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A Make-to-Stock Mountain-Type Inventory Model

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Abstract

We consider the buffer content of a fluid queue or storage process. The buffer content varies in a way that depends on the state of an underlying three-state Markov process. In state 0 the buffer content increases at a rate $\alpha(x)$ that is a function of the current buffer level x; in states 1 and 2 it decreases linearly, with different speeds. We study the steady-state buffer content, by using level crossing theory and by exploiting relations between the fluid queue and queues with instantaneous input and/or output.

1 Introduction

In this paper we consider a storage process in which the buffer content is governed by an underlying three-state Markov process. In state 0 the buffer content increases at a rate $\alpha(x)$ that is a function of the current buffer level x; in states 1 and 2 it decreases linearly, with different speeds. The two distinguishing features of the storage model under consideration are: (i) the input process is non-instantaneous, and (ii) it is workload-dependent. Below we discuss each of these features in the context of the literature.

In classic queueing models, the workload increases instantaneously when a customer arrives, and it decreases linearly between arrivals. In the related literature of dam and storage processes, the input process is sometimes assumed to be *non-instantaneous* and thus the buffer content decreases and increases linearly in an alternating manner; an early example is Gaver and Miller [9]. In the early seventies, motivated by performance issues in communication networks, fluid queues fed by on/off sources started to attract attention: buffers fed by a number of independent sources which alternate between *on* (send fluid in at a constant rate) and *off*. Some of the key papers on this topic are by Anick, Mitra and Sondhi [2], Cohen [8], Kaspi and Rubinovitch [13], Kosten [15, 16, 17], Kosten and Vrieze [18], Rogers [22] and Rubinovitch [23]. We refer to Kulkarni [19] for a survey; another survey by Boxma and Dumas [4] focuses on fluid queues with an emphasis on long-tailed on-periods and its effect on the buffer content.

Early papers on queues with *workload-dependent* service speed are (again) Gaver and Miller [9], who allowed the (constant) service speed to depend on whether the buffer content level is below or above a certain threshold, and Harrison and Resnick [10], who allowed a more general state-dependent service speed $r(x)$ at level x. Bekker et al. [3] consider workload-dependent service speeds *and* arrival rates. Fluid queues with state-dependent production and release rates were, among others studied in [5], [12] and [24].

We are aware of only one study in which the background process can be in more than two different states and one of the inflow or outflow rates depends on the buffer content (Scheinhardt et al. [24]), but the authors restrict themselves to a finite buffer content. Our paper is related to [6]. In that paper, the buffer content varies linearly in all three states. However, one of the states there may have a general distribution.

Storage processes are relevant for many application areas: next to classical examples from water storage, production-inventory and communication systems, there is a growing interest in storage processes which model energy storage. In the latter application area, it seems natural to have a background process that determines the speed at which the buffer content increases/decreases. In the storage of wind energy, for example, different weather conditions may be represented by different states of an underlying (semi-)Markov process. In this application area, it may also be important to allow for an input rate which depends on the buffer content; indeed, if the buffer content becomes too large, it may become useless to store more energy, and there might also be more loss of energy.

A second motivating example comes from a production environment. Consider two identical machines which continuously produce a certain fluid that goes into a buffer. The buffer is depleted at a fixed demand rate. Each machine is independently subject to breakdowns. If one machine is broken, a repairman fixes it. If both machines are broken, an extra repairman has to be hired. We can distinguish three states, corresponding to the number of broken machines. In state 0, both machines are working and the buffer content increases. In state 1, one machine is broken, and the buffer content decreases at some fixed rate. In state 2, both machines are broken and the buffer content decreases at a faster fixed rate (the demand rate). In state 0, production may be regulated, i.e., the machines work at a reduced speed, so as to avoid too high holding costs. The net increase rate then is a function of the buffer content level. Under exponentiality assumptions regarding the breakdown and repair times, the underlying process is Markovian.

Our model contains several ingredients that make it of general interest: a non-instantaneous input process, an input rate that is a function of the buffer content, and a non-trivial three-state underlying Markov process. Various choices of the model parameters yield special cases which have been studied before, like ordinary two-state fluid models (let $\alpha(x) \equiv \alpha$) and two-state models with one workload-dependent input rate (let the rate of one of the states 1 or 2 go to infinity). We show how one can use level crossing theory to reduce the study of the buffer content to that of the buffer content process restricted to particular states of the Markov process. The evolution of the buffer content during the time periods in which the Markov process is in other states, is replaced by a jump upward or downward. This gives rise to (more classical) queues with instantaneous input and/or work removal. The paper of Kella and Whitt [14] establishes relations between a broad class of fluid queues and queues with instantaneous input.

The paper is organized as follows. Section 2 contains a detailed model description. In that section we also argue, using level crossing theory, that in order to determine the steady-state buffer content distribution, it suffices to obtain the steady-state buffer content distribution during two out of the three periods of the underlying Markov process. Sections 3 and 4 are successively devoted to the steady-state buffer content distribution during 0-periods and 1-periods. A key role in the analysis is played by a homogeneous second-order differential equation for the density of the buffer content process in state 0. We show, using the Maple computer algebra system [20], that this differential equation can be solved explicitly for particular choices of $\alpha(x)$. Numerical results are presented in Section 5. The paper concludes in Section 6 with a brief summary and suggestions for possible extensions.

2 Model Description

Consider a three-state Markov process $\mathbf{J} = \{J(t) : t \geq 0\}$ with state space $\{0, 1, 2\}$ and generator

$$\begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} -\mu & \mu & 0 \\ \lambda & -(\lambda+\eta) & \eta \\ 0 & \nu & -\nu \end{bmatrix}.$$

Using the balance equation approach, we find the steady-state probabilities for this process as,

$$p_0 = \frac{\nu\lambda}{\eta\mu + \nu\mu + \nu\lambda}, \quad p_1 = \frac{\nu\mu}{\eta\mu + \nu\mu + \nu\lambda}, \quad p_2 = \frac{\eta\mu}{\eta\mu + \nu\mu + \nu\lambda}. \tag{1}$$

Next consider a buffer with content process $\{Y(t), t \geq 0\}$, that is governed by the underlying Markov process $\mathbf{J}$ in the following way.

- If $J(t) = 2$ then the buffer content decreases at a constant rate b.
- If $J(t) = 1$ then the buffer content decreases at a constant rate $b - a > 0$.
- If $J(t) = 0$ then the buffer content grows at a rate $\alpha(x) > 0$ when $Y(t) = x$.
- If $Y(t) = 0$, then the buffer content stays at 0 in states 1 and 2; when $\mathbf{J}$ returns to state 0, the buffer content starts to increase again.

The above model corresponds to the machine production example of Section 1. In particular, State 2 corresponds to two broken machines, and the buffer depletes at the full demand rate b. State 1 corresponds to one broken machine, while the other machine produces at full speed a which is not sufficient to compensate the demand (depletion) rate b, so the net output rate is $b - a > 0$. If the buffer content becomes zero, it stays that way until both machines are back in operation; backorders are not allowed. State 0 corresponds to both machines working and so the buffer content grows. The rates λ, μ, ν, η correspond to the following: When one machine is broken λ is the repair rate. If both machines are broken, the total repair rate is $\nu > \lambda$. The breakdown rate of a machine is η. In principle, the rate to leave state 0, μ, equals 2η in the production example.

It is clear that level 0 is reachable from any starting state $x > 0$. Furthermore, the 'mountain' process $\mathbf{Y} = \{Y(t) : t \geq 0\}$ is a regenerative process whose cycle is terminated at the end of the unsatisfied demand period (idle period) and the two-dimensional process $(\mathbf{Y}, \mathbf{J})$ is a Markov process. We assume that the conditions for stability are fulfilled so that $Y = \lim_{t\to\infty} Y(t)$ represents the equilibrium random variable of the mountain (the latter limit is defined in terms of weak convergence). We refer to [7] for an extensive discussion of stability conditions for queueing models with work-modulated arrival and/or service times; one may translate the increments during 0-periods in our model into service requirements that depend on the work found upon arrival, and subsequently use the stability conditions from Model 1 of [7].

Clearly, the distribution $F_Y(\cdot)$ of Y has an atom at 0, designated by π_Y, and for all $x > 0$, $F_Y(x)$ is an absolutely continuous distribution with density $f_Y(x)$ so that

$$F_Y(x) = \pi_Y + \int_0^x f_Y(y)\, dy. \tag{2}$$

We focus on the density $f_Y(\cdot)$ of the mountain process. By the law of total probability,

$$f_Y(x) = f_0(x)p_0 + f_1(x)p_1 + f_2(x)p_2, \tag{3}$$

where $f_i(\cdot)$ is the conditional density of $\mathbf{Y}$ given that i machines are broken and p_i is the equilibrium probability that i machines are broken; $p_0 + p_1 + p_2 = 1$.

Using the same argument as in [6] we conclude that the direct computation of $f_2(x)$ is redundant because by level crossing theory (LCT),

$$\alpha(x)f_0(x) = (b-a)f_1(x) + bf_2(x), \tag{4}$$

so that by (4) we only have to compute separately $f_0(x)$ and $f_1(x)$:

$$f_Y(x) = f_0(x)\left[p_0 + p_2\frac{\alpha(x)}{b}\right] + f_1(x)\left[p_1 - p_2\left(1 - \frac{a}{b}\right)\right]. \tag{5}$$

A typical realization of $\mathbf{Y}$ is shown in Figure 1, along with the corresponding buffer content processes $\mathbf{V_0}$ and $\mathbf{V_1}$ where $\mathbf{Y}$ is restricted to the time epochs in which $J(t) = 0$ and $J(t) = 1$, respectively. The probabilities p_i in (3) are the steady-state probabilities of the Markov chain $\mathbf{J}$; they are given in (1). In the next two sections we successively determine $f_0(x)$ and $f_1(x)$, thus obtaining $f_Y(x)$ via (5).

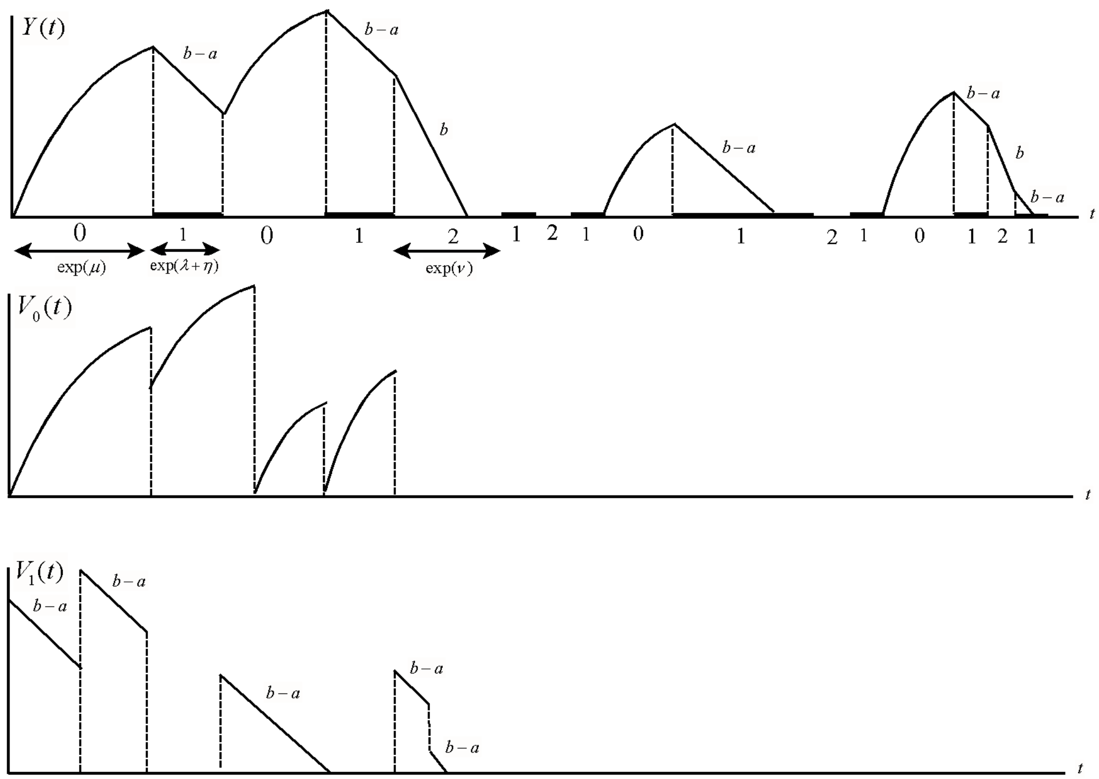

Figure 1: The sample paths of $Y(t)$, $V_0(t)$ and $V_1(t)$.

3 Computation of $f_0(\cdot)$

To compute $f_0(x)$ we have to compute the steady-state density of the process $\mathbf{V}_0 = \{V_0(t) : t \geq 0\}$ where $\mathbf{V}_0$ is generated by $\mathbf{Y}$ by deleting the time periods in which $\mathbf{Y}$ is decreasing and gluing together the time periods in which $\mathbf{Y}$ is increasing. As a result of this construction, $\mathbf{V}_0$ is a production process that increases according to the state-dependent production rate $\alpha(\cdot)$ and between negative jumps. There are no idle periods. The negative jumps are i.i.d. random variables having a special phase-type distribution $G(\cdot)$ whose LST is given by

$$G^*(\theta) = \frac{\lambda(\nu + b\theta)}{(\lambda + \eta + (b-a)\theta)(\nu + b\theta) - \eta\nu}. \tag{6}$$

This is derived by observing that an arbitrary negative jump consists of $N+1$ jumps related to type-1 periods, which are $\exp((\lambda+\eta)/(b-a))$ distributed, and N jumps related to type-2 periods, which are $\exp(\nu/b)$ distributed, where N is geometrically distributed with parameter

$\eta/(\lambda+\eta)$. Inversion of (6) yields,

$$1 - G(x) = C_1 e^{-d_1 x} + C_2 e^{-d_2 x}, \tag{7}$$

where,

$$d_{1,2} = \frac{1}{2}\left(\frac{\lambda+\eta}{b-a} + \frac{\nu}{b}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\lambda+\eta}{b-a} + \frac{\nu}{b}\right)^2 - 4\frac{\lambda\nu}{(b-a)b}},$$

and,

$$C_1 = 1 - C_2, \quad C_2 = \frac{\lambda/(b-a) - d_1}{d_2 - d_1}.$$

By LCT we have the balance equation [see also Equation (2) of Harrison and Resnick [10]:

$$\alpha(x) f_0(x) = \mu \int_x^\infty [1 - G(w-x)] f_0(w)\, dw. \tag{8}$$

In order to solve the integral equation (8) for the unknown function $f_0(x)$, we convert it to an equivalent ordinary differential equation with suitable boundary conditions. To that end, we first multiply both sides of (8) by $e^{-d_1 x}$, and differentiate both sides w.r.t. x. Dividing the result by $e^{-d_1 x}$ results in the integro-differential equation,

$$f_0(x)[\alpha'(x) - d_1\alpha(x) + \mu(C_1 + C_2)] + \alpha(x) f_0'(x) = \mu C_2 (d_2 - d_1) e^{d_2 x} \int_x^\infty e^{-d_2 w} f(w)\, dw. \tag{9}$$

Now multiplying both sides of (9) by $e^{-d_2 x}$ and once again differentiating both sides of the resulting equation and finally dividing the result by $e^{-d_2 x}$ and collecting the terms gives

$$A_1(x) f_0''(x) + A_2(x) f_0'(x) + A_3(x) f_0(x) = 0 \tag{10}$$

where

$$\begin{aligned}
A_1(x) &= \alpha(x) \\
A_2(x) &= 2\alpha'(x) - \alpha(x)(d_1 + d_2) + \mu(C_1 + C_2) \\
A_3(x) &= [\alpha''(x) - d_1\alpha'(x)] - d_2[\alpha'(x) - d_1\alpha(x) + \mu(C_1 + C_2)] + \mu C_2 (d_2 - d_1).
\end{aligned}$$

The two natural boundary conditions for this ODE are, (i) $\int_0^\infty f_0(x)\, dx = 1$, and (ii) $\alpha(0) f_0(0) = \mu \int_0^\infty f_0(w)[1 - G(w)]\, dw$; i.e., (8) should hold for $x = 0$.

We consider two relevant choices of $\alpha(x)$, for which $f_0(x)$ can be determined explicitly which will be discussed in Section 5: (i) $\alpha(x) = Be^{-rx}$, and (ii) $\alpha(x) = 1/(x+s)$. We will show in Section 5 that the general solution of (10) when $\alpha(x) = Be^{-rx}$ is given in terms of Whittaker functions, and when $\alpha(x) = 1/(x+s)$, the solution is found in terms of the hypergeometric functions (Abramowitz and Stegun [1, Ch. 13]).

Remark 1 Another case for which (8) can be handled is the case where $\alpha(x) \equiv 0$ for $x \geq K$, for some finite positive K. This corresponds to a finite buffer of size K, that is, the buffer level always fluctuates between 0 and K. Introducing $\hat{f}_0(x) = f_0(K-x)$, one may then convert (8) into a Volterra integral equation for $\hat{f}_0$ that can be solved using Picard iteration.

4 Computation of $f_1(\cdot)$

To compute $f_1(x)$ we have to compute the steady-state density of the process $\mathbf{V}_1 = \{V_1(t) : t \geq 0\}$ where $\mathbf{V}_1$ is generated by $\mathbf{Y}$ by deleting the time periods in which both machines are broken or both are working. Note that $\mathbf{V}_1$ can be interpreted as a special work process of a single server queue with Poisson jumps and additional *negative customers*; so the jumps can be positive as well as negative. The negative jumps are independent $\exp(\nu/b)$ distributed random variables but the positive jumps are state dependent. The balance equation obtained from LCT is

$$\begin{aligned}(b-a)f_1(x) \quad &+ \eta \int_x^\infty e^{-(\nu/b)(w-x)} f_1(w)\ dw \\ &= \lambda \int_0^x e^{-\mu(A(x)-A(w))} f_1(w)\ dw + Ce^{-\mu A(x)},\end{aligned} \tag{11}$$

where

$$A(x) = \int_0^x \frac{1}{\alpha(w)}\ dw. \tag{12}$$

The two terms in the lefthand side correspond to the two possibilities to downcross level x: (i) while one machine is working or (ii) with a downward jump (when the one working machine breaks down). The two terms in the righthand side correspond to the two possibilities to upcross level x: with a jump from some level $w \in (0,x)$ or with a jump from 0. As solving (11) is laborious, we prefer to follow another approach; one that is based on an idea in [6]. Let us denote the steady-state value of the $\mathbf{V}_0$-process just before and just after a jump by Z_0 and W_0, respectively. Let us further denote the value of the $\mathbf{V}_1$-process just before an upward jump by W_1.

Step 1 During 0-periods, the $\mathbf{V}_0$-process increases monotonously at rate $\alpha(x)$ when the level is x. PASTA implies that the steady-state distribution of the $\mathbf{V}_0$-process equals that at the end of the 0-period, i.e., after an $\exp(\mu)$ distributed amount of time. Hence $Z_0 \stackrel{d}{=} V_0$.

Step 2 $W_0 \stackrel{d}{=} \max(Z_0 - G, 0)$, where the distribution of G is the H_2-distribution given in (7).

Step 3 The values of the $\mathbf{V}_1$-process just before upward jumps can be matched one-to-one with the values of the $\mathbf{V}_0$-process just after downward jumps. Hence $W_1 \stackrel{d}{=} W_0$.

Step 4 PASTA implies that the distribution of the $\mathbf{V}_1$-process *just before upward jumps* is the same as the *steady-state* distribution of the $\mathbf{V}_1$-process. Hence $V_1 \stackrel{d}{=} W_1$.

Combining Steps 1-4 yields the following lemma:

Lemma 1

$$V_1 \stackrel{d}{=} \max(V_0 - G, 0). \quad \blacksquare \tag{13}$$

Thus, the knowledge of $f_0(\cdot)$ yields $f_1(\cdot)$. In particular,

$$f_1(x) = \int_{t=0}^{\infty} f_0(x+t)\ dG(t),$$

and,

$$\Pr(V_1 = 0) = \int_{x=0}^{\infty} f_0(x)(1 - G(x))\ dx.$$

5 Numerical Examples

In this section we present a numerical study of the two cases presented in Section 2 where the $\alpha(x)$ function assumes different forms.

5.1 Case 1: $\alpha(x) = Be^{-rx}$

As shown in Section 3, the computation of $f_0(x)$ involves solving a second order ordinary differential equation given in (10). Solving this ODE with the help of the computer algebra system Maple [11] we obtain

$$\begin{aligned} f_0(x) &= c_1 \mathcal{W}_M\left(\frac{1}{2}\phi_1, \frac{1}{2}\phi_2, \phi_3 e^{rx}\right) \times \exp\left(-\frac{1}{2}\frac{\phi_4 e^{rx} + \phi_5 x}{\phi_6}\right) \\ &\quad + c_2 \mathcal{W}_W\left(\frac{1}{2}\phi_1, \frac{1}{2}\phi_2, \phi_3 e^{rx}\right) \times \exp\left(-\frac{1}{2}\frac{\phi_4 e^{rx} + \phi_5 x}{\phi_6}\right), \end{aligned}$$

where ϕ_i, $i = 1, \ldots, 6$ are functions of the problem data $(a, b, \eta, \mu, \lambda, \nu, B, r)$ and where c_1 and c_2 are the arbitrary constants to be determined using the boundary conditions. The independent solutions $\mathcal{W}_W(\xi, \rho, x)$ and $\mathcal{W}_M(\xi, \rho, x)$ are the "Whittaker W" and "Whittaker M" functions, respectively (see Abramowitz and Stegun [1, Ch. 13]). These special functions are given as,

$$\begin{aligned}
\mathcal{W}_W(\xi, \rho, x) &= e^{-x/2} x^{1/2+\rho} K\left(\frac{1}{2} + \rho - \xi, 1 + 2\rho, x\right), \\
\mathcal{W}_M(\xi, \rho, x) &= e^{-x/2} x^{1/2+\rho} H\left(\frac{1}{2} + \rho - \xi, 1 + 2\rho, x\right),
\end{aligned}$$

with the Kummer function $K(a_1, a_2, x)$ being one of the independent solutions of another 2nd order ODE [i.e., $xy''(x) + (a_2 - x)y'(x) - a_1 y(x) = 0$] and $H(a_1, a_2, x)$ as the hypergeometric function

$$H(a_1, a_2, x) = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + n)/\Gamma(a_1)}{\Gamma(a_2 + n)/\Gamma(a_2)} \left(\frac{x^n}{n!}\right)$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ is the gamma function evaluated at $z > 0$; (Abramowitz and Stegun [1, Ch. 13]). We should point out that although these special functions are defined in terms of infinite sums and solution of a differential equation, they have been implemented in computer algebra systems such as Maple [11] which makes their pointwise evaluations relatively straightforward.

Recall that since $f_0(x)$ is a proper density, it must satisfy the condition $\int_0^\infty f_0(x)\, dx = 1$. Thus, in principle, determination of the coefficients c_1 and c_2 requires the integration of the Whittaker functions. Since this is quite a challenging operation, we follow an alternate route and solve the ODE in (10) using numerical techniques in the following example.

Consider now the problem with the parameter values $(B, r) = (\frac{1}{2}, 1)$, $(\eta, \mu, \lambda, \nu) = (1, 2, 1, 3)$ and $(a, b) = (1, \frac{3}{2})$. With these values we find $(d_1, d_2) = (5.236, 0.764)$, and $(C_1, C_2) = (0.276, 0.724)$ from which we can calculate the c.d.f. $G(t)$ given in (7). To solve the ODE (10) numerically for $f_0(x)$, we impose the boundary conditions $\int_0^\infty f_0(x)\, dx = 1$, $\alpha(0) f_0(0) = \mu \int_0^\infty [1 - G(w)] f_0(w)\, dw$, $f_0(0) > 0$, and $f_0(x_{\max}) = 0$ where $x_{\max}$ is a large but finite constant. We implement the "shooting method" of solving of boundary value problems (Roberts and Shipman [21]) and find that at $f_0(0) = 2.820$ the boundary conditions are satisfied and the solution for $f_0(x)$ is found as in Figure 2.

Now, to determine $f_1(x)$, we use the relation

$$f_1(x) = \int_0^\infty f_0(x + t)\, dG(t) = \int_0^\infty f_0(x + t) g(t)\, dt, \tag{14}$$

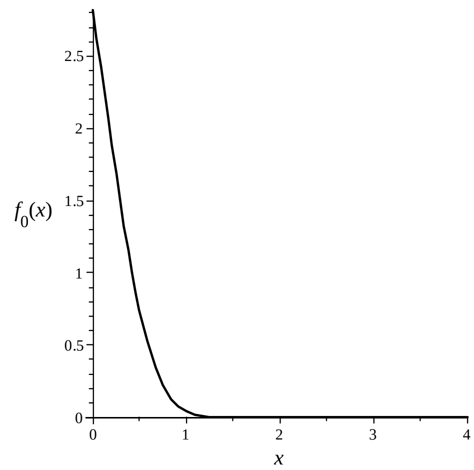

Figure 2: The density $f_0(x)$ in Case 1.

where, from (7), we find $G(x) = 1 - (C_1 e^{-d_1 x} + C_2 e^{-d_2 x})$, and $G'(x) = g(x) = C_1 d_1 e^{-d_1 x} + C_2 d_2 e^{-d_2 x}$. Performing the integration (numerically) in (14) for each value of x, we find $f_1(x)$ as in Figure 3.

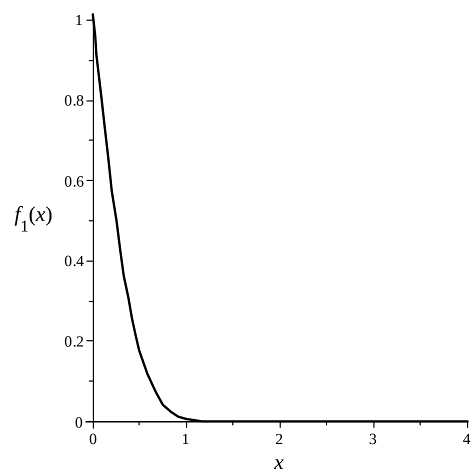

Figure 3: The improper density $f_1(x)$ in Case 1.

We note that since $f_1(x)$ is not a proper density, in this case we find $\int_0^\infty f_1(x)\, dx = 0.295$. The atom π_1 which would make $f_1(x)$ a density is thus $\pi_1 = 1 - 0.295 = 0.705$ which can also be calculated from (13) as $\pi_1 = \Pr(G > V_0) = \Pr(V_1 = 0) = \int_0^\infty f_0(t)[1 - G(t)]\, dt = 0.705$.

Using now the result for the steady state probabilities in (1) of the Markov process $\mathbf{J}$, we find

$$p_0 = \frac{3}{11}, \quad p_1 = \frac{6}{11}, \quad p_2 = \frac{2}{11}, \tag{15}$$

from which we can determine the improper density $f_Y(x)$ in (5) which is given in Figure 4.

Since $\int_0^\infty f_Y(x)\ dx = 0.463$, the atom π_Y in (2) is found as $\pi_Y = 1 - 0.463 = 0.537$, thus, $F_Y(x) = 0.537 + \int_0^x f_Y(y)\ dy$ which can be evaluated numerically for each x.

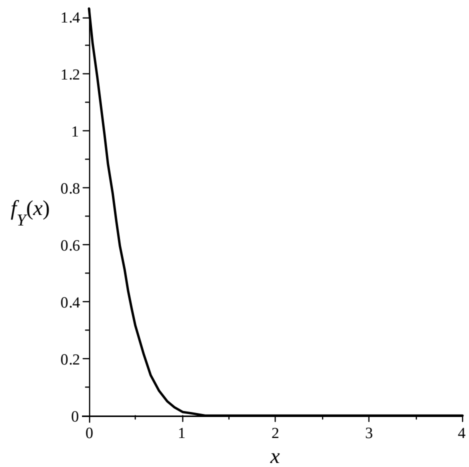

Figure 4: The improper density $f_Y(x)$ in Case 1.

Since the atom π_Y and the improper density $f_Y(x)$ are available, we can also calculate the mean and variance of Y easily. In particular, we have,

$$\begin{aligned} E(Y) &= 0 \cdot \pi_Y + \int_0^\infty x f_Y(x)\ dx = 0.115 \\ \mathrm{Var}(Y) &= [0 - E(Y)]^2 \cdot \pi_Y + \int_0^\infty [x - E(Y)]^2 f_Y(x)\ dx = 0.035 \end{aligned}$$

and so $\sqrt{\mathrm{Var}(Y)} = 0.186$.

As a simple check of the above calculations, we observe that the second boundary condition $\alpha(0) f_0(0) = \mu \int_0^\infty f_0(w)[1 - G(w)]\ dw$ is satisfied since $\alpha(0) = \frac{1}{2}$, $f_0(0) = 2.820$, $\mu = 2$, and $\int_0^\infty f_0(w)[1 - G(w)]\ dw = 0.705$.

5.2 Case 2: $\alpha(x) = 1/(x + s)$

For this case, we compute the density $f_0(x)$ by solving the second order ODE given in (10) with $\alpha(x) = 1/(x + s)$. We obtain the solution in terms of the generalized hypergeometric functions $F(\cdot, \cdot, x)$ (Abramowitz and Stegun [1, Ch. 13]),

$$\begin{aligned} f_0(x) &= c_1 F\left(\kappa_1, \frac{1}{2}, \kappa_2(x)\right) \exp(-\kappa_3(x)) \\ &\quad + c_2 F\left(\kappa_4, \frac{3}{2}, \kappa_2(x)\right) \kappa_5(x) \exp(-\kappa_3(x))(x + s) \end{aligned}$$

where $\kappa_i(x)$, $i = 1, \ldots, 5$ are functions of the problem data $(a, b, \eta, \mu, \lambda, \nu, B, r)$ and where c_1 and c_2 are the arbitrary constants to be determined using the boundary conditions. Similar to the implementation of the Whittaker functions, the generalized hypergeometric function is also implemented in Maple which makes the pointwise evaluation of these functions relatively straightforward. But, as before, since the solution must satisfy the condition $\int_0^\infty f_0(x)\, dx = 1$, and since it becomes difficult to integrate generalized hypergeometric function, we choose an alternate route and determine the density $f_0(x)$ by numerically solving the ODE in (10).

We use the same set of parameters as in Case 1, i.e., $(\eta, \mu, \lambda, \nu) = (1, 2, 1, 3)$ and $(a, b) = (1, \frac{3}{2})$, and consider $\alpha(x) = 1/(x + 2)$ (so $s = 2$). Using again the shooting method and solving the ODE (10) with $\alpha(x) = 1/(x + s)$ numerically with $f_0(0) = 2.725$, we find the density $f_0(x)$ in Figure 5.

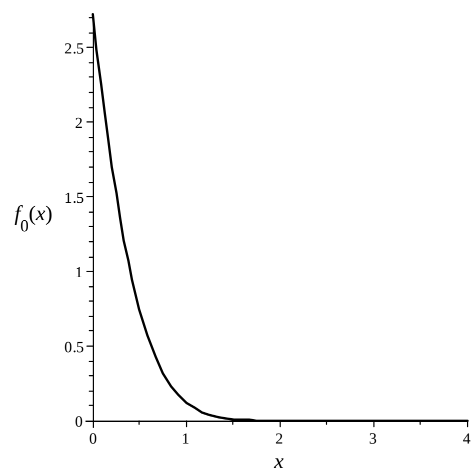

Figure 5: The density $f_0(x)$ in Case 2.

To find the improper density $f_1(x)$, we again use the relation (14) and evaluate this function for each value of x. The resulting solution is plotted in Figure 6. The integral of this improper density is found as $\int_0^\infty f_1(x)\, dx = 0.319$ implying that the atom is $\pi_1 = 1 - 0.319 = 0.681$ which also follows from (13) as $\pi_1 = \Pr(G > V_0) = \Pr(V_1 = 0) = \int_0^\infty f_0(t)[1 - G(t)]\, dt = 0.681$.

Since the steady state probabilities are given in (15), improper density $f_Y(x)$ in (5) is found as in Figure 7.

The integral of the improper density $f_Y(x)$ is computed as $\int_0^\infty f_Y(x)\, dx = 0.480$ from which we obtain the atom of Y as $\pi_Y = 1 - 0.480 = 0.520$. As before, the c.d.f. of Y can be evaluated numerically for each x via $F_Y(x) = 0.520 + \int_0^x f_Y(y)\, dy$. Finally, we compute the mean and variance of Y and find $E(Y) = 0.144$ and $\text{Var}(Y) = 0.056$, with $\sqrt{\text{Var}(Y)} = 0.238$.

To check the above calculations, we note that since $\alpha(0) = \frac{1}{2}$, $f_0(0) = 2.725$, $\mu = 2$, and

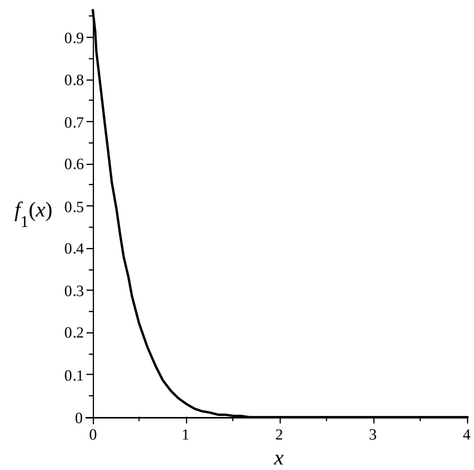

Figure 6: The improper density $f_1(x)$ in Case 2.

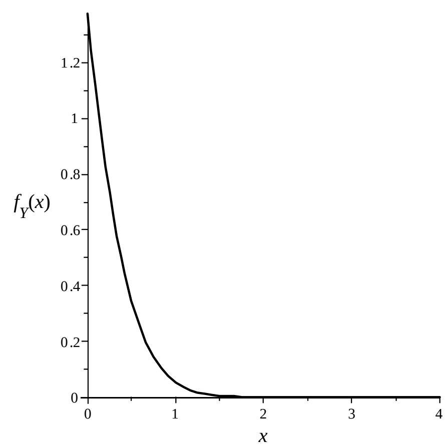

Figure 7: The improper density $f_Y(x)$ in Case 2.

$\int_0^\infty f_0(w)[1 - G(w)]\, dw = 0.681$, the second boundary condition is also satisfied.

6 Conclusions and Suggestions for Further Research

We have studied a make-to-stock inventory model, in which the buffer content varies in a way that depends on the state of an underlying three-state Markov process. We have derived the buffer content distribution for the case in which the buffer content increases at some level-dependent rate $\alpha(\cdot)$ in state 0, and decreases linearly in states 1 and 2.

We finally mention several interesting and relevant possibilities for further research. (i) Other choices of $\alpha(\cdot)$. (ii) In some applications, one might have a level-dependent rate *down.* (iii) If $b < a$, then there are two states "up" and one state "down." (iv) It would be interesting to allow for more than 3 states of the underlying Markov process. (v) The results of the present study might be used for optimization and control purposes. For example, one might want to choose $\alpha(\cdot)$ in order to optimize some cost function.

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