EURANDOM PREPRINT SERIES
2011-018

The precise tail behavior of the total progeny of a killed branching random walk

Elie Aïdékon, Yueyun Hu, and Olivier Zindy
ISSN 1389-2355

THE PRECISE TAIL BEHAVIOR OF THE TOTAL PROGENY OF A KILLED BRANCHING RANDOM WALK

ELIE AIDEKON, YUEYUN HU, AND OLIVIER ZINDY

March 1, 2011

Abstract. Consider a branching random walk on the real line with a killing barrier at zero: starting from a nonnegative point, particles reproduce and move independently, but are killed when they touch the negative half-line. The population of the killed branching random walk dies out almost surely in both critical and subcritical cases, where by subcritical case we mean that the rightmost particle of the branching random walk without killing has a negative speed and by critical case when this speed is zero. We investigate the total progeny of the killed branching random walk and give its precise tail distribution both in the critical and subcritical cases, which solves an open problem of D. Aldous [4].

1. Introduction

We consider a one-dimensional discrete-time branching random walk V on the real line $\mathbb{R}$. At the beginning, there is a single particle located at the origin 0. Its children, who form the first generation, are positioned according to a certain point process $\mathscr{L}$ on $\mathbb{R}$. Each of the particles in the first generation independently gives birth to new particles that are positioned (with respect to their birth places) according to a point process with the same law as $\mathscr{L}$; they form the second generation. And so on. For any $n \geq 1$, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation.

Clearly, the particles of the branching random walk V form a Galton–Watson tree, which we denote by $\mathcal{T}$. Call $\varnothing$ the root. For every vertex $u \in \mathcal{T}$, we denote by $|u|$ its generation (then $|\varnothing| = 0$) and by $(V(u), |u| = n)$ the positions of the particles in the n-th generation. Then $\mathscr{L} = \sum_{|u|=1} \delta_{\{V(u)\}}$. The tree $\mathcal{T}$ will encode the genealogy of our branching random walk.

It will be more convenient to consider a branching random walk V starting from an arbitrary $x \in \mathbb{R}$ [namely, $V(\varnothing) = x$], whose law is denoted by $\mathbf{P}_x$ and the corresponding expectation by $\mathbf{E}_x$. For simplification, we write $\mathbf{P} \equiv \mathbf{P}_0$ and $\mathbf{E} \equiv \mathbf{E}_0$. Let $\nu := \sum_{|u|=1} 1$ be the number of particles in the first generation and denote by $\nu(u)$ the number of children of $u \in \mathcal{T}$.

Assume that $\mathbf{E}[\nu] > 1$, namely the Galton–Watson tree $\mathcal{T}$ is supercritical, then the system survives with positive probability $\mathbf{P}(\mathcal{T} = \infty) > 0$. Let us define the logarithmic generating function for the branching walk:

$$\psi(t) := \log \mathbf{E}\Big[\sum_{|u|=1} e^{tV(u)}\Big] \in (-\infty, +\infty], \qquad t \in \mathbb{R}.$$

2000 *Mathematics Subject Classification.* 60F05,60J80.

Key words and phrases. Killed branching random walk, total progeny, spinal decomposition,Yaglom-type theorem, time reversed random walk.

This research was supported by the french ANR project MEMEMO and the Netherlands Organisation for Scientific Research (NWO).

We shall assume that ψ is finite on an open interval containing 0 and that supp$\mathscr{L}\cap(0,\infty)\neq\emptyset$ [the later condition is to ensure that V can visit $(0,\infty)$ with positive probability, otherwise the problem that we shall consider becomes of a different nature]. Assume that there exists $\varrho_*>0$ such that

$$\psi(\varrho_*)=\varrho_*\psi'(\varrho_*). \tag{1.1}$$

We also assume that ψ is finite on an open set containing $[0,\varrho_*]$. The condition (1.1) is not restrictive: For instance, if we denote by $m^*=\operatorname{esssup}\operatorname{supp}\mathscr{L}$, then (1.1) is satisfied if either $m^*=\infty$ or $m^*<\infty$ and $\mathbf{E}\sum_{|u|=1}1_{\{V(u)=m^*\}}<1$, see Jaffuel [17] for detailed discussions.

Recall that (Kingman [22], Hammersley [13], Biggins [7]) conditioned on $\{\mathcal{T}=\infty\}$,

$$\lim_{n\to\infty}\frac{1}{n}\max_{|u|=n}V(u)=\psi'(\varrho_*),\qquad \text{a.s.}, \tag{1.2}$$

where ϱ_* is given in (1.1). According to $\psi'(\varrho_*)=0$ or $\psi'(\varrho_*)<0$, we call the case critical or subcritical. Conditioned on $\{\mathcal{T}=\infty\}$, the rightmost particle in the branching random walk without killing has a negative speed in the subcritical case, while in the critical case it converges almost surely to $-\infty$ in the logarithmical scale (see [15] and [2] for the precise statement of the rate of almost sure convergence).

We now place a killing barrier at zero: any particle which enters $(-\infty,0)$ is removed and does not produce any offspring. Hence at every generation $n\geq 0$, survive only the particles that always stayed nonnegative up to time n. Denote by $\mathscr{Z}$ the set of all lived particles of the killed branching walk:

$$\mathscr{Z}:=\Big\{u\in\mathcal{T}\,:\,V(v)\geq 0,\qquad\forall\,v\in[\varnothing,u]\Big\},$$

where $[\varnothing,u]$ denotes the shortest path in the tree $\mathcal{T}$ from u to the root $\varnothing$. We are interested in the total progeny

$$Z:=\#\mathscr{Z}.$$

Then $Z<\infty, a.s.$, in both critical and subcritical cases. David Aldous made the following conjecture:

Conjecture (D.Aldous [4]):

- **(i) (critical case):** If $\psi'(\varrho_*)=0$, then $\mathbf{E}[Z]<\infty$ and $\mathbf{E}[Z\log Z]=\infty$.
- **(ii) (subcritical case):** If $\psi'(\varrho_*)<0$, then there exists some constant $b>1$ such that $\mathbf{P}(Z>n)=n^{-b+o(1)}$ as $n\to\infty$.

Let us call iid case if $\mathscr{L}$ is of form: $\mathscr{L}=\sum_{i=1}^{\nu}\delta_{\{X_i\}}$ with $(X_i)_{i\geq1}$ a sequence of i.i.d. real-valued variables, independent of ν. There are several previous works on the critical and iid case: when (X_i) are Bernoulli random variables, Pemantle [29] obtained the precise asymptotic of $\mathbf{P}(Z=n)$ as $n\to\infty$, where the key ingredient of his proof is the recursive structure of the system inherited from the Bernoulli variables (X_i). For general random variables (X_i), Addario-Berry and Broutin [1] recently confirmed Aldous' conjecture (i); This was improved later by Aïdékon [3] who proved that for a regular tree $\mathcal{T}$ (namely when ν equals some integer), for any fixed $x\geq 0$,

$$c_1R(x)e^{\varrho_*x}\leq\liminf_{n\to\infty}n(\log n)^2\mathbf{P}_x(Z>n)\leq\limsup_{n\to\infty}n(\log n)^2\mathbf{P}_x(Z>n)\leq c_2R(x)e^{\varrho_*x},$$

where $c_2>c_1>0$ are two constants and $R(x)$ is some renewal function which will be defined later. For the continuous setting, the branching Brownian motion, Maillard [27] solved the question by analytic tools, using link with the F-KPP equation. Berestycki et al. [5] looked at the genealogy of the branching Brownian motion with absorption in the near-critical case.

In this paper, we aim at the exact tail behavior of Z both in critical and subcritical cases and for a general point process $\mathscr{L}$.

Before the statement of our result, we remark that in the subcritical case ($\psi'(\varrho_*) < 0$), there are two real numbers ϱ_- and ϱ_+ such that $0 < \varrho_- < \varrho_* < \varrho_+$ and

$$\psi(\varrho_-) = \psi(\varrho_+) = 0,$$

[the existence of ϱ_+ follows from the assumption that $\mathrm{supp}\mathscr{L} \cap (0,\infty) \neq \emptyset$].

In the critical case, we suppose that

$$\mathbf{E}\left[\nu^{1+\delta^*}\right] < \infty, \qquad \sup_{\theta\in[-\delta^*,\varrho_*+\delta^*]} \psi(\theta) < \infty, \qquad \text{for some } \delta^* > 0. \tag{1.3}$$

In the subcritical case, we suppose that

$$\mathbf{E}\left[\sum_{|u|=1} (1 + e^{\varrho_- V(u)})\right]^{\frac{\varrho_+}{\varrho_-}+\delta^*} < \infty, \qquad \sup_{\theta\in[-\delta^*,\varrho_++\delta^*]} \psi(\theta) < \infty, \tag{1.4}$$

for some $\delta^* > 0$. In both cases, we always assume that there is no lattice that supports $\mathscr{L}$ almost surely.

Our result on the total progeny reads as follows.

Theorem 1 (Tail of the total progeny). *Assume (1.1) and that*

$$\mathbf{E}[\nu^\alpha] < \infty, \qquad \text{for some } \begin{cases} \alpha > 2, & \text{in the critical case;} \\ \alpha > 2\frac{\varrho_+}{\varrho_-}, & \text{in the subcritical case.} \end{cases} \tag{1.5}$$

(i) (Critical case) If $\psi'(\varrho_*) = 0$ and (1.3) holds, then there exists a constant $c_{crit} > 0$ such that for any $x \geq 0$,

$$\mathbf{P}_x\Big(Z > n\Big) \sim c_{crit}\, R(x)\, e^{\varrho_* x} \frac{1}{n(\log n)^2}, \qquad n \to \infty,$$

where $R(x)$ is a renewal function defined in (3.20).

(ii) (Subcritical case) If $\psi'(\varrho_*) < 0$ and (1.4) holds, then there exists a constant $c_{sub} > 0$ such that for any $x \geq 0$,

$$\mathbf{P}_x\Big(Z > n\Big) \sim c_{sub} R(x) e^{\varrho_+ x} n^{-\frac{\varrho_+}{\varrho_-}}, \qquad n \to \infty,$$

where $R(x)$ is a renewal function defined in (3.20).

The values of c_{crit} and c_{sub} are given in Lemma 2. Let us make some remarks on the assumptions (1.3) and (1.4).

Remark 1 (iid case). *If $\mathscr{L} = \sum_{i=1}^{\nu} \delta_{\{X_i\}}$ with $(X_i)_{i\geq 1}$ a sequence of i.i.d. real-valued variables, independent of ν, then (1.3) holds if and only if for some $\delta > 0$, $\mathbf{E}[\nu^{1+\delta}] < \infty$ and $\sup_{\theta\in[-\delta,\varrho_*+\delta]} \mathbf{E}\left[e^{\theta X_1}\right] < \infty$ while (1.4) holds if and only if $\mathbf{E}[\nu^{\frac{\varrho_+}{\varrho_-}+\delta}] < \infty$ and $\sup_{\theta\in[-\delta,\varrho_++\delta]} \mathbf{E}\left[e^{\theta X_1}\right] < \infty$ for some $\delta > 0$.*

Remark 2. *By Hölder's inequality, elementary computations show that (1.3) is equivalent to $\mathbf{E}\left[\sum_{|u|=1}(1 + e^{\varrho_* V(u)})\right]^{1+\delta} < \infty$ and $\sup_{\theta\in[-\delta,\varrho_*+\delta]} \psi(\theta) < \infty$, for some $\delta > 0$.*

To explain the strategy of the proof of Theorem 1, we introduce at first some notations: for any vertex $u \in \mathcal{T}$ and $a \in \mathbb{R}$, we define

$$\tau_a^+(u) := \inf\{0 \leq k \leq |u| \,:\, V(u_k) > a\}, \tag{1.6}$$

$$\tau_a^-(u) := \inf\{0 \leq k \leq |u| \,:\, V(u_k) < a\}, \tag{1.7}$$

with convention $\inf \emptyset := \infty$ and for $n \geq 1$ and for any $|u| = n$, we write $\{u_0 = \varnothing, u_1, ..., u_n\} = [\varnothing, u]$ the shortest path from the root $\varnothing$ to u(u_k is the ancestor of k-th generation of u).

By using these notations, the total progeny set $\mathscr{Z}$ of the killed branching random walk can be represented as follows:

$$\mathscr{Z} = \{u \in \mathcal{T} : \tau_0^-(u) > |u|\}.$$

For $a \leq x$, we define $\mathcal{L}[a]$ as the set of individuals of the (non-killed) branching random walk which lie below a for its first time (see Figure 1):

$$\mathcal{L}[a] := \{u \in \mathcal{T} \,:\, |u| = \tau_a^-(u)\}, \qquad a \leq x. \tag{1.8}$$

Since the whole system goes to $-\infty$, $\mathcal{L}[a]$ is well defined. In particular, $\mathcal{L}[0]$ is the set of leaves of the progeny of the killed branching walk. As an application of a general fact for a wide class of graphs, we can compare the set of leaves $\mathcal{L}[0]$ with $\mathscr{Z}$. Then it is enough to investigate the tail asymptotics of $\#\mathcal{L}[0]$.

To state the result for $\#\mathcal{L}[0]$, we shall need an auxiliary random walk S, under a probability $\mathbf{Q}$, which are defined respectively in (3.17) and in (3.16) with the parameter there $\varrho = \varrho_*$ in the critical case, and $\varrho = \varrho_+$ in the subcritical case. We mention that under $\mathbf{Q}$, the random walk S is recurrent in the critical case and transient in the subcritical case. Let us also consider the renewal function $R(x)$ associated to S (see (3.20)) and τ_0^- the first time when S becomes negative (see (3.8)). For notational simplification, let us write $\mathbf{Q}[\xi]$ for the expectation of ξ under $\mathbf{Q}$. Then, we have the following theorem.

Theorem 2 (Tail of the number of leaves)**.** *Assume (1.1).*

(i) Critical case : if $\psi'(\varrho_*) = 0$ and (1.3) holds, then for any $x \geq 0$, we have when $n \to \infty$

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \sim c'_{crit} R(x) e^{\varrho_* x} \frac{1}{n(\log n)^2},$$

where $c'_{crit} := \mathbf{Q}[e^{-\varrho_* S_{\tau_0^-}}] - 1$.

(ii) Subcritical case : If $\psi'(\varrho_*) < 0$ and (1.4) holds, then we have for any $x \geq 0$ when $n \to \infty$,

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \sim c'_{sub} R(x) \mathrm{e}^{\varrho_+ x} n^{-\frac{\varrho_+}{\varrho_-}},$$

for some constant $c'_{sub} > 0$.

We stress that $\mathbf{Q}$, S, and $R(\cdot)$ depend on the parameter $\varrho = \varrho_*$ (critical case) or $\varrho = \varrho_+$ (subcritical case). If $\sum_{|u|=1}(1 + e^{\varrho - V(u)})$ has some larger moments, then we can give, as in the critical case (i), a probabilistic interpretation of the constant c'_{sub} in the subcritical case.

Lemma 1. *Under (1.1) with $\psi'(\varrho_*) < 0$ and (1.4). Let us assume furthermore that*

$$\mathbf{E}\left[\sum_{|u|=1} (1 + e^{\varrho - V(u)})\right]^{\frac{\varrho_+}{\varrho_-}+1+\delta} < \infty, \qquad \textit{for some } \delta > 0, \tag{1.9}$$

then

$$c'_{sub} = c_{\varrho_-}\, (c^*_{sub})^{\varrho_+/\varrho_-}\, \mathbf{Q}(\tau_0^- = \infty),$$

where c_{ϱ_-} and c^*_{sub} are given respectively by (7.16) and Lemma 21 [$\mathbf{Q}(\tau_0^- = \infty) > 0$ since the random walk S under $\mathbf{Q}$ drifts to ∞].

The next lemma establishes the relation between $\#\mathcal{L}[0]$ and the total progeny $Z = \#\mathscr{Z}$. Recall that $\mathbf{E}[\nu] > 1$.

Lemma 2. *Assume (1.5). Then Theorem 2 implies Theorem 1 with*
(i) in the critical case: $c_{crit} = (\mathbf{E}[\nu] - 1)^{-1} c'_{crit}$,
(ii) in the subcriticase: $c_{sub} = (\mathbf{E}[\nu] - 1)^{-\frac{\varrho_+}{\varrho_-}} c'_{sub}$.

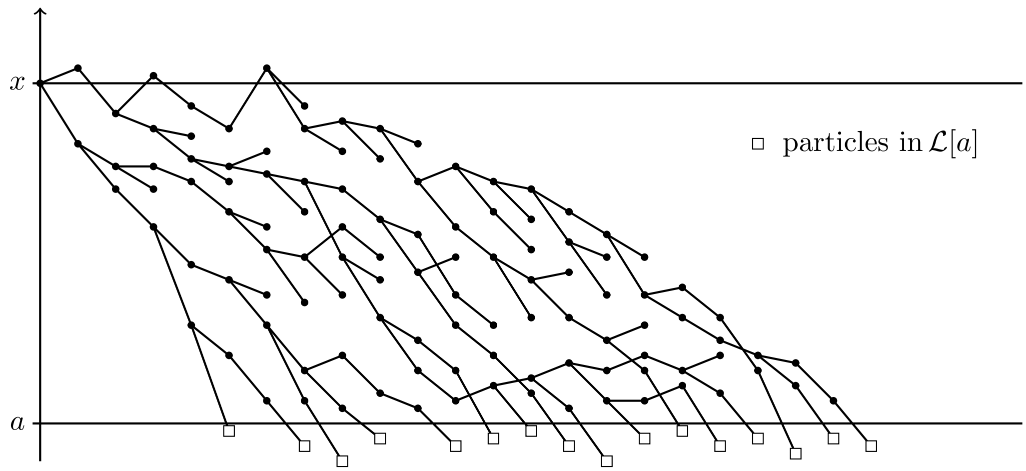

FIGURE 1. The set $\mathcal{L}[a]$

The above lemma will be proven in Section 2, and the rest of this paper is devoted to the proof of Theorem 2. To this end, we shall investigate the maximum of the killed branching random walk and its progeny. Define for any $L > 0$,

$$H(L) := \sum_u 1_{\{\tau_0^-(u) > \tau_L^+(u) = |u|\}} = \#\mathscr{H}(L), \qquad L > 0, \tag{1.10}$$

where

$$\mathscr{H}(L) := \{u \in \mathcal{T} : \tau_0^-(u) > \tau_L^+(u) = |u|\} \tag{1.11}$$

denotes the set of particles of the branching random walk on $[0, L]$ with two killing barriers which were absorbed at level L [then $\mathscr{H}(L) \subset \mathscr{L}$]. Finally, we define

$$Z[0, L] := \sum_u 1_{\{\tau_0^-(u) = |u| < \tau_L^+(u)\}}, \qquad L > 0, \tag{1.12}$$

the number of particles (leaves) which touch 0 before L, see Figure 2.

The following result may have independent interest: The first two parts give a precise estimate on the probability that a level t is reached by the killed branching random walk. In the third part, conditioning on the event that the level t is reached, we establish the convergence in distribution of the overshoots at level t seen as a random point process.

Theorem 3. *Assume (1.1).*

(i) *Assuming $\psi'(\varrho_*) = 0$ (critical case) and (1.3), we have*

$$\mathbf{P}_x(H(t) > 0) \sim \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x) e^{\varrho_* x} \frac{e^{-\varrho_* t}}{t}, \qquad t \to \infty,$$

where $\mathbf{Q}$ is defined in (3.16), the random variable $\Re$ is given in (5.27) with $\varrho = \varrho_*$ and $C_R > 0$ is a constant given in (3.21).

(ii) *Assuming $\psi'(\varrho_*) < 0$ (subcritical case) and (1.4), we have*

$$\mathbf{P}_x(H(t) > 0) \sim \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x) e^{\varrho_+ x} e^{-\varrho_+ t}, \qquad t \to \infty,$$

where $\mathbf{Q}$ is defined in (3.16), the random variable $\Re$ is given in (5.27) with $\varrho = \varrho_+$ and $C_R > 0$ is a constant given in (3.21).

(iii) *In both cases and under $\mathbf{P}_x(\cdot \mid H(t) > 0)$, the point process $\mu_t := \sum_{u \in \mathscr{H}(t)} \delta_{\{V(u) - t\}}$ converges in distribution toward a point process $\widehat{\mu}_\infty$ on $(0, \infty)$, where $\widehat{\mu}_\infty$ is distributed as μ_∞ under the probability measure $\frac{\Re^{-1}}{\mathbf{Q}[\Re^{-1}]} \cdot \mathbf{Q}$, with μ_∞ defined in (5.26).*

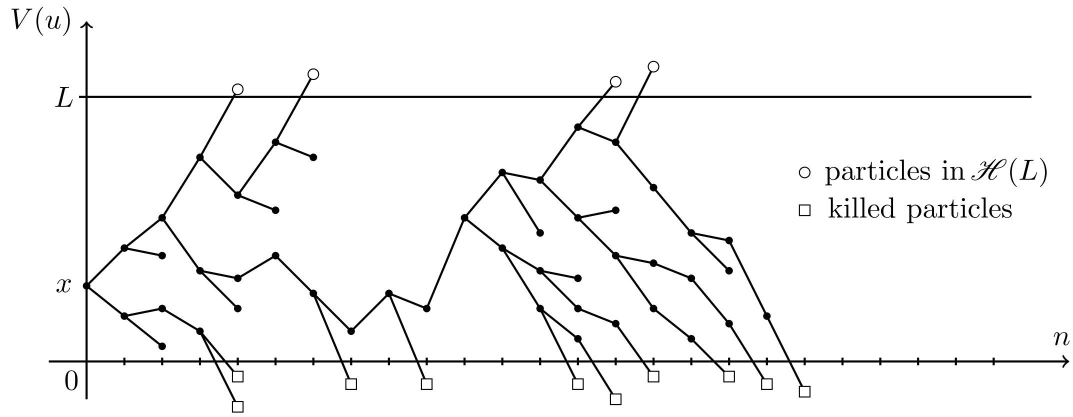

FIGURE 2. The set $\mathscr{H}(L)$

The Yaglom-type result Theorem 3 plays a crucial role in the proof of Theorem 2. Loosely speaking, to make the total progeny Z (or the set of leaves $\mathcal{L}[0]$) as large as possible, the branching walk will reach some level L as high as possible and the descendants of all particles hitting L will make the main contribution in $\#\mathcal{L}[0]$. We control the error terms by computing the moments of $Z[0, L]$ which are the most technical parts in the proof of Theorem 2.

In the computations of moments of $Z[0, L]$, we have to distinguish the contributions of *good* particles from *bad* particles. By good particle, we mean that its children do not make extraordinary jumps (and the number of its children is not too big). Then the number of good particles will have high moments, however that of bad particles only have low moments. To describe separately the numbers of good and bad particles in $Z[0, L]$, we shall modify the Yaglom-type result Theorem 3 (iii) as follows.

Denote by Ω_f the set of σ-finite measures on $\mathbb{R}$. For any individual $u \neq \varnothing$, let $\overleftarrow{u}$ be the parent of u and define

$$\Delta V(u) := V(u) - V(\overleftarrow{u}).$$

Let us fix a measurable function $\mathscr{B} : \Omega_f \to \mathbb{R}_+$ and write by a slightly abuse of notation

$$\mathscr{B}(u) \equiv \mathscr{B}\left(\sum_{\overleftarrow{v}=\overleftarrow{u}, v\neq u} \delta_{\{\Delta V(v)\}}\right), \qquad \forall\, u \in \mathcal{T}\backslash\{\varnothing\},$$

and $\mathscr{B}(u) = 0$ if u does not have any brothers. We assume some integrability: there exists some $\delta_1 > 0$ such that

$$\mathbf{E}\left[\sum_{|u|=1} (1 + 1_{\{\varrho=\varrho_*\}}|V(u)|)e^{\varrho V(u)}\mathscr{B}(u)^{\delta_1}\right] < \infty, \tag{1.13}$$

where $\varrho = \varrho_*$ if $\psi'(\varrho_*) = 0$ and $\varrho = \varrho_+$ if $\psi'(\varrho_*) < 0$. For the functions $\mathscr{B}$ appearing in this paper, for instance, $\mathscr{B}(\theta) = (\frac{1}{\lambda}\int(1 + e^{\varrho x})\theta(\mathrm{d}x))^2$ in the critical case and $\mathscr{B}(\theta) = (\frac{1}{\lambda}\int \theta(\mathrm{d}x)(1 + e^{\varrho_- x}))^{1/\varrho_-}$ in the subcritical case (see Sections 6 and 7 where the constant λ is introduced) for $\theta \in \Omega_f$, (1.13) will always be a consequence of (1.3) or (1.4) by taking a sufficiently small δ_1.

Define for $u \in \mathcal{T}$,

$$\beta_L(u) := \inf\{1 \le j \le |u| : \mathscr{B}(u_j) > e^{L-V(u_{j-1})}\}, \qquad L > 0, \tag{1.14}$$

with the convention that $\inf \emptyset = \infty$. We consider

$$\mathscr{H}_{\mathscr{B}}(L) \;:=\; \left\{u \in \mathcal{T} : \tau_0^-(u) > \tau_L^+(u) = |u|, \beta_L(u) = \infty\right\}.$$

In other words, $\mathscr{H}_{\mathscr{B}}(L)$ only contains those particles u in $\mathscr{H}(L)$ such that $\mathscr{B}(u_j), j \le |u|$, are not very large. Obviously, $\mathscr{H}_{\mathscr{B}} \equiv \mathscr{H}$ if $\mathscr{B} = 0$. We get an extension of Theorem 3 (iii) as follows:

Proposition 1. *Assuming (1.13) and the hypothesis of Theorem 3. Under* $\mathbf{P}_x(\cdot \,|\, H(t) > 0)$, *the point process* $\mu_{\mathscr{B},t} := \sum_{u\in \mathscr{H}_{\mathscr{B}}(t)} \delta_{\{V(u)-t\}}$ *converges in distribution toward a point process* $\widehat{\mu}_{\mathscr{B},\infty}$ *on* $\mathbb{R}$, *where* $\widehat{\mu}_{\mathscr{B},\infty}$ *is distributed as* $\mu_{\mathscr{B},\infty}$ *under the probability measure* $\frac{\Re^{-1}}{\mathbf{Q}[\Re^{-1}]} \cdot \mathbf{Q}$, *with* $\mu_{\mathscr{B},\infty}$ *defined in (5.24).*

To prove Theorems 2, 3 and Proposition 1, we shall develop a spinal decomposition for the killed branching random walk up to some stopping lines. Viewed from the stopping lines, the branching walk on the spine behaves as a two-dimensional Markov chain: The first coordinate is a real-valued random walk (sometimes conditioned to stay positive) until some first passage times, and the second coordinate takes values in the space of point measures, whose laws we describe through a family of Palm measures. As the parameter of the stopping lines goes to infinity, we shall also need some accurate estimates on the real-valued random walk and establish a convergence in law for the time-reversal random walk, in both transient and recurrent cases.

The rest of this paper is organized as follows:

- Section 2: we prove Lemma 2. Then the rest of this paper is devoted to the proofs of Theorems 2, 3, Lemma 1 and Proposition 1.
- Section 3: we develop the spinal decompositions for the killed and non-killed branching random walks, which are the main theoretical tools in the proofs.
- Section 4: we collect several preliminary results on the one-dimensional real-valued random walk, both in recurrent and transient cases; in particular, we establish a result of convergence in law for a time reversal random walk. The proofs of these results are postponed in Section 8.
- Section 5: by admitting three technical lemmas (whose proofs are postponed in Section 8), we prove Theorem 3 and Proposition 1.
- Sections 6 and 7: based on Theorem 3 and Proposition 1, we prove Theorem 2 in the critical and subcritical cases respectively. We also prove Lemma 1 in this section.
- Section 8 contains the proofs of the technical lemmas stated in Sections 4 and 5.

Throughout this paper, we adopt the following notations: For a point process $\Theta = \sum_{i=1}^m \delta_{\{x_i\}}$, we write $\langle f, \Theta\rangle = \sum_{i=1}^m f(x_i)$. Unless stated otherwise, we denote by c or c' (possibly with some subscript) some unimportant positive constants whose values may change from one paragraph to another, and by $f(t) \sim g(t)$ as $t \to t_0 \in [0,\infty]$ if $\lim_{t\to t_0} \frac{f(t)}{g(t)} = 1$; We also write $\mathbf{E}[X, A] \equiv \mathbf{E}[X 1_A]$ when A is an event and $\mathbf{E}[X]^k = \mathbf{E}[X^k] \neq (\mathbf{E}[X])^k$ when X does not have a short expression.

2. From the number of leaves to the total progeny of the killed branching walk: Proof of Lemma 2

We recall that our branching random walk starts from $x \ge 0$. We introduced for $u \in \mathcal{T}$, $\tau_a^-(u) := \inf\{0 \le k \le |u| \,:\, V(u_k) < a\}$, and

$$\mathcal{L}[a] := \{u \in \mathcal{T} \,:\, |u| = \tau_a^-(u)\}, \qquad a \le x.$$

Proof of Lemma 2. We equip the tree $\mathcal{T}$ with the lexicographical order. Let U_k be the k-th vertex for this order in the set $\mathscr{Z}$ of the living particles. It is well defined until $k = Z$ when all living particles have been explored. For $k \in [1, Z]$, we introduce

$$Y_k := 1 + \sum_{i=1}^{k} (\nu(U_i) - 1)$$

and we notice that $Y_Z = \#\mathcal{L}[0]$ [This can be easily checked by using an argument of recurrence on the maximal generation of the individuals of $\mathscr{Z}$]. We extend the definition of Y_k to $k > Z$, by $Y_{k+1} := Y_k + \nu_k - 1$ where ν_k is taken from a family $\{\nu_i, \, i \geq 1\}$ of i.i.d random variables distributed as $\nu(\varnothing)$ and independent of our branching random walk. We claim that $(Y_k, \, k \geq 1)$ is a random walk. To see this, observe that we can construct the killed branching random walk in the following way. Let $(\mathscr{L}_i^{(c)}, \, i \geq 1)$ be i.i.d copies of $\mathscr{L}$. At step 1, the root $\varnothing =: U_1$ located at x generates the point process $\mathscr{L}_1^{(c)}$. If all the children are killed, we stop the construction. Otherwise, we call U_2 the first vertex for the lexicographical order that is alive. Then, U_2 generates the point process $\mathscr{L}_2^{(c)}$, and we continue similarly. The process that we get has the law of the killed branching random walk. In particular, if $\nu_i^{(c)}$ denotes the number of points of $\mathscr{L}_i^{(c)}$, then $(Y_k, \, k \geq 1)$ has the law of $(\sum_{i=1}^k (\nu_i^{(c)} - 1), \, k \geq 1)$ which is a random walk by construction. This proves the claim. We suppose that Theorem 2 holds and we want to deduce Theorem 1. Let us look at the upper bound of $\mathbf{P}_x(Z > n)$. Let $m := \mathbf{E}[\nu] > 1$ and take $\varepsilon \in (0, m-1)$. We have

$$\begin{aligned} \mathbf{P}_x(\#\mathcal{L}[0] \leq (m-1-\varepsilon)n, \, Z > n) &= \mathbf{P}_x(Y_Z \leq (m-1-\varepsilon)n, \, Z > n) \\ &= \sum_{k>n} \mathbf{P}_x(Y_k \leq (m-1-\varepsilon)n, \, Z = k) \\ &\leq \sum_{k>n} \mathbf{P}_x(Y_k \leq (m-1-\varepsilon)k), \end{aligned}$$

which is exponentially small by Cramér's bound. By Theorem 2, $\mathbf{P}_x(\#\mathcal{L}[0] > n)$ decreases polynomially. Therefore,

$$\begin{aligned} \mathbf{P}_x(Z > n) &\leq \mathbf{P}_x(\#\mathcal{L}[0] > (m-1-\varepsilon)n) + \mathbf{P}_x(\#\mathcal{L}[0] \leq (m-1-\varepsilon)n, \, Z > n) \\ &= \mathbf{P}_x(\#\mathcal{L}[0] > (m-1-\varepsilon)n)(1 + o(1)). \end{aligned}$$

Letting n go to ∞, then $\varepsilon \to 0$ yields the upper bound. For the lower bound, we take $\varepsilon > 0$ and we observe that,

$$\begin{aligned} \mathbf{P}_x(\#\mathcal{L}[0] > (m-1+\varepsilon)n, \, Z \leq n) &= \mathbf{P}_x(Y_Z > (m-1+\varepsilon)n, \, Z \leq n) \\ &\leq \mathbf{P}_x(\max_{1 \leq k \leq n} (Y_k - (m-1)k) > \varepsilon n). \end{aligned}$$

Let $\alpha > 2$ in the critical case and $\alpha > 2\varrho_+/\varrho_-$ in the subcritical case. By Doob's inequality,

$$\mathbf{E}\big| \max_{1 \leq k \leq n} (Y_k - (m-1)k) \big|^\alpha \leq \frac{\alpha^\alpha}{(\alpha-1)^\alpha} \mathbf{E}\big|(Y_n - (m-1)n)\big|^\alpha \leq c(\alpha) \, \mathbf{E}(\sum_{i=1}^n (\nu_i - m)^2)^{\alpha/2},$$

for some constant $c = c(\alpha) > 0$. By convexity,

$$\mathbf{E}(\sum_{i=1}^n (\nu_i - m)^2)^{\alpha/2} \leq n^{\frac{\alpha}{2}-1} \mathbf{E} \sum_{i=1}^n |\nu_i - m|^\alpha = n^{\alpha/2} \mathbf{E}|\nu - m|^\alpha.$$

It follows that

$$\mathbf{P}_x(\#\mathcal{L}[0] > (m-1+\varepsilon)n, \, Z \leq n) \leq \frac{c\mathbf{E}|\nu - m|^\alpha}{\varepsilon^\alpha} \, n^{-\alpha/2}.$$

Therefore,

$$\mathbf{P}_x(Z > n) \geq \mathbf{P}_x(\#\mathcal{L}[0] > (m-1+\varepsilon)n) - \frac{c\mathbf{E}|\nu - m|^\alpha}{\varepsilon^\alpha} \, n^{-\alpha/2},$$

which proves the lower bound by taking $n \to \infty$ then $\varepsilon \to 0$. □

3. Spinal decomposition

3.1. **Spinal decomposition of a branching random walk (without killing).** We begin with a general formalism of the spinal decomposition for a branching random walk. This decomposition has already been used in the literature by many authors in various forms, see e.g. Lyons, Pemantle and Peres [26], Lyons [25] and Biggins and Kyprianou [9].

There is a one-to-one correspondence between the branching random walk $(V(u)_{u\in\mathcal{T}})$ and a marked tree $\{(u, V(u)) : u \in \mathcal{T}\}$. For $n \geq 1$, let $\mathscr{F}_n$ be the sigma-algebra generated by the branching random walk in the first n generations. For any $u \in \mathcal{T}\backslash\{\varnothing\}$, denote by $\overleftarrow{u}$ the parent of u. Write as before $[\varnothing, u] = \{u_0 := \varnothing, u_1, ..., u_{|u|}\}$ the shortest path from the root $\varnothing$ to u(with $|u_i| = i$ for any $0 \leq i \leq |u|$).

Let $h : \mathcal{T} \to [0, \infty)$ be measurable such that $h(\varnothing) > 0$ and for any $x \in \mathbb{R}$, $v \in \mathcal{T}$ with $|v| = n \geq 0$,

$$\mathbf{E}_x\Big[\sum_{\overleftarrow{u}=v} h(u) \,\Big|\, \mathscr{F}_n\Big] = \lambda\, h(v), \tag{3.1}$$

where $\lambda > 0$ is some positive constant. Let $\mathscr{H}_+ := \{u \in \mathcal{T} : h(u) > 0\}$. In our examples of h in this paper, $\lambda = 1$, $h(u) = f(V(u))$ or $h(u) = f(V(u_1), ..., V(u_{|u|}))$ for some non-random function f, and $\mathscr{H}_+$ equals either $\mathcal{T}$ or $\mathscr{Z}$ the set of progeny of the killed branching walk.

Define

$$W_n := \frac{1}{h(\varnothing)\lambda^n} \sum_{|u|=n} h(u), \qquad n \geq 0.$$

Fix $x \in \mathbb{R}$. Clearly by (3.1), (W_n) is a $(\mathbf{P}_x, (\mathscr{F}_n))$-martingale.

On the enlarged probability space formed by marked trees with distinguished rays, we may construct a probability $\mathbf{Q}_x^{(h)}$ and an infinite ray $\{w_0 = \varnothing, w_1, w_2, ...\}$ such that for any $n \geq 1$, $\overleftarrow{w}_n = w_{n-1}$, and

$$\mathbf{Q}_x^{(h)}\Big(w_n = u \,\Big|\, \mathscr{F}_\infty\Big) = \frac{h(u)}{h(\varnothing)\lambda^n\, W_n}, \qquad \forall\, |u| = n, \tag{3.2}$$

and

$$\frac{d\mathbf{Q}_x^{(h)}}{d\mathbf{P}_x}\Big|_{\mathscr{F}_n} = W_n. \tag{3.3}$$

To construct $\mathbf{Q}_x^{(h)}$, we follow Lyons [25] under a slightly more general framework: Let $\mathscr{L} := \sum_{|u|=1} \delta_{\{V(u)\}}$. For any $y \in \mathscr{H}_+$, denote by $\widetilde{\mathscr{L}_y}$ a random variable whose law has the Radon-Nikodym density W_1 with respect to the law of $\mathscr{L}$ under $\mathbf{P}_y$. Put one particle $w_0 = \varnothing$ at $x \in \mathscr{H}_+$. Generate offsprings and displacements according to an independent copy of $\widetilde{\mathscr{L}_x}$. Let $\{|u| = 1\}$ be the set of the children of w_0. We choose $w_1 = u$ according to the probability $\frac{h(u)}{h(w_0)\,\lambda\, W_1}$. All children $u \neq w_1$ give rise to independent branching random walks of law $\mathbf{P}_{V(u)}$, while conditioned on $V(w_1) = y$, w_1 gives offsprings and displacements according to an independent copy of $\widetilde{\mathscr{L}_y}$. We choose w_2 among the children of w_1 in the same size-biased way, and so on. Denote by $\mathbf{Q}_x^{(h)}$ the joint law of the marked tree $(V(u))_{|u|\geq 0}$ and the infinite ray $\{w_0 = \varnothing, w_1, ..., w_n, ..\}$. Then $\mathbf{Q}_x^{(h)}$ satisfies (3.3) and (3.2), which can be checked in the same way as in Lyons [25].

Under $Q_x^{(h)}$, we write, for $k \ge 1$,

$$\mho_k := \{u : |u| = k, \overleftarrow{u} = w_{k-1}, u \neq w_k\}. \tag{3.4}$$

In words, $\mho_k$ is the set of children of w_{k-1} except w_k, or equivalently, the set of the brothers of w_k, and is possibly empty. Define $S_0 := V(\varnothing)$ and

$$S_n := V(w_n), \qquad \Theta_n := \sum_{u \in \mho_n} \delta_{\{\Delta V(u)\}}, \qquad n \ge 1, \tag{3.5}$$

where we recall that $\Delta V(u) := V(u) - V(\overleftarrow{u})$. Finally, let us introduce the following sigma-field:

$$\mathscr{G}_n := \sigma\Big\{(\Delta V(u),\ u \in \mho_k),\ V(w_k),\ w_k,\ \mho_k,\ 1 \le k \le n\Big\}. \tag{3.6}$$

Then $\mathscr{G}_\infty$ is the sigma-field generated by all random variables related to the spine $\{w_k, k \ge 0\}$. Let us write $v < u$ if v is an ancestor of u [then $v \le u$ if $v < u$ or $v = u$]. By the standard '*words*'-representation in a tree, $u < v$ if and only if the word v is a concatenation of the word u with some word s, namely $v = us$ with $|s| \ge 1$.

The promised spinal decomposition is as follows. Since it differs only slightly from the spinal decomposition presented in Lyons [25] and Biggins and Kyprianou [9], we feel free to omit the proof.

Proposition 2. *Assume (3.1) and fix $x \in \mathscr{H}_+$. Under probability $\mathbf{Q}_x^{(h)}$,*

(i) *for each $n \ge 1$, conditionally on $\mathscr{G}_{n-1}$ and on $\{S_{n-1} = y\}$, the point process $(V(u), \overleftarrow{u} = w_{n-1})$ is distributed as $\widetilde{\mathscr{L}_y}$. In particular, the process $(S_n, \Theta_n)_{n\ge0}$ is Markovian. Moreover, $(S_n)_{n\ge0}$ is also a Markov chain and satisfies*

$$\mathbf{Q}_x^{(h)}\Big(f(S_n) \,\Big|\, S_{n-1} = y, \mathscr{G}_{n-1}\Big) = \frac{1}{\lambda}\mathbf{E}_y\Big[\sum_{|u|=1} f(V(u))\frac{h(u)}{h(\varnothing)}\Big],$$

for any nonnegative measurable function f, $n \ge 1$ and $y \in \mathscr{H}_+$.

(ii) *Conditionally on $\mathscr{G}_\infty$, the shifted branching random walks $\{V(vu) - V(v)\}_{|u|\ge0}$, for all $v \in \bigcup_{k=1}^\infty \mho_k$, are independent, and have the same law as $\{V(u)\}_{|u|\ge0}$ under $\mathbf{P}_0$.*

Remark that under $\mathbf{Q}_x^{(h)}$, $\{w_n, n \ge 0\}$ lives in $\mathscr{H}_+$ with probability one. We can extend Proposition 2 to the so-called stopping lines. Recall (1.6) and (1.7). For $0 \le x < t$, we consider the stopping line

$$\mathscr{C}_t := \{u \in \mathcal{T} : \tau_t^+(u) = |u|\}. \tag{3.7}$$

Note that for any $v \in \mathcal{T}$, $|v| < \tau_t^+(v)$ means that $\sup_{0\le i\le|v|} V(v_i) \le t$ (see Figure 3). The process $\{V(u)\}_{|u|\le\tau_t^+(u)}$ can be interpreted as the branching random walk stopped by the line $\mathscr{C}_t$. Recalling (1.11), we remark that $\mathscr{C}_t \cap \mathscr{Z} = \mathscr{H}(t)$, where as before $\mathscr{Z}$ denotes the set of progeny of the killed branching random walk.

Let $\mathscr{F}_{\mathscr{C}_t} := \sigma\{(u, V(u)) : u \in \mathcal{T}, |u| \le \tau_t^+(u)\}$ be the σ-field generated by the branching walk V up to the stopping line $\mathscr{C}_t$. Assuming (3.1), we define

$$W_{\mathscr{C}_t} := \frac{1}{h(\varnothing)} \sum_{u \in \mathscr{C}_t} h(u)\lambda^{-|u|}.$$

Define two family of stopping times for the process $(S_n := V(w_n), n \ge 0)$ under $\mathbf{Q}_x^{(h)}$,

$$\tau_a^+ := \inf\{k \ge 0 : S_k > a\}, \qquad \tau_a^- := \inf\{k \ge 0 : S_k < a\}, \qquad \forall\, a \in \mathbb{R}, \tag{3.8}$$

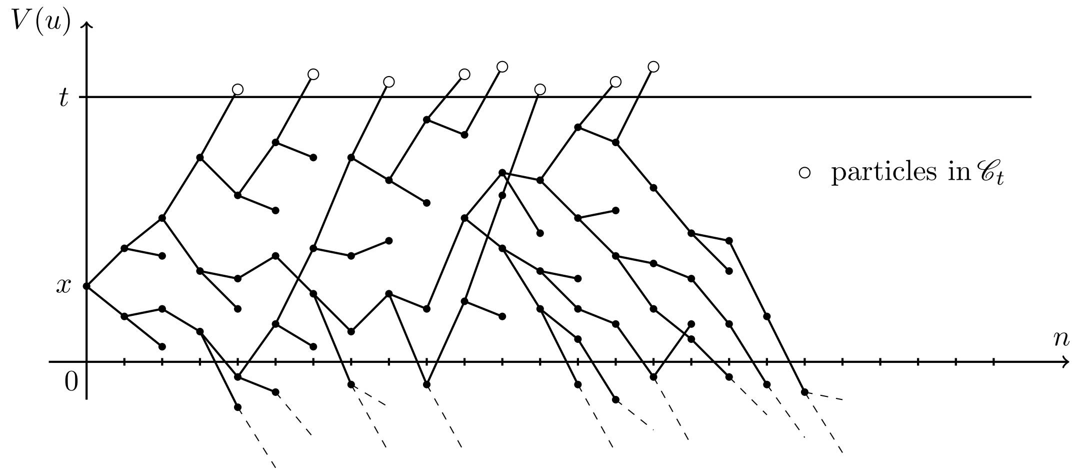

FIGURE 3. The set $\mathscr{C}_t$

with the usual convention $\inf \emptyset = \infty$ and the corresponding overshoot and undershoot processes

$$T_a^+ := S_{\tau_a^+} - a, \qquad T_a^- := a - S_{\tau_a^-}, \qquad \forall\, a \in \mathbb{R}. \tag{3.9}$$

Analogously to (3.6), we introduce the sigma-field

$$\mathscr{G}_{\mathscr{C}_t} := \sigma\Big\{(\Delta V(u),\ u \in \mho_k),\ V(w_k),\ w_k,\ \mho_k,\ 1 \le k \le \tau_t^+,\ \tau_t^+\Big\}, \tag{3.10}$$

generated by all information related to the spine $[\varnothing, w(\tau_t^+)]$. Similarly, we recall $\mathcal{L}[a]$ in (1.8) and define $\mathscr{F}_{\mathcal{L}[a]}$, $W_{\mathcal{L}[a]}$, $\mathscr{G}_{\mathcal{L}[a]}$ as before. The next result describes the decomposition along the spine $[\varnothing, w(\tau_t^+)]$ (resp. $[\varnothing, w(\tau_a^-)]$).

Proposition 3. *Assume (3.1) and let $x \in \mathscr{H}_+$. Take $t \ge x$. Suppose that h is such that $\mathbf{Q}_x^{(h)}(\tau_t^+ < \infty) = 1$. Then,*

$$\frac{d\mathbf{Q}_x^{(h)}}{d\mathbf{P}_x}\Big|_{\mathscr{F}_{\mathscr{C}_t}} = W_{\mathscr{C}_t}. \tag{3.11}$$

(i) *Under probability $\mathbf{Q}_x^{(h)}$, conditionally on $\mathscr{G}_{\mathscr{C}_t}$ and on $\{V(v) = x_v, v \in \bigcup_{k=1}^{\tau_t^+} \mho_k\}$, the shifted branching random walks $\{V(vu) - V(v)\}_{u\,:\,|vu| \le \tau_t^+(vu)}$, stopped by the line $\mathscr{C}_t$, are independent, and have the same law as $\{V(u)\}_{|u| \le \tau_{t-x_v}^+(u)}$ under $\mathbf{P}_0$, stopped by the line $\mathscr{C}_{t-x_v}$.*

(ii) *The distribution of the spine within $\mathscr{C}_t$ is given by*

$$\mathbf{Q}_x^{(h)}\Big(w_{\tau_t^+} = u \,|\, \mathscr{F}_{\mathscr{C}_t}\Big) = \frac{h(u)\lambda^{-|u|}}{h(\varnothing) W_{\mathscr{C}_t}}, \qquad \forall\, u \in \mathscr{C}_t.$$

(iii) *For any bounded measurable function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ and for any bounded $\mathscr{F}_{\mathscr{C}_t}$-measurable random variable Φ_t,*

$$\mathbf{E}_x\left[\sum_{u \in \mathscr{C}_t} \frac{h(u)}{h(\varnothing)\lambda^{|u|}}\, f(V(u_i), 0 \le i \le |u|)\, \Phi_t\right] = \mathbf{Q}_x^{(h)}\left[f(S_i, 0 \le i \le \tau_t^+)\, \Phi_t\right].$$

Similarly, take $a \le x$ and assume that h is such that $\mathbf{Q}_x^{(h)}(\tau_a^- < \infty) = 1$. Then the analog holds for $\mathscr{C}_t$ replaced by $\mathcal{L}[a]$ (and τ_t^+ by τ_a^-).

Remark 3. *If $\mathbf{Q}_x^{(h)}(\tau_t^+ < \infty) = 1$ for all t, then $W_{\mathscr{C}_t}$ is a $(\mathbf{P}_x, \mathscr{F}_{\mathscr{C}_t})$-martingale by Lemma 6.1 and Theorem 6.1 in* [9]. *The equivalent holds for $\mathcal{L}[a]$.*

Proof of Proposition 3. It is enough to prove that for any $g : \mathcal{T} \to \mathbb{R}$ measurable and bounded,

(3.12)
$$\mathbf{E}_x\left[\sum_{u\in\mathscr{C}_t} \frac{h(u)}{h(\varnothing)\lambda^{|u|}} f(V(u_i), 0\le i\le |u|)\, g(u)\, \Phi_t\right] = \mathbf{Q}_x^{(h)}\left[f(S_i, 0\le i\le \tau_t^+)\, g(w_{\tau_t^+})\, \Phi_t\right].$$

In fact, the Part (iii) follows from (3.12), and by taking $f \equiv g \equiv 1$ in (3.12) we get (3.11); Taking $f \equiv 1$ in (3.12) and using (3.11), we get the Part (ii); Finally since τ_t^+ is a stopping time for $(S_k)_k$, the Part (i) follows easily from Proposition 2.

To check (3.12), it is enough to show that for any $N \ge 1$, (3.12) holds for all Φ_t of form $\Phi_{t,N} := F(u, V(u), u\in\mathcal{T}, |u| \le \tau_t^+(u)\wedge N)$ with a bounded measurable function F. Notice that the left-hand-side of (3.12) equals

$$\sum_{n=0}^{\infty}\mathbf{E}_x\left[\sum_{|u|=n} 1_{\{\tau_t^+(u)=n\}}\frac{h(u)}{h(\varnothing)\lambda^n} f(V(u_i), 0\le i\le n)\, g(u)\, \Phi_{t,N}\right] := \sum_{n=0}^{\infty}(3.13)_n, \tag{3.13}$$

with obvious definition of $(3.13)_n$. If $n \ge N$, since $\Phi_{t,N}$ is measurable with respect to $\mathscr{F}_N$, we deduce from (3.2) and the absolute continuity (3.3) that

$$(3.13)_n = \mathbf{Q}_x^{(h)}\left[1_{\{\tau_t^+=n\}} f(S_i, 0\le i\le n) g(w_n)\Phi_{t,N}\right].$$

For $n < N$, we deduce from the branching property along the stopping line $\mathscr{C}_t$ (see Jagers [18]) that

$$\begin{aligned}(3.13)_n &= \mathbf{E}_x\Big[\sum_{|u|=n} 1_{\{\tau_t^+(u)=n\}} f(V(u_i), 0\le i\le n) g(u)\Phi_{t,N} \sum_{|v|=N, u<v}\frac{h(v)}{h(\varnothing)\lambda^N}\Big]\\ &= \mathbf{E}_x\Big[\sum_{|v|=N} 1_{\{\tau_t^+(v)=n\}} f(V(v_i), 0\le i\le n) g(v_n)\Phi_{t,N}\frac{h(v)}{h(\varnothing)\lambda^N}\Big]\\ &= \mathbf{Q}_x^{(h)}\Big[1_{\{\tau_t^+=n\}} f(S_i, 0\le i\le n) g(w_n)\Phi_{t,N}\Big],\end{aligned}$$

by using again (3.2) and the absolute continuity (3.3) at N. Noting that $f(S_i, 0 \le i \le n)g(w_n) = f(S_i, 0\le i\le \tau_t^+)\, g(w_{\tau_t^+})$ on $\{\tau_t^+ = n\}$, we take the sum of $(3.13)_n$ over all n and obtain (3.12). The proof for $\mathcal{L}[a]$ works by analogy. □

Let us present below a particular example of h and the corresponding laws of $(\Theta_n, S_n)_{n\ge 0}$. Recall (1.1). Define

$$h(u) := \begin{cases} e^{\varrho_* V(u)}, & \text{if } \psi'(\varrho_*) = 0,\\ e^{\varrho_+ V(u)}, & \text{if } \psi'(\varrho_*) < 0,\end{cases} \qquad u\in\mathcal{T}. \tag{3.14}$$

Since $\psi(\varrho_*) = 0$ in the critical case and $\psi(\varrho_+) = 0$ in the subcritical case, the function h satisfies (3.1) with $\lambda = 1$ and $\mathscr{H}_+ = \mathcal{T}$. We mention that in the subcritical case, since $\psi(\varrho_-) = 0$, the function $u \to e^{\varrho_- V(u)}$ also satisfies (3.1) with $\lambda = 1$. This fact will be explored in Section 7 for the definition of $\mathbf{Q}^{(\varrho_-)}$, the measure satisfying (3.3) with $h(u) = e^{\varrho_- V(u)}$.

Write for any $x \in \mathbb{R}$, $\mathbf{Q}_x \equiv \mathbf{Q}_x^{(h)}$ the probability with the choice of h given in (3.14). For simplification, let

$$\varrho := \begin{cases} \varrho_*, & \text{if } \psi'(\varrho_*) = 0 \text{ (critical case)};\\ \varrho_+, & \text{if } \psi'(\varrho_*) < 0 \text{ (subcritical case)}.\end{cases} \tag{3.15}$$

Then for any $x \in \mathbb{R}$, $\mathbf{Q}_x$ satisfies

$$\frac{d\mathbf{Q}_x}{d\mathbf{P}_x}\Big|_{\mathscr{F}_n} = e^{-\varrho x} \sum_{|u|=n} e^{\varrho V(u)}. \tag{3.16}$$

We shall write $\mathbf{Q} \equiv \mathbf{Q}_0$ when $x = 0$. The following description of the law of $(S_n, \Theta_n)_{n\ge 0}$ under $\mathbf{Q}_x$ is an easy consequence of Proposition 2 (i).

Corollary 1. *Recall (3.15) and (3.5). Fix $x \in \mathbb{R}$.*

(i) Under $\mathbf{Q}_x$, $(S_n - S_{n-1}, \Theta_n)_{n\ge 1}$ are i.i.d. under $\mathbf{Q}_x$ whose common law is determined by

$$\mathbf{Q}_x\Big[f(S_n - S_{n-1}) e^{-\langle g, \Theta_n\rangle}\Big] = \mathbf{E}\Big[\sum_{|u|=1} e^{\varrho V(u)} f(V(u)) e^{-\sum_{v\neq u, |v|=1} g(V(v))}\Big],$$

for any $n \ge 1$, any measurable functions $f, g : \mathbb{R} \to \mathbb{R}_+$. In particular, the process $(S_n)_{n\ge 0}$ is a random walk on $\mathbb{R}$, starting from $S_0 = x$, with step distribution given by

$$\mathbf{Q}_x\Big[f(S_n - S_{n-1})\Big] = \mathbf{E}\Big[\sum_{|u|=1} f(V(u))\, e^{\varrho V(u)}\Big], \qquad n \ge 1. \tag{3.17}$$

(ii) For any $n \ge 1$ and any measurable function $F : \mathbb{R}^{n+1} \to \mathbb{R}_+$,

$$\mathbf{E}_x\Big[\sum_{|u|=n} F(V(u_i), 0 \le i \le n)\Big] = e^{\varrho x}\, \mathbf{Q}_x\Big[e^{-\varrho S_n}\, F(S_i, 0 \le i \le n)\Big].$$

(iii) For any $n \ge 1$, and any $|u| = n$,

$$\mathbf{Q}_x(w_n = u \,|\, \mathscr{F}_n) = \frac{e^{\varrho\, V(u)}}{\sum_{|v|=n} e^{\varrho\, V(v)}}.$$

Remark that by (3.17), $\mathbf{Q}[S_1] = 0$ and $\mathbf{Q}[S_1^2] = \psi''(\varrho_*) > 0$ in the critical case, while $\mathbf{Q}[S_1] = \psi'(\varrho_+) > 0$ in the subcritical case.

3.2. **Spinal decomposition for a killed branching random walk.** Before introducing a change of measure related to the killed branching walk, we recall some elementary facts on the Palm distribution of the point process $\mathscr{L} = \sum_{|u|=1} \delta_{\{V(u)\}}$ under $\mathbf{P}$. Let $\mathbf{E}\big(\mathscr{L}(dx)\big)$ be the intensity measure of $\mathscr{L}$, namely for any measurable function $f : \mathbb{R} \to \mathbb{R}_+$,

$$\int_{\mathbb{R}} f(x)\mathbf{E}\big(\mathscr{L}(dx)\big) = \mathbf{E}[\sum_{|u|=1} f(V(u))].$$

Clearly $\mathbf{E}\big(\mathscr{L}(dx)\big)$ is σ-finite since ψ is well-defined on some interval. Then there exists a family $(\Xi_x, x \in \mathbb{R})$, called reduced Palm distributions, of distributions of random point measures on $\mathbb{R}$ such that

$$\int_{\Omega_f} F(x, \theta)\Xi_x(d\theta) = \frac{\mathbf{E}\Big[F(x, \mathscr{L} - \delta_{\{x\}})\, \mathscr{L}(dx)\Big]}{\mathbf{E}\big(\mathscr{L}(dx)\big)}, \qquad \mathbf{E}\big(\mathscr{L}(dx)\big)\text{-p.p. } x. \tag{3.18}$$

for any measurable $F : \mathbb{R} \times \Omega_f(\mathbb{R}) \to \mathbb{R}_+$, and where Ω_f denotes the set of σ-finite measures on $\mathbb{R}$. See Kallenberg [20], Chapter 10 for more details. Roughly saying, Ξ_x is the distribution of $\mathscr{L} - \delta_{\{x\}}$ conditioned on that $\mathscr{L}$ charges x.

In this subsection, let $((S_n), \mathbf{Q}_x)$ be as in Corollary 1 and (3.16). Based on Corollary 1 (i) (with $n = 1$ and $x = 0$), elementary computations give that for any measurable $f, g : \mathbb{R} \to \mathbb{R}_+$,

$$\mathbf{Q}\Big[f(S_1) e^{-\langle g, \Theta_1\rangle}\Big] = \int_{\mathbb{R}} \mathbf{E}\big(\mathscr{L}(dx)\big) e^{\varrho x} f(x) \int_{\Omega_f} e^{-\langle g, \theta\rangle} \Xi_x(d\theta).$$

It follows immediately from (3.17) that the law of S_1 under $\mathbf{Q}$ is given by $\mathbf{Q}(S_1 \in dx) = \mathbf{E}\big(\mathscr{L}(dx)\big)e^{\varrho x}$. Hence for any measurable $f, g : \mathbb{R} \to \mathbb{R}_+$,

$$\mathbf{Q}\Big[f(S_1)e^{-\langle g, \Theta_1\rangle}\Big] = \int_{\mathbb{R}} \mathbf{Q}\big(S_1 \in dx\big) f(x) \int_{\Omega_f} e^{-\langle g, \theta\rangle} \Xi_x(d\theta). \tag{3.19}$$

In words, Ξ_x is the law of Θ_1 conditioned on $\{S_1 = x\}$ under $\mathbf{Q}$.

Now, we are interested in a change of measure in the killed branching random walk. To introduce the corresponding density, we consider $R(\cdot)$ the renewal function of the random walk $(S_n)_{n\geq 0}$ under $\mathbf{Q}$. More precisely, for $x > 0$, $R(x)$ is defined by the expected number (under $\mathbf{Q}$) of visits to $[-x, 0]$ before first returning to $[0, \infty)$, i.e. $R(0) = 1$, and

$$R(x) := \mathbf{Q}\left[\sum_{j=0}^{\tau^*-1} 1_{\{-x \leq S_j\}}\right], \qquad \forall x > 0, \tag{3.20}$$

with $\tau^* := \inf\{j \geq 1 : S_j \geq 0\}$. We extend the definition of R on the whole real line by letting $R(x) = 0$ for all $x < 0$.

Recall that $\mathbf{Q}[S_1] = 0$ in the critical case and $\mathbf{Q}[S_1] > 0$ in the subcritical case. It is known (see Lemma 3) that the following limits exist and equal to some positive constants:

$$C_R := \begin{cases} \lim_{x\to\infty} \frac{R(x)}{x} = \frac{1}{\mathbf{Q}[-S_{\tau_0^-}]}, & \text{if } \psi'(\varrho_*) = 0 \text{ (critical case)}, \\ \lim_{x\to\infty} R(x) = \frac{1}{\mathbf{Q}(\tau_0^- = \infty)}, & \text{if } \psi'(\varrho_*) < 0 \text{ (subcritical case)}, \end{cases} \tag{3.21}$$

with τ_0^- defined in (3.8). Recall (3.15). Define

$$h_+(u) := R(V(u))e^{\varrho V(u)}\, 1_{\{V(u_1)\geq 0, \ldots, V(u_n)\geq 0\}}, \qquad |u| = n,\ u \in \mathcal{T}.$$

It is well-known that $(R(S_n)1_{\{\tau_0^- > n\}}, n \geq 0)$ is a $\mathbf{Q}_x$-martingale for any $x \geq 0$. Then h_+ satisfies (3.1) with $\lambda = 1$. Note that in this case, $\mathscr{H}_+ = \{u \in \mathcal{T} : V(u_0) \geq 0, ..., V(u_{|u|}) \geq 0\} = \mathscr{Z}$ is exactly the set of progeny of the killed branching walk.

Let $\mathbf{Q}_x^+$ be the probability satisfying (3.3) and (3.2) with $h = h_+$:

$$\frac{d\mathbf{Q}_x^+}{d\mathbf{P}_x}\Big|_{\mathscr{F}_n} := \frac{e^{-\varrho x}}{R(x)} \sum_{|u|=n,\, u\in\mathscr{Z}} R(V(u))e^{\varrho V(u)} =: M_n^*, \qquad x \geq 0,\ n \geq 1. \tag{3.22}$$

with $M_0^* := 1$. Write for simplification $\mathbf{Q}^+ = \mathbf{Q}_0^+$. Recalling (3.5), we deduce from Proposition 2 the following result, see Figure 4 below.

Corollary 2. *Recall (3.15). Fix $x \geq 0$. Under $\mathbf{Q}_x^+$,*

(a) $(S_n)_{n\geq 0}$ is a $(\mathscr{G}_n)$-Markov chain: for any $n \geq 1$, $y > 0$, and for any measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbf{Q}_x^+\Big[f(S_n)\big|\, \mathscr{G}_{n-1}, S_{n-1} = y\Big] = \mathbf{Q}_y\Big[\frac{R(S_1)}{R(y)} f(S_1)1_{\{S_1\geq 0\}}\Big].$$

In words, under $\mathbf{Q}_x^+$, the process $(S_n, n \geq 0)$ has the same law as the random walk $(S_n, n \geq 0)$ under $\mathbf{P}_x$, conditioned to stay nonnegative.

(b) Conditioned on $(S_n)_{n\geq 0}$, the point processes $(\Theta_n)_{n\geq 1}$ are independent and each Θ_n is distributed as $\Xi_{S_n - S_{n-1}}$.

(c) For any nonnegative function F and any $n \geq 0$,

$$\mathbf{E}_x\Big[\sum_{u\in\mathscr{Z}, |u|=n} F(V(u_i), 0 \leq i \leq |u|)\Big] = R(x)e^{\varrho x}\, \mathbf{Q}_x^+\Big[\frac{e^{-\varrho S_n}}{R(S_n)}\, F(S_i, 0 \leq i \leq n)\Big].$$

Proof of Corollary 2: The formula many-to-one (c) is routine. Let us only check (a) and (b): By Proposition 2 (i), we get that for any $n \geq 1$,

$$
\begin{aligned}
& \mathbf{Q}_x^+\Big[e^{-\langle g,\Theta_n\rangle} f(S_n)\Big|\, \mathscr{G}_{n-1}, S_{n-1}=y\Big] \\
(3.23) \quad &= \mathbf{E}_y\Big[\sum_{|u|=1} \frac{1}{R(y)e^{\varrho y}} e^{\varrho V(u)} R(V(u)) 1_{\{V(u)\geq 0\}} f(V(u)) e^{-\sum_{v\neq u,|v|=1} g(V(v)-y)}\Big] \\
(3.24) \quad &= \mathbf{Q}_y\Big[\frac{R(S_1)}{R(y)} 1_{\{S_1\geq 0\}} f(S_1) e^{-\langle g,\Theta_1\rangle}\Big] \\
(3.25) \quad &= \mathbf{Q}\Big[\frac{R(y+S_1)}{R(y)} 1_{\{y+S_1\geq 0\}} f(y+S_1) e^{-\langle g,\Theta_1\rangle}\Big],
\end{aligned}
$$

by using Corollary 1 (i). Taking $g=0$ in (3.24) yields the assertions in (a). Taking $n=1$ gives the joint law of (S_1,Θ_1) under $\mathbf{Q}_x^+$. Let $p(dz)=\mathbf{Q}(S_1\in dz)$ be the law of S_1 under $\mathbf{Q}$. Recall that Ξ_z is the law of Θ_1 conditioning on $\{S_1=z\}$ under $\mathbf{Q}$. Then for any event $A\in\mathscr{G}_{n-1}$, we deduce from (3.25) that

$$
\begin{aligned}
& \mathbf{Q}_x^+\Big[e^{-\langle g,\Theta_n\rangle} f(S_n) 1_A\Big] \\
&= \mathbf{Q}_x^+\Big[1_A \int_{\mathbb{R}} p(dz)\frac{R(S_{n-1}+z)}{R(S_{n-1})} 1_{\{S_{n-1}+z\geq 0\}} f(S_{n-1}+z) \int_{\Omega_f} \Xi_z(d\theta) e^{-\langle g,\theta\rangle}\Big] \\
&= \mathbf{Q}_x^+\Big[1_A\, f(S_n) \int_{\Omega_f} \Xi_{S_n-S_{n-1}}(d\theta) e^{-\langle g,\theta\rangle}\Big],
\end{aligned}
$$

by using (a) for the last equality. This together with the Markov property of (S_n) with respect to $(\mathscr{G}_n)$, imply that for any $n\geq 1$ and $g:\mathbb{R}\to\mathbb{R}_+$,

$$
\mathbf{Q}_x^+\Big[e^{-\langle g,\Theta_n\rangle}\Big|\, \mathscr{G}_{n-1}, (S_j)_{j\geq 0}\Big] = \int_{\Omega_f} \Xi_{S_n-S_{n-1}}(d\theta) e^{-\langle g,\theta\rangle},
$$

proving (b). □

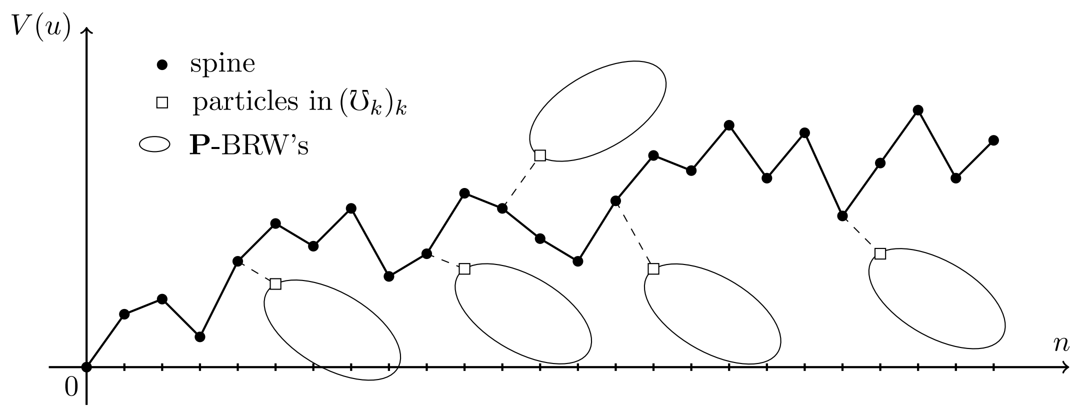

FIGURE 4. Spinal decomposition under $\mathbf{Q}_0^+$

Remark 4. *If we assume that $\mathscr{L}=\sum_{i=1}^{\nu}\delta_{\{X_i\}}$ with $(X_i)_{i\geq 1}$ a sequence of i.i.d. real-valued variables of the same law as X, independent of ν, then the expectation in (3.23) equals to*

$$
\sum_{n\geq 0} \mathbf{P}\big(\nu=k\big) k \mathbf{E}\Big[\frac{R(X+y)}{R(y)} e^{\varrho X} 1_{\{X+y\geq 0\}} f(X+y)\Big] \Big(\mathbf{E} e^{-g(X)}\Big)^{k-1},
$$

which implies that under $\mathbf{Q}_x^+$ for each $n\geq 1$, conditionally on $\mathscr{G}_{n-1}$ and on $\{S_{n-1}=y\}$, S_n and Θ_n are independent and Θ_n is distributed as $\sum_{i=1}^{\widetilde{\nu}-1}\delta_{X_i}$, with $\widetilde{\nu}$ the size-biased of ν, $\mathbf{Q}_x^+(\widetilde{\nu}=k)=k\mathbf{P}(\nu=k)/\mathbf{E}[\nu], k\geq 1$, and independent of $(X_i)_{i\geq 1}$.

We may extend Corollary 2 to the stopping lines. Remark that $\mathscr{C}_t \cap \mathscr{Z} = \mathscr{H}(t)$ (see (3.7) and (1.11)). We deduce from Proposition 3 the following result:

Corollary 3. *Recall (3.15) and (3.8). Fix $0 \le x < t$. We have*

$$\frac{d\mathbf{Q}_x^+}{d\mathbf{P}_x}\Big|_{\mathscr{F}_{\mathscr{C}_t}} = \frac{e^{-\varrho x}}{R(x)} \sum_{u \in \mathscr{H}(t)} R(V(u)) e^{\varrho V(u)} =: M^*_{\mathscr{C}_t}. \tag{3.26}$$

(i) *Under probability $\mathbf{Q}_x^+$, conditionally on $\mathscr{G}_{\mathscr{C}_t}$ and on $\{V(v) = x_v, v \in \bigcup_{k=1}^{\tau_t^+} \mho_k\}$, the shifted branching random walks $\{V(vu) - V(v)\}_{u\,:\,|vu| \le \tau_t^+(vu)}$, stopped by the line $\mathscr{C}_t$, are independent, and have the same law as $\{V(u)\}_{|u| \le \tau^+_{t-x_v}(u)}$ under $\mathbf{P}_0$, stopped by the line $\mathscr{C}_{t-x_v}$.*

(ii) *Moreover, for any measurable function $F : \mathbb{R}^{\mathbb{N}_+} \to \mathbb{R}_+$,*

$$\mathbf{E}_x\Big[\sum_{u \in \mathscr{C}_t \cap \mathscr{Z}} F(V(u_i), 0 \le i \le |u|)\Big] = R(x) e^{\varrho x}\, \mathbf{Q}_x^+\Big[\frac{e^{-\varrho S_{\tau_t^+}}}{R(S_{\tau_t^+})} 1_{\{\tau_t^+ < \tau_0^-\}} F(S_i, 0 \le i \le \tau_t^+)\Big].$$

4. One-dimensional real-valued random walks

In this section we collect some preliminary results for a one-dimensional random walk $(S_n)_{n\ge 0}$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Most of the results in this section will be applied to the random walk S defined in (3.17) under $\mathbf{Q}$ in Section 3. For the sake of clarity of presentation, the technical proofs are postponed to Section 8.

4.1. Time-reversal random walks.

Let $((S_n)_n, \mathbb{P}_x)$ be a real-valued random walk starting from $x \in \mathbb{R}$. We write $\mathbb{P} = \mathbb{P}_0$. Assume that $\mathbb{E}[S_1] \ge 0$ and $\mathbb{E}[|S_1|^{3+\delta}] < \infty$ for some $\delta > 0$. In words, we consider random walks that do not drift to $-\infty$. Moreover we assume that the distribution of S_1 is non-arithmetic. Let us adopt the same notations τ^+, τ^- and $R(\cdot)$ as in Section 3:

$$\tau_a^+ := \inf\{k \ge 0 \,:\, S_k > a\}, \qquad \tau_a^- := \inf\{k \ge 0 \,:\, S_k < a\}, \tag{4.1}$$

and the overshoot $T_a^+ := S_{\tau_a^+} - a > 0$ and the undershoot $T_a^- := a - S_{\tau_a^-} > 0$. Let $R(\cdot)$ be as in (3.20) the renewal function of $(S_n)_{n\ge 0}$ under $\mathbb{P}$. i.e. with $\tau^* := \inf\{j \ge 1 : S_j \ge 0\}$,

$$R(x) := \mathbb{E}\left[\sum_{j=0}^{\tau^*-1} 1_{\{-x \le S_j\}}\right], \qquad \forall x > 0,$$

and $R(0) = 1$.

Following [6], we introduce the law of the random walk conditioned to stay nonnegative. To this aim, we see $(S_n)_{n\ge 0}$ under $\mathbb{P}_x$ as a Markov chain with transition function $\mu(y, \mathrm{d}z) := \mathbb{P}(y + S_1 \in \mathrm{d}z)$. We denote by $\mathbb{P}_x^+$ the h-transform of $\mathbb{P}_x$ by the function R. That is, $\mathbb{P}_x^+$ is a probability measure under which $(S_n)_{n\ge 0}$ is a homogeneous Markov chain on the nonnegative real numbers, with transition function

$$\mu_R(y, \mathrm{d}z) := \frac{R(z)}{R(y)} \mu(y, \mathrm{d}z), \qquad y, z \ge 0. \tag{4.2}$$

It is well known that $\mathbb{P}^+$-almost surely $S_n \to \infty$ when $n \to \infty$. When $(S_n)_{n\ge 0}$ drifts to ∞ (i.e. when $\mathbb{E}[S_1] > 0$), $\mathbb{P}^+$ is the law of the random walk conditioned to stay nonnegative in the usual sense, i.e. $\mathbb{P}^+(\cdot) = \mathbb{P}(\cdot \,|\, S_1 \ge 0, \dots, S_n \ge 0, \dots)$.

We denote by $(\sigma_n, H_n)_{n\ge 1}$ the strict ascending ladder epochs and ladder heights of S. Some results from random walk theory are important in the proofs presented here and recorded in the following lemma.

Lemma 3. *Assume that* $\mathbb{E}[S_1] \geq 0$, $\mathbb{E}[|S_1|^{3+\delta}] < \infty$ *for some* $\delta > 0$ *and that the distribution of* S_1 *is non-arithmetic. Then,*

(i) T_t^+ *converges in law to a finite random variable when* t *tends to infinity.*
(ii) $(T_t^+, t \geq 0)$ *is bounded in* L^p *for all* $1 < p < 1 + \delta$.
(iii) $S_{\tau_t^+}/t$ *converges in probability to* 1 *when* t *tends to infinity.*
(iv)
- *If* $\mathbb{E}[S_1] = 0$, *there exists a constant* $C_R \in (0, \infty)$ *such that* $R(x)/x \longrightarrow C_R$ *when* $x \to \infty$. *In this case,* $C_R = \frac{1}{\mathbb{E}[T_0^-]} = \frac{1}{\mathbb{E}[-S_{\tau_0^-}]}$.
- *If* $\mathbb{E}[S_1] > 0$, *there exists a constant* $C_R \in (0, \infty)$ *such that* $R(x) \longrightarrow C_R$ *when* $x \to \infty$. *In this case,* $C_R = \frac{1}{\mathbb{P}(\tau_0^- = \infty)}$.

(v)
- *If* $\mathbb{E}[S_1] = 0$, *then* $\mathbb{P}(\tau_t^+ < \tau_0^-) \sim \frac{1}{C_R t}$ *when* $t \to \infty$.
- *If* $\mathbb{E}[S_1] > 0$, *then* $\mathbb{P}(\tau_t^+ < \tau_0^-) \to \frac{1}{C_R}$ *when* $t \to \infty$.

Proof: Notice that T_t^+ is also the overshoot of the random walk (H_n) above the level t. In the case $\mathbb{E}[S_1] = 0$, Doney [11] implies that H_1 has a finite $(2+\delta)$-moment which in view of Lorden ([24], Theorem 3, applied to (H_n)), implies that $(T_t^+, t \geq 0)$ is bounded in L^p for all $1 < p < 1 + \delta$. In the case $\mathbb{E}[S_1] > 0$, again by Lorden ([24], Theorem 3, applied to (S_n)), $(T_t^+, t \geq 0)$ is bounded in L^p for all $1 < p < 2 + \delta$. This provides Part (ii) of the lemma. Moreover, we see that in both cases, $H_1 = T_0^+$ has a finite expectation and obviously is non-arithmetic, then a refinement of the renewal theorem gives Part (i) of the Lemma (Feller [12], pp. 370 equation (4.10)). For both cases, Part (iii) is a consequence of Part (ii). To show (iv), we recall the duality lemma:

$$R(x) = 1 + \sum_{n=1}^{\infty} \mathbb{P}\left(H_n^- \leq x\right), \qquad x > 0,$$

where $(H_n^-, n \geq 1)$ denotes the (strict) ascending ladder heights of $-S$ (in particular, $H_1^- = T_0^-$ the undershoot at 0). In the case $\mathbb{E}[S_1] = 0$, Part (iv) is a consequence of the renewal theorem (see Feller [12] pp. 360) with $C_R = \frac{1}{\mathbb{E}[T_0^-]}$ while Part (v) is obtained by applying the optional stopping theorem to the martingale $(S_k, 0 \leq k \leq \tau_t^+ \wedge \tau_0^-)$ (the uniform integrability is guaranteed by (ii), see [3], Lemma 2.2). In the case $\mathbb{E}[S_1] > 0$, Part (iv) and (v) follow from the duality lemma: $C_R = \mathbb{E}[\tau^*] = \lim_{x \to \infty} R(x) = 1 + \sum_{n=1}^{\infty} \mathbb{P}(H_n^- < \infty) = 1 + \sum_{n=1}^{\infty} \mathbb{P}(\tau_0^- < \infty)^n = \frac{1}{\mathbb{P}(\tau_0^- = \infty)}$. □

We recall now Tanaka's construction (see Figure 5) of the random walk conditioned to stay positive. Let us recall that $(\sigma_n, H_n)_{n \geq 1}$ are the strict ascending ladder epochs and ladder heights of S and let $(w_i)_{i \geq 1}$ be independent copies of the segment of the random walk $(S_n)_{n \geq 0}$ up to time $\sigma := \sigma_1$ viewed from (σ, S_σ) in reversed time and reflected in the x-axis; that is, $(w_i)_{i \geq 0}$ are independent copies of

$$(0, S_\sigma - S_{\sigma-1}, S_\sigma - S_{\sigma-2}, \ldots, S_\sigma - S_1, S_\sigma). \tag{4.3}$$

We write now $w_i = (w_i(\ell)\,;\, \ell = 0, 1, 2, \ldots \sigma^{(i)})$ to identify the components of w_i. In [31], Tanaka shows that the random walk conditioned to stay positive can be constructed by gluing the w_i's together, each starting from the end of the previous one. More formally, let $(\sigma_n^+, H_n^+)_{n \geq 1}$ be the renewal process formed from the independent random variables $(\sigma^{(i)}, w_i(\sigma^{(i)}))$, that is

$$(\sigma_n^+, H_n^+) = (\sigma^{(1)} + \cdots + \sigma^{(n)}, w_1(\sigma^{(1)}) + \cdots + w_n(\sigma^{(n)})), \qquad n \geq 1. \tag{4.4}$$

Then, Tanaka's result says that the random walk conditioned to stay positive can be constructed via the process $(\zeta_n)_{n \geq 0}$ given by

$$\zeta_n = H_k^+ + w_{k+1}(n - \sigma_k^+) \qquad \sigma_k^+ < n \leq \sigma_{k+1}^+. \tag{4.5}$$

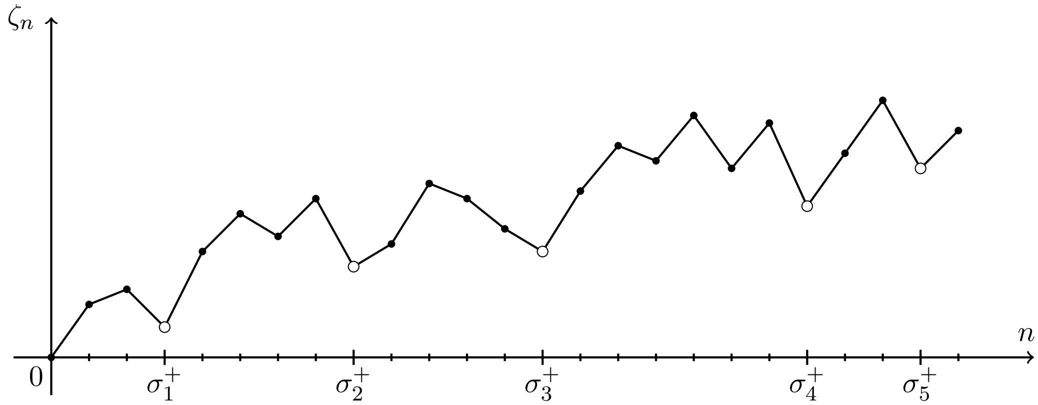

FIGURE 5. Tanaka's construction

Finally we introduce a process $(\hat{S}_n)_{n\geq 0}$ (obtained by modifying slightly the random walk conditioned to stay positive) which will be the limit process that appears in the following lemma. Let $\widetilde{\sigma} := \sup\{n \geq 1 : \zeta_n = \min_{1\leq i\leq n} \zeta_i\}$ and observe that $\widetilde{\sigma}$ is almost surely finite since $\zeta_n \to \infty$. Then $(\hat{S}_n)_{n\geq 0}$ is defined by

$$\mathbb{E}\Big[F((\hat{S}_n)_{n\geq 0})\Big] = \frac{1}{\mathbb{E}\,[H_1]}\mathbb{E}\Big[\zeta_{\widetilde{\sigma}}\, F((\zeta_n)_{n\geq 0})\Big], \tag{4.6}$$

for any test function F. Observe that Tanaka's construction implies $\mathbb{E}[\zeta_{\widetilde{\sigma}}] = \mathbb{E}\,[H_1]$. Moreover we introduce $\hat{\sigma} := \sup\{n \geq 1 : \hat{S}_n = \min_{1\leq i\leq n} \hat{S}_i\}$ which is almost surely finite since $\hat{S}_n \to \infty$.

Lemma 4. *Assume that $\mathbb{E}[S_1] \geq 0$, $\mathbb{E}[|S_1|^{3+\delta}] < \infty$ for some $\delta > 0$ and that the distribution of S_1 is non-arithmetic. Recall (4.1) and fix an arbitrary integer $K \geq 1$. Let $F : \mathbb{R}_+^* \times \mathbb{R}_+^K \to \mathbb{R}$ be a bounded and measurable function. Suppose that for any $z \in \mathbb{R}_+^K$, the set $\{x \in \mathbb{R}_+^* : F(\cdot, z)$ is not continuous at $x\}$ is at most countable [which may depend on z]. Then*

(i)
$$\lim_{t\to\infty} \mathbb{E}\Big[F(T_t^+, (S_{\tau_t^+} - S_{\tau_t^+ - j})_{1\leq j\leq K}) \,\big|\, \tau_t^+ > K\Big] = \mathbb{E}\Big[F(U\hat{S}_{\hat{\sigma}}, (\hat{S}_j)_{1\leq j\leq K})\Big],$$
where $(\hat{S}_n)_{n\geq 0}$ is the process defined by (4.6) and U is a uniform random variable on $[0,1]$ independent of $(\hat{S}_n)_{n\geq 0}$.

(ii)
$$\lim_{t\to\infty} \mathbb{E}^+\Big[F(T_t^+, (S_{\tau_t^+} - S_{\tau_t^+ - j})_{0\leq j\leq K}) \,\big|\, \tau_t^+ > K\Big] = \mathbb{E}\Big[F(U\hat{S}_{\hat{\sigma}}, (\hat{S}_j)_{1\leq j\leq K})\Big],$$
where $\mathbb{E}^+$ denotes the expectation with respect to the probability measure $\mathbb{P}^+$.

As a consequence, under $\mathbb{P}(\cdot \,|\, \tau_t^+ > K)$ or under $\mathbb{P}^+(\cdot \,|\, \tau_t^+ > K)$, the random vector $(T_t^+, (S_{\tau_t^+} - S_{\tau_t^+ - j})_{0\leq j\leq K})$ converges in distribution toward $(U\hat{S}_{\hat{\sigma}}, (\hat{S}_j)_{0\leq j\leq K})$ when $t \to \infty$. We also note that the conditioning with respect to the event $\{\tau_t^+ > K\}$ is just technical since this event is asymptotically typical (indeed almost surely $\tau_t^+ \to \infty$ when $t \to \infty$).

Proof. See Section 8. □

We end this subsection by an estimate on a random walk with positive drift:

Lemma 5. *Assume that $\mathbb{E}[S_1] > 0$, $\mathbb{E}[S_1^2] < \infty$. Let $(a_i, S_i - S_{i-1})_{i\geq 1}$ be an i.i.d. sequence such that $a_i \geq 0$ almost surely. For any $p \geq 1$ such that $\mathbb{E}[a_1^p] < \infty$ and for any $\kappa > 0$, there* *exists some constant $c_{p,\kappa} > 0$ such that*

$$\mathbb{E}_x \left[\sum_{k=0}^{\tau_t^+ -1} a_{k+1}\, e^{\kappa\,(S_k - t)} \right]^p \le c_{p,\kappa}, \qquad \forall t > 0,\, \forall x \le t. \tag{4.7}$$

Proof. See Section 8. □

4.2. **Centered random walks.** Let $((S_n)_{n\ge 0}, \mathbb{P}_x)$ be a real-valued random walk starting from $x \in \mathbb{R}$. We write $\mathbb{P} = \mathbb{P}_0$. Assume that

$$\mathbb{E}[S_1] = 0, \quad Var(S_1) > 0, \quad \mathbb{E}\left[e^{uS_1}\right] < \infty,\ \forall u \in (-(1+\eta), \eta), \tag{4.8}$$

for some $\eta > 0$. Recall that $\mathbb{P}(\tau_L^+ < \tau_0^-)$ is of order $\frac{1}{L}$ as $L \to \infty$ (cf. Lemma 3). We have the following estimate.

Lemma 6. *Under (4.8). For any $\delta > 0$, there exist some constants $c > 1$ and $c' = c'(\delta) > 1$ such that for all $L \ge 1$, $0 \le a \le L$,*

$$\mathbb{E}_a\Big[e^{-S_{\tau_0^-}} 1_{\{\tau_0^- < \tau_L^+\}}\Big] \;\le\; c\frac{L-a+1}{L}, \tag{4.9}$$

$$\mathbb{E}_a\Big[\sum_{j=0}^{\tau_L^+ -1} e^{-\delta(L-S_j)}\Big] + \mathbb{E}_a\Big[\sum_{j=0}^{\tau_0^- -1} e^{-\delta S_j}\Big] \;\le\; c', \tag{4.10}$$

$$\mathbb{E}_a\left[e^{S_{\tau_0^- -1} - S_{\tau_0^-}}\right] \;\le\; c, \tag{4.11}$$

$$\mathbb{E}_a\Big[\sum_{0\le j<\tau_0^- \wedge \tau_L^+} e^{-\delta S_j}\Big] \;\le\; c'\,\frac{L-a+1}{L}, \tag{4.12}$$

$$\mathbb{E}_a\Big[\sum_{0\le j<\tau_0^- \wedge \tau_L^+} e^{-\delta(L-S_j)}\Big] \;\le\; c'\,\frac{a+1}{L}, \tag{4.13}$$

$$\mathbb{E}_a\Big[e^{-S_{\tau_0^-}} 1_{\{\tau_0^- < \tau_L^+\}} \sum_{0\le j<\tau_0^-} e^{-\delta(L-S_j)}\Big] \;\le\; c'\,\frac{a+1}{L^2}. \tag{4.14}$$

Remark: A weaker assumption $\sup_{-\eta\le u\le \eta} \mathbb{E}\left[e^{uS_1}\right] < \infty$ is enough to get (4.10), (4.11), (4.12) and (4.13).

Proof. See Section 8. □

4.3. **Random walks with negative drift.** In this subsection, we give estimates on transient random walks. We take again $((S_n)_{n\ge 0}, \mathbb{P}_x)$ a random walk, but we suppose now that $\mathbb{E}[S_1] < 0$, hence the random walk drifts to $-\infty$. We suppose that there exist γ, η_1, $\eta_2 > 0$ such that

$$\mathbb{E}[e^{\gamma S_1}] = 1, \qquad \mathbb{E}[e^{uS_1}] < \infty,\ \forall\, u \in (-\eta_1, \gamma + \eta_2). \tag{4.15}$$

Then,

$$\mathbb{P}(\tau_a^- < \tau_0^+) \to \mathbb{P}(\tau_0^+ = \infty) > 0, \qquad a \to -\infty. \tag{4.16}$$

By Theorem 1 of [16], if S_1 is non-arithmetic, then

$$\mathbb{P}(\tau_a^+ < \tau_0^-) \sim c_6\, e^{-\gamma a}, \qquad a \to +\infty, \tag{4.17}$$

for some constant $c_6 > 0$. We end this section by two lemmas:

Lemma 7. *Under (4.15). For any $r > 0$, we can find some positive constants c, c', c'' such that for any $a \geq 0$, $L > 1$,*

$$\mathbb{E}_a\Big[e^{-r\, S_{\tau_0^-}}\Big] \leq c(r) \qquad \text{if } r < \eta_1, \tag{4.18}$$

$$\mathbb{E}_a\Big[\sum_{0\leq \ell<\tau_L^+} (1+L-S_\ell)^\alpha e^{r\, S_\ell}\Big] \leq c'(r,\alpha) e^{\gamma(a-L)} e^{r\, L} \qquad \text{if } r > \gamma,\ \alpha \geq 0. \tag{4.19}$$

$$\mathbb{E}_a\Big[\sum_{\ell=0}^{\min(\tau_0^-,\tau_L^+)} (1+L-S_\ell)^\alpha\, e^{\gamma S_\ell}\Big] \leq c''\, e^{\gamma a}(1+L-a)^{1+\alpha}, \quad a\in[0,L],\ \alpha\geq 0. \tag{4.20}$$

Proof. See Section 8. □

Lemma 8. *Under (4.15). Fix some $0 \leq \eta < \eta_1$, $b > 0$ and $\alpha \geq 0$. Assume that $(S_n - S_{n-1}, a_n)_{n\geq 1}$ are i.i.d. with $a_1 \geq 0$ almost surely.*

(i) For any $0 \leq p < \gamma/b$ satisfying $\mathbb{E}\left[(1+1_{\{S_1<0\}}e^{-\eta S_1})a_1^p\right] < \infty$, there exists some constant $c_p > 0$ such that

$$\mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\Big(\sum_{\ell=1}^{\tau_0^-} e^{bS_{\ell-1}}a_\ell\Big)^p\right] \leq c_p\, e^{b\,p\,x}, \qquad x\geq 0. \tag{4.21}$$

(ii) Assume $p \geq 1$ is such that $\mathbb{E}\left[(1+1_{\{S_1<0\}}e^{-\eta S_1})a_1^p\right] < \infty$ and $\mathbb{E}[e^{p\,b\,S_1}] < \infty$. There exists some constant $c_{p,\eta,\alpha} > 0$ such that for all $L > 0$ and $0 \leq x \leq L$,

$$\mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\Big(\sum_{\ell=1}^{\min(\tau_0^-,\tau_L^+)} (1+L-S_{\ell-1})^\alpha e^{bS_{\ell-1}}a_\ell\Big)^p\right]$$

$$\leq c_{p,\eta,\alpha} \times \begin{cases} (1+L-x)^{\alpha p}\, e^{p\,bx}, & \text{if } p<\gamma/b,\\ e^{\gamma x}\,(1+L-x)^{1+\alpha\, p} & \text{if } p=\gamma/b,\\ e^{\gamma(x-L)+p\,bL} & \text{if } p>\gamma/b.\end{cases} \tag{4.22}$$

Proof. See Section 8. □

5. Maximum of the killed branching random walk: Proofs of Theorem 3 and Proposition 1

Let us first recall the following criterion for convergence in distribution of point processes which can be found in Resnick [30] (see pp. 153, proposition 3.19). Let E be a polish space. Then, let us define the Laplace transform of a point process θ with probability measure $\mathbb{P}$ by

$$\Psi_{\mathbb{P}}(f) := \int \exp\left\{-\int f\,\mathrm{d}\theta\right\}\mathrm{d}\mathbb{P}(\theta) = \int \exp\left\{-\langle f,\theta\rangle\right\}\mathrm{d}\mathbb{P}(\theta), \tag{5.1}$$

where f is a positive measurable function from E to $\mathbb{R}$. Let $C_K^+(E)$ be the space of continuous functions from E to $\mathbb{R}_+$ with compact support. Then we have

$$\lim_{n\to\infty}\Psi_{\mathbb{P}_n}(f) = \Psi_{\mathbb{P}}(f), \qquad \forall f\in C_K^+(E), \tag{5.2}$$

if and only if

$$\mathbb{P}_n \overset{\text{(vague)}}{\longrightarrow} \mathbb{P}, \qquad n\to\infty, \tag{5.3}$$

which is the same as the convergence in distribution of the point processes.

Recall the real-valued random walk (S_n) defined in Corollary 1. In order to treat both critical and subcritical cases in the same proof, we introduce the following function defined on $\mathbb{R}_+$ by

$$\mathscr{R}(t) := \begin{cases} t, & \text{if } \psi'(\varrho_*) = 0, \\ 1, & \text{if } \psi'(\varrho_*) < 0, \end{cases}, \qquad \varrho := \begin{cases} \varrho_*, & \text{if } \psi'(\varrho_*) = 0, \\ \varrho_+, & \text{if } \psi'(\varrho_*) < 0, \end{cases}, \tag{5.4}$$

and observe that the renewal function $R(\cdot)$, associated with the random walk $(S_n, \mathbf{Q})$ defined by (3.20), satisfies that (see (3.21))

$$R(t) \sim C_R \mathscr{R}(t), \qquad t \to \infty.$$

We take the notation of Theorem 3 and Proposition 1. The key step is to prove that for any $f \in C_K^+(\mathbb{R})$ and when $t \to \infty$, we have

$$\mathbf{E}_x\left[e^{-\langle f, \mu_{\mathscr{B},t}\rangle} 1_{\{H(t)>0\}}\right] \sim \frac{R(x)e^{\varrho x}}{C_R \mathscr{R}(t) e^{\varrho t}} \mathbf{Q}\left[\frac{e^{-\langle f, \mu_{\mathscr{B},\infty}\rangle}}{\Re}\right]. \tag{5.5}$$

We recall from (3.26) that $M^*_{\mathscr{C}_t} = \frac{e^{-\varrho x}}{R(x)} \sum_{u \in \mathscr{H}(t)} R(V(u)) e^{\varrho V(u)}$, where $\mathscr{H}(t)$ denotes the set of those $u \in \mathscr{L}$ satisfying $\tau_t^+(u) = |u|$ (see (1.11)). Then $H(t) > 0$ if and only if $M^*_{\mathscr{C}_t} > 0$. It follows that

$$\mathbf{E}_x\left[e^{-\langle f, \mu_{\mathscr{B},t}\rangle} 1_{\{H(t)>0\}}\right] = \mathbf{E}_x\left[\frac{M^*_{\mathscr{C}_t}}{M^*_{\mathscr{C}_t}} e^{-\langle f, \mu_{\mathscr{B},t}\rangle} 1_{\{H(t)>0\}}\right] = \mathbf{Q}_x^+\left[\frac{e^{-\langle f, \mu_{\mathscr{B},t}\rangle}}{M^*_{\mathscr{C}_t}}\right]. \tag{5.6}$$

We will now use the so-called "decomposition along the spine" (w_k) (under $\mathbf{Q}_x^+$). Recalling that $\mho_k = \{u : |u| = k, \overleftarrow{u} = w_{k-1}, x \neq w_k\}$, we have

$$\langle f, \mu_{\mathscr{B},t}\rangle = f(T_t^+) 1_{\{\beta_t(w_{\tau_t^+}) = \infty\}} + \sum_{1 \le k \le \tau_t^+} \sum_{u \in \mho_k} 1_{\{\beta_t(u) = \infty\}} \langle f, \mu_{\mathscr{B},t}^{(V(u))}\rangle, \tag{5.7}$$

where $T_t^+ = S_{\tau_t^+} - t$ denotes the overshoot of S above the level t (see (3.9)), and for any $u \in \mathcal{T}$ the point process $\mu_{\mathscr{B},t}^{(V(u))}$ is associated to the subtree $\mathcal{T}^{(u)}$ (rooted at u) of $\mathcal{T}$ and defined by

$$\mu_{\mathscr{B},t}^{(V(u))} := \sum_{v \in \mathcal{T}^{(u)} \cap \mathscr{H}_\beta(t)} \delta_{\{V(v)-t\}}, \qquad \mu_t^{(V(u))} := \sum_{v \in \mathcal{T}^{(u)} \cap \mathscr{H}(t)} \delta_{\{V(v)-t\}}. \tag{5.8}$$

Recall that $R(s) \sim C_R \mathscr{R}(s)$ when $s \to \infty$. Since $V(u) > t$ for all $u \in \mathscr{H}(t)$, we get that, under $\mathbf{Q}_x^+$,

$$M^*_{\mathscr{C}_t} \sim \frac{e^{-\varrho x}}{R(x)} C_R \mathscr{R}(t) e^{\varrho t} \sum_{u \in \mathscr{H}(t)} \mathscr{R}\left(1 + \frac{V(u)-t}{t}\right) e^{\varrho(V(u)-t)}, \qquad t \to \infty. \tag{5.9}$$

Then repeating the spinal decomposition arguments for the above sum $\sum_{u \in \mathscr{H}(t)}$ we obtain

$$\mathbf{E}_x\left[e^{-\langle f, \mu_t\rangle} 1_{\{H(t)>0\}}\right] \sim \frac{R(x)e^{\varrho x}}{C_R \mathscr{R}(t) e^{\varrho t}} \mathbf{Q}_x^+\left[\frac{I_\beta(t)}{J(t)}\right], \tag{5.10}$$

with

$$\begin{aligned} I_\beta(t) &:= \exp\Big\{ - f(T_t^+) 1_{\{\beta_t(w_{\tau_t^+}) = \infty\}} - \sum_{1 \le k \le \tau_t^+} \sum_{u \in \mho_k} 1_{\{\beta_t(u) = \infty\}} \langle f, \mu_{\mathscr{B},t}^{(V(u))}\rangle \Big\}, \\ J(t) &:= \mathscr{R}\left(1 + \frac{T_t^+}{t}\right) e^{\varrho T_t^+} + \sum_{1 \le k \le \tau_t^+} \sum_{u \in \mho_k} \int \mathscr{R}\left(1 + \frac{z}{t}\right) e^{\varrho z} \mu_t^{(V(u))}(\mathrm{d}z). \end{aligned}$$

Therefore, to prove (5.5) we only have to show that

$$\lim_{t\to\infty} \mathbf{Q}_x^+\left[\frac{I_\beta(t)}{J(t)}\right] = \mathbf{Q}\left[\frac{e^{-\langle f,\mu_{\mathscr{B},\infty}\rangle}}{\Re}\right]. \tag{5.11}$$

Note that $I_\beta(t)\in[0,1]$ and $J(t)\geq 1$, hence $\frac{I_\beta(t)}{J(t)}\in[0,1]$. Recalling the convergence in law of the process $(t-S_{\tau_t^+-j})_{0\leq j\leq K}$ for any fixed $K\geq 1$ (see Lemma 4), we will restrict the sums over k in $I_\beta(t)$ and $J(t)$ to k's between τ_t^+-K and τ_t^+. To this aim let us introduce $H^u(t)$ the number of descendants of u that reach t before 0 (with the convention $H^u(t)=1$ if $V(u)>t$). The following lemma ensures that with probability close to 1, $\sum_{1\leq k\leq\tau_t^+-K}\sum_{u\in\mho_k} H^u(t)=0$ [the sum is 0 if $\tau_t^+\leq K$]:

Lemma 9. *We have*

(i) $\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{Q}_x^+\left(\sum_{k=1}^{\tau_t^+-K}\sum_{u\in\mho_k}H^u(t)\geq 1\right)=0,$

(ii) $\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{Q}_x^+\left(\beta_t(w_{\tau_t^+})\leq\tau_t^+-K\right)=0.$

Proof of Lemma 9. See Subsection 8.4. □

Notice that $\lim_{t\to\infty}\mathbf{Q}_x^+(\tau_t^+>K)=1$ and that on $\{\beta_t(w_{\tau_t^+})>\tau_t^+-K,\tau_t^+>K\}$,

$$\beta_t(u)=\inf\{\tau_t^+-K<j\leq|u|:\mathscr{B}(u_j)>e^{t-V(u_{j-1})}\}=:\beta_t^K(u),$$

for any $u=w_{\tau_t^+}$ or $u\in\mho_k$ with $\tau_t^+-K<k\leq\tau_t^+$. The advantage of $\beta_t^K(u)$ is that $\beta_t^K(u)$ only locally depends on the spines around τ_t^+. Therefore (5.11) will be a consequence of

$$\lim_{K\to\infty}\lim_{t\to\infty}\mathbf{Q}_x^+\left[\frac{I'_\beta(t,K)}{J'(t,K)}1_{\{\tau_t^+>K\}}\right]=\mathbf{Q}\left[\frac{e^{-\langle f,\mu_{\mathscr{B},\infty}\rangle}}{\Re}\right], \tag{5.12}$$

with

$$\begin{aligned} I'_\beta(t,K) &:= \exp\Big\{-f(T_t^+)1_{\{\beta_t^K(w_{\tau_t^+})=\infty\}}-\sum_{\tau_t^+-K<k\leq\tau_t^+}\sum_{u\in\mho_k}1_{\{\beta_t^K(u)=\infty\}}\langle f,\mu_{\mathscr{B},t}^{(V(u))}\rangle\Big\},\\ J'(t,K) &:= \mathscr{R}\left(1+\frac{T_t^+}{t}\right)e^{\varrho T_t^+}+\sum_{\tau_t^+-K<k\leq\tau_t^+}\sum_{u\in\mho_k}\int\mathscr{R}\left(1+\frac{z}{t}\right)e^{\varrho z}\,\mu_t^{(V(u))}(\mathrm{d}z). \end{aligned}$$

Recall (5.8) that the measures $\mu_{\mathscr{B},t}^{(V(u))}$ in the previous expressions are associated with the branching random walk killed at 0. Now, we want to replace the measures $\mu_{\mathscr{B},t}^{(V(u))}$ by the same measures $\widetilde{\mu}_{\mathscr{B},t}^{(V(u))}$ but associated with the non-killed branching random walk:

$$\widetilde{\mu}_{\mathscr{B},t}^{(V(u))}:=\sum_{v\in\mathcal{T}^{(u)}\cap\mathscr{C}_t}1_{\{\beta_t(v)=\infty\}}\,\delta_{\{V(v)-t\}},\qquad \widetilde{\mu}_t^{(V(u))}:=\sum_{v\in\mathcal{T}^{(u)}\cap\mathscr{C}_t}\delta_{\{V(v)-t\}}, \tag{5.13}$$

where we recall that $v\in\mathcal{T}^{(u)}\cap\mathscr{C}_t$ if and only if v is a descendant of u and $\tau_t^+(v)=|v|$ (see (3.7) for the definition of $\mathscr{C}_t$). The following lemma confirms that we can replace $\mu^{(V(u))}$ by $\widetilde{\mu}^{(V(u))}$ with probability close to 1:

Lemma 10. *Let us define for $t>0$ and $K\geq 1$*

$$\Gamma(t,K):=\{\tau_t^+>K\}\cap\Big\{(\mu_{\mathscr{B},t}^{(V(u))},\mu_t^{(V(u))})=(\widetilde{\mu}_{\mathscr{B},t}^{(V(u))},\widetilde{\mu}_t^{(V(u))}),\ \forall\,u\in\mho_k,\ \forall\,k\in(\tau_t^+-K,\tau_t^+]\Big\}.$$

Then for any $K\geq 1$, we have

$$\lim_{t\to\infty}\mathbf{Q}_x^+\Big(\Gamma(t,K)^c\Big)=0.$$

Proof of Lemma 10. See Subsection 8.4. □

By Lemmas 9 and 10, to prove (5.5) it is enough to show that

$$\lim_{K\to\infty}\lim_{t\to\infty}\mathbf{Q}_x^+\left[\frac{\widetilde{I}_\beta(t,K)}{\widetilde{J}(t,K)}1_{\{\tau_t^+>K\}}\right]=\mathbf{Q}\left[\frac{e^{-\langle f,\mu_{\mathscr{B},\infty}\rangle}}{\Re}\right], \tag{5.14}$$

where $\widetilde{I}_\beta(t,K)$ and $\widetilde{J}(t,K)$ are as $I'_\beta(t,K)$ and $J'(t,K)$ but with $\widetilde{\mu}^{(V(u))}$ in lieu of $\mu^{(V(u))}$:

$$\begin{aligned}\widetilde{I}_\beta(t,K) &:= \exp\Big\{-f(T_t^+)1_{\{\beta_t^K(w_{\tau_t^+})=\infty\}}-\sum_{\tau_t^+-K<k\le\tau_t^+}\sum_{u\in\mho_k}1_{\{\beta_t^K(u)=\infty\}}\langle f,\widetilde{\mu}_{\mathscr{B},t}^{(V(u))}\rangle\Big\},\\ \widetilde{J}(t,K) &:= \mathscr{R}\left(1+\frac{T_t^+}{t}\right)e^{\varrho T_t^+}+\sum_{\tau_t^+-K<k\le\tau_t^+}\sum_{u\in\mho_k}\int\mathscr{R}\left(1+\frac{z}{t}\right)e^{\varrho z}\,\widetilde{\mu}_t^{(V(u))}(\mathrm{d}z).\end{aligned}$$

Let us now introduce a family of point processes denoted by $(\overline{\mu}_{\mathscr{B},y},\overline{\mu}_y)_{y\in\mathbb{R}}$, which are associated to the non-killed branching random walk V under $\mathbf{P}$ and are defined by

$$\overline{\mu}_{\mathscr{B},y}:=\begin{cases}\sum_{v\in\mathscr{C}_y}1_{\{\beta_y(v)=\infty\}}\delta_{\{V(v)-y\}}, & \text{if}\quad y\ge0,\\ \delta_{\{-y\}}, & \text{if}\quad y<0,\end{cases} \tag{5.15}$$

and

$$\overline{\mu}_y:=\begin{cases}\sum_{v\in\mathscr{C}_y}\delta_{\{V(v)-y\}}, & \text{if}\quad y\ge0,\\ \delta_{\{-y\}}, & \text{if}\quad y<0.\end{cases} \tag{5.16}$$

where $\mathscr{C}_y$ was defined in (3.7); in particular, $\{V(v)-y,v\in\mathscr{C}_y\}$ denotes exactly the set of overshoots of the (non-killed) branching random walk V above the level y. By Part (i) of Corollary 3, under $\mathbf{Q}^+$, conditionally on $\mathscr{G}_{\mathscr{C}_t}$ and on $\{V(u)=x_u,u\in\mho_k,1\le k\le\tau_t^+\}$, the family $\{(\widetilde{\mu}_{\mathscr{B},t}^{(V(u))},\widetilde{\mu}_t^{(V(u))}),u\in\mho_k,1\le k\le\tau_t^+\}$ is independent and satisfies

$$\Big((\widetilde{\mu}_{\mathscr{B},t}^{(V(u))},\,\widetilde{\mu}_t^{(V(u))}),\ \text{under } \mathbf{Q}_x^+\Big)\overset{\text{law}}{=}\big((\overline{\mu}_{\mathscr{B},t-x_u},\,\overline{\mu}_{t-x_u}),\ \text{under } \mathbf{P}\big). \tag{5.17}$$

For convenience of notations, let us introduce

$$S_i^{(t)} := S_{\tau_t^+}-S_{\tau_t^+-i},\qquad 1\le i\le\tau_t^+, \tag{5.18}$$

$$\Theta_i^{(t)} := \Theta_{\tau_t^+-i+1},\qquad 1\le i\le\tau_t^+. \tag{5.19}$$

Recall that $T_t^+:=S_{\tau_t^+}-t$ denotes the overshoot of S over t. Thus, (5.17) yields that on $\{\tau_t^+>K\}$,

$$\mathbf{Q}_x^+\left[\frac{\widetilde{I}_\beta(t,K)}{\widetilde{J}(t,K)}\,\middle|\,\mathscr{G}_{\mathscr{C}_t}\right]\overset{\text{a.s.}}{=}\varphi_{t,K}\left(T_t^+,S_1^{(t)},\dots,S_K^{(t)},\Theta_1^{(t)},\dots,\Theta_K^{(t)}\right), \tag{5.20}$$

where for any $t_0>0,s_1,...,s_K>0$ and the point measures $\theta^{(i)}$, $1\le i\le K$, of form $\theta^{(i)}=\sum_{j=1}^{m^{(i)}}\delta_{x_j^{(i)}}$, we define

$$\mathscr{D}_{i,K}:=\bigcap_{j=i}^{K}\left\{\mathscr{B}(\theta^{(j)})\le e^{-t_0+s_j}\right\},\qquad 1\le i\le K,$$

and

$$\begin{aligned}&\varphi_{t,K}\left(t_0,s_1,\dots,s_K,\theta^{(1)},\dots,\theta^{(K)}\right)\\ := \quad &\mathbf{E}\left[\frac{\exp\Big\{-f(t_0)1_{\mathscr{D}_{1,K}}-\sum_{i=1}^K 1_{\mathscr{D}_{i,K}}\sum_{j=1}^{m^{(i)}}\langle f,\,\overline{\mu}^{(i,j)}_{\mathscr{B},s_i-t_0-x^{(i)}_j}\rangle\Big\}}{\mathscr{R}\left(1+\frac{t_0}{t}\right)e^{\varrho t_0}+\sum_{i=1}^K\sum_{j=1}^{m^{(i)}}\int\mathscr{R}\left(1+\frac{z}{t}\right)e^{\varrho z}\,\overline{\mu}^{(i,j)}_{s_i-t_0-x^{(i)}_j}(\mathrm{d}z)}\right],\end{aligned}$$

and with (under $\mathbf{P}$) $((\overline{\mu}^{(i,j)}_{\mathscr{B},y},\overline{\mu}^{(i,j)}_y),y\in\mathbb{R})_{i,j\ge 1}$ i.i.d. copies of $((\overline{\mu}_{\mathscr{B},y},\overline{\mu}_y),y\in\mathbb{R})$. Then, applying Part (b) of Corollary 2 to (5.20) implies that on $\{\tau_t^+>K\}$,

$$\mathbf{Q}_x^+\left[\frac{\widetilde{I}(t,K)}{\widetilde{J}(t,K)}\,\middle|\, S_k,0\le k\le\tau_t^+,\tau_t^+\right]\overset{\text{a.s.}}{=}\widetilde{\varphi}_{t,K}\left(T_t^+,S_1^{(t)},\dots,S_K^{(t)}\right),\tag{5.21}$$

with

$$\widetilde{\varphi}_{t,K}\left(t_0,s_1,\dots,s_K\right) \;:=\; \int\prod_{i=1}^K \Xi_{s_i-s_{i-1}}(d\theta^{(i)})\,\varphi_{t,K}\left(t_0,s_1,\dots,s_K,\theta^{(1)},\dots,\theta^{(K)}\right),$$

with $s_0:=0$ for notational convenience. Now for any $(t_0,s_1,\dots,s_K)\in\mathbb{R}_+^*\times\mathbb{R}_+^K$ and for any family $(\theta^{(i)})_{1\le i\le K}$ of point processes $\theta^{(i)}:=\sum_{j=1}^{m^{(i)}}\delta_{x^{(i)}_j}$, let us define

$$\begin{aligned}\varphi_{\infty,K}\left(t_0,s_1,\dots,s_K,\theta^{(1)},\dots,\theta^{(K)}\right) &:= \lim_{t\to\infty}\varphi_{t,K}\left(t_0,s_1,\dots,s_K,\theta^{(1)},\dots,\theta^{(K)}\right),\\ \widetilde{\varphi}_{\infty,K}\left(t_0,s_1,\dots,s_K\right) &:= \lim_{t\to\infty}\widetilde{\varphi}_{t,K}\left(t_0,s_1,\dots,s_K\right),\end{aligned}$$

and observe that these limits exist by the dominated convergence theorem, which also yields that

$$\begin{aligned}&\widetilde{\varphi}_{\infty,K}\left(t_0,s_1,\dots,s_K\right)\\ &= \int\prod_{i=1}^K\Xi_{s_i-s_{i-1}}(d\theta^{(i)})\,\varphi_{\infty,K}\left(t_0,s_1,\dots,s_K,\theta^{(1)},\dots,\theta^{(K)}\right)\\ &= \int\prod_{i=1}^K\Xi_{s_i-s_{i-1}}(d\theta^{(i)})\,\mathbf{E}\left[\frac{\exp\Big\{-f(t_0)1_{\mathscr{D}_{1,K}}-\sum_{i=1}^K 1_{\mathscr{D}_{i,K}}\sum_{j=1}^{m^{(i)}}\langle f,\,\overline{\mu}^{(i,j)}_{\mathscr{B},s_i-t_0-x^{(i)}_j}\rangle\Big\}}{e^{\varrho t_0}+\sum_{i=1}^K\sum_{j=1}^{m^{(i)}}\int e^{\varrho z}\,\overline{\mu}^{(i,j)}_{s_i-t_0-x^{(i)}_j}(\mathrm{d}z)}\right].\end{aligned}\tag{5.22}$$

The next step is to replace $\widetilde{\varphi}_{t,K}$ by $\widetilde{\varphi}_{\infty,K}$:

Lemma 11. *Fix $K\ge 1$. Then we have*

$$\lim_{t\to\infty}\mathbf{Q}_x^+\left[\left|\widetilde{\varphi}_{t,K}(T_t^+,S_1^{(t)},\dots,S_K^{(t)})-\widetilde{\varphi}_{\infty,K}(T_t^+,S_1^{(t)},\dots,S_K^{(t)})\right|1_{\{\tau_t^+>K\}}\right]=0.\tag{5.23}$$

Proof of Lemma 11. See Subsection 8.4. □

Note that since $\widetilde{\varphi}_{t,K}(\cdot)$ and $\widetilde{\varphi}_{\infty,K}(\cdot)$ differ only if $\psi'(\varrho_*)=0$, the previous result is not trivial only in the critical case.

Finally thanks to (5.21) and Lemma 11, the double limits (5.14) will be a consequence of the following lemma.

Lemma 12. *We have*

$$\lim_{K\to\infty}\lim_{t\to\infty}\mathbf{Q}_x^+\left[\widetilde{\varphi}_{\infty,K}(T_t^+,S_1^{(t)},\dots,S_K^{(t)})1_{\{\tau_t^+>K\}}\right]=\mathbf{Q}\left[\frac{e^{-\langle f,\,\mu_{\mathscr{B},\infty}\rangle}}{\Re}\right],$$

where

$$\mu_{\mathscr{B},\infty} := \delta_{U\hat{S}_{\hat{\sigma}}} 1_{\mathscr{D}_1} + \sum_{i=1}^{\infty} 1_{\mathscr{D}_i} \sum_{j=1}^{\widetilde{\nu}_i} \overline{\mu}^{(i,j)}_{\mathscr{B},\hat{S}_i - U\hat{S}_{\hat{\sigma}} - \widetilde{X}_j^{(i)}}, \tag{5.24}$$

$$\mathscr{D}_i := \bigcap_{j=i}^{\infty} \left\{ \mathscr{B}(\widetilde{\Theta}_j) \le e^{-U\hat{S}_{\hat{\sigma}} + \hat{S}_j} \right\}, \qquad \forall\, i \ge 1, \tag{5.25}$$

$$\mu_{\infty} := \delta_{U\hat{S}_{\hat{\sigma}}} + \sum_{i=1}^{\infty} \sum_{j=1}^{\widetilde{\nu}_i} \overline{\mu}^{(i,j)}_{\hat{S}_i - U\hat{S}_{\hat{\sigma}} - \widetilde{X}_j^{(i)}}, \tag{5.26}$$

$$\Re := e^{\varrho U\hat{S}_{\hat{\sigma}}} + \sum_{i=1}^{\infty} \sum_{j=1}^{\widetilde{\nu}_i} \int e^{\varrho z}\, \overline{\mu}^{(i,j)}_{\hat{S}_i - U\hat{S}_{\hat{\sigma}} - \widetilde{X}_j^{(i)}}(\mathrm{d}z) = \int e^{\varrho z}\, \mu_{\infty}(\mathrm{d}z), \tag{5.27}$$

and $\varrho = \varrho_*$ *if* $\psi'(\varrho_*) = 0$, $\varrho = \varrho_+$ *if* $\psi'(\varrho_*) < 0$, *and under* $\mathbf{Q}$,

- *the* $((\overline{\mu}^{(i,j)}_{\mathscr{B},y}, \overline{\mu}^{(i,j)}_y), y \in \mathbb{R})_{i,j\ge 1}$ *are i.i.d. with common distribution that of* $((\overline{\mu}_{\mathscr{B},y}, \overline{\mu}_y), y \in \mathbb{R})$ *under* $\mathbf{P}$ *(see (5.16)), and are independent of everything else;*
- *the process* $(\hat{S}_n)_n$ *(as well as the associated random time* $\hat{\sigma}$*) and the random variable* U *are introduced in Lemma 4 (see Subsection 4.1),*
- *conditionally on* $\{\hat{S}_n, n \ge 0\}$, *the random point processes* $\widetilde{\Theta}_i := \sum_{j=1}^{\widetilde{\nu}_i} \delta_{\widetilde{X}_j^{(i)}}$ *for* $i \ge 1$ *are independent and* $\widetilde{\Theta}_i$ *is distributed as* $\Xi_{\hat{S}_{i-1} - \hat{S}_i}$ *(see (3.18) and Corollary 2 for the Palm measures* $(\Xi_z, z \in \mathbb{R})$*).*

Proof of Lemma 12. See Subsection 8.4. □

Proof of Theorem 3 and Proposition 1: Assembling (5.21), Lemma 11 and Lemma 12 imply (5.14), hence (5.5): namely for any $f \in C_K^+(\mathbb{R})$ and when $t \to \infty$, we have

$$\mathbf{E}_x \left[e^{-\langle f, \mu_{\mathscr{B},t}\rangle} 1_{\{H(t)>0\}} \right] \sim \frac{R(x) e^{\varrho x}}{C_R \mathscr{R}(t) e^{\varrho t}} \mathbf{Q} \left[\frac{e^{-\langle f, \mu_{\mathscr{B},\infty}\rangle}}{\Re} \right] = \frac{R(x) e^{\varrho x}}{C_R \mathscr{R}(t) e^{\varrho t}} \mathbf{Q}[\Re^{-1}]\, \mathbf{Q}\left[e^{-\langle f, \widehat{\mu}_{\mathscr{B},\infty}\rangle} \right],$$

by the definition of $\widehat{\mu}_{\mathscr{B},\infty}$. Taking $f = 0$ in the above asymptotical equivalence yields Part (i) and Part (ii) of Theorem 3 while Proposition 1 is a consequence of Part (i) and Part (ii) together with (5.5). Finally, taking $\mathscr{B} \equiv 0$ in Proposition 1 gives Part (iii), which completes the proof of Theorem 3. □

6. Proof of Theorem 2: The critical case

We look at the critical case $\psi'(\varrho_*) = 0$. By linear transformation on V, we may assume that $\varrho_* = 1$ in the whole section without any loss of generality. We investigate the tail distribution of the number of leaves $\#\mathcal{L}[0]$ (see (1.8) for the definition). We will see that when $\mathcal{L}[0]$ is large, the main contribution comes from particles that reached a critical height L. For integrability reasons, we will also restrict to *good* particles whose brothers do not display atypical jumps, and are not too many. We denote by $\mho(v) := \{u \in \mathcal{T} : \overleftarrow{u} = \overleftarrow{v}\,;\, u \ne v\}$ the set of brothers of v ($\overleftarrow{v}$ denotes as before the parent of v in the tree $\mathcal{T}$). For $\lambda > 1, L > 1$ (typically λ is a large constant whereas $L \to \infty$), we say that

$$u \in \mathbb{B}(L,\lambda) \text{ if there exists some } 1 \le j \le |u| : \sum_{v \in \mho(u_j)} (1 + e^{\Delta V(v)}) > \lambda e^{\frac{L - V(u_{j-1})}{2}}$$

and $u \in \mathbb{G}(L,\lambda)$ if such j does not exist. In words, $\mathbb{G}(L,\lambda)$ collects *good* particles in the sense that their large moments are finite, however $\mathbb{B}(L,\lambda)$ is a set of *bad* particles for which only low moments exist. Recall (1.12) that $Z[0,L]=\sum_u 1_{\{\tau_0^-(u)=|u|<\tau_L^+(u)\}}$ counts the number of leaves in the killed branching random walk that did not touch the level L. Let us decompose $Z[0,L]$ as the sums over good particles and bad particles:

$$Z[0,L]=Z_g[0,L]+Z_b[0,L]$$

with

$$Z_g[0,L]:=\sum_{u\in\mathbb{G}(L,\lambda)}1_{\{\tau_0^-(u)=|u|<\tau_L^+(u)\}},\quad Z_b[0,L]:=\sum_{u\in\mathbb{B}(L,\lambda)}1_{\{\tau_0^-(u)=|u|<\tau_L^+(u)\}}.$$

The following lemma shows that we can neglect the number of *bad* particles.

Lemma 13. *For $\delta>0$ small enough, there exist constants $c=c(\delta)>0$ and $c'=c'(\delta)>0$ such that for $x\geq 0$, , $\lambda\geq 1$ and $L\geq 1$*

$$\mathbf{E}_x\left[Z_b[0,L]\right]\leq c\lambda^{-\delta}\frac{(1+x)e^x}{L^2}+ce^xe^{-c'L}. \tag{6.1}$$

For $\delta>0$ small enough, there exists a constant $c=c(\delta)>0$ such that for $x\geq 0$, $\lambda\geq 1$, $L\geq 1$ and $B\geq 0$,

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}Z^{(u)}[0,L+B]\right]\leq c\lambda^{-\delta}\frac{1+B}{L+B}\frac{(1+x)e^x}{L} \tag{6.2}$$

where $Z^{(u)}[0,L+B]$ is the number of leaves in $\mathcal{L}[0]$ that are descendants of u and did not cross level $L+B$.

Proof. We prove separately (6.1) and (6.2). The notation c denotes a constant that can change value from line to line.

Proof of Equation (6.1).
By Proposition 3 (applied to $\mathcal{L}[0]$ and $h(y):=e^y$), we see that

$$\begin{aligned}\mathbf{E}_x\left[Z_b[0,L]\right] &= e^x\mathbf{Q}_x\left[\frac{1}{\sum_{u\in\mathcal{L}[0]}e^{V(u)}}Z_b[0,L]\right]\\ &= e^x\mathbf{Q}_x\left[\sum_{u\in\mathcal{L}[0]}\frac{e^{V(u)}}{\sum_{u\in\mathcal{L}[0]}e^{V(u)}}e^{-V(u)}1_{\{\tau_0^-(u)<\tau_L^+(u)\}}1_{\{u\in\mathbb{B}(L,\lambda)\}}\right].\end{aligned}$$

The weight $\frac{e^{V(u)}}{\sum_{u\in\mathcal{L}[0]}e^{V(u)}}$ is the probability that the vertex u is the spine, see Proposition 3. Therefore,

$$\mathbf{E}_x\left[Z_b[0,L]\right]=e^x\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}1_{\{w_{\tau_0^-}\in\mathbb{B}(L,\lambda)\}}\right],$$

where τ_0^- (resp. τ_L^+) is the hitting time of $(-\infty,0)$ (resp. $(L,+\infty)$) by the random walk S. Let $\delta\in(0,1)$, and, for $k\geq 1$, $a_k:=\sum_{u\in\mho_k}\{1+e^{\Delta V(u)}\}\leq e^{\frac{L-S_{k-1}}{2}}$ (we recall that $\mho_k:=\mho(w_k)$). From the definition of $\mathbb{B}(L,\lambda)$, we observe that

$$1_{\{w_{\tau_0^-}\in\mathbb{B}(L,\lambda)\}}\leq\sum_{k=1}^{\tau_0^-}1_{\{a_k>\lambda e^{(L-S_{k-1})/2}\}}\leq\sum_{k=1}^{\tau_0^-}\min\left(a_k^\delta\lambda^{-\delta}e^{-\delta\frac{L-S_{k-1}}{2}},1\right).$$

It yields that

$$\mathbf{E}_x\left[Z_b[0,L]\right]\leq e^x\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{k=1}^{\tau_0^-}\min\left(a_k^\delta\lambda^{-\delta}e^{-\delta\frac{L-S_{k-1}}{2}},1\right)\right]. \tag{6.3}$$

We first consider the term corresponding to $k = \tau_0^-$, i.e

$$\begin{aligned} &\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\min\left(a^\delta_{\tau_0^-}\lambda^{-\delta}e^{-\delta\frac{L-S_{\tau_0^--1}}{2}},1\right)\right] \\ \leq\ &\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}\min\left(a^\delta_{\tau_0^-}\lambda^{-\delta}e^{-\delta\frac{L-S_{\tau_0^--1}}{2}},1\right)\right]. \end{aligned}$$

We know that $(S_n)_n$ is under $\mathbf{Q}$ a centered random walk. Assumption (1.3) ensures that $\mathbf{Q}\left[e^{-(1+\eta)S_1}\right]$ is finite if η is small enough. In turn, it implies (see (8.15)) that $\mathbf{Q}_x\left[e^{-(1+\eta)S_{\tau_0^-}}\right] \leq c$ for small $\eta > 0$, and any $x \geq 0$. We also have $\mathbf{Q}_x\left[e^{S_{\tau_0^--1}-S_{\tau_0^-}}\right] \leq c$ by (4.11). Then it is not hard to see that $\mathbf{Q}_x[e^{-S_{\tau_0^-}}1_{\mathcal{E}^c}] \leq c'e^{-\eta'\delta L}$, for some constant $\eta' > 0$ where $\mathcal{E} := \{S_{\tau_0^-} \geq -\delta L/8, S_{\tau_0^--1} \leq L/2\}$. Therefore, we can restrict to the event $\mathcal{E}$, on which $e^{-S_{\tau_0^-}} \leq e^{\delta L/8}$, and $e^{-\delta\frac{L-S_{\tau_0^--1}}{2}} \leq e^{-\delta L/4}$. It yields that

$$\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}\min\left(a^\delta_{\tau_0^-}\lambda^{-\delta}e^{-\delta\frac{L-S_{\tau_0^--1}}{2}},1\right)\right] \leq c'e^{-\eta'\delta L} + \lambda^{-\delta}e^{-\delta\frac{L}{8}}\mathbf{Q}_x[a^\delta_{\tau_0^-}].$$

Observe that

$$\mathbf{Q}_x[a^\delta_{\tau_0^-}] = \sum_{j=1}^\infty \mathbf{Q}_x\left[1_{\{j-1<\tau_0^-\}}\mathbf{Q}_{S_{j-1}}[1_{\{S_1<0\}}a_j^\delta]\right],$$

by the Markov property at $j-1$. For $y := S_{j-1} \geq 0$,

$$\mathbf{Q}_y[1_{\{S_1<0\}}a_j^\delta] \leq \mathbf{Q}_y[e^{-\frac{1}{2}S_1}a_j^\delta] = e^{-\frac{1}{2}y}\,\mathbf{Q}[e^{-\frac{1}{2}S_1}a_j^\delta].$$

By Cauchy-Schwarz' inequality and (1.3) we have $\mathbf{Q}[e^{-S_1/2}a_j^\delta] \leq c$ if $\delta > 0$ is chosen small enough. Therefore,

$$\mathbf{Q}_x[a^\delta_{\tau_0^-}] \leq c\sum_{j=1}^\infty \mathbf{Q}_x\left[1_{\{j-1<\tau_0^-\}}e^{-\frac{1}{2}S_{j-1}}\right],$$

which is uniformly bounded by (4.10). Hence, we showed that

$$\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}a^\delta_{\tau_0^-}\lambda^{-\delta}e^{-\delta\frac{L-S_{\tau_0^--1}}{2}}\right] \leq ce^{-\eta''\delta L}. \tag{6.4}$$

We consider now the terms corresponding to $k < \tau_0^-$ in (6.3). By the Markov property at time k, we get

$$\begin{aligned} &\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{k<\tau_0^-<\tau_L^+\}}a_k^\delta\lambda^{-\delta}e^{-\delta\frac{L-S_{k-1}}{2}}\right] \\ \leq\ &\lambda^{-\delta}\mathbf{Q}_x\left[1_{\{k<\tau_0^-<\tau_L^+\}}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{2}}\right]\sup_{y\geq 0}\mathbf{Q}_y[e^{-S_{\tau_0^-}}] \\ =\ &c\lambda^{-\delta}\mathbf{Q}_x\left[1_{\{k<\tau_0^-<\tau_L^+\}}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{2}}\right], \end{aligned}$$

again by (8.15). By the Markov property at time $k-1$, we observe that the last expectation is $\mathbf{Q}_x\left[1_{\{k<\tau_0^-<\tau_L^+\}}e^{-\delta\frac{L-S_{k-1}}{2}}\right]\mathbf{Q}[a_1^\delta]$. Summing over $k\geq 1$, we deduce that

$$\begin{aligned}
&\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{k=1}^{\tau_0^--1}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{2}}\right]\\
\leq\ & c\mathbf{Q}_x\left[1_{\{\tau_0^-<\tau_L^+\}}\sum_{k=1}^{\tau_0^--1}e^{-\delta\frac{L-S_{k-1}}{2}}\right].
\end{aligned}$$

By (4.14), we have $\mathbf{Q}_x\left[1_{\{\tau_0^-<\tau_L^+\}}\sum_{k=0}^{\tau_0^--1}e^{-\delta\frac{L-S_{k-1}}{2}}\right]\leq c\frac{1+x}{L^2}$ for some $c=c(\delta)$. We obtain that

$$\lambda^{-\delta}\mathbf{Q}_x\left[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{k=1}^{\tau_0^--1}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{2}}\right]\leq c'\lambda^{-\delta}\frac{1+x}{L^2}.$$

Then (6.1) follows from Equations (6.3) and (6.4).

Proof of Equation (6.2).
By the branching property, we have

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}Z^{(u)}[0,L+B]\right]=\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}f(V(u))\right],$$

with $f(y):=\mathbf{E}_y[Z[0,L+B]]$. Using the measure $\mathbf{Q}_y$, Proposition 3 implies that $f(y)=e^y\mathbf{Q}_y[e^{-V(w_{\tau_0^-})}1_{\{\tau_0^-<\tau_{L+B}^+\}}]$ which is smaller than $c\frac{1+(L+B-y)_+}{L+B}e^y$ by (4.9). It follows that

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}Z^{(u)}[0,L+B]\right]\leq\frac{c(1+B)}{L+B}\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}e^{V(u)}\right]. \tag{6.5}$$

By Proposition 3 with $\mathscr{C}_L$ and $h(y):=e^y$, we observe that

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}e^{V(u)}\right]\ =\ e^x\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}1_{\{w_{\tau_L^+}\in\mathbb{B}(L,\lambda)\}}\right].$$

As before, we have for $\delta>0$,

$$1_{\{w_{\tau_L^+}\text{ is}\in\mathbb{B}(\mathrm{L},\lambda)\}}\leq\lambda^{-\delta}\sum_{k=1}^{\tau_L^+}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{2}}$$

where $a_k:=\sum_{u\in\mho_k}\{1+e^{\Delta V(u)}\}$. Hence,

$$\begin{aligned}
\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}1_{\{w_{\tau_L^+}\in\mathbb{B}(L,\lambda)\}}\right]\ &\leq\ \lambda^{-\delta}\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}\sum_{k=1}^{\tau_L^+}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{4}}\right]\\
&=\ \lambda^{-\delta}\sum_{k\geq1}\mathbf{Q}_x\left[1_{\{k\leq\tau_L^+<\tau_0^-\}}a_k^\delta e^{-\delta\frac{L-S_{k-1}}{4}}\right].
\end{aligned}$$

Using the Markov property at time $k-1$ for every $k \geq 1$ yields that

$$\begin{aligned}\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}1_{\{w_{\tau_L^+}\in\mathbb{B}(L,\lambda)\}}\right] &\leq c'\lambda^{-\delta}\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}\sum_{k=1}^{\tau_L^+}e^{-\delta\frac{L-S_{k-1}}{2}}\right]\\ &= c'\lambda^{-\delta}\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}\sum_{k=1}^{\tau_L^+}e^{-\delta\frac{L-S_{k-1}}{2}}\right]\end{aligned}$$

with $c' = \mathbf{Q}[a_1^\delta] < \infty$ if $\delta > 0$ is small enough by (1.3). We get by equation (4.13)

$$\mathbf{Q}_x\left[1_{\{\tau_L^+<\tau_0^-\}}1_{\{w_{\tau_L^+}\in\mathbb{B}(L,\lambda)\}}\right] \leq c\lambda^{-\delta}\frac{1+x}{L}.$$

Going back to (6.5), we obtain

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)}1_{\{u\in\mathbb{B}(L,\lambda)\}}Z^{(u)}[0,L+B]\right] \leq c\lambda^{-\delta}\frac{(1+B)}{L+B}\frac{(1+x)e^x}{L},$$

proving (6.2). □

We are going to re-prove the following estimate of Aïdékon [3] but in a more general setting. We recall that $\mathcal{L}[0,L]$ is the set of leaves in $\mathcal{L}[0]$ which did not hit $(L,+\infty)$, and $Z[0,L] := \#\mathcal{L}[0,L]$. We call similarly $\mathcal{L}_g[0,L]$ the leaves in $\mathcal{L}[0,L]$ which are in $\mathbb{G}(L,\lambda)$, hence we have $Z_g[0,L] := \#\mathcal{L}_g[0,L]$ the number of such leaves.

Lemma 14. *Fix $\lambda \geq 1$ and assume that $\psi'(\varrho_*) = 0$ with $\varrho_* = 1$. Under (1.3), there exists some constant $c > 0$ such that for all $L \geq 1$, and $0 \leq x \leq L$,*

$$\mathbf{E}_x\Big[(Z_g[0,L])^2\Big] \leq c\lambda(1+x)e^x\frac{e^L}{L^3}.$$

Proof: Writing $Z_g[0,L] = \sum_{v\in\mathcal{L}[0]}e^{V(v)}1_{\{\tau_L^+(v)>|v|\}}e^{-V(v)}1_{\{v\in\mathbb{G}(L)\}}$, we deduce from Proposition 3 (applied to $\mathcal{L}[0]$ and $h(x) := e^x$) that

$$\mathbf{E}_x\Big[(Z_g[0,L])^2\Big] = e^x\mathbf{Q}_x\Big[Z_g[0,L]e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}1_{\{w_{\tau_0^-}\in\mathbb{G}(L,\lambda)\}}\Big]. \tag{6.6}$$

We decompose $Z_g[0,L]$ along the spine $(w_n,\, n\geq 0)$ as follows:

$$Z_g[0,L] \leq 1 + \sum_{k=1}^{\tau_0^-}\sum_{u\in\mho_k}Z^{(u)}[0,L],$$

where $Z^{(u)}[0,L] := \sum_{v\in\mathcal{T}^{(u)}}1_{\{\tau_0^-(v)=|v|<\tau_L^+(v)\}}$ denotes the number of descendants of u, touching 0 before L [$\mathcal{T}^{(u)}$ means as before the subtree rooted at u]. We have an inequality here since we dropped the condition that the particles must be in $\mathbb{G}(L,\lambda)$. By Proposition 2, under $\mathbf{Q}_x$, conditioned on $\mathscr{G}_\infty := \sigma\{\omega_j, S_j, \mho_j, (V(u), u\in\mho_j), j\geq 0\}$, $(Z^{(u)}[0,L])_{u\in\mho_j, j\leq\tau_0^-}$ are independent and each $Z^{(u)}[0,L]$ is distributed as $(Z[0,L], \mathbf{P}_{V(u)})$. In particular,

$$\mathbf{Q}_x[Z_g[0,L]\,|\,\mathscr{G}_\infty] \leq 1 + \sum_{k=1}^{\tau_0^-}\sum_{u\in\mho_k}\mathbf{E}_{V(u)}[Z[0,L]].$$

Proposition 3 implies as well that for any $z\in\mathbb{R}$,

$$\mathbf{E}_z[Z[0,L]] = e^z\mathbf{Q}_z\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\Big],$$

which is zero if $z > L$ and is 1 if $z < 0$. By (4.9), we get that

$$\mathbf{E}_z[Z[0,L]] \le ce^z \frac{L-z+1}{L} 1_{\{z\in[0,L]\}} + 1_{\{z<0\}}.$$

Hence,

$$\mathbf{Q}_x\Big[Z_g[0,L]\,\Big|\,\mathscr{G}_\infty\Big] \le 1 + \sum_{k=1}^{\tau_0^-}\sum_{u\in\mho_k}\left(ce^{V(u)}\frac{L-V(u)+1}{L}1_{\{V(u)\in[0,L]\}} + 1_{\{V(u)<0\}}\right).$$

For $k < \tau_L^+$, we observe that [recalling $S_{k-1} = V(w_{k-1})$]

$$\begin{aligned} &\sum_{u\in\mho_k} e^{V(u)}\frac{L-V(u)+1}{L}1_{\{V(u)\in[0,L]\}} \\ &= e^{S_{k-1}}\sum_{u\in\mho_k} e^{\Delta V(u)}\frac{L-V(u)+1}{L}1_{\{V(u)\in[0,L]\}} \\ &\le c\frac{L-S_{k-1}+1}{L}e^{S_{k-1}}a_k \end{aligned}$$

with $a_k := \sum_{u\in\mho_k}\{1+e^{\Delta V(u)}\}$. If $w_{\tau_0^-} \in \mathbb{G}(L,\lambda)$, it follows that for any $k < \tau_0^-$,

$$\sum_{u\in\mho_k} e^{V(u)}\frac{L-V(u)+1}{L}1_{\{V(u)\in[0,L]\}} \le \lambda e^L \frac{L-S_{k-1}+1}{L}e^{\frac{S_{k-1}-L}{2}}.$$

Similarly, we observe that $\sum_{u\in\mho_k} 1_{\{V(u)<0\}} \le a_k \le \lambda e^{\frac{L}{2}}$. Therefore, if $w_{\tau_0^-} \in \mathbb{G}(L,\lambda)$, then

$$\mathbf{Q}_x\Big[Z_g[0,L]\,\Big|\,\mathscr{G}_\infty\Big] \le 1 + c\lambda\frac{e^L}{L}\sum_{k=1}^{\tau_0^-}(L-S_{k-1}+1)e^{\frac{S_{k-1}-L}{2}}.$$

Equation (6.6) implies that

$$\begin{aligned} \mathbf{E}_x\Big[(Z_g[0,L])^2\Big] \;\le\; & e^x\mathbf{Q}_x\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\Big] + \\ & c\lambda\frac{e^{x+L}}{L}\mathbf{Q}_x\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{k=1}^{\tau_0^-}(L-S_{k-1}+1)e^{\frac{S_{k-1}-L}{2}}\Big]. \end{aligned}$$

The right-hand side is smaller than $e^x(1+c'(1+x)\lambda\frac{e^{x+L}}{L^3})$ by (4.14). It completes the proof of the lemma. □

We look now at the progeny of a particle which went far to the right. Recall the derivative martingale

$$\partial W_n := -\sum_{|u|=n} V(u)e^{V(u)}, \qquad n\ge 0.$$

According to Theorems 5.1 and 5.2 in Biggins and Kyprianou [9], under $\mathbf{P}$, ∂W_n converges almost surely to ∂W_∞ which has infinite mean and is almost surely positive on $\{\mathcal{T}=\infty\}$.

Lemma 15. *Assuming $\psi'(\varrho_*) = 0$ with $\varrho_* = 1$. Under (1.3), as $t\to\infty$, the law of $\#\mathcal{L}[0]$ under $\mathbf{P}_t$, normalized by e^t/t converges in distribution to $c^*\,\partial W_\infty$, with*

$$c^* := \frac{\mathbf{Q}\big[e^{-S_{\tau_0^-}}-1\big]}{-\mathbf{Q}\big[S_{\tau_0^-}\big]}. \tag{6.7}$$

Proof: By linear translation, it is equivalent to prove that under $\mathbf{P}_0$, $\#\mathcal{L}[-t]$ normalized by e^t/t converges in law to $c^*\,\partial W_\infty$. If we define

$$\partial W_{\mathcal{L}[-t]} := -\sum_{u\in\mathcal{L}[-t]} V(u)e^{V(u)},$$

then $\partial W_{\mathcal{L}[-t]}$ converges almost surely to ∂W_∞ (cf. Biggins and Kyprianou [9], Theorem 5.3). We write

$$\partial W_{\mathcal{L}[-t]} = te^{-t}\,(\sum_{u\in\mathcal{L}[-t]} e^{V(u)+t} + \frac{1}{t}\eta_t), \tag{6.8}$$

with $\eta_t = -\sum_{u\in\mathcal{L}[-t]}(V(u)+t)e^{V(u)+t}$. At this stage, we may apply a result of Nerman [28] for the asymptotic behavior of $\frac{1}{\#\mathcal{L}[-t]}\sum_{u\in\mathcal{L}[-t]} e^{V(u)+t}$: Let $\xi := \sum_{u\in\mathcal{L}[0]}\delta_{\{-V(u)\}}$ be the point process formed by the (non-killed) branching walk V stopped at the line $\mathcal{L}[0]$. Generate a branching random walk $(V_\xi(u), u\in\mathcal{T}_\xi)$ from the point process ξ, where $V_\xi, \mathcal{T}_\xi$ are related to ξ in the same way as $V, \mathcal{T}$ are to $\mathscr{L}$. Define $\mathcal{L}_\xi[a] := \{u\in\mathcal{T}_\xi\,:\,|u| = \tau_a^+(u)\}$ for all $a>0$. Clearly $\mathcal{L}_\xi[t] = \mathcal{L}[-t]$ and

$$\frac{\sum_{u\in\mathcal{L}[-t]} e^{V(u)+t}}{\#\mathcal{L}[-t]} = \frac{\sum_{u\in\mathcal{T}_\xi}\psi_u(t-\sigma_u)}{\sum_{u\in\mathcal{T}_\xi}\phi_u(t-\sigma_u)},$$

where for any $u\in\mathcal{T}_\xi$, $\sigma_u := -V_\xi(u)$ and

$$\psi_u(x) := 1_{\{x\ge 0\}}\sum_{\overleftarrow{v}=u} e^{x-(\sigma_v-\sigma_u)}1_{\{\sigma_v-\sigma_u>x\}}, \quad \phi_u(x) := 1_{\{x\ge 0\}}\sum_{\overleftarrow{v}=u} 1_{\{\sigma_v-\sigma_u>x\}}.$$

Applying Theorem 6.3 in Nerman [28] (with $\alpha=1$ and $\lambda_u=\infty$ there) gives that conditioned on $\{\mathcal{T}=\infty\}$, almost surely, when t tends to infinity

$$\frac{\sum_{u\in\mathcal{T}_\xi}\psi_u(t-\sigma_u)}{\sum_{u\in\mathcal{T}_\xi}\phi_u(t-\sigma_u)} \to \frac{\mathbf{E}\Big[\sum_{|v|=1,v\in\mathcal{T}_\xi} e^{-\sigma_v}\sigma_v\Big]}{\mathbf{E}\Big[\sum_{|v|=1,v\in\mathcal{T}_\xi}(1-e^{-\sigma_v})\Big]}.$$

Observe that $\mathbf{E}\Big[\sum_{|v|=1,v\in\mathcal{T}_\xi} e^{-\sigma_v}\sigma_v\Big] = -\mathbf{E}\Big[\sum_{u\in\mathcal{L}[0]} e^{V(u)}V(u)\Big] = -\mathbf{Q}\Big[S_{\tau_0^-}\Big]$, and similarly, $\mathbf{E}\Big[\sum_{|v|=1,v\in\mathcal{T}_\xi}(1-e^{-\sigma_v})\Big] = \mathbf{Q}\Big[e^{-S_{\tau_0^-}}\Big]-1$. Therefore conditioned on $\{\mathcal{T}=\infty\}$, almost surely

$$\frac{\sum_{u\in\mathcal{L}[-t]} e^{V(u)+t}}{\#\mathcal{L}[-t]} \to \frac{\mathbf{Q}\Big[S_{\tau_0^-}\Big]}{1-\mathbf{Q}\Big[e^{-S_{\tau_0^-}}\Big]}, \qquad t\to\infty.$$

On the other hand, following the same strategy, we get that conditioned on $\{\mathcal{T}=\infty\}$, we have almost surely

$$\frac{\eta_t}{\#\mathcal{L}[-t]} \to \frac{\mathbf{Q}\Big[(S_{\tau_0^-})^2/2\Big]}{\mathbf{Q}\Big[e^{-S_{\tau_0^-}}\Big]-1}, \qquad t\to\infty.$$

Dividing both sides of (6.8) by $\#\mathcal{L}[-t]$, and using the fact that $\partial W_{\mathcal{L}[-t]}$ goes to ∂W_∞, we deduce the lemma. $\square$

We also need the following simple technical lemma whose proof is postponed in Section 8:

Lemma 16. *On some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $\sum_{i=1}^{\xi} \delta_{\{Y_i\}}$ be a point process on $\mathbb{R}_+$. Let $(\Gamma_i, i \geq 1)$ be a sequence of i.i.d. random variables on $\mathbb{R}_+$, independent of $\sigma\{\xi, Y_i, 1 \leq i \leq \xi\}$. Assume that for some $p > 0$ and $a > 0$,*

$$\mathbb{P}\Big(\Gamma_1 > t\Big) = (a + o(1))t^{-p}, \qquad t \to \infty.$$

(i) If $p = 1$ and if there exists some $\delta > 0$ such that $\mathbb{E}\left[\sum_{i=1}^{\xi} Y_i^{1+\delta}\right] < \infty$, then

$$\lim_{t\to\infty} t\,\mathbb{P}\Big(\sum_{i=1}^{\xi} Y_i\,\Gamma_i > t\Big) = a\,\mathbb{E}\Big[\sum_{i=1}^{\xi} Y_i\Big].$$

(ii) If $p > 1$ and if there exists some $\delta > 0$ such that $\mathbb{E}\Big[\sum_{i=1}^{\xi}(1 + Y_i)\Big]^{p+\delta} < \infty$, then

$$\lim_{t\to\infty} t^p\,\mathbb{P}\Big(\sum_{i=1}^{\xi} Y_i\,\Gamma_i > t\Big) = a\,\mathbb{E}\Big[\sum_{i=1}^{\xi} Y_i^p\Big].$$

In the critical case, the branching random walk goes to $-\infty$. In particular, almost surely, $H(L) = 0$ if L is large enough. Fix $\lambda \geq 1$. For $L \geq 1$, let $\mu_{\lambda,L} := \sum_{u\in\mathscr{H}(L)} \delta_{\{V(u)-L\}} 1_{\{u\in\mathbb{G}(L,\lambda)\}}$. Then Proposition 1 implies that $\mu_{\lambda,L}$ under $\mathbf{P}(\cdot \,|\, H(L) > 0)$ converges when $L \to \infty$ to $\hat{\mu}_{\mathscr{B},\infty}$ defined in Proposition 1 with $\mathscr{B}(u) := \lambda^{-2}(\sum_{v\in\mho(u)}\{1 + e^{\Delta V(v)}\})^2$. We will write $\hat{\mu}_{\lambda,\infty} := \sum_{i=1}^{\hat{\zeta}_\lambda} \delta_{x_i}$ instead of $\hat{\mu}_{\mathscr{B},\infty}$. Since the measures $\hat{\mu}_{\lambda,\infty}$ are increasing in λ, we can assume that the labelling $(x_i)_i$ does not depend on $\lambda \geq 1$. We write similarly $\mu_{\lambda,\infty} := \sum_{i=1}^{\zeta_\lambda} \delta_{x_i}$ for the measure $\mu_{\mathscr{B},\infty}$ given by Proposition 1, and we know that the Radon-Nykodym derivative of $\hat{\mu}_{\lambda,\infty}$ with respect to $\mu_{\lambda,\infty}$ is $\frac{\Re^{-1}}{\mathbf{Q}[\Re^{-1}]}$. Notice that if $\hat{\zeta}_\lambda = 0$, then $\hat{\mu}_{\lambda,\infty}$ is the measure zero.

Lemma 17. *Assuming $\psi'(\varrho_*) = 0$ with $\varrho_* = 1$ and (1.3). Fix $\lambda \geq 1$ and let $\hat{\mu}_{\lambda,\infty}$ and $\mu_{\lambda,\infty}$ be as above. Under $\mathbf{Q}$, let $(\partial W_\infty^{(i)}, i \geq 1)$ be a sequence of i.i.d. random variables, independent of $\hat{\mu}_{\lambda,\infty}$ and of common law that of ∂W_∞ under $\mathbf{P}$. For any $\lambda \geq 1$, we have*

$$\lim_{t\to\infty} t\mathbf{Q}\left(\sum_{i=1}^{\hat{\zeta}_\lambda} e^{x_i}\partial W_\infty^{(i)} > t\right) = \frac{\mathbf{Q}[\Re^{-1}\sum_{i=1}^{\zeta_\lambda} e^{x_i}]}{\mathbf{Q}[\Re^{-1}]}. \tag{6.9}$$

Moreover, for any $c > 0$,

$$\lim_{\lambda\to\infty} \lambda^2\mathbf{Q}\left(\sum_{i=1}^{\hat{\zeta}_\lambda} e^{x_i}\partial W_\infty^{(i)} > c\,\lambda^2\right) = \frac{1}{c\mathbf{Q}[\Re^{-1}]}. \tag{6.10}$$

Proof of Lemma 17: For any $i \geq 1$, by Theorem 2.5 (i) of Liu [23],

$$\mathbf{Q}\big(\partial W_\infty^{(i)} > t\big) = \mathbf{P}\big(\partial W_\infty > t\big) \sim \frac{1}{t}, \qquad t \to \infty. \tag{6.11}$$

In order to prove (6.9), we shall apply Lemma 16 (i) and it is enough to show that there exists some $\delta > 0$ such that $\mathbf{Q}\left[\sum_{i=1}^{\hat{\zeta}_\lambda}(1 + e^{x_i})^{1+\delta}\right] < \infty$. Remark that $\hat{\mu}_{\lambda,\infty}$ has the support contained in $\mathbb{R}_+$, hence for $\delta > 0$, $\mathbf{Q}\left[\sum_{i=1}^{\hat{\zeta}_\lambda}(1 + e^{x_i})^{1+\delta}\right] \leq 2^{1+\delta}\mathbf{Q}\left[\sum_{i=1}^{\hat{\zeta}_\lambda} e^{(1+\delta)x_i}\right]$. We are going to prove a stronger statement: for $\widehat{\mu}_\infty$ the point process defined in Theorem (3) (iii), we have

$$\mathbf{Q}\left[\int \widehat{\mu}_\infty(dx)e^{(1+\delta)x}\right] < \infty. \tag{6.12}$$

The statement (6.12) implies the corresponding integrability for $\hat{\mu}_{\lambda,\infty}$ since $\hat{\mu}_{\lambda,\infty}$ is stochastically dominated by $\widehat{\mu}_\infty$. To prove (6.12), we consider $\chi(L) := \sum_{u\in\mathscr{H}(L)} e^{(1+\delta)(V(u)-L)}$ and prove first that, under $\mathbf{P}\big(\cdot\,|\mathscr{H}(L)\neq\emptyset\big)$, $\chi(L)$ converges in law to $\int \widehat{\mu}_\infty(dx)e^{(1+\delta)x}$. In order to apply the convergence in law of Theorem 3 (iii), we need some tightness result. We claim that

$$\sup_{L\geq 1}\mathbf{P}_x\big(\exists i\in[|1,H(L)|]\,:\,V(u^{(i)})-L>K\,|\,H(L)>0\big)=o_K(1), \tag{6.13}$$

where we order the set of particles in $\mathscr{H}(L)$ (eventually empty) in an arbitrary way: $\mathscr{H}(L)=\{u^{(i)},1\leq i\leq H(L)\}$. Markov inequality yields that the probability term in (6.13) is smaller than

$$e^{-K}e^{-L}\mathbf{E}_x[\sum_{u\in\mathscr{H}(L)}e^{V(u)}]\mathbf{P}_x(H(L)>0)^{-1}\leq ce^{-K}L\mathbf{E}_x[\sum_{u\in\mathscr{H}(L)}e^{V(u)}], \tag{6.14}$$

where the inequality is a consequence of Theorem 3 (i). To prove the claimed tightness result it is sufficient to show that there exists some constant $c>0$ such that for any $x\geq 0$ and $L\geq\max(1,x)$ we have

$$\mathbf{E}_x[\sum_{u\in\mathscr{H}(L)}e^{V(u)}]\leq c(1+x)\frac{e^x}{L}. \tag{6.15}$$

To see it, we change of measure from $\mathbf{P}_x$ to $\mathbf{Q}_x$ by Proposition 3 (applied to $\mathscr{C}_L$ and $h(x):=e^x$) and find that

$$\mathbf{E}_x[\sum_{u\in\mathscr{H}(L)}e^{V(u)}]=e^x\mathbf{Q}_x\left(\tau_L^+<\tau_0^-\right).$$

Then (8.17) implies (6.15). Assembling (6.14) and (6.15) yields (6.13) and allows us to apply Theorem 3 (iii) to obtain the convergence in distribution, under $\mathbf{P}\big(\cdot\,|\mathscr{H}(L)\neq\emptyset\big)$, of $\chi(L)$ toward $\int\widehat{\mu}_\infty(dx)e^{(1+\delta)x}$.

Then (6.12) will hold once we have checked that $\mathbf{E}(\chi(L)\,\big|\,\mathscr{H}(L)\neq\emptyset)$ is bounded on L. By Theorem 3 (i) with $\varrho_*=1$, it is enough to show that

$$\mathbf{E}\Big[\chi(L)\Big]\leq c\,\frac{e^{-L}}{L}. \tag{6.16}$$

But by the change of measure,

$$\mathbf{E}\Big[\chi(L)\Big]=e^{-L}\mathbf{Q}\left[e^{\delta(S_{\tau_L^+}-L)},\tau_L^+<\tau_0^-\right].$$

The above expectation $\mathbf{Q}[\cdot]$ is less than $\frac{c}{L}$ by applying (4.9) to the random walk $(\delta(L-S_j))_{j\geq 0}$ (the integrability is guaranteed if δ is sufficiently small). This proves (6.16) and a fortiori (6.9).

Remark that by (6.12) and Lemma 16 (i), if we write $\widehat{\mu}_\infty=\sum_{i=1}^{\widehat{\zeta}}\delta_{\{x_i\}}$, then

$$\mathbf{Q}\left(\sum_{i=1}^{\hat{\zeta}}e^{x_i}\partial W_\infty^{(i)}>t\right)\sim\frac{\mathbf{Q}[\Re^{-1}\sum_{i=1}^{\zeta}e^{x_i}]}{\mathbf{Q}[\Re^{-1}]}\frac{1}{t}=\frac{1}{\mathbf{Q}[\Re^{-1}]}\frac{1}{t},\qquad t\to\infty$$

since $\Re=\sum_{i=1}^{\zeta}e^{x_i}$ by definition, see (5.27). We have already observed that $\widehat{\mu}_{\lambda,\infty}$ is stochastically non-decreasing in λ and is dominated by $\widehat{\mu}_\infty$ [$\widehat{\mu}_\infty$ corresponds to $\widehat{\mu}_{\lambda,\infty}$ with $\lambda=\infty$]. Then $\limsup_{\lambda\to\infty}\lambda^2\mathbf{Q}\left(\sum_{i=1}^{\hat{\zeta}_\lambda}e^{x_i}\partial W_\infty^{(i)}>c\,\lambda^2\right)\leq\limsup_{\lambda\to\infty}\lambda^2\mathbf{Q}\left(\sum_{j=1}^{\hat{\zeta}}e^{y_i}\partial W_\infty^{(i)}>c\,\lambda^2\right)$ which is $\frac{1}{c\mathbf{Q}[\Re^{-1}]}$, yielding the upper bound in (6.10).

For the lower bound, let $\lambda_0 > 1$ and by the monotonicity in $\hat{\mu}_\lambda$,

$$\begin{aligned}\liminf_{\lambda\to\infty} \lambda^2 \mathbf{Q}\left(\sum_{i=1}^{\hat{\zeta}_\lambda} e^{x_i}\partial W_\infty^{(i)} > c\,\lambda^2\right) &\geq \liminf_{\lambda\to\infty} \lambda^2 \mathbf{Q}\left(\sum_{i=1}^{\hat{\zeta}_{\lambda_0}} e^{x_i}\partial W_\infty^{(i)} > c\,\lambda^2\right) \\ &= \frac{\mathbf{Q}[\Re^{-1}\sum_{i=1}^{\zeta_{\lambda_0}} e^{x_i}]}{c\mathbf{Q}[\Re^{-1}]},\end{aligned}$$

by applying (6.9) to $\widehat{\mu}_{\lambda_0,\infty}$. Letting $\lambda_0 \to \infty$ and noting that $\sum_{i=1}^{\zeta_{\lambda_0}} e^{x_i} = \int e^x \mu_{\lambda_0,\infty}(dx) \to \Re$, this gives the lower bound of (6.10). □

Recall that we obtained the existence of some constant $c > 0$ such that for any $x \geq 0$, $L \geq 0$ with $L \geq \max(1, x)$ we have

$$\mathbf{E}_x[\sum_{u\in\mathscr{H}(L)} e^{V(u)}] \leq c(1+x)\frac{e^x}{L}. \tag{6.17}$$

We now have all the ingredients to prove Theorem 2 in the critical case.

Proof of Theorem 2 (i), (critical case):

Lower bound of Theorem 2 (i): Recall that we have assumed $\varrho_* = 1$ by linear transformation. Fix a constant $A > 0$. Consider $n \to \infty$ and let $L_{n,A} := \log n + \log\log n - A$. We recall from (1.10) that $H(L_{n,A}) = \#\mathscr{H}(L_{n,A})$ is the number of particles that hit level $L_{n,A}$ before touching 0. We call $H_g(L_{n,A}) := \#\mathscr{H}_g(L_{n,A})$ the number of particles in $\mathscr{H}(L_{n,A})$ which are in $\mathbb{G}(L_n, \lambda)$ with $\lambda := e^{\frac{A}{2}}$. We order the set of particles in $\mathscr{H}_g(L_{n,A})$ (eventually empty) in an arbitrary way: $\mathscr{H}_g(L_{n,A}) = \{u^{(i)}, 1 \leq i \leq H_g(L_{n,A})\}$. Denote by $\#\mathcal{L}^{(i)}[0]$ the number of descendants of the i-th particle $u^{(i)}$ which are absorbed at 0. Then,

$$\begin{aligned}\mathbf{P}_x\left(\#\mathcal{L}[0] > n\right) &\geq \mathbf{P}_x\left(\sum_{i=1}^{H_g(L_{n,A})} \#\mathcal{L}^{(i)}[0] > n\right) \\ &= \mathbf{P}_x\left(H(L_{n,A}) > 0\right) \mathbf{P}_x\left(\sum_{i=1}^{H_g(L_{n,A})} \#\mathcal{L}^{(i)}[0] > n \,\Big|\, H(L_{n,A}) > 0\right).\end{aligned} \tag{6.18}$$

By Theorem 3 (i), $\mathbf{P}_x\big(H(L_{n,A}) > 0\big) \sim \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x)\, e^x\, \frac{e^{-L_{n,A}}}{L_{n,A}}$ as $n \to \infty$. On the other hand, conditioned on $\mathscr{H}_g(L_{n,A})$ and on $\{V(u^{(i)}), 1 \leq i \leq H_g(L_{n,A})\}$, $(\#\mathcal{L}^{(i)}[0])_{1\leq i\leq H_g(L_{n,A})}$ are independent, and each $\#\mathcal{L}^{(i)}[0]$ is distributed as $\#\mathcal{L}[0]$ under $\mathbf{P}_{V(u^{(i)})}$.

By Lemma 15, if we denote by $B^{(i)} := \#\mathcal{L}^{(i)}[0]e^{-V(u^{(i)})}V(u^{(i)})$, then conditioned on $\mathscr{H}_g(L_{n,A})$ and on $\{V(u^{(i)}), 1 \leq i \leq H_g(L_{n,A})\}$, for each i, $B^{(i)}$ converges in law to $c^*\partial W_\infty^{(i)}$ as $n \to \infty$, where $\partial W_\infty^{(i)}, i \geq 1$, is a sequence of i.i.d. random variables of common law that of $(\partial W_\infty, \mathbf{P})$, and independent of $\mu_{L_{n,A}}$. We may assume by Skorohod's representation theorem that for each i, $B^{(i)}$ converges almost surely to $c^*\partial W_\infty^{(i)}$.

Let $\varepsilon \in (0,1)$. First, we want to show that we can restrict to the event $E(L_{n,A}) := \{B^{(i)} > (1-\varepsilon)\, c^*\, \partial W_\infty^{(i)}\,;\, \forall i : 1 \leq i \leq H_g(L_{n,A})\}$. We have

$$\begin{aligned}&\mathbf{P}_x(E(L_{n,A})^c \,|\, H(L_{n,A}) > 0) \\ \leq\ &\mathbf{E}_x[H_g(L_{n,A}) \,\big|\, H(L_{n,A}) > 0] \sup_{z\geq L_{n,A}} \mathbf{P}_z(ze^{-z}\#\mathcal{L}[0] < (1-\varepsilon)c^*\partial W_\infty) \\ =:\ &\mathbf{E}_x[H_g(L_{n,A}) \,\big|\, H(L_{n,A}) > 0]\, \eta_{L_{n,A}}.\end{aligned}$$

The term $\eta_{L_{n,A}}$ goes to zero as $n \to \infty$ by Lemma 15. By (6.17) and Theorem 3 (i), we have $\mathbf{E}_x[H_g(L_{n,A}) \big| H(L_{n,A}) > 0] \leq e^{-L_{n,A}} \mathbf{E}_x[\sum_{u \in \mathscr{H}(L_{n,A})} e^{V(u)} \big| H(L_{n,A}) > 0] \leq c$ for some positive constant $c = c(x)$ which depends on x. Hence, $\mathbf{P}_x(E(L_{n,A})^c | H(L_{n,A}) > 0) = o_{L_{n,A}}(1)$. We have

$$\begin{aligned} & \mathbf{P}_x\Big(\sum_{i=1}^{H_g(L_{n,A})} \#\mathcal{L}^{(i)}[0] > n \,\big|\, H(L_{n,A}) > 0 \Big) \\ = \ & \mathbf{P}_x\Big(\sum_{i=1}^{H_g(L_{n,A})} \frac{e^{V(u^{(i)})}}{V(u^{(i)})} B^{(i)} > n \,\big|\, H(L_{n,A}) > 0 \Big) \\ \geq \ & \mathbf{P}_x\Big(\sum_{i=1}^{H_g(L_{n,A})} \frac{e^{V(u^{(i)})}}{V(u^{(i)})} B^{(i)} > n, \, E(L_{n,A}) \,\big|\, H(L_{n,A}) > 0 \Big). \end{aligned} \tag{6.19}$$

Observe that

$$\begin{aligned} & \mathbf{P}_x \left(\sum_{i=1}^{H_g(L_{n,A})} \frac{e^{V(u^{(i)})}}{V(u^{(i)})} B^{(i)} > n \, , \, E(L_{n,A}) \,\middle|\, H(L_{n,A}) > 0 \right) \\ \geq \ & \mathbf{P}_x \left(\sum_{i=1}^{H_g(L_{n,A})} \frac{e^{V(u^{(i)})}}{V(u^{(i)})} \partial W_\infty^{(i)} > \frac{n}{c^*(1-\varepsilon)} \, , \, E(L_{n,A}) \,\middle|\, H(L_{n,A}) > 0 \right) \\ \geq \ & \mathbf{P}_x \left(\sum_{i=1}^{H_g(L_{n,A})} \frac{e^{V(u^{(i)})}}{V(u^{(i)})} \partial W_\infty^{(i)} > \frac{n}{c^*(1-\varepsilon)} \,\middle|\, H(L_{n,A}) > 0 \right) + o_{L_{n,A}}(1), \end{aligned} \tag{6.20}$$

where $o_{L_{n,A}}(1) \to 0$ as $L_{n,A} \to \infty$. In order to apply the convergence in law of Proposition 1, we need some tightness result. Recalling (6.13), it is sufficient to show that

$$\sup_{L \geq 1} \mathbf{P}_x \big(\exists i \in [|1, H(L)|] \, : \, \partial W_\infty^{(i)} > K \,|\, H(L) > 0 \big) = o_K(1).$$

Since the $\partial W_\infty^{(i)}$'s are i.i.d. copies of ∂W_∞ and independent of $\mu_{L_{n,A}}$, Markov inequality yields that the probability term in the previous equation is smaller than

$$K^{-1/2} \mathbf{E}_x[H(L) \,|\, H(L) > 0] \mathbf{E}[\sqrt{\partial W_\infty}] = O(K^{-1/2}),$$

by using (6.17), Theorem 3 (i) and (6.11). This yields the claimed tightness and allows us to apply Proposition 1 to get

$$\begin{aligned} & \lim_{n \to \infty} \mathbf{P}_x \left(\sum_{i=1}^{H_g(L_{n,A})} \frac{e^{V(u^{(i)})}}{V(u^{(i)})} \partial W_\infty^{(i)} > \frac{n}{c^*(1-\varepsilon)} \,\middle|\, H(L_{n,A}) > 0 \right) \\ = \ & \mathbf{Q} \Big(\sum_{i=1}^{\hat{\zeta}_\lambda} e^{x_i} \partial W_\infty^{(i)} > \frac{e^A}{c^*(1-\varepsilon)} \Big), \end{aligned} \tag{6.21}$$

where $\hat{\mu}_{\lambda,\infty} := \sum_{i=1}^{\hat{\zeta}_\lambda} \delta_{x_i}$ is the point process defined before Lemma 17, and we recall that $\lambda := e^{\frac{A}{2}}$. By (6.18), (6.19), (6.20), (6.21) and the definition of $L_{n,A}$, we deduce that for any $A > 0$,

$$\liminf_{n \to \infty} n (\log n)^2 \, \mathbf{P}_x \left(\mathcal{L}[0] > n \right) \geq \frac{\mathbf{Q}\left[\Re^{-1} \right]}{C_R} \, R(x) \, e^x \, e^A \, \mathbf{Q} \left(\sum_{i=1}^{\hat{\zeta}_\lambda} e^{x_i} \partial W_\infty^{(i)} > \frac{\lambda^2}{c^*(1-\varepsilon)} \right).$$

We let $\varepsilon \to 0$ to get

$$\liminf_{n\to\infty} n(\log n)^2\, \mathbf{P}_x\Big(\#\mathcal{L}[0] > n\Big) \geq R(x)e^x C(A),$$

with $C(A) := \frac{\mathbf{Q}[\Re^{-1}]}{C_R}\, e^A\, \mathbf{Q}\Big(\sum_{i=1}^{\hat\zeta_\lambda} e^{x_i} c^* \partial W_\infty^{(i)} > \lambda^2\Big)$.

By Lemma 17, we have $C(A) \to \frac{c^*}{C_R}$ as $A \to \infty$, which leads to

$$\liminf_{n\to\infty} n(\log n)^2\, \mathbf{P}_x\Big(\#\mathcal{L}[0] > n\Big) \geq R(x)e^x \frac{c^*}{C_R}.$$

We notice that we showed in fact that, for any $A > 0$,

$$\liminf_{n\to\infty} n(\log n)^2\, \mathbf{P}_x\left(\sum_{i=1}^{H_g(L_{n,A})} \#\mathcal{L}^{(i)}[0] > n\right) \geq R(x)e^x C(A).$$

Repeating the same argument with this time $E'(L_{n,A}) := \{B^{(i)} < (1+\varepsilon)\partial W_\infty^{(i)}\,;\, \forall i : 1 \leq i \leq H_g(L_{n,A})\}$ yields that $C(A)$ is also a limsup. Therefore,

$$\lim_{n\to\infty} n(\log n)^2\, \mathbf{P}_x\Big(\sum_{i=1}^{H_g(L_{n,A})} \#\mathcal{L}^{(i)}[0] > n\Big) = R(x)e^x C(A), \tag{6.22}$$

with $C(A) \to \frac{c^*}{C_R}$ as $A \to \infty$.

Upper bound of Theorem 2 (i). Let $\eta > 0$ and $\varepsilon > 0$. We take again $L_{n,A} := \log n + \log\log n - A$ and $\lambda := e^{\frac{A}{2}}$. Markov inequality with (6.1) implies that if A is taken large enough,

$$\limsup_{n\to\infty} n(\log n)^2 \mathbf{P}_x\left(Z_b[0, L_{n,A}] > \eta n\right) \leq \varepsilon.$$

By Theorem 3 (i), we can choose $B > 0$ large enough such that

$$\limsup_n\; n(\log n)^2 \mathbf{P}_x(H(L_n + B) > 0) \leq \varepsilon. \tag{6.23}$$

On the other hand, by (6.2) and Markov inequality, we obtain that for A large enough,

$$\begin{aligned} &\limsup_n\; n(\log n)^2 \mathbf{P}_x\left(\sum_{u\in\mathscr{H}(L_{n,A})} 1_{\{u\in\mathbb{B}(L_{n,A},\lambda)\}} \#\mathcal{L}^{(u)}[0] > \eta n,\; H(L_n+B) = 0\right) \\ \leq\;& \limsup_n\; n(\log n)^2 \frac{1}{\eta n} \mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L_{n,A})} 1_{\{u\in\mathbb{B}(L_{n,A},\lambda)\}} Z^{(u)}[0, L_n + B]\right] \leq \varepsilon \end{aligned} \tag{6.24}$$

where the notation $Z^{(u)}[,]$ was introduced in Lemma 13. Finally, it yields that

$$\limsup_n\; n(\log n)^2 \mathbf{P}_x\left(\sum_{u\in\mathscr{H}(L_{n,A})} 1_{\{u\in\mathbb{B}(L_{n,A},\lambda)\}} \#\mathcal{L}^{(u)}[0] > \eta n\right) \leq 2\varepsilon. \tag{6.25}$$

We now show that the "good particles" which never touch $L_{n,A}$ are negligible when A is large. We recall that $Z_g(0, L_{n,A})$ is the number of particles in $\mathbb{G}(L_n, \lambda)$ that touch 0 before $L_{n,A}$. By Lemma 14,

$$\mathbf{E}_x\Big[Z_g(0, L_{n,A})^2\Big] \leq c(1+x)\, e^x\, \lambda \frac{e^{L_{n,A}}}{L_{n,A}^3}\,.$$

Therefore, by the choice of $L_{n,A}$ and λ we have that for any fixed $\eta > 0$,

$$\limsup_{n\to\infty} n(\log n)^2 \mathbf{P}_x\Big(Z_g[0, L_{n,A}] > \eta n\Big) \le \frac{c(1+x)e^x e^{-\frac{A}{2}}}{\eta^2},$$

which is less than ε if A is large enough. By triangular inequality, for any $0 < \eta < 1/3$ and any $\varepsilon > 0$, we deduce that if A is large enough

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \le \mathbf{P}_x\left(\sum_{i=1}^{H_g(L_{n,A})} \#\mathcal{L}^{(i)}[0] > (1-3\eta)n\right) + 4\varepsilon.$$

From this and (6.22), by letting $A \to \infty$ and $\eta \to 0$, we deduce the upper bound

$$\limsup_{n\to\infty} n(\log n)^2\, \mathbf{P}_x\Big(\#\mathcal{L}[0] > n\Big) \le R(x)e^x \frac{c^*}{C_R}.$$

Thus we have

$$\lim_{n\to\infty} n(\log n)^2\, \mathbf{P}_x\Big(\#\mathcal{L}[0] > n\Big) = R(x)e^x c'_{crit},$$

with $c'_{crit} = \frac{c^*}{C_R}$. Finally, we recall that C_R is the limit of $R(x)/x$ as $x \to \infty$, $R(x)$ being the renewal function for the descending ladder heights. The renewal theorem implies that $C_R = \mathbf{Q}[-S_{\tau_0^-}]^{-1}$. Hence, from the value of c^* in (6.7), we end up with $c'_{crit} = \mathbf{Q}[e^{-S_{\tau_0^-}} - 1]$ indeed. □

7. Proof of Theorem 2: The subcritical case

We treat here the subcritical case $\psi'(\varrho_*) < 0$. Define a new probability measure $\mathbf{Q}^{(\varrho_-)}$ by (3.3) with $h(u) = \mathrm{e}^{\varrho_- V(u)}$ for all $u \in \mathcal{T}$. Then for any $x \in \mathbb{R}$,

$$\frac{d\mathbf{Q}_x^{(\varrho_-)}}{d\mathbf{P}_x}\Big|_{\mathscr{F}_n} = e^{-\varrho_- x} \sum_{|u|=n} e^{\varrho_- V(u)}, \qquad n \ge 0.$$

We recall that $\mathbf{Q}$ satisfies (3.16) with $\varrho = \varrho_+$.

Applying Proposition 2, we see that the trajectory of the spine (S_n) is a random walk that drifts to $+\infty$ under $\mathbf{Q}$, and drifts to $-\infty$ under $\mathbf{Q}^{(\varrho_-)}$, in fact, $\mathbf{Q}[S_1] = \psi'(\varrho_+) > 0$ and $\mathbf{Q}^{(\varrho_-)}[S_1] = \psi'(\varrho_-) < 0$. In particular (see (4.16) and (4.17), changing S_1 in $-S_1$ for $\mathbf{Q}^{(\varrho_-)}$), we deduce the existence of $C_R^{(\varrho_-)} > 0$ such that

$$\mathbf{Q}^{(\varrho_-)}(\tau_L^+ < \tau_0^-) \sim \frac{1}{C_R^{(\varrho_-)}} e^{(\varrho_- - \varrho_+)L}, \qquad \mathbf{Q}(\tau_L^+ < \tau_0^-) \sim \frac{1}{C_R}, \qquad L \to \infty, \tag{7.1}$$

(the second equivalence follows from Lemma 3). The strategy of the proof of Theorem 2 (ii) is in the same spirit as in the critical case (i). Recall (1.8) that $\mathcal{L}[0]$ denotes the set of leaves of the killed branching random walk. We give first an estimate on the moments of $\#\mathcal{L}[0]$.

Lemma 18. *For any integer $k < \frac{\varrho_+}{\varrho_-}$, there exists some constant $c_k > 0$ such that for any $x \ge 0$*

$$\mathbf{E}_x[(\#\mathcal{L}[0])^k] \le c_k e^{k\varrho_- x}.$$

Proof of Lemma 18. We give a proof by induction on k. Changing measure from $\mathbf{P}_x$ to $\mathbf{Q}_x^{(\varrho_-)}$ with Proposition 3 (with $\mathcal{L}[0]$ and $h(u) = e^{\varrho_- V(u)}$ for $u \in \mathcal{T}$) yields the identity

$$\mathbf{E}_x\left[(\#\mathcal{L}[0])^k\right] = e^{\varrho_- x}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}} (\#\mathcal{L}[0])^{k-1}\right]. \tag{7.2}$$

By (4.18), the case $k = 1$ holds. Suppose that it is true for $k-1 \geq 1$, and that $2 \leq k < \frac{\varrho_+}{\varrho_-}$. We decompose $\#\mathcal{L}[0]$ along the spine

$$\#\mathcal{L}[0] = 1 + \sum_{\ell=1}^{\tau_0^-} \sum_{u \in \mho_\ell} \#\mathcal{L}^{(u)}[0],$$

where $\#\mathcal{L}^{(u)}[0]$ is the number of particles descendants of u absorbed at 0. We mention that if $V(u) < 0$, then $\#\mathcal{L}^{(u)}[0] = 1$. Conditionally on $\mathcal{G}_\infty$, $(\#\mathcal{L}^{(u)}[0])_{u \in \mho_{j+1}}$, $0 \leq j < \tau_0^-$, are independent and each $\#\mathcal{L}^{(u)}[0]$ is distributed as $(\#\mathcal{L}[0], \mathbf{P}_{V(u)})$. By the triangle inequality,

$$\mathbf{Q}_x^{(\varrho_-)} \left[(\#\mathcal{L}[0])^{k-1} \mid \mathcal{G}_\infty \right]^{1/(k-1)} \leq 1 + \sum_{\ell=1}^{\tau_0^-} \sum_{u \in \mho_\ell} \mathbf{Q}_x^{(\varrho_-)} \left[\left(\#\mathcal{L}^{(u)}[0] \right)^{k-1} \mid \mathcal{G}_\infty \right]^{1/(k-1)}.$$

For each ℓ and $u \in \mho_\ell$, we have from our induction assumption

(7.3)
$$\mathbf{Q}_x^{(\varrho_-)} \left[\left(\#\mathcal{L}^{(u)}[0] \right)^{k-1} \Big| \mathcal{G}_\infty \right] \leq 1_{\{V(u)<0\}} + 1_{\{V(u)\geq 0\}} c_{k-1} e^{\varrho_-(k-1)V(u)} \leq c \left(1 + e^{\varrho_- V(u)} \right)^{k-1}.$$

Therefore we get

$$\mathbf{Q}_x^{(\varrho_-)} \left[(\#\mathcal{L}[0])^{k-1} \Big| \mathcal{G}_\infty \right]^{1/(k-1)} \leq 1 + c' \sum_{\ell=1}^{\tau_0^-} \sum_{u \in \mho_\ell} 1 + e^{\varrho_- V(u)}.$$

In view of (7.2), we deduce that

$$\begin{aligned} \mathbf{E}_x \left[(\#\mathcal{L}[0])^k \right] &\leq c\, e^{\varrho_- x} + c\, e^{\varrho_- x}\, \mathbf{Q}_x^{(\varrho_-)} \left[e^{-\varrho_- S_{\tau_0^-}} \left(\sum_{\ell=1}^{\tau_0^-} \sum_{u \in \mho_\ell} \{1 + e^{\varrho_- V(u)}\} \right)^{k-1} \right] \\ &\leq c\, e^{\varrho_- x} + c\, e^{\varrho_- x}\, \mathbf{Q}_x^{(\varrho_-)} \left[e^{-\varrho_- S_{\tau_0^-}} \left(\sum_{\ell=1}^{\tau_0^-} e^{\varrho_- S_{\ell-1}} a_\ell \right)^{k-1} \right], \end{aligned}$$

where for any $\ell \geq 1$, $a_\ell := \sum_{u \in \mho_\ell} \{1 + e^{\varrho_- \Delta V(u)}\}$. Plainly Corollary 1 also holds with $\varrho = \varrho_-$, which implies that under $\mathbf{Q}_x^{(\varrho_-)}$, the random variables $(S_\ell - S_{\ell-1}, a_\ell)_{\ell \geq 1}$ are i.i.d. (whose law does not depend on x). Moreover

$$\mathbf{Q}^{(\varrho_-)}[(1 + 1_{\{S_1<0\}} e^{-\varrho_- S_1})\, a_1^{k-1}] \leq \mathbf{E} \left[\sum_{|u|=1} (1 + e^{\varrho_- V(u)}) \right]^k < \infty,$$

by (1.4). Applying (4.21) with $b = \varrho_-$, $p = k-1$, $\gamma = \varrho_+ - \varrho_-$ (recalling that $\varrho_+/\varrho_- > k \geq 2$), we get $\mathbf{Q}_x^{(\varrho_-)} \left[e^{-\varrho_- S_{\tau_0^-}} \left(\sum_{\ell=1}^{\tau_0^-} e^{\varrho_- S_{\ell-1}} a_\ell \right)^{k-1} \right] \leq c e^{(k-1)\varrho_- x}$, proving the lemma. □

We introduce the analog of *good* and *bad* particles in the subcritical case, and we feel free to use the same notation. For $\lambda > 1, L > 1$, we say now that

$$u \in \mathbb{B}(L, \lambda) \text{ if there exists some } 1 \leq j \leq |u| : \sum_{v : \overleftarrow{v} = u_{j-1}} (1 + e^{\varrho_- \Delta V(v)}) > \lambda e^{\varrho_-(L - V(u_{j-1}))},$$

and $u \in \mathbb{G}(L,\lambda)$ otherwise, and we define again

$$Z_g[0,L] := \sum_{u\in\mathbb{G}(L,\lambda)} 1_{\{\tau_0^-(u)=|u|<\tau_L^+(u)\}}, \quad Z_b[0,L] := \sum_{u\in\mathbb{B}(L,\lambda)} 1_{\{\tau_0^-(u)=|u|<\tau_L^+(u)\}}.$$

Recall the notation δ^* in (1.4).

Lemma 19. *Let $k^* := \lfloor \frac{\varrho_+}{\varrho_-} \rfloor + 1$ be the smallest integer such that $k^* > \frac{\varrho_+}{\varrho_-}$. Let $0 < \delta_2 < \min(\frac{\delta^*}{2}, k^* - \frac{\varrho_+}{\varrho_-})$.*

(i) There exists some constant $c > 0$ such that for any $L > x \ge 0$,

$$\mathbf{E}_x\left[Z_g[0,L]^{k^*}\right] \le c\,\lambda^{k^*-\frac{\varrho_+}{\varrho_-}-\delta_2}\, e^{\varrho_+ x} e^{(\varrho_- k^* - \varrho_+)L}.$$

(ii) For $q := \frac{\varrho_+}{\varrho_-} + \delta_2$, there exists some constant $c' := c'(\lambda, q) > 0$ such that for any $L > x \ge 0$,

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)\cap\mathbb{G}(L,\lambda)} e^{\varrho_- V(u)}\right]^q \le c'\, e^{\varrho_+ x}\, e^{(q\varrho_- - \varrho_+)L}.$$

(iii) If we assume (1.9), then

$$\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)} e^{\varrho_- V(u)}\right]^{k^*} \le c\, e^{\varrho_+ x}\, e^{(k^*\varrho_- - \varrho_+)L}, \qquad 0 \le x < L.$$

Proof of Lemma 19.

(i): Let k be an integer. By changing of measure from $\mathbf{P}_x$ to $\mathbf{Q}_x^{(\varrho_-)}$, we obtain

$$\mathbf{E}_x[(Z_g[0,L])^k] = e^{\varrho_- x}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}} 1_{\{w_{\tau_0^-}\in\mathbb{G}(L,\lambda)\}}(Z_g[0,L])^{k-1},\ \tau_0^- < \tau_L^+\right]. \tag{7.4}$$

By decomposing the tree $\mathcal{T}$ along the spine (w_ℓ), we get that

$$Z_g[0,L] \le Z[0,L] = 1 + \sum_{\ell=1}^{\tau_0^-}\sum_{u\in\mho_\ell} Z^{(u)}[0,L], \tag{7.5}$$

where $Z^{(u)}[0,L] := \sum_{v\in\mathcal{T}^{(u)}} 1_{\{\tau_0^-(v)=|v|<\tau_L^+(v)\}}$ denotes the number of descendants of u, touching 0 before L [$\mathcal{T}^{(u)}$ means as before the subtree rooted at u]. By Proposition 2, under $\mathbf{Q}_x$, conditioned on $\mathscr{G}_\infty = \sigma\{\omega_j, S_j, \mho_j, (V(u), u\in\mho_j), j\ge 0\}$, the random variables $(Z^{(u)}[0,L])_{u\in\mho_\ell,\ell\le\tau_0^-}$ are independent and each $Z^{(u)}[0,L]$ is distributed as $(Z[0,L], \mathbf{P}_{V(u)})$. Conditioning and using the triangle inequality, we have

$$\left(\mathbf{Q}_x^{(\varrho_-)}\left[(Z_g[0,L])^{k-1}\,|\,\mathscr{G}_\infty\right]\right)^{1/(k-1)} \le 1 + \sum_{\ell=1}^{\tau_0^-}\sum_{u\in\mho_\ell}\left(\mathbf{Q}_x^{(\varrho_-)}\left[\left(Z^{(u)}[0,L]\right)^{k-1}\,\Big|\,\mathscr{G}_\infty\right]\right)^{1/(k-1)}. \tag{7.6}$$

Assume $k < (\varrho_+/\varrho_-)+1$. From Lemma 18, since $Z^{(u)}[0,L] \le \#\mathcal{L}^{(u)}[0]$ and $k-1 < \varrho_+/\varrho_-$, we know that

$$\left(\mathbf{Q}_x^{(\varrho_-)}\left[\left(Z^{(u)}[0,L]\right)^{k-1}\,|\,\mathscr{G}_\infty\right]\right)^{1/(k-1)} \le c\, e^{\varrho_- V(u)} + 1_{\{V(u)<0\}},$$

where the indicator comes from $Z^{(u)}[0,L]=1$ if $V(u)<0$. It follows that

$$\begin{aligned}\mathbf{E}_x[(Z_g[0,L])^k] &\le c\,e^{\varrho_- x}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\right]+\\ & c\,e^{\varrho_- x}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}1_{\{w_{\tau_0^-}\in\mathbb{G}(L,\lambda),\tau_0^-<\tau_L^+\}}\left(\sum_{\ell=1}^{\tau_0^-}\sum_{u\in\mho_\ell}(1+e^{\varrho_- V(u)})\right)^{k-1}\right]\\ &=: c\,e^{\varrho_- x}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\right]+c\,e^{\varrho_- x}A_{(7.7)}, \qquad (7.7)\end{aligned}$$

with some larger constant $c>0$ and the obvious definition of $A_{(7.7)}$ for the remaining expectation under $\mathbf{Q}_x^{(\varrho_-)}$. By (4.18), see also Theorem 4 in [24] applied to $-S$ at τ_x^+, $\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\right]\le c$. Therefore we have shown that for all $k<(\varrho_+/\varrho_-)+1$,

$$\mathbf{E}_x[(Z_g[0,L])^k]\le c' e^{\varrho_- x}+c\,e^{\varrho_- x}A_{(7.7)}. \qquad (7.8)$$

To estimate $A_{(7.7)}$, let us adopt the notation a_ℓ: for any $\ell\ge 1$, $a_\ell:=\sum_{u\in\mho_\ell}(1+e^{\varrho_-\Delta V(u)})$, hence $\sum_{\ell=1}^{\tau_0^-}\sum_{u\in\mho_\ell}(1+e^{\varrho_- V(u)})\le\sum_{\ell=1}^{\tau_0^-}e^{\varrho_- S_{\ell-1}}a_\ell$. On $\{w_{\tau_0^-}\in\mathbb{G}(L,\lambda)\}$, $a_\ell\le\lambda^s e^{s\,\varrho_-(L-S_{\ell-1})}\,a_\ell^{1-s}$ for any $0<s<1$. It follows that

$$A_{(7.7)}\le\lambda^{s(k-1)}\,e^{s\varrho_-(k-1)L}\,\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\left(\sum_{\ell=1}^{\tau_0^-}e^{\varrho_-(1-s)S_{\ell-1}}a_\ell^{1-s}\right)^{k-1},\ \tau_0^-<\tau_L^+\right], \qquad (7.9)$$

for any $0<s<1$ and $k<(\varrho_+/\varrho_-)+1$.

If ϱ_+/ϱ_- is not an integer, then $k^*<\frac{\varrho_+}{\varrho_-}+1$ and (7.9) holds for $k=k^*$. Take

$$s=\frac{k^*-\frac{\varrho_+}{\varrho_-}-\delta_2}{k^*-1}. \qquad (7.10)$$

Notice that

$$\mathbf{Q}_x^{(\varrho_-)}\left[(1+1_{\{S_1<0\}}e^{-\varrho_- S_1})\,a_1^{(1-s)(k^*-1)}\right]\le\mathbf{E}\left[\sum_{|u|=1}(1+e^{\varrho_- V(u)})\right]^{(1-s)(k^*-1)+1}<\infty,$$

$$\mathbf{Q}^{(\varrho_-)}\left[e^{(1-s)\varrho_-(k^*-1)S_1}\right]=e^{\psi(\varrho_-(1+(1-s)(k^*-1)))}<\infty,$$

by (1.4). Under $\mathbf{Q}^{(\varrho_-)}$, $(S_\ell-S_{\ell-1},a_\ell^{1-s})_{\ell\ge1}$ are i.i.d. Applying (4.22) to the expectation term $\mathbf{Q}_x^{(\varrho_-)}[\cdot]$ in (7.9) with $\gamma=\varrho_+-\varrho_-,b=\varrho_-(1-s),\eta=\varrho_-,p=k^*-1$ and noticing that $pb>\gamma$, we get that if we take $k=k^*$ in (7.7), then

$$\begin{aligned}A_{(7.7)} &\le c\,\lambda^{s(k^*-1)}\,e^{s\varrho_-(k^*-1)L}\,e^{(\varrho_+-\varrho_-)(x-L)+(k^*-1)(\varrho_--s\varrho_-)L}\\ &= c\lambda^{s(k^*-1)}\,e^{(\varrho_+-\varrho_-)(x-L)+(k^*-1)\varrho_- L}.\end{aligned}$$

This estimate with (7.8) prove (i) in the case that ϱ_+/ϱ_- is not an integer.

It remains to treat the case when ϱ_+/ϱ_- is an integer. Then $k^*=\frac{\varrho_+}{\varrho_-}+1$. Applying (7.7) to $k=k^*-1$ (which is less than $\frac{\varrho_+}{\varrho_-}+1$), we have that

$$\mathbf{E}_x[(Z_g[0,L])^{k^*-1}]\le c'e^{\varrho_- x}+c\,e^{\varrho_- x}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\left(\sum_{\ell=1}^{\tau_0^-}e^{\varrho_- S_{\ell-1}}\,a_\ell\right)^{k^*-2},\ \tau_0^-<\tau_L^+\right],$$

which by an application of (4.22) with $\alpha = 0, \gamma = \varrho_+ - \varrho_-, b = \varrho_-, p = k^* - 2 = \gamma/b$ [it is easy to check the integrability hypothesis in Lemma 8 (ii)], yields that

$$\mathbf{E}_x[(Z_g[0,L])^{k^*-1}] \le c(1+L-x)e^{\varrho_+ x}, \qquad 0 \le x \le L.$$

Moreover, $\mathbf{E}_x[(Z_g[0,L])^{k^*-1}]$ is 1 if $x < 0$ and 0 if $x > L$. Going back to (7.6) and (7.4) with now $k = k^*$, we obtain that

$$\mathbf{E}_x[(Z_g[0,L])^{k^*}] \le c e^{\varrho_- x} \mathbf{Q}_x^{(\varrho_-)} \left[1 + e^{-\varrho_- S_{\tau_0^-}} 1_{\{w_{\tau_0^-} \in \mathbb{G}(L,\lambda)\}} A^{k^*-1},\ \tau_0^- < \tau_L^+\right]$$

with

$$A := \sum_{\ell=1}^{\tau_0^-} \sum_{u \in \mho_\ell} \left((1+L-V(u))^{\frac{\varrho_-}{\varrho_+}} e^{\varrho_- V(u)} 1_{\{V(u)\in[0,L]\}} + 1_{\{V(u)<0\}} \right).$$

Observe that on $\{\ell \le \tau_0^- < \tau_L^+\}$,

$$\begin{aligned} &\sum_{u \in \mho_\ell} (1+L-V(u))^{\frac{\varrho_-}{\varrho_+}} e^{\varrho_- V(u)} 1_{\{V(u)\in[0,L]\}} + 1_{\{V(u)<0\}} \\ \le\ & c(1+L-S_{\ell-1})^{\frac{\varrho_-}{\varrho_+}} e^{\rho_- S_{\ell-1}} \sum_{u\in\mho_\ell} (1+e^{\varrho_- \Delta V(u)}), \end{aligned}$$

which in turn is bounded by $c(1+L-S_{\ell-1})^{\frac{\varrho_-}{\varrho_+}} e^{\rho_- S_{\ell-1}} \lambda^s e^{s\varrho_-(L-S_{\ell-1})} a_\ell^{1-s}$ since $w_{\tau_0^-} \in \mathbb{G}(L,\lambda)$, where $0 < s < 1$ is as in (7.10). It follows that

$$\begin{aligned} &\mathbf{E}_x[(Z_g[0,L])^{k^*}] \le c' \lambda^{s(k^*-1)} e^{s\varrho_-(k^*-1)L}\, e^{\varrho_- x} \times \\ &\mathbf{Q}_x^{(\varrho_-)} \left[e^{-\varrho_- S_{\tau_0^-}} \left(\sum_{\ell=1}^{\tau_0^-} (1+L-S_{\ell-1})^{\frac{\varrho_-}{\varrho_+}} a_\ell^{1-s}\, e^{\varrho_-(1-s)S_{\ell-1}} \right)^{k^*-1},\ \tau_0^- < \tau_L^+ \right]. \end{aligned}$$

Again, we apply (4.22) to $(S_\ell - S_{\ell-1}, a_\ell^{1-s})_{\ell\ge1}$ with $\gamma = \varrho_+ - \varrho_-, b = \varrho_-(1-s), \eta = \varrho_-, p = k^*-1 > \gamma/b$ [the integrability hypothesis can be easily checked as before], which yields that $\mathbf{E}_x[(Z_g[0,L])^{k^*}] \le c' \lambda^{s(k^*-1)}\, e^{\varrho_+ x + (k^*\varrho_- - \varrho_+)L}$, proving (i) in the case that ϱ_+/ϱ_- is an integer.

(ii): Write in this proof $\Lambda := \sum_{u\in\mathscr{H}(L)\cap\mathbb{G}(L,\lambda)} e^{\varrho_- V(u)}$. Instead of $\mathbf{Q}_x^{(\varrho_-)}$, we shall make use of the probability $\mathbf{Q}$ defined in (3.16) with $\varrho = \varrho_+$ for the change of measure. We stress that under $\mathbf{Q}$, (S_n) drifts to $+\infty$.

Firstly, we prove by induction on k that for any $1 \le k \le k^*-1$, there exists some constant $c_k = c_k(\lambda) > 0$ such that

$$\mathbf{E}_x\left[\Lambda^k\right] \le c_k\, e^{\varrho_+ x}\, e^{(k\varrho_- - \varrho_+)L}. \tag{7.11}$$

By the change of measure, we get that for $k \ge 1$,

$$\begin{aligned} \mathbf{E}_x[\Lambda^k] &= e^{\varrho_+ x} \mathbf{Q}_x \left[e^{(\varrho_- - \varrho_+) S_{\tau_L^+}} 1_{\{w_{\tau_L^+}\in\mathbb{G}(L,\lambda)\}} \Lambda^{k-1},\ \tau_L^+ < \tau_0^- \right] \\ &= e^{\varrho_+ x + (\varrho_- - \varrho_+)L} \mathbf{Q}_x \left[e^{(\varrho_- - \varrho_+) T_L^+} 1_{\{w_{\tau_L^+}\in\mathbb{G}(L,\lambda)\}} \Lambda^{k-1},\ \tau_L^+ < \tau_0^- \right], \end{aligned} \tag{7.12}$$

where $T_L^+ := S_{\tau_L^+} - L > 0$. This yields the case $k = 1$ of (7.11).

Assume $2 \le k \le k^*-1$ and that (7.11) holds for $1, ..., k-1$. Exactly as before, we decompose Λ along the spine up to τ_L^+, apply the triangular inequality and arrive at

$$(\mathbf{Q}_x[\Lambda^{k-1}|\mathscr{G}_\infty])^{1/(k-1)} \le e^{\varrho_- S_{\tau_L^+}} + \sum_{\ell=1}^{\tau_L^+} \sum_{u\in\mho_\ell} (\mathbf{Q}_x[(\Lambda^{(u)})^{k-1}|\mathscr{G}_\infty])^{1/(k-1)},$$

where $\Lambda^{(u)} := \sum_{v\in\mathcal{T}^{(u)}\cap\mathscr{H}(L)\cap\mathbb{G}(L,\lambda)} e^{\varrho_- V(v)}$ with $\mathcal{T}^{(u)}$ the subtree rooted at u. By Proposition 2, under $\mathbf{Q}_x$ and conditioning on $\mathscr{G}_\infty$, each $\Lambda^{(u)}$ is distributed as $(\Lambda, \mathbf{P}_{V(u)})$. Hence by induction assumption, $(\mathbf{Q}_x[(\Lambda^{(u)})^{k-1}|\mathscr{G}_\infty])^{1/(k-1)} \le c_{k-1}^{\frac{1}{k-1}} e^{\frac{\varrho_+(V(u)-L)}{k-1}} e^{\varrho_- L}$. Then,

$$(\mathbf{Q}_x[\Lambda^{k-1}|\mathscr{G}_\infty])^{1/(k-1)} \le e^{\varrho_- S_{\tau_L^+}} + c_{k-1}^{\frac{1}{k-1}} e^{\varrho_- L} \sum_{\ell=1}^{\tau_L^+} \sum_{u\in\mho_\ell} e^{\frac{\varrho_+ \Delta V(u)}{k-1}} e^{\frac{\varrho_+(S_{\ell-1}-L)}{k-1}}.$$

Notice that $\frac{\varrho_+}{k-1} \ge \varrho_-$ and that on $\{w_{\tau_L^+} \in \mathbb{G}(L,\lambda)\}$,

$$\sum_{u\in\mho_\ell} e^{\frac{\varrho_+}{k-1}\Delta V(u)} \le a_\ell \max_{u\in\mho_\ell} e^{(\frac{\varrho_+}{k-1}-\varrho_-)\Delta V(u)} \le (a_\ell)^{1-s} \lambda^{\frac{\varrho_+}{\varrho_-(k-1)}-(1-s)} e^{(\frac{\varrho_+}{k-1}-(1-s)\varrho_-)(L-S_{\ell-1})},$$

with $s := \frac{k^*-\frac{\varrho_+}{\varrho_-}-\delta_2}{k^*-1}$. We mention that the above inequality holds for $k=k^*$.

Going back to (7.12), we obtain that [we keep the density there $e^{(\varrho_--\varrho_+)T_L^+}$ only for $e^{\varrho_- S_{\tau_L^+}}$ and use the inequality $(x+y)^{k-1} \le 2^{k-1}(x^{k-1}+y^{k-1})$]

$$\begin{aligned}&\mathbf{E}_x[\Lambda^k]\\ \le\ & c(\lambda) e^{\varrho_+ x+(\varrho_--\varrho_+)L)} e^{\varrho_-(k-1)L} \left(\mathbf{Q}_x[e^{(k\varrho_--\varrho_+)T_L^+}] + \mathbf{Q}_x\Big[\sum_{\ell=1}^{\tau_L^+} (a_\ell)^{1-s} e^{(1-s)\varrho_-(S_{\ell-1}-L)}\Big]^{k-1} \right).\end{aligned}$$

Remark that $\mathbf{Q}_x[e^{(k\varrho_--\varrho_+)T_L^+}] = \mathbf{Q}[e^{(k\varrho_--\varrho_+)T_{L-x}^+}]$ is bounded by some constant since we have $\mathbf{Q}[e^{(k\varrho_--\varrho_++\delta)S_1}] = \exp\{\psi(k\varrho_-+\delta)\} < \infty$ if $\delta>0$ is sufficiently small [here we use the fact that $k \le k^*-1$]. By Lemma 5, the above expectation $\mathbf{Q}_x[\cdots]^{k-1}$ is uniformly bounded, which proves (7.11).

To control $\mathbf{E}_x[\Lambda^q]$, we use the change of measure:

$$\mathbf{E}_x[\Lambda^q] = e^{\varrho_+ x+(\varrho_--\varrho_+)L} \mathbf{Q}_x\left[e^{(\varrho_--\varrho_+)T_L^+} 1_{\{w_{\tau_L^+}\in\mathbb{G}(L,\lambda)\}} \Lambda^{q-1},\ \tau_L^+ < \tau_0^- \right].$$

Since $q<k^*$, $(\mathbf{Q}_x[\Lambda^{q-1}|\mathscr{G}_\infty])^{1/(q-1)} \le (\mathbf{Q}_x[\Lambda^{k^*-1}|\mathscr{G}_\infty])^{1/(k^*-1)}$. From (7.11) with $k=k^*-1$ there, we use the same arguments as before and get that

$$\begin{aligned}&\mathbf{E}_x[\Lambda^q]\\ \le\ & c e^{\varrho_+ x+(\varrho_--\varrho_+)L} e^{\varrho_-(q-1)L} \left(\mathbf{Q}_x[e^{(q\varrho_--\varrho_+)T_L^+}] + \mathbf{Q}_x\Big[\sum_{\ell=1}^{\tau_L^+} (a_\ell)^{1-s} e^{(1-s)\varrho_-(S_{\ell-1}-L)}\Big]^{q-1} \right).\end{aligned}$$

Again, $\mathbf{Q}_x[e^{(q\varrho_--\varrho_+)T_L^+}]$ is bounded by some constant since $\mathbf{Q}[e^{(q\varrho_--\varrho_++\delta)S_1}] = \exp(\psi(q\varrho_-+\delta) < \infty$ if $\delta>0$ is sufficiently small. By Lemma 5, the above expectation $\mathbf{Q}_x[\cdots]^{q-1}$ is uniformly bounded, which proves (ii).

(iii) The proof goes in the same spirit as that of (i) and (ii): Let $\chi(L) := \sum_{u\in\mathscr{H}(L)} e^{\varrho_-(V(u)-L)}$ and we prove by induction that for any $1 \le k \le k^*$,

$$\mathbf{E}_x\left[\chi(L)^k\right] \le c_k\, e^{\varrho_+(x-L)}, \qquad x\in\mathbb{R}. \tag{7.13}$$

The case $k = 1$ is obvious by the change of measure. Assume (7.13) for $k-1$ and $2 \le k \le k^*$. By repeating the same arguments as in (ii), we get that

$$\mathbf{E}_x[\chi(L)^k] \le c\, e^{\varrho_-(x-L)} \times$$
$$\left(\mathbf{Q}_x^{(\varrho_-)}[e^{(k-1)\varrho_- T_L^+}, \tau_L^+ < \tau_0^-] + \mathbf{Q}_x^{(\varrho_-)}\left[\Big(\sum_{\ell=1}^{\tau_L^+}\sum_{u\in\mho_\ell} e^{\frac{\varrho_+}{k-1}(V(u)-L)}\Big)^{k-1}, \tau_L^+ < \tau_0^-\right]\right). \tag{7.14}$$

By the absolute continuity between $\mathbf{Q}_x^{(\varrho_-)}$ and $\mathbf{Q}_x$,

$$\begin{aligned}
\mathbf{Q}_x^{(\varrho_-)}[e^{(k-1)\varrho_- T_L^+}, \tau_L^+ < \tau_0^-] &= e^{(\varrho_+-\varrho_-)x-(k-1)\varrho_- L}\mathbf{Q}_x[e^{(k\varrho_- -\varrho_+)S_{\tau_L^+}}, \tau_L^+ < \tau_0^-] \\
&= e^{(\varrho_+-\varrho_-)(x-L)}\mathbf{Q}_x[e^{(k\varrho_- -\varrho_+)T_L^+}, \tau_L^+ < \tau_0^-] \\
&\le ce^{(\varrho_+-\varrho_-)(x-L)},
\end{aligned}$$

where the term $\mathbf{Q}_x[e^{k\varrho_- -\varrho_+)T_L^+}]$ is uniformly bounded, since for $k \le k^*$ and sufficiently small $\delta_4 > 0$, $\mathbf{Q}[e^{(k\varrho_- -\varrho_+ +\delta_4)S_1}] = e^{\psi(k\varrho_- +\delta_4)} < \infty$ by (1.9).

It remains to control the second expectation term $\mathbf{Q}_x^{(\varrho_-)}$ in (7.14). Let $b_\ell := \sum_{u\in\mho_\ell} e^{\frac{\varrho_+}{k-1}\Delta V(u)}$, for $\ell \ge 1$. Under $\mathbf{Q}_x^{(\varrho_-)}$, $(S_\ell - S_{\ell-1}, b_\ell)_{\ell\ge1}$ are i.i.d. and

$$\mathbf{Q}^{(\varrho_-)}[b_1^{k-1}] = \mathbf{E}\left[(\sum_{|u|=1} e^{\varrho_- V(u)})(\sum_{v\ne u} e^{\frac{\varrho_+}{k-1}V(v)})^{k-1}\right] \le \mathbf{E}\left[\sum_{|u|=1} e^{\varrho_- V(u)}\right]^{1+\frac{\varrho_+}{\varrho_-}},$$

since $\varrho_- < \frac{\varrho_+}{k-1}$. Then $\mathbf{Q}^{(\varrho_-)}[b_1^{k-1}] < \infty$ by (1.9). Going back to (7.14), we see that the expectation term $\mathbf{Q}_x[(\cdot\cdot)^{k-1}, \tau_L^+ < \tau_0^-]$ equals

$$\mathbf{Q}_x^{(\varrho_-)}\left[\Big(\sum_{\ell=1}^{\tau_L^+} b_\ell\, e^{\frac{\varrho_+}{k-1}(S_{\ell-1}-L)}\Big)^{k-1}, \tau_L^+ < \tau_0^-\right] \le c'\, e^{(\varrho_+-\varrho_-)(x-L)},$$

by applying (4.22) to $(S_\ell - S_{\ell-1}, b_\ell)_{\ell\ge1}$ with $\gamma = \varrho_+ - \varrho_-$, $b = \varrho_+/(k-1)$ and $p = k-1$. This proves (7.13) hence (iii). □

The next lemma controls the number of bad particles.

Lemma 20. *Let $r = \frac{\varrho_+}{\varrho_-} - 1 + \frac{\delta^*}{2}$ (with δ^* as in (1.4)).*

(i) There exists some constant $c = c(r) > 0$ such that for all $0 \le x \le L$,

$$\mathbf{E}_x\left[Z_b[0,L]\right] \le c\,\lambda^{-r}\, e^{\varrho_+ x} e^{(\varrho_- -\varrho_+)L}.$$

(ii) Denote by $\mathcal{L}_{b,L}[0] := \{v \in \mathcal{L}[0] : \exists\, u \in \mathscr{H}(L)\cap\mathbb{B}(L,\lambda) \text{ with } u < v\}$ the set of leaves which are descendants of some element of $\mathscr{H}(L)\cap\mathbb{B}(L,\lambda)$. Then for any $0 \le x \le L$,

$$\mathbf{E}_x\left[\#\mathcal{L}_{b,L}[0]\right] \le c\,\lambda^{-r}\, e^{\varrho_+ x} e^{(\varrho_- -\varrho_+)L}.$$

Proof of Lemma 20:

(i) By changing the measure from $\mathbf{P}_x$ to $\mathbf{Q}_x^{(\varrho_-)}$:

$$\mathbf{E}_x\left[Z_b[0,L]\right] = e^{\varrho_- x}\,\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\, 1_{\{w_{\tau_0^-}\in\mathbb{B}(L,\lambda)\}}, \tau_0^- < \tau_L^+\right].$$

Let us write $a_j := \sum_{u\in\mho_j}(1+e^{\varrho_-\Delta V(u)})$, $j\ge 1$, in this proof. Then

$$1_{\{w_{\tau_0^-}\in\mathbb{B}(L,\lambda)\}}\le\sum_{j=1}^{\tau_0^-}\lambda^{-r}\,a_j^r\,e^{-r\varrho_-(L-S_{j-1})},\tag{7.15}$$

which yields that

$$\begin{aligned}\mathbf{E}_x\left[Z_b[0,L]\right] &\le \lambda^{-r}e^{\varrho_- x}\,\mathbf{Q}_x^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\sum_{j=1}^{\tau_0^-}a_j^r\,e^{-r\varrho_-(L-S_{j-1})},\tau_0^-<\tau_L^+\right]\\ &\le c\lambda^{-r}e^{\varrho_- x}\,e^{(\varrho_+-\varrho_-)(x-L)},\end{aligned}$$

by applying (4.22) to $\gamma=\varrho_+-\varrho_-,p=1$ and $b=r\varrho_->\gamma$ [the integrability hypothesis is satisfied thanks to (1.4) and the choice of r: $\mathbf{Q}^{(\varrho_-)}\left[(1+1_{\{S_1<0\}}e^{-\varrho_- S_1})\,a_1^r\right]\le\mathbf{E}\left[\sum_{|u|=1}(1+e^{\varrho_- V(u)})\right]^{r+1}<\infty$, and $\mathbf{Q}^{(\varrho_-)}\left[e^{r\varrho_- S_1}\right]=e^{\psi(\varrho_-(1+r))}<\infty$]. This proves (i).

(ii) Remark that $\#\mathcal{L}_{b,L}[0]=\sum_{u\in\mathscr{H}(L)\cap\mathbb{B}(L,\lambda)}\#\mathcal{L}^{(u)}[0]$, where $\mathcal{L}^{(u)}[0]$ denotes the set of leaves which are descendants of u. By the branching property, conditioned on $\mathscr{H}(L)\cap\mathbb{B}(L,\lambda)$, $(\#\mathcal{L}^{(u)}[0])_{u\in\mathscr{H}(L)\cap\mathbb{B}(L,\lambda)}$ are independent and are distributed as $\#\mathcal{L}[0]$ under $\mathbf{P}_{V(u)}$. It follows from Lemma 18 (with $k=1$) that

$$\mathbf{E}_x\left(\#\mathcal{L}_{b,L}[0]\right)\le c\mathbf{E}_x\left[\sum_{u\in\mathscr{H}(L)\cap\mathbb{B}(L,\lambda)}e^{\varrho_- V(u)}\right]=ce^{\varrho_- x}\,\mathbf{Q}_x^{(\varrho_-)}\left[w_{\tau_L^+}\in\mathbb{B}(L,\lambda),\tau_L^+<\tau_0^-\right],$$

by the change of measure from $\mathbf{P}_x$ to $\mathbf{Q}_x^{(\varrho_-)}$. By (7.15) (with τ_L^+ instead of τ_0^-), the above probability under $\mathbf{Q}_x^{(\varrho_-)}$ is less than

$$\begin{aligned}&\lambda^{-r}\,\mathbf{Q}_x^{(\varrho_-)}\left[\sum_{j=1}^{\tau_L^+}a_j^r\,e^{-r\varrho_-(L-S_{j-1})},\tau_L^+<\tau_0^-\right]\\ \le\ &\lambda^{-r}\sum_{j\ge1}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-r\varrho_-(L-S_{j-1})},j\le\min(\tau_L^+,\tau_0^-)\right]\mathbf{Q}_x^{(\varrho_-)}[a_j^r],\end{aligned}$$

since for each j, a_j is independent of $(S_{j-1},j\le\min(\tau_L^+,\tau_0^-))$; moreover $\mathbf{Q}_x^{(\varrho_-)}[a_j^r]=\mathbf{Q}^{(\varrho_-)}[a_j^r]=c'<\infty$ as in (i). Then we have

$$\mathbf{E}_x\left[Z_b[0,L]\right]\le cc'e^{\varrho_- x}\lambda^{-r}\sum_{j\ge1}\mathbf{Q}_x^{(\varrho_-)}\left[e^{-r\varrho_-(L-S_{j-1})},j\le\min(\tau_L^+,\tau_0^-)\right],$$

which by an application of (4.19) (with $r\varrho_->\gamma:=\varrho_+-\varrho_-$) gives (ii). □

Let $M_\infty^{(\varrho_-)}$ be the almost sure limit of $M_n^{(\varrho_-)}:=\sum_{|u|=n}e^{\varrho_- V(u)}$. By [8],[25], $M_\infty^{(\varrho_-)}$ is almost surely positive on the event $\{\mathcal{T}=\infty\}$. From [23], we know that there exists a constant c_{ϱ_-} such that

$$\mathbf{P}(M_\infty^{(\varrho_-)}>t)\sim c_{\varrho_-}t^{-\varrho_+/\varrho_-},\qquad t\to\infty.\tag{7.16}$$

We mention that the constant c_{ϱ_-} is given in [19], Theorem 4.10:

$$c_{\varrho_-}=\frac{1}{\varrho_+\psi'(\varrho_+)}\mathbf{E}\Big[(\sum_{|u|=1}e^{\varrho_- u}M_\infty^{(\varrho_-,u)})^{\varrho_+/\varrho_-}-\sum_{|u|=1}e^{\varrho_+ u}(M_\infty^{(\varrho_-,u)})^{\varrho_+/\varrho_-}\Big],$$

where under $\mathbf{P}$ and conditioned on $\{V(u),|u|=1\}$, $(M_\infty^{(\varrho_-,u)})_{|u|=1}$ are i.i.d. copies of $M_\infty^{(\varrho_-)}$.

Lemma 21 (Subcritical case)**.** *As $t \to \infty$, the law of $\#\mathcal{L}[0]$ under $\mathbf{P}_t$, the number of descendants absorbed at 0 of a particle starting from t, normalized by $e^{\varrho_- t}$ converges in distribution to $c^*_{sub} M_\infty^{(\varrho_-)}$ where*

$$c^*_{sub} = \frac{\mathbf{Q}^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\right] - 1}{\varrho_- \, \mathbf{Q}^{(\varrho_-)}\left[-S_{\tau_0^-}\right]}.$$

Proof of Lemma 21: The proof goes in the same way as that of Lemma 15, we only point out the main difference and omit the details. Recall that $\mathcal{L}[a] := \{u \in \mathcal{T} : |u| = \tau_a^-(u)\}$. By linear translation, it is enough to prove that $e^{-\varrho_- t}\#\mathcal{L}[-t]$ converges in law to $c^*_{sub} M_\infty^{(\varrho_-)}$. Let $M^{(\varrho_-)}_{\mathcal{L}[-t]} := \sum_{u\in\mathcal{L}[-t]} e^{\varrho_- V(u)}$, which converges almost surely to $M_\infty^{(\varrho_-)}$. On the other hand, we have $M^{(\varrho_-)}_{\mathcal{L}[-t]} = e^{-\varrho_- t}\sum_{u\in\mathcal{L}[-t]} e^{\varrho_-(V(u)+t)}$. Just like the proof of Lemma 15, we apply Theorem 6.3 in Nerman [28] (with $\alpha = \varrho_-$ there) and obtain that on $\{\mathcal{T} = \infty\}$, almost surely

$$\frac{\sum_{u\in\mathcal{L}[-t]} e^{\varrho_-(V(u)+t)}}{\#\mathcal{L}[-t]} \to \varrho_- \frac{\mathbf{Q}^{(\varrho_-)}\left[-S_{\tau_0^-}\right]}{\mathbf{Q}^{(\varrho_-)}\left[e^{-\varrho_- S_{\tau_0^-}}\right] - 1}, \qquad t \to \infty.$$

which easily yields the lemma. □

Lemma 22. *Let $\widehat{\mu}_{\lambda,\infty} := \sum_{i=1}^{\hat\zeta_\lambda} \delta_{\{x_i\}}$ be the point process defined in Proposition 1 associated with $\mathscr{B}(\theta) := (\frac{1}{\lambda}\int \theta(dx)(1+e^{\varrho_- x}))^{1/\varrho_-}$ for $\theta \in \Omega_f$. Let $(M_\infty^{(\varrho_-,i)}, i \ge 1)$ be a sequence of i.i.d. random variables of common law that of $(M_\infty^{(\varrho_-)}, \mathbf{P})$, independent of $\widehat{\mu}_{\lambda,\infty}$. As $t \to \infty$, we have*

$$\mathbf{Q}\left(\sum_{i=1}^{\hat\zeta_\lambda} e^{\varrho_- x_i} M_\infty^{(\varrho_-,i)} > t\right) \sim c_{\varrho_-}\,\mathbf{Q}\left[\int \widehat{\mu}_{\lambda,\infty}(dx) e^{\varrho_+ x}\right] t^{-\varrho_+/\varrho_-}.$$

We mention that as $\lambda \to \infty$, $\mathbf{Q}\left[\int \widehat{\mu}_{\lambda,\infty}(dx)e^{\varrho_+ x}\right] \to \frac{1}{\mathbf{Q}[\Re^{-1}]}$ by (5.24) and (5.27).

Proof. Let $\Lambda_{L,\lambda} := \sum_{u\in\mathscr{H}(L)\cap\mathbb{G}(L,\lambda)} e^{\varrho_-(V(u)-L)}$. By Proposition 1, under $\mathbf{P}_x(\cdot|H(L)>0)$, $\Lambda_{L,\lambda}$ converges in law to $\int \mu_{\lambda,\infty}(dx)e^{\varrho_- x} = \sum_{i=1}^{\hat\zeta_\lambda} e^{\varrho_- x_i}$ (some tightness is required here but we omit the details since the arguments are similar to the critical case). By Lemma 19 (ii), the family $(\Lambda_{L,\lambda}, \mathbf{P}_x(\cdot|H(L)>0))$ is bounded in L^q with $q = \frac{\rho_+}{\rho_-} + \delta_2$, hence

$$\mathbf{Q}\left[\sum_{i=1}^{\hat\zeta_\lambda} e^{\varrho_- x_i}\right]^q < \infty. \tag{7.17}$$

This together with (7.16) allows us to apply Lemma 16 to $p = \frac{\varrho_+}{\varrho_-}$ and yields the desired asymptotic. □

We now prove Theorem 2 in the subcritical case.

Proof of Theorem 2 (ii):

Lower bound of Theorem 2 (ii): The proof of the lower bound goes in the same way as that of Theorem 2 (i) by using Proposition 1 and Lemma 21. Let $A > 0$. Consider $n \to \infty$, let $L_A := \frac{1}{\varrho_-}\log n - A$ and $\lambda := e^{\varrho_- A}$. We keep the same notations $H_g(L_A), (\#\mathcal{L}^{(i)}[0], 1 \le i \le H_g(L_A))$. We define as well $B^{(i)} := \#\mathcal{L}^{(i)}[0]e^{-\varrho_- V(u^{(i)})}$ for $u^{(i)} \in \mathscr{H}(L_A)$, and $E(L_A)$

the event that $B^{(i)} > (1-\varepsilon) M_\infty^{\varrho_-,i}$, $\forall i$ with small $\varepsilon>0$. Repeating the proof of the lower bound of Theorem 2 (i), and using Proposition 1 and Lemma 21, we get that for any $A>0$,

$$\begin{aligned}
&\liminf_{n\to\infty} n^{\frac{\varrho_+}{\varrho_-}} \mathbf{P}_x\Big(\sum_{i=1}^{H_g(L_A)} \#\mathcal{L}^{(i)}[0] > n\Big) \\
\geq\ & \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x)\, e^{\varrho_+ x}\, e^{\varrho_+ A}\, \mathbf{Q}\Big(\sum_{i=1}^{\hat\zeta_\lambda} e^{\varrho_- x_i} M_\infty^{(\varrho_-,i)} > \frac{1}{c^*_{sub}} e^{\varrho_- A}\Big) \\
=:\ & \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x) e^{\varrho_+ x} C_s(A), \qquad (7.18)
\end{aligned}$$

where $\widehat\mu_{A,\infty} := \sum_{i=1}^{\hat\zeta_\lambda} \delta_{\{x_i\}}$ is the point process as in Lemma 22 (with $\lambda := e^{\varrho_- A}$ there) and c^*_{sub} is defined in Lemma 21. The same also holds for the upper bound, hence for any $A>0$,

$$\lim_{n\to\infty} n^{\frac{\varrho_+}{\varrho_-}} \mathbf{P}_x\Big(\sum_{i=1}^{H_g(L_A)} \#\mathcal{L}^{(i)}[0] > n\Big) = \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x) e^{\varrho_+ x} C_s(A). \qquad (7.19)$$

Since $\mathbf{P}_x\big(\#\mathcal{L}[0]>n\big) \geq \mathbf{P}_x\big(\sum_{i=1}^{H(L_A)} \#\mathcal{L}^{(i)}[0] > n\big)$, we get that for any $A>0$,

$$\liminf_{n\to\infty} n^{\varrho_+/\varrho_-} \mathbf{P}_x\big(\#\mathcal{L}[0]>n\big) \geq \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x) e^{\varrho_+ x} C_s(A). \qquad (7.20)$$

Upper bound of Theorem 2 (ii): By Lemma 20 and Lemma 19 (i) with $L := L_A = \frac{1}{\varrho_-}\log n - A$, $\lambda := e^{\varrho_- A}$, we obtain the following estimate: For any $\varepsilon>0$,

$$\mathbf{P}_x\Big(Z_g[0,L_A] \geq \varepsilon n\Big) \leq (\varepsilon n)^{-k^*} c\, e^{A(\varrho_- k^* - \varrho_+ - \delta_2 \varrho_-)} e^{\varrho_+ x + (\varrho_- k^* - \varrho_+) L_A} = c_{\varepsilon,x}\, n^{-\varrho_+/\varrho_-}\, e^{-\delta_2 \varrho_- A},$$

and

$$\mathbf{P}_x\Big(Z_b[0,L_A] \geq \varepsilon n\Big) \leq \frac{1}{\varepsilon n} c e^{-A(\varrho_+ - \varrho_- + \delta^* \varrho_-/2)} e^{\varrho_+ x + (\varrho_- - \varrho_+) L_A} = c_{\varepsilon,x}\, n^{-\varrho_+/\varrho_-}\, e^{-\delta^* \varrho_- A/2},$$

with the same estimate for $\mathbf{P}_x\Big(\mathcal{L}_{b,L_A}[0] \geq \varepsilon n\Big)$. Since $Z[0,L_A] = Z_g[0,L_A] + Z_b[0,L_A]$, we obtain that for any $\varepsilon>0$,

$$\limsup_{A\to\infty} \limsup_{n\to\infty} n^{\varrho_+/\varrho_-} \mathbf{P}_x\left(Z[0,L_A] + \mathcal{L}_{b,L_A}[0] \geq 3\varepsilon n\right) = 0.$$

From here and using the fact that $\#\mathcal{L}[0] = Z[0,L_A] + \mathcal{L}_{b,L_A}[0] + \sum_{i=1}^{H_g(L_A)} \#\mathcal{L}^{(i)}[0]$, we deduce from (7.19) that for any $A>0$,

$$\limsup_{n\to\infty} n^{\varrho_+/\varrho_-} \mathbf{P}_x\big(\#\mathcal{L}[0]>n\big) \leq \frac{\mathbf{Q}[\Re^{-1}]}{C_R} R(x) e^{\varrho_+ x} C_s(A) + o_A(1),$$

with $o_A(1) \to 0$ as $A\to\infty$ (in fact exponentially fast). This together with the lower bound (7.20) yields that $\lim_{n\to\infty} n^{\varrho_+/\varrho_-} \mathbf{P}_x\big(\#\mathcal{L}[0]>n\big)$ exists and is finite. Then, a fortiori, $\lim_{A\to\infty} C_s(A)$ also exists and is some finite constant. This proves Theorem 2 (ii). □

We end this section by giving the proof of Lemma 1.

Proof of Lemma 1: By (3.21), $C_R = 1/\mathbf{Q}(\tau_0^- = \infty)$. Recall (7.18). It suffices to show that

$$\lim_{A\to\infty} C_s(A) = \frac{c_{\varrho_-}}{\mathbf{Q}[\Re^{-1}]} (c^*_{sub})^{\varrho_+/\varrho_-}. \qquad (7.21)$$

The lower bound follows from the monotonicity: the random point measure $\widehat{\mu}_{A,\infty}$ is stochastically increasing in A; Then for any $A > A_0$,

$$C_S(A) \geq e^{\varrho_+ A}\, \mathbf{Q}\Big(\sum_{i=1}^{\hat{\xi}_{\lambda_0}} e^{\varrho_- z_i} M_\infty^{(\varrho_-,i)} > \frac{1}{c^*_{sub}} e^{\varrho_- A}\Big),$$

where $\widehat{\mu}_{A_0,\infty} = \sum_{i=1}^{\xi_{\lambda_0}} \delta_{\{z_i\}}$. By Lemma 22 with $\lambda_0 = e^{\varrho_- A_0}$ there, we get that for any $A_0 > 0$,

$$\liminf_{A\to\infty} C_s(A) \geq c_{\varrho_-}\, \mathbf{Q}\left[\int \widehat{\mu}_{A_0,\infty}(dx) e^{\varrho_+ x}\right] (c^*_{sub})^{\varrho_+/\varrho_-}.$$

Letting $A_0 \to \infty$, the above expectation term converges to $1/\mathbf{Q}[\Re^{-1}]$ and proves the lower bound.

To derive the upper bound, by Lemma 19 (iii) and Theorem 3 (ii), we get that under $\mathbf{P}(\cdot|\mathscr{H}(L) > 0)$, $\sum_{u\in\mathscr{H}(L)} e^{\varrho_-(V(u)-L)}$ is bounded in L^{k^*} and converges in law to $\sum_{i=1}^{\hat{\zeta}_\infty} e^{\varrho_- x_i}$, where $\widehat{\mu}_\infty = \sum_{i=1}^{\hat{\zeta}_\infty} \delta_{\{x_i\}}$. Therefore

$$\mathbf{Q}\left[\sum_{i=1}^{\hat{\zeta}_\infty} e^{\varrho_- x_i}\right]^{k^*} < \infty,$$

which in view of Lemma 16 and (7.16) yields, as $A \to \infty$,

$$e^{\varrho_+ A}\, \mathbf{Q}\Big(\sum_{i=1}^{\hat{\zeta}_\infty} e^{\varrho_- x_i} M_\infty^{(\varrho_-,i)} > \frac{1}{c^*_{sub}} e^{\varrho_- A}\Big) \to \frac{c_{\varrho_-}}{\mathbf{Q}[\Re^{-1}]} (c^*_{sub})^{\varrho_+/\varrho_-}.$$

Since $\widehat{\mu}_\infty$ stochastically dominates $\widehat{\mu}_{A,\infty}$, this gives the desired upper bound for $C_s(A)$ and completes the proof of the lemma. □

8. Proofs of the technical Lemmas

8.1. **Proof of Lemma 4.** Obviously we may assume that $\|F\|_\infty \leq 1$ throughout the proof of (i) and (ii).

Proof of Part (i). Since $\mathbb{P}(\tau_t^+ > K) \to 1$ as $t \to \infty$, it is enough to show that

$$\lim_{t\to\infty} \mathbb{E}\Big[1_{\{\tau_t^+ > K\}} F(T_t^+, (S_{\tau_t^+} - S_{\tau_t^+ - j})_{1\leq j\leq K})\Big] = \mathbb{E}\Big[F(U\hat{S}_{\hat{\sigma}}, (\hat{S}_j)_{1\leq j\leq K})\Big]. \tag{8.1}$$

Recall that $(\sigma_n, H_n)_{n\geq 1}$ are the strict ascending ladder epochs and ladder heights of S. Since for some (unique) $n \geq 1$, $\tau_t^+ = \sigma_n$ and $T_t^+ = H_n - t$, we can write

$$\begin{aligned} B_t &:= \mathbb{E}\Big[1_{\{\tau_t^+ > K\}} F(T_t^+, (S_{\tau_t^+} - S_{\tau_t^+ - j})_{1\leq j\leq K})\Big] \\ &= \sum_{n\geq 1} \mathbb{E}\Big[1_{\{H_{n-1}\leq t < H_n\}} 1_{\{K<\sigma_n\}} F(H_n - t, (S_{\sigma_n} - S_{\sigma_n - j})_{1\leq j\leq K})\Big]. \end{aligned}$$

Let us choose some integer $m > K$. Notice that $\sigma_n - \sigma_{n-m} > K$ and $\sigma_n > K$ for $n \geq m$. Since the previous sum for $n < m$ is smaller than $\mathbb{P}(H_m > t)$ which tends to 0 when t tends to infinity, we get

$$\begin{aligned} B_t &= \sum_{n\geq m} \mathbb{E}\Big[1_{\{H_{n-1}\leq t < H_n\}} F(H_n - t, (S_{\sigma_n} - S_{\sigma_n - j})_{1\leq j\leq K})\Big] + o_t(1) \\ &=: B'_t + o_t(1), \end{aligned}$$

with $|o_t(1)| \le \mathbf{P}(H_m > t) \to 0$ as $t \to \infty$. Applying the strong Markov property at the stopping time σ_{n-m}, we obtain that

$$\begin{aligned} B'_t &= \sum_{n\ge m} \mathbb{E}\Big[1_{\{H_{n-m}\le t\}} \mathbb{E}_{H_{n-m}}\left[1_{\{H_{m-1}\le t<H_m\}} F(H_m - t, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})\right]\Big] \\ &= \sum_{n\ge m} \mathbb{E}\left[1_{\{H_{n-m}\le t\}} g(t - H_{n-m})\right], \end{aligned}$$

with

$$g(x) := \mathbb{E}\Big[1_{\{H_{m-1}\le x<H_m\}} F(H_m - x, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})\Big], \qquad \forall\, x \ge 0.$$

Therefore

$$B'_t = \int_0^t g(t-x)\mathrm{d}u(x), \tag{8.2}$$

with $u(x) = \sum_{n\ge 0} \mathbb{P}(H_n \le x)$. Let us check that g is directly Riemann integrable on $\mathbb{R}_+$. Recall that a function g is directly Riemann integrable (see Feller [12], pp. 362) if g is continuous almost everywhere and satisfies

$$\sum_{n=0}^{\infty} \sup_{n\le x\le n+1} |g(x)| < \infty. \tag{8.3}$$

Observe first that $\|F\|_\infty \le 1$ implies $\|g\|_\infty \le 1$. Now recall that H_1 is integrable. Therefore,

$$\sum_{n\ge 0} \sup_{n\le x\le n+1} |g(x)| \le \sum_{n\ge 0} \mathbb{P}(H_m \ge n) = 1 + \mathbb{E}\left[H_m\right] = 1 + m\mathbb{E}[H_1] < \infty,$$

yielding (8.3). Now we prove that g is a.e. continuous. For $z \in \mathbb{R}_+^K$, denote by $D(z) \subset \mathbb{R}_+^*$ the set on which $F(\cdot, z)$ is discontinuous. By assumption, $D(z)$ is at most countable for any real z, hence $D((S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})$ is a random set (maybe empty) at most countable; The same is true for the random set

$$\Upsilon := \bigcup_{n=1}^{\infty} \Big\{ H_n - z : z \in D((S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K}) \cup \{0\} \Big\}.$$

In other words, we may represent Υ by a sequence of random variables taking values in $\mathbb{R}$. It follows that

$$\mathscr{D} := \Big\{ y : \mathbb{P}\big(y \in \Upsilon\big) > 0 \Big\} \quad \text{ is at most countable.}$$

We claim that for any $x \in \mathbb{R}_+^* \backslash \mathscr{D}$, g is continuous at x. In fact, for any sequence $(x_n)_n$ such that $x_n \to x$ as $n \to \infty$, let $\xi_n := 1_{\{H_{m-1}\le x_n<H_m\}} F(H_m - x_n, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})$ and $\xi := 1_{\{H_{m-1}\le x<H_m\}} F(H_m - x, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})$, we shall show that as $n \to \infty$,

$$\xi_n \to \xi, \qquad a.s., \tag{8.4}$$

which in view of the dominated convergence theorem, implies that $g(x_n) \to g(x)$ and the desired continuity of g at x. To prove (8.4), firstly we remark that

$$\limsup_{n\to\infty} |1_{\{H_{m-1}\le x_n<H_m\}} - 1_{\{H_{m-1}\le x<H_m\}}| \le 1_{\{H_{m-1}=x\}} + 1_{\{H_m = x\}} = 0, \quad a.s., \tag{8.5}$$

since $x \notin \mathscr{D}$ [hence a fortiori $\mathbb{P}(H_n = x) = 0$ for all $n \ge 1$]. Secondly,

$$\mathbb{P}(H_m - x \in D((S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})) \le \mathbb{P}(x \in \Upsilon) = 0,$$

since $x \notin \mathscr{D}$. In words, almost surely, $H_m - x \notin D((S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})$, which implies that $F(\cdot, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})$ is continuous at $H_m - x$; hence $F(H_m - x_n, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K}) \to F(H_m - x, (S_{\sigma_m} - S_{\sigma_m - j})_{1\le j\le K})$ a.s. when $n \to \infty$. This and (8.5) yield (8.4) and the continuity of g on $\mathbb{R}_+^* \backslash \mathscr{D}$. Then g is directly Riemann integrable.

Going back to (8.2), we apply the renewal theorem (see Feller [12], pp. 363) and obtain that

$$\lim_{t\to\infty} B'_t = \frac{1}{\mathbb{E}\,[H_1]}\int_0^\infty g(x)\mathrm{d}x,$$

which implies

$$\begin{aligned}\lim_{t\to\infty} B_t &= \frac{1}{\mathbb{E}\,[H_1]}\mathbb{E}\Big[\int_0^{H_m-H_{m-1}} F\big(H_m-H_{m-1}-x,(S_{\sigma_m}-S_{\sigma_m-j})_{1\le j\le K}\big)dx\Big]\\ &= \frac{1}{\mathbb{E}\,[H_1]}\mathbb{E}\Big[(H_m-H_{m-1})F\big(U(H_m-H_{m-1}),(S_{\sigma_m}-S_{\sigma_m-j})_{1\le j\le K}\big)\Big],\end{aligned}$$

by using the independent uniform variable U.

Finally since the random segments $\big\{(S_{\sigma_k+j}-S_{\sigma_k})_{0\le j\le \sigma_{k+1}-\sigma_k}\,;\,0\le k<m\big\}$ are i.i.d., Tanaka's construction (see (4.5)) implies that under $\mathbb{P}$ the segment of the random walk $(S_n)_{n\ge0}$ up to time σ_m viewed from (σ_m,S_{σ_m}) in reversed time and reflected in the x-axis, i.e. $(S_{\sigma_m}-S_{\sigma_m-j})_{0\le j\le K}$, has the same law as $(\zeta_j)_{0\le j\le K}$. Moreover since with this "partial" construction H_m-H_{m-1} corresponds to the value of the reversed and reflected process at time $\widetilde{\sigma}=\sup\{n\ge1\,:\,\zeta_n=\min_{1\le i\le n}\zeta_i\}$, we obtain that

$$\begin{aligned}&\frac{1}{\mathbb{E}\,[H_1]}\mathbb{E}\Big[(H_m-H_{m-1})F\big(U(H_m-H_{m-1}),(S_{\sigma_m}-S_{\sigma_m-j})_{1\le j\le K}\big)\Big]\\ =\;&\frac{1}{\mathbb{E}\,[H_1]}\mathbb{E}\Big[\zeta_{\widetilde{\sigma}}\,F\big(U\zeta_{\widetilde{\sigma}},(\zeta_j)_{1\le j\le K}\big)\Big]=\mathbb{E}\Big[F\big(U\hat{S}_{\hat{\sigma}},(\hat{S}_j)_{1\le j\le K}\big)\Big],\end{aligned}$$

by using (4.6). This proves (8.1) and the part (i) of the lemma.

Proof of Part (ii) Write for notational convenience $\widetilde{S}_j^{(t)}:=S_{\tau_t^+}-S_{\tau_t^+-j}$ when $1\le j\le\tau_t^+$. Note that Part (i) of the lemma implies

$$\lim_{L\to\infty}\mathbb{E}\left[1_{\{K<\tau_L^+\}}F(T_L^+,(\widetilde{S}_j^{(L)})_{1\le j\le K})\right]=\mathbb{E}\left[F(U\hat{S}_{\hat{\sigma}},(\hat{S}_j)_{1\le j\le K})\right]=:C_F. \tag{8.6}$$

Using the absolute continuity between $\mathbb{P}^+$ and $\mathbb{P}$ up to the stopping time τ_t^+ [the martingale $(R(S_j)1_{(j<\tau_t^+)},j\le\tau_t^+)$ is uniformly integrable thanks to Lemma 3 (ii) and (iv)], we can write

$$\mathbb{E}^+\left[1_{\{K<\tau_t^+\}}F(T_t^+,(\widetilde{S}_j^{(t)})_{1\le j\le K})\right]=\mathbb{E}\left[R(S_{\tau_t^+})1_{\{K<\tau_t^+<\tau_0^-\}}F(T_t^+,(\widetilde{S}_j^{(t)})_{1\le j\le K})\right].$$

We treat first the case $\mathbf{E}[S_1]=0$. Combining Parts (iii) and (iv) of Lemma 3, we deduce from the above equality that as $t\to\infty$,

$$\mathbb{E}^+\left[1_{\{K<\tau_t^+\}}F(T_t^+,(\widetilde{S}_j^{(t)})_{1\le j\le K})\right]\sim C_R t\,\mathbb{E}\left[1_{\{K<\tau_t^+<\tau_0^-\}}F(T_t^+,(\widetilde{S}_j^{(t)})_{1\le j\le K})\right]=:A_t.$$

Let us now introduce $\ell_t:=t-2t^\gamma$ with $(1+\delta/2)^{-1}<\gamma<1$ and observe that $\tau_{\ell_t}^+<\tau_0^-$ on the event $\{\tau_t^+<\tau_0^-\}$. Recalling that Part (ii) of Lemma 3 says that $(T_t^+,t\ge0)$ is bounded in L^p for all $1<p<1+\delta$, we get $\mathbb{P}(T_{\ell_t}^+>t^\gamma)\le c\,t^{-\gamma p}=o(t^{-1})$ by choosing p such that $\gamma p>1$. Therefore we obtain

$$\begin{aligned}A_t &= C_R t\,\mathbb{E}\left[1_{\{K<\tau_t^+<\tau_0^-\}}1_{\{S_{\tau_{\ell_t}^+}\le t-t^\gamma\}}F(T_t^+,(\widetilde{S}_j^{(t)})_{1\le j\le K})\right]+o_t(1)\\ &= A'_t+A''_t+o_t(1),\end{aligned}$$

where $o_t(1) \to 0$ as $t \to \infty$ and

$$A'_t := C_R t\, \mathbb{E}\left[1_{\{\tau_t^+ < \tau_0^-\}} 1_{\{S_{\tau^+_{\ell_t}} \le t - t^\gamma\}} 1_{\{\tau_t^+ - \tau^+_{\ell_t} > K\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{1\le j\le K})\right],$$

$$A''_t := C_R t\, \mathbb{E}\left[1_{\{K<\tau_t^+ < \tau_0^-\}} 1_{\{S_{\tau^+_{\ell_t}} \le t - t^\gamma\}} 1_{\{\tau_t^+ - \tau^+_{\ell_t} \le K\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{1\le j\le K})\right].$$

Applying the strong Markov property at the stopping time $\tau^+_{\ell_t}$ yields

$$A'_t = C_R t\, \mathbb{E}\Big[1_{\{\tau^+_{\ell_t} < \tau_0^-\}} 1_{\{S_{\tau^+_{\ell_t}} \le t-t^\gamma\}} f(S_{\tau^+_{\ell_t}})\Big],$$

where

$$f(x) := \mathbb{E}_x\left[1_{\{K<\tau_t^+<\tau_0^-\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{1\le j\le K})\right]. \tag{8.7}$$

Then, writing

$$\mathbb{E}_x[1_{\{K<\tau_t^+\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{1\le j\le K})] = \mathbb{E}[1_{\{K<\tau_L^+\}} F(T_L^+, (\widetilde{S}_j^{(L)})_{1\le j\le K})],$$

with $L = t - x$, Equation (8.6) yields

$$\max_{x\in[\ell_t; t-t^\gamma]} \left|\mathbb{E}_x\left[1_{\{K<\tau_t^+\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{0\le j\le K})\right] - C_F\right| \longrightarrow 0, \qquad t\to\infty. \tag{8.8}$$

from which we deduce

$$\max_{x\in[\ell_t; t-t^\gamma]} |f(x) - C_F| \longrightarrow 0, \qquad t\to\infty,$$

since uniformly in $x \ge \ell_t$, $\mathbb{P}_x(\tau_0^- < \tau_t^+) = \mathbb{P}(\tau^-_{-x} < \tau^+_{t-x}) \le \mathbb{P}(\tau^-_{-\ell_t} < \tau^+_{t^\gamma}) = o_t(1)$. Furthermore, observing that $\mathbb{P}(\tau^+_{\ell_t} < \tau_0^-) \sim \frac{1}{C_R t}$ (see Part (v) of Lemma 3 and recall that $\ell_t = t - 2t^\gamma$ with $\gamma < 1$) and $\mathbb{P}(t - S_{\tau^+_{\ell_t}} \le t^\gamma) = \mathbb{P}(T^+_{\ell_t} > t^\gamma) = o(t^{-1})$ imply $\mathbb{P}(\tau^+_{\ell_t} < \tau_0^-\,;\, S_{\tau^+_{\ell_t}} \le t - t^\gamma) \sim 1/C_R t$, when t tends to infinity, we obtain

$$A'_t \longrightarrow C_F, \qquad t\to\infty. \tag{8.9}$$

Similarly, the strong Markov property applied at the stopping time $\tau^+_{\ell_t}$ implies

$$A''_t \le C_R t\, \mathbb{E}\Big[1_{\{\tau^+_{\ell_t}<\tau_0^-\}} 1_{\{S_{\tau^+_{\ell_t}} \le t-t^\gamma\}} \mathbb{P}_{S_{\tau^+_{\ell_t}}}(\tau_t^+ \le K)\Big].$$

Moreover, observe that

$$\sup_{x\le t-t^\gamma} \mathbb{P}_x(\tau_t^+ \le K) \le \mathbb{P}_{t-t^\gamma}(\tau_t^+ \le K) = \mathbb{P}(\tau^+_{t^\gamma} \le K) = o_t(1), \tag{8.10}$$

which implies

$$A''_t \le C_R t\, \mathbb{P}(\tau^+_{\ell_t} < \tau_0^-\,,\, S_{\tau^+_{\ell_t}} \le t - t^\gamma)\, \mathbb{P}(\tau^+_{t^\gamma} \le K) = o_t(1), \tag{8.11}$$

by recalling that $\mathbb{P}(\tau^+_{\ell_t} < \tau_0^-\,;\, S_{\tau^+_{\ell_t}} \le t - t^\gamma) \sim \frac{1}{C_R t}$. Combining (8.9), (8.11) and recalling (8.7), we obtain $A_t \to C_F$, when $t \to \infty$, which concludes the proof of Part (ii) in the case $\mathbb{E}[S_1] = 0$.

The case $\mathbb{E}[S_1] > 0$ is similar but easier. Indeed, combining Parts (iii) and (iv) of Lemma 3 implies

$$\mathbb{E}^+\left[1_{\{K<\tau_t^+\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{1\le j\le K})\right] \sim C_R\, \mathbb{E}\left[1_{\{K<\tau_t^+<\tau_0^-\}} F(T_t^+, (\widetilde{S}_j^{(t)})_{1\le j\le K})\right] =: \widetilde{A}_t.$$

Recalling that $\ell_t = t - 2t^\gamma$ and that Part (ii) of Lemma 3 implies $\mathbb{P}(T^+_{\ell_t} > t^\gamma) = o_t(1)$, we get

$$\begin{aligned}\widetilde{A}_t &= C_R\,\mathbb{E}\left[1_{\{K<\tau_t^+<\tau_0^-\}}1_{\{S_{\tau^+_{\ell_t}}\le t-t^\gamma\}}F(T_t^+,(\widetilde{S}^{(t)}_j)_{1\le j\le K})\right]+o_t(1)\\ &= C_R\,\mathbb{E}\left[1_{\{\tau_t^+<\tau_0^-\}}1_{\{S_{\tau^+_{\ell_t}}\le t-t^\gamma\}}1_{\{\tau_t^+-\tau^+_{\ell_t}>K\}}F(T_t^+,(\widetilde{S}^{(t)}_j)_{1\le j\le K})\right]+o_t(1),\end{aligned}\tag{8.12}$$

the last equality being a consequence of (8.10), which still holds in the case $\mathbb{E}[S_1] > 0$. Then, the strong Markov property yields

$$\widetilde{A}_t = C_R\,\mathbb{E}\Big[1_{\{\tau^+_{\ell_t}<\tau_0^-\}}1_{\{S_{\tau^+_{\ell_t}}\le t-t^\gamma\}}f(S_{\tau^+_{\ell_t}})\Big]+o_t(1),\tag{8.13}$$

where we recall that the function f is defined by (8.7). Now the strategy is exactly the same as for the previous case. Indeed, since $\mathbb{P}_x(\tau_0^- < \tau_t^+) = o_t(1)$ (uniformly in $x \ge \ell_t$) is still true, (8.6) implies $\max_{x\in[\ell_t;t-t^\gamma]}|f(x) - C_F| \to 0$, when t tends to ∞. Combining this with Part (v) of Lemma 3 (which implies $\mathbb{P}(\tau^+_{\ell_t} < \tau_0^-\,;\, S_{\tau^+_{\ell_t}} \le t - t^\gamma) \to 1/C_R$) yields $\widetilde{A}_t \to C_F$, when $t \to \infty$. This concludes the proof of Part (ii) of the lemma and completes the proof of Lemma 4. □

Proof of Lemma 5: We may assume that p equals some integer, say, $m \ge 1$. Indeed, for any $m-1 < p \le m$, by the concavity,

$$\mathbb{E}_x\left[\sum_{k=0}^{\tau_t^+-1} a_{k+1}\,e^{\kappa(S_k-t)}\right]^p \le \mathbb{E}_x\left[\sum_{k=0}^{\tau_t^+-1}(a_{k+1})^{p/m}\,e^{\kappa p(S_k-t)/m}\right]^m.$$

Applying (4.7) to $((a_{k+1})^{p/m}, S_k - S_{k-1})$ with integer m yields the general case p.

Now, we consider $p = m$ is some integer and prove (4.7). Firstly,

$$\mathbb{E}\left[\sum_{k=0}^{\tau_t^+-1} e^{\kappa(S_k-t)}\right] \le \sum_{k=0}^{\infty}\mathbb{E}\left[1_{\{\overline{S}_k\le t\}}e^{\kappa(\overline{S}_k-t)}\right] = \int_0^t e^{-\kappa(t-y)}\,\mathrm{d}u(y),$$

where $\overline{S}_k := \max\{S_j : 0 \le j \le k\}$ and

$$u(y) := \sum_{n=0}^{\infty}\mathbb{P}\Big(\overline{S}_n \le y\Big), \qquad y \ge 0.$$

Remark that u is finite and satisfies the following renewal equation (see Heyde [14], Theorem 1):

$$u(y) = 1_{\{0\le y\}} + F * u(y), \qquad y \ge 0,$$

with $F(s) := \mathbb{P}(S_1 \le s), s \in \mathbb{R}$. According to the renewal theorem (see Heyde [14], Theorem 2 or Feller [12] pp. 362 (1.17) and pp. 381), $\int_0^t e^{-\kappa(t-y)}\,\mathrm{d}u(y) = O(1)$ as $t \to \infty$ (the limit exists in the non-arithmetic case). By linear transformation, we obtain that for any $\kappa > 0$, $\mathbb{E}_x\left[\sum_{k=0}^{\tau_t^+-1} e^{\kappa(S_k-t)}\right]$ is uniformly bounded for all $x \le t$.

We now prove the lemma by induction on m. By independence, $\mathbb{E}_x\left[\sum_{k=0}^{\tau_t^+-1} a_{k+1}\,e^{\kappa(S_k-t)}\right] = \sum_{k\ge 0}\mathbb{E}_x\left[e^{\kappa(S_k-t)}, k < \tau_t^+ - 1\right]\mathbb{E}[a_1]$ is bounded by some constant (the law of a_{k+1} does not depend on x), this proves the lemma in the case $m = 1$.

Let $m \geq 2$ and assume that the lemma holds for $1, ..., m-1$. Write $\chi_i := \sum_{k=i}^{\tau_t^+-1} a_{k+1} e^{\kappa(S_k - t)}$ for $0 \leq i < \tau_t^+$ and $\chi_{\tau_t^+} := 0$. Remark that

$$(\chi_0)^m = \sum_{i=0}^{\tau_t^+-1} [(\chi_i)^m - (\chi_{i+1})^m] = \sum_{j=0}^{m-1} C_m^j \sum_{i=0}^{\tau_t^+-1} a_{i+1}^{m-j} \, e^{(m-j)\kappa(S_i - t)} \, (\chi_{i+1})^j.$$

Applying the Markov property at $i+1$, we get

$$\begin{aligned} \mathbb{E}_x \left[\chi_0^m\right] &= \sum_{j=0}^{m-1} C_m^j \mathbb{E}_x \left[\sum_{i=0}^{\tau_t^+-1} a_{i+1}^{m-j} \, e^{(m-j)\kappa(S_i - t)} \, \mathbb{E}_{S_i}\left[(\chi_{i+1})^j\right] \right] \\ &\leq c \sum_{j=0}^{m-1} \mathbb{E}_x \left[\sum_{i=0}^{\tau_t^+-1} a_{i+1}^{m-j} \, e^{(m-j)\kappa(S_i - t)} \right], \end{aligned}$$

since by the induction hypothesis $\mathbb{E}_{S_i}\left[(\chi_{i+1})^{m-j}\right]$ is bounded by some constant. The last expectation is again uniformly bounded (the case $m = 1$ of the lemma), which proves that the lemma holds for m, as desired. □

8.2. **Proof of Lemma 6.** For $a \in \mathbb{R}$, denote as before by $T_a^+ := S_{\tau_a^+} - a > 0$ (resp. $T_a^- := a - S_{\tau_a^-} > 0$) the overshoot (resp. undershoot) at level a. Clearly the overshoot T_a^+ is also the overshoot at the level a for the strict ascending ladder heights (H_n). By the assumption (4.8), $\max(S_1, 0)$ has finite η-exponential moment. This in view of Doney [11] implies that $\mathbb{E}[e^{\delta H_1}] < \infty$ for any $0 < \delta < \eta$. Applying Chang ([10], Proposition 4.2) shows that for any $0 < \delta < \eta$, there exist some constant $c = c(\delta) > 0$ such that for all $b \geq a, x > 0$,

$$\mathbb{P}_a(T_b^+ > x) \leq c e^{-\delta x}. \tag{8.14}$$

Similarly for the undershoot $T_a^- > 0$: since $\max(-S_1, 0)$ has a finite $(1+\eta)$-exponential moment, we get that for any $0 < \delta < \eta$,

$$\mathbb{P}_b(T_a^- > x) \leq c e^{-(1+\delta)x}, \qquad \forall\, a \leq b, \forall\, x > 0. \tag{8.15}$$

By (8.14) and (8.15), $\max_{0 \leq k \leq \tau_0^- \wedge \tau_L^+} |S_k| \leq L + T_L^+ + T_0^-$ is integrable under $\mathbb{P}_a$. By applying the optional stopping theorem, we get

$$a = \mathbb{E}_a\Big[S_{\tau_0^- \wedge \tau_L^+}\Big] = \mathbb{E}_a\Big[(S_{\tau_0^-} - S_{\tau_L^+}) 1_{\{\tau_0^- < \tau_L^+\}}\Big] + \mathbb{E}_a[S_{\tau_L^+}].$$

Observe that $\mathbb{E}_a[S_{\tau_L^+}] = L + \mathbb{E}_a[T_L^+] \leq L + \frac{c}{\delta}$ by (8.14). Since $S_{\tau_0^-} - S_{\tau_L^+} < -L$, we obtain

$$\mathbb{P}_a\Big(\tau_0^- < \tau_L^+\Big) \leq \frac{L - a + c'}{L}, \qquad \forall 0 \leq a \leq L. \tag{8.16}$$

Exactly doing the same and using (8.15), we get

$$\mathbb{P}_a\Big(\tau_0^- > \tau_L^+\Big) \leq \frac{a + c'}{L}, \qquad \forall\, 0 \leq a \leq L. \tag{8.17}$$

Let us also mention that by considering the martingale $(S_j^2 - Var(S_1) j)_{j \geq 1}$, which is uniformly integrable on $[0, \tau_0^- \wedge \tau_L^+]$, we can find some constant $c > 0$ such that for all $L > 1$ and $0 \leq a \leq L$,

$$\mathbb{E}_a\Big[\tau_0^- \wedge \tau_L^+\Big] \leq c L^2. \tag{8.18}$$

(i) Proof of (4.9): If $L-a\ge\frac{L}{3}$, we deduce from (8.15) that $\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\Big]\le\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}\Big]\le c$ which is less than $c'\frac{L-a+1}{L}$ if $c'\ge 3c$.

Let $0<L-a<\frac{L}{3}$. Note that under $\mathbb{P}_a$, $\tau_0^-<\tau_L^+$ implies that $\tau_{L/2}^-\le\tau_0^-<\tau_L^+$. Then by the strong Markov property at $\tau_{L/2}^-$,

$$\begin{aligned}\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\Big] &= \mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_{L/2}^-\le\tau_0^-<\tau_L^+\}}\Big]\\ &= \mathbb{E}_a\Big[1_{\{\tau_{L/2}^-<\tau_L^+\}}\mathbb{E}_{S_{\tau_{L/2}^-}}\big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\big]\Big]\\ &\le \mathbb{E}_a\Big[1_{\{\tau_{L/2}^-<\tau_L^+\}}\big(c+e^{-S_{\tau_{L/2}^-}}1_{\{S_{\tau_{L/2}^-}<0\}}\big)\Big],\end{aligned}$$

where we use the fact that for all $z:=S_{\tau_{L/2}^-}\ge 0$, $\mathbb{E}_z\big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\big]\le\mathbb{E}_z[e^{-S_{\tau_0^-}}]\le c$ by (8.15). Since $S_{\tau_{L/2}^-}<0$ means that $T_{L/2}^-\ge L/2$, we deduce from (8.15) that

$$\mathbb{E}_a\Big[e^{-S_{\tau_{L/2}^-}}1_{\{S_{\tau_{L/2}^-}<0\}}\Big]=\mathbb{E}_a\Big[e^{\frac{L}{2}+T_{L/2}^-}1_{\{T_{L/2}^-\ge L/2\}}\Big]\le ce^{-\delta L/2}.$$

This together with (8.16) give that

$$\begin{aligned}\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\Big] &\le c\mathbb{P}_a\Big(\tau_{L/2}^-<\tau_L^+\Big)+ce^{-\delta L/2}\\ &= c\mathbb{P}_{a-L/2}\Big(\tau_0^-<\tau_{L/2}^+\Big)+ce^{-\delta L/2}\\ &\le c\frac{L-a+c'}{(L/2)}+ce^{-\delta L/2}\\ &\le c''\frac{L-a+1}{L}.\end{aligned}$$

(ii) Proof of (4.10): Let us show that $\mathbb{E}[\sum_{j=0}^{\tau_0^--1}e^{-\delta S_j}]<\infty$:

$$\mathbb{E}\Big[\sum_{j=0}^{\tau_0^--1}e^{-\delta S_j}\Big]=\sum_{j\ge0}\mathbb{E}\Big[e^{-\delta S_j},\,j<\tau_0^-\Big]\le\sum_{j\ge0}c\,(1+j)^{-3/2}<\infty,$$

where we used Theorem 4 (and Theorem 6 if S_1 is lattice) of [32] for the bound of $\mathbb{E}\Big[e^{-\delta S_j},\,j<\tau_0^-\Big]$. Let $(H_n^-,\sigma_n^-)_{n\ge0}$ be the strict ascending ladder heights and epochs of $-S$ (with $\sigma_0^-:=0$). For $a>0$, we notice that

$$\begin{aligned}\mathbb{E}_a\Big[\sum_{j=0}^{\tau_0^--1}e^{-\delta S_j}\Big] &= \mathbb{E}\Big[\sum_{j=0}^{\tau_{-a}^--1}e^{-\delta(a+S_j)}\Big]\\ &= \sum_{n=0}^{\infty}\mathbb{E}\Big[\sum_{\sigma_n^-\le j<\sigma_{n+1}^-}e^{-\delta(a+S_j)}\,1_{\{H_n^-\le a\}}\Big]\\ &= \sum_{n=0}^{\infty}\mathbb{E}\Big[e^{-\delta(a-H_n^-)}\,1_{\{H_n^-\le a\}}\Big]\,\mathbb{E}\Big[\sum_{j=0}^{\tau_0^--1}e^{-\delta S_j}\Big],\end{aligned}$$

by applying the strong Markov property at σ_n^-. We showed that $\mathbb{E}\Big[\sum_{j=0}^{\tau_0^- -1} e^{-\delta S_j}\Big] < \infty$. On the other hand, Lemma 5 applied to the random walk $(H_n^-)_{n\ge 0}$ says that

$$\sup_{a>0}\sum_{n=0}^{\infty}\mathbb{E}\Big[e^{-\delta(a-H_n^-)}\,1_{\{H_n^-\le a\}}\Big] < \infty.$$

Hence $\sup_{a\ge 0}\mathbb{E}_a\Big[\sum_{j=0}^{\tau_0^- -1} e^{-\delta S_j}\Big] < \infty$. Similary, by considering the random walk $L - S.$, we get that $\mathbb{E}_a\Big[\sum_{j=0}^{\tau_L^+ -1} e^{-\delta(L-S_j)}\Big]$ is uniformly bounded by some constant. This proves (4.10).

(iii) Proof of (4.11). Discussing on the value of the time τ_0^- then using the Markov property, we have

$$\begin{aligned}\mathbb{E}_a\Big[e^{S_{\tau_0^- -1}-S_{\tau_0^-}}\Big] &\le \sum_{k\ge 1}\mathbb{E}_a\Big[e^{S_{k-1}-S_k}1_{\{\tau_0^-=k\}}\Big]\\ &= \sum_{k\ge 1}\mathbb{E}_a\Big[h(-S_{k-1})1_{\{\tau_0^-\ge k\}}\Big]\end{aligned}$$

where for any $y\in\mathbb{R}$, $h(y) := \mathbb{E}[e^{-S_1}1_{\{S_1\le y\}}] \le e^{\delta y}\mathbb{E}[e^{-(1+\delta)S_1}] = ce^{\delta y}$ for $\delta > 0$ small enough. Hence,

$$\mathbb{E}_a\Big[e^{S_{\tau_0^- -1}-S_{\tau_0^-}}\Big] \le c\mathbb{E}_a\Big[\sum_{k=0}^{\tau_0^- -1} e^{-\delta S_k}\Big]$$

and (4.11) follows from (4.10).

(iv) Proof of (4.12) and (4.13): Clearly (4.13) follows from (4.12) by considering the random walk $(L-S_j)_{j\ge 0}$. It suffices to prove (4.12). If $L - a \ge L/3$, there is nothing to prove since $\mathbb{E}_a\Big[\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\Big] \le \mathbb{E}_a\Big[\sum_{0\le j<\tau_0^-} e^{-\delta S_j}\Big]$ is less than some constant by (4.10).

Considering $L - a < L/3$. We have

$$\begin{aligned}&\mathbb{E}_a\Big[\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\Big]\\ =\;& \mathbb{E}_a\Big[1_{\{\tau_{L/2}^-\ge\tau_0^-\wedge\tau_L^+\}}\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\Big] + \mathbb{E}_a\Big[1_{\{\tau_{L/2}^-<\tau_0^-\wedge\tau_L^+\}}\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\Big]\\ \le\;& \mathbb{E}_a\Big[e^{-\delta L/2}\,\tau_0^-\wedge\tau_L^+\Big] + \mathbb{E}_a\Big[1_{\{\tau_{L/2}^-<\tau_0^-\wedge\tau_L^+\}}\sum_{\tau_{L/2}^-\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\Big]\\ \le\;& cL^2e^{-\delta L/2} + \mathbb{E}_a\Big[1_{\{\tau_{L/2}^-<\tau_0^-\wedge\tau_L^+\}}\mathbb{E}_{S_{\tau_{L/2}^-}}\big[\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\big]\Big],\end{aligned}$$

by using (8.18) and the strong Markov property at $\tau_{L/2}^-$. Let $x := S_{\tau_{L/2}^-} < L/2$. If $x < 0$, then under $\mathbb{P}_x$, $\tau_0^- = 0$ and $\mathbf{E}_x\big[\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\big] = 0$, whereas if $0 \le x < L/2$,

$\mathbb{E}_x\big[\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\big] \le c$ by (4.10). Then we get

$$\begin{aligned}\mathbb{E}_a\Big[\sum_{0\le j<\tau_0^-\wedge\tau_L^+} e^{-\delta S_j}\Big] &\le cL^2e^{-\delta L/2}+c\mathbb{P}_a\Big(\tau_{L/2}^-<\tau_0^-\wedge\tau_L^+\Big)\\ &\le cL^2e^{-\delta L/2}+c\mathbb{P}_a\Big(\tau_{L/2}^-<\tau_L^+\Big)\\ &\le cL^2e^{-\delta L/2}+c\,\frac{L-a+c'}{L/2},\end{aligned}$$

by using (8.16). This proves (4.12).

(v) Proof of (4.14): By monotonicity, it is sufficient to prove (4.14) for $0<\delta<\eta$. Then, notice that

$$\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{0\le j<\tau_0^-}e^{-\delta(L-S_j)}\Big]=\sum_{n=1}^{\infty}\mathbb{E}_a\Big[1_{\{n\le\tau_L^+\wedge\tau_0^-,\,S_n<0\}}e^{-S_n}\sum_{0\le j<n}e^{-\delta(L-S_j)}\Big].$$

Applying the Markov property of S at $n-1$ and using the fact that for all $x\ge 0$, $\mathbb{E}_x[e^{-S_1}1_{\{S_1<0\}}]=\mathbb{E}[e^{-x-S_1}1_{\{S_1<-x\}}]\le c(\delta)e^{-(1+\delta)x}$ by (4.8) (recall that $0<\delta<\eta$), we get that

$$\begin{aligned}&\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{0\le j<\tau_0^-}e^{-\delta(L-S_j)}\Big]\\ \le\;& c\sum_{n=1}^{\infty}\mathbb{E}_a\Big[1_{\{n\le\tau_L^+\wedge\tau_0^-\}}e^{-(1+\delta)S_{n-1}}\sum_{0\le j<n}e^{-\delta(L-S_j)}\Big]\\ =\;& c\sum_{j=0}^{\infty}\mathbb{E}_a\Big[1_{\{j<\tau_L^+\wedge\tau_0^-\}}e^{-\delta(L-S_j)}\mathbb{E}_{S_j}\big[\sum_{0\le m<\tau_L^+\wedge\tau_0^-}e^{-(1+\delta)S_m}\big]\Big],\end{aligned}\tag{8.19}$$

where the last equality follows from the Markov property at j. Applying (4.12) and (4.13), we get that

$$\begin{aligned}&\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\sum_{0\le j<\tau_0^-}e^{-\delta(L-S_j)}\Big]\\ \le\;& c\sum_{j=0}^{\infty}\mathbb{E}_a\Big[1_{\{j<\tau_L^+\wedge\tau_0^-\}}e^{-\delta(L-S_j)}c\frac{L-S_j+1}{L}\Big]\\ \le\;& \frac{c'}{L}\mathbb{E}_a\Big[\sum_{0\le j<\tau_L^+\wedge\tau_0^-}e^{-\frac{\delta}{2}(L-S_j)}\Big]\le c\,\frac{a+1}{L^2},\end{aligned}$$

proving (4.14).

□

We mention that (8.19) also holds with $\delta=0$, which implies that

$$\mathbb{E}_a\Big[e^{-S_{\tau_0^-}}1_{\{\tau_0^-<\tau_L^+\}}\tau_0^-\Big]\le c\mathbb{E}_a[\tau_0^-\wedge\tau_L^+]\le c'L^2,\qquad\forall L\ge1, 0\le a\le L.\tag{8.20}$$

8.3. **Proof of Lemmas 7 and 8.** Keeping the notation T_a^- for the undershoot at level a, we have as before for any $0<r<\eta_1$,

$$\mathbb{P}_b(T_a^->x)\le c(r)e^{-r\,x},\qquad\forall\,a\le b,\,\forall\,x>0.\tag{8.21}$$

Proof of Lemma 7:

(i) Proof of (4.18). It is a straightforward consequence of (8.21).

(ii) Proof of (4.19). Let us introduce the tilted measure $\tilde{\mathbb{P}}_a$ defined by $\frac{d\tilde{\mathbb{P}}_a}{d\mathbb{P}_a}\big|_{\sigma(S_0,\cdots,S_n)} := e^{\gamma(S_n-S_0)}$. Under $\tilde{\mathbb{P}}_a$, the random walk drifts to $+\infty$. We write

$$\begin{aligned}\mathbb{E}_a\Big[\sum_{0\le \ell<\tau_L^+}(1+L-S_\ell)^\alpha e^{r\,S_\ell}\Big] &= \sum_{\ell\ge 0}\mathbb{E}_a[(1+L-S_\ell)^\alpha e^{r\,S_\ell}\,1_{\{\ell<\tau_L^+\}}]\\ &= e^{\gamma a}\sum_{\ell\ge 0}\tilde{\mathbb{E}}_a[(1+L-S_\ell)^\alpha e^{(r-\gamma)S_\ell}1_{\{\ell<\tau_L^+\}}]\\ &= e^{\gamma a}e^{(r-\gamma)L}\tilde{\mathbb{E}}_a\Big[\sum_{0\le\ell<\tau_L^+}(1+L-S_\ell)^\alpha e^{(r-\gamma)(S_\ell-L)}\Big]\\ &\le ce^{\gamma a}e^{(r-\gamma)L}\tilde{\mathbb{E}}_a\Big[\sum_{0\le\ell<\tau_L^+}e^{(r-\gamma)(S_\ell-L)/2}\Big].\end{aligned}$$

Therefore, we only have to show that

$$\sup_{a\ge 0}\tilde{\mathbb{E}}_a\Big[\sum_{0\le\ell<\tau_L^+}e^{(r-\gamma)(S_\ell-L)/2}\Big]\le c,$$

which is done by the same argument as in the proof of (4.10).

(iii) Proof of (4.20). We have

$$\begin{aligned}\mathbb{E}_a\Big[\sum_{\ell=0}^{\min(\tau_0^-,\tau_L^+)}(1+L-S_\ell)^\alpha e^{\gamma S_\ell}\Big] &= e^{\gamma a}\tilde{\mathbb{E}}_a\Big[\sum_{\ell=0}^{\min(\tau_0^-,\tau_L^+)}(1+L-S_\ell)^\alpha\Big]\\ &= e^{\gamma a}\tilde{\mathbb{E}}\Big[\sum_{\ell=0}^{\min(\tau_{-a}^-,\tau_{L-a}^+)}(1+L-a-S_\ell)^\alpha\Big].\end{aligned}$$

Remark that $(1+L-a-S_\ell)^\alpha \le c(1+L-a)^\alpha + c|S_\ell|^\alpha 1_{\{S_\ell<0\}}$ and that $\tilde{\mathbb{E}}\Big[\sum_{\ell\ge 0}|S_\ell|^\alpha 1_{\{S_\ell<0\}}\Big]<\infty$ (indeed observe that for any $\gamma'\in(0,\gamma)$ there exists $c(\alpha,\gamma')$ such that $\sum_{\ell\ge0}|S_\ell|^\alpha 1_{\{S_\ell<0\}}\le c(\alpha,\gamma')\sum_{\ell\ge 0}e^{-\gamma' S_\ell}$, whose expectation under $\tilde{\mathbb{P}}$ is finite, see Kesten [21]). Therefore, we get

$$\tilde{\mathbb{E}}\Big[\sum_{\ell=0}^{\min(\tau_{-a}^-,\tau_{L-a}^+)}(1+L-a-S_\ell)^\alpha\Big]\le c'(1+L-a)^\alpha\,\tilde{\mathbb{E}}\left[\tau_{L-a}^+\right]+c'\le c(1+L-a)^{\alpha+1},$$

which completes the proof of the lemma. □

Proof of Lemma 8: Firstly, we remark that it is enough to prove the lemma for integer p. In fact, let $k-1<p\le k$ with some integer k and assume that (i) holds for k in lieu of p. Then by concavity,

$$\mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\Big(\sum_{\ell=1}^{\tau_0^-}e^{bS_{\ell-1}}a_\ell\Big)^p\right]\le \mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\Big(\sum_{\ell=1}^{\tau_0^-}e^{\frac{pb}{k}S_{\ell-1}}(a_\ell)^{p/k}\Big)^k\right].$$

Applying (i) to $(S_\ell-S_{\ell-1},a_\ell^{p/k})$ with $\frac{pb}{k}$ in lieu of b gives (4.21). The same is true for (ii).

Now we assume p integer and we shall use the Markov property to expand the power. Let either $\chi:=\tau_0^-$ or $\chi:=\min(\tau_0^-,\tau_L^+)$ and consider a measurable function $f:\mathbb{R}\to\mathbb{R}_+$. Define

$$A_{\chi,f}(x,k):=\mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\Big(\sum_{\ell=1}^{\chi}f(S_{\ell-1})\,a_\ell\Big)^k\right],\qquad k\ge 0,\ x\in\mathbb{R},$$

and we mention that $A_{\chi,f}(x,0) = e^{-\eta x}$ if $x < 0$, $A_\chi(x,k) = 0$ if $x < 0$ and $k \geq 1$. Let $k \geq 1$ and $Y_i := \sum_{\ell=i}^{\chi} f(S_{\ell-1})\, a_\ell$ for $1 \leq i \leq \tau_0^-$, $Y_{\chi+1} := 0$. Then

$$Y_1^k = \sum_{i=1}^{\chi}(Y_i^k - Y_{i+1}^k) = \sum_{r=1}^{k} C_k^r \sum_{i=1}^{\chi}(f(S_{i-1}))^r (a_i)^r (Y_{i+1})^{k-r}.$$

Applying the Markov property at i gives that

$$\begin{aligned} A_\chi(x,k) &= \sum_{r=1}^{k} C_k^r \sum_{i=1}^{\infty} \mathbb{E}_x\left[1_{\{i\leq\chi\}}(f(S_{i-1}))^r (a_i)^r A_{\chi,f}(S_i, k-r)\right] \\ &= B_\chi(x,k) + C_\chi(x,k), \end{aligned} \tag{8.22}$$

with

$$\begin{aligned} B_\chi(x,k) &:= \sum_{r=1}^{k} C_k^r \sum_{i=1}^{\infty} \mathbb{E}_x\left[1_{\{i\leq\chi, S_i\geq 0\}}(f(S_{i-1}))^r (a_i)^r A_{\chi,f}(S_i, k-r)\right], \\ C_\chi(x,k) &:= \sum_{i=1}^{\infty} \mathbb{E}_x\left[1_{\{i\leq\chi, S_i< 0\}}(f(S_{i-1}))^k (a_i)^k e^{-\eta S_i}\right]. \end{aligned}$$

In the rest of the proof of the lemma, we shall use twice the notations $A_\chi(x,k)$, $B_\chi(x,k)$, $C_\chi(x,k)$ but without the subscript χ and take $\chi = \tau_0^-$, $f(y) = e^{by}$ in the proof of (i) and $\chi = \min(\tau_0^-, \tau_L^+)$, $f = (L-y+1)^\alpha e^{by}$ in the proof of (ii).

Proof of (i): Let in this proof $A(x,k) = \mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\left(\sum_{\ell=1}^{\tau_0^-} e^{bS_{\ell-1}} a_\ell\right)^k\right]$. We prove (4.21) by induction on k.

The case $k = 0$ follows from (4.18). Let $1 \leq k < \gamma/b$ and assume that we know that $A(x,j) \leq c_j\, e^{j\,b\,x}$ for all $0 \leq j \leq k-1$ and $x \geq 0$. We have to show that $A(x,k) \leq c_k\, e^{k\,b\,x}$.

Using the induction hypothesis, $A(S_\ell, k-r) \leq c_{k-r} e^{(k-r)b\,S_\ell}$ if $S_\ell \geq 0$. From (8.22), we have

$$\begin{aligned} B(x,k) &\leq c\sum_{r=1}^{k}\sum_{\ell\geq 1}\mathbb{E}_x\left[e^{kb\,S_{\ell-1}}(a_\ell)^r\, e^{(k-r)b\,\Delta S_\ell},\ \ell \leq \tau_0^-\right] \\ &\leq c\sum_{r=1}^{k}\sum_{\ell\geq 1}\mathbb{E}_x\left[e^{kb\,S_{\ell-1}}(a_\ell)^r\, e^{(k-r)b\,\Delta S_\ell}\right], \end{aligned}$$

with $\Delta S_\ell := S_\ell - S_{\ell-1}$ for $\ell \geq 1$. By the independence of $(a_\ell, \Delta S_\ell)$, we get that

$$\begin{aligned} B(x,k) &\leq c\sum_{r=1}^{k}\mathbb{E}_x\left[(a_1)^r\, e^{(k-r)b\,\Delta S_1}\right]\sum_{\ell\geq 1}\mathbb{E}_x\left[e^{kb\,S_{\ell-1}}\right] \\ &= ce^{kb\,x}\sum_{r=1}^{k}\mathbb{E}\left[(a_1)^r\, e^{(k-r)b\,S_1}\right]\sum_{\ell\geq 1}\left(\mathbb{E}\left[e^{kb\,S_1}\right]\right)^{\ell-1}. \end{aligned}$$

Observe that

$$\sum_{r=1}^{k}\mathbb{E}\left[(a_1)^r\, e^{(k-r)b\,S_1}\right] \leq \mathbb{E}\left[(a_1 + e^{b\,S_1})^k\right] \leq 2^k\left(\mathbb{E}\left[a_1^k\right] + \mathbb{E}\left[e^{kb\,S_1}\right]\right) < \infty,$$

and $\mathbb{E}\left[e^{kb\,S_1}\right] < 1$ since $k < \gamma/b$. Hence $B(x,k) \leq c_k\, e^{kb\,x}$.

It remains to deal with $C(x,k)$. Observe from (8.22) that

$$C(x,k) = \sum_{i=1}^{\infty} \mathbb{E}_x\left[e^{bkS_{i-1}}(a_i)^k 1_{\{\tau_0^->i-1\}} 1_{\{S_i<0\}} e^{-\eta S_i}\right]$$

$$= \sum_{i=1}^{\infty} \mathbb{E}_x\left[e^{bkS_{i-1}} 1_{\{\tau_0^->i-1\}} \mathbb{E}_{S_{i-1}}[1_{\{S_1<0\}}(a_1)^k e^{-\eta S_1}]\right],$$

by the Markov property at $i-1$. Since $y := S_{i-1} > 0$,

$$\mathbb{E}_y[1_{\{S_1<0\}}(a_1)^k e^{-\eta S_1}] = e^{-\eta y}\mathbb{E}[1_{\{S_1<-y\}}(a_1)^k e^{-\eta S_1}] \le \mathbb{E}[1_{\{S_1<0\}}(a_1)^k e^{-\eta S_1}].$$

It follows that

$$C(x,k) \le c\sum_{i=1}^{\infty} \mathbb{E}_x\left[e^{bkS_{i-1}}\right] \le c' e^{bkx},$$

since $bk < \gamma$. This yields that $A(x,k) = B(x,k) + C(x,k) \le ce^{bkx}$ proving (4.21).

Proof of (ii): Write in this proof

$$A(x,j) := \mathbb{E}_x\left[e^{-\eta S_{\tau_0^-}}\Big(\sum_{\ell=1}^{\min(\tau_0^-,\tau_L^+)} (1+L-S_{\ell-1})^\alpha e^{bS_{\ell-1}} a_\ell\Big)^j\right], \quad x\in\mathbb{R}, j\ge 0.$$

We mention that $A(x,0) = e^{-\eta x}$ if $x<0$ and for $j\ge 1$, $A(x,j) = 0$ if $x<0$ or $x>L$.

From (8.22), $A(x,k) = B(x,k) + C(x,k)$ with

$$B(x,k) = \sum_{r=1}^{k} C_k^r \sum_{j\ge1} \mathbb{E}_x\left[(1+L-S_{j-1})^{\alpha r}\, e^{rbS_{j-1}}(a_j)^r A(S_j,k-r)\, 1_{\{j<\min(\tau_0^-,\tau_L^+)\}}\right], \tag{8.23}$$

$$C(x,k) = \sum_{i=1}^{\infty} \mathbb{E}_x\left[(L-S_{i-1}+1)^{\alpha k} a_i^k e^{bkS_{i-1}} e^{-\eta S_i}\, 1_{\{i=\tau_0^-<\tau_L^+\}}\right]. \tag{8.24}$$

We now prove (4.22) by induction on p, where p equals some integer $m\ge 1$.

Firstly, let $m<\gamma/b$ and assume (4.22) holds for all $A(x,j)$ with $0\le j\le m-1$. By (8.23),

$$B(x,m) \le c\sum_{r=1}^{m}\sum_{j\ge1} \mathbb{E}_x\left[(1+L-S_{j-1})^{\alpha r}\, e^{rbS_{j-1}}(a_j)^r(1+L-S_j)^{\alpha(m-r)}\, e^{b(m-r)S_j},\, j<\tau_L^+\right].$$

Write as before $\Delta S_j = S_j - S_{j-1}$. Notice that for any $j<\tau_L^+$, $(1+L-S_j)^{\alpha(m-r)}\, e^{b(m-r)\Delta S_j} \le c + c(1+L-S_{j-1})^{\alpha(m-r)}\, e^{b(m-r)\Delta S_j}$. By the independence of $(a_j, \Delta S_j)$, it is easy to see that the above expectation under $\mathbb{E}_x$ is less than

$$c\,\mathbb{E}[a_1^r(1+e^{b(m-r)S_1})]\,\mathbb{E}_x\left[(1+L-S_{j-1})^{\alpha m}\, e^{mbS_{j-1}},\, j<\tau_L^+\right],$$

which implies that

$$B(x,m) \le c'\sum_{j\ge1}\mathbb{E}_x\left[(1+L-S_{j-1})^{\alpha m}\, e^{mbS_{j-1}},\, j<\tau_L^+\right]$$

$$= c' e^{mbx}\sum_{j\ge1}\mathbb{E}\left[(1+L-x-S_{j-1})^{\alpha m}\, e^{mbS_{j-1}},\, j<\tau_{L-x}^+\right]$$

$$\le c(1+L-x)^{\alpha m} e^{mbx}, \tag{8.25}$$

where the last estimate follows from the facts that for $j<\tau_{L-x}^+$, $(1+L-x-S_{j-1})^{\alpha m} \le c(1+L-x)^{\alpha m} + c|S_{j-1}|^{\alpha m}$ and that $\sum_{j\ge1}\mathbb{E}\left[|S_{j-1}|^{\alpha m}\, e^{mbS_{j-1}}\right] < \infty$ (since $mb<\gamma$).

By the Markov property at $i-1$,

$$C(x,m) = \sum_{i=1}^{\infty} \mathbb{E}_x \left[(L - S_{i-1} + 1)^{\alpha m} e^{bmS_{i-1}} \mathbb{E}_{S_{i-1}}[1_{\{S_1<0\}} a_1^m e^{-\eta S_1}],\ i-1 < \tau_0^- < \tau_L^+ \right].$$

As in the proof of (i), $\mathbb{E}_{S_{i-1}}[1_{\{S_1<0\}} a_1^m e^{-\eta S_1}]$ is less than some constant, hence

$$\begin{aligned} C(x,m) &\le c \sum_{i=1}^{\infty} \mathbb{E}_x \left[(L - S_{i-1} + 1)^{\alpha m} e^{bmS_{i-1}},\ i-1 < \tau_0^- < \tau_L^+ \right] \\ &\le c'(1 + L - x)^{\alpha m} e^{mbx}, \end{aligned} \tag{8.26}$$

by (8.25). Therefore, $A(x,m) = B(x,m) + C(x,m) \le c(1 + L - x)^{\alpha m} e^{mbx}$ proving the case m.

Considering now the case when $\gamma/b = m$ is an integer. Since $m - r < \gamma/b$ for any $1 \le r \le m$, $B(y, m-r) \le c_{m-r,\alpha}(1 + L - y)^{\alpha(m-r)} e^{(m-r)y}$ for $0 \le y \le L$. By (8.23),

$$\begin{aligned} &B(x,m) \\ \le\ & c \sum_{r=1}^{m} \sum_{j\ge 1} \mathbb{E}_x \left[(1 + L - S_{j-1})^{\alpha r}\, e^{rbS_{j-1}} (a_j)^r (1 + L - S_j)^{\alpha(m-r)} e^{(m-r)S_j} 1_{\{j < \min(\tau_0^-, \tau_L^+)\}} \right]. \end{aligned}$$

Repeating the same argument as before, we get that

$$B(x,m) \le c' \mathbb{E}_x \left[\sum_{j=1}^{\min(\tau_0^-, \tau_L^+)} (1 + L - S_{j-1})^{\alpha m}\, e^{mbS_{j-1}} \right] \le c\,(1 + L - x)^{1+\alpha m},$$

by (4.20). According to (8.26), we get the same estimate for $C(x,m)$, which proves the case $m = \gamma/b$.

It remains to deal with the case $m > \gamma/b$. Let $m_1 := \lfloor \gamma/b \rfloor + 1$ be the least integer larger than γ/b and assume that $\mathbb{E}[a_1^{m_1}] < \infty$, $\mathbb{E}[e^{b(m_1-1)S_1}] < \infty$. We check that (4.22) is satisfied for $m = m_1$: applying (8.23) and using the already proved results for $A(x, m_1 - r)$ (since $m_1 - r \le \gamma/b$), we get that $B(x, m_1)$ is bounded by

$$c \sum_{r=1}^{m_1} \sum_{j\ge 1} \mathbb{E}_x \left[(1 + L - S_{j-1})^{\alpha r}\, e^{rbS_{j-1}} (a_j)^r (1 + L - S_j)^{1+\alpha(m_1-r)}\, e^{b(m-r)S_j} 1_{\{j<\tau_L^+\}} \right],$$

(the extra 1 in the power comes from the possible $m_1 - 1 = \gamma/b$). As before, we get that

$$B(x, m_1) \le c' \sum_{j\ge 1} \mathbb{E}_x \left[(1 + L - S_{j-1})^{1+\alpha m_1}\, e^{m_1 bS_{j-1}},\ j < \tau_L^+ \right] \le c\, e^{\gamma(x-L) + m_1 bL},$$

by applying (4.20). The same estimate holds for $C(x, m_1)$ by using (8.25). This proves that (4.22) holds for $m = m_1$. The other $m > m_1$ can be treated by induction on m and by using the same arguments as before, we omit the details. □

8.4. **Proofs of Lemmas 9, 10, 11 and 12:** We give in this subsection the proofs of these lemmas used in the proof of Theorem 3.

Proof of Lemma 9: Write in this proof

$$A_{(8.27)} := \left\{ \sum_{k=1}^{\tau_t^+ - K} \sum_{u\in\mho_k} H^u(t) > 0 \right\}, \qquad B_{(8.27)} := \left\{ \beta_t(w_{\tau_t^+}) \le \tau_t^+ - K \right\}. \tag{8.27}$$

Let us first observe that Markov inequality together with Part (i) of Corollary 3 imply

$$\mathbf{Q}_x^+\left(A_{(8.27)}\,\middle|\,\mathscr{G}_\infty\right) \le \sum_{k=1}^{\tau_t^+-K}\sum_{u\in\mho_k}\pi(V(u),t), \tag{8.28}$$

with

$$\pi(x,t) := \mathbf{E}_x\left[H(t)\right]1_{\{x\le t\}} + 1_{\{x>t\}}.$$

Furthermore, Part (ii) of Corollary 3 yields for any $x \le t$

$$\mathbf{E}_x\left[H(t)\right] = R(x)e^{\varrho x}\,\mathbf{Q}_x^+\left[\frac{e^{-\varrho S_{\tau_t^+}}}{R(S_{\tau_t^+})}1_{\{\tau_t^+<\tau_0^-\}}\right] \le \frac{R(x)}{R(t)}e^{\varrho x}e^{-\varrho t} \le e^{\varrho(x-t)},$$

from which we deduce that $\pi(x,t) \le e^{\varrho(x-t)}1_{\{x\le t\}} + 1_{\{x>t\}} \le e^{\varrho(x-t)}$. Therefore, we obtain

$$\mathbf{Q}_x^+\left(A_{(8.27)}\,\middle|\,\mathscr{G}_\infty\right) \le \sum_{k=0}^{\tau_t^+-K-1} e^{\varrho(S_k-t)}\sum_{u\in\mho_{k+1}} e^{\varrho\Delta V(u)}.$$

On the other hand, by the definition of $\beta_t(w_{\tau_t^+})$ (see (1.14)),

$$1_{B_{(8.27)}} \le \sum_{k=0}^{\tau_t^+-K-1} e^{\varrho(S_k-t)}(\mathscr{B}(w_{k+1}))^{\varrho}.$$

It follows that

$$\mathbf{Q}_x^+\left(A_{(8.27)}\cup B_{(8.27)}\,\middle|\,\mathscr{G}_\infty\right) \le \sum_{k=0}^{\tau_t^+-K-1} e^{\varrho(S_k-t)}b_{k+1} := \Upsilon(t), \tag{8.29}$$

with $b_{k+1} := \sum_{u\in\mho_{k+1}} e^{\varrho\Delta V(u)} + (\mathscr{B}(w_{k+1}))^{\varrho}$. Recall that under $\mathbf{Q}_x^+$, $(S_k, b_k)_{k\ge 0}$ is a Markov chain, see Proposition 2. Fix a $\lambda > 0$. Then the following double limits equal zero:

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{Q}_x^+\left(\exists k < \tau_t^+ - K : \, t - S_k < \lambda,\, \tau_t^+ > K\right) = 0. \tag{8.30}$$

In fact, let t be large and observe that

$$\mathbf{Q}_x^+\left(\exists k < \tau_t^+ - K : \, t - S_k < \lambda,\, \tau_t^+ > K\right) \le \mathbf{Q}_x^+\left(\tau_{t-\lambda}^+ + K < \tau_t^+\right)$$

which by the Markov property at $\tau_{t-\lambda}^+$, is less than $\sup_{t-\lambda<y<t}\mathbf{Q}_y^+\left(K < \tau_t^+\right)$. By the absolute continuity between $\mathbf{Q}_y^+$ and $\mathbf{Q}_y$,

$$\mathbf{Q}_y^+\left(K < \tau_t^+\right) = \mathbf{Q}_y\left[1_{\{K<\tau_t^+\wedge\tau_0^-\}}\frac{R(S_K)}{R(y)}\right] \le \frac{R(t)}{R(y)}\,\mathbf{Q}_y(\tau_t^+ > K) = \frac{R(t)}{R(y)}\,\mathbf{Q}(\tau_{t-y}^+ > K).$$

It follows that

$$\limsup_{t\to\infty}\mathbf{Q}_x^+\left(\exists k < \tau_t^+ - K : \, t - S_k < \lambda,\; \tau_t^+ > K\right) \le \mathbf{Q}(\tau_\lambda^+ > K)\limsup_{t\to\infty}\frac{R(t)}{R(t-\lambda)} = \mathbf{Q}(\tau_\lambda^+ > K),$$

which goes to 0 as $K \to \infty$. This proves (8.30).

Let

$$E_1(t,K) := \left\{\forall k < \tau_t^+ - K : t - S_k \ge \lambda, \tau_t^+ > K\right\}.$$

Since $\mathbf{Q}_x^+(\tau_t^+ > K) \to 1$ as $t \to \infty$, which in view of (8.30) yields that for any small $\varepsilon > 0$, there exists some $K_0 = K_0(\varepsilon,\lambda) > 0$ such that for all $K \ge K_0$, there exists some $t_0(K,\varepsilon,\lambda)$ satisfying

$$\mathbf{Q}_x^+(E_1(t,K)^c) \le \varepsilon, \qquad \forall\, t \ge t_0. \tag{8.31}$$

We claim that there exists some small $\delta > 0$ such that

$$\sup_{z\ge 0} \mathbf{Q}_z^+[b_1^\delta] < \infty, \tag{8.32}$$

$$\limsup_{t\to\infty} \mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1} e^{\kappa\,(S_k-t)}\right] < \infty, \tag{8.33}$$

for any $\kappa > 0$.

Admitting for the moment (8.32) and (8.33), we prove the lemma as follows: define

$$E_2(t,K) := \bigcap_{k=0}^{\tau_t^+-K-1} \left\{ b_{k+1} \le e^{\frac{\varrho}{2}(t-S_k)} \right\} \cap \{\tau_t^+ > K\}.$$

By (8.29) and on $E_2(t,K)\cap E_1(t,K)$ which is $\mathscr{G}_\infty$-measurable,

$$\mathbf{Q}_x^+\left(A_{(8.27)} \cup B_{(8.27)} \,\middle|\, \mathscr{G}_\infty\right) \le \Upsilon(t) \le \sum_{k=0}^{\tau_t^+-K-1} e^{\frac{\varrho}{2}(S_k-t)},$$

which is less than $e^{-\varrho\,\lambda/4}\sum_{k=0}^{\tau_t^+-K-1} e^{\frac{\varrho}{4}(S_k-t)}$ since on $E_1(t,K)$, $S_k - t \le -\lambda$ for $k < \tau_t^+ - K$. This with (8.31) imply that for all $t \ge t_0$,

$$\begin{aligned} &\mathbf{Q}_x^+\left(A_{(8.27)} \cup B_{(8.27)}\right) \\ \le\ & \varepsilon + \mathbf{Q}_x^+\Big(E_2(t,K)^c \cap E_1(t,K)\Big) + e^{-\varrho\,\lambda/4}\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1} e^{\frac{\varrho}{4}(S_k-t)}\right]. \end{aligned} \tag{8.34}$$

On the other hand, fix the constant $\delta > 0$ in (8.32), we have

$$\begin{aligned} \mathbf{Q}_x^+\Big(E_2(t,K)^c \cap E_1(t,K)\Big) &\le \mathbf{Q}_x^+\left[1_{E_1(t,K)} \sum_{k<\tau_t^+-K} (b_{k+1})^\delta\, e^{-\frac{\delta\varrho}{2}(t-S_k)}\right] \\ &\le e^{-\delta\varrho\lambda/4}\,\mathbf{Q}_x^+\left[1_{E_1(t,K)} \sum_{k<\tau_t^+-K} (b_{k+1})^\delta\, e^{-\frac{\delta\varrho}{4}(t-S_k)}\right] \\ &\le e^{-\delta\varrho\lambda/4}\,\mathbf{Q}_x^+\left[\sum_{k<\tau_t^+} (b_{k+1})^\delta\, e^{-\frac{\delta\varrho}{4}(t-S_k)}\right]. \end{aligned}$$

Applying the Markov property at k gives that

$$\begin{aligned} \mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1} e^{\frac{\delta\varrho}{4}(S_k-t)}(b_{k+1})^\delta\right] &= \sum_{k=0}^{\infty} \mathbf{Q}_x^+\left[1_{\{k<\tau_t^+\}} e^{\frac{\delta\varrho}{4}(S_k-t)} \mathbf{Q}_{S_k}^+(b_1^\delta)\right] \\ &\le \sup_{z\ge 0}\mathbf{Q}_z^+[b_1^\delta]\,\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1} e^{\frac{\delta\varrho}{4}(S_k-t)}\right]. \end{aligned}$$

By (8.32) and (8.33), we get some constant c independent of λ and t [the constant c may depend on x, δ] such that $\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1} e^{\frac{\delta\varrho}{4}(S_k-t)}(b_{k+1})^\delta\right] \le c$ and $\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1} e^{\frac{\varrho}{4}(S_k-t)}\right] \le c$. Going back to (8.34), we obtain that for all $K \ge K_0$,

$$\limsup_{t\to\infty} \mathbf{Q}_x^+\left(A_{(8.27)} \cup B_{(8.27)}\right) \le \varepsilon + ce^{-\delta\varrho\lambda/4} + c\,e^{-\varrho\,\lambda/4}.$$

Letting $\lambda \to \infty$ and $\varepsilon \to 0$, we get that

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{Q}_x^+\left(A_{(8.27)}\cup B_{(8.27)}\right)=0.$$

It remains to show (8.32) and (8.33). By (3.22),

$$\begin{aligned}\mathbf{Q}_z^+[b_1^\delta] &= \mathbf{E}_z\Big[\frac{e^{-\varrho z}}{R(z)}\sum_{|u|=1}1_{\{V(u)\ge 0\}}R(V(u))e^{\varrho V(u)}\big(\sum_{v\neq u}e^{\varrho(V(v)-z)}+\mathscr{B}(u)^\varrho\big)^\delta\Big]\\ &= \mathbf{E}\Big[\frac{1}{R(z)}\sum_{|u|=1}1_{\{V(u)\ge -z\}}R(V(u)+z)e^{\varrho V(u)}\big(\sum_{v\neq u}e^{\varrho V(v)}+\mathscr{B}(u)^\varrho\big)^\delta\Big]\\ &\le \begin{cases} c\mathbf{E}\Big[\sum_{|u|=1}(1+|V(u)|)e^{\varrho V(u)}\big((\sum_{|v|=1}e^{\varrho V(v)})^\delta+\mathscr{B}(u)^{\delta\varrho}\big)\Big], & \text{(critical case)},\\ c\mathbf{E}\Big[\big(\sum_{|u|=1}e^{\varrho V(u)}\big)^{1+\delta}+\big(\sum_{|u|=1}e^{\varrho V(u)}\big)\mathscr{B}(u)^{\delta\varrho}\Big], & \text{(subcritical case)},\end{cases}\end{aligned}$$

since $R(z)\sim C_R z$ in the critical case and $R(z)\sim C_R$ in the subcritical case as $z\to\infty$. If $\delta>0$ is sufficiently small, the later expectations are finite by (1.13) together with (1.3) and (1.4) respectively, which yields (8.32).

To show (8.33), we deduce from the absolute continuity between $\mathbf{Q}_x^+$ and $\mathbf{Q}_x$ that

$$\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1}e^{\kappa\,(S_k-t)}\right] = \sum_{k=0}^{\infty}\mathbf{Q}_x\left[1_{\{k<\tau_t^+\wedge\tau_0^-\}}e^{\kappa\,(S_k-t)}\frac{R(S_k)}{R(x)}\right]. \tag{8.35}$$

Let us distinguish the critical and subcritical cases: In the critical case, $\mathbf{Q}[S_1]=0$ and $R(z)\sim C_R z$ as $z\to\infty$. There exists some constant c such that for all $t\ge 1$, the RHS of (8.35) is less than

$$ct\sum_{k=0}^{\infty}\mathbf{Q}_x\left[1_{\{k<\tau_t^+\wedge\tau_0^-\}}e^{\kappa(S_k-t)}\right]=ct\,\mathbf{Q}_x\left[\sum_{k=0}^{\tau_t^+\wedge\tau_0^--1}e^{\kappa(S_k-t)}\right].$$

Applying (4.13) with $L=t$ and $\delta=\kappa$ [this δ has nothing to do with that in (8.32)] gives that $\mathbf{Q}_x\left[\sum_{k=0}^{\tau_t^+\wedge\tau_0^--1}e^{\kappa(S_k-t)})\right]\le c_5\frac{x+1}{t}$. Hence $\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1}e^{\kappa(S_k-t)}\right]\le c(x+1)$ for all $t\ge 1$. This proves (8.33) in the critical case.

In the subcritical case, we note that $\mathbf{Q}[S_1]>0$ and $R(\cdot)$ is bounded. By (8.35), we get that for some constant $c>0$,

$$\mathbf{Q}_x^+\left[\sum_{k=0}^{\tau_t^+-1}e^{\kappa(S_k-t)}\right]\le c\sum_{k=0}^{\infty}\mathbf{Q}_x\left[1_{\{k<\tau_t^+\}}e^{\kappa(S_k-t)}\right],$$

which, according to Lemma 5 is uniformly bounded by some constant. This completes the proof of (8.33) and hence that of Lemma 9. □

Proof of Lemma 10: Observe that

$$\{\tau_t^+>K\}\cap\Gamma^c(t,K)\subset\bigcup_{k\in(\tau_t^+-K,\tau_t^+]}\bigcup_{u\in\mho_k}\Big\{\exists v\in\mathcal{T}^{(u)}:|u|\le\tau_0^-(v)<\tau_t^+(v)=|v|\Big\}.$$

Recall that $\mathscr{G}_{\mathscr{C}_t} = \sigma\Big\{(\Delta V(u), u \in \mho_k), V(w_k), w_k, \mho_k, 1 \le k \le \tau_t^+\Big\}$. For any event $F \in \mathscr{G}_{\mathscr{C}_t}$, we deduce from Corollary 3 that

$$\mathbf{Q}_x^+\left(\{\tau_t^+ > K\} \cap \Gamma^c(t,K)\right) \le \mathbf{Q}_x^+(F^c) + \mathbf{Q}_x^+\left[1_F \sum_{k\in(\tau_t^+-K,\tau_t^+]} \sum_{u\in\mho_k} f(V(u))\right],$$

with $f(y) := \mathbf{P}_y(\exists v : \tau_0^-(v) < \tau_t^+(v) = |v|) = \mathbf{P}(\exists v : \tau_{-y}^-(v) < \tau_{t-y}^+(v) = |v|)$ [we mention that $f(y) = 0$ if $y > t$]. For any $y \le t$, by the branching property at $\tau_{-y}^-(v)$, $f(y) \le \sup_{z\le -y} \mathbf{P}_z(\exists u : \tau_{t-y}^+(u) < \infty) = \mathbf{P}(\exists u : \tau_t^+(u) < \infty) := \eta(t)$ which converges to 0 since the (non-killed) branching random walk V goes to $-\infty$. Therefore,

$$\mathbf{Q}_x^+\left(\{\tau_t^+ > K\} \cap \Gamma^c(t,K)\right) \le \mathbf{Q}_x^+(F^c) + \eta(t)\,\mathbf{Q}_x^+\left[1_F \sum_{k\in(\tau_t^+-K,\tau_t^+]} \#\mho_k\right].$$

Consider an arbitrary $\varepsilon > 0$. By Lemma 4 (ii), $(S_{\tau_t^+} - S_{\tau_t^+-i}, 1 \le i \le K)$ converges in law, hence there exists some $\lambda = \lambda(\varepsilon, K) > 0$ such that for all large t (in particular, $t > 4\lambda$),

$$\mathbf{Q}_x^+(F_1) := \mathbf{Q}_x^+\left(\{\tau_t^+ > K\} \cap \bigcap_{k\in(\tau_t^+-K,\tau_t^+]} \Big\{S_k > t-\lambda,\ |S_k - S_{k-1}| \le \lambda\Big\}\right) > 1-\varepsilon,$$

with obvious definition of the event F_1. Let $C > 0$ and define

$$F_2 := F_1 \cap \Big\{\forall k \in (\tau_t^+ - K, \tau_t^+] : \#\mho_k \le C\Big\}.$$

Hence for all sufficiently large t, $\mathbf{Q}_x^+(\tau_t^+ \le K) \le \varepsilon$ and

$$\begin{aligned} \mathbf{Q}_x^+\left(\Gamma^c(t,K)\right) &\le 2\varepsilon + \mathbf{Q}_x^+(F_1 \cap F_2^c) + \eta(t)\,\mathbf{Q}_x^+\left[1_{F_2} \sum_{k\in(\tau_t^+-K,\tau_t^+]} \#\mho_k\right] \\ &\le 2\varepsilon + \mathbf{Q}_x^+(F_1 \cap F_2^c) + C\,K\,\eta(t), \end{aligned} \tag{8.36}$$

with $\eta(t) \to 0$ as $t \to \infty$. By (1.3) and (1.4), we can find a sufficiently small $\delta > 0$ such that $\mathbf{Q}[(\#\mho_1)^\delta] = \mathbf{E}\big[(\nu-1)^\delta \sum_{|u|=1} e^{\varrho V(u)}\big] := c < \infty$. Observe that

$$\begin{aligned} \mathbf{Q}_x^+(F_1 \cap F_2^c) &\le C^{-\delta}\mathbf{Q}_x^+\left[1_{F_1} \sum_{k\in(\tau_t^+-K,\tau_t^+]} (\#\mho_k)^\delta\right] \\ &\le C^{-\delta} \sum_{k\ge1} \mathbf{Q}_x^+\left[1_{\{|S_k-S_{k-1}|\le\lambda, S_{k-1}>t-\lambda, \tau_t^+\ge k\}}\,(\#\mho_k)^\delta\right] \\ &= C^{-\delta} \sum_{k\ge1} \mathbf{Q}_x\left[\frac{R(S_k)}{R(x)} 1_{\{|S_k-S_{k-1}|\le\lambda, S_{k-1}>t-\lambda, k\le\tau_t^+\wedge\tau_0^-\}}\,(\#\mho_k)^\delta\right] \\ &\le C^{-\delta} \sum_{k\ge1} \frac{R(t+\lambda)}{R(x)} \mathbf{Q}_x\left[1_{\{S_{k-1}>t-\lambda, k\le\tau_t^+\wedge\tau_0^-\}}\,(\#\mho_k)^\delta\right], \end{aligned}$$

since R is non-decreasing and $S_k \le t+\lambda$. By Corollary 1 (i), under $\mathbf{Q}_x$, $\#\mho_k$ is independent of $\{S_{k-1} > t-\lambda, k \le \tau_t^+ \wedge \tau_0^-\}$ and has the same law as $\#\mho_1$; moreover $\mathbf{Q}_x[(\#\mho_1)^\delta] =$

$\mathbf{Q}[(\#\mho_1)^\delta] =: c < \infty$. Using the fact that $R(t+\lambda) \le 2R(t-\lambda)$ for all large t, we have

$$\begin{aligned} \mathbf{Q}_x^+(F_1 \cap F_2^c) &\le c\,C^{-\delta} \sum_{k\ge 1} \frac{R(t+\lambda)}{R(x)} \mathbf{Q}_x \left[1_{\{S_{k-1}>t-\lambda, k\le \tau_t^+ \wedge \tau_0^-\}} \right] \\ &\le 2c\,C^{-\delta} \sum_{k\ge 1} \mathbf{Q}_x \left[\frac{R(S_{k-1})}{R(x)} 1_{\{S_{k-1}>t-\lambda, k\le \tau_t^+ \wedge \tau_0^-\}} \right] \\ &= 2c\,C^{-\delta}\, \mathbf{Q}_x^+ \left[\sum_{k=1}^{\tau_t^+} 1_{\{S_{k-1}>t-\lambda\}} \right]. \end{aligned}$$

Observe that $\mathbf{Q}_x^+ \left[\sum_{k=1}^{\tau_t^+} 1_{\{S_{k-1}>t-\lambda\}} \right] \le \mathbf{Q}_x^+ \left[\sum_{k=1}^{\tau_t^+} e^{\varrho(S_{k-1}-(t-\lambda))} \right]$ which by (8.33) is smaller than some constant $c = c(\lambda, x) < \infty$. Going back to (8.36), we get that

$$\mathbf{Q}_x^+ \left(\Gamma^c(t,K)\right) \le 2\varepsilon + 2c\,C^{-\delta} + C\,K\,\eta(t).$$

Letting $t \to \infty$, $C \to \infty$ and then $\varepsilon \to 0$ (δ being fixed), we prove Lemma 10. □

Proof of Lemma 11: Firstly, note that there is nothing to prove in the subcritical case [since $\mathscr{R}(t) \equiv 1$ by (5.4)]. It remains to consider the critical case, thus $\varrho = \varrho_*$ and $\mathscr{R}(t) = t$ for all $t \ge 0$. For notational convenience, write

$$\begin{aligned} A &:= \exp\Big\{ -f(t_0) 1_{\mathscr{D}_{1,K}} - \sum_{i=1}^K 1_{\mathscr{D}_{i,K}} \sum_{j=1}^{m^{(i)}} \langle f, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}} \rangle \Big\}, \\ B &:= e^{\varrho_* t_0} + \sum_{i=1}^K \sum_{j=1}^{m^{(i)}} \int e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z), \\ D &:= t_0 e^{\varrho_* t_0} + \sum_{i=1}^K \sum_{j=1}^{m^{(i)}} \int z e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z). \end{aligned}$$

Then

$$\begin{aligned} \varphi_{t,K}\left(t_0, s_1, \ldots, s_K, \theta^{(1)}, \ldots, \theta^{(K)}\right) &= \mathbf{E}\Big[\frac{A}{B + \frac{1}{t} D} \Big], \\ \varphi_{\infty,K}\left(t_0, s_1, \ldots, s_K, \theta^{(1)}, \ldots, \theta^{(K)}\right) &= \mathbf{E}\Big[\frac{A}{B} \Big]. \end{aligned}$$

Since $f \ge 0$, $A \le 1$, and we get that

$$|\varphi_{t,K}\left(t_0, s_1, \ldots, s_K, \theta^{(1)}, \ldots, \theta^{(K)}\right) - \varphi_{\infty,K}\left(t_0, s_1, \ldots, s_K, \theta^{(1)}, \ldots, \theta^{(K)}\right)| \le \frac{1}{t} \mathbf{E}\Big[\frac{D}{B^2} \Big].$$

We are going to prove that

$$\frac{D}{B^2} \le \frac{1}{\varrho_*}, \qquad a.s.$$

Indeed, notice firstly that the non-killed branching random walk V goes to $-\infty$, $\mu^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z)$ is an a.s. finite measure on $\mathbb{R}_+$, and $t_0 e^{\varrho_* t_0} \le \frac{1}{\varrho_*} e^{2\varrho_* t_0}$ for any $t_0 > 0$. Secondly, let $\zeta_{i,j} := \sup\{a > 0 : \int_{[a,\infty)} \mu^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z) > 0\}$. Note that $\zeta_{i,j} \le \frac{1}{\varrho_*} e^{\varrho_* \zeta_{i,j}} \le \frac{1}{\varrho_*} \int e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z)$. It follows that $\int z e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z) \le \zeta_{i,j} \int e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z) \le \frac{1}{\varrho_*} \left(\int e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z) \right)^2$.

Hence

$$D \le \frac{1}{\varrho_*} e^{2\varrho_* t_0} + \frac{1}{\varrho_*} \sum_{i=1}^{K} \sum_{j=1}^{m^{(i)}} \Big(\int e^{\varrho_* z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z)\Big)^2 \le \frac{B^2}{\varrho_*},$$

yielding that $\big|\widetilde{\varphi}_{t,K}(T_t^+, S_1^{(t)}, \dots, S_K^{(t)}) - \widetilde{\varphi}_{\infty,K}(T_t^+, S_1^{(t)}, \dots, S_K^{(t)})\big| \le \frac{1}{t\varrho_*}$ and proving Lemma 11. □

Proof of Lemma 12: We prove the following stronger statement: For any $K \ge 1$,

$$\begin{aligned} &\lim_{t\to\infty} \mathbf{Q}_x^+ \Big[\widetilde{\varphi}_{\infty,K}(T_t^+, S_1^{(t)}, \dots, S_K^{(t)}) 1_{\{\tau_t^+ > K\}} \Big] \\ (8.37) \qquad &= \mathbf{Q}\left[\frac{\exp\Big\{ - f(U\hat{S}_{\hat{\sigma}}) 1_{\mathscr{D}_{1,K}} - \sum_{i=1}^K 1_{\mathscr{D}_{i,K}} \sum_{j=1}^{\widetilde{\nu}_i} \langle f, \overline{\mu}^{(i,j)}_{\hat{S}_i - U\hat{S}_{\hat{\sigma}} - \widetilde{X}_j^{(i)}} \rangle \Big\}}{e^{\varrho U \hat{S}_{\hat{\sigma}}} + \sum_{i=1}^K \sum_{j=1}^{\widetilde{\nu}_i} \int e^{\varrho z}\, \overline{\mu}^{(i,j)}_{\hat{S}_i - U\hat{S}_{\hat{\sigma}} - \widetilde{X}_j^{(i)}}(\mathrm{d}z)} \right], \end{aligned}$$

which implies Lemma 12 by letting $K \to \infty$. Define

$$\begin{aligned} \pounds_i(\mathbf{s}, \theta) &:= \min_{i \le j \le K} \left(s_j - \log \mathscr{B}(\theta_j) \right), \qquad 1 \le i \le K, \\ A(t_0, \mathbf{s}, \theta) &:= \exp\Big\{ - f(t_0) 1_{\{\pounds_1(\mathbf{s},\theta) \ge t_0\}} - \sum_{i=1}^K 1_{\{\pounds_i(\mathbf{s},\theta) \ge t_0\}} \sum_{j=1}^{m^{(i)}} \langle f, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}} \rangle \Big\}, \\ B(t_0, \mathbf{s}, \theta) &:= \mathrm{e}^{\varrho t_0} + \sum_{i=1}^K \sum_{j=1}^{m^{(i)}} \int e^{\varrho z}\, \overline{\mu}^{(i,j)}_{s_i - t_0 - x_j^{(i)}}(\mathrm{d}z), \end{aligned}$$

for $\mathbf{s} := (s_1, ..., s_K)$, $\theta := (\theta_1, ..., \theta_K)$, with $\theta_i = \sum_{j=1}^{m^{(i)}} \delta_{\{x_j^{(i)}\}}$, $1 \le i \le K$. Denote by $\Theta(\mathbf{s})$ a random variable taking values in $\Omega_f^{\otimes K}$ with law $\prod_{i=1}^K \Xi_{s_i - s_{i-1}}(d\theta^{(i)})$. Then [recalling $s_0 := 0$]

$$\begin{aligned} \widetilde{\varphi}_{\infty,K}(t_0, \mathbf{s}) &= \int \mathbf{E}\Big[\frac{A(t_0, \mathbf{s}, \theta)}{B(t_0, \mathbf{s}, \theta)} \Big] \prod_{i=1}^K \Xi_{s_i - s_{i-1}}(d\theta^{(i)}) \\ &= \mathbf{E}\Big[\frac{A(t_0, \mathbf{s}, \Theta(\mathbf{s}))}{B(t_0, \mathbf{s}, \Theta(\mathbf{s}))} \Big], \qquad (t_0, \mathbf{s}) \in \mathbb{R}_+^* \times \mathbb{R}_+^K. \end{aligned}$$

Plainly the function $\widetilde{\varphi}_{\infty,K}$ is bounded by 1. Therefore Lemma 12 will be a consequence of Lemma 4 if we have checked that for any fixed $\mathbf{s} \in \mathbb{R}_+^K$, the function $t_0 \to \widetilde{\varphi}_{\infty,K}(t_0, \mathbf{s})$ is continuous excepted from a set at most countable.

To this end, we study at first the continuity of $y \to \langle f, \overline{\mu}_y^{(i,j)} \rangle$ which are i.i.d. copies of $\langle f, \overline{\mu}_y \rangle$. Recall that $\langle f, \overline{\mu}_y \rangle = \sum_{u \in \mathscr{C}_y} f(V(u) - y)$ for any fixed $y > 0$. Let us consider $\widetilde{\tau}_t^+(u) := \inf\{k : V(u_k) \ge t\}$ and define the associated optional line $\widetilde{\mathscr{C}_t}$ just like (3.7). By the definition of the stopping line $\widetilde{\mathscr{C}_y}$ and the continuity of f, we immediately obtain

$$(8.38) \qquad \limsup_{k\to\infty} |\langle f, \overline{\mu}_{y_k} \rangle - \langle f, \overline{\mu}_y \rangle| \le f(0) \sum_u 1_{\{\widetilde{\tau}_y^+ = |u|,\, V(u) = y\}} = f(0) \sum_{u \in \widetilde{\mathscr{C}_y}} 1_{\{V(u) = y\}},$$

for any sequence $(y_k)_k$, such that $y_k \to y$ when $k \to \infty$. On the other hand, Corollary 1 (ii) also holds for this family of optional lines by replacing n by $\widetilde{\tau}_t^+$. Then we take the expectation (under $\mathbf{P}$) in (8.38) and obtain that

$$(8.39) \qquad \mathbf{E}\left[\limsup_{k\to\infty} |\langle f, \overline{\mu}_{y_k} \rangle - \langle f, \overline{\mu}_y \rangle| \right] \le f(0) e^{-\varrho y}\, \mathbf{Q}(S_{\widetilde{\tau}_y^+} = y).$$

where $\widetilde{\tau}_y^+ := \inf\{n \ge 0 : S_n \ge y\}$. Denoting as before by $(H_n)_{n\ge1}$ the (strict) ascending ladder heights of S, we remark that

$$\Lambda_1 := \Big\{y : \mathbf{Q}(S_{\widetilde{\tau}_y^+} = y) > 0\Big\} \subset \bigcup_{n=1}^{\infty} \Big\{y : \mathbf{Q}(H_n = y) > 0\Big\} \text{ is countable.}$$

Then by (8.39), $y \to \langle f, \overline{\mu}_y \rangle$ is continuous (in L^1 hence a fortiori in probability) on $y \notin \Lambda_1$. The same holds for $y \to \langle f, \overline{\mu}_y^{(i,j)} \rangle$ with any $i, j \ge 1$. Now we write explicitly $\Theta(\mathbf{s})$ by a random vector $\Theta(\mathbf{s}) = (\theta_1, ..., \theta_K)$ with $\theta_i := \sum_{j=1}^{M^{(i)}} \delta_{\{X_j^{(i)}\}}$ and the associated random variables $\mathcal{L}_i(\mathbf{s}, \theta)$, $1 \le i \le K$ [The random variables $M^{(i)}$ take values in $\mathbb{N}$, $X_j^{(i)}$ in $\mathbb{R}$, and $\mathcal{L}_i(\mathbf{s}, \theta)$ in $\mathbb{R} \cup \{\infty\}$]. Observe that all the following three events are countable:

$$\begin{aligned}
\Lambda_2 &:= \bigcup_{i=1}^{K} \Big\{x : \mathbf{P}\big(x = X_j^{(i)}, \text{ for some } 1 \le j \le M^{(i)}\big) > 0\Big\}, \\
\Lambda_3 &:= \bigcup_{i=1}^{K} \Big\{x : \mathbf{P}\big(x = \mathcal{L}_i(\mathbf{s}, \theta)\big) > 0\Big\}, \\
\Lambda_4 &:= \Lambda_3 \cup \bigcup_{i=1}^{K} \Big\{s_i - x - y : x \in \Lambda_2, y \in \Lambda_1\Big\}.
\end{aligned}$$

We claim that $\varphi_{\infty,K}(t_0, \mathbf{s})$ is continuous on $t_0 \notin \Lambda_4$. To check this, we fix $t_0 \notin \Lambda_3$ and take a sequence $t_n \to t_0$ as $n \to \infty$. Let

$$E := \bigcup_{i=1}^{K} \bigcup_{j=1}^{M_j^{(i)}} \{X_j^{(i)} \in s_i - t_0 - \Lambda_1\} \cup \{\mathcal{L}_i(\mathbf{s}, \theta) = t_0\}.$$

Since $t_0 \notin \Lambda_4$, we deduce from the definition of Λ_2 that $\mathbf{P}(E) = 0$. Observe that on E^c, $s_i - t_0 - X_j^{(i)} \notin \Lambda_1$ and $t_0 \ne \mathcal{L}_i(\mathbf{s}, \theta)$, hence $A(t_n, \mathbf{s}, \theta) 1_{E^c} \to A(t_0, \mathbf{s}, \theta) 1_{E^c}$ in probability. In other words, $A(t_n, \mathbf{s}, \theta) \to A(t_0, \mathbf{s}, \theta)$ in probability and the same holds for $B(t_n, \mathbf{s}, \theta)$. By the dominated convergence theorem, when $n \to \infty$,

$$\varphi_{\infty,K}(t_n, \mathbf{s}) = \mathbf{E}\Big[\frac{A(t_n, \mathbf{s}, \Theta(\mathbf{s}))}{B(t_n, \mathbf{s}, \Theta(\mathbf{s}))}\Big] \to \mathbf{E}\Big[\frac{A(t_0, \mathbf{s}, \Theta(\mathbf{s}))}{B(t_0, \mathbf{s}, \Theta(\mathbf{s}))}\Big] = \varphi_{\infty,K}(t_0, \mathbf{s}),$$

proving the desired continuity at any $t_0 \notin \Lambda_3$. Then we can apply Lemma 4 and get Lemma 12. □

8.5. **Proof of Lemma 16.** Throughout the proof, $\delta > 0$ is taken to be sufficiently small.

Proof of (i): Let us write $f(x) := -\log \mathbb{E} e^{-x\Gamma_1}$ for $x \ge 0$; By Tauberian theorem,

$$f(x) \sim a \frac{x}{\log(1/x)}, \qquad x \to 0.$$

Let $A_x := \{\max_{1 \le i \le \xi} Y_i \le x^{-1+\frac{\delta}{2}}\}$ ($\max_\emptyset = 0$). Then for $x > 0$,

$$\mathbb{P}\Big(A_x^c\Big) \le \mathbb{E} \sum_{i=1}^{\xi} x^{(1+\delta)(1-\frac{\delta}{2})} Y_i^{1+\delta} = c\, x^{(1+\delta)(1-\frac{\delta}{2})} = o(x^{1+\delta/3}), \quad x \to 0,$$

since $\delta > 0$ is small. By independence of (Γ_i), we have

$$\mathbb{E}\left[e^{-x\sum_{i=1}^{\xi} Y_i \Gamma_i}\right] = \mathbb{E}\Big[e^{-\sum_{i=1}^{\xi} f(xY_i)}\Big] = \mathbb{E} \exp\left[-\sum_{i=1}^{\xi} f(xY_i) 1_{A_x}\right] + o(x^{1+\delta/3}). \tag{8.40}$$

Define

$$\Upsilon_x := \frac{\log(1/x)}{x} \sum_{i=1}^{\xi} f(xY_i)\, 1_{A_x}\,, \qquad 0 < x < 1.$$

Plainly as $x \to 0$, $\Upsilon_x \to a \sum_{i=1}^{\xi} Y_i$ almost surely. Notice that on A_x, $xY_i \le x^{\delta/2}$, which together with the asymptotic of f implies that for all $0 < x < x_0$ with x_0 sufficiently small, $f(xY_i) \le 2a \frac{xY_i}{\log(1/(xY_i))} \le \frac{4a}{\delta} \frac{xY_i}{\log(1/x)}$, for all $1 \le i \le \xi$. Hence

$$\frac{\log(1/x)}{x} \left(1 - e^{-\frac{x}{\log(1/x)} \Upsilon_x}\right) \le \Upsilon_x \le \frac{4a}{\delta} \sum_{i=1}^{\xi} Y_i.$$

By the dominated convergence theorem,

$$\frac{\log(1/x)}{x} \left(1 - \mathbb{E} \exp\left[- \sum_{i=1}^{\xi} f(xY_i) 1_{A_x}\right]\right) \to a\, \mathbb{E} \sum_{i=1}^{\xi} Y_i.$$

This and (8.40) yield that as $x \to 0$, $\frac{\log(1/x)}{x} \left(1 - \mathbb{E}\left[e^{-x \sum_{i=1}^{\xi} Y_i \Gamma_i}\right]\right) \to a\, \mathbb{E} \sum_{i=1}^{\xi} Y_i$ which implies (i) by Tauberian theorem.

Proof of (ii): Define $W := \sum_{i=1}^{\xi} Y_i$ and let $\lambda > 1$ and $0 < \varepsilon < a/2$. By conditioning on $(Y_i)_{1 \le i \le \xi}$ and using the tail of Γ_i, we have that for large t,

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=1}^{\xi} Y_i \Gamma_i > t\Big) &\ge \mathbb{P}\Big(\max_{1 \le i \le \xi} (Y_i \Gamma_i) > t, W \le \lambda\Big) \\
&\ge \mathbb{E}\Big[1_{\{W \le \lambda\}} \big(1 - \prod_{i=1}^{\xi} (1 - \frac{(a-\varepsilon) Y_i^p}{t^p})\big)\Big] \\
&\ge (a - 2\varepsilon) \mathbb{E}\Big[1_{\{W \le \lambda\}} \sum_{i=1}^{\xi} Y_i^p\Big]\, t^{-p},
\end{aligned}$$

which implies that

$$\liminf_{t \to \infty} t^p\, \mathbb{P}\Big(\sum_{i=1}^{\xi} Y_i \Gamma_i > t\Big) \ge (a - 2\varepsilon) \mathbb{E}\Big[1_{\{W \le \lambda\}} \sum_{i=1}^{\xi} Y_i^p\Big].$$

Letting $\varepsilon \to 0$ and then $\lambda \to \infty$ yields the lower bound.

To prove the upper bound, we remark that by considering $\frac{c+Y_i}{c}$ instead of Y_i (with $c > 0$), we can assume without loss of generality that almost surely $Y_i \ge 1$ (if $i \le \xi$).

By the Markov inequality (δ being small),

$$\mathbb{P}(W > t^{1-\delta/2}) \le t^{-(p+\delta)(1-\delta/2)} \mathbb{E}[W^{p+\delta}] = o(t^{-p}). \tag{8.41}$$

Let $\varepsilon > 0$ be small and define
(8.42)

$$A_{(8.42)} := \Big\{ \max_{1 \le i \le \xi} (Y_i \Gamma_i) \le \varepsilon t \Big\},\ B_{(8.42)} := \Big\{ \sum_{i=1}^{\xi} Y_i \Gamma_i \ge t \Big\},\ C_{(8.42)} := \Big\{ W \le t^{1-\delta/2} \Big\}.$$

By conditioning on $\mathbb{Y} := \sigma\{Y_i, 1 \le i \le \xi, \xi\}$, we get that

$$\mathbb{P}\Big(A_{(8.42)} \cap B_{(8.42)} \cap C_{(8.42)}\Big) \le t^{-p-\delta} \mathbb{E}\Big[1_{C_{(8.42)}} \mathbb{E}\big[(\sum_{i=1}^{\xi} Y_i \Gamma_i)^{p+\delta} 1_{A_{(8.42)}} | \mathbb{Y}\big]\Big].$$

By convexity, $(\sum_{i=1}^{\xi} y_i\Gamma_i)^{p+\delta} \le (\sum_{i=1}^{\xi} y_i)^{p+\delta-1}\sum_{i=1}^{\xi} y_i\Gamma_i^{p+\delta}$ for any $y_i \ge 0$. Observe that by using the tail of Γ_i,

$$\mathbb{E}\Big[\Gamma_i^{p+\delta}1_{\{\Gamma_i\le \frac{\varepsilon t}{y_i}\}}\Big] \le \int_0^{\varepsilon t/y_i}(p+\delta)x^{p+\delta-1}\mathbb{P}(\Gamma_i > x)dx \le \frac{2(p+\delta)}{\delta}(\varepsilon t/y_i)^{\delta},$$

for all large t and $y_i \le t^{1-\delta/2}$. It follows that for any $0<\varepsilon<1$,

$$\mathbb{P}\Big(A_{(8.42)}\cap B_{(8.42)}\cap C_{(8.42)}\Big) \le c_{p,\delta}\, t^{-p}\,\varepsilon^{\delta}\,\mathbb{E}\Big[W^{p+\delta-1}\sum_{i=1}^{\xi} Y_i^{1-\delta}\Big]. \tag{8.43}$$

Since $Y_i \ge 1$, the above expectation is less than $\mathbb{E}[W^{p+\delta}]$ which is finite.

Pick up $1<q<p$ and $p-q<1/2$. Using the Markov inequality and conditioning on $\mathbb{Y}$, we obtain

$$\begin{aligned}
& \mathbb{P}\Big(\{\exists i\le\xi : \varepsilon t < \Gamma_i Y_i < (1-\varepsilon)t\}\cap B_{(8.42)}\cap C_{(8.42)}\Big)\\
\le\ & \mathbb{P}\Big(\{\exists i\le \xi : \Gamma_i Y_i > \varepsilon t\,,\ \sum_{j\ne i} Y_j\Gamma_j > \varepsilon t\}\cap C_{(8.42)}\Big)\\
\le\ & (\varepsilon t)^{-1-q}\,\mathbb{E}\left[\sum_{i=1}^{\xi} Y_i\Gamma_i(\sum_{j\ne i} Y_j\Gamma_j)^q 1_{C_{(8.42)}}\right]\\
\le\ & (\varepsilon t)^{-1-q}\,\mathbb{E}\left[\sum_{i=1}^{\xi} Y_i\,(\sum_{k\ne i} Y_k)^{q-1}\,(\sum_{j\ne i} Y_j\Gamma_j^q\Gamma_i)\,1_{C_{(8.42)}}\right]\\
\le\ & (\varepsilon t)^{-1-q}\,\mathbb{E}[\Gamma_1]\,\mathbb{E}[\Gamma_1^q]\,\mathbb{E}\left[W^{1+q}1_{C_{(8.42)}}\right],
\end{aligned}$$

since $(\sum_{j\ne i} Y_j\Gamma_j)^q \le (\sum_{k\ne i} Y_k)^{q-1}(\sum_{j\ne i} Y_j\Gamma_j^q)$ for all i by the convexity inequality and since the Γ_j's are i.i.d. and independent from $\mathbb{Y}$. Furthermore, observe that $\mathbb{E}\left[W^{1+q}1_{C_{(8.42)}}\right] \le \mathbb{E}\left[W^{p+\delta}\right]\, t^{(1+q-p-\delta)(1-\delta/2)}$. Therefore, we obtain

$$\mathbb{P}\Big(\{\exists i\le\xi : \varepsilon t < \Gamma_i Y_i < (1-\varepsilon)t\}\cap B_{(8.42)}\cap C_{(8.42)}\Big) \le c_{\varepsilon,q}t^{-p-(1+q-p)\delta/2}.$$

This combined with (8.41) and (8.43) yields that, for all large t,

$$\begin{aligned}
\mathbb{P}\Big(B_{(8.42)}\Big) &\le \mathbb{P}\Big(\max_{1\le i\le\xi}(Y_i\Gamma_i) > (1-\varepsilon)\,t, C_{(8.42)}\Big) + c'_{p,\delta}t^{-p}\,\varepsilon^{\delta} + o(t^{-p})\\
&\le \mathbb{E}\Big[\sum_{i=1}^{\xi}\frac{(a+\varepsilon)Y_i^p}{(1-\varepsilon)^p t^p}1_{\{W\le t^{1-\delta/2}\}}\Big] + c'_{p,\delta}t^{-p}\,\varepsilon^{\delta} + o(t^{-p}).
\end{aligned}$$

It follows that

$$\limsup_{t\to\infty} t^p\mathbb{P}\Big(\sum_{i=1}^{\xi} Y_i\,\Gamma_i > t\Big) \le \mathbb{E}\Big[\sum_{i=1}^{\xi}\frac{(a+\varepsilon)Y_i^p}{(1-\varepsilon)^p}\Big] + c'_{p,\delta}\,\varepsilon^{\delta},$$

where $\delta>0$ is fixed. Letting $\varepsilon\to 0$ yields the upper bound and completes the proof of the Lemma. □

Acknowledgements: We thank Pascal Maillard for helpful discussions.

References

[1] Addario-Berry, L. and Broutin, N. (2009+). Total progeny in killed branching random walk. To appear in *Probability Theory and Related Fields.* `arXiv:0908.1083`.

[2] Addario-Berry, L. and Reed, B. (2009). Minima in branching random walks. *Ann. Probab.*, **37**, 1044–1079.

[3] Aïdékon, E. (2010). Tail asymptotics for the total progeny of the critical killed branching random walk. *Elec. Comm. Probab.*, **15**, 522–533.

[4] Aldous, D. Power laws and killed branching random walks, *URL http://www.stat.berkeley. edu/ aldous/Research/OP/brw.html.*

[5] Berestycki, J., Berestycki, N. and Schweinsberg, J. (2010+). The genealogy of branching Brownian motion with absorption. Preprint, `arXiv:1001.2337`.

[6] Bertoin, J. and Doney, R.A. (1994). On conditioning a random walk to stay nonnegative. *Ann. Probab.*, **22**, 2152–2167.

[7] Biggins, J. D. (1976). The first- and last-birth problems for a multitype age-dependent branching process. *Adv. Appl. Probab.*, **8**, 446–459.

[8] Biggins, J. D.(1977). Martingale convergence in the branching random walk. *J. Appl. Probab.*, **142**, 25–37.

[9] Biggins, J. D. and Kyprianou, A. E. (2004). Measure change in multitype branching. *Adv. in Appl. Probab.*, **36**, 544–581.

[10] Chang, J. T. (1994). Inequalities for the Overshoot. *Ann. Appl. Probab.*, **4**, 1223–1233.

[11] Doney, R.A. (1980). Moments of Ladder Heights in Random Walks. *J. Appl. Probab.*, **17**, 248–252.

[12] Feller, W. (1971). *An Introduction to Probability Theory and its Applications,* Vol. II. (2nd ed.). Wiley, New York.

[13] Hammersley, J. M. (1974). Postulates for subadditive processes. *Ann. Probab.*, **2**, 652–680.

[14] Heyde, C. C. (1967). Asymptotic renewal results for a natural generalization of classical renewal theory. *J. Roy. Statist. Soc.*, **29**, 141–150.

[15] Hu, Y. and Shi, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.*, **37**, 742–789.

[16] Iglehart, D.L. (1972). Extreme values in the GI/G/1 queue. *Ann. Math. Statist.*, **43**, 627–635.

[17] Jaffuel, B. (2009+). The critical barrier for the survival of the branching random walk with absorption. Preprint. `ArXiv math.PR/0911.2227`.

[18] Jagers, P. (1989). General branching processes as Markov fields. *Stochastic Process. Appl.*, **32**, 183–212.

[19] Jelenković, P.R. and Olvera-Cravioto, M. (2010+). Implicit renewal theory and power tails on trees. Preprint. `arXiv:1006.3295v1`.

[20] Kallenberg, O. (1976). *Random Measures.* Academie-Verlag, Berlin.

[21] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Act. Math.*, **131**, 207–248.

[22] Kingman, J. F. C. (1975). The first birth problem for an age-dependent branching process. *Ann. Probab.*, **3**, 790–801.

[23] Liu, Q. (2000). On generalized multiplicative cascades, *Stochastic Process. Appl.*, **86**, 263–286.

[24] Lorden, G. (1970). On Excess Over the Boundary. *Ann. Math. Statist.*, **41**, 520–527.

[25] Lyons, R. (1997). A simple path to Biggins' martingale convergence for branching random walk. *Classical and modern branching processes (Minneapolis, MN, 1994)*, **84**, 217–221.

[26] Lyons, R., Pemantle, R. and Peres, Y. (1995). Conceptual proofs of $L\log L$ criteria for mean behavior of branching processes. *Ann. Probab.*, **23**, 1125–1138.

[27] Maillard, P. (2010+). The number of absorbed individuals in branching Brownian motion with a barrier. Preprint. `arXiv:1004.1426v2`.

[28] Nerman, O. (1981). On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **57**, 365–395.

[29] Pemantle, R. (1999). Critical killed branching process tail probabilities. *Manuscript.*

[30] Resnick, S. I. (1987). *Extreme values, regular variation and point processes.* Springer-Verlag (New-York).

[31] Tanaka, H. (1989). Time reversal of random walks in one dimension. *Tokyo J. Math.*, **12**, 159–174.

[32] Vatutin, V.A. and Wachtel, V. (2009). Local probabilities for random walks conditioned to stay positive. *Probab. Theory Relat. Fields*, **143**, 177-217.

Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

E-mail address: elie.aidekon@gmail.com

Département de Mathématiques, CNRS UMR 7539, Université Paris XIII, 93430 Villetaneuse, France

E-mail address: yueyun@math.univ-paris13.fr

Laboratoire de Probabilités et Modèles Aléatoires, CNRS UMR 7599, Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail address: olivier.zindy@upmc.fr