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P.M. van de Ven, J.S.H. van Leeuwaarden, D. Denteneer, A.J.E.M. Janssen  
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# Spatial fairness in linear random-access networks

P.M. van de Ven<sup>1,2</sup>                      D. Denteneer<sup>3</sup>  
J.S.H. van Leeuwen<sup>1,2</sup>                A.J.E.M. Janssen<sup>1,2</sup>

## Abstract

Random-access networks may exhibit severe unfairness in throughput, in the sense that some nodes receive consistently higher throughput than others. Recent studies show that this unfairness is due to local differences in the neighborhood structure: nodes with fewer neighbors receive better access. We study the unfairness in saturated linear networks, and adapt the random-access CSMA protocol to remove the unfairness completely, by choosing the activation rates of nodes as a specific function of the number of neighbors. We then investigate the consequences of this choice of activation rates on the network-average saturated throughput, and we show that these rates perform well in non-saturated settings.

## 1 Introduction

Random-access protocols such as CSMA [15] have gained much popularity for their ability to regulate the access of network nodes to a shared medium in a fully distributed fashion, and are for example used in the IEEE 802.11 standard. A major drawback of these protocols, however, is that they can exhibit severe unfairness, in the sense that some of the nodes get starved, while others receive good access. We propose a way of compensating for the possible unfairness by enhancing the random-access protocols with local information about the immediate neighborhood of nodes.

Unfairness in wireless networks is an active topic of research. Wang and Kar [22] considered three nodes on a line that only block their direct neighbors, and showed that the middle node is starved when the activation rates of all three nodes increase. Such unfairness has been studied for more general networks by Durvy, Dousse and Thiran [6, 7] and Denteneer, Borst, Van de Ven and Hiertz [4]. We study a similar model as in [6, 7, 4], i.e., a network with  $n$  nodes on a line, in which active nodes block a certain subset of other nodes. In particular, unblocked nodes become active, and active nodes deactivate, after exponential times, and an active node blocks the first  $\beta$  nodes on both sides. We say that nodes that might block each other are *neighbors*.

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<sup>1</sup>Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

<sup>2</sup>Eurandom, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

<sup>3</sup>Philips Research Europe, 5656 AE Eindhoven, The Netherlands

To deal with the unfairness, we shall modify the model in [6, 7, 4] in one important way: instead of letting nodes activate at the same rate, we allow for node-specific activation rates. By choosing the activation rate  $\nu_i$  of node  $i$  as a particular function of the number of its neighbors, we can guarantee that all nodes in the network have the same throughput, completely removing the unfairness. Our main contribution is that we prove that this fair choice of activation rates  $\nu_i^*$ , takes on the extremely simple form

$$\nu_i^* = \alpha(1 + \alpha)^{\gamma(i) - \gamma(1)} \quad (1)$$

with  $\gamma(i)$  the number of neighbors of node  $i$ , and  $\alpha$  any positive constant. These node-dependent activation rates are still in line with the distributed nature of the random-access protocol, as  $\nu_i^*$  only requires the number of neighbors, which a node can obtain locally by sensing its direct environment. By choosing the activation rates according to (1) we essentially ensure long-term fairness. The first result in this direction is due to Kelly [13], who considered a tree network (of which the linear network is a special case) with nearest-neighbor blocking ( $\beta = 1$ ). The results in this paper are only for linear networks, although the Markov random field approach from [13] allows us to extend our results to various other topologies, see Section 6.

In the classical models the activation rates (back-off times) are assumed to be equal, and fixed over time. In our model we allow different, but fixed, activation rates. When the activation rates are allowed to be adapted over time, there are simple necessary and sufficient conditions for stability to be achievable. Several authors have proposed clever backlog-based algorithms for adapting activation rates that achieve stability whenever it is feasible to do so at all [10, 12, 11, 16, 17, 19].

The paper is structured as follows. In Section 2 we introduce the linear network in more detail. In Section 3 we study some of the key features of the unfairness that arises when all nodes have equal activation rates. In Section 4 we prove that the activation rates in (1) yield equal throughputs. In Section 5 we investigate the impact of the rates in (1) on the network-average throughput. Section 6 presents some conclusions and further research directions.

## 2 Model description

We consider  $n$  nodes on a line, and assume that all nodes are saturated (have an infinite supply of packets available for transmission). A transmitting node blocks the first  $\beta$  nodes on both sides. When node  $i$  is blocked, it remains silent until all nodes within distance  $\beta$  are inactive, at which point it tries to activate after an exponentially distributed (back-off) time with mean  $1/\nu_i$ . Node  $i$  activates if it is still unblocked when the back-off timer runs out. If a node finds itself blocked when the back-off timer expires, it waits until all neighboring nodes are inactive once more and then repeats the back-off procedure. Without loss of generality, we assume that transmissions last for an exponentially distributed

time with unit mean. Under these assumptions, the  $n$ -dimensional process that describes the activity of nodes is a continuous-time Markov process.

Each state of the Markov process is described as

$$\omega = (\omega_1, \dots, \omega_n) \in \{0, 1\}^n,$$

where  $\omega_i = 1$  when node  $i$  is active. Let  $\Omega \subseteq \{0, 1\}^n$  be the set of all feasible states. Call  $\omega \in \Omega$  *feasible* if no two 1's in  $\omega$  are  $\beta$  positions or less apart, i.e.,  $\omega_i \omega_k = 0$  if  $1 \leq |i - k| \leq \beta$ .

The Markov process that describes the activity of nodes is then fully specified by the state space  $\Omega$  and the transition rates

$$r(\omega, \omega') = \begin{cases} \nu_i & \text{if } \omega' = \omega + e_i, \\ 1 & \text{if } \omega' = \omega - e_i, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Here  $e_i$  denotes the vector with all zeros except for a 1 at position  $i$ .

Alternatively, we can express the set of feasible states as all states that satisfy a certain system of linear equations. Let  $A$  be an  $(n - \beta) \times n$  matrix where each row contains  $\beta + 1$  consecutive 1's, in the following way:

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ & & \ddots & & & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{pmatrix}. \quad (3)$$

Now we can write the state space as  $\Omega = \{\omega \in \{0, 1\}^n \mid A\omega \leq C\}$ , where  $C$  is the all-1 vector (of size  $n - \beta$ ). This characterization has a natural interpretation as a set of capacity constraints, and nodes can activate only when enough capacity is available. We allocate unit capacity to each node, and use the convention that whenever a node is active it uses its own capacity, as well as the capacity of all its neighbors to the left. The  $i$ th row of  $A$  thus represents the capacity required when node  $i$  is active. The constraints that arise from the first  $\beta$  nodes on the line are redundant, and ignoring these leads to the matrix  $A$  in (3).

From the description of the state space as a set of capacity constraints, it is clear that our model belongs to the general class of loss networks, see Kelly [14]. Loss networks are known to be reversible, and possess product-form solutions. For our Markov process, this product-form solution is given by the stationary measure  $\pi$  on  $\Omega$  for which

$$\pi(\omega) = \begin{cases} Z_n^{-1} \prod_{i=1}^n \nu_i^{\omega_i} & \text{if } \omega \in \Omega, \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

and where  $Z_n$  is the normalization constant that makes  $\pi$  a probability measure. This result is well known in the context of wireless networks, see e.g. [2, 4, 6, 22].

Our main concern is with the long-term behavior of nodes, characterized by their throughputs. A common throughput-degrading phenomenon in wireless networks is collisions, which may occur when multiple nearby nodes transmit simultaneously, causing these transmissions to fail. However, we assume that the blocking range  $\beta$  is large enough to rule out collisions, so any activity of a node contributes to its throughput. Although a model without collisions might seem limited, numerous simulation studies show that choosing the blocking range just large enough to preclude collisions gives very good performance, see e.g. [9, 21, 24, 26]. In fact, [20] shows that this choice is throughput-optimal when the activation rates are sufficiently large. We study the throughput vector  $\theta = (\theta_1, \dots, \theta_n)$ , where  $\theta_i$  represents the fraction of time node  $i$  is active. Denote the total number of feasible states by  $K$ , let  $\Omega = \{\Omega_1, \dots, \Omega_K\}$ , and introduce the  $n \times K$  incidence matrix  $X$  such that  $X_{ik} = 1$  when the  $i$ th element in the state  $\Omega_k$  equals 1. Then

$$\theta = X \cdot \Pi, \quad (5)$$

with  $\Pi = (\pi(\Omega_1), \dots, \pi(\Omega_K))$ .

By exploiting the structure of the network, we can construct alternative expressions for the throughput in (5). Specifically, we shall make use of the observation that if node  $i$  is active, nodes to the left of  $i$  behave independently from nodes to the right of  $i$ . This leads to the following theorem.

**Theorem 1.** *Define the sequence  $(Z_i)_{i=-\infty}^{\infty}$  such that  $Z_i = 1$  for  $i \leq 0$ , and*

$$Z_i = 1 + \nu_1 + \dots + \nu_i, \quad i = 1, 2, \dots, \beta + 1, \quad (6)$$

$$Z_i = Z_{i-1} + \nu_i Z_{i-\beta-1}, \quad i = \beta + 2, \beta + 3, \dots \quad (7)$$

*Let the vector of activation rates  $\nu = (\nu_1, \dots, \nu_n)$  be symmetric, i.e.,  $\nu_i = \nu_{n+1-i}$ ,  $i = 1, \dots, n$ . Then*

$$\theta_i = \nu_i \frac{Z_{i-\beta-1} Z_{n-i-\beta}}{Z_n}, \quad i = 1, \dots, n. \quad (8)$$

*Proof.* By conditioning on whether node  $i$  is active, we can decompose the activity of the network into two parts, separated by this active node (see [2], Equation (15)),

$$\theta_i = \nu_i \frac{Z_{1:i-\beta-1} Z_{i+\beta+1:n}}{Z_{1:n}}, \quad (9)$$

where  $Z_{i:j}$  is the normalization constant of a network consisting only of nodes  $i, \dots, j$ . For simplicity we denote  $Z_i := Z_{1:i}$ , and the symmetry of  $\nu$  implies

$$Z_{i:n} = Z_{1:n-i+1}. \quad (10)$$

Substituting (10) into (9) yields the expression for  $\theta_i$  in (8). By conditioning on the activity of node  $i$ , we immediately get the recursion relation (7).  $\square$

### 3 Unfairness

We now venture deeper into the problem of unfairness, and assume for now that all nodes have equal activation rates  $\nu_i = \sigma$ . As observed by Durvy *et al.* [6] and Denteneer *et al.* [4], the throughput distribution in this case is highly unfair, in the sense that some nodes have a larger throughput than others. This unfairness can be explained by the *node-in-the-middle* phenomenon discussed for example in Wang and Kar [22] and Garetto, Salonidis and Knightly [8] for the case  $n = 3$ ,  $\beta = 1$ : the middle node is in an unfavorable position as it has to wait for both outer nodes to deactivate, whereas these outer nodes only compete for transmission with the middle node.

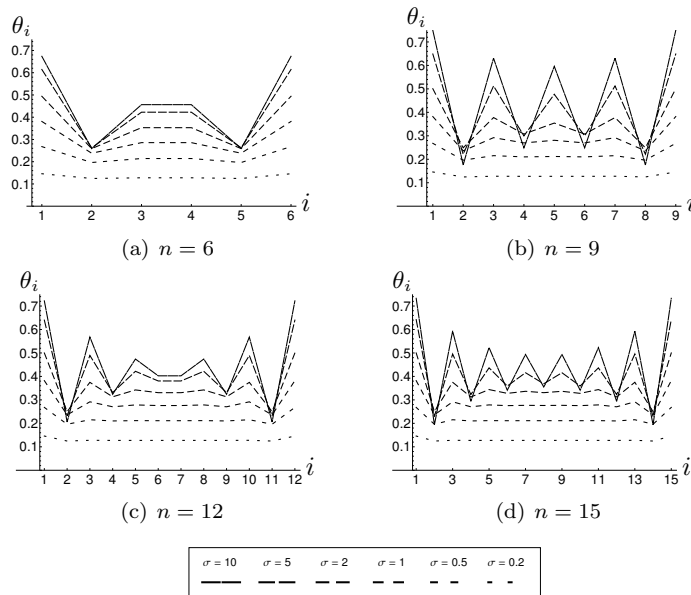


Figure 1: The per-node throughput for  $\beta = 1$  and various values of  $n$  and  $\sigma$

In order to compute the throughput in (8), we need to compute the  $Z_i$  from (6) and (7). When all nodes have equal activation rate  $\sigma$ , the generating function  $G_Z(x)$  of the  $Z_i$  is given by (see Van de Ven, Janssen and Van Leeuwen [20])

$$G_Z(x) = \sum_{i=0}^{\infty} Z_i x^i = \frac{x - 1 + \sigma x^{\beta+1} - \sigma x}{(x - 1)(1 - x - \sigma x^{\beta+1})}. \quad (11)$$

Let  $\lambda_0, \dots, \lambda_\beta$  denote the  $\beta + 1$  roots of

$$\lambda^{\beta+1} - \lambda^\beta - \sigma = 0. \quad (12)$$

These roots can be shown to be distinct, and from Rouché's theorem it follows that there exists a unique positive real root  $\lambda_0$  such that  $\lambda_0 > |\lambda_j|$ ,  $j = 1, \dots, \beta$ .

We can obtain the  $Z_i$  by applying partial fraction expansion to (11), which gives ([20], Proposition 1)

$$Z_i = \sum_{j=0}^{\beta} c_j \lambda_j^i, \quad i = 0, 1, \dots \quad (13)$$

Of course, in this particular case of equal  $\sigma$ , the  $Z_i$  satisfy a homogeneous recursion equation per (7) and hence it is no surprise that they have the form (13). The  $c_j$  follow from (6) and their particular form is given in [20] as

$$c_j = \frac{\lambda_j^{\beta+1}}{(\beta+1)\lambda_j - \beta}. \quad (14)$$

Moreover,

$$Z_i = c_0 \lambda_0^i (1 + o(1)), \quad i \rightarrow \infty, \quad (15)$$

and we shall use this asymptotic relation at several places, see (17) and (59).

For ease of presentation, we restrict ourselves to  $\beta = 1$  in the remainder of this section. Figures 1(a)-1(d) show the per-node throughput for various values of  $n$  and  $\sigma$ . All figures display a similar pattern, with the outer nodes having the highest throughput. Moreover, all figures are symmetric, and exhibit some form of oscillatory behavior. These observations are formalized in the following result.

**Proposition 1.** *For  $\nu_i = \sigma > 0$ ,  $i = 1, \dots, n$  and  $\beta = 1$ , the throughput has the following properties:*

(i) *Symmetric:*  $\theta_i = \theta_{n-i+1}$ ,  $i = 1, 2, \dots, n$ .

(ii) *Alternating and converging:*  $(-1)^i (\theta_{i+1} - \theta_i)$  is positive and decreasing for  $i = 1, 2, \dots, \lfloor n/2 \rfloor$ .

Proposition 1 is proven in A.

In Figure 1 we see that for  $\beta = 1$ , the largest difference in throughput is between nodes 1 and 2. Proposition 1(ii) confirms that this is the most unfair situation, and it persists even in larger networks where the node-in-the-middle problem is mitigated by the activity of the remaining nodes. In fact, for large networks we have the following result.

**Proposition 2.** *For  $\nu_i = \sigma > 0$ ,  $i = 1, \dots, n$  and  $\beta = 1$ , we have*

$$\frac{\theta_1}{\theta_2} \sim \lambda_0 = \frac{1 + \sqrt{1 + 4\sigma}}{2}, \quad n \rightarrow \infty. \quad (16)$$

*Proof.* We have from (8) that  $\theta_1 = Z_{n-2}/Z_n$  and  $\theta_2 = Z_{n-3}/Z_n$ . Using (15) we obtain

$$\frac{\theta_1}{\theta_2} \sim \lambda_0, \quad n \rightarrow \infty. \quad (17)$$

For  $\beta = 1$  we can explicitly solve (12) to obtain  $\theta_1/\theta_2 \sim \lambda_0 = \frac{1}{2}(1 + \sqrt{1 + 4\sigma})$ .  $\square$

We note that for  $\beta = 1$ , the  $Z_i$  satisfy a three-term recursion reminiscent of that satisfied by the Chebyshev polynomials  $U_n$  of the second kind. Accordingly, we have

$$Z_i = (-\sigma)^{\frac{1}{2}(i+1)} U_{i+1}(\sqrt{-1/4\sigma}) = \sum_{j=0}^{\lfloor \frac{i+1}{2} \rfloor} \binom{i+1-j}{j} \sigma^j. \quad (18)$$

The latter expression can be interpreted as the summation over all possible combinations of nodes that can be active simultaneously.

Figure 1 shows another interesting property of this network. Increasing  $\sigma$  leads in many cases to a higher throughput for each of the nodes. However, we also observe that there exists a critical value  $\sigma^*$ , such that at least one of the throughputs  $\theta_i$  decreases as  $\sigma$  increases beyond  $\sigma^*$ . The characterization of this critical value is a possible topic for future research.

Results similar to those presented in this section can be obtained for  $\beta \geq 2$ . As an example, Figures 2(a)-2(b) show the per-node throughput for  $n = 9$  and  $\beta = 2, 3$ . Both figures exhibit similar oscillatory behavior as observed for  $\beta = 1$ , although the oscillation period increases with  $\beta$ .

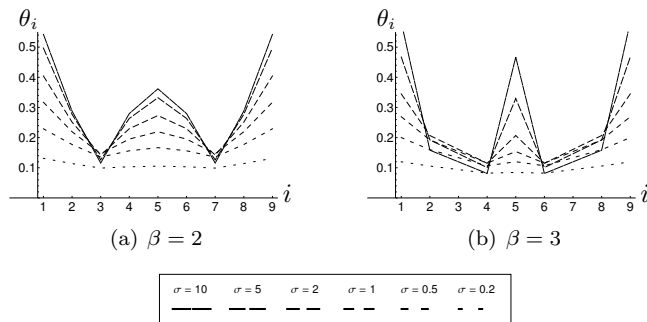


Figure 2: The per-node throughput for  $n = 9$  and various values of  $\beta$  and  $\sigma$

## 4 Fairness

In this section we present a way to completely remove the unfairness that was discussed in Section 3. In order to do so, we choose node-dependent activation rates  $\nu_i$  such that all nodes have equal throughput ( $\theta_1 = \theta_2 = \dots = \theta_n$ ). From (4) and (5) we see that in order to meet this objective we have to solve a system of  $n$  nonlinear equations. It seems that in general this system cannot be solved directly. We therefore choose a more indirect approach, and we first consider two special cases that can be solved explicitly. The insight obtained from these exact solutions is then used to guess the general solution to the system of nonlinear equations.



The first case is when  $\beta = n - 2$ , so that all but the two outer nodes will block the entire network.

**Proposition 3.** *For linear networks with three or more nodes, and  $\beta = n - 2$ , setting  $\nu_1 = \nu_n = \alpha$  and  $\nu_i = \alpha(1 + \alpha)$  for all other nodes yields equal throughputs*

$$\theta_i = \frac{\alpha}{1 + (n - 1)\alpha}, \quad i = 1, \dots, n. \quad (19)$$

*Proof.* From (8) we see that

$$\theta_1 = Z_n^{-1} \nu_1 (1 + \nu_n), \quad (20)$$

$$\theta_i = Z_n^{-1} \nu_i, \quad i = 2, 3, \dots, n - 1, \quad (21)$$

$$\theta_n = Z_n^{-1} \nu_n (1 + \nu_1). \quad (22)$$

The inherent symmetry of the model allows us to set  $\nu_1 = \nu_n$ . Moreover, for the throughput of the other nodes to be equal, we require  $\nu_2 = \dots = \nu_{n-1} = \nu_1(1 + \nu_1)$ . If we set  $\nu_1 = \alpha$ , and substitute this into (20)-(22), we get a throughput of

$$\theta_i = Z_n^{-1} \alpha(1 + \alpha). \quad (23)$$

The normalization constant  $Z_n$  can be determined by summing over all feasible states:

$$\begin{aligned} Z_n &= 1 + \sum_{i=1}^n \nu_i + \nu_1 \nu_n = 1 + (n - 2)\alpha(1 + \alpha) + 2\alpha + \alpha^2 \\ &= (1 + \alpha)(1 + (n - 1)\alpha). \end{aligned} \quad (24)$$

Substituting (24) into (23) yields (19).  $\square$

The case  $n = 5$ ,  $\beta = 3$  of Proposition 3 was considered in [4]. The second special case corresponds to  $n = 2(\beta + 1)$ , so that a node blocks at least half of the network.

**Proposition 4.** *For linear networks with  $n = 2m$  nodes,  $m \in \mathbb{N}$ , and  $\beta = m - 1$ , setting  $\nu_i = \alpha(1 + \alpha)^{i-1}$  for  $i = 1, \dots, m$  yields equal throughputs*

$$\theta_i = \frac{\alpha}{1 + m\alpha}, \quad i = 1, \dots, n. \quad (25)$$

*Proof.* To achieve equal throughputs, we see from (4) and (5) that for the case at hand we should solve the system of equations

$$\begin{aligned} \nu_1 + \nu_1(\nu_{m+1} + \dots + \nu_n) &= \nu_2 + \nu_2(\nu_{m+2} + \dots + \nu_n) \\ &= \nu_3 + \nu_3(\nu_{m+3} + \dots + \nu_n) \\ &\vdots \\ &= \nu_m + \nu_m \nu_n. \end{aligned} \quad (26)$$

Indeed, the throughput of node  $i$  can be written as a sum over all states in which node  $i$  is active. Using symmetry, (26) can be written as

$$\begin{aligned}\nu_1 + \nu_1(\nu_1 + \cdots + \nu_m) &= \nu_2 + \nu_2(\nu_1 + \cdots + \nu_{m-1}) \\ &= \nu_3 + \nu_3(\nu_1 + \cdots + \nu_{m-2}) \\ &\vdots \\ &= \nu_m + \nu_m \nu_1.\end{aligned}\tag{27}$$

Let  $\nu_1 = \alpha > 0$ . The solution of (27) is easily seen to be

$$\nu_i = \alpha(1 + \alpha)^{i-1}, \quad i = 1, \dots, m,\tag{28}$$

and hence

$$\theta_i = Z_n^{-1} \alpha(1 + \alpha)^m.\tag{29}$$

Summing over all possible states yields

$$\begin{aligned}Z_n &= 1 + \sum_{i=1}^n \nu_i + \sum_{i=1}^m \nu_i \sum_{j=i+m}^n \nu_j \\ &= 1 + \sum_{i=1}^m \nu_i + \sum_{i=1}^m \nu_i (1 + \nu_1 + \cdots + \nu_{m-i-1}) \\ &= 1 + ((1 + \alpha)^m - 1) + m\alpha(1 + \alpha)^m \\ &= (1 + m\alpha)(1 + \alpha)^m.\end{aligned}\tag{30}$$

Substituting (30) into (29) gives (25).  $\square$

It is clear that the complexity of the system of equations governed by (5) reduces considerably for the choices of  $\beta$  discussed in Propositions 3 and 4. For general  $\beta$  this system remains rather complicated. However, we can use Propositions 3 and 4 to make an educated guess about the general solution. First observe that the fair activation rates in Propositions 3 and 4 only depend on the number of neighbors (nodes within  $\beta$  hops) that each node has. Denote by  $\gamma(i)$  the number of neighbors of node  $i$ , let  $\alpha > 0$ , and choose activation rates  $\nu_i^*$  as in (1). We see that this choice is consistent with the fair activation rates in Propositions 3 and 4. We now show that  $\nu_i^*$  indeed achieves fairness for all  $\beta$ . To this end, we first show that when the activation rates are chosen according to (1), the recursive relations (6) and (7) for the normalization constant  $Z_i$  have a closed-form solution.

**Lemma 1.** *Let  $\alpha > 0$  and choose  $\nu_i^*$  as in (1). Then*

$$Z_i = (1 + \alpha)^i, \quad i = 1, 2, \dots, n - \beta.\tag{31}$$

*Proof.* Substituting (31) into (6) gives, for  $i \leq \beta + 1$ ,

$$Z_i = 1 + \alpha + \alpha(1 + \alpha) + \cdots + \alpha(1 + \alpha)^{i-1} = (1 + \alpha)^i.\tag{32}$$

Substituting (31) into (7) gives

$$Z_i = (1 + \alpha)^{i-1} + \alpha(1 + \alpha)^\beta(1 + \alpha)^{i-\beta-1} = (1 + \alpha)^i, \quad (33)$$

for  $i \geq \beta + 2$ .  $\square$

With Lemma 1 we are now in position to prove our main result.

**Theorem 2.** *Let  $\alpha > 0$ ,  $\beta \leq n - 1$  and choose  $\nu_i^*$  as in (1). Then*

$$\theta_i = \frac{\alpha}{1 + (1 + \beta)\alpha}, \quad i = 1, \dots, n. \quad (34)$$

*Proof.* To prove this result we substitute the normalization constants from Lemma 1 into the expression for the throughput in (8). We distinguish between different values of  $i$ .

For  $i \geq \beta + 1$  and  $i \leq n - \beta$  we see that  $\nu_i^* = \alpha(1 + \alpha)^\beta$  and

$$Z_{i-\beta-1} = (1 + \alpha)^{i-\beta-1}, \quad Z_{n-i-\beta} = (1 + \alpha)^{n-i-\beta}. \quad (35)$$

Similarly, for  $i \geq \beta + 1$  and  $i \geq n - \beta + 1$  we have  $\nu_i^* = \alpha(1 + \alpha)^{n-i}$  and

$$Z_{i-\beta-1} = (1 + \alpha)^{i-\beta-1}, \quad Z_{n-i-\beta} = 1. \quad (36)$$

For  $i \leq \beta$  and  $i \leq n - \beta$  we have  $\nu_i^* = \alpha(1 + \alpha)^{i-1}$  and

$$Z_{i-\beta-1} = 1, \quad Z_{n-i-\beta} = (1 + \alpha)^{n-i-\beta}. \quad (37)$$

Finally, for  $i \leq \beta$  and  $i \geq n - \beta + 1$  we have  $\nu_i^* = \alpha(1 + \alpha)^{n-\beta-1}$  and

$$Z_{i-\beta-1} = 1, \quad Z_{n-i-\beta} = 1. \quad (38)$$

Substituting (35)-(38) into (8) yields

$$\theta_i = Z_n^{-1} \alpha(1 + \alpha)^{n-\beta-1}. \quad (39)$$

We next consider the normalization constant. Let  $m$  such that  $n = \beta + m$ , then by (7),

$$Z_n = Z_{n-1} + \nu_n^* Z_{n-\beta-1}, \quad (40)$$

which gives upon iteration

$$Z_n = Z_{n-\beta} + \sum_{i=1}^{\beta} \nu_{n+1-i}^* Z_{n-\beta-i}. \quad (41)$$

Substituting (31) into (41) yields

$$\begin{aligned} Z_n &= (1 + \alpha)^{n-\beta} + \sum_{i=1}^{\min\{m, \beta\}} \alpha(1 + \alpha)^{i-1} (1 + \alpha)^{n-\beta-i} + \sum_{i=m+1}^{\beta} \alpha(1 + \alpha)^{n-\beta-i} \\ &= (1 + \alpha)^{n-\beta-1} (1 + (\beta + 1)\alpha). \end{aligned} \quad (42)$$

Combining (42) and (39) leads to (34).  $\square$

To better understand why the rates (1) only depend on the number of neighbors of each node, we study the rates in the two limiting regimes of light traffic ( $\alpha \downarrow 0$ ) and heavy traffic ( $\alpha \rightarrow \infty$ ). First rewrite (1) as

$$\nu_i^* = \alpha \sum_{j=0}^{\gamma(i)-\gamma(1)} \binom{\gamma(i)-\gamma(1)}{j} \alpha^j, \quad i = 1, \dots, n. \quad (43)$$

When  $\alpha$  is small, nodes activate slowly, and few nodes will be active simultaneously. In fact, the Markov process spends most of its time in states with at most one active node, and node interaction (blocking) is negligible. This is reflected in the light-traffic activation rates that follow immediately from (43):

$$\nu_i^* = \alpha + (\gamma(i) - \gamma(1))\alpha^2 + O(\alpha^3), \quad \alpha \downarrow 0. \quad (44)$$

Hence, for small  $\alpha$ ,  $\nu_i^* \approx \alpha$ , which is the same for all nodes. Indeed, if at most one node is active (as is the case for  $\alpha$  small), there is no blocking, and therefore no need to discriminate between nodes. As  $\alpha$  increases, states with two active nodes are increasingly likely, and nodes may now block their neighbors (all nodes within distance  $\beta$ ). This is accounted for in the activation rate by the term  $(\gamma(i) - \gamma(1))\alpha^2$ , which is linear in the number of neighbors. Thus in light traffic, only the number of neighbors is of importance, rather than the structure of the entire network. This reasoning extends to more general networks.

Next, we consider large activation rates, and we compare the equal rates (all nodes activate with rate  $\sigma \rightarrow \infty$ ) with the fair rates (1) (with  $\alpha \rightarrow \infty$ ). In both cases, nodes activate almost instantaneously when they get the chance to do so, i.e., when all neighbors are inactive. Consequently, the only states that have positive limiting probability are those consisting of maximal independent sets of active nodes. The distribution according to which these maximal states occur depends on the choice of activation rates.

First consider the case of equal activation rates  $\nu_i = \sigma$ ,  $i = 1, \dots, n$ . We have seen in Section 3 that this creates unfairness, and that the unfairness increases with  $\sigma$ . In particular, we see from (4) that the only states that have positive probability for  $\sigma \rightarrow \infty$  are those of maximum size, i.e., states with  $\lceil n/(\beta + 1) \rceil$  active nodes. Thus, for  $\sigma \rightarrow \infty$ ,

$$\pi(\omega) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } \omega \in \mathcal{M}, \\ 0 & \text{otherwise,} \end{cases} \quad (45)$$

with  $\mathcal{M} \subset \Omega$  the set of states of maximum size and  $|\mathcal{M}|$  the cardinality of this set. The throughput of each node is thus determined by the number of maximum states it is contained in, which is not necessarily the same for all nodes.

For the fair activation rates (43), we see that

$$\nu_i^* = \alpha^{\gamma(i)-\gamma(1)+1} + O(\alpha^{\gamma(i)-\gamma(1)}), \quad \alpha \rightarrow \infty. \quad (46)$$

So the activation rate of a node is characterized by the leading exponent  $\gamma(i) - \gamma(1) + 1$ , and the limiting probability of a state is determined by the sum

of these exponents over all active nodes. In fact, the only states that have positive limiting probability for  $\alpha \rightarrow \infty$  are those that maximize the sum of the exponents of  $\alpha$  over all active nodes. It turns out that there are  $\beta + 1$  such states, with active nodes  $\{i, i + (\beta + 1), \dots, i + (\beta + 1)\tau_i\}$ ,  $i = 1, \dots, \beta + 1$ , with  $\tau_i = (\lceil \frac{n+1-i}{\beta+1} \rceil - 1)$ . Each such state has limiting probability

$$\pi(\omega) = Z_n^{-1} \prod_{j=0}^{\tau_i} \nu_{i+(\beta+1)j} = Z_n^{-1} (\alpha^{n-\beta} + O(\alpha^{n-\beta-1})) = \frac{1}{\beta+1}, \quad \alpha \rightarrow \infty, \quad (47)$$

because  $Z_n = (\beta + 1)\alpha^{n-\beta} + O(\alpha^{n-\beta-1})$ , as  $\alpha \rightarrow \infty$ . Contrary to the case of equal rates, we see that each node appears in exactly one state with positive limiting probability. This explains the equal throughputs in heavy traffic.

This result strongly depends on the structure of the network, as the maximal independent sets may change drastically with the addition or the removal of even a single node. As a result, the simple, locally determined heavy-traffic activation rates (46) may only hold for linear networks.

## 5 Network-average throughput

The fair rates  $\nu_i^*$  in (1) are designed to remove the unfairness that arises when all nodes have equal activation rates  $\sigma$ . In order to compare the two schemes, we want to set their respective parameters  $\alpha$  and  $\sigma$  such that the average per-node throughputs are equal. In a network with  $\nu_i = \sigma > 0$ ,  $i = 1, \dots, n$ , write  $Z_i(\sigma)$  and  $\theta_i(\sigma)$  for the normalization constant of a network with  $i$  nodes, and the throughput of node  $i$ , respectively. Let  $\bar{\theta}_n(\sigma) := n^{-1} \sum_{i=1}^n \theta_i(\sigma)$  denote the average per-node throughput in a network with  $n$  nodes.

In Section 4 we showed that all nodes have equal throughput  $\alpha/(1+\alpha(\beta+1))$  when using the fair activation rates. When all nodes have equal activation rates, a closed-form expression for the throughput does not seem available. However, we can express the average throughput in terms of the normalization constant  $Z_n$ .

**Proposition 5.** *Let  $\nu_i = \sigma$ ,  $i = 1, \dots, n$ . The average per-node throughput is given by*

$$\bar{\theta}_n(\sigma) = \frac{\sigma}{nZ_n(\sigma)} \frac{dZ_n(\sigma)}{d\sigma}. \quad (48)$$

*Proof.* We have from (8) with  $\nu_i = \sigma$  that

$$\bar{\theta}_n(\sigma) = \frac{\sigma}{nZ_n} \sum_{i=1}^n Z_{i-\beta-1} Z_{n-i-\beta}. \quad (49)$$

We compute, using the definition of  $Z_i$  in Theorem 1,

$$\sum_{n=1}^{\infty} \left( \sum_{i=1}^n Z_{i-\beta-1} Z_{n-i-\beta} \right) x^n = x \left( \frac{x^\beta - 1}{x - 1} + x^\beta G_Z(x) \right)^2, \quad (50)$$

with  $G_Z(x)$  the generating function (11). Some rewriting then gives

$$\sum_{n=1}^{\infty} \left( \sum_{i=1}^n Z_{i-\beta-1} Z_{n-i-\beta} \right) x^n = \frac{x}{(1-x-\sigma x^{\beta+1})^2}. \quad (51)$$

On the other hand, we compute that

$$\frac{d}{d\sigma} [G_Z(x)] = \frac{x}{(1-x-\sigma x^{\beta+1})^2} \quad (52)$$

and the result follows.  $\square$

By Proposition 5 and the partial fraction expansion (13) of  $Z_i$ , we can express the average per-node throughput in terms of the roots  $\lambda_0, \dots, \lambda_\beta$  of (12).

**Proposition 6.** *Let  $\nu_i = \sigma$ ,  $i = 1, \dots, n$ . The average per-node throughput is given by*

$$\bar{\theta}_n(\sigma) = \frac{\sigma P}{n Q}, \quad (53)$$

where

$$P = \sum_{j=0}^{\beta} \frac{\lambda_j^{n+1}}{(\beta+1)\lambda_j - \beta} \left( \frac{n+\beta+1}{(\beta+1)\lambda_j - \beta} - \frac{(\beta+1)\lambda_j}{((\beta+1)\lambda_j - \beta)^2} \right), \quad Q = \sum_{j=0}^{\beta} \frac{\lambda_j^{n+\beta+1}}{(\beta+1)\lambda_j - \beta}. \quad (54)$$

*Proof.* By (13) and (14) we have

$$Z_n(\sigma) = \sum_{j=0}^{\beta} \frac{\lambda_j^{n+\beta+1}}{(\beta+1)\lambda_j - \beta}, \quad (55)$$

where  $\lambda_j$  are the  $(\beta+1)$  roots  $\lambda$  of (12). By implicit differentiation of (12) with respect to  $\sigma$  we find

$$\frac{d\lambda_j}{d\sigma} = \frac{1}{\lambda_j^{\beta-1}} \frac{1}{(\beta+1)\lambda_j - \beta}. \quad (56)$$

Then from (55) and (56) we get

$$\begin{aligned} \frac{dZ_n(\sigma)}{d\sigma} &= \sum_{j=0}^{\beta} \left( \frac{(n+\beta+1)\lambda_j^{n+\beta}}{(\beta+1)\lambda_j - \beta} - \frac{(\beta+1)\lambda_j^{n+\beta+1}}{((\beta+1)\lambda_j - \beta)^2} \right) \frac{d\lambda_j}{d\sigma} \\ &= \sum_{j=0}^{\beta} \frac{\lambda_j^{n+1}}{(\beta+1)\lambda_j - \beta} \left( \frac{n+\beta+1}{(\beta+1)\lambda_j - \beta} - \frac{(\beta+1)\lambda_j}{((\beta+1)\lambda_j - \beta)^2} \right). \end{aligned} \quad (57)$$

The result follows from substituting (55) and (57) into (48).  $\square$

When the network grows large ( $n \rightarrow \infty$ ) the root of largest modulus,  $\lambda_0$ , becomes dominant, and (53)-(54) simplifies.

**Corollary 1.** Let  $\nu_i = \sigma$ ,  $i = 1, \dots, n$ . The limiting average per-node throughput  $\bar{\theta}(\sigma) := \lim_{n \rightarrow \infty} \bar{\theta}_n(\sigma)$  is given by

$$\bar{\theta}(\sigma) = \frac{\lambda_0 - 1}{(\beta + 1)\lambda_0 - \beta}. \quad (58)$$

*Proof.* We have, as  $n \rightarrow \infty$ ,

$$P = \frac{\lambda_0^{n+1}}{(\beta + 1)\lambda_0 - \beta} \frac{n}{(\beta + 1)\lambda_0 - \beta} (1 + o(1)), \quad Q = \frac{\lambda_0^{n+\beta+1}}{(\beta + 1)\lambda_0 - \beta} (1 + o(1)). \quad (59)$$

Hence

$$\bar{\theta}_n(\sigma) = \frac{\sigma P}{n Q} = \frac{\sigma \lambda_0^{-\beta}}{(\beta + 1)\lambda_0 - \beta} (1 + o(1)), \quad (60)$$

and the result follows as  $\sigma \lambda_0^{-\beta} = \lambda_0 - 1$  by (12).  $\square$

The limiting expression (58) occurs in a variety of contexts in [7, 20, 18, 1, 25]. When  $\beta\sigma \rightarrow \infty$ , the throughput (58) simplifies even further.

**Corollary 2.** Let  $\nu_i = \sigma$ ,  $i = 1, \dots, n$  and let  $n \rightarrow \infty$ . The limiting average throughput satisfies

$$\bar{\theta}(\sigma) = \frac{1}{\beta + 1} (1 + o(1)), \quad \beta\sigma \rightarrow \infty. \quad (61)$$

*Proof.* By rewriting (58) we have

$$\bar{\theta}(\sigma) = \frac{1}{\beta + 1} \frac{1}{1 + \frac{1}{(\beta+1)(\lambda_0-1)}}. \quad (62)$$

Consequently, for  $\bar{\theta}(\sigma) = \frac{1}{\beta+1} (1 + o(1))$  to hold, it is necessary and sufficient that  $(\beta + 1)(\lambda_0 - 1) \rightarrow \infty$ . Recall from (12) that  $\lambda_0$  is such that

$$\lambda_0^\beta (\lambda_0 - 1) = \sigma. \quad (63)$$

Let  $M > 0$  be some positive constant, and assume that  $\beta\sigma \leq M$ . Then

$$\beta \lambda_0^\beta (\lambda_0 - 1) = \beta\sigma \leq M, \quad (64)$$

and so  $\beta(\lambda_0 - 1) \leq M$ . Conversely, assume that  $\beta(\lambda_0 - 1) \leq K$  for some positive constant  $K > 0$ . Then

$$\beta\sigma = \beta(\lambda_0 - 1)\lambda_0^\beta \leq \beta(\lambda_0 - 1)\exp(\beta(\lambda_0 - 1)) \leq K e^K. \quad (65)$$

Hence

$$\beta(\lambda_0 - 1) \text{ bounded} \Leftrightarrow \beta\sigma \text{ bounded}. \quad (66)$$

It follows that a sufficient condition for  $(\beta+1)(\lambda_0-1) \rightarrow \infty$  is that  $\beta\sigma \rightarrow \infty$ .  $\square$

Corollary 2 implies for  $\beta$  fixed and  $\sigma \rightarrow \infty$  that  $\bar{\theta}(\sigma) \rightarrow \frac{1}{\beta+1}$ . Thus for  $n \rightarrow \infty$ , both the equal and fair activation rates can achieve the maximum throughput by letting  $\sigma \rightarrow \infty$  and  $\alpha \rightarrow \infty$ , respectively. This can be explained by the observation that for both sets of activation rates, the system spends almost all the time in maximum-size independent sets of active nodes, thus maximizing spatial reuse.

Next, we fix  $\sigma > 0$  and search for  $\alpha = \alpha_n(\sigma)$  such that

$$\bar{\theta}_n(\sigma) = \frac{\alpha}{1 + \alpha(\beta + 1)}, \quad (67)$$

so the network-average throughput is identical for the fair rates and equal rates. For  $\alpha(\sigma) := \lim_{n \rightarrow \infty} \alpha_n(\sigma)$  we can make this comparison explicit. By equating (34) and (58) and solving for  $\alpha$ , we have  $\alpha(\sigma) = \lambda_0 - 1$ .

It is intuitively clear that imposing fairness may compromise the throughput. From (34) it is seen that the fair per-node throughputs are bounded above by  $\frac{1}{\beta+1}$ , and that this upper bound can be approached arbitrarily closely by letting  $\alpha \rightarrow \infty$ . Corollary 1 shows that, as  $n \rightarrow \infty$ , the average throughputs in the fair case and unfair case are equal when  $\alpha$  is taken to be  $\lambda_0 - 1$ . The maximum activation rate in this limiting case equals

$$\nu_{\max}^* = \alpha(1 + \alpha)^\beta = \nu_i, \quad \beta + 1 \leq i \leq n - \beta \quad (68)$$

as is seen from (1). This maximum equals  $\sigma$  by (63) since  $1 + \alpha = \lambda_0$ . Hence, as  $n \rightarrow \infty$ , the fair case achieves the same average throughput with activation rates that are less than or equal to those in the unfair case. The situation is considerably more complicated when we keep  $n$  bounded. Then it may well occur that  $\bar{\theta}_n(\sigma)$  exceeds  $\frac{1}{\beta+1}$ , which is the upper bound for the throughput achievable by the fair scheme. In Figure 3 we have plotted curves  $\alpha = \alpha_n(\sigma)$ , for the case that  $n = 10$  and various values of  $\beta$  in the range  $0 \leq \sigma \leq 20$ . The occurrence of the asymptotes for the curves with  $\beta = 2, 5$  shows that  $\bar{\theta}_n(\sigma) > \frac{1}{\beta+1}$  when  $\sigma$  is to the right of the asymptote. In the cases that  $\beta = 1, 4, 9$ , there are no asymptotes, which can be explained by the observation made in Section 4 that a system operating with equal rates  $\sigma \rightarrow \infty$  has at all times  $\lceil \frac{n}{\beta+1} \rceil$  nodes activated. When  $\frac{n}{\beta+1}$  is integer, the resulting average throughput does not exceed the upper bound  $\frac{1}{\beta+1}$ , and no asymptote occurs.

## 6 Conclusions and outlook

In this paper we studied the unfairness in linear random-access networks. We proposed node-specific activation rates (1) as a function of the number of neighbors, and showed that these rates provide equal throughput for all nodes. The rate of a node increases with its number of neighbors. Intuitively, this structure can be explained by the observation that, as the number of neighbors increases, a node needs a higher activation rate to retain its throughput. Consequently, the rates (1), which are exact in linear networks, might serve as a heuristic in more complex networks.



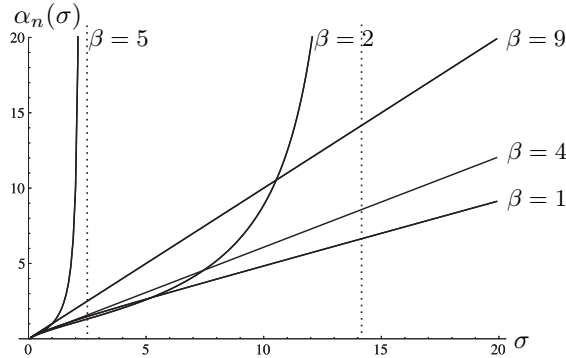


Figure 3:  $\alpha_n(\sigma)$  plotted against  $\sigma$  for various values of  $\beta$  and  $n = 10$

Finding exact expressions for the activation rates that provide strict fairness for networks beyond the linear network is challenging. Kelly [13] obtained results for trees with  $\beta = 1$ , and finds that rates such as in (1), where nodes on the leaves of the tree have lower rates than those in the stem of the tree, provide strict fairness. For such trees, it seems possible to extend Kelly's result to the  $\beta$ -hop blocking situation. Also other regular topologies such as certain grids appear to admit similar analysis.

Regular networks like grids or trees may not always be a good representation of topologies encountered in practice, which in general are less structured. The results obtained in this paper, however, rely heavily on the diagonal structure of the capacity matrix  $A$  in (3), which only exists for certain well-structured networks. For more general networks, and hence more general matrices  $A$ , the objective of equal throughputs boils down to solving the system of nonlinear equations that follows from (5). In fact, (5) can be described in terms of the mapping (with  $\nu = (\nu_1, \dots, \nu_n) \in (0, \infty)^n$ )

$$\nu \mapsto \theta(\nu) = X \cdot \Pi,$$

with  $\theta(\nu) \in (0, \infty)^n$ . It can be shown that the mapping  $\theta$  is globally invertible on its range, and that  $\eta(\nu) = Z_n(\nu)\theta(\nu)$  is globally invertible on  $(0, \infty)^n$ . Thus, given a vector  $c \in (0, \infty)^n$ , there is a unique  $\nu = \nu(c) \in (0, \infty)^n$  such that  $\eta(\nu) = c$ . When  $c(s) = s(1, 1, \dots, 1)^t$  this corresponds to all nodes having equal throughput  $\theta_i(s) = s/Z_n(\nu(c(s)))$ . See [3] for a preliminary analysis in terms of fixed-point equations. The full analysis is rather involved and will appear elsewhere.

Throughout this paper we have assumed all nodes to be saturated, and we derived fair activation rates that give equal throughput to all nodes. Alternatively, we may consider a model where packets arrive over time, and nodes may not always have packets to transmit. An active node transmits one packet, which takes an exponential time with mean 1. After finishing this transmission, the node has to compete for access again. The dynamics are the same as in the

saturated scenario, except that when a node has no packets for transmission, it will remain inactive until it receives a new packet. We argue that the fair activation rates also perform well in such unsaturated settings.

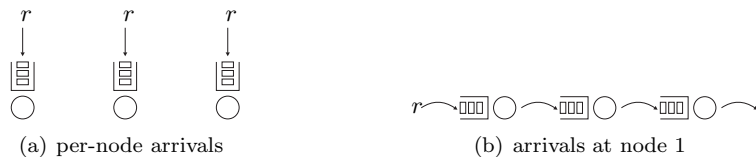


Figure 4: Two unsaturated models

First consider the network in Figure 4(a), where packets arrive at each node according to an independent Poisson process with rate  $r$ , and leave the system once transmitted. Nodes activate according to the fair activation rates (1) with parameter  $\alpha > 0$ . This model reduces to the saturated model when  $r \rightarrow \infty$ . Simulations suggest that the system is stable whenever  $r < \alpha/(1 + \alpha(\beta + 1))$ , the saturation throughput.

Next, consider the network in Figure 4(b), where packets arrive at node 1 according to a Poisson process with rate  $r$ , and are routed along nodes  $2, \dots, n$ . When node  $n$  finishes a transmission, the corresponding packet exits the system. The throughput of node  $n$  is of special interest, as it represents the end-to-end throughput of the network, that is, the rate at which packets leave the network. If  $\theta_n = r$ , the system will eventually empty. If  $\theta_n < r$ , packets arrive at a higher rate than the network can sustain, and packets will accumulate at certain bottleneck nodes. Figure 5 shows simulation results for the end-to-end throughput of this network, plotted against the arrival rate  $r$  for  $n = 5$ ,  $\beta = 1$ . The thick, gray line corresponds to the network where all nodes have equal activation rate  $\sigma = 6$ , and the black line shows the throughput of a network with fair activation rates (1) and  $\alpha = 11.68$ . The values of  $\sigma$  and  $\alpha$  are chosen such that the average per-node throughput is equal for both activation schemes, as prescribed by (67). The network with equal activation rates performs poorly. When the arrival rate grows beyond a certain threshold, node 2 saturates and the throughput drops, see e.g. Xu and Saadawi [23] and Dousse [5]. The network with fair rates, on the other hand, can sustain higher arrival rates and experiences no throughput drop when in overload. In fact, the end-to-end throughput approaches the per-node throughput in the corresponding saturated network (indicated by the dashed horizontal line). So the network again appears stable whenever  $r < \alpha/(1 + \alpha(\beta + 1))$ .

To describe the state of the above non-saturated networks we have to take into account the queue length at each node, as well as node activity. The stochastic processes that describe both the activity of nodes and their queue lengths, are once again Markov processes, but do not have the appealing product-form stationary distribution encountered in the saturated network. In fact, these Markov processes appear rather intractable, and even proving the stability con-

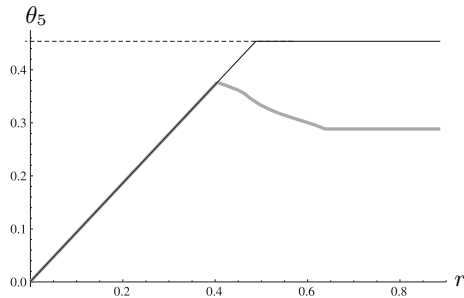


Figure 5: The end-to-end throughput of a network with equal activation rates (thick, gray) and fair activation rates (black), plotted against the arrival rate at node 1

ditions remains an open problem.

## A Proof of Proposition 1

We first establish an auxiliary result. Define  $a(i, l, n)$  as the number of states in which exactly  $l$  nodes are active, including node  $i$ . For successive nodes, the following relations hold.

**Lemma 2.** For  $n \in \mathbb{N}, i \leq \lceil \frac{n}{2} \rceil - 1$ ,

$$a(i, l, n) = a(i + 1, l, n), \quad l \leq i, \quad (69)$$

$$a(i, l, n) > a(i + 1, l, n), \quad i \text{ odd}, i < l \leq \lceil n/2 \rceil, \quad (70)$$

$$a(i, l, n) < a(i + 1, l, n), \quad i \text{ even}, i < l \leq \lceil n/2 \rceil. \quad (71)$$

*Proof.* The proof is by induction on  $i$ . Conditioning on activity of node 1 and node  $n$  yields the relations

$$a(i, l, n) = a(i - 2, l - 1, n - 2) + a(i - 1, l, n - 1), \quad (72)$$

$$a(i, l, n) = a(i, l - 1, n - 2) + a(i, l, n - 1), \quad (73)$$

with boundary conditions

$$a(0, l, n) = 0 \text{ for all } n \text{ and } l;$$

$$a(1, l, n) = 1 \text{ for } l > 0 \text{ and all } n;$$

$$a(1, l, n) = 0 \text{ for } l \leq 0 \text{ and all } n.$$

Hence, the initialization step of the induction is

$$a(0, l, n) < a(1, l, n), \quad 0 < l < \lceil n/2 \rceil,$$

$$a(0, l, n) = a(1, l, n), \quad l \leq 0.$$

Consider odd  $i \leq \lceil n/2 \rceil - 2$ , let  $i + 1 < l < \lceil n/2 \rceil$ , and assume  $a(i, l, n) > a(i + 1, l, n)$ . Using (72) and (73) we get

$$\begin{aligned} a(i + 1, l, n) &= a(i + 1, l - 1, n - 2) + a(i + 1, l, n - 1) \\ &< a(i, l - 1, n - 2) + a(i + 1, l, n - 1) = a(i + 2, l, n). \end{aligned}$$

This proves assertion (70). Assertions (69) and (71) can be proved in a similar manner.  $\square$

We now use Lemma 2 to prove Proposition 1.

*Proof.* (Proposition 1) Assertion (i) can be shown by rewriting the throughput as follows:

$$\theta_i = Z_n^{-1} \sum_l a(i, l, n) \sigma^l = Z_n^{-1} \sum_l a(n - i + 1, l, n) \sigma^l = \theta_{n-i+1}. \quad (74)$$

To prove assertion(ii) we first show that  $(-1)^i(\theta_{i+1} - \theta_i)$  is positive. That is,

$$\begin{aligned} (-1)^i(\theta_{i+1} - \theta_i) &= (-1)^i Z_n^{-1} \sum_l (a(i + 1, l, n) - a(i, l, n)) \sigma^l \\ &= (-1)^i Z_n^{-1} \sum_{l=i+1}^{\lceil n/2 \rceil} (a(i + 1, l, n) - a(i, l, n)) \sigma^l > 0, \end{aligned} \quad (75)$$

where the inequality follows from Lemma 2. Using (75), Proposition 1(ii) follows from

$$\begin{aligned} (-1)^i(\theta_{i+1} - \theta_i) &= (-1)^i \left( \theta_{i+1} - Z_n^{-1} \sum_l a(i, l, n) \sigma^l \right) \\ &= (-1)^i \left( \theta_{i+1} - Z_n^{-1} \sum_l (a(i, l - 1, n - 2) + a(i, l, n - 1)) \sigma^l \right) \\ &> (-1)^i \left( \theta_{i+1} - Z_n^{-1} \sum_l (a(i, l - 1, n - 2) + a(i + 1, l, n - 1)) \sigma^l \right) \\ &= (-1)^i \left( \theta_{i+1} - Z_n^{-1} \sum_l a(i + 2, l, n) \sigma^l \right) \\ &= (-1)^{i+1} (\theta_{i+2} - \theta_{i+1}). \end{aligned}$$

This completes the proof.  $\square$

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