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Summary. We consider the minimum of a super-critical branching random walk. In [1], Addario-Berry and Reed proved the tightness of the minimum centered around its mean value. We show that a convergence in law holds, giving the analog of a well-known result of Bramson [9] in the case of the branching Brownian motion.

1 Introduction

We consider a branching random walk defined as follows. The process starts with one particle located at 0. At time 1, the particle dies and gives birth to a point process $\mathcal{L}$. Then, at each time $n \in \mathbb{N}$, the particles of generation n die and give birth to independent copies of the point process $\mathcal{L}$, translated to their position. If $\mathbb{T}$ is the genealogical tree of the process, we see that $\mathbb{T}$ is a Galton-Watson tree, and we denote by $|x|$ the generation of the vertex $x \in \mathbb{T}$ (the ancestor is the only particle at generation 0). For each $x \in \mathbb{T}$, we denote by $V(x) \in \mathbb{R}$ its position on the real line. With this notation, $(V(x), |x| = 1)$ is distributed as $\mathcal{L}$. The collection of positions $(V(x),\, x \in \mathbb{T})$ defines our branching random walk.

We assume that we are in the boundary case (in the sense of [7])

$$\mathbf{E}\left[\sum_{|x|=1} 1\right] > 1, \qquad \mathbf{E}\left[\sum_{|x|=1} \mathrm{e}^{-V(x)}\right] = 1, \qquad \mathbf{E}\left[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\right] = 0. \tag{1.1}$$

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Every branching random walk satisfying mild assumptions can be reduced to this case by some renormalization. We refer to Appendix A in [16] for a precise discussion. Notice that we allow $\mathbf{E}[\sum_{|x|=1} 1] = \infty$, and more generally $\mathbf{P}(\sum_{|x|=1} 1 = \infty) > 0$. We are interested in the minimum at time n

$$M_n := \min\{ V(x), \, |x| = n\}.$$

Writing for $y \in \mathbb{R} \cup \{\pm\infty\}$, $y_+ := \max(y, 0)$, we introduce the random variables

$$X := \sum_{|x|=1} \mathrm{e}^{-V(x)}, \qquad \tilde{X} := \sum_{|x|=1} V(x)_+ \mathrm{e}^{-V(x)}. \tag{1.2}$$

We assume that

- the distribution of $\mathcal{L}$ is non-lattice,
- we have

$$\mathbf{E}\left[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)} \right] < \infty. \tag{1.3}$$

$$\mathbf{E}\left[X (\ln_+ X)^2 \right] < \infty, \qquad \mathbf{E}\left[\tilde{X} \ln_+ \tilde{X} \right] < \infty. \tag{1.4}$$

These assumptions are discussed after Theorem 1.1. Under (1.1), the minimum M_n goes to infinity, as it can be easily seen from the fact that $\sum_{|u|=n} \mathrm{e}^{-V(u)}$ goes to zero ([20]). The law of large numbers for the speed of the minimum goes back to the works of Hammersley [14], Kingman [17] and Biggins [5], and we know that $\frac{M_n}{n}$ converges almost surely to 0 in the boundary case. The second order was recently found separately by Hu and Shi [15], and Addario-Berry and Reed [1], and is proved to be equal to $\frac{3}{2}\ln n$ in probability, though there exist almost sure fluctuations (Theorem 1.2 in [15]). In [1], the authors computed the expectation of M_n to within $O(1)$, and showed, under suitable assumptions, that the sequence of the minimum is tight around its mean. Through recursive equations, Bramson and Zeitouni [10] obtained the tightness of M_n around its median, when assuming some properties on the decay of the tail distribution. In the particular case where the step distribution is log-concave, the convergence in law of M_n around its median was proved earlier by Bachmann [4]. The aim of this paper is to get the convergence of the minimum M_n centered around $\frac{3}{2}\ln n$ for a general class of branching random walks. This is the analog of the seminal work from Bramson [9], to which our approach bears some resemblance. To state our result, we introduce the *derivative martingale*, defined for any $n \geq 0$ by

$$\partial W_n := \sum_{|x|=n} V(x)\mathrm{e}^{-V(x)}. \tag{1.5}$$

From [6] (and Proposition A.1 in the Appendix), we know that the martingale converges almost surely to some limit ∂W_∞, which is strictly positive on the set of non-extinction of $\mathbb{T}$. Notice that under (1.1), the tree $\mathbb{T}$ has a positive probability to survive. If the process is extinct at time n, we set $M_n := \infty$ (or $\min \emptyset := \infty$ in the definition of M_n).

Theorem 1.1 *There exists a constant $C^* \in (0, \infty)$ such that for any real x,*

$$\lim_{n\to\infty} \mathbf{P}\left(M_n \geq \frac{3}{2}\ln n + x\right) = \mathbf{E}\left[\mathrm{e}^{-C^* \mathrm{e}^x \partial W_\infty}\right]. \tag{1.6}$$

Remark 1. We can see our theorem as the analog of the result of Lalley and Sellke [19] in the case of the branching Brownian motion : the minimum converges to a random shift of the Gumble distribution.

Remark 2. The condition of non-lattice distribution is necessary since it is hopeless to have a convergence in law around $\frac{3}{2}\ln n$ in general. We do not know if an analogous result holds in the lattice case. Without (1.3), we can expect to have still a convergence in law but centered around $\kappa \ln n$ for some constant $\kappa \neq 3/2$. This comes from the different behaviour of the probability to remain positive for one-dimensional random walks with infinite variance. Finally, the condition (1.4) appears naturally for ∂W_∞ not being identically zero (see [6], Theorem 5.2).

The proof of the theorem is divided into three steps. First, we look at the tail distribution of the minimum M_n^{kill} of the branching random walk killed below zero, i.e $M_n^{\mathrm{kill}} := \min\{V(x),\, V(x_k) \geq 0,\, \forall 0 \leq k \leq |x|\}$, where x_k denotes the ancestor of x at generation k.

Proposition 1.2 *There exists a constant $C_1 > 0$ such that*

$$\limsup_{z\to\infty} \limsup_{n\to\infty} \left| \mathrm{e}^z \mathbf{P}\left(M_n^{\mathrm{kill}} \leq \frac{3}{2}\ln n - z\right) - C_1 \right| = 0.$$

This allows us to get the tail distribution of M_n in a second stage.

Proposition 1.3 *We have*

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\frac{\mathrm{e}^z}{z}\mathbf{P}\left(M_n\le\frac{3}{2}\ln n-z\right)-C_1c_0\right|=0$$

where C_1 is the constant in Proposition 1.2, and $c_0>0$ is defined in (2.10).

Looking at the set of particles that cross a high level $A>0$ for the first time, we then deduce the theorem for the constant $C^*=C_1c_0$.

The paper is organized as follows. Section 2 introduces a useful and well-known tool, the many-to-one lemma. Then, Sections 3, 4 and 5 contain respectively the proofs of Proposition 1.2, Proposition 1.3 and Theorem 1.1.

Throughout the paper, $(c_i)_{i\ge0}$ denote positive constants. We write $\mathbf{E}[f,A]$ for $\mathbf{E}[f\mathbf{1}_A]$, and we set $\sum_\emptyset:=0$, $\prod_\emptyset:=1$.

2 The many-to-one lemma

For $a\in\mathbb{R}$, we denote by $\mathbf{P}_a$ the probability distribution associated to the branching random walk starting from a, and $\mathbf{E}_a$ the corresponding expectation. Under (1.1), there exists a centered random walk $(S_n,n\ge0)$ such that for any $n\ge1$, $a\in\mathbb{R}$ and any measurable function $g:\mathbb{R}^n\to[0,\infty)$,

$$\mathbf{E}_a\Big\{\sum_{|x|=n}g(V(x_1),\cdots,V(x_n))\Big\}=\mathbf{E}_a\Big\{\mathrm{e}^{S_n-a}g(S_1,\cdots,S_n)\Big\}\tag{2.1}$$

where, under $\mathbf{P}_a$, we have $S_0=a$ almost surely. We will write $\mathbf{P}$ and $\mathbf{E}$ instead of $\mathbf{P}_0$ and $\mathbf{E}_0$ for brevity. In particular, under (1.3), S_1 has a finite variance $\sigma^2:=\mathbf{E}[S_1^2]=\mathbf{E}[\sum_{|x|=1}V(x)^2\mathrm{e}^{-V(x)}]$. Equation (2.1) is called in the litterature the many-to-one lemma and can be seen as a consequence of Proposition 2.1 below.

2.1 Lyons' change of measure

We introduce the *additive martingale*

$$W_n:=\sum_{|u|=n}\mathrm{e}^{-V(u)}\tag{2.2}$$

and we define for any $a \in \mathbb{R}$ a probability measure $\mathbf{Q}_a$ such that for any $n \geq 0$,

$$\mathbf{Q}_a\,|_{\mathscr{F}_n} := \mathrm{e}^a W_n \bullet \mathbf{P}_a\,|_{\mathscr{F}_n} \tag{2.3}$$

where $\mathscr{F}_n$ denotes the sigma-algebra generated by the positions $(V(x),\, |x| \leq n)$ up to time n. We will write $\mathbf{Q}$ instead of $\mathbf{Q}_0$.

To give the description of the branching random walk under $\mathbf{Q}_a$, we introduce the point process $\hat{\mathcal{L}}$ with Radon-Nykodin derivative $\sum_{i\in\mathcal{L}} \mathrm{e}^{-V(i)}$ with respect to the law of $\mathcal{L}$, and we define the following process. At time 0, the population is composed of one particle w_0 located at $V(w_0) = a$. Then, at each step n, particles of generation n die and give birth to independent point processes distributed as $\mathcal{L}$, except for the particle w_n which generates a point process distributed as $\hat{\mathcal{L}}$. The particle w_{n+1} is chosen among the children of w_n with probability proportional to $\mathrm{e}^{-V(x)}$. We denote by $\mathcal{B}_a := (V(x))$ the family of the positions of this system. We still call $\mathbb{T}$ the genealogical tree of the process, so that $(w_n)_{n\geq 0}$ is a ray of $\mathbb{T}$, which we will call the *spine*. This change of probability was used in [20], see also [15]. We refer to [21] for the case of the Galton–Watson tree, to [12] for the analog for the branching Brownian motion, and to [6] for spine decompositions in various types of branching.

Proposition 2.1 ([20],[15]) *(i)Under $\mathbf{Q}_a$, the branching random walk has the distribution of $\mathcal{B}_a$.*
(ii) For any $|x| = n$, we have

$$\mathbf{Q}_a\{w_n = x \,|\, \mathscr{F}_n\} = \frac{\mathrm{e}^{-V(x)}}{W_n}. \tag{2.4}$$

(iii) The spine process $(V(w_n),\, n \geq 0)$ has the distribution of the centered random walk $(S_n,\, n \geq 0)$ under $\mathbf{P}_a$.

Before closing this subsection, we collect some elementary facts about centered random walks with finite variance.

There exists a constant $\alpha_1 > 0$ such that for any $x \geq 0$ and $n \geq 1$

$$\mathbf{P}_x(\min_{j\leq n} S_j \geq 0) \leq \alpha_1(1+x)n^{-1/2}. \tag{2.5}$$

There exists a constant $\alpha_2 > 0$ such that for any $b \geq a \geq 0$, $x \geq 0$ and $n \geq 1$

$$\mathbf{P}_x(S_n \in [a,b],\, \min_{j\leq n} S_j \geq 0) \leq \alpha_2(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.6}$$

Let $0 < \Lambda < 1$. There exists a constant $\alpha_3 = \alpha_3(\Lambda) > 0$ such that for any $b \geq a \geq 0$, $x, y \geq 0$ and $n \geq 1$

$$\begin{aligned} & \mathbf{P}_x(S_n \in [y+a, y+b],\ \min_{j \leq n} S_j \geq 0,\ \min_{\Lambda n \leq j \leq n} S_j \geq y) \\ \leq\ & \alpha_3(1+x)(1+b-a)(1+b)n^{-3/2}. \end{aligned} \tag{2.7}$$

Let $(a_n,\, n \geq 0)$ be a non-negative sequence such that $\lim_{n\to\infty} \frac{a_n}{n^{1/2}} = 0$. There exists a constant $\alpha_4 > 0$ such that for any $a \in [0, a_n]$ and $n \geq 1$

$$\mathbf{P}(S_n \in [a, a+1],\ \min_{j \leq n} S_j \geq 0,\ \min_{n/2 < j \leq n} S_j \geq a) \geq \alpha_4 n^{-3/2}. \tag{2.8}$$

Equation (2.5) comes from [18]. Equations (2.6) and (2.7) are for example Lemmas A.1 and A.3 in [3]. Equation (2.8) is Lemma A.3 of [2]: even if the uniformity in $a \in [0, a_n]$ is not stated there, it follows directly from the proof.

2.2 A convergence in law for the one-dimensional random walk

We recall that (S_n) is a centered random walk under $\mathbf{P}$, with finite variance $\mathbf{E}[S_1^2] = \sigma^2 \in (0, \infty)$. We introduce its renewal function $R(x)$ which is zero if $x < 0$, 1 if $x = 0$, and for $x > 0$

$$R(x) := \sum_{k \geq 0} \mathbf{P}(S_k \geq -x,\ S_k < \min_{0 \leq j \leq k-1} S_j). \tag{2.9}$$

Similarly, we define $R_-(x)$ as the renewal function associated to $-S$. It is known (see [22]) that there exists $c_0 > 0$ such that

$$\lim_{x \to \infty} \frac{R(x)}{x} = c_0. \tag{2.10}$$

Since $\mathbf{E}[S_1] = 0$ and $\mathbf{E}[S_1^2] < \infty$, there exist $C_-, C_+ > 0$ such that

$$\begin{aligned} \mathbf{P}\left(\min_{1 \leq i \leq n} S_i \geq 0\right) &\sim \frac{C_+}{\sqrt{n}}, \\ \mathbf{P}\left(\max_{1 \leq i \leq n} S_i \leq 0\right) &\sim \frac{C_-}{\sqrt{n}} \end{aligned}$$

as $n \to \infty$ ([18]).

Lemma 2.2 *Let $(r_n)_{n\geq 0}$ be a sequence of numbers such that $\lim_{n\to\infty}\frac{r_n}{n^{1/2}}=0$. Let $F:\mathbb{R}_+\to\mathbb{R}$ be a Riemann integrable function. We suppose that there exists a non-increasing function $\overline{F}:\mathbb{R}_+\to\mathbb{R}$ such that $|F(x)|\leq \overline{F}(x)$ for any $x\geq 0$ and $\int_{x\geq 0} x\overline{F}(x)<\infty$. Then, as $n\to\infty$,*

$$\mathbf{E}\left[F(S_n-y),\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y\right]\sim\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}n^{-3/2}\int_{x\geq 0}F(x)R_-(x)dx \tag{2.11}$$

uniformly in $y\in[0,r_n]$.

Proof. Let $\varepsilon>0$. Since $|F(x)|\leq\overline{F}(x)$ and $\overline{F}$ is non-increasing, we have for any integer $M\geq 1$,

$$\begin{aligned}&\mathbf{E}\left[|F(S_n-y)|,\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y,S_n\geq y+M\right]\\ \leq\ &\sum_{j\geq M}\overline{F}(j)\mathbf{P}\left(\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y,S_n\in[y+j,y+j+1)\right).\end{aligned}$$

For $j\geq 1$, we have by (2.6),

$$\mathbf{P}\left(\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y,S_n\in[y+j,y+j+1)\right)\leq c_1\frac{j}{n^{3/2}}.$$

It yields that

$$\mathbf{E}\left[|F(S_n-y)|,\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y,S_n\geq y+M\right]\leq\frac{c_1}{n^{3/2}}\sum_{j\geq M}\overline{F}(j)j$$

which is less than $\varepsilon n^{-3/2}$ for $M\geq 1$ large enough. Therefore, we can restrict to F with compact support. By approximating F by scale functions (F is Riemann integrable by assumption), we only prove (2.11) for $F(x)=\mathbf{1}_{\{x\in[0,a]\}}$, for any $a\geq 0$. We have

$$\begin{aligned}&\mathbf{E}\left[F(S_n-y),\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y\right]\\ =\ &\mathbf{P}(\min_{k\in[0,n]}S_k\geq 0,\ \min_{k\in[n/2,n]}S_k\geq y,\ S_n\leq y+a).\end{aligned}$$

Applying the Markov property at time $n/2$ (we assume that $n/2$ is integer for simplicity), we obtain that

$$\mathbf{E}\left[F(S_n-y),\min_{k\in[0,n]}S_k\geq 0,\min_{k\in[n/2,n]}S_k\geq y\right]=\mathbf{E}\left[\phi(S_{n/2}),\min_{k\in[0,n/2]}S_k\geq 0\right] \tag{2.12}$$

where

$$\phi(x) := \mathbf{P}_x \left(\min_{k\in[0,n/2]} S_k \geq y,\ S_{n/2} \leq y + a \right).$$

We estimate $\phi(x)$. Reversing time, we notice that

$$\phi(x) = \mathbf{P} \left(\min_{k\in[0,n/2]} (-S_k) \geq -S_{n/2} - x + y \geq -a \right).$$

We introduce the strict descending ladder heights and times (H_ℓ, T_ℓ) of $-S$ defined by $H_0 := 0$, $T_0 := 0$ and for any $\ell \geq 0$,

$$\begin{aligned} T_{\ell+1} &:= \min\{k \geq T_\ell + 1 \,:\, (-S_k) < H_\ell\}, \\ H_{\ell+1} &:= -S_{T_{\ell+1}}. \end{aligned}$$

Since $\mathbf{E}[S_1] = 0$, we have $T_\ell < \infty$ for any $\ell \geq 0$. We observe that $R_-(x) = \sum_{\ell\geq 0} \mathbf{P}(H_\ell \geq -x)$. Discussing on the time ℓ such that $H_\ell = \min_{k\in[0,n/2]}(-S_k)$, we have

$$\begin{aligned} &\mathbf{P} \left(\min_{k\in[0,n/2]} (-S_k) \geq -S_{n/2} - x + y \geq -a \right) \\ = &\sum_{\ell\geq 0} \mathbf{P} \left(T_\ell \leq n/2, H_\ell \geq -S_{n/2} - x + y \geq -a, \min_{k\in[T_\ell,n/2]} (-S_k) \geq H_\ell \right). \end{aligned}$$

Hence,

$$\phi(x) = \sum_{\ell\geq 0} \mathbf{P} \left(T_\ell \leq n/2, H_\ell \geq -S_{n/2} - x + y \geq -a, \min_{k\in[T_\ell,n/2]} (-S_k) \geq H_\ell \right). \tag{2.13}$$

By the Markov property at time T_ℓ, we see that

$$\begin{aligned} &\mathbf{P} \left(H_\ell \geq -S_{n/2} - x + y \geq -a, \min_{k\in[T_\ell,n/2]} (-S_k) \geq H_\ell \,\middle|\, (H_\ell,\, T_\ell) \right) \\ = &\ \mathbf{1}_{\{H_\ell \geq -a\}} \mathbf{P} \left(\min_{j\in[0,\frac{n}{2}-t]} (-S_j) \geq 0, -S_{\frac{n}{2}-t} \in [x-y-a-h, x-y] \right)_{h=H_\ell, t=T_\ell}. \end{aligned}$$

Let $\psi(x) := x\mathrm{e}^{-x^2/2}\mathbf{1}_{\{x\geq 0\}}$. By Theorem 1 of [11], we check that

$$\begin{aligned} &\mathbf{1}_{\{h\geq -a\}} \mathbf{P} \left(\min_{j\in[0,\frac{n}{2}-t]} (-S_j) \geq 0, -S_{\frac{n}{2}-t} \in [x-y-a-h, x-y] \right) \\ = &\ \mathbf{1}_{\{h\geq -a\}} \frac{2C_-}{\sigma n} (h+a) \psi \left(\frac{x}{\sigma\sqrt{n/2}} \right) + \mathbf{1}_{\{h\geq -a\}} o(n^{-1}) \end{aligned}$$

uniformly in $x \in \mathbb{R}$, $t \le n^{1/2}$, $h \in [-a,0]$ and $y \in [0,r_n]$. To deal with $t \in [n^{1/2}, n/2]$, we see that

$$\mathbf{1}_{\{h\ge -a\}}\mathbf{P}\left(\min_{j\in[0,\frac{n}{2}-t]}(-S_j)\ge 0, -S_{\frac{n}{2}-t}\in[x-y-a-h,x-y]\right) = \mathbf{1}_{\{h\ge -a\}}O(1)\left(\frac{n}{2}-t+1\right)^{-1}$$

again by Theorem 1 of [11]. Going back to (2.13), it implies that

$$\begin{aligned}\phi(x) &= o(n^{-1}) + \frac{2C_-}{\sigma n}\psi\left(\frac{x}{\sigma\sqrt{n/2}}\right)\sum_{\ell\ge 0}\mathbf{E}\left[(H_\ell+a)\mathbf{1}_{\{H_\ell\ge -a,\, T_\ell\le n^{1/2}\}}\right]\\ &\quad + O(1)\sum_{\ell\ge 0}\mathbf{E}\left[\frac{H_\ell+a}{\frac{n}{2}-T_\ell+1}\mathbf{1}_{\{H_\ell\ge -a,\, T_\ell\in(n^{1/2},n/2]\}}\right]\\ &= o(n^{-1}) + \frac{2C_-}{\sigma n}\psi\left(\frac{x}{\sigma\sqrt{n/2}}\right)\sum_{\ell\ge 0}\mathbf{E}\left[(H_\ell+a)\mathbf{1}_{\{H_\ell\ge -a\}}\right]\\ &\quad + O(1)\sum_{\ell\ge 0}\mathbf{E}\left[\frac{H_\ell+a}{\frac{n}{2}-T_\ell+1}\mathbf{1}_{\{H_\ell\ge -a,\, T_\ell\in(n^{1/2},n/2]\}}\right].\end{aligned}$$

We want to show that the term in the last line is $o(n^{-1})$ as well. We observe that

$$\mathbf{E}\left[\frac{H_\ell+a}{\frac{n}{2}-T_\ell+1}\mathbf{1}_{\{H_\ell\ge -a,\, T_\ell\in(n^{1/2},n/2]\}}\right] \le a\mathbf{E}\left[\frac{1}{\frac{n}{2}-T_\ell+1}\mathbf{1}_{\{H_\ell\ge -a,\, T_\ell\in(n^{1/2},n/2]\}}\right].$$

Since $\sum_{\ell\ge 0}\mathbf{P}\left(H_\ell\ge -a,\, T_\ell=k\right)\le \mathbf{P}_a(S_k\in[0,a],\, \min_{j\le k}S_j\ge 0)$, we obtain by (2.6) that

$$\sum_{\ell\ge 0}\mathbf{P}\left(H_\ell\ge -a,\, T_\ell=k\right)\le \alpha_2(1+a)^3k^{-3/2}.$$

It yields that

$$\begin{aligned}\sum_{\ell\ge 0}\mathbf{E}\left[\frac{H_\ell+a}{\frac{n}{2}-T_\ell+1}\mathbf{1}_{\{H_\ell\ge -a,\, T_\ell\in(n^{1/2},n/2]\}}\right] &\le a\alpha_2(1+a)^3\sum_{k=\lfloor n^{1/2}\rfloor+1}^{\lfloor n/2\rfloor}k^{-3/2}\frac{1}{\frac{n}{2}-k+1}\\ &= o(n^{-1})\end{aligned}$$

indeed. Therefore

$$\phi(x) = o(n^{-1}) + \frac{2C_-}{\sigma n}\psi\left(\frac{x}{\sigma\sqrt{n/2}}\right)\sum_{\ell\ge 0}\mathbf{E}\left[(H_\ell+a)\mathbf{1}_{\{H_\ell\ge -a\}}\right]$$

uniformly in $x\ge 0$. Equation (2.12) becomes

$$\begin{aligned}&\mathbf{E}\left[F(S_n-y),\, \min_{k\in[0,n]}S_k\ge 0,\, \min_{k\in[n/2,n]}S_k\ge y\right]\\ &= o(n^{-3/2}) + \frac{2C_-}{\sigma n}\mathbf{E}\left[\psi\left(\frac{S_{n/2}}{\sigma\sqrt{n/2}}\right),\, \min_{k\in[0,n/2]}S_k\ge 0\right]\sum_{\ell\ge 0}\mathbf{E}\left[(H_\ell+a)\mathbf{1}_{\{H_\ell\ge -a\}}\right]\end{aligned}$$

where we used (2.5). We know (see [8]) that $S_n/(\sigma n^{1/2})$ conditioned on $\min_{k\in[0,n]} S_k$ being non-negative converges to the Rayleigh distribution. Therefore,

$$\mathbf{E}\left[\psi\left(\frac{S_{n/2}}{\sigma\sqrt{n/2}}\right),\ \min_{k\in[0,n/2]} S_k \geq 0\right] \sim \frac{C_+}{2\sqrt{n}}\sqrt{\frac{\pi}{2}}.$$

We end up with

$$\begin{aligned}&\mathbf{E}\left[F(S_n - y),\ \min_{k\in[0,n]} S_k \geq 0,\ \min_{k\in[n/2,n]} S_k \geq y\right]\\ =\ & o(n^{-3/2}) + \frac{C_-C_+}{\sigma n^{3/2}}\sqrt{\frac{\pi}{2}}\sum_{\ell\geq 0}\mathbf{E}\left[(H_\ell + a)\mathbf{1}_{\{H_\ell \geq -a\}}\right].\end{aligned}$$

We recall that $\sum_{\ell\geq 0}\mathbf{P}(H_\ell \geq -a) = R_-(a)$ by definition. We check that

$$\sum_{\ell\geq 0}\mathbf{E}\left[(H_\ell + a)\mathbf{1}_{\{H_\ell\geq -a\}}\right] = \int_{x\geq 0} F(x)R_-(x)dx,$$

which completes the proof. □

3 The minimum of a killed branching random walk

It reveals useful to study first the killed branching random walk. We consider only individuals that stay above 0, and we investigate the behaviour of the minimal position

(3.1) $$M_n^{\text{kill}} := \inf\{V(u),\ |u| = n,\ V(u_k) \geq 0,\ \forall\, 0 \leq k \leq n\}.$$

[$\inf\emptyset := \infty$]. If $M_n^{\text{kill}} < \infty$, i.e. if the killed branching random walk survives until time n, we denote by $m^{\text{kill},(n)}$ a vertex chosen uniformly in the set $\{u : |u| = n,\ V(u) = M_n^{\text{kill}},\ V(u_k) \geq 0,\ \forall\, 0 \leq k \leq n\}$ of the particles that realize the minimum. We will see that the typical order of M_n^{kill} is $\frac{3}{2}\ln n$. It will be convenient to use the following notation, for $z \geq 0$:

(3.2) $$a_n(z) := \frac{3}{2}\ln n - z,$$

(3.3) $$I_n(z) := [a_n(z) - 1, a_n(z)).$$

The section is devoted to the proof of the following proposition.

Proposition 3.1 *For any $\varepsilon > 0$, there exist $A > 0$ and $N \geq 1$ such that for any $n \geq N$ and $z \in [A, \ln(n)]$,*

$$\left|\mathrm{e}^z\,\mathbf{P}(M_n^{\text{kill}} \in I_n(z)) - C_2\right| \leq \varepsilon$$

where C_2 is some positive constant.

Corollary 3.2 *Let $C_1 := \frac{C_2}{1-\mathrm{e}^{-1}}$. For any $\varepsilon > 0$, there exist $A > 0$ and $N \geq 1$ such that for any $n \geq N$ and $z \in [A, (\ln n)/2]$,*

$$\left| \mathrm{e}^z \, \mathbf{P}\left(M_n^{\mathrm{kill}} \leq \frac{3}{2}\ln n - z \right) - C_1 \right| \leq \varepsilon.$$

Proposition 1.2 follows.

Assuming that Proposition 3.1 holds, let us see how it implies the corollary.

Proof of Corollary 3.2. Let $\varepsilon > 0$. We have by equation (2.1),

$$\begin{aligned} \mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u) \leq \ln n, \, \min_{0\leq j\leq n} V(u_j) \geq 0\}} \right] &= \mathbf{E}\left[\mathrm{e}^{S_n}, \, S_n \leq \ln n, \, \min_{0 \leq j \leq n} S_j \geq 0 \right] \\ &\leq n \mathbf{P}\left(S_n \leq \ln n, \, \min_{0\leq j\leq n} S_j \geq 0 \right). \end{aligned}$$

By (2.6), we have $\mathbf{P}\left(S_n \leq \ln n, \, \min_{0\leq j\leq n} S_j \geq 0\right) \leq c_2 \frac{1+(\ln n)^2}{n^{3/2}}$. Hence, there exists N_1 such that for any $n \geq N_1$

$$\mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u) \leq \ln n, \, \min_{0\leq j\leq n} V(u_j) \geq 0\}} \right] \leq \varepsilon.$$

We observe that $\mathbf{P}(M_n^{\mathrm{kill}} \leq \ln n)$ is less than the left-hand side. Therefore, $\mathbf{P}(M_n^{\mathrm{kill}} \leq \ln n) \leq \varepsilon$ for $n \geq N_1$. Let A and N be as in Proposition 3.1. We have for $n \geq N$ and $z \in [A, \ln n]$,

$$\left| \mathrm{e}^z \, \mathbf{P}(M_n^{\mathrm{kill}} \in I_n(z)) - C_2 \right| \leq \varepsilon.$$

We obtain that, for $z \in [A, (\ln n)/2]$,

$$\left| \mathrm{e}^z \, \mathbf{P}\left(M_n^{\mathrm{kill}} \in \left[\frac{3}{2}\ln n - z - \lfloor (\ln n)/2 \rfloor - 1, \frac{3}{2}\ln n - z \right) \right) - \sum_{k=0}^{\lfloor (\ln n)/2 \rfloor} \mathrm{e}^{-k} C_2 \right| \leq \sum_{k=0}^{\lfloor (\ln n)/2 \rfloor} \mathrm{e}^{-k} \varepsilon.$$

Hence, for $n \geq \max(N_1, N)$, and $z \in [A, (\ln n)/2]$

$$\left| \mathrm{e}^z \, \mathbf{P}\left(M_n^{\mathrm{kill}} < \frac{3}{2}\ln n - z \right) - \sum_{k\geq 0} \mathrm{e}^{-k} C_2 \right| \leq \sum_{k\geq 0} \mathrm{e}^{-k} \varepsilon + C_2 \sum_{k > (\ln n)/2} \mathrm{e}^{-k} + \varepsilon.$$

Take N_2 such that if $n \geq N_2$, then $C_2 \sum_{k>(\ln n)/2} \mathrm{e}^{-k} \leq \varepsilon$. We obtain for $n \geq \max(N_1, N, N_2)$, and $z \in [A, (\ln n)/2]$

$$\left| \mathrm{e}^z \, \mathbf{P}\left(M_n^{\mathrm{kill}} < \frac{3}{2}\ln n - z \right) - \frac{C_2}{1 - \mathrm{e}^{-1}} \right| \leq \varepsilon \Big(\frac{1}{1 - \mathrm{e}^{-1}} + 2 \Big)$$

which completes the proof. □

3.1 Tightness of the minimum

We want to estimate the probability of the event $\{M_n^{\rm kill} \in I_n(z)\}$. The first lemma gives information on the path of particles located in $I_n(z)$. Roughly speaking, we show that they typically follow excursions over the curve $k \to \frac{3}{2}\ln k$.

Lemma 3.3 *Let $0<\Lambda<1$. There exist constants $c_3, c_4>0$ such that for any $n\ge 1$, $L\ge 0$, $x\ge 0$ and $z\ge 0$,*

$$
\begin{aligned}
&\mathbf{P}_x\Big(\exists |u|=n\,:\, V(u)\in I_n(z),\ \min_{k\in[0,n]} V(u_k)\ge 0,\ \min_{k\in[\Lambda n,n]} V(u_k)\in I_n(z+L)\Big) \\
&\le\ c_3(1+x)\mathrm{e}^{-c_4 L}\mathrm{e}^{-x-z}.
\end{aligned} \tag{3.4}
$$

Proof. Let E be the event in (3.4), and for $0\le k\le n$

$$
d_k = d_k(n,z,L) := \begin{cases} 0, & \text{if } 0\le k\le \Lambda n,\\ \max(\frac{3}{2}\ln n - z - L - 1, 0), & \text{if } \Lambda n < k \le 2n. \end{cases} \tag{3.5}
$$

Discussing on the time when the minimum $\min_{k\in[\Lambda n,n]} V(u_k)$ is reached, we observe that $E\subset \bigcup_{k\in[\Lambda n,n]} E_k$ where we defined $E_k := \bigcup_{|u|=n} E_k(u)$ and for any $|u|=n$,

$$
E_k(u) := \Big\{ V(u_\ell)\ge d_\ell\,,\ \forall\, 0\le \ell\le n,\ V(u)\in I_n(z),\ V(u_k)\in I_n(z+L)\Big\}.
$$

Similarly, let

$$
E_k(S) := \Big\{ S_\ell \ge d_\ell, \forall\, 0\le\ell\le n,\ S_n\in I_n(z),\ S_k\in I_n(z+L)\Big\}.
$$

We notice that $\mathbf{P}_x(E_k)\le \mathbf{E}_x\left[\sum_{|u|=n}\mathbf{1}_{E_k(u)}\right]$ which is $\mathbf{E}_x\left[\mathrm{e}^{S_n-x}\mathbf{1}_{E_k(S)}\right]$ by (2.1). In particular,

$$
\mathbf{P}_x(E_k)\le n^{3/2}\mathrm{e}^{-x-z}\mathbf{P}_x(E_k(S)). \tag{3.6}
$$

We need to estimate $\mathbf{P}_x(E_k(S))$. By the Markov property at time k,

$$
\begin{aligned}
\mathbf{P}_x(E_k(S)) \ \le\ & \mathbf{P}_x\left(S_\ell\ge d_\ell\,,\ \forall\, 0\le \ell\le k, S_k\in I_n(z+L)\right)\\
& \times \mathbf{P}\left(S_{n-k}\in[L-1,L+1],\ \min_{\ell\in[0,n-k]} S_\ell\ge 0\right).
\end{aligned}
$$

For the second term of the right-hand side, we know from (2.6) that there exists a constant $c_5>0$ such that

$$
\mathbf{P}\left(S_{n-k}\in[L-1,L+1],\ \min_{\ell\in[0,n-k]} S_\ell\ge 0\right)\le c_5(n-k+1)^{-3/2}(1+L). \tag{3.7}
$$

To bound the first term, we have to discuss on the value of k. Suppose that $\frac{\Lambda+1}{2}n \le k \le n$. We have by (2.7)

$$\mathbf{P}_x\left(S_\ell \ge d_\ell,\, \forall\, 0 \le \ell \le k, S_k \in I_n(z+L)\right) \le c_6 \frac{(1+x)}{n^{3/2}}. \tag{3.8}$$

If $\Lambda n \le k < \frac{\Lambda+1}{2}n$, we simply write

$$\begin{aligned}\mathbf{P}_x\left(S_\ell \ge d_\ell,\, \forall\, 0 \le \ell \le k, S_k \in I_n(z+L)\right) &\le \mathbf{P}_x\left(S_k \in I_n(z+L),\ \min_{\ell\in[0,k]} S_\ell \ge 0\right)\\ &\le c_7(1+x)\ln(n)n^{-3/2} \qquad (3.9)\end{aligned}$$

by (2.6) . From (3.7), (3.8) and (3.9), there exists a constant $c_8 > 0$ such that

$$\sum_{k\in[\Lambda n, n-a]} \mathbf{P}_x(E_k(S)) \le c_8(1+x)(1+L)\frac{a^{-1/2}}{n^{3/2}}$$

for any $a \ge 1$. By (3.6), it implies that

$$\sum_{k\in[\Lambda n, n-a]} \mathbf{P}_x(E_k) \le c_8(1+x)(1+L)\mathrm{e}^{-x-z}a^{-1/2}. \tag{3.10}$$

It remains to bound $\mathbf{P}_x(E_k)$ for $n-a < k \le n$. We observe that

$$\mathbf{P}_x(E_k) \le \mathbf{P}_x\left(\exists\, |u| = k\,:\, V(u_\ell) \ge d_\ell\,,\, \forall\, 0 \le \ell \le k, V(u) \in I_n(z+L)\right).$$

By an application of (2.1), we have

$$\mathbf{P}_x(E_k) \le n^{3/2}\mathrm{e}^{-x-z-L}\mathbf{P}_x\left(S_\ell \ge d_\ell\,,\, \forall\, 0 \le \ell \le k, S_k \in I_n(z+L)\right)$$

which is $\le c_9\mathrm{e}^{-x-z-L}(1+x)$ by (2.7) (for $k \ge (1+\Lambda)n/2$ for example). It follows that,

$$\sum_{k\in[n-a,n]} \mathbf{P}_x(E_k) \le c_9(1+a)(1+x)\mathrm{e}^{-x-z-L}. \tag{3.11}$$

Equations (3.10) and (3.11) yield that, for $a \in [1, (1-\Lambda)n/2]$,

$$\mathbf{P}_x(E) \le \sum_{k\in[\Lambda n, n]} \mathbf{P}_x(E_k) \le (1+x)\mathrm{e}^{-x-z}\Big\{c_8(1+L)a^{-1/2} + c_9(1+a)\mathrm{e}^{-L}\Big\}.$$

We take $a = \mathrm{e}^{\alpha L}$ with $\alpha > 0$ to complete the proof. □

Recall that $a_n(z) := \frac{3}{2}\ln n - z$, and $I_n(z) := [a_n(z) - 1, a_n(z))$.

Definition 3.4 *For $|u| = n$, we say that $u \in \mathcal{Z}^{z,L}$ if*

$$V(u) \in I_n(z), \ \min_{k\in[0,n]} V(u_k) \geq 0, \ \min_{k\in[n/2,n]} V(u_k) \geq a_n(z+L).$$

In words, $u \in \mathcal{Z}^{z,L}$ loosely means that a particle is located around $\frac{3}{2}\ln n - z$, and made an excursion above the curve $k \to \frac{3}{2}\ln k - z - L$. We easily deduce from Lemma 3.3 that for any $\varepsilon > 0$, there exists $L > 0$ large enough such that for any $n \geq 1$ and $z \geq 0$,

$$\mathbf{P}\left(\exists |u| = n \,:\, u \notin \mathcal{Z}^{z,L},\, V(u) \leq \frac{3}{2}\ln n - z\right) \leq \varepsilon \mathrm{e}^{-z}. \tag{3.12}$$

Equivalently, with high probability, any particle located below $\frac{3}{2}\ln n - z$ made an excursion above the curve $k \to \frac{3}{2}\ln k - z - L$. We show now that $\mathbf{P}(M_n^{\mathrm{kill}} \leq \frac{3}{2}\ln n - z)$ has an exponential decay as $z \to \infty$.

Lemma 3.5 *There exist $c_{10}, c_{11} > 0$ such that for any $n \geq 1$ and $z \in [0, (3/2)\ln n - 1]$*

$$\mathrm{e}^z \mathbf{P}\left(M_n^{\mathrm{kill}} \leq \frac{3}{2}\ln n - z\right) \in [c_{10}, c_{11}].$$

Proof. The proof relies on usual first and second moment arguments. By equation (3.12), there exists $L > 0$ such that for any $z \geq 0$ and $n \geq 1$, we have $\mathbf{P}(m^{\mathrm{kill},(n)} \notin \mathcal{Z}^{z,L},\, M_n^{\mathrm{kill}} \in I_n(z)) \leq \mathrm{e}^{-z}$. Let $(d_k)_{0\leq k\leq n}$ be, as defined by (3.5) in the case $\Lambda = 1/2$,

$$d_k = d_k(n,z,L) := \begin{cases} 0, & \text{if } 0 \leq k \leq \frac{n}{2}, \\ \max(\frac{3}{2}\ln n - z - L - 1, 0), & \text{if } \frac{n}{2} < k \leq n. \end{cases} \tag{3.13}$$

We have by (2.1),

$$\begin{aligned} \mathbf{P}(m^{\mathrm{kill},(n)} \in \mathcal{Z}^{z,L},\, M_n^{\mathrm{kill}} \in I_n(z)) &\leq \mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{u\in\mathcal{Z}^{z,L}\}}\right] \\ &= \mathbf{E}\left[\mathrm{e}^{S_n},\, S_k \geq d_k,\, \forall\, 0 \leq k \leq n, S_n \in I_n(z)\right] \\ &\leq n^{3/2}\mathrm{e}^{-z}\mathbf{P}\Big\{S_k \geq d_k, \forall\, 0 \leq k \leq n, S_n \in I_n(z)\Big\}. \end{aligned}$$

By (2.7), the right-hand side is less than $c_{12}(L)\mathrm{e}^{-z}$. We obtain that

$$\mathbf{P}(M_n^{\mathrm{kill}} \in I_n(z)) \leq (c_{12}+1)\mathrm{e}^{-z}.$$

This implies the upper bound. To prove the lower bound, we introduce

$$e_k = e_k^{(n)} := \begin{cases} k^{1/12}, & \text{if } 0 \leq k \leq \frac{n}{2}, \\ (n-k)^{1/12}, & \text{if } \frac{n}{2} < k \leq n. \end{cases}$$

We say that $|u|=n$ is a good vertex if $u\in\mathcal{Z}^{z,0}$ and

$$\sum_{v\in\Omega(u_k)} \mathrm{e}^{-(V(v)-d_k)}\Big\{1+(V(v)-d_k)_+\Big\} \le B\mathrm{e}^{-e_k} \qquad \forall\, 1\le k\le n, \tag{3.14}$$

where $\Omega(y)$ stands for the set of brothers of y, i.e the particles $x\neq y$ which share the same parent as y in the tree $\mathbb{T}$. By (2.8), there exists $c_{13}>0$ such that $\mathbf{Q}(w_n\in\mathcal{Z}^{z,0})\ge 2c_{13}n^{-3/2}$. Then, by Lemma C.1, we can choose $B>0$ such that for any $n\ge 1$ and $z\in[0,(3/2)\ln n-1]$

$$\mathbf{Q}(w_n \text{ is a good vertex}) \ge c_{13}n^{-3/2}.$$

Let Good_n be the number of good vertices at generation n. We have by definition of the measure $\mathbf{Q}$ and Proposition 2.1 (ii),

$$\begin{aligned}
\mathbf{E}\left[\mathrm{Good}_n\right] &= \mathbf{E}_{\mathbf{Q}}\left[\frac{1}{W_n}\sum_{|u|=n}\mathbf{1}_{\{u \text{ is a good vertex}\}}\right]\\
&= \mathbf{E}_{\mathbf{Q}}\left[\mathrm{e}^{V(w_n)},\, w_n \text{ is a good vertex}\right]\\
&\ge n^{3/2}\mathrm{e}^{-z-1}\mathbf{Q}\left(w_n \text{ is a good vertex}\right)\\
&\ge c_{13}\mathrm{e}^{-z-1}. \qquad (3.15)
\end{aligned}$$

We look at the second moment. We use again Proposition 2.1 (ii) to see that

$$\begin{aligned}
\mathbf{E}\left[(\mathrm{Good}_n)^2\right] &= \mathbf{E}_{\mathbf{Q}}\left[\mathrm{e}^{V(w_n)}\mathrm{Good}_n,\, w_n \text{ is a good vertex}\right]\\
&\le n^{3/2}\mathrm{e}^{-z}\mathbf{E}_{\mathbf{Q}}\left[\mathrm{Good}_n,\, w_n \text{ is a good vertex}\right].
\end{aligned}$$

Let Y_n be the number of vertices $|u|=n$ such that $u\in\mathcal{Z}^{z,0}$. We notice that $Y_n\ge\mathrm{Good}_n$, hence

$$\mathbf{E}\left[(\mathrm{Good}_n)^2\right]\le n^{3/2}\mathrm{e}^{-z}\mathbf{E}_{\mathbf{Q}}\left[Y_n,\, w_n \text{ is a good vertex}\right].$$

We decompose Y_n along the spine. We get

$$Y_n = \mathbf{1}_{\{w_n\in\mathcal{Z}^{z,0}\}} + \sum_{k=1}^{n}\sum_{u\in\Omega(w_k)} Y_n(u)$$

where $Y_n(u)$ is the number of vertices $|v|=n$ which are descendants of u and such that $v\in\mathcal{Z}^{z,0}$. Therefore,

$$\begin{aligned}
\mathbf{E}\left[(\mathrm{Good}_n)^2\right] \le\ & n^{3/2}\mathrm{e}^{-z}\Big\{\mathbf{Q}(w_n \text{ is a good vertex})\\
&+\sum_{k=1}^{n}\mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} Y_n(u), w_n \text{ is a good vertex}\right]\Big\}.
\end{aligned}$$

Let $\mathcal{G}_\infty := \sigma\{w_j, V(w_j), \Omega(w_j), (V(u))_{u\in\Omega(w_j)}, j \geq 1\}$ be the sigma-algebra generated by the spine and its brothers. By Proposition 2.1 (i), we know that the branching random walk rooted at $u \in \Omega(w_k)$ has the same law under $\mathbf{P}$ and $\mathbf{Q}$. We take now d_k equal to 0 if $k \leq n/2$ and $\max((3/2)\ln n - z - 1, 0)$ if $n/2 < k \leq n$ (or, equivalently, we take $d_k := d_k(n, z, 0)$ in (3.13)). For $u \in \Omega(w_k)$, we have $Y_n(u) = 0$ if there exists $j \leq |u|$ such that $V(u_j) \leq d_j$. Otherwise, we have by (2.1),

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}}\left[Y_n(u)\,|\,\mathcal{G}_\infty\right] &= \mathbf{E}_{V(u)}\left[\sum_{|v|=n-k} \mathbf{1}_{\{V(v_j)\geq d_{k+j},\,\forall\, 0\leq j\leq n-k,\, V(v)\in I_n(z)\}}\right] \\ &= \mathrm{e}^{-V(u)}\mathbf{E}_{V(u)}\left[\mathrm{e}^{S_{n-k}}, S_j \geq d_{k+j}, \forall\, 0 \leq j \leq n-k, S_{n-k} \in I_n(z)\right].\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}}[Y_n(u)\,|\,\mathcal{G}_\infty] &\leq n^{3/2}\mathrm{e}^{-z-V(u)}\mathbf{P}_{V(u)}\left(S_j \geq d_{k+j}, \forall\, 0 \leq j \leq n-k, S_{n-k} \in I_n(z)\right) \\ &=: n^{3/2}\mathrm{e}^{-z-V(u)}\mathrm{p}(V(u), k, n).\end{aligned}$$

We obtain that

$$\begin{aligned}\mathbf{E}\left[(\mathrm{Good}_n)^2\right] \leq\; & n^{3/2}\mathrm{e}^{-z}\Big\{\mathbf{Q}(w_n \text{ is a good vertex}) + \\ & n^{3/2}\mathrm{e}^{-z}\sum_{k=1}^{n}\mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} \mathrm{e}^{-V(u)}\mathrm{p}(V(u), k, n), w_n \text{ is a good vertex}\right]\Big\}. \qquad (3.16)\end{aligned}$$

We want to bound $\mathrm{p}(r, k, n)$ for $r \in \mathbb{R}$. We have to split the cases $k \leq n/2$ and $n/2 < k \leq n$. Suppose first that $k \leq n/2$. Then $\mathrm{p}(r, k, n) = 0$ if $r < 0$. If $r \geq 0$, we apply (2.7) to see that

$$\mathrm{p}(r, k, n) \leq c_{14}(r + 1)n^{-3/2}.$$

It implies that

$$\begin{aligned}&\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} \mathrm{e}^{-V(u)}\mathrm{p}(V(u), k, n), w_n \text{ is a good vertex}\right] \\ \leq\; & c_{14}n^{-3/2}\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} \mathrm{e}^{-V(u)}(1 + V(u)_+), w_n \text{ is a good vertex}\right] \\ \leq\; & c_{14}Bn^{-3/2}\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\mathrm{e}^{-e_k}\mathbf{Q}\left(w_n \text{ is a good vertex}\right)\end{aligned}$$

where the last inequality comes from the property (3.14) satisfied by a good vertex. When $n/2 < k \leq n$, we simply write $\mathrm{p}(r,k,n) \leq 1$ and we get

$$\begin{aligned}
&\sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n} \mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} \mathrm{e}^{-V(u)}\mathrm{p}(V(u),k,n),\ w_n \text{ is a good vertex}\right] \\
\leq\ & \sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n} \mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} \mathrm{e}^{-V(u)},\ w_n \text{ is a good vertex}\right] \\
=\ & n^{-3/2}\mathrm{e}^{z} \sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n} \mathbf{E}_{\mathbf{Q}}\left[\sum_{u\in\Omega(w_k)} \mathrm{e}^{-(V(u)-d_k)},\ w_n \text{ is a good vertex}\right] \\
\leq\ & Bn^{-3/2}\mathrm{e}^{z} \sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n} \mathrm{e}^{-e_k}\mathbf{Q}\left(w_n \text{ is a good vertex}\right)
\end{aligned}$$

by (3.14). Going back to (3.16), we end up with

$$\begin{aligned}
\mathbf{E}\left[(\mathrm{Good}_n)^2\right] &\leq n^{3/2}\mathrm{e}^{-z}\Big\{1 + c_{15}\sum_{k=1}^{n} \mathrm{e}^{-e_k}\Big\}\mathbf{Q}\left(w_n \text{ is a good vertex}\right) \\
&\leq c_{16}n^{3/2}\mathrm{e}^{-z}\mathbf{Q}\left(w_n \text{ is a good vertex}\right).
\end{aligned}$$

Now, observe that $\mathbf{Q}(w_n \text{ is a good vertex}) \leq \mathbf{Q}(w_n \in \mathcal{Z}^{z,0}) \leq c_{17}n^{-3/2}$ by Definition 3.4 and equation (2.7). Hence

$$\mathbf{E}\left[(\mathrm{Good}_n)^2\right] \leq c_{18}\mathrm{e}^{-z}. \tag{3.17}$$

By the Paley-Zygmund inequality, we have $\mathbf{P}(\mathrm{Good}_n \geq 1) \geq \frac{\mathbf{E}[\mathrm{Good}_n]^2}{\mathbf{E}[(\mathrm{Good}_n)^2]}$ which is greater than $c_{19}\mathrm{e}^{-z}$ by (3.15) and (3.17). We conclude by observing that if $\mathrm{Good}_n \geq 1$ then $M_n^{\mathrm{kill}} \leq \frac{3}{2}\ln n - z$. □

3.2 Proof of Proposition 3.1

Lemma 3.5 already gives the good rate of decay, but we want to strenghten it into an equivalent as $z \to \infty$. We recall that $m^{\mathrm{kill},(n)}$ is chosen uniformly among the particles that realize the minimum. We introduced the notation $\mathcal{Z}^{z,L}$ in Definition 3.4. By (3.12), we can assume that $m^{\mathrm{kill},(n)} \in \mathcal{Z}^{z,L}$ when $M_n^{\mathrm{kill}} \in I_n(z)$. The first step of the proof is to give a representation of the probability $\mathbf{P}\left(M_n^{\mathrm{kill}} \in I_n(z),\, m^{\mathrm{kill},(n)} \in \mathcal{Z}^{z,L}\right)$ in terms of the spine decomposition presented in Proposition 2.1.

Lemma 3.6 *For any $z \geq 0$, $L \geq 0$, and $n \geq 1$, we have*

$$\mathbf{P}\left(M_n^{\text{kill}} \in I_n(z),\, m^{\text{kill},(n)} \in \mathcal{Z}^{z,L}\right) = \mathbf{E}_{\mathbf{Q}}\left[\frac{\mathrm{e}^{V(w_n)}\mathbf{1}_{\{V(w_n)=M_n^{\text{kill}}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}}\}}},\, w_n \in \mathcal{Z}^{z,L}\right]. \tag{3.18}$$

Proof. We observe that

$$\begin{aligned}\mathbf{P}\left(M_n^{\text{kill}} \in I_n(z),\, m^{\text{kill},(n)} \in \mathcal{Z}^{z,L}\right) &= \mathbf{E}\left[\sum_{|u|=n}\mathbf{1}_{\{u=m^{\text{kill},(n)},u\in\mathcal{Z}^{z,L}\}}\right] \\ &= \mathbf{E}\left[\frac{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}},\,u\in\mathcal{Z}^{z,L}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}}\}}}\right].\end{aligned}$$

Using the measure $\mathbf{Q}$, it follows from Proposition 2.1 (ii) that

$$\mathbf{E}\left[\frac{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}},\,u\in\mathcal{Z}^{z,L}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}}\}}}\right] = \mathbf{E}_{\mathbf{Q}}\left[\frac{\mathrm{e}^{V(w_n)}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}}\}}}\mathbf{1}_{\{V(w_n)=M_n^{\text{kill}},w_n\in\mathcal{Z}^{z,L}\}}\right],$$

which completes the proof. □

For b integer, we define the event $\mathcal{E}_n$ by

$$\mathcal{E}_n = \mathcal{E}_n(z,b) := \{\forall\, k \leq n-b,\, \forall v \in \Omega(w_k),\, \min_{u\geq v, |u|=n} V(u) > a_n(z)\} \tag{3.19}$$

where, as before, $\Omega(w_k)$ denotes the set of brothers of w_k. On the event $\mathcal{E}_n \cap \{M_n^{\text{kill}} \in I_n(z)\}$, we are sure that any particle located at the minimum separated from the spine after the time $n-b$. The following lemma will be proved in subsection 3.3.

Lemma 3.7 *Let $\eta > 0$ and $L > 0$. There exist $A > 0$ and $B \geq 1$ such that for any $b \geq B$, $n \geq 1$ and $z \geq A$,*

$$\mathbf{Q}((\mathcal{E}_n)^c,\, w_n \in \mathcal{Z}^{z,L}) \leq \eta n^{-3/2}. \tag{3.20}$$

Let, for $x \geq 0$, $L > 0$, and $b \geq 1$

$$F_{L,b}(x) := \mathbf{E}_{\mathbf{Q}_x}\left[\frac{\mathrm{e}^{V(w_b)-L}\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}},\, \min_{k\in[0,b]} V(w_k) \geq 0,\, V(w_b) \in [L-1, L)\right]. \tag{3.21}$$

We stress that M_b which appears in the definition of $F_{L,b}(x)$ is the minimum at time b of the **non-killed** branching random walk. Then, define

$$C_{L,b} := \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} \int_{x\geq 0} F_{L,b}(x) R_-(x) dx, \tag{3.22}$$

where C_-, C_+ and $R_-(x)$ were defined in subsection 2.2. We recall that, by Proposition 2.1 (iii), the spine has the law of $(S_n)_{n\ge 0}$. In (3.21), we see that $\frac{\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}$ is smaller than 1, and $\mathrm{e}^{V(w_b)-L}\le 1$. Hence, $|F_{L,b}(x)|\le \mathbf{P}(S_b\le L-x)=:\overline{F}(x)$ which is non-increasing in x, and $\int_{x\ge 0}\overline{F}(x)xdx=\frac{1}{2}\mathbf{E}[(L-S_b)^2\mathbf{1}_{\{S_b\le L\}}]<\infty$. Moreover, observe that

$$F_{L,b}(x)=\mathbf{E_Q}\left[\frac{\mathrm{e}^{V(w_b)+x-L}\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}\mathbf{1}_{\{\min_{k\in[0,b]}V(w_k)\ge -x,\,V(w_b)\in[-x+L-1,-x+L)\}}\right].$$

The fraction in the expectation is smaller than 1. Using the identity $|\mathbf{1}_E-a\mathbf{1}_F|\le 1-a+|\mathbf{1}_E-\mathbf{1}_F|$ for $a\in(0,1)$, it yields that for $x\ge 0$, $\varepsilon>0$ and any $y\in[x,x+\varepsilon]$,

$$\begin{aligned}
&|F_{L,b}(y)-F_{L,b}(x)|\\
\le\;& \mathbf{E_Q}\left[\left|\mathrm{e}^{-(y-x)}\mathbf{1}_{\{\min_{k\in[0,b]}V(w_k)\ge -y,\,V(w_b)+y-L\in[-1,0)\}}-\mathbf{1}_{\{\min_{k\in[0,b]}V(w_k)\ge -x,\,V(w_b)+x-L\in[-1,0)\}}\right|\right]\\
\le\;& 1-\mathrm{e}^{-\varepsilon}+E_{\mathbf{Q}}\left[\mathbf{1}_{\{\min_{k\in[0,b]}V(w_k)+x\in[-\varepsilon,0)\}}+\mathbf{1}_{\{V(w_b)+x-L\in[-1-\varepsilon,-1)\cup(-\varepsilon,0]\}}\right].
\end{aligned}$$

We easily deduce that $F_{L,b}$ is Riemann integrable. Therefore, $F_{L,b}$ satisfies the conditions of Lemma 2.2.

We want to prove that the expectation in (3.18) behaves like e^{-z} with some constant factor, as $z\to\infty$. By Lemma 3.7, we can restrict to the event $\mathcal{E}_n$. The next lemma shows that the expectation on this event is then equivalent to $C_{L,b}\mathrm{e}^{-z}$.

Lemma 3.8 *Let $L>0$ and $\eta>0$. Let A and B be as in Lemma 3.7. For any $b\ge B$, we have for n large enough, and $z\in[A,\ln n]$,*

$$\left|\mathrm{e}^{z}\mathbf{E_Q}\left[\frac{\mathrm{e}^{V(w_n)}\mathbf{1}_{\{V(w_n)=M_n^{\mathrm{kill}}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}\}}},\,w_n\in\mathcal{Z}^{z,L},\,\mathcal{E}_n\right]-C_{L,b}\right|\le\eta. \tag{3.23}$$

Proof. Let L, η, A, B be as in the lemma. Take $n\ge 1$, $b\ge B$ and $z\ge A$. We denote by $\mathbf{Q}_{(3.23)}$ the expectation in (3.23). By the Markov property at time $n-b$ (for $n>2b$), we have

$$\mathbf{Q}_{(3.23)}=\mathbf{E_Q}\Big[F^{\mathrm{kill}}(V(w_{n-b})),\,V(w_\ell)\ge d_\ell,\,\forall\,\ell\le n-b,\,\mathcal{E}_n\Big]$$

where $d_\ell:=0$ if $0\le\ell\le n/2$ and $d_\ell:=\max(a_n(z)-L,0)$ if $n/2<\ell\le n$, and F^{kill} is defined by

$$F^{\mathrm{kill}}(x):=E_{\mathbf{Q}_x}\left[\frac{\mathrm{e}^{V(w_b)}\mathbf{1}_{\{V(w_b)=M_b^{\mathrm{kill}}\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b^{\mathrm{kill}}\}}},\,\min_{k\in[0,b]}V(w_k)\ge a_n(z+L),\,V(w_b)\in I_n(z)\right]. \tag{3.24}$$

Notice that $F^{\mathrm{kill}}(x) \le n^{3/2}\mathrm{e}^{-z}\mathbf{Q}_x(\min_{k\in[0,b]} V(w_k) \ge a_n(z+L),\, V(w_b)\in I_n(z))$. Hence

$$\begin{aligned}
&\Big|\mathbf{Q}_{(3.23)} - \mathbf{E}_{\mathbf{Q}}\Big[F^{\mathrm{kill}}(V(w_{n-b})),\, V(w_\ell)\ge d_\ell,\, \forall\, \ell \le n-b\Big]\Big| \\
=\ & \mathbf{E}_{\mathbf{Q}}\Big[F^{\mathrm{kill}}(V(w_{n-b})),\, V(w_\ell)\ge d_\ell,\, \forall\, \ell\le n-b,\, (\mathcal{E}_n)^c\Big] \\
\le\ & n^{3/2}\mathrm{e}^{-z}\mathbf{E}_{\mathbf{Q}}\Big[\mathbf{Q}_{V(w_{n-b})}(\min_{k\in[0,b]} V(w_k)\ge a_n(z+L),\, V(w_b)\in I_n(z))\mathbf{1}_{\{V(w_\ell)\ge d_\ell,\forall\,\ell\le n-b\},\,(\mathcal{E}_n)^c}\Big].
\end{aligned}$$

By the Markov property, the term

$$\mathbf{E}_{\mathbf{Q}}\Big[\mathbf{Q}_{V(w_{n-b})}(\min_{k\in[0,b]} V(w_k)\ge a_n(z+L),\, V(w_b)\in I_n(z))\mathbf{1}_{\{V(w_\ell)\ge d_\ell,\forall\,\ell\le n-b\},\,(\mathcal{E}_n)^c}\Big]$$

is equal to $\mathbf{Q}\Big[w_n\in\mathcal{Z}^{z,L},\, (\mathcal{E}_n)^c\Big] \le \eta n^{-3/2}$ by our choice of A and B. Therefore,

$$\Big|\mathbf{Q}_{(3.23)} - \mathbf{E}_{\mathbf{Q}}\Big[F^{\mathrm{kill}}(V(w_{n-b})),\, V(w_\ell)\ge d_\ell,\, \forall\,\ell\le n-b\Big]\Big| \le \eta\mathrm{e}^{-z}. \tag{3.25}$$

Recall the definition of $F_{L,b}$ in (3.21). We would like to replace $F^{\mathrm{kill}}(x)$ by $n^{3/2}\mathrm{e}^{-z}F_{L,b}(x - a_n(z+L))$. We notice that

$$\begin{aligned}
& n^{3/2}\mathrm{e}^{-z}F_{L,b}(x-a_n(z+L)) \\
=\ & \mathbf{E}_{\mathbf{Q}_x}\left[\frac{\mathrm{e}^{V(w_b)}\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}},\, \min_{k\in[0,b]} V(w_k)\ge a_n(z+L),\, V(w_b)\in I_n(z)\right].
\end{aligned}$$

We observe that the only difference with (3.24) is that the branching random walk is not killed anymore. Since $\left|\frac{\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}} - \frac{\mathbf{1}_{\{V(w_b)=M_b^{\mathrm{kill}}\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b^{\mathrm{kill}}\}}}\right|$ is always smaller than 1 and is equal to zero if no particle touched the barrier 0, we have that, for any $H\ge 0$ such that $H\le a_n(z+L)$,

$$\left|\frac{\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}} - \frac{\mathbf{1}_{\{V(w_b)=M_b^{\mathrm{kill}}\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b^{\mathrm{kill}}\}}}\right| \le \mathbf{1}_{\{\exists |u|\le b:\, V(u)\le a_n(z+L+H)\}}.$$

Consequently,

$$\begin{aligned}
& \Big|F^{\mathrm{kill}}(x) - n^{3/2}\mathrm{e}^{-z}F_{L,b}(x-a_n(z+L))\Big| \\
\le\ & \mathbf{E}_{\mathbf{Q}_x}\left[\mathrm{e}^{V(w_b)}\mathbf{1}_{\{\exists|u|\le b:\, V(u)\le a_n(z+L+H)\}},\, \min_{k\in[0,b]} V(w_k)\ge a_n(z+L),\, V(w_b)\in I_n(z)\right] \\
\le\ & n^{3/2}\mathrm{e}^{-z}\mathbf{E}_{\mathbf{Q}_x}\left[\mathbf{1}_{\{\exists|u|\le b:\, V(u)\le a_n(z+L+H)\}},\, \min_{k\in[0,b]} V(w_k)\ge a_n(z+L),\, V(w_b)\in I_n(z)\right] \\
=\ & n^{3/2}\mathrm{e}^{-z}G_H(x-a_n(z+L))
\end{aligned}$$

with

$$G_H(y) := \mathbf{Q}_y\left(\{\exists |u|\le b\,:\, V(u)\le -H\}\cap\{\min_{k\in[0,b]} V(w_k)\ge 0,\, V(w_b)\in[L-1,L)\}\right).$$

It shows that, for any $H\in[0,a_n(z+L)]$,

$$\begin{aligned}&\mathbf{E_Q}\Big[\Big|F^{\rm kill}(V(w_{n-b})) - n^{3/2}\mathrm{e}^{-z}F_{L,b}(V(w_{n-b})-a_n(z+L))\Big|\mathbf{1}_{\{V(w_\ell)\ge d_\ell,\,\forall\,\ell\le n-b\}}\Big]\\ \le\;& n^{3/2}\mathrm{e}^{-z}\mathbf{E_Q}\Big[G_H(V(w_{n-b})-a_n(z+L))\mathbf{1}_{\{V(w_\ell)\ge d_\ell,\,\forall\,\ell\le n-b\}}\Big].\end{aligned}$$

We choose H such that $\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\int_{y\ge 0}G_H(y)R_-(y)dy\le \eta/2$. The function G_H satisfies the conditions of Lemma 2.2 for the same reasons than $F_{L,b}$ does. By Lemma 2.2, it yields that

$$\mathbf{E_Q}\Big[\Big|F^{\rm kill}(V(w_{n-b})) - n^{3/2}\mathrm{e}^{-z}F_{L,b}(V(w_{n-b})-a_n(z+L))\Big|\mathbf{1}_{\{V(w_\ell)\ge d_\ell,\,\forall\,\ell\le n-b\}}\Big]\le \eta\mathrm{e}^{-z}$$

for n large enough and $z\in[0,\ln n]$. Combined with (3.25), we get

(3.26)
$$\Big|\mathbf{Q}_{(3.23)} - n^{3/2}\mathrm{e}^{-z}\mathbf{E_Q}\Big[F_{L,b}(V(w_{n-b})-a_n(z+L)),\, V(w_\ell)\ge d_\ell,\,\forall\, 0\le\ell\le n-b\Big]\Big|\le 2\eta\mathrm{e}^{-z}.$$

Recall the definition of $C_{L,b}$ in (3.22). We apply again Lemma 2.2 to see that

$$\mathbf{E_Q}\Big[F_{L,b}(V(w_{n-b})-a_n(z+L)),\, V(w_\ell)\ge d_\ell,\,\forall\, 0\le\ell\le n-b\Big]\sim\frac{C_{L,b}}{n^{3/2}}$$

as $n\to\infty$ uniformly in $z\in[0,\ln n]$. Consequently, we have for n large enough and $z\in[0,\ln n]$,

$$\Big|n^{3/2}\mathrm{e}^{-z}\mathbf{E_Q}\Big[F_{L,b}(V(w_{n-b})-a_n(z+L)),\, V(w_\ell)\ge d_\ell,\,\forall\, 0\le\ell\le n-b\Big]-\mathrm{e}^{-z}C_{L,b}\Big|\le\eta\mathrm{e}^{-z}.$$

The lemma follows from (3.26). □

We now have the tools to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\varepsilon>0$. By (3.12), there exists $L_0\ge 0$ such that for any $L\ge L_0$, $z\ge 0$ and $n\ge 1$

$$\mathbf{P}(m^{{\rm kill},(n)}\notin\mathcal{Z}^{z,L},\, M_n^{\rm kill}\in I_n(z))\le\varepsilon\mathrm{e}^{-z}.$$

By Lemma 3.6, it yields that for $L\ge L_0$

$$\left|\mathbf{P}(M_n^{\rm kill}\in I_n(z)) - \mathbf{E_Q}\left[\frac{\mathrm{e}^{V(w_n)}\mathbf{1}_{\{V(w_n)=M_n^{\rm kill}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\rm kill}\}}},\, w_n\in\mathcal{Z}^{z,L}\right]\right|\le\varepsilon\mathrm{e}^{-z}.$$

For $\eta > 0$ and $L \geq L_0$, take $B = B(L) \geq 1$ and $A = A(L) > 0$ as in Lemma 3.7. We have then

$$\mathbf{E_Q}\left[\frac{\mathrm{e}^{V(w_n)}\mathbf{1}_{\{V(w_n)=M_n^{\text{kill}}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}}\}}},\, w_n \in \mathcal{Z}^{z,L},\, \mathcal{E}_n^c\right] \leq n^{3/2}\mathrm{e}^{-z}\mathbf{Q}(w_n \in \mathcal{Z}^{z,L},\, \mathcal{E}_n^c) \leq \eta \mathrm{e}^{-z}$$

for $b \geq B$, $z \geq A$ and $n \geq 1$. Consequently,

$$\left|\mathbf{P}(M_n^{\text{kill}} \in I_n(z)) - \mathbf{E_Q}\left[\frac{\mathrm{e}^{V(w_n)}\mathbf{1}_{\{V(w_n)=M_n^{\text{kill}}\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\text{kill}}\}}},\, w_n \in \mathcal{Z}^{z,L},\, \mathcal{E}_n\right]\right| \leq (\varepsilon+\eta)\mathrm{e}^{-z}.$$

By Lemma 3.8, we get that for $L \geq L_0$, $b \geq B(L)$, n large enough and $z \in [A(L), \ln n]$,

$$\left|\mathrm{e}^{z}\mathbf{P}(M_n^{\text{kill}} \in I_n(z)) - C_{L,b}\right| \leq (\varepsilon + 2\eta). \tag{3.27}$$

We still call $\mathbf{Q}_{(3.23)}$ the expectation in the left-hand side of (3.23). We introduce

$$\begin{aligned} C_{L,b}^- &:= \liminf_{z\to\infty}\liminf_{n\to\infty} \mathrm{e}^{z}\mathbf{Q}_{(3.23)}, \\ C_{L,b}^+ &:= \limsup_{z\to\infty}\limsup_{n\to\infty} \mathrm{e}^{z}\mathbf{Q}_{(3.23)}. \end{aligned}$$

In particular, taking the limits in $n \to \infty$ then $z \to \infty$ in (3.23), we have, for $b \geq B(L)$

$$C_{L,b} - \eta \leq C_{L,b}^- \leq C_{L,b}^+ \leq C_{L,b} + \eta.$$

Notice that $\mathcal{E}_n$ (hence $\mathbf{Q}_{(3.23)}$) is increasing in b. It implies that $C_{L,b}^-$ and $C_{L,b}^+$ are both increasing in b. Let C_L^- and C_L^+ be respectively the (possibly zero or infinite) limits of $C_{L,b}^-$ and $C_{L,b}^+$ when $b \to \infty$. By (3.27), we know that $C_{L,b} \leq \mathrm{e}^{z}\mathbf{P}(M_n^{\text{kill}} \in I_n(z)) + \varepsilon + 2\eta$ for $b \geq B(L)$, hence $C_{L,b} \leq c_{11} + \varepsilon + 2\eta$ by Lemma 3.5. It implies that C_L^- and C_L^+ are finite and bounded uniformly in $L \geq L_0$. We have then

$$\limsup_{b\to\infty} C_{L,b} - \eta \leq C_L^- \leq C_L^+ \leq \liminf_{b\to\infty} C_{L,b} + \eta.$$

Letting η go to 0, it yields that $C_{L,b}$ has a limit as $b \to \infty$, that we denote by $C(L) = C_L^- = C_L^+$. Similarly, we see that $\mathbf{Q}_{(3.23)}$ is increasing in L. It gives that $C(L)$ admits a limit as $L \to \infty$, that we denote by C_2, which is necessarily finite. However we do not know yet if $C_2 > 0$. Let $L > L_0$ such that $|C_2 - C(L)| \leq \eta$ and $b \geq B(L)$ such that $|C_{L,b} - C(L)| \leq \eta$. Then, by (3.27), there exists $N \geq 1$ such that for any $n \geq N$ and $z \in [A(L), \ln n]$, we have

$$\left|\mathrm{e}^{z}\mathbf{P}(M_n^{\text{kill}} \in I_n(z)) - C_2\right| \leq \varepsilon + 4\eta.$$

It remains to show that $C_2 > 0$. We see that, necessarily,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\mathrm{e}^{z}\mathbf{P}(M_n^{\text{kill}} \leq \frac{3}{2}\log n - z) - \frac{C_2}{1-\mathrm{e}^{-1}}\right| = 0.$$

We know then that $C_2 > 0$ by the lower bound obtained in Lemma 3.5. □

3.3 Proof of Lemma 3.7

We present here the postponed proof of Lemma 3.7.

Proof of Lemma 3.7. We follow the same strategy as for Lemma 3.5. Let $\eta > 0$. To avoid superfluous notation, we prove the lemma for $L = 0$ (the general case works similarly). Recall the definition of $\mathcal{E}_n = \mathcal{E}_n(z,b)$ in (3.19). We want to show that $\mathbf{P}(\mathcal{E}_n^c,\, w_n \in \mathcal{Z}^{z,0}) \le \eta n^{-3/2}$ when b and z are large enough. As before, we take the numbers $(d_k, 0 \le k \le n)$ and $(e_k, 0 \le k \le n)$ such that $d_k := 0$ if $0 \le k \le n/2$, $d_k := \max(\frac{3}{2}\ln n - z - 1, 0)$ if $n/2 < k \le n$ and

$$e_k = e_k^{(n)} := \begin{cases} k^{1/12}, & \text{if } 0 \le k \le \frac{n}{2}, \\ (n-k)^{1/12}, & \text{if } \frac{n}{2} < k \le n. \end{cases}$$

We say again that $|u| = n$ is a good vertex if $u \in \mathcal{Z}^{z,0}$ and

$$\sum_{v \in \Omega(u_k)} \mathrm{e}^{-(V(v)-d_k)} \Big\{ 1 + (V(v) - d_k)_+ \Big\} \le B\mathrm{e}^{-e_k} \qquad \forall\, 1 \le k \le n$$

with B such that, for $n \ge 1$ and $z \ge 0$

$$\mathbf{Q}(w_n \in \mathcal{Z}^{z,0},\, w_n \text{ is not a good vertex}) \le \frac{\eta}{n^{3/2}} \tag{3.28}$$

(see Lemma C.1). Let as before $\Omega(w_k)$ be the set of brothers of w_k and $\mathcal{G}_\infty$ be the sigma-algebra generated by $\{w_k,\, V(w_k),\, \Omega(w_k),\, (V(u))_{u\in\Omega(w_k)},\, k \ge 1\}$. Recall the law of the branching random walk under $\mathbf{Q}$ in Proposition 2.1 (i). For $\mathcal{E}_n$ to happen, every brother of the spine at generation less than $n - b$ must have its descendants at time n greater than $a_n(z)$. In other words,

$$\mathbf{Q}((\mathcal{E}_n)^c,\, w_n \text{ is a good vertex}) = \mathbf{Q}\left[1 - \prod_{k=1}^{n-b} \prod_{u \in \Omega(w_k)} p\,(u,z),\, w_n \text{ is a good vertex} \right] \tag{3.29}$$

where $p\,(u,z) := \mathbf{P}_{V(u)}(M_{n-|u|}^{\text{kill}} > a_n(z))$ is the probability that the killed branching random walk rooted at u has its minimum greater than $a_n(z)$ at time $n - |u|$. From Lemma 3.3, we see that, if $|u| \le n/2$, then

$$1 - p\,(u,z) \le c_{20}(1 + V(u)_+)\mathrm{e}^{-z-V(u)}.$$

Since w_n is a good vertex, we have for $k \le n/2$ (hence $d_k = 0$), $\sum_{u\in\Omega(w_k)}(1+V(u)_+)\mathrm{e}^{-V(u)} \le B\mathrm{e}^{-e_k} = B\mathrm{e}^{-k^{1/12}}$. It implies that for z large enough, and $1 \le k \le n/2$,

$$\prod_{u\in\Omega(w_k)} p\,(u,z) \ge \exp\left(-c_{21}\mathrm{e}^{-z}\mathrm{e}^{-k^{1/12}} \right).$$

It yields that

$$\prod_{k=1}^{\lfloor n/2 \rfloor} \prod_{u\in\Omega(w_k)} p\,(u,z) \geq \exp\left(-c_{21}\mathrm{e}^{-z}\sum_{k=1}^{\lfloor n/2 \rfloor}\mathrm{e}^{-k^{1/12}}\right) \geq \exp(-c_{22}\mathrm{e}^{-z}).$$

Therefore, there exists $A_1 > 0$ such that for any $z \geq A_1$, and $n \geq 1$

$$\prod_{k=1}^{\lfloor n/2 \rfloor} \prod_{u\in\Omega(w_k)} p\,(u,z) \geq (1-\eta)^{1/2}. \tag{3.30}$$

If $k > n/2$, we simply observe that if $M_\ell^{\mathrm{kill}} \leq x$, a fortiori $M_\ell \leq x$. Since W_n (defined in (2.2)) is a martingale, we have $1 = \mathbf{E}[W_\ell] \geq \mathbf{E}[\mathrm{e}^{-M_\ell}] \geq \mathrm{e}^{-x}\mathbf{P}(M_\ell \leq x)$ for any $\ell \geq 1$ and $x \in \mathbb{R}$. We get that

$$1 - p\,(u,z) \leq \mathbf{P}(M_{n-|u|} < a_n(z) - V(u)) \leq \mathrm{e}^{a_n(z)}\mathrm{e}^{-V(u)}.$$

We rewrite it (we have $z \geq 0$), $1 - p\,(u,z) \leq n^{3/2}\mathrm{e}^{-V(u)} = \mathrm{e}^{-(V(u)-d_k)}$ for $n/2 < k \leq n$. Since w_n is a good vertex, we get that $\prod_{u\in\Omega(w_k)} p\,(u,z) \geq \mathrm{e}^{-c_{23}\mathrm{e}^{-e_k}} = \mathrm{e}^{-c_{23}(n-k)^{1/12}}$. Consequently,

$$\prod_{k=\lfloor n/2 \rfloor+1}^{n-b} \prod_{u\in\Omega(w_k)} p\,(u,z) \geq \mathrm{e}^{-c_{23}\sum_{k=\lfloor n/2 \rfloor+1}^{n-b}\mathrm{e}^{-(n-k)^{1/12}}}.$$

It yields that there exists $B \geq 1$ such that for any $b \geq B$ and any $n \geq 1$, we have,

$$\prod_{k=\lfloor n/2 \rfloor+1}^{n-b} \prod_{u\in\Omega(w_k)} p\,(u,z) \geq (1-\eta)^{1/2}. \tag{3.31}$$

In view of (3.30) and (3.31), we have for $b \geq B$, $z \geq A_1$ and $n \geq 1$, $\prod_{k=1}^{n-b}\prod_{u\in\Omega(w_k)} p\,(u,z) \geq (1-\eta)$. Plugging it into (3.29) yields that

$$\mathbf{Q}((\mathcal{E}_n)^c,\, w_n \text{ is a good vertex}) \leq \eta\mathbf{Q}\,(w_n \text{ is a good vertex }) \leq \eta\mathbf{Q}\left(w_n \in \mathcal{Z}^{z,0}\right).$$

It follows from (3.28) that

$$\mathbf{Q}((\mathcal{E}_n)^c,\, w_n \in \mathcal{Z}^{z,0}) \leq \eta(\mathbf{Q}\left(w_n \in \mathcal{Z}^{z,0}\right) + n^{-3/2}).$$

Remember that the spine behaves as a centered random walk. Then apply (2.7) to see that $\mathbf{Q}\,(w_n \in \mathcal{Z}^{z,0}) \leq c_{24}n^{-3/2}$, which completes the proof of the lemma. □

4 Tail distribution of the minimum of the BRW

We prove a slightly stronger version of Proposition 1.3.

Proposition 4.1 *Let C_1 be as in Proposition 1.2 and c_0 as in (2.10). For any $\varepsilon > 0$, there exists $N \geq 1$ and $A > 0$ such that for any $n \geq N$ and $z \in [A, 2\ln\ln n]$,*

$$\Big|\frac{\mathrm{e}^z}{z}\mathbf{P}(M_n \leq \frac{3}{2}\ln n - z) - C_1 c_0\Big| \leq \varepsilon.$$

We introduce some notation. To go from the tail distribution of M_n^{kill} to the one of M_n, we have to control excursions inside the negative axis that can appear at the beginning of the branching random walk. For $z \geq A \geq 0$ and $n \geq 1$, we define the set

$$\mathcal{S}_A := \{u \in \mathbb{T} \,:\, \min_{k \leq |u|-1} V(u_k) > V(u) \geq A - z, |u| \leq (\ln n)^{10}\}. \tag{4.1}$$

We notice that $\mathcal{S}_A$ depends on n and z, but we omit to write this dependency in the notation for sake of concision. For $z \geq 0$ and $u \in \mathcal{S}_A$, we define the indicator $\mathcal{B}_{n,z}(u)$ equal to 1 if and only if the branching random walk emanating from u and killed below $V(u)$ has its minimum below $\frac{3}{2}\ln n - z$. Equivalently,

Definition 4.2 *For $u \in \mathcal{S}_A$, we call $\mathcal{B}_{n,z}(u)$ the indicator of the event that there exists $|v| = n$, $v > u$ such that $V(v_\ell) \geq V(u)$, $\forall |u| \leq \ell \leq n$ and $V(v) \leq \frac{3}{2}\ln n - z$.*

Finally, let for $|v| \geq 1$,

$$\xi(v) := \sum_{w \in \Omega(v)} (1 + (V(w) - V(\overleftarrow{v}))_+)\mathrm{e}^{-(V(w) - V(\overleftarrow{v}))} \tag{4.2}$$

where $\overleftarrow{v}$ denotes the parent of v (and $y_+ := \max(y, 0)$). To avoid some extra integrability conditions, we are led to consider vertices $u \in \mathcal{S}_A$ which behave 'nicely', meaning that $\xi(u_k)$ is not too big along the path $\{u_1, \ldots, u_{|u|} = u\}$. The first subsection controls the set $\mathcal{S}_A$. Proposition 1.3 is then proved in subsection 4.2.

4.1 The branching random walk at the beginning

We will see that $\mathbf{P}(M_n \leq \frac{3}{2}\ln n - z)$ is comparable to the probability that there exists $u \in \mathcal{S}_A$ such that $\mathcal{B}_{n,z}(u) = 1$. The lemmas in this section are used to give an equivalent of this probability. As usual, we will use a second moment argument. Lemmas 4.3 and 4.4 give bounds respectively on the first moment and second moment of the number of such vertices u.

Lemma 4.3 *(i) Recall that $R(x)$ is the renewal function of $(S_n)_{n\geq 0}$ defined in (2.9). Let $\varepsilon > 0$ and C_1 be the constant in Proposition 1.2. There exists $A \geq 0$ such that for n large enough and $z \in [A, \ln^{1/5}(n)]$,*

$$\left| \frac{e^z}{R(z-A)} \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \right] - C_1 \right| \leq \varepsilon. \tag{4.3}$$

(ii) For any $|u| \geq 1$, let $\mathcal{T}(u) := \{\forall 1 \leq k \leq |u| : \xi(u_k) < e^{(V(u_{k-1})+z-A)/2}\}$. We have

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \mathbf{1}_{\mathcal{T}(u)^c} \right] = o(z)e^{-z}$$

uniformly in $A \geq 0$ and $n \geq 1$.

Proof. Let $k \leq (\ln n)^{10}$. By the Markov property at time k, we have

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \mathbf{1}_{\{|u|=k\}} \right] = \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathbf{1}_{\{|u|=k\}} \mathbf{P}\left(M^{\text{kill}}_{n-k} \leq a_n(z+r) \right)_{r=V(u)} \right] \tag{4.4}$$

where we recall that M^{kill}_{n-k} is the minimum of the branching random walk killed below zero at time $n-k$. We observe that $V(u) \in [A-z, 0]$ when $u \in \mathcal{S}_A$. We check by Corollary 3.2 that there exist $A > 0$ and $N \geq 1$ such that for any $n \geq N$, $k \leq (\ln(n))^{10}$ and $z+r \in [A, \ln(n)/2]$,

$$\left| e^{z+r} \mathbf{P}\left(M^{\text{kill}}_{n-k} \leq a_n(z+r) \right) - C_1 \right| \leq \varepsilon.$$

Plugging it into (4.4), it implies that, for $n \geq N$, $k \leq (\ln n)^{10}$ and $z \in [A, \ln(n)/4]$,

$$\left| e^z \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \mathbf{1}_{\{|u|=k\}} \right] - C_1 \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} e^{-V(u)} \mathbf{1}_{\{|u|=k\}} \right] \right| \leq \varepsilon \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} e^{-V(u)} \mathbf{1}_{\{|u|=k\}} \right].$$

From the definition of $\mathcal{S}_A$, we observe that by (2.1), $\mathbf{E}\left[\sum_{u\in\mathcal{S}_A} e^{-V(u)} \mathbf{1}_{\{|u|=k\}}\right] = \mathbf{P}(S_k \geq A-z,\, S_k < S_\ell,\, \forall\, 0 \leq \ell < k-1)$. Hence, we can rewrite the inequality above as

$$\begin{aligned} &\left| e^z \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \mathbf{1}_{\{|u|=k\}} \right] - C_1 \mathbf{P}(S_k \geq A-z,\, S_k < S_\ell,\, \forall\, 0 \leq \ell < k-1) \right| \\ \leq\;& \varepsilon \mathbf{P}(S_k \geq A-z,\, S_k < S_\ell,\, \forall\, 0 \leq \ell < k-1). \end{aligned}$$

By definition of the renewal function $R(x)$, we have $R(z-A) = \sum_{k\geq 0} \mathbf{P}(S_k \geq A-z,\, S_k < S_\ell,\, \forall\, 0 \leq \ell < k-1)$. Therefore, summing over $k \leq (\ln n)^{10}$ (and since $|u| \leq (\ln n)^{10}$ if

$u \in \mathcal{S}_A$), we get

$$\begin{aligned}&\left|\mathrm{e}^z\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u)\right]-C_1R(z-A)\right|\\ \le\ & \varepsilon R(z-A)+C_1\sum_{k>(\ln n)^{10}}\mathbf{P}(S_k\ge A-z,\, S_k<S_\ell,\,\forall\, 0\le \ell\le k-1).\end{aligned}$$

Observe that

$$\begin{aligned}\mathbf{P}(S_k\ge A-z,\, S_k<S_\ell,\,\forall\, 0\le\ell\le k-1) &\le \mathbf{P}(S_k\in(A-z,0],\, \min_{\ell<k}S_\ell\ge A-z)\\ &\le c_{24}(1+z-A)^3(1+k)^{-3/2}\\ &\le c_{24}(1+\ln n)^3(1+k)^{-3/2}\end{aligned}$$

by (2.6), for $n\ge 1$ and $z\in[A,\ln n]$. Therefore, $\sum_{k>(\ln n)^{10}}\mathbf{P}(S_k\ge A-z,\, S_k<S_\ell,\,\forall\, 0\le \ell\le k-1)\le c_{25}\ln(n)^{-2}\le\varepsilon$ for n large enough. Since $R(z-A)$ is always bigger than 1, we obtain for $n\ge N$, and $z\in[A,\ln n]$,

$$\left|\mathrm{e}^z\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u)\right]-C_1R(z-A)\right| \;\le\; \varepsilon R(z-A)(1+C_1).$$

This ends the proof of (i). Similarly, we have by the Markov property

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u)\mathbf{1}_{\mathcal{T}(u)^c}\right]=\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathbf{1}_{\mathcal{T}(u)^c}\mathbf{P}\left(M^{\mathrm{kill}}_{n-|u|}\le a_n(z+r)\right)_{r=V(u)}\right].$$

By Lemma 3.5, we have for any $n\ge 1$, $k\le(\ln n)^{10}$ and $z+r\ge 0$

$$\mathbf{P}\left(M^{\mathrm{kill}}_{n-k}\le a_n(z+r)\right)\le c_{26}\mathrm{e}^{-z-r}.$$

Remember that if $u\in\mathcal{S}_A$, then $|u|\le(\ln n)^{10}$ and $z+V(u)\ge A\ge 0$. It implies that

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u)\mathbf{1}_{\mathcal{T}(u)^c}\right]\le c_{26}\mathrm{e}^{-z}\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathbf{1}_{\mathcal{T}(u)^c}\mathrm{e}^{-V(u)}\right]. \tag{4.5}$$

At this stage, we make use of the measure $\mathbf{Q}$. We have

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}_A}\mathbf{1}_{\mathcal{T}(u)^c}\mathrm{e}^{-V(u)}\right]=\sum_{k=0}^{\lfloor(\ln n)^{10}\rfloor}\mathbf{Q}(w_k\in\mathcal{S}_A,\mathcal{T}(w_k)^c). \tag{4.6}$$

We see that $\mathbf{Q}(w_0 \in \mathcal{S}_A, \mathcal{T}(w_0)^c) \leq \mathbf{Q}(\mathcal{T}(w_0)^c) = 0$. For $k \geq 1$, we have by definition of the event $\mathcal{T}(w_k)$ that $\mathbf{1}_{\mathcal{T}(w_k)^c} \leq \sum_{\ell=1}^k \mathbf{1}_{\{\xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}}$. It follows that

$$\mathbf{Q}(w_k \in \mathcal{S}_A, \mathcal{T}(w_k)^c) \leq \sum_{\ell=1}^{k} \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right).$$

Together with equations (4.5) and (4.6), it gives that

$$\mathbf{E}\left[\sum_{u \in \mathcal{S}_A} \mathcal{B}_{n,z}(u) \mathbf{1}_{\mathcal{T}(u)^c}\right] \leq c_{26} \mathrm{e}^{-z} \sum_{\ell=1}^{\lfloor (\ln n)^{10} \rfloor} \sum_{k=\ell}^{\lfloor (\ln n)^{10} \rfloor} \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right).$$

In order to prove (ii), it is enough to show that

$$\sum_{\ell \geq 1} \sum_{k \geq \ell} \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right) = o(z) \tag{4.7}$$

uniformly in $A \geq 0$. The left-hand side is 0 if $z < A$. Therefore, we will assume that $z \geq A$. For $k \geq \ell$, notice that if $w_k \in \mathcal{S}_A$, then necessarily $\min_{j \leq \ell} V(w_j) \geq A - z$, $V(w_k) \geq A - z$ and $V(w_k) < \min_{\ell \leq j \leq k-1} V(w_j)$ (in particular, k is a ladder epoch for the random walk started at $V(w_\ell)$). It implies that

$$\begin{aligned} & \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right) \\ \leq\ & \mathbf{Q}\left(\xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2},\, \min_{j \leq \ell} V(w_j) \geq A - z,\, A - z \leq V(w_k) < \min_{\ell \leq j \leq k-1} V(w_j)\right). \end{aligned}$$

Summing over $k \geq \ell$, we get

$$\begin{aligned} & \sum_{k \geq \ell} \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right) \\ \leq\ & \mathbf{E}_{\mathbf{Q}}\left[\mathbf{1}_{\{\xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j \leq \ell} V(w_j) \geq A-z\}} \sum_{k \geq \ell} \mathbf{1}_{\{A-z \leq V(w_k) < \min_{\ell \leq j \leq k-1} V(w_j)\}}\right]. \end{aligned}$$

By the Markov property at time ℓ, we recognize in the term $\sum_{k \geq \ell} \mathbf{1}_{\{A-z \leq V(w_k) < \min_{\ell \leq j \leq k-1} V(w_j)\}}$ the number of strict descending ladder heights above level $A - z$ when starting from $V(w_\ell)$. Consequently,

$$\begin{aligned} & \sum_{k \geq \ell} \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right) \\ \leq\ & \mathbf{E}_{\mathbf{Q}}\left[\mathbf{1}_{\{\xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j \leq \ell} V(w_j) \geq A-z\}} R(z - A + V(w_\ell))\right]. \end{aligned}$$

We know from (2.10) that there exists $c_{27} > 0$ such that $R(x) \le c_{27}(1+x)$ for any $x \ge 0$. Thus, $R(z - A + V(w_\ell)) \le c_{27}(1 + z - A + V(w_{\ell-1}))_+ + c_{27}(V(w_\ell) - V(w_{\ell-1}))_+$. Also, we obviously have $\min_{j \le \ell} V(w_j) \le \min_{j \le \ell-1} V(w_j)$. It yields that

$$\sum_{k \ge \ell} \mathbf{Q}\left(w_k \in \mathcal{S}_A,\, \xi(w_\ell) \ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\right) \le c_{27}(f(\ell) + g(\ell))$$

where

$$\begin{aligned} f(\ell) &:= \mathbf{E_Q}\left[\mathbf{1}_{\{\xi(w_\ell) \ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j \le \ell-1} V(w_j) \ge A - z\}} (z - A + V(w_{\ell-1}))\right], \\ g(\ell) &:= \mathbf{E_Q}\left[\mathbf{1}_{\{\xi(w_\ell) \ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j \le \ell-1} V(w_j) \ge A - z\}} (V(w_\ell) - V(w_\ell - 1))_+\right]. \end{aligned}$$

Equation (4.7) boils down to

$$\sum_{\ell \ge 1} (f(\ell) + g(\ell)) = o(z). \tag{4.8}$$

Let (ξ, Δ) be a generic random variable independent of all the random variables used so far, and distributed as $(\xi(w_1), V(w_1))$ (under $\mathbf{Q}$). Using the Markov property at time $\ell - 1$ in $f(\ell)$, we get

$$f(\ell) = \mathbf{E_Q}\left[\mathbf{1}_{\{\xi \ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j \le \ell-1} V(w_j) \ge A - z\}} (z - A + V(w_{\ell-1}))\right].$$

Summing over ℓ (and replacing $\ell - 1$ by ℓ) yields that

$$\sum_{\ell \ge 1} f(\ell) = \mathbf{E_Q}\left[\sum_{\ell \ge 0} \mathbf{1}_{\{V(w_\ell) + z - A \le 2\ln(\xi)\}} \mathbf{1}_{\{\min_{j \le \ell} V(w_j) \ge A - z\}} (z - A + V(w_\ell))\right]$$

By Lemma B.2 (i), we have for any $x \ge 0$

$$\begin{aligned} &\mathbf{E_Q}\left[\sum_{\ell \ge 0} \mathbf{1}_{\{V(w_\ell) + z - A \le x\}} \mathbf{1}_{\{\min_{j \le \ell} V(w_j) \ge A - z\}} (z - A + V(w_\ell))\right] \\ &\le c_{28}(1+x)^2(1 + \min(x, z - A)) \\ &\le c_{28}(1+x)^2(1 + \min(x, z)). \end{aligned}$$

We deduce that, with the notation of (1.2),

$$\begin{aligned} \sum_{\ell \ge 1} f(\ell) &\le c_{28}\mathbf{E_Q}[(1 + 2\ln_+ \xi)^2(1 + \min(2\ln_+ \xi, z))] \\ &= c_{28}\mathbf{E}[X(1 + 2\ln_+(X + \tilde{X}))^2(1 + \min(2\ln_+(X + \tilde{X}), z))] \\ &= o(z) \end{aligned} \tag{4.9}$$

under (1.4) by Lemma B.1 (ii). We consider now $g(\ell)$. We have similarly

$$\sum_{\ell\geq 1} g(\ell) = \mathbf{E}_{\mathbf{Q}}\left[\Delta_+ \sum_{\ell\geq 0} \mathbf{1}_{\{V(w_\ell)+z-A\leq 2\ln(\xi)\}} \mathbf{1}_{\{\min_{j\leq \ell} V(w_j)\geq A-z\}}\right].$$

From Lemma B.2 (i), we get

$$\begin{aligned}\sum_{\ell\geq 1} g(\ell) &\leq c_{28}\mathbf{E}_{\mathbf{Q}}[\Delta_+(1+2\ln_+\xi)(1+\min(2\ln\xi, z))] \\ &= c_{28}\mathbf{E}[\tilde{X}(1+2\ln_+(X+\tilde{X}))(1+\min(2\ln(X+\tilde{X}), z))] \\ &= o(z) \qquad (4.10)\end{aligned}$$

by Lemma B.1 (ii). Equations (4.9) and (4.10) imply (4.8). □

We compute the second moment in the following lemma.

Lemma 4.4 *Recall the notation $\mathcal{S}_A$ in (4.1), $B_{n,z}(u)$ in Definition 4.2 and $\mathcal{T}(u)$ in Lemma 4.3. There exists a constant $c_{29} > 0$ such that for any $z \geq A \geq 0$, and $n \geq 1$,*

$$\mathbf{E}\left[U^2\right] - \mathbf{E}\left[U\right] \leq c_{29}\mathrm{e}^{-z}\mathrm{e}^{-A} \qquad (4.11)$$

where $U := \sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u)\mathbf{1}_{\mathcal{T}(u)}$.

Proof. Let U be as in the lemma. We observe that

$$U^2 - U = \sum_{u\neq v\in\mathcal{S}_A} \mathcal{B}_{n,z}(u)\mathcal{B}_{n,z}(v)\mathbf{1}_{\mathcal{T}(u),\mathcal{T}(v)}$$

where $u \neq v \in \mathcal{S}_A$ is a short way to write $u \in \mathcal{S}_A, v \in \mathcal{S}_A, u \neq v$. It follows that

$$\begin{aligned}\mathbf{E}[U^2 - U] &\leq \mathbf{E}\left[\sum_{u\neq v} \mathcal{B}_{n,z}(u)\mathcal{B}_{n,z}(v)\mathbf{1}_{\{u,v\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(u)}\right] \\ &\leq 2\mathbf{E}\left[\sum_{u\neq v, |u|\geq|v|} \mathcal{B}_{n,z}(u)\mathcal{B}_{n,z}(v)\mathbf{1}_{\{u,v\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(u)}\right].\end{aligned}$$

For $|u| \geq |v|$, and $u \neq v$, notice that $\mathcal{B}_{n,z}(u)$ depends on the branching random walk rooted at u, whereas $\mathcal{B}_{n,z}(v)\mathbf{1}_{\{u\in\mathcal{S}_A\}}$ is independent of it (even if v is a (strict) ancestor of u). Therefore, by the branching property,

$$\mathbf{E}[U^2 - U] \leq 2\mathbf{E}\left[\sum_{u\neq v, |u|\geq|v|} \Phi(V(u), n-|u|)\mathcal{B}_{n,z}(v)\mathbf{1}_{\{u,v\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(u)}\right]$$

where, for any $r \geq 0$ and $\ell \leq n$

$$\Phi(r,\ell) := \mathbf{P}\left(M_\ell^{\text{kill}} \leq a_n(z+r)\right). \tag{4.12}$$

By Lemma 3.5, we have $\Phi(V(u), n-|u|) \leq c_{30}\mathrm{e}^{-z-V(u)}$ for $|u| = o(n)$, which is the case when $u \in \mathcal{S}_A$ by definition. It gives that

$$\begin{aligned} \mathbf{E}[U^2 - U] &\leq c_{30}\mathrm{e}^{-z}\mathbf{E}\left[\sum_{u\neq v, |u|\geq|v|} \mathrm{e}^{-V(u)}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{u,v\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(u)}\right] \\ &\leq c_{30}\mathrm{e}^{-z}\sum_{k\geq 0}\mathbf{E}\left[\sum_{|u|=k}\mathrm{e}^{-V(u)}\mathbf{1}_{\{u\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(u)}\sum_{v\neq u, |v|\leq k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}\right]. \end{aligned} \tag{4.13}$$

The weight $\mathrm{e}^{-V(u)}$ hints for a change of measure from $\mathbf{P}$ to $\mathbf{Q}$. For any $k \geq 0$, we have by Proposition 2.1 (ii)

$$\begin{aligned} &\mathbf{E}\left[\sum_{|u|=k}\mathrm{e}^{-V(u)}\mathbf{1}_{\{u\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(u)}\sum_{v\neq u, |v|\leq k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}\right] \\ &= \mathbf{E}_{\mathbf{Q}}\left[\mathbf{1}_{\{w_k\in\mathcal{S}_A\}}\mathbf{1}_{\mathcal{T}(w_k)}\sum_{v\neq w_k, |v|\leq k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}\right]. \end{aligned} \tag{4.14}$$

We have to discuss on the location of the vertex v with respect to w_k. We say that $u \nsim v$ if v is not an ancestor of u, nor u is an ancestor of v. If $v \neq w_k$ and $|v| \leq k$, then either $v \nsim u$, or $v = w_\ell$ for some $\ell < k$. In view of (4.13) and (4.14), the lemma will be proved once the following two estimates are shown:

$$\sum_{k\geq 1}\mathbf{E}_{\mathbf{Q}}\left[\sum_{v\nsim w_k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}, w_k\in\mathcal{S}_A, \mathcal{T}(w_k)\right] \leq c_{31}\mathrm{e}^{-A}, \tag{4.15}$$

$$\sum_{k\geq 1}\sum_{\ell=0}^{k-1}\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell),\, w_k\in\mathcal{S}_A,\, w_\ell\in\mathcal{S}_A,\, \mathcal{T}(w_k)\right] \leq c_{32}\mathrm{e}^{-A}. \tag{4.16}$$

Proof of equation (4.15).
Decomposing the sum $\sum_{v\nsim w_k}$ along the spine, we see that

$$\sum_{v\nsim w_k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}} = \sum_{\ell=1}^{k}\sum_{x\in\Omega(w_\ell)}\sum_{v\geq x}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}, \tag{4.17}$$

where $\Omega(w_\ell)$ is as usual the set of brothers of w_ℓ. The branching random walk rooted at $x \in \Omega(w_\ell)$ has the same law under $\mathbf{P}$ and $\mathbf{Q}$. Let as before $\mathcal{G}_\infty := \sigma\{w_j, \Omega(w_j), V(w_j), V(x), x \in \Omega(w_j),\, j \geq 0\}$ be the sigma-algebra associated to the spine and its brothers. We have, for $x \in \Omega(w_\ell)$

$$\mathbf{E}_{\mathbf{Q}}\left[\sum_{v \geq x} \mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}} \,\Big|\, \mathcal{G}_\infty\right] = \mathbf{E}_{\mathbf{Q}}\left[\sum_{v\geq x} \Phi(V(v), n-|v|)\mathbf{1}_{\{v\in\mathcal{S}_A\}} \,\Big|\, \mathcal{G}_\infty\right] \tag{4.18}$$

with the notation of (4.12), which is

$$\leq c_{29}\mathrm{e}^{-z}\mathbf{E}_{\mathbf{Q}}\left[\sum_{v\geq x}\mathrm{e}^{-V(v)}\mathbf{1}_{\{v\in\mathcal{S}_A\}} \,\Big|\, \mathcal{G}_\infty\right]$$

by Lemma 3.5. We observe now that if $v \geq x$ and $v \in \mathcal{S}_A$, then $\min_{|x|\leq j\leq |v|-1} V(v_j) > V(v) \geq A - z$. Therefore

$$\mathbf{E}_{\mathbf{Q}}\left[\sum_{v\geq x}\mathrm{e}^{-V(v)}\mathbf{1}_{\{v\in\mathcal{S}_A\}} \,\Big|\, \mathcal{G}_\infty\right] \leq \mathbf{E}_{V(x)}\left[\sum_{v\in\mathbb{T}}\mathrm{e}^{-V(v)}\mathbf{1}_{\{\min_{j\leq|v|-1} V(v_j)>V(v)\geq A-z\}}\right].$$

By (2.1), we have

$$\begin{aligned}\mathbf{E}_{V(x)}\left[\sum_{v\in\mathbb{T}}\mathrm{e}^{-V(v)}\mathbf{1}_{\{\min_{j\leq|v|-1} V(v_j)>V(v)\geq A-z\}}\right] &= \mathrm{e}^{-V(x)}\mathbf{E}\left[\sum_{i\geq 0}\mathbf{1}_{\{\min_{j\leq i-1} S_j > S_i \geq A-z-r\}}\right]_{r=V(x)} \\ &= \mathrm{e}^{-V(x)}R(z-A+V(x))\end{aligned}$$

by definition of the renewal function R in (2.9). Going back to (4.18), we get that for any $x \in \Omega(w_\ell)$,

$$\mathbf{E}_{\mathbf{Q}}\left[\sum_{v\geq x}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}} \,\Big|\, \mathcal{G}_\infty\right] \leq c_{29}\mathrm{e}^{-z}\mathrm{e}^{-V(x)}R(z-A+V(x)).$$

In view of (4.17), we have

$$\begin{aligned}&\sum_{k\geq 0}\mathbf{E}_{\mathbf{Q}}\left[\sum_{v\nsim w_k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}, w_k\in\mathcal{S}_A,\, \mathcal{T}(w_k)\right] \\ \leq\;& c_{29}\mathrm{e}^{-z}\sum_{k\geq 1}\sum_{\ell=1}^{k}\mathbf{E}_{\mathbf{Q}}\left[\sum_{x\in\Omega(w_\ell)}\mathrm{e}^{-V(x)}R(z-A+V(x)),\, w_k\in\mathcal{S}_A, \mathcal{T}(w_k)\right].\end{aligned} \tag{4.19}$$

We know that (2.10) implies $R(x) \leq c_{27}(x_+ + 1)$. We observe that, for $a \geq 1$

$$\sum_{x\in\Omega(w_j)} (a + V(x)_+)\mathrm{e}^{-V(x)} \leq (a + V(w_{j-1})_+)\mathrm{e}^{-V(w_{j-1})}\xi(w_j). \tag{4.20}$$

First by (4.20) then by definition of $\mathcal{T}(w_k)$, it yields that for $k \geq \ell \geq 1$

$$\begin{aligned}
&\mathbf{E_Q}\left[\sum_{x\in\Omega(w_\ell)} \mathrm{e}^{-V(x)}R(z - A + V(x)),\, w_k \in \mathcal{S}_A, \mathcal{T}(w_k)\right]\\
\leq\ & c_{27}\mathbf{E_Q}\left[(z - A + V(w_\ell - 1) + 1)\mathrm{e}^{-V(w_{\ell-1})}\xi(w_\ell),\, w_k \in \mathcal{S}_A, \mathcal{T}(w_k)\right]\\
\leq\ & c_{27}\mathrm{e}^{(z-A)/2}\mathbf{E_Q}\left[\mathrm{e}^{-V(w_{\ell-1})/2}(z - A + V(w_{\ell-1}) + 1),\, w_k \in \mathcal{S}_A\right].
\end{aligned}$$

By Proposition 2.1 (iii), we have

$$\begin{aligned}
&\sum_{k\geq 1}\sum_{\ell=1}^{k}\mathbf{E_Q}\left[\mathrm{e}^{-V(w_{\ell-1})/2}(z - A + V(w_{\ell-1}) + 1),\, w_k \in \mathcal{S}_A\right]\\
=\ & \sum_{k=1}^{\lfloor(\ln n)^{10}\rfloor}\sum_{\ell=1}^{k}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z - A + S_{\ell-1} + 1),\, \min_{j\leq k-1} S_j > S_k \geq A - z\right]\\
\leq\ & \sum_{k\geq 1}\sum_{\ell=1}^{k}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z - A + S_{\ell-1} + 1),\, \min_{j\leq k-1} S_j > S_k \geq A - z\right]
\end{aligned}$$

Equation (4.19) becomes

$$\begin{aligned}
&\sum_{k\geq 1}\mathbf{E_Q}\left[\sum_{v\nsim w_k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}, w_k \in \mathcal{S}_A,\, \mathcal{T}(w_k)\right]\\
\leq\ & c_{29}c_{27}\mathrm{e}^{-z}\mathrm{e}^{(z-A)/2}\sum_{k\geq 1}\sum_{\ell=1}^{k}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z - A + S_{\ell-1} + 1),\, \min_{j\leq k-1} S_j > S_k \geq A - z\right].
\end{aligned} \tag{4.21}$$

We observe that

$$\begin{aligned}
&\sum_{k\geq 1}\sum_{\ell=1}^{k}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z - A + S_{\ell-1} + 1),\, \min_{j\leq k-1} S_j > S_k \geq A - z\right]\\
=\ & \sum_{\ell\geq 1}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z - A + S_{\ell-1} + 1)\sum_{k\geq \ell}\mathbf{1}_{\{\min_{j\leq k-1} S_j > S_k \geq A - z\}}\right].
\end{aligned}$$

By the Markov property at time $\ell - 1$, we get

$$\begin{aligned}
&\sum_{k\geq 1}\sum_{\ell=1}^{k}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z-A+S_{\ell-1}+1),\ \min_{j\leq k-1}S_j > S_k \geq A-z\right]\\
\leq\ &\sum_{\ell\geq 1}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z-A+S_{\ell-1}+1)R(S_{\ell-1}+z-A),\ \min_{j\leq \ell-1}S_j \geq A-z\right]\\
\leq\ & c_{27}\sum_{\ell\geq 1}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z-A+S_{\ell-1}+1)^2,\ \min_{j\leq \ell-1}S_j \geq A-z\right].
\end{aligned}$$

By Lemma B.2 (iii), we have

$$\sum_{\ell\geq 1}\mathbf{E}\left[\mathrm{e}^{-S_{\ell-1}/2}(z-A+S_{\ell-1}+1)^2,\ \min_{j\leq \ell-1}S_j \geq A-z\right] \leq c_{33}\mathrm{e}^{(z-A)/2}.$$

Consequently, by (4.21)

$$\sum_{k\geq 1}\mathbf{E_Q}\left[\sum_{v\sim w_k}\mathcal{B}_{n,z}(v)\mathbf{1}_{\{v\in\mathcal{S}_A\}}, w_k\in\mathcal{S}_A,\ \mathcal{T}(w_k)\right] \leq c_{34}\mathrm{e}^{-z}\mathrm{e}^{(z-A)/2}\mathrm{e}^{(z-A)/2} = c_{34}\mathrm{e}^{-A}.$$

Equation (4.15) follows.

Proof of equation (4.16)
We have

$$\begin{aligned}
&\sum_{k\geq 1}\sum_{\ell=0}^{k-1}\mathbf{E_Q}\left[\mathcal{B}_{n,z}(w_\ell),\ w_k\in\mathcal{S}_A,\ w_\ell\in\mathcal{S}_A,\ \mathcal{T}(w_k)\right]\\
=\ &\sum_{\ell\geq 0}\sum_{k>\ell}\mathbf{E_Q}\left[\mathcal{B}_{n,z}(w_\ell),\ w_k\in\mathcal{S}_A,\ w_\ell\in\mathcal{S}_A,\ \mathcal{T}(w_k)\right]\\
=\ &\sum_{\ell\geq 0}\mathbf{E_Q}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\right].
\end{aligned}$$

Let t_ℓ be the first time t after ℓ such that $V(w_t) < V(w_\ell)$. If $k > \ell$ and $w_k \in \mathcal{S}_A$, then $V(w_k) < V(w_\ell)$, which means that necessarily $k \geq t_\ell$ (and $t_\ell < (\ln n)^{10}$). Moreover, we have $\mathcal{T}(w_i) \subset \mathcal{T}(w_j)$ if $i \leq j$. Thus,

$$\begin{aligned}
\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)} &= \mathbf{1}_{\{w_{t_\ell}\in\mathcal{S}_A,\, t_\ell<(\ln n)^{10}\}}\sum_{k\geq t_\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\\
&\leq \mathbf{1}_{\{w_{t_\ell}\in\mathcal{S}_A,\, t_\ell<(\ln n)^{10}\}\cap\mathcal{T}(w_{t_\ell})}\sum_{k\geq t_\ell}\mathbf{1}_{\{\min_{t_\ell\leq j<k}V(w_j)>V(w_k)\geq A-z\}}.
\end{aligned}$$

We observe that $\mathcal{B}_{n,z}(w_\ell)$ is a function of the branching random walk killed below $V(w_\ell)$ and therefore is independent of the subtree rooted at w_{t_ℓ}. As a result, applying the branching property, we get

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\right]\\
\leq\ &\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{S}_A,\,t_\ell<(\ln n)^{10}\}\cap\mathcal{T}(w_{t_\ell})}\sum_{k\geq t_\ell}\mathbf{1}_{\{\min_{t_\ell\leq j<k}V(w_j)>V(w_k)\geq A-z\}}\right]\\
=\ &\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{S}_A,\,t_\ell<(\ln n)^{10}\}\cap\mathcal{T}(w_{t_\ell})}R(z-A+V(w_{t_\ell}))\right].
\end{aligned}$$

We have $V(w_{t_\ell})<V(w_\ell)$. Since R is a non-decreasing function, we obtain

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\right]\\
\leq\ &\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{S}_A,\,t_\ell<(\ln n)^{10}\}\cap\mathcal{T}(w_{t_\ell})}R(z-A+V(w_\ell))\right].
\end{aligned}$$

We can now apply the Markov property at time ℓ. It yields that

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\right]\\
\leq\ &\mathbf{E}_{\mathbf{Q}}\left[\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}R(z-A+V(w_\ell))\tilde{\Phi}(V(w_\ell),n-\ell)\right]\\
=\ &\mathbf{1}_{\{\ell<(\ln n)^{10}\}}\mathbf{E}_{\mathbf{Q}}\left[\mathbf{1}_{\{\min_{j<\ell}V(w_j)>V(w_\ell)\geq A-z\}}R(z-A+V(w_\ell))\tilde{\Phi}(V(w_\ell),n-\ell)\right]
\end{aligned}$$

where, if $\tau_0^-:=\min\{j\geq 0\,:\,V(w_j)<0\}$, then

$$\tilde{\Phi}(r,i):=\mathbf{Q}\left(\tau_0^-<(\ln n)^{10},\,M_i^{\text{kill}}\leq a_n(z+r),\,\xi(w_j)\leq \mathrm{e}^{(r+V(w_{j-1})+z-A)/2},\,\forall\,1\leq j\leq\tau_0^-\right).$$

By Proposition 2.1 (iii), it implies that

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\right]\\
\leq\ &\mathbf{1}_{\{\ell<(\ln n)^{10}\}}\mathbf{E}\left[\mathbf{1}_{\{\min_{j<\ell}S_j>S_\ell\geq A-z\}}R(z-A+S_\ell)\tilde{\Phi}(S_\ell,n-\ell)\right].
\end{aligned}\tag{4.22}$$

Let us estimate $\tilde{\Phi}(r,i)$ for $i>n/2$. We have to decompose along the spine. Notice that if $M_i^{\text{kill}}\leq a_n(z+r)$, then there must be some $j<\tau_0^-$ and $x\in\Omega(w_j)$ such that there exists a line of descent from x which stays above 0 and ends below $a_n(z+r)$ at time i. Therefore,

$$\begin{aligned}&\tilde\Phi(r,i)\\ \le\ & \sum_{j=1}^{\lfloor(\ln n)^{10}\rfloor}\mathbf{E}_{\mathbf{Q}}\left[\sum_{x\in\Omega(w_j)}\mathbf{P}_{V(x)}(M_{i-j}^{\mathrm{kill}}\le a_n(z+r)),\,\xi(w_j)\le \mathrm{e}^{(r+V(w_{j-1})+z-A)/2},\,j<\tau_0^-\right].\end{aligned}$$

By Lemma 3.3, we get that for $i>n/2$

$$\begin{aligned}\tilde\Phi(r,i) &\le c_{35}\mathrm{e}^{-z-r}\sum_{j=1}^{\lfloor(\ln n)^{10}\rfloor}\mathbf{E}_{\mathbf{Q}}\left[\sum_{x\in\Omega(w_j)}(1+V(x)_+)\mathrm{e}^{-V(x)},\,\xi(w_j)\le \mathrm{e}^{(r+V(w_{j-1})+z-A)/2},\,j<\tau_0^-\right]\\ &\le c_{35}\mathrm{e}^{-z-r}\sum_{j=1}^{\lfloor(\ln n)^{10}\rfloor}\mathbf{E}_{\mathbf{Q}}\left[\mathrm{e}^{-V(w_{j-1})}(1+V(w_{j-1}))\mathrm{e}^{(r+V(w_{j-1})+z-A)/2},\,j<\tau_0^-\right],\end{aligned}$$

by (4.20). It follows that

$$\begin{aligned}\tilde\Phi(r,i) &\le c_{35}\mathrm{e}^{-A}\mathrm{e}^{-(r+z-A)/2}\sum_{j\ge1}\mathbf{E}\left[\mathrm{e}^{-S_{j-1}/2}(1+S_{j-1}),\,j<\tau_0^-\right]\\ &= c_{36}\mathrm{e}^{-A}\mathrm{e}^{-(r+z-A)/2},\end{aligned}$$

by Lemma B.2 (ii). Going back to (4.22), (notice that $n-\ell>n-(\ln n)^{10}$), we obtain that

$$\begin{aligned}&\mathbf{E}_{\mathbf{Q}}\left[\mathcal{B}_{n,z}(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}_A\}}\sum_{k>\ell}\mathbf{1}_{\{w_k\in\mathcal{S}_A\}\cap\mathcal{T}(w_k)}\right]\\ \le\ & c_{36}\mathrm{e}^{-A}\mathbf{E}\left[\mathbf{1}_{\{\min_{j<\ell}S_j>S_\ell\ge A-z\}}R(z-A+S_\ell)\mathrm{e}^{-(S_\ell+z-A)/2}\right].\end{aligned}$$

Summing over $\ell\ge1$, then applying Lemma B.2 (iii) completes the proof of (4.16). □

4.2 Proof of Proposition 4.1

We can now prove Proposition 4.1.

Proof of Proposition 4.1. Let $\varepsilon > 0$. We see that for any $r \geq 0$,

$$\begin{aligned}
&\mathbf{P}\left(\exists |u| \geq (\ln n)^{10} \,:\, V(u) \in [-r,0],\, \min_{j\leq |u|} V(u_j) \geq -r\right) \\
\leq &\sum_{k\geq (\ln n)^{10}} \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\{V(u)\in[-r,0],\, \min_{j\leq k} V(u_j)\geq -r\}}\right] \\
= &\sum_{k\geq (\ln n)^{10}} \mathbf{E}\left[\mathrm{e}^{S_k}, S_k \in [-r,0],\, \min_{j\leq k} S_j \geq -r\right] \\
\leq &\sum_{k\geq (\ln n)^{10}} \mathbf{P}(S_k \in [-r,0], \min_{j\leq k} S_j \geq -r)
\end{aligned}$$

by (2.1). We notice that $\mathbf{P}(S_k \in [-r,0], \min_{j\leq k} S_j \geq -r) \leq c_{37}(1+r)^2 k^{-3/2}$ by (2.6). Therefore,

$$\mathbf{P}\left(\exists |u| \geq (\ln n)^{10} \,:\, V(u) \in [-r,0],\, \min_{j\leq |u|} V(u_j) \geq -r\right) \leq c_{38}(1+r)^2(\ln n)^{-5}. \tag{4.23}$$

We also observe that

$$\begin{aligned}
\mathbf{P}(\exists\, u \in \mathbb{T} \,:\, V(u) \leq -r) \;&\leq\; \sum_{n\geq 0} \mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u)\leq -r, V(u_k)>-r, \forall\, k<n\}}\right] \\
&=\; \sum_{n\geq 0} \mathbf{E}\left[\mathrm{e}^{S_n}, S_n \leq -r, S_k > -r\, \forall k < n\right] \\
&\leq\; \mathrm{e}^{-r}.
\end{aligned} \tag{4.24}$$

On the event $\{\forall\, |u| \geq (\ln n)^{10} \,:\, V(u) \geq 0\} \cap \{\forall u \in \mathbb{T},\, V(u) \geq A - z\}$, we observe that $M_n \leq \frac{3}{2}\ln n - z$ if and only if $\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \geq 1$ (recall the definition of $\mathcal{B}_{n,z}$ and $\mathcal{S}_A$ in Definition 4.2 and in (4.1)). It yields that, for $n \geq 1$ and $z \geq A$,

$$\left|\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) - \mathbf{P}\left(\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \geq 1\right)\right| \leq c_{38}(1+z-A)^2(\ln n)^{-5} + \mathrm{e}^{A-z}.$$

Let us look at the upper bound. We have $\mathbf{P}\left(\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u) \geq 1\right) \leq \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u)\right]$. Therefore (take $A \geq 1$)

$$\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) \leq c_{38} z^2 (\ln n)^{-5} + \mathrm{e}^{A-z} + \mathbf{E}\left[\sum_{u\in\mathcal{S}_A} \mathcal{B}_{n,z}(u)\right].$$

Lemma 4.3 implies that for $n \geq N_1$ and $z \in [A_1, (\ln n)^{1/5}]$,

$$\frac{\mathrm{e}^z}{R(z-A_1)}\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) - C_1 \leq c_{38}\frac{\mathrm{e}^z}{R(z-A_1)}z^2(\ln n)^{-5} + \frac{\mathrm{e}^{A_1}}{R(z-A_1)} + \varepsilon.$$

Since $R(x) \sim c_0$ at infinity by (2.10), we have for $n \geq N_1$ and $z \in [A_2, (\ln n)^{1/5}]$

$$\frac{\mathrm{e}^z}{c_0 z}\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) - C_1 \leq c_{38} z\mathrm{e}^z(\ln n)^{-5} + \frac{\mathrm{e}^{A_1}}{c_0 z} + 2\varepsilon.$$

We deduce that for $n \geq N_2$ and $z \in [A_3, \ln\ln n]$

$$\frac{\mathrm{e}^z}{c_0 z}\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) - C_1 \leq 4\varepsilon.$$

This proves the upper bound. Similarly, we have for the lower bound

$$\begin{aligned}\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) &\geq \mathbf{P}\left(\sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u) \geq 1\right) - c_{38}z^2(\ln n)^{-5} - \mathrm{e}^{A-z}\\ &\geq \mathbf{P}(\sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u)\mathbf{1}_{\mathcal{T}(u)} \geq 1) - c_{38}z^2(\ln n)^{-5} - \mathrm{e}^{A-z}.\end{aligned}$$

If we write as in Lemma 4.4, $U := \sum_{u\in\mathcal{S}_A}\mathcal{B}_{n,z}(u)\mathbf{1}_{\mathcal{T}(u)}$, then by the Paley-Zygmund formula, we have $\mathbf{P}(U \geq 1) \geq \frac{\mathbf{E}[U]^2}{\mathbf{E}[U^2]}$. By Lemma 4.3, we know that $\frac{\mathrm{e}^z}{R(z-A_4)}\mathbf{E}[U] \geq C_1 - \varepsilon$ for $n \geq N_2$ and $z \in [A_4, (\ln n)^{1/5}]$. By Lemma 4.4, we have that $\mathbf{E}[U^2] \leq (1+\varepsilon)\mathbf{E}[U]$ if A_4 is taken large enough. Hence, $\frac{\mathrm{e}^z}{R(z-A_4)}\mathbf{P}(U \geq 1) \geq \frac{\mathrm{e}^z}{R(z-A_4)}(1+\varepsilon)^{-1}\mathbf{E}[U] \geq (1+\varepsilon)^{-1}(C_1-\varepsilon)$. It yields that

$$\frac{\mathrm{e}^z}{R(z-A_4)}\mathbf{P}(M_n \leq \frac{3}{2}\ln n - z) \geq (1+\varepsilon)^{-1}(C_1-\varepsilon) - c_{38}z^2(\ln n)^{-5} - \mathrm{e}^{A_4-z}.$$

From here, we conclude as before to see that for $n \geq N_2$ and $z \in [A_5, \ln\ln n]$,

$$\frac{\mathrm{e}^z}{c_0 z}\mathbf{P}(M_n \leq \frac{3}{2}\ln n - z) \geq C_1 - c_{39}\varepsilon.$$

The proposition follows. □

5 Proof of Theorem 1.1

For $\beta \geq 0$, we look at the branching random walk killed below $-\beta$. The population at time n of this process is $\{|u| = n \,:\, V(u_k) \geq -\beta, \forall\, k \leq n\}$. We define the associated martingale

(5.1) $$D_n^{(\beta)} := \sum_{|u|=n} R(\beta + V(u))\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\geq-\beta,\, k\leq n\}}.$$

Since $D_n^{(\beta)}$ is non-negative, it has a limit almost surely and we denote by $D_\infty^{(\beta)}$ this limit. Under (1.3) and (1.4), we know by Proposition A.1 that $D_\infty^{(\beta)} > 0$ almost surely on the event of non-extinction for the killed branching random walk. For $A \geq 0$, let $\mathcal{Z}[A]$ denote the set of particles absorbed at level A, i.e.

$$\mathcal{Z}[A] := \{u \in \mathbb{T} \,:\, V(u) \geq A,\, V(u_k) < A \,\forall\, k < |u|\}.$$

By Theorem 7 of [6], we know that $\sum_{u\in\mathcal{Z}[A]} R(\beta + V(u))\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta,\, k\leq n\}}$ converges to $D_\infty^{(\beta)}$ almost surely as $A \to \infty$. Recall that $R(x) \sim c_0 x$ at infinity by (2.10). On the event $\{\min_{u\in\mathbb{T}} V(u) \geq -\beta\}$, we see that necessarily $D_\infty^{(\beta)} = c_0 \partial W_\infty$ almost surely, and $\sum_{u\in\mathcal{Z}[A]} R(\beta + V(u))\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta,\, k\leq n\}} \sim c_0 \sum_{u\in\mathcal{Z}[A]}(\beta + V(u))\mathrm{e}^{-V(u)}$ as $A \to \infty$. Again by Theorem 7 of [6], we have $\lim_{A\to\infty}\sum_{u\in\mathcal{Z}[A]} \mathrm{e}^{-V(u)} = 0$ almost surely. We deduce that

$$\lim_{A\to\infty} \sum_{u\in\mathcal{Z}[A]} V(u)\mathrm{e}^{-V(u)} = \partial W_\infty \tag{5.2}$$

on the event $\{\min_{u\in\mathbb{T}} V(u) \geq -\beta\}$, and therefore almost surely by making $\beta \to \infty$. We can now prove the convergence in law.

Proof of Theorem 1.1. Fix $x \in \mathbb{R}$ and let $\varepsilon > 0$. For any $A > 0$, we have for n large enough

$$\begin{aligned} \mathbf{P}(\exists\, u \in \mathcal{Z}[A] \,:\, |u| \geq (\ln n)^{10}) &\leq \varepsilon, \\ \mathbf{P}(\exists\, u \in \mathcal{Z}[A] \,:\, V(u) \geq \ln\ln n) &\leq \varepsilon. \end{aligned}$$

Take $A > 0$. Let $\mathcal{Y}_A := \{\max_{u\in\mathcal{Z}[A]} |u| \leq (\ln(n))^{10},\, \max_{u\in\mathcal{Z}[A]} V(u) \leq \ln\ln n\}$. We observe that

$$\begin{aligned} \mathbf{P}(M_n \geq \frac{3}{2}\ln n + x) &\geq \mathbf{P}(M_n \geq \frac{3}{2}\ln n + x,\, \mathcal{Y}_A) \\ &= \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} \mathbf{P}(M_{n-t} \geq \frac{3}{2}\ln(n) + x - r)_{r=V(u),t=|u|},\, \mathcal{Y}_A\right]. \end{aligned}$$

By Proposition 4.1, there exists A large enough and $N \geq 1$ such that for any $n \geq N$, $t \leq (\ln n)^{10}$ and $z \in [A - x, \ln\ln n - x]$,

$$\left|\frac{\mathrm{e}^z}{z}\mathbf{P}(M_{n-t} \leq \frac{3}{2}\ln(n) - z) - C_1 c_0\right| \leq \varepsilon. \tag{5.3}$$

We get that

$$\mathbf{P}(M_n \geq \frac{3}{2}\ln n + x) \geq \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} (1 - (C_1c_0 + \varepsilon)(V(u) - x)\mathrm{e}^{x - V(u)}),\, \mathcal{Y}_A\right].$$

Since $\mathbf{P}(\mathcal{Y}_A^c) \le 2\varepsilon$ for n large enough, we have for n large enough

$$\mathbf{P}(M_n \ge \frac{3}{2}\ln n + x) \ge \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} (1-(C_1c_0+\varepsilon)(V(u)-x)\mathrm{e}^{x-V(u)})\right] - 2\varepsilon.$$

In particular,

$$\liminf_{n\to\infty} \mathbf{P}(M_n \ge \frac{3}{2}\ln n + x) \ge \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} (1-(C_1c_0+\varepsilon)(V(u)-x)\mathrm{e}^{x-V(u)})\right] - 2\varepsilon.$$

We make A go to infinity. We have almost surely by (5.2) and the fact that $\sum_{u\in\mathcal{Z}[A]}\mathrm{e}^{-V(u)}$ vanishes

$$\lim_{A\to\infty} \sum_{u\in\mathcal{Z}[A]} \ln(1-(C_1c_0+\varepsilon)(V(u)-x)\mathrm{e}^{x-V(u)}) = -(C_1c_0+\varepsilon)\mathrm{e}^x\partial W_\infty. \tag{5.4}$$

By dominated convergence, we deduce that

$$\liminf_{n\to\infty} \mathbf{P}(M_n \ge \frac{3}{2}\ln n + x) \ge \mathbf{E}\left[\exp(-(C_1c_0+\varepsilon)\mathrm{e}^x\partial W_\infty)\right] - 2\varepsilon,$$

which gives the lower bound by letting $\varepsilon \to 0$. The upper bounds works similarly. Let A be such that (5.3) is satisfied for n large enough. We observe that, for n large enough,

$$\begin{aligned}
\mathbf{P}(M_n \ge \frac{3}{2}\ln n + x) &\le \mathbf{P}(M_n \ge \frac{3}{2}\ln n + x, \mathcal{Y}_A) + 2\varepsilon \\
&= \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} \mathbf{P}(M_{n-t} \ge 3/2\ln(n) + x - r)_{r=V(u),t=|u|}, \mathcal{Y}_A\right] + 2\varepsilon \\
&\le \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} \mathbf{P}(M_{n-t} \ge 3/2\ln(n) + x - r)_{r=V(u),t=|u|}\right] + 2\varepsilon.
\end{aligned}$$

Using (5.3), we end up with

$$\limsup_{n\to\infty} \mathbf{P}(M_n \ge \frac{3}{2}\ln n + x) \le \mathbf{E}\left[\prod_{u\in\mathcal{Z}[A]} (1-(C_1c_0-\varepsilon)(V(u)-x)\mathrm{e}^{x-V(u)})\right] + 2\varepsilon.$$

From here, we proceed as for the lower bound. □

A The derivative martingale

We work under (1.1), (1.3) and (1.4) but we drop the assumption that $\mathcal{L}$ is non-lattice. The renewal function $R(x)$ was defined in (2.9). For any $\beta \geq 0$, let

$$D_n^{(\beta)} := \sum_{|u|=n} R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta,\,\forall\, k\leq n\}}$$

be the (non-negative) martingale associated to the branching random walk killed below $-\beta$ and we denote by $D_\infty^{(\beta)}$ its limit. The question of the convergence in L^1 was adressed in [6], where the authors give almost optimal conditions for the convergence to hold. However, we deal with slightly weaker conditions, so we have to prove the convergence in our case.

Proposition A.1 *Assume (1.1), (1.3) and (1.4).*
(i) For any $\beta \geq 0$, $D_n^{(\beta)}$ converges in L^1 to $D_\infty^{(\beta)}$.
(ii) We have $D_\infty^{(\beta)} > 0$ almost surely on the event of non-extinction of the branching random walk killed below $-\beta$.
(iii) We have $\partial W_\infty > 0$ almost surely on the event of non-extinction of $\mathbb{T}$.

Proof. We adapt the proof of [6] (see [20] for the case of the additive martingale). For any $y \geq 0$, let $\mathbf{Q}_y^{(\beta)}$ defined by

$$\frac{d\mathbf{Q}_y^{(\beta)}}{d\mathbf{P}_y}\Big|_{\mathcal{F}_n} := \frac{D_n^{(\beta)}}{R(y+\beta)\mathrm{e}^{-y}}.$$

We write $\mathbf{Q}^{(\beta)}$ for $\mathbf{Q}_0^{(\beta)}$. Then, under $\mathbf{Q}_y^{(\beta)}$, the branching random walk has the following spine decomposition (we refer to [3] for a more precise description). The spine w_0 starts at $V(w_0) = y$. At time 1 it gives birth to a point process distributed as $(V(x), |x| = 1)$ under $\mathbf{Q}_y^{(\beta)}$. Then the spine w_1 at time 1 is chosen proportionally to $R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u)\geq -\beta\}}$ among the children of w_0. At each time n, the spine w_n produces an independent point process distributed as $(V(x), |x| = 1)$ under $\mathbf{Q}_{V(w_n)}^{(\beta)}$, while the other particles $|u| = n$ generate independent point processes distributed as $(V(x), |x| = 1)$ under $\mathbf{P}_{V(u)}$. The spine w_{n+1} at time $n+1$ is chosen proportionally to the weight $R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta,\,\forall\, k\leq n\}}$ among the children of w_n. Under $\mathbf{Q}_y^{(\beta)}$, the spine process $(V(w_n),\, n \geq 0)$ is distributed as the random walk $(S_n)_{n\geq 0}$ conditioned to stay above $-\beta$, i.e, for any measurable non-negative function $F : \mathbb{R}^{n+1} \to \mathbb{R}_+$,

$$\mathbf{E}_{\mathbf{Q}_y^{(\beta)}}\left[F(V(w_0),\ldots,V(w_n))\right] = \frac{1}{R(y+\beta)}\mathbf{E}_y\left[F(S_0,\ldots,S_n)R(\beta+S_n),\, \min_{k\leq n} S_k \geq -\beta\right].$$

We observe now that if $\limsup_{n\to\infty} D_n^{(\beta)} < \infty$, $\mathbf{Q}^{(\beta)}$-a.s then $D_n^{(\beta)}$ converges in mean under $\mathbf{P}$ (it is actually an equivalence, see [20] for example). Let

$$\mathcal{G}_\infty := \sigma\{w_j, V(w_j), \Omega(w_j), (V(u))_{u\in\Omega(w_j)},\, j\geq 1\}$$

be the sigma-algebra of the spine and its brothers. We have

$$\mathbf{E}_{\mathbf{Q}^{(\beta)}}[D_n^{(\beta)}\,|\,\mathcal{G}_\infty] = R(\beta+V(w_n))\mathrm{e}^{-V(w_n)} + \sum_{k=1}^{n}\sum_{x\in\Omega(w_k)} R(\beta+V(x))\mathrm{e}^{-V(x)}\mathbf{1}_{\{V(x_j)\geq-\beta,\forall j\leq k\}}.$$

We know that $V(w_n)\to\infty$ $\mathbf{Q}^{(\beta)}$-almost surely, therefore $R(\beta+V(w_n))\mathrm{e}^{-V(w_n)}$ goes to zero as $n\to\infty$. Furthermore, we see that $1/D_n^{(\beta)}$ is under $\mathbf{Q}^{(\beta)}$ a positive supermartingale, and therefore converges as $n\to\infty$. We still denote by $D_\infty^{(\beta)}$ the (possibly infinite) limit of $D_n^{(\beta)}$ under $\mathbf{Q}^{(\beta)}$. We already know that $R(x)\leq c_{27}(1+x_+)$ for any $x\in\mathbb{R}$. Then, by Fatou's lemma

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}^{(\beta)}}[D_\infty^{(\beta)}] &\leq \liminf_{n\to\infty}\mathbf{E}_{\mathbf{Q}^{(\beta)}}[D_n^{(\beta)}]\\ &\leq c_{27}\sum_{k\geq1}\sum_{x\in\Omega(w_k)}(1+(\beta+V(x))_+)\mathrm{e}^{-V(x)}. \end{aligned}\tag{A.1}$$

To prove (i), it remains to show that the right-hand side of the last inequality is finite. We observe that

$$\sum_{k\geq1}\sum_{x\in\Omega(w_k)}(1+(\beta+V(x))_+)\mathrm{e}^{-V(x)} \leq A_1 + A_2 \tag{A.2}$$

with

$$A_1 := \sum_{k\geq1}(1+\beta+V(w_{k-1}))\mathrm{e}^{-V(w_{k-1})}\sum_{x\in\Omega(w_k)}\mathrm{e}^{-(V(x)-V(w_{k-1}))}, \tag{A.3}$$

$$A_2 := \sum_{k\geq1}\mathrm{e}^{-V(w_{k-1})}\sum_{x\in\Omega(w_k)}(V(x)-V(w_{k-1}))_+\mathrm{e}^{-(V(x)-V(w_{k-1}))}. \tag{A.4}$$

Let us consider A_1. We recall that $X := \sum_{|x|=1}\mathrm{e}^{-V(x)}$, $\tilde{X} := \sum_{|x|=1}V(x)_+\mathrm{e}^{-V(x)}$ and we introduce $X' := \sum_{|x|=1}R(\beta+V(x))\mathrm{e}^{-V(x)}\mathbf{1}_{\{V(x)\geq-\beta\}}$. We have for any $z\in\mathbb{R}$ and $a\geq-\beta$

$$\begin{aligned}\mathbf{Q}_a^{(\beta)}(X>z) &= \frac{1}{R(a+\beta)\mathrm{e}^{-a}}\mathbf{E}_a\left[X'\mathbf{1}_{\{X>z\}}\right]\\ &\leq c_{40}\mathbf{E}\left[X\mathbf{1}_{\{X>z\}}\right] + c_{41}\frac{1}{1+a+\beta}\mathbf{E}\left[\tilde{X}\mathbf{1}_{\{X>z\}}\right]\\ &=: c_{40}h_1(z) + c_{41}\frac{1}{1+a+\beta}h_2(z).\end{aligned}\tag{A.5}$$

We deduce by the Markov property at time $k-1$ that

$$\begin{aligned} & \mathbf{Q}^{(\beta)}\left(\sum_{x\in\Omega(w_k)} \mathrm{e}^{-(V(x)-V(w_{k-1})} \ge \mathrm{e}^{V(w_{k-1})/2}\right) \\ \le\ & \mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[c_{40} h_1(\mathrm{e}^{V(w_{k-1})/2}) + c_{41}\frac{1}{1+V(w_{k-1})+\beta} h_2(\mathrm{e}^{V(w_{k-1})/2})\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \text{(A.6)}\qquad & \sum_{k\ge 1}\mathbf{Q}^{(\beta)}\left(\sum_{x\in\Omega(w_k)} \mathrm{e}^{-(V(x)-V(w_{k-1})} \ge \mathrm{e}^{V(w_{k-1})/2}\right) \\ \le\ & c_{40}\sum_{\ell\ge 0}\mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[h_1(\mathrm{e}^{V(w_\ell)/2})\right] + c_{41}\sum_{\ell\ge 0}\mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[\frac{1}{1+V(w_\ell)+\beta} h_2(\mathrm{e}^{V(w_\ell)/2})\right]. \end{aligned}$$

We estimate $\sum_{\ell\ge 0}\mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[h_1(\mathrm{e}^{V(w_\ell)/2})\right]$. Going back to the measure $\mathbf{P}$, we have

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[h_1(\mathrm{e}^{V(w_\ell)/2})\right] &= \frac{1}{R(\beta)}\mathbf{E}\left[R(\beta+S_\ell) h_1(\mathrm{e}^{S_\ell/2}),\, \min_{j\le \ell} S_j \ge -\beta\right] \\ &= \frac{1}{R(\beta)}\mathbf{E}\left[R(\beta+S_\ell) X \mathbf{1}_{\{S_\ell \le 2\ln X\}},\, \min_{j\le \ell} S_j \ge -\beta\right], \end{aligned}$$

where X is independent of the random walk $(S_n,\, n\ge 0)$. Conditioning on X, then using Lemma B.2 (i), we get that

$$\text{(A.7)}\qquad \sum_{\ell\ge 0}\mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[h_1(\mathrm{e}^{V(w_\ell)/2})\right] \le \frac{c_{42}}{R(\beta)}\mathbf{E}[X(1+\ln_+ X)^2)]$$

which is finite by (1.4). Similarly,

$$\mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[\frac{1}{1+V(w_\ell)+\beta} h_2(\mathrm{e}^{V(w_\ell)/2})\right] \le c_{43}\mathbf{E}\left[\tilde{X}\mathbf{1}_{\{S_\ell\le 2\ln X\}},\, \min_{j\le \ell} S_j \ge -\beta\right].$$

Lemma B.2 (i) implies that

$$\text{(A.8)}\qquad \sum_{\ell\ge 0}\mathbf{E}_{\mathbf{Q}^{(\beta)}}\left[\frac{1}{1+V(w_\ell)+\beta} h_2(\mathrm{e}^{V(w_\ell)/2})\right] \le c_{44}\mathbf{E}\left[\tilde{X}(1+\ln_+ X)\right] < \infty$$

under (1.4) by Lemma B.1 (i). Equations (A.6) , (A.7) and (A.8) give that

$$\text{(A.9)}\qquad \sum_{k\ge 1}\mathbf{Q}^{(\beta)}\left(\sum_{x\in\Omega(w_k)} \mathrm{e}^{-(V(x)-V(w_{k-1})} \ge \mathrm{e}^{V(w_{k-1})/2}\right) < \infty.$$

By the Borel-Cantelli lemma, we obtain that

$$(1+\beta+V(w_{k-1}))\mathrm{e}^{-V(w_{k-1})}\sum_{x\in\Omega(w_k)}\mathrm{e}^{-(V(x)-V(w_{k-1})}\leq(1+\beta+V(w_{k-1}))\mathrm{e}^{-V(w_{k-1})/2}$$

for k large enough almost surely. From (A.3), we deduce that $A_1<\infty$. We proceed similarly for A_2, replacing in (A.5) $\mathbf{1}_{\{X>z\}}$ by $\mathbf{1}_{\{\tilde{X}>z\}}$. By analogy, we find that $A_2<\infty$ if $\mathbf{E}[X(1+\ln_+\tilde{X}]^2]$ and $\mathbf{E}[\tilde{X}(1+\ln_+\tilde{X})]$ are finite. This is the case by (1.4) and Lemma B.1 (i). Equations (A.1) and (A.2) yield that $D_\infty^{(\beta)}<\infty$ $\mathbf{Q}^{(\beta)}$-a.s, which ends the proof of (i). We prove now (iii). It is well-known (see Theorem 12 of [6]) that $\mathbf{P}(\partial W_\infty>0\,|\,\mathrm{non-extinction})$ is 0 or 1. We have $\mathbf{P}(D_\infty^{(0)}>0)>0$ by (i). Since $R(x)\leq c_{27}(1+x_+)$, we see that $D_\infty^{(0)}\leq c_{27}\partial W_\infty$, and therefore $\mathbf{P}(\partial W_\infty>0)>0$. Hence, we have $\partial W_\infty>0$ almost surely on the event of non-extinction. We can now prove (ii). Let $\beta\geq 0$. On the event of non-extinction of the branching random walk killed below β, we can find a vertex u (in the killed branching random walk) such that $\min\{V(v),v>u\}\geq V(u)$ and $\#\{v\in\mathbb{T}:v>u\}=\infty$. The sum $\sum_{v\geq u,|v|=n}R(\beta+V(v))\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v_k)\geq-\beta,\forall k\leq n\}}$ converges then to $c_0\partial W_\infty(u)$ where $\partial W_\infty(u)$ is the almost sure limit of $\sum_{v\geq u,|v|=n}V(v)\mathrm{e}^{-V(v)}$. We know from (iii) that $\partial W_\infty(u)>0$, hence $\sum_{v\geq u,|v|=n}R(\beta+V(v))\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v_k)\geq-\beta,\forall k\leq n\}}$ has a positive limit. Since $D_n^{(\beta)}\geq\mathrm{e}^{-V(u)}\sum_{v\geq u,|v|=n}R(\beta+V(v))\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v_k)\geq-\beta,\forall k\leq n\}}$, we have that $D_\infty^{(\beta)}>0$. $\square$

B Auxiliary estimates

Lemma B.1 *Let X and $\tilde{X}$ be non-negative random variables such that (1.4) holds.*
(i) We have

$$\mathbf{E}\left[X(\ln_+\tilde{X})^2\right]<\infty,\qquad\mathbf{E}\left[\tilde{X}\ln_+X\right]<\infty.$$

(ii) As $z\to\infty$,

$$\begin{aligned}\mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2\min(\ln_+(X+\tilde{X}),z)\right]&=o(z),\\ \mathbf{E}\left[\tilde{X}\ln_+(X+\tilde{X})\min(\ln_+(X+\tilde{X}),z)\right]&=o(z).\end{aligned}$$

Proof. We prove (i). We claim that for any $x,\tilde{x}\geq 0$

$$x(\ln_+\tilde{x})^2\leq 3x(\ln_+x)^2+2\mathrm{e}^{-1}\tilde{x}\ln_+\tilde{x}. \tag{B.1}$$

If $\tilde{x}\leq x$, the inequality is immediate. Therefore, we suppose that $\tilde{x}>x$ and we write $z=\tilde{x}/x$. Equation (B.1) can be rewritten

$$(\ln z)^2+2\ln z\ln_+x\leq 3(\ln_+x)^2+2\mathrm{e}^{-1}z\ln z+2\mathrm{e}^{-1}z\ln_+x.$$

We check that $(\ln z)^2 \leq 2e^{-1} z \ln z$ and $2 \ln z \ln_+ x \leq 2e^{-1} z \ln_+ x$ for $z \geq 1$, which ends the proof of (B.1). It yields that

$$\mathbf{E}\left[X(\ln_+ \tilde{X})^2\right] \leq 3\mathbf{E}\left[X(\ln_+ X)^2\right] + 2e^{-1}\mathbf{E}\left[\tilde{X} \ln_+ \tilde{X}\right]$$

which is finite under (1.4). Also, $\tilde{X} \ln_+ X \leq \max(\tilde{X} \ln_+ \tilde{X}, X \ln_+ X)$, hence $\mathbf{E}[\tilde{X} \ln_+(X)] < \infty$. We turn now to the proof of (ii). Let $\varepsilon > 0$. We observe that

$$\begin{aligned} & \mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z)\right] \\ = & \mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z),\, \ln_+(X+\tilde{X}) \geq \varepsilon z\right] \\ & + \mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z),\, \ln_+(X+\tilde{X}) < \varepsilon z\right]. \end{aligned}$$

On one hand,

$$\begin{aligned} & \mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z),\, \ln_+(X+\tilde{X}) \geq \varepsilon z\right] \\ \leq & z\mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2,\, \ln_+(X+\tilde{X}) \geq \varepsilon z\right] \\ = & z o_z(1) \end{aligned}$$

since $\mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2\right] < \infty$. On the other hand,

$$\mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z),\, \ln_+(X+\tilde{X}) < \varepsilon z\right] \leq \varepsilon z \mathbf{E}\left[X(\ln_+ X+\tilde{X})^2\right].$$

Thus, $\mathbf{E}\left[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z)\right] \leq (1+\mathbf{E}[X(\ln_+ X+\tilde{X})^2])\varepsilon z$ for z large enough, and is therefore $o(z)$. We show similarly that $\mathbf{E}\left[\tilde{X} \ln_+(X+\tilde{X}) \min(\ln_+(X+\tilde{X}), z)\right] = o(z)$. □

Let $(S_n)_{n\geq 0}$ be a one-dimensional random walk, with $\mathbf{E}[S_1] = 0$ and $\mathbf{E}[(S_1)^2] < \infty$.

Lemma B.2 *(i) There exists a constant $c_{45} > 0$ such that for any $\alpha \geq 0$, $z \geq 0$ and $x \geq 0$*

$$\mathbf{E}_z\left[\sum_{\ell \geq 0} \mathbf{1}_{\{S_\ell \leq x\}} \mathbf{1}_{\{\min_{j\leq \ell} S_j \geq 0\}} S_\ell^\alpha\right] \leq c_{45}(1+x)^{1+\alpha}(1+\min(x,z)).$$

(ii) We have

$$\mathbf{E}\left[\sum_{\ell \geq 0} e^{-S_\ell/2} \mathbf{1}_{\{\min_{j\leq \ell} S_j \geq 0\}}\right] = c_{46} < \infty.$$

(iii) There exists a constant $c_{47} > 0$ such that for any $z \geq 0$,

$$\mathbf{E}_z \left[\sum_{\ell \geq 0} \mathrm{e}^{-S_\ell/2} \mathbf{1}_{\{\min_{j\leq \ell} S_j \geq 0\}} \right] \leq c_{47}.$$

Proof. We observe that, for $\alpha \geq 0$,

$$\mathbf{E}_z \left[S_\ell^\alpha \mathbf{1}_{\{S_\ell \leq x,\, \min_{j\leq \ell} S_j \geq 0\}} \right] \leq x^\alpha \mathbf{P}_z \left(S_\ell \leq x,\, \min_{j \leq \ell} S_j \geq 0 \right).$$

To prove (i), it remains to show that $\sum_{\ell \geq 0} \mathbf{P}_z \left(S_\ell \leq x,\, \min_{j\leq\ell} S_j \geq 0 \right) \leq c_{45}(1+x)(1+\min(x,z))$. Suppose that $x < z$. If τ_x^- denotes the first passage time at level x of $(S_n)_{n\geq 0}$, we have

$$\begin{aligned} \sum_{\ell\geq 0} \mathbf{P}_z \left(S_\ell \leq x,\, \min_{j\leq \ell} S_j \geq 0 \right) &= \mathbf{E}_z \left[\sum_{\ell \geq \tau_x^-} \mathbf{1}_{\{S_\ell \leq x,\, \min_{j\leq\ell} S_j \geq 0\}} \right] \\ &\leq \mathbf{E} \left[\sum_{\ell\geq 0} \mathbf{1}_{\{S_\ell \leq x,\, \min_{j\leq \ell} S_j \geq -x\}} \right] \end{aligned}$$

where we used the Markov property at time τ_x^-. We have

$$\begin{aligned} \sum_{\ell \geq 0} \mathbf{P} \left(S_\ell \leq x,\, \min_{j\leq\ell} S_j \geq -x \right) &\leq 1 + x^2 + \sum_{\ell > x^2} \mathbf{P} \left(S_\ell \leq x,\, \min_{j \leq \ell} S_j \geq -x \right) \\ &\leq 1 + x^2 + c_{48} \sum_{\ell > x^2} (1+x)^3 \ell^{-3/2} \\ &\leq c_{49}(1+x)^2 \qquad \text{(B.2)} \end{aligned}$$

by (2.6). Suppose now that $x \geq z$. Then,

$$\begin{aligned} &\sum_{\ell \geq 0} \mathbf{P}_z \left(S_\ell \leq x,\, \min_{j\leq\ell} S_j \geq 0 \right) \\ \leq\ &\sum_{\ell \leq x^2} \mathbf{P}_z \left(\min_{j\leq \ell} S_j \geq 0 \right) + \sum_{\ell > x^2} \mathbf{P}_z \left(S_\ell \leq x,\, \min_{j\leq\ell} S_j \geq 0 \right). \end{aligned}$$

From (2.5), we know that $\mathbf{P}_z \left(\min_{j\leq\ell} S_j \geq 0 \right) \leq c_{50}(1+z)(1+\ell)^{-1/2}$, whereas, by (2.6),

$$\mathbf{P}_z \left(S_\ell \leq x,\, \min_{j\leq\ell} S_j \geq 0 \right) \leq c_{51}(1+z)(1+x)^2(1+\ell)^{-3/2}.$$

We get

$$\begin{aligned}\sum_{\ell\geq 0}\mathbf{P}_z\left(S_\ell\leq x,\, \min_{j\leq \ell}S_j\geq 0\right) &\leq c_{50}\sum_{\ell\leq x^2}\frac{1+z}{\sqrt{1+\ell}}+c_{51}\sum_{\ell>x^2}(1+z)(1+x)^2(1+\ell)^{-3/2} \\ &\leq c_{52}(1+z)(1+x). \qquad \text{(B.3)}\end{aligned}$$

From (B.2) when $x<z$ and (B.3) when $x\geq z$, we have for $x,z\geq 0$,

$$\sum_{\ell\geq 0}\mathbf{P}_z\left(S_\ell\leq x,\, \min_{j\leq\ell}S_j\geq 0\right)\leq (c_{49}+c_{52})(1+x)(1+\min(x,z)).$$

This ends the proof of (i). We turn to the statement (ii). We have

$$\sum_{\ell\geq 0}\mathbf{E}\left[\mathrm{e}^{-S_\ell/2}\mathbf{1}_{\{\min_{j\leq\ell}S_j\geq 0\}}\right]=\sum_{\ell\geq 0}\sum_{i\geq 0}\mathrm{e}^{-i/2}\mathbf{P}(S_\ell\in[i,i+1),\,\min_{j\leq\ell}S_j\geq 0).$$

By (2.6), $\mathbf{P}(S_\ell\in[i,i+1),\,\min_{j\leq\ell}S_j\geq 0)\leq c_{53}(1+i)(1+\ell)^{-3/2}$, which completes the proof of (ii). Let $(\tilde T_k,\tilde H_k,k\geq 0)$ be the strict descending ladder epochs and heights of $(S_n)_{n\geq 0}$, i.e. $\tilde T_0:=0$, $\tilde H_0:=S_0$ and for any $k\geq 1$, $\tilde T_k:=\min\{j>\tilde T_{k-1}\,:\,S_j<H_{k-1}\}$, $\tilde H_k:=S_{\tilde T_k}$. By applying the Markov property at the times $(\tilde T_k,\,k\geq 0)$, we observe that

$$\mathbf{E}_z\left[\sum_{\ell\geq 0}\mathrm{e}^{-S_\ell/2}\mathbf{1}_{\{\min_{j\leq\ell}S_j\geq 0\}}\right]=c_{46}\mathbf{E}_z\left[\sum_{k\geq 0}\mathrm{e}^{-\tilde H_k/2}\mathbf{1}_{\{\tilde H_k\geq 0\}}\right]$$

where c_{46} is the constant of (ii). The fact that $Z(z):=\mathbf{E}_z\left[\sum_{k\geq 0}\mathrm{e}^{-\tilde H_k/2}\mathbf{1}_{\{\tilde H_k\geq 0\}}\right]$ is bounded in $z\geq 0$ then comes from the renewal theorem: let $U(dy)$ denotes the renewal measure of $(\tilde H_k,\,k\geq 0)$, i.e. $U(dy):=\sum_{k\geq 0}\mathbf{P}(\tilde H_k\in dy)$. Then $Z(z)=\int_{-z}^0\mathrm{e}^{-(z+y)/2}U(dy)$ which is bounded by the renewal theorem (see Section XI.1 of [13]). □

For $\alpha>0$, $a\geq 0$, $n\geq 1$ and $0\leq i\leq n$, we define

$$k_i:=\begin{cases} i^\alpha, & \text{if } 0\leq i\leq\lfloor n/2\rfloor,\\ a+(n-i)^\alpha, & \text{if } \lfloor n/2\rfloor<i\leq n.\end{cases} \qquad \text{(B.4)}$$

Lemma B.3 *Let $\alpha\in(0,1/6)$ and $\varepsilon>0$.*
(i) There exist $d>0$ and $c_{53}>0$ such that for $u\geq 0$, $a\geq 0$ and $n\geq 1$,

$$\mathbf{P}\Big\{\exists 0\leq i\leq n:\, S_i\leq k_i-d,\,\min_{j\leq n}S_j\geq 0,\,\min_{\lfloor n/2\rfloor<j\leq n}S_j\geq a,\,S_n\leq a+u\Big\} \qquad \text{(B.5)}$$

$$\leq\ (1+u)^2\Big\{\frac{\varepsilon}{n^{3/2}}+c_{53}\frac{(n^\alpha+a)^2}{n^{2-\alpha}}\Big\},$$

where k_i is given by (B.4).

Proof. We treat $n/2$ as an integer. Let E be the event in (B.5). We have $\mathbf{P}(E) \le \sum_{i=1}^n \mathbf{P}(E_i)$ where

$$E_i := \{S_i \le k_i - d, \min_{j\le n} S_j \ge 0, \min_{\lambda n<j\le n} S_j \ge a, \ S_n \le a+u\}.$$

We first treat the case $i \le n/2$, so that $k_i = i^\alpha$. By the Markov property at time $i\ge 1$ and (2.7), we have

$$\mathbf{P}(E_i) \le \frac{c_{54}(1+u)^2}{n^{3/2}} \mathbf{E}\left[(1+S_i)\mathbf{1}_{\{S_i\le i^\alpha, \min_{j\le i} S_j \ge 0\}}\right]$$

which is smaller than $\frac{c_{55}(1+u)^2}{n^{3/2}} \frac{(1+i^\alpha)^3}{i^{3/2}}$ by (2.6). It yields that, if L is greater than some constant L_0 (which does not depend on d), we have

$$\sum_{i=L}^{\lambda n} \mathbf{P}(E_i) \le (1+u)^2 \frac{\varepsilon}{n^{3/2}}, \tag{B.6}$$

$[\sum_{i=x}^y := 0$ if $x > y$.$]$ We treat the case $n/2 < i \le n$. We have by the Markov property at time i and (2.6),

$$\mathbf{P}(E_i) \le \frac{c_{56}(1+u)^2}{(n-i+1)^{3/2}} \mathbf{E}\left[(1+S_i-a)\mathbf{1}_{\{S_i\le a+(n-i)^\alpha, \min_{j\le n} S_j\ge 0, \min_{\lambda n<j\le i} S_j \ge a\}}\right].$$

If $i\ge 2n/3$, we use (2.7) to see that $\mathbf{P}(E_i) \le c_{57}(1+u)^2 \frac{(1+n-i)^{3\alpha-\frac{3}{2}}}{n^{3/2}}$. Therefore, if $L\ge L_1$, (L_1 does not depend on d),

$$\sum_{i=\lfloor 2n/3\rfloor}^{n-L} \mathbf{P}(E_i) \le (1+u)^2 \frac{\varepsilon}{n^{3/2}}. \tag{B.7}$$

If $n/2 < i < 2n/3$, we simply write

$$\begin{aligned}
\mathbf{P}(E_i) &\le \frac{c_{58}(1+u)^2}{(n-i+1)^{3/2}} \mathbf{E}\left[(1+S_i-a)\mathbf{1}_{\{a\le S_i\le a+(n-i)^\alpha, \min_{j\le i} S_j\ge 0\}}\right] \\
&\le c_{59}(1+u)^2 \frac{(n-i)^\alpha}{(n-i+1)^{3/2}} \mathbf{P}(a\le S_i \le a+(n-i)^\alpha, \min_{j\le i} S_j \ge 0) \\
&\le c_{60}(1+u)^2 \frac{n^\alpha (a+n^\alpha)^2}{n^3}
\end{aligned}$$

by (2.6). We deduce that

$$\sum_{i=n/2}^{\lfloor 2n/3\rfloor} \mathbf{P}(E_i) \le c_{61}(1+u)^2 \frac{(n^\alpha+a)^2}{n^{2-\alpha}}. \tag{B.8}$$

Notice that our choice of L does not depend on the constant d. Thus, we are allowed to choose $d \geq L^\alpha$, for which $\mathbf{P}(E_i) = 0$ if $i \in [1, L] \cup [n-L, n]$. We obtain by (B.6),(B.7) and (B.8)

$$\sum_{i=1}^{n} \mathbf{P}(E_i) \leq (1+u)^2 \Big\{ 2\frac{\varepsilon}{n^{3/2}} + c_{61} \frac{(n^\alpha + a)^2}{n^{2-\alpha}} \Big\},$$

hence $\mathbf{P}(E) \leq (1+u)^2 \Big\{ 2\frac{\varepsilon}{n^{3/2}} + c_{61} \frac{(n^\alpha + a)^2}{n^{2-\alpha}} \Big\}$ indeed. □

C The good vertex

We recall the definition of a good vertex. Let $z \geq 0$ and $L \geq 0$. We define for $n \geq 1$ and $k \leq n$

$$d_k := \begin{cases} 0, & \text{if } 0 \leq k \leq \frac{n}{2}, \\ \max(\frac{3}{2} \ln n - z - L - 1, 0), & \text{if } \frac{n}{2} < k \leq n. \end{cases}$$

Let also

$$e_k = e_k^{(n)} := \begin{cases} k^{1/12}, & \text{if } 0 \leq k \leq \frac{n}{2}, \\ (n-k)^{1/12}, & \text{if } \frac{n}{2} < k \leq n. \end{cases}$$

For $|u| = n$, we say that $u \in \mathcal{Z}^{z,L}$ if $V(u_k) \geq d_k$ for $k \leq n$ and $V(u) \in I_n(z)$. We say that u is a good vertex if $u \in \mathcal{Z}^{z,L}$ and for any $1 \leq k \leq n$,

$$\text{(C.1)} \qquad \sum_{v \in \Omega(u_k)} \mathrm{e}^{-(V(v)-d_k)} \Big\{ 1 + (V(v) - d_k)_+ \Big\} \leq B \mathrm{e}^{-e_k}.$$

We defined the probability $\mathbf{Q}$ in (2.3) and the spine $(w_n, n \geq 0)$ in subsection 2.1.

Lemma C.1 *Fix $L \geq 0$. For any $\varepsilon > 0$, we can find B large enough in (C.1) such that $\mathbf{Q}(w_n$ is not a good vertex, $w_n \in \mathcal{Z}^{z,L}) \leq \varepsilon n^{-3/2}$ for any $n \geq 1$ and $z \geq 0$.*

Proof. Fix $L \geq 0$ and let $\varepsilon > 0$. We do as if $n/2$ is an integer. By Lemma B.3, there exists $c_{62} = c_{62}(L) > 0$ and $N = N(L)$ such that for $n \geq N$ and $z \geq 0$

$$\mathbf{Q}\left(w_n \in \mathcal{Z}^{z,L}, \exists 0 \leq j \leq n-1 \,:\, V(w_j) \leq d_{j+1} + 2e_{j+1} - c_{62} \right) \leq \frac{\varepsilon}{n^{3/2}}.$$

We see that, for any $1 \leq k \leq n$,

$$\left\{ \sum_{v \in \Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)} \Big\{ 1 + (V(v) - d_k)_+ \Big\} > B\mathrm{e}^{-e_k},\, V(w_{k-1}) \geq d_k + 2e_k - c_{62} \right\}$$

$$\subset \left\{ \sum_{v \in \Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)} \Big\{ 1 + (V(v) - d_k)_+ \Big\} > B\mathrm{e}^{-(V(w_{k-1}) - d_k + c_{62})/2} \right\}.$$

Therefore, for $n \ge N$ and $z \ge 0$

$$\text{(C.2)}\quad \mathbf{Q}(w_n \text{ is not a good vertex, } w_n \in \mathcal{Z}^{z,L}) \le \frac{\varepsilon}{n^{3/2}} + \sum_{k=1}^n \mathbf{Q}\left(\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}\Big\{1+(V(v)-d_k)_+\Big\} > B\mathrm{e}^{-(V(w_{k-1})-d_k+c_{62})/2},\, w_n\in \mathcal{Z}^{z,L}\right).$$

We only have to show that we can find B large enough such that

$$\sum_{k=1}^n \mathbf{Q}\left(\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}\Big\{1+(V(v)-d_k)_+\Big\} > B\mathrm{e}^{-(V(w_{k-1})-d_k)/2},\, w_n\in \mathcal{Z}^{z,L}\right) \le \frac{\varepsilon}{n^{3/2}}.$$

Let $1 \le k \le n/2$, hence $d_k = 0$. By the Markov property at time k, we get

$$\begin{aligned} &\mathbf{Q}\left(\sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\, w_n\in \mathcal{Z}^{z,L}\right)\\ \le\ & \mathbf{E_Q}\left[\lambda(V(w_k),k,n),\ \sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\, V(w_j)\ge 0,\,\forall\, j\le k\right]\end{aligned}$$

where $\lambda(r,k,n) := \mathbf{Q}_r(V(w_j) \ge d_{j+k},\, \forall\, j \le n-k,\, V(w_{n-k}) \in I_n(z))$. We get by (2.7), $\lambda(r,k,n) \le c_{63} n^{-3/2}(1+r_+)$. It yields that

$$\begin{aligned} \text{(C.3)}\quad & n^{3/2}\mathbf{Q}\left(\sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\, w_n\in \mathcal{Z}^{z,L}\right)\\ \le\ & c_{63}\mathbf{E_Q}\left[1+V(w_k)_+,\ \sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\, V(w_j)\ge 0,\,\forall\, j\le k\right].\end{aligned}$$

We see that

$$\begin{aligned} &\sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big(1+V(v)_+\Big)\\ \le\ & \mathrm{e}^{-V(w_{k-1})}\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-V(w_{k-1}))}\Big\{1+V(w_{k-1})_+ + (V(v)-V(w_{k-1}))_+\Big\}\\ \le\ & \mathrm{e}^{-V(w_{k-1})}(1+V(w_{k-1})_+)\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-V(w_{k-1}))}\Big\{1+(V(v)-V(w_{k-1}))_+\Big\}.\end{aligned}$$

With the notation of (4.2), we have then

$$\sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big(1+V(v)_+\Big) \le \mathrm{e}^{-V(w_{k-1})}(1+V(w_{k-1})_+)\xi(w_k).$$

It yields that

$$\begin{aligned}
&\mathbf{E_Q}\left[1+V(w_k)_+,\ \sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\ V(w_j)\geq 0,\ \forall\, j\leq k\right]\\
\leq\ &\mathbf{E_Q}\left[1+V(w_k)_+,\ \xi(w_k) > B\frac{\mathrm{e}^{V(w_{k-1})/2}}{1+V(w_{k-1})},\ V(w_j)\geq 0,\ \forall\, j\leq k\right].
\end{aligned}$$

On the other hand, we have

$$1+V(w_k)_+ \leq 1+V(w_{k-1})_+ + (V(w_k)-V(w_{k-1}))_+.$$

Let (ξ,Δ) be generic random variables distributed as $\left(\sum_{|x|=1}(1+V(x)_+)\mathrm{e}^{-V(x)}, V(w_1)_+\right)$ under $\mathbf{Q}$, and independent of the other random variables. By the Markov property at time $k-1$, we obtain that

$$\begin{aligned}
&\mathbf{E_Q}\left[1+V(w_k)_+,\ \sum_{v\in\Omega(w_k)} \mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\ V(w_j)\geq 0,\ \forall j\leq k\right]\\
\leq\ &\mathbf{E_Q}\left[\kappa(V(w_{k-1})),\ V(w_j)\geq 0,\ \forall\, j\leq k-1\right]
\end{aligned}$$

with, for $x\geq 0$, $\kappa(x) := (1+x)\mathbf{1}_{\{\xi > B\mathrm{e}^{x/2}/(1+x)\}} + \Delta_+\mathbf{1}_{\{\xi > B\mathrm{e}^{x/2}/(1+x)\}}$. Taking $\tilde{B} = c_{64}B$ slightly bigger than B, we can assume that $\kappa(x)\leq (1+x)\mathbf{1}_{\{\xi>\tilde{B}\mathrm{e}^{x/3}\}} + \Delta_+\mathbf{1}_{\{\xi>\tilde{B}\mathrm{e}^{x/3}\}}$. In view of (C.3), it follows that

$$\sum_{k\leq n/2}\mathbf{Q}\left(\sum_{v\in\Omega(w_k)}\mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\ w_n\in\mathcal{Z}^{z,L}\right)\leq c_{63}n^{-3/2}(D_1+D_2)$$

where

$$\begin{aligned}
D_1 &:= \sum_{k\geq 0}\mathbf{E_Q}\left[(1+V(w_k))\mathbf{1}_{\{V(w_k)\leq 3(\ln\xi-\ln\tilde{B})\}},\ \min_{j\leq k}V(w_j)\geq 0\right]\\
D_2 &:= \sum_{k\geq 0}\mathbf{E_Q}\left[\Delta_+\mathbf{1}_{\{V(w_k)\leq 3(\ln\xi-\ln\tilde{B})\}},\ \min_{j\leq k}V(w_j)\geq 0\right].
\end{aligned}$$

We recall that by Proposition 2.1 $(V(w_n),\, n\geq 0)$ is distributed as $(S_n,\, n\geq 0)$ (under $\mathbf{P}$). Notice that in the definition of D_1, the term inside the expectation is 0 if $\tilde{B}>\xi$. Therefore, we can add the indicator that $\tilde{B}\leq\xi$. By Lemma B.2 (i), we get that

$$D_1\leq c_{65}\mathbf{E_Q}\left[\mathbf{1}_{\{\tilde{B}\leq\xi\}}(1+(\ln\xi-\ln\tilde{B})_+)^2\right]\leq c_{65}\mathbf{E_Q}\left[\mathbf{1}_{\{\tilde{B}\leq\xi\}}(1+\ln_+\xi)^2\right].$$

Observe that $\xi = X + \tilde{X}$ with the notation of (1.2). Going back to the measure $\mathbf{P}$, we get

$$D_1 \leq c_{65}\mathbf{E}\left[X\mathbf{1}_{\{\tilde{B}\leq X+\tilde{X}\}}(1+\ln_+(X+\tilde{X}))^2\right] \leq \varepsilon$$

for B (or $\tilde{B} := c_{64}B$) large enough since $\mathbf{E}\left[X(1+\ln_+(X+\tilde{X}))^2\right] < \infty$ by (1.4) and Lemma B.1 (i). Similarly,

$$D_2 \leq c_{66}\mathbf{E}\left[\tilde{X}\mathbf{1}_{\{\tilde{B}\leq X+\tilde{X}\}}(1+\ln_+(X+\tilde{X}))\right] \leq \varepsilon$$

for B large enough. Therefore,

$$\sum_{k\leq n/2}\mathbf{Q}\left(\sum_{v\in\Omega(w_k)}\mathrm{e}^{-V(v)}\Big\{1+V(v)_+\Big\} > B\mathrm{e}^{-V(w_{k-1})/2},\, w_n\in\mathcal{Z}^{z,L}\right) \leq 2\frac{\varepsilon}{n^{3/2}}.$$

The case $n/2 < k < n$ can be treated with the same strategy by reversing time, and we feel free to skip this case. We find that

$$\sum_{k=\lfloor n/2\rfloor+1}^{n}\mathbf{Q}\left(\sum_{v\in\Omega(w_k)}\mathrm{e}^{-(V(v)-d_k)}\Big\{1+(V(v)-d_k)_+\Big\} > B\mathrm{e}^{-(V(w_{k-1})-d_k)/2},\, w_n\in\mathcal{Z}^{z,L}\right) \leq 2\frac{\varepsilon}{n^{3/2}}$$

for B large enough. Going back to (C.2), it yields that $\mathbf{Q}(w_n$ is not a good vertex, $w_n \in \mathcal{Z}^{z,L}) \leq 5\varepsilon n^{-3/2}$ for $n \geq N$ and $z \geq 0$. Choosing $B \geq N$, the inequality holds for any $n \geq 1$ and $z \geq 0$. □

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