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Summary. We consider a Galton–Watson tree with offspring distribution ν of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the n-th generation and taking the limit $n \to \infty$. In the case $E[\nu \ln(\nu)] < \infty$, this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to $\ln(m)$ ([2], [9]). When $E[\nu \ln(\nu)] = \infty$, we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres [10] .

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d'enfants ν a une moyenne finie. La mesure uniforme sur la frontière de l'arbre s'obtient en chargeant chaque sommet de la n-ième génération avec une masse 1, puis en prenant la limite $n \to \infty$. Dans le cas $E[\nu \ln(\nu)] < \infty$, cette mesure a été très étudiée, et l'on sait que la dimension de Hausdorff de la mesure est égale à $\ln(m)$ ([2], [9]). Lorsque $E[\nu \ln(\nu)] = \infty$, nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres [10] .

Keywords: Galton–Watson tree, Hausdorff dimension.

AMS subject classifications: 60J80, 28A78.

1 Introduction

Let $\mathcal{T}$ be a Galton–Watson tree of root e, associated to the offspring distribution $q := (q_k, k \geq 0)$. We denote by GW the distribution of $\mathcal{T}$ on the space of rooted trees, and ν a generic random variable on $\mathbb{N}$ with distribution q. We suppose that $q_0 = 0$ and $m := \sum_{k\geq 0} kq_k \in (1, \infty)$:

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the tree has no leaf (hence survives forever) and is not degenerate. For any vertex u, we write $|u|$ for the height of vertex u ($|e| = 0$), $\nu(u)$ for the number of children of u, and Z_n is the population at height n. We define $S(\mathcal{T})$ as the set of all infinite self-avoiding paths of $\mathcal{T}$ starting from the root and we define a metric on $S(\mathcal{T})$ by $d(r, r') := e^{-|r \wedge r'|}$ where $r \wedge r'$ is the highest vertex belonging to r and r'. The space $S(\mathcal{T})$ is called boundary of the tree, and elements of $S(\mathcal{T})$ are called rays.

When $E[\nu \ln(\nu)] < \infty$, it is well-known that the martingale $m^{-n} Z_n$ converges in L^1 and almost surely to a positive limit ([4]). Seneta [13] and Heyde [3] proved that in the general case (i.e allowing $E[\nu \ln(\nu)]$ to be infinite), there exist constants $(c_n)_{n \geq 0}$ such that

(a) $W_\infty := \lim_{n \to \infty} \frac{Z_n}{c_n}$ exists a.s.

(b) $W_\infty > 0$ a.s.

(c) $\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = m$.

In particular, for each vertex $u \in \mathcal{T}$, if $Z_k(u)$ stands for the number of descendants v of u such that $|v| = |u| + k$, we can define

$$W_\infty(u) := \lim_{k \to \infty} \frac{Z_k(u)}{c_k}$$

and we notice that $m^{-n} \sum_{|u|=n} W_\infty(u) = W_\infty(e)$.

Definition. The uniform measure (also called branching measure) is the unique Borel measure on $S(\mathcal{T})$ such that

$$\mathbf{UNIF}(\{r \in S(\mathcal{T}),\, r_n = u\}) := \frac{m^{-n} W_\infty(u)}{W_\infty(e)}$$

for any integer n and any vertex u of height n.

We observe that, for any vertex u of height n,

$$\mathbf{UNIF}(\{r \in S(\mathcal{T}),\, r_n = u\}) = \lim_{k \to \infty} \frac{Z_k(u)}{Z_{n+k}}\,.$$

Therefore the uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of **UNIF**, defined by

$$\dim(\mathbf{UNIF}) := \min\{\dim(E),\, \mathbf{UNIF}(E) = 1\}$$

where the minimum is taken over all subsets $E \subset S(\mathcal{T})$ and $\dim(E)$ is the Hausdorff dimension of set E. The case $E[\nu \ln(\nu)] < \infty$ has been well studied. In [2] and [9], it is shown that $\dim(\mathbf{UNIF}) = \ln(m)$ almost surely. A description of the multifractal spectrum is available in [5],[9],[12],[14]. The case $E[\nu \ln(\nu)] = \infty$ presented as Question 3.1 in [10] was left open. This case is proved to display an extreme behaviour.

Theorem 1.1. *If* $E[\nu \ln(\nu)] = \infty$*, then* $\dim(\mathbf{UNIF}) = 0$ *for* GW*-a.e tree* $\mathcal{T}$*.*

The drop in the dimension comes from bursts of offspring at some places of the tree $\mathcal{T}$. Namely, for **UNIF**-a.e. ray r, the number of children of r_n will be greater than $(m - o(1))^n$ for infinitely many n. To prove it, we work with a particular measure Q, under which the distribution of the numbers of children of a uniformly chosen ray is more tractable. Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3.

2 A spine decomposition

For $k \geq 1$ and $s \in (0, 1)$, we call $\phi_k(s)$ the probability generating function of Z_k

$$\phi_k(s) := E[s^{Z_k}] \, .$$

We denote by $\phi_k^{-1}(s)$ the inverse map on $(0, 1)$ and we let $s \in (0, 1)$. Then $M_n := \phi_n^{-1}(s)^{Z_n}$ defines a martingale and converges in L^1 to some $M_\infty > 0$ a.s ([3]). Therefore we can take in (a)

$$c_n := \frac{-1}{\ln(\phi_n^{-1}(s))}$$

which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(e)}$. For any vertex u at generation n, we define similarly

$$M_\infty(u) := \frac{1}{\phi_n^{-1}(s)} e^{-m^{-n} W_\infty(u)} = e^{1/c_n} e^{-m^{-n} W_\infty(u)}$$

which is the limit of the martingale $M_k(u) := e^{1/c_n} e^{-Z_k(u)/c_{n+k}}$. In [6], Lynch introduces the so-called derivative martingale

$$\partial M_n := e^{1/c_n} \frac{Z_n}{\phi_n'(\phi_n^{-1}(s))} M_n$$

and shows that the derivative martingale also converges almost surely and in L^1 (∂M_n is in fact bounded). Moreover the limit ∂M_∞ is positive almost surely. We deduce that the ratio $\phi'_n(\phi_n^{-1}(s))/c_n$ converges to some positive constant. In particular, it follows from (c) that

$$\lim_{n\to\infty} \frac{\phi'_{n+1}(\phi_{n+1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} = m. \tag{2.1}$$

We are interested in the probability measure Q on the space of rooted trees defined by

$$\frac{dQ}{d\mathrm{GW}} := \partial M_\infty \,.$$

Let us describe this change of measure. We call a marked tree a couple (T, r) where T is a rooted tree and r a ray of the tree T. Let $(\mathbb{T}, \xi)$ be a random variable in the space of all marked trees (equipped with some probability $\mathrm{P}(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level k and on the location of the ray at level k, (which we denote respectively by $\mathbb{T}_k$ and ξ_k),

- the number of children of the vertices at generation k are independent
- the vertex ξ_k has a number $\nu(\xi_k)$ of children such that for any ℓ

$$\mathrm{P}(\nu(\xi_k) = \ell) = \tilde{q}_\ell^s := q_\ell \, \ell \exp\left(-\frac{\ell-1}{c_{k+1}}\right) \frac{\phi'_k(\phi_k^{-1}(s))}{\phi'_{k+1}(\phi_{k+1}^{-1}(s))} \tag{2.2}$$

- the number of children of a vertex $u \neq \xi_k$ at generation k verifies for any ℓ

$$\mathrm{P}(\nu(u) = \ell) = \tilde{q}_\ell := q_\ell \, e^{1/c_k} \exp\left(-\frac{\ell}{c_{k+1}}\right) \tag{2.3}$$

- the vertex ξ_{k+1} is chosen uniformly among the children of ξ_k

As often in the literature, we will call the ray ξ the spine. We refer to [8], [7] for motivation on spine decompositions. In our case, we can see $\mathbb{T}$ as a Galton-Watson tree in varying environment and with immigration. The fact that (2.2) and (2.3) define probabilities come from the equations (remember that by definition $e^{-1/c_k} = \phi_k^{-1}(s)$)

$$\begin{aligned} E\left[(\phi_{k+1}^{-1}(s))^\nu\right] &= \phi_k^{-1}(s)\,, \\ E\left[\nu(\phi_{k+1}^{-1}(s))^{\nu-1}\right] &= \frac{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}{\phi'_k(\phi_k^{-1}(s))}\,. \end{aligned}$$

We mention that in [8], a similar decomposition was presented using the martingale $\frac{Z_n}{m^n}$. In this case, the offspring distribution of the spine is the size-biased distribution $(\frac{\ell q_\ell}{m})_{\ell\geq 0}$ whereas the other particles generate offspring according to the original distribution q. In particular, the offspring distributions do not depend on the generation. When $E[\nu\ln(\nu)]<\infty$, the process, which is a Galton–Watson process with immigration, has a distribution equivalent to GW. It is no longer true when $E[\nu\ln(\nu)]=\infty$, in which case the spine can give birth to a super-exponential number of children.

Proposition 2.1. *Under Q, the tree $\mathcal{T}$ has the distribution of $\mathbb{T}$. Besides, for* P*-almost every tree $\mathbb{T}$, the distribution of ξ conditionally on $\mathbb{T}$ is the uniform measure* **UNIF**.

Proof. For any tree T, we define T_n the tree T obtained by keeping only the n-first generations. Let T be a tree. We will prove by induction that, for any integer n and any vertex u at generation n,

$$\mathrm{P}(\mathbb{T}_n = T_n,\, \xi_n = u) = \frac{\partial M_n}{Z_n}\mathrm{GW}(\mathcal{T}_n = T_n)\,. \tag{2.4}$$

For $n=0$, it is straightforward since $\mathbb{T}_0$ and $\mathcal{T}_0$ are reduced to the root. We suppose that this is true for $n-1$, and we prove it for n. Let $\overleftarrow{u}$ denote the parent of u, and, for any vertex v at height $n-1$, let $k(v)$ denote the number of children of v in the tree T. We have

$$\begin{aligned}
& \mathrm{P}(\mathbb{T}_n = T_n,\, \xi_n = u \,|\, \mathbb{T}_{n-1} = T_{n-1},\, \xi_{n-1} = \overleftarrow{u}) \\
= \;& \frac{1}{k(\overleftarrow{u})}\frac{\tilde{q}^s_{k(\overleftarrow{u})}}{\tilde{q}_{k(\overleftarrow{u})}}\prod_{|v|=n-1}\tilde{q}_{k(v)} \\
= \;& \frac{e^{1/c_n}}{e^{1/c_{n-1}}}\frac{\phi'_{n-1}(\phi_{n-1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))}\frac{e^{Z_{n-1}/c_{n-1}}}{e^{Z_n/c_n}}\prod_{|v|=n-1} q_{k(v)} \\
= \;& \frac{e^{1/c_n}}{e^{1/c_{n-1}}}\frac{\phi'_{n-1}(\phi_{n-1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))}\frac{e^{Z_{n-1}/c_{n-1}}}{e^{Z_n/c_n}}\mathrm{GW}(\mathcal{T}_n = T_n \,|\, \mathcal{T}_{n-1} = T_{n-1})\,.
\end{aligned}$$

We use the induction assumption to get

$$\mathrm{P}(\mathbb{T}_n = T_n,\, \xi_n = u) = e^{1/c_n}\frac{1}{\phi'_n(\phi_n^{-1}(s))}e^{\frac{-Z_n}{c_n}}\mathrm{GW}(\mathcal{T}_n = T_n)$$

which proves (2.4). Summing over the n-th generation of T gives

$$\mathrm{P}(\mathbb{T}_n = T_n) = \partial M_n \mathrm{GW}(\mathcal{T}_n = T_n) = Q(\mathcal{T} = T_n)\,.$$

This computation also shows that $\mathrm{P}(\xi_n = u \,|\, \mathbb{T}_n) = 1/Z_n$ which implies that ξ is uniformly distributed on the boundary $S(\mathbb{T})$. □

Remark A. For u a vertex of $\mathbb{T}$ at generation n, call $\mathbb{T}(u)$ the subtree rooted at u. A similar computation shows that if $u \notin \xi$, then the distribution P_u of $\mathbb{T}(u)$ (conditionally on $\mathbb{T}_n$ and on ξ_n) verifies

$$\frac{d\mathrm{P}_u}{d\mathrm{GW}} = M_\infty(u)\,.$$

3 Proof of Theorem 1.1

The following proposition shows that in the tree $\mathcal{T}$, there exist infinitely many times when the ball $\{r \in S(\mathcal{T}) \,:\, r_n = \xi_n\}$ has a 'big' weight.

Proposition 3.1. *Suppose that $E[\nu \ln(\nu)] = \infty$. Then we have* P*-a.s.*

$$\limsup_{n\to\infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m)\,.$$

Proof. Let $1 < a < b < m$ and $n \geq 0$. We get from (2.2)

$$\begin{aligned}
\mathrm{P}\left(\nu(\xi_n) \in (a^n, b^n)\right) &= \frac{\phi_n'(\phi_n^{-1}(s))}{\phi_{n+1}'(\phi_{n+1}^{-1}(s))} E\left[\nu e^{-(\nu-1)/c_{n+1}},\, \nu \in (a^n, b^n)\right] \\
&\geq \frac{\phi_n'(\phi_n^{-1}(s))}{\phi_{n+1}'(\phi_{n+1}^{-1}(s))} e^{-b^n/c_n} E\left[\nu,\, \nu \in (a^n, b^n)\right]\,.
\end{aligned}$$

From (c) and (2.1), we deduce that for n large enough, we have

$$\mathrm{P}\left(\nu(\xi_n) \in (a^n, b^n)\right) \geq \frac{1}{2m} E\left[\nu,\, \nu \in (a^n, b^n)\right]\,.$$

Therefore, under the condition $E[\nu \ln(\nu)] = \infty$, we have

$$\sum_{n\geq 0} \mathrm{P}(\nu(\xi_n) \in (a^n, b^n)) = \infty\,. \tag{3.1}$$

Let $H(\xi_n) := \{u \in \mathbb{T} : u \text{ child of } \xi_n,\, u \neq \xi_{n+1}\}$. By Remark A, we have

$$\mathrm{P}\left(\sum_{u\in H(\xi_n)} W_\infty(u) \leq a^n \,\middle|\, \mathbb{T}_{n+1}, \xi_{n+1}\right) = E_{\mathrm{GW}}\left[\prod_{u\in\mathcal{H}} M_\infty(u), \sum_{u\in\mathcal{H}} W_\infty(u) \leq a^n\right]_{\mathcal{H}=H(\xi_n)}.$$

Since $M_\infty(u) \le e^{1/c_n}$ for any $|u| = n$, we get

$$\mathrm{P}\left(\sum_{u\in H(\xi_n)} W_\infty(u) \le a^n \,\middle|\, \mathbb{T}_{n+1}, \xi_{n+1}\right) \le e^{(\nu(\xi_n)-1)/c_n}\mathrm{GW}\left(\sum_{u\in\mathcal{H}} W_\infty(u) \le a^n\right)_{\mathcal{H}=H(\xi_n)}.$$

Let $(W_\infty^i, i \ge 1)$ be independent random variables distributed as $W_\infty(e)$ under GW. It follows that on the event $\{\nu(\xi_n) \in (a^n, b^n)\}$, we have

$$\mathrm{P}\left(\sum_{u\in H(\xi_n)} W_\infty(u) \le a^n \,\middle|\, \mathbb{T}_{n+1}, \xi_{n+1}\right) \le e^{(b^n-1)/c_n}\mathrm{GW}\left(\sum_{i=1}^{a^n} W_\infty^i \le a^n\right) =: d_n\,.$$

We obtain that

$$\begin{aligned}\mathrm{P}\left(\sum_{u\in H(\xi_n)} W_\infty(u) > a^n\right) &\ge \mathrm{P}\left(\sum_{u\in H(\xi_n)} W_\infty(u) > a^n,\, \nu(\xi_n) \in (a^n, b^n)\right)\\ &\ge \mathrm{P}\left(\nu(\xi_n) \in (a^n, b^n)\right)(1-d_n)\,.\end{aligned}$$

By (c), $e^{(b^n-1)/c_n}$ goes to 1. Furthermore, we know from [13] that $E_{\mathrm{GW}}[W_\infty(e)] = \infty$, which ensures by the law of large numbers that d_n goes to 0. By equation (3.1), we deduce that

$$\sum_{n\ge 0}\mathrm{P}\left(\sum_{u\in H(\xi_n)} W_\infty(u) > a^n\right) = \infty\,.$$

We use the Borel-Cantelli lemma to see that $\sum_{u\in H(\xi_n)} W_\infty(u) > a^n$ infinitely often. Since $W_\infty(\xi_n) \ge \frac{1}{m}\sum_{u\in H(\xi_n)} W_\infty(u)$, we get that $W_\infty(\xi_n) \ge a^n/m$ for infinitely many n, P-a.s. Hence

$$\limsup_{n\to\infty} \frac{1}{n}\ln(W_\infty(\xi_n)) \ge \ln(a)\,.$$

Let a go to m to have the lower bound. Since $W_\infty(\xi_n) \le \sum_{|u|=n} W_\infty(u) = m^n W_\infty(e)$, we have $\limsup_{n\to\infty}\frac{1}{n}\ln(W_\infty(\xi_n)) \le \ln(m)$ hence Proposition 3.1. □

We turn to the proof of the theorem.

Proof of Theorem 1.1. By Proposition 3.1, we have

$$\mathrm{P}\left(\limsup_{n\to\infty}\frac{1}{n}\ln(W_\infty(\xi_n)) = \ln(m)\right) = 1\,.$$

In particular, for P-a.e. $\mathbb{T}$,

$$\mathrm{P}\left(\limsup_{n\to\infty}\frac{1}{n}\ln(W_\infty(\xi_n))=\ln(m)\,\Big|\,\mathbb{T}\right)=1\,.$$

By Proposition 2.1, the distribution of ξ given $\mathbb{T}$ is **UNIF**. Therefore, for P-a.e. $\mathbb{T}$,

$$\mathbf{UNIF}\left(r\in S(\mathbb{T})\,:\,\limsup_{n\to\infty}\frac{1}{n}\ln(W_\infty(r_n))=\ln(m)\right)=1\,. \tag{3.2}$$

Again by Proposition 2.1, the distribution of $\mathbb{T}$ is the one of $\mathcal{T}$ under Q. We deduce that (3.2) holds for Q-a.e. tree $\mathcal{T}$. Since Q and GW are equivalent, equation (3.2) holds for GW-a.e. tree $\mathcal{T}$. We call the Hölder exponent of **UNIF** at ray r^* the quantity

$$\text{Hö}(\mathbf{UNIF})(r^*):=\liminf_{n\to\infty}\frac{-1}{n}\ln\left(\mathbf{UNIF}(\{r\in S(\mathcal{T})\,:\,r_n=r_n^*\})\right)\,.$$

By definition of **UNIF**, we can rewrite it

$$\text{Hö}(\mathbf{UNIF})(r^*)=\liminf_{n\to\infty}\frac{-1}{n}\ln\left(m^{-n}W_\infty(r_n^*)/W_\infty(e)\right)\,.$$

Therefore, for **UNIF**-a.e. ray r, Hö(**UNIF**)$(r)=0$. By Theorem 14.15 of [11] (or § 14 of [1]), it implies that dim(**UNIF**) $=0$ GW-almost surely. □

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