, we have

$$
P \left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \mid T_{n+1}, \xi_{n+1} \right) = E_{GW} \left[ \prod_{u \in H} M_\infty(u), \sum_{u \in H} W_\infty(u) \leq a^n \right]_{H=H(\xi_n)}.
$$
Since $M_\infty(u) \leq e^{1/c_n}$ for any $|u| = n$, we get
\[
P \left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \bigg| T_{n+1}, \xi_{n+1} \right) \leq e^{(\nu(\xi_n)-1)/c_n} GW \left( \sum_{u \in H} W_\infty(u) \leq a^n \right) \quad_{H=H(\xi_n)}.
\]

Let $(W_i^\infty, i \geq 1)$ be independent random variables distributed as $W_\infty(e)$ under $GW$. It follows that on the event $\{\nu(\xi_n) \in (a^n, b^n)\}$, we have
\[
P \left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \bigg| T_{n+1}, \xi_{n+1} \right) \leq e^{(b^n-1)/c_n} GW \left( \sum_{i=1}^{a^n} W_i^\infty \leq a^n \right) =: d_n.
\]

We obtain that
\[
P \left( \sum_{u \in H(\xi_n)} W_\infty(u) > a^n \right) \geq P \left( \sum_{u \in H(\xi_n)} W_\infty(u) > a^n, \nu(\xi_n) \in (a^n, b^n) \right) \geq P \left( \nu(\xi_n) \in (a^n, b^n) \right) (1 - d_n).
\]

By (c), $e^{(b^n-1)/c_n}$ goes to 1. Furthermore, we know from [13] that $E_{GW}[W_\infty(e)] = \infty$, which ensures by the law of large numbers that $d_n$ goes to 0. By equation (3.1), we deduce that
\[
\sum_{n \geq 0} P \left( \sum_{u \in H(\xi_n)} W_\infty(u) > a^n \right) = \infty.
\]

We use the Borel-Cantelli lemma to see that $\sum_{u \in H(\xi_n)} W_\infty(u) > a^n$ infinitely often. Since $W_\infty(\xi_n) \geq \frac{1}{m} \sum_{u \in H(\xi_n)} W_\infty(u)$, we get that $W_\infty(\xi_n) \geq a^n/m$ for infinitely many $n$, P.a.s. Hence
\[
\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) \geq \ln(a).
\]

Let $a$ go to $m$ to have the lower bound. Since $W_\infty(\xi_n) \leq \sum_{|u|=n} W_\infty(u) = m^n W_\infty(e)$, we have $\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) \leq \ln(m)$ hence Proposition 3.1. □

We turn to the proof of the theorem.

**Proof of Theorem 1.1.** By Proposition 3.1, we have
\[
P \left( \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \right) = 1.
\]
In particular, for P-a.e. $T$,

$$P \left( \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \mid T \right) = 1.$$

By Proposition 2.1, the distribution of $\xi$ given $T$ is UNIF. Therefore, for P-a.e. $T$,

$$(3.2) \quad \text{UNIF} \left( r \in S(T) : \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(r_n)) = \ln(m) \right) = 1.$$

Again by Proposition 2.1, the distribution of $T$ is the one of $\mathcal{T}$ under $Q$. We deduce that (3.2) holds for $Q$-a.e. tree $\mathcal{T}$. Since $Q$ and GW are equivalent, equation (3.2) holds for GW-a.e. tree $\mathcal{T}$. We call the Hölder exponent of $\text{UNIF}$ at ray $r^*$ the quantity

$$\text{Hö}(\text{UNIF})(r^*) := \liminf_{n \to \infty} -\frac{1}{n} \ln \left( \text{UNIF}(\{r \in S(\mathcal{T}) : r_n = r_n^*\}) \right).$$

By definition of $\text{UNIF}$, we can rewrite it

$$\text{Hö}(\text{UNIF})(r^*) = \liminf_{n \to \infty} -\frac{1}{n} \ln \left( m^{-n}W_\infty(r_n^*)/W_\infty(e) \right).$$

Therefore, for $\text{UNIF}$-a.e. ray $r$, $\text{Hö}(\text{UNIF})(r) = 0$. By Theorem 14.15 of [11] (or § 14 of [1]), it implies that $\text{dim}(\text{UNIF}) = 0$ GW-almost surely. □

Acknowledgements. The author thanks Russell Lyons for useful comments on the work. This work was supported in part by the Netherlands Organisation for Scientific Research (NWO).

References


