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THRESHOLD STRATEGIES FOR RISK PROCESSES AND THEIR RELATION TO QUEUEING THEORY

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Abstract

We consider a risk model with threshold strategy, where the insurance company pays off a certain percentage of the income as dividend whenever the current surplus is larger than a given threshold. We investigate the ruin time, ruin probability and the total dividend, using methods and results from queueing theory.

Keywords: Queues, M/G/1, G/M/1, risk processes, ruin theory, threshold strategy, dividend

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1. Introduction

In this paper we consider a risk model with threshold strategy, where the insurance company pays off a certain percentage of the income as dividends whenever the current surplus is larger than a given threshold b. Such a risk model has been studied by several authors; see in particular [7] and references therein. Suppose that $(T_k)_{k\geq 1}$ denotes the arrival times of the claims. We assume that the interarrival times are i.i.d. and have an exponential distribution with mean $1/\mu$. The surplus process $(R(t))_{t\geq 0}$ increases linearly with slope one if $R(t) < b$ and with slope $1-\gamma$, if $R(t) \geq b$ (the so-called constant barrier case $\gamma = 1$ was studied by Lin et al. [8] and Li and Garrido [6]).

The claim sizes are assumed to be i.i.d. with common distribution function $G(\cdot)$, having mean $1/\lambda$.

Let $\tau = \inf\{t : R(t) < 0 | R(0) = x\}$ denote the ruin time and let $\psi(x) = \mathbb{P}(\tau < \infty | R(0) = x)$ be the ruin probability. We write $\overline{\psi}(x) = 1 - \psi(x)$ for the survival probability and let $\rho = \mu/\lambda$. We distinguish three cases (ignoring the more delicate boundary cases):

1. $\rho > 1$. In this case $R(t) \to -\infty$ as $t \to \infty$ and $\psi(x) = 1$.

2. $1-\gamma < \rho < 1$. In this case $\psi(x) = 1$.

3. $\rho < 1-\gamma$. In this case $R(t) \to \infty$ as $t \to \infty$ and $\psi(x) < 1$.

The paper is organized as follows. The ruin time distribution is studied in Section 2, and the survival probability (for Case 3) in Section 3. Section 4 considers the distribution of the total dividend until ruin, when ruin is certain (so for Cases 1 and 2).

Remark 1: It should be noted that the computation of the total dividend paid until ruin under Cases 1 and 3 is of special interest from an operations research point of view. For example, consider a simple objective function

$$Q(x,b,\gamma) = \gamma E\left(\int_0^\tau \mathbb{1}(R(t) > b)dt\right) + \pi E\left(R(\tau)\right) \tag{1}$$

where the initial capital $R(0) = x$, the rate γ and the switchover level b are the control parameters of the problem. The first term on the right of (1) is the income from dividend until ruin and the second term is the penalty on the deficit at ruin where π is the penalty on one unit of deficit (clearly, $R(\tau) < 0$). The trade-off between the profit functional $E\left(\int_0^\tau \mathbb{1}(R(t) > b)dt\right)$ and the cost functional $E\left(R(\tau)\right)$ is very intuitive.

Remark 2: The main contribution of our paper is methodological. In studying an important risk model, we repeatedly establish a link to queueing models. For example, the surplus process in the risk model is interpreted as the attained waiting time process in a $G/M/1$ queue; and also various links with the $M/G/1$ queue are established. This allows us to use queueing-theoretic methods and concepts, like level crossings and busy period arguments, to study the key performance measures of the risk model. Not all our results are new. In particular, the key quantities of Sections 2 and 3 have been studied in [7] (see its Corollary 7.1 for the Laplace transform of the ruin time, and its Corollary 6.1 for the probability of ruin). However, the methods in [7] are very different from our methods, and the results are also presented in a different form.

2. The ruin time

We consider the Laplace-Stieltjes transform (LST) $\varphi_\alpha(x) = \mathbb{E}[e^{-\alpha\tau}\mathbb{1}(\tau < \infty)|R(0) = x]$ of the ruin time (notice that $\varphi_0(x) = \psi(x)$), which is a special case of the so called Gerber-Shiu function. It has been shown in [5] that

$$(1-\gamma\mathbb{1}(x \geq b))\varphi_\alpha'(x) \quad = \quad (\mu+\alpha)\varphi_\alpha(x) - \mu\overline{G}(x) - \mu\int_0^x \varphi_\alpha(x-y)\,dG(y), \tag{2}$$

where we write $\overline{G}(x) := 1 - G(x)$. See also the discussion of this equation in [7].

We note the following duality with the workload process $(V(t))_{t\geq 0}$ of an M/G/1 queue with arrival rate μ, service time distribution $G(\cdot)$ and state dependent service rate $1-\gamma\mathbb{1}(x \geq b)$. According to [2] we have that $\mathbb{P}(V(t) > x|V(0) = 0) = \mathbb{P}(\tau \leq t|R(0) = x)$. Writing $F(t,x) = \mathbb{P}(V(t) \leq x|V(0) = 0)$ for the distribution function of the workload at time t, we have

$$\begin{aligned}
\varphi_\alpha(x) &= \int_0^\infty e^{-\alpha t}d\mathbb{P}(\tau \leq t|R(0) = x) = \int_0^\infty \alpha e^{-\alpha t}\mathbb{P}(\tau \leq t|R(0) = x)\,dt \\
&= 1 - \int_0^\infty \alpha e^{-\alpha t}F(t,x)\,dt = 1 - \mathbb{E}[F(T_\alpha, x)],
\end{aligned}$$

where T_α denote an independent exponential random variable with mean $1/\alpha$. It is shown in [4] that

$$(1-\gamma\mathbb{1}(x \geq b))\tfrac{\partial}{\partial x}F(t,x) = \tfrac{\partial}{\partial t}F(t,x) + \mu F(t,x) - \mu\int_0^x G(x-y)\,F(t,dy).$$

Multiplying by $\alpha e^{-\alpha t}$ on both sides and integrating over $(0,\infty)$ yields (2).

To solve (2), suppose that $\varphi_\alpha^-(x)$ is a function that satisfies the equation

$$\tfrac{d}{dx}\varphi_\alpha^-(x) = (\mu+\alpha)\varphi_\alpha^-(x) - \mu\overline{G}(x) - \mu\int_0^x \varphi_\alpha^-(x-y)\,dG(y), \tag{3}$$

not only for $x < b$, but for all $x \geq 0$. Let $\Psi_\alpha^-(s) = \int_0^\infty e^{-sx}\varphi_\alpha^-(x)\,dx$ denote its Laplace transform and $G^*(s)$ the LST of $G(\cdot)$, then

$$s\Psi_\alpha^-(s) - \varphi_\alpha^-(0) = (\mu+\alpha)\Psi_\alpha^-(s) - \mu\frac{1-G^*(s)}{s} - \mu\Psi_\alpha^-(s)G^*(s),$$

and hence

$$\Psi_\alpha^-(s) \quad = \quad \frac{\varphi_\alpha^-(0) - \mu\frac{1-G^*(s)}{s}}{s - \mu(1-G^*(s)) - \alpha}. \tag{4}$$

Inversion of (4) yields $\varphi_\alpha^-(x)$ for $x \leq b$, which is equal to $\varphi_\alpha(x)$ there. Explicit inversion is, e.g., possible if $G^*(s)$ is a rational LST in which the denominator is of degree n. It is then easily seen that $\Psi_\alpha^-(s)$ is a rational LST in which the denominator is of degree $n+1$.

Next we turn to $x \geq b$. Define the partial transform $\Psi_\alpha^+(s) = \int_b^\infty e^{-sx}\varphi_\alpha(x)\,dx$. Multiplying both sides of (2) by e^{-sx} and integrating from b to ∞ yields

$$\begin{aligned}(1-\gamma)s\Psi_\alpha^+(s) - (1-\gamma)\varphi_\alpha(b) \quad &= \quad (\mu+\alpha)\Psi_\alpha^+(s) - \mu\int_b^\infty \overline{G}(x)e^{-sx}\,dx \\ &\quad -\mu G^*(s)\Psi_\alpha^+(s) - J(s,x),\end{aligned}$$

where $J(s,x) = \mu G^*(s)\int_0^b \varphi_\alpha(x)e^{-sx}\,dx - \mu\int_0^b\int_0^x \varphi_\alpha(x-u)\,dG(u)e^{-sx}\,dx$ can be evaluated from (4). Hence

$$\Psi_\alpha^+(s) \quad = \quad \frac{(1-\gamma)\varphi_\alpha(b) - \mu\int_b^\infty \overline{G}(x)e^{-sx}\,dx - J(s,x)}{s\big(1-\gamma-\alpha-\mu(1-G^*(s))\big)},$$

where all terms in the righthand side are, at least formally, known.

In what follows we present an alternative approach to obtain the LST of the ruin time in the case of a threshold strategy. Let $x < b$ and let U_1 denote the time of the first upcrossing of level b by the process R. Define

$$B_1 \quad = \quad \inf\{t > U_1 : R(t) < b\} - U_1,$$

with $\inf\emptyset = \infty$, and let I_1 denote the underflow at the moment of first downcrossing of level b. If $\tau \geq U_1$, then the ruin time consists of U_1 plus, independently, $B_1 + \tau(b - I_1)$, where $\tau(b-I_1)$ denotes the ruin time starting from level $b - I_1$. Hence

$$\mathbb{E}[e^{-\alpha\tau}|R(0)=x] = \phi_*(x,\alpha) + \phi^*(x,\alpha)\mathbb{E}[e^{-\alpha(B_1+\tau(b-I_1))}], \tag{5}$$

where

$$\phi_*(x,\alpha) = \mathbb{E}[\mathbb{1}(\tau < U_1)e^{-\alpha\tau}|R(0)=x] \quad \text{and} \quad \phi^*(x,\alpha) = \mathbb{E}[\mathbb{1}(\tau \geq U_1)e^{-\alpha U_1}|R(0)=x].$$

Notice that, in the last term in the righthand side of (5), we were allowed to omit the condition $R(0) = x$. The two functionals $\phi_*(x, \alpha)$ and $\phi^*(x, \alpha)$ have been investigated in [10]. The functional $\phi^*(x, \alpha)$ matches the functional designated by $\Gamma_U^*(\theta \mid \beta_1, \beta_2)$ in [10] and the functional $\phi_*(x, \alpha)$ matches the functional $\Gamma_L^*(\theta \mid \beta_1, \beta_2)$ in [10].

It remains to evaluate the rightmost term $\mathbb{E}[e^{-\alpha(B_1+\tau(b-I_1))}]$ in (5). Defining $b_\alpha(u) = \mathbb{E}[e^{-\alpha B_1}(I_1 < u)]$,

$$\begin{aligned}\mathbb{E}[e^{-\alpha B_1 - \alpha\tau(b-I_1)}] &= \int_0^b \mathbb{E}[e^{-\alpha\tau(b-u)}]db_\alpha(u) + \int_b^\infty db_\alpha(u) \\ &= \int_0^b \big(\mathbb{E}[e^{-\alpha\tau(b-u)}] - 1\big)db_\alpha(u) + \mathbb{E}[e^{-\alpha B_1}]. \qquad (6)\end{aligned}$$

The LST of B_1 and the joint LST of B_1 and I_1 are obtained by relating the ruin model to a $G/M/1$ queue with interarrival time distribution $G(\cdot)$, exponential service times with rate $\mu/(1-\gamma)$ and service speed 1: the intervals between claims become service times with adapted rate $\mu/(1-\gamma)$, and the claim sizes become interarrival times. This amounts to a geometric transformation in which $P_1 := (1-\gamma)B_1$ becomes a busy period in the $G/M/1$ queue, and I_1 becomes the subsequent idle period. We have

$$\mathbb{E}[e^{-\alpha P_1 - \beta I_1}] = \frac{\mu}{1-\gamma}\frac{G^*(z(\alpha)+\alpha) - G^*(\beta)}{\beta - \alpha - z(\alpha)} = \frac{\frac{\mu}{1-\gamma}(1 - G^*(\beta)) - z(\alpha)}{\beta - \alpha - z(\alpha)}, \qquad (7)$$

where $z(\alpha)$ is the unique zero of $\frac{\mu}{1-\gamma}(1 - G^*(\alpha + z)) - z$ in the right half plane (see Equation (20) in [1], or [11]). Hence

$$\mathbb{E}[e^{-\alpha B_1 - \beta I_1}] = \frac{\frac{\mu}{1-\gamma}(1 - G^*(\beta)) - z(\alpha/(1-\gamma))}{\beta - \alpha/(1-\gamma) - z(\alpha/(1-\gamma))}. \qquad (8)$$

Since $\mathbb{E}[e^{-\alpha B_1 - \beta I_1}] = \int_0^\infty e^{-\beta u} db_\alpha(u)$, $b_\alpha(\cdot)$ can be calculated by using the inversion formula for Laplace-Stieltjes transforms.

Finally, a brief remark about the case $x \geq b$. In this case, the ruin time is the sum of two components: the time until the risk process first hits b via a claim that takes it to 0 or to some value $y \in (0, b)$, plus (in the latter case) the time to ruin when starting in $y < b$. The first component is the ruin time starting at $x - b$ in a risk model with slope $1 - \gamma$. The second component follows from the above reasoning, after one has determined the distribution of the undershoot y; the latter distribution is given in Theorem 1 of [9].

3. Ruin probability

We now focus on the case $\rho < 1 - \gamma$, since then ruin is not certain: $\overline{\psi}(x) = \Pr(\tau = \infty | R(0) = x) > 0$. In fact, we are interested in computing the latter probability.

In what follows, let $\theta(x, y)$ be the probability that $R(t)$ reaches y before time τ, given $R(0) = x \leq y$. Then (see Lemma 3 in [1]), for $y \leq b$,

$$\theta(x, y) = \frac{F(x)}{F(y)}, \qquad (9)$$

where $F(x)$ is the stationary distribution of the workload process of an M/G/1 queue with service times distributed according to $G(\cdot)$ and Poisson arrivals with intensity

μ. Also, by the duality explained in Section 2, $\overline{\psi}(x) = F_\gamma(x)$, where F_γ is now the stationary distribution of $V(t)$, the workload process of an M/G/1 type dam with release rate $1-\gamma$ whenever the dam content exceeds b. The LST of $F_\gamma(x)$ has been derived in [4]. Below we derive an explicit expression for $F_\gamma(x)$ (instead of an expression for its LST), by deriving such an expression for $\bar{\psi}(x)$. The following result is essentially Corollary 6.1 in [7], but with a new and, we believe, insightful proof and form of expression. More specifically, the expressions appearing in our Theorem 1 have a probabilistic interpretation which is different from the context in [7]. Compare e.g. our first relation in Theorem 1 with the respective equation $\overline{\psi}_1(u) = q(b)\overline{\psi}_{1,\infty}(u)$ in [7], where the quantity $q(b)$ is equal to our $\overline{\psi}(b)/F(b)$ and $\overline{\psi}_{1,\infty}(u)$ equals $F(b)\theta(u,b)$. Whereas in [7] renewal-type equations are examined to derive Theorem 5.1 and then Corollary 6.1, we invoke earlier results from queueing theory, which may not be known to be useful in this connection. To formulate the result, we need yet another $M/G/1$ workload distribution: $F^{(\gamma)}$ is the stationary distribution of the workload in the $M/G/1$ queue with arrival rate λ, service time distribution $G(\cdot)$ and service speed $1-\gamma$ (so $F^{(0)}(\cdot) \equiv F(\cdot)$).

Theorem 1. *We have $\overline{\psi}(x) = \theta(x,b)\overline{\psi}(b)$ for $x < b$ and*

$$\overline{\psi}(x) = F^{(\gamma)}(x-b) + \overline{F}^{(\gamma)}(x-b)\overline{\psi}(b)\int_0^b \theta(b-u,b)\, H_{x-b}(du) \tag{10}$$

for $x \geq b$, where the survival probability at b is given by

$$\overline{\psi}(b) = F(b)\frac{1-\rho-\gamma}{1-\rho-\gamma F(b)}, \tag{11}$$

and where $H_t(\cdot)$ is the distribution function of the remaining lifetime at time t in a renewal process with renewal times distributed according to $G_{eq}(z) = \lambda\int_0^z \overline{G}(u)\,du$.

Proof. We first determine $\overline{\psi}(b)$. Starting in b, the process either tends to infinity without ever reaching b again, the probability being $1-\rho_\gamma$ (where $\rho_\gamma := \rho/(1-\gamma)$), or with probability ρ_γ it returns to b and jumps below b. The distribution of the undershoot is given by the equilibrium distribution G_{eq} of G ($p = 1$ case on p. 36 in [11]). Given the undershoot equals u, the probability to reach b again before time τ is $\theta(b-u,b)$. At the moment at which $R(t)$ upcrosses b, we have the same situation as before: with probability $1-\rho_\gamma$ the process tends to infinity without returning to b and with probability $\rho_\gamma\int_0^b \theta(b-u,b)\,dG_{eq}(u)$ the process returns to b and upcrosses b again before time τ.

In Formula (5.110) on p. 297 of Cohen [3] the following is shown for the $GI/G/1$ queue with load $\rho < 1$: The steady-state workload, when positive, is in distribution equal to the sum of two independent quantities, viz., the steady-state waiting time and the residual service time. In the $M/G/1$ case this implies, using PASTA, that $F(y) = 1-\rho+\rho\int_0^y F(y-u)dG_{eq}(u)$. Hence in particular

$$\int_0^b \theta(b-u,b)\; dG_{eq}(u) \quad = \quad \frac{F(b)+\rho-1}{\rho F(b)}.$$

Altogether we have

$$\overline{\psi}(b) = 1-\rho_\gamma+\rho_\gamma\int_0^b \theta(b-u,b)dG_{eq}(u)\overline{\psi}(b),$$

so

$$\overline{\psi}(b) = \frac{1-\rho_\gamma}{1-\rho_\gamma \frac{F(b)+\rho-1}{\rho F(b)}} = F(b)\frac{1-\rho_\gamma}{F(b)(1-\frac{1}{1-\gamma})+(\frac{1}{1-\gamma}-\rho_\gamma)} = F(b)\frac{1-\rho-\gamma}{1-\rho-\gamma F(b)}.$$

If $x < b$ then in order to have an infinite ruin time, the process has to reach b before it reaches 0, the probability being $\theta(x,b)$; this yields $\overline{\psi}(x) = \theta(x,b)\overline{\psi}(b)$.

If $x \geq b$ then $\tau = \infty$ if either of the following events occurs. (i) The process tends to infinity without reaching b again. The probability of this event is $F^{(\gamma)}(x-b)$ (see the formula for $\theta(x,b)$ and let $b \to \infty$). (ii) The process goes down to b again, thereby jumping below b, and afterwards crosses b again before time τ. Since the undershoot below b has the distribution $H_{x-b}(x)$ (see Theorem 1 in [9]) it follows that the probability of this event is $\overline{F}^{(\gamma)}(x-b)\int_0^b \theta(b-u,b)\,dH_{x-b}(u)$. Hence (10) follows.

4. Total dividends

Again let $R(0) = x < b$. We assume in this section that $\rho > 1-\gamma$, so that ruin is certain. The total dividends until ruin are given by (suppressing x in the notation)

$$D(\tau) = \gamma \int_0^\tau \mathbb{1}(R(s) > b)\,ds.$$

Note that $D(t) + R(t)$ increases with slope one between the claims. Below we shall derive an expression for the LST $\mathbb{E}[e^{-\alpha D(\tau)}]$.

Let $U_0 = 0$ and let U_n denote the time of the nth upcrossing of the process R of the level b. Define

$$B_n \quad = \quad \inf\{t > U_n : R(t) < b\} - U_n,$$

with $\inf \emptyset = \infty$. By the strong Markov property, $B_1, B_2, ...$ are i.i.d.; note that $U_1, U_2, \ldots$ are independent and that $U_2, U_3, \ldots$ are identically distributed. Let $N = \sup\{n : U_n < \tau\}$ denote the number of upcrossings before ruin. Then

$$D(\tau) = \gamma(B_1 + ... + B_N). \tag{12}$$

Clearly N is independent of $B_1, B_2, \ldots$ and

$$\mathbb{P}(N = n) = \begin{cases} p_0, & n = 0, \\ (1-p_0)(1-p)^{n-1}p, & n = 1, 2, ... \,, \end{cases}$$

where $p_0 = 1 - \theta(x,b)$ is the conditional probability that level 0 is reached before level b is upcrossed, given the starting state is x. Since $\rho > 1-\gamma$ we cannot apply Formula (9) here. Instead, let $R_0(t)$ be the surplus process of a risk model with $\gamma = 0$ and let τ_0 denote the first downcrossing time of level $-x$. We assume that $R_0(0) = 0$ so that R_0 becomes a Lévy process. It is known that the probability distribution of the maximum, $\mathbb{P}(\max_{0 \leq t < \infty} R_0(t) \leq x)$, is equal to the limit distribution of the reflected process $A_0(t) = R_0(t) - \inf\{R_0(s) : 0 \leq s \leq t\}$ as $t \to \infty$ and is hence equal to an exponential distribution with mean $1/\eta$, where $\eta \in (0,\mu)$ solves $\eta = \mu(1 - G^*(\eta))$ (cf.

[1]). Hence $\mathbb{P}(\max_{0\le t<\infty} R_0(t) \ge b-x) = e^{-\eta(b-x)}$ and

$$\begin{aligned} e^{-\eta(b-x)} &= \mathbb{P}(\{\max_{0\le t<\tau_0} R_0(t) \ge b-x\} \cup \{\max_{\tau_0\le t<\infty} R_0(t) \ge b-x\}) \\ &= \mathbb{P}(\max_{0\le t<\tau_0} R_0(t) \ge b-x) + \mathbb{P}(\max_{\tau_0\le t<\infty} R_0(t) \ge b-x) \\ &\quad -\mathbb{P}(\{\max_{0\le t<\tau_0} R_0(t) \ge b-x\} \cap \{\max_{\tau_0\le t<\infty} R_0(t) \ge b-x\}). \end{aligned} \tag{13}$$

It follows from the strong Markov property of R_0 that

$$\mathbb{P}(\max_{\tau_0\le t<\infty} R_0(t) \ge b-x) = \mathbb{P}(\max_{0\le t<\infty} R_0(t) \ge b+I_x) = \mathbb{E}[e^{-\eta(b+I_x)}],$$

where I_x denotes a random variable, independent of R_0, with the same distribution as $-x-R_0(\tau_0)$. Since $\theta(x,b)$ is the probability that R_0 reaches $b-x$ before time τ_0 and since $\mathbb{P}(\max_{\tau_0\le t<\infty} R_0(t) \ge b-x | \max_{0\le t<\tau_0} R_0(t) \ge b-x)$ is equal to $\mathbb{E}[e^{-\eta(b+I_b)}]$ (here $\tau_0<\infty$ a.s. is used), it follows from (13) that

$$e^{-\eta(b-x)} = \theta(x,b) + \mathbb{E}[e^{-\eta(b+I_x)}] - \mathbb{E}[e^{-\eta(b+I_b)}]\theta(x,b).$$

Since $p_0 = 1-\theta(x,b)$ it then follows that

$$p_0 = 1 - \frac{e^{\eta x} - \mathbb{E}[e^{-\eta I_x}]}{e^{\eta b} - \mathbb{E}[e^{-\eta I_b}]}. \tag{14}$$

We note that a formula for $\mathbb{E}[e^{-\eta I_x}]$ is given in [1] and that one might use Theorem 4 in [1] to obtain formula (14).

To compute the parameter p (i.e., the probability that level 0 is reached before level b is upcrossed, after a downward jump through b), recall that the negative overflows $I_n = b - R(U_n + B_n)$ at a moment of downcrossing of level b have the same law as that of the idle period in the $G/M/1$ queue. Its LST is obtained by taking $\alpha = 0$ in (7) or (8). Let $F_I(\cdot)$ be the distribution with that LST. Then,

$$p = 1 - F_I(b) + \int_0^b [1-\theta(b-y,b)]dF_I(y). \tag{15}$$

It now follows from (12) that

$$\begin{aligned} \mathbb{E}[e^{-\alpha D(\tau)}] &= \mathbb{E}\big[\mathbb{E}[e^{-\alpha D(\tau)} \mid N]\big] = \mathbb{E}[\mathbb{E}[e^{-\alpha\gamma B_1}]^N] \\ &= p_0 + (1-p_0)\frac{p\mathbb{E}[e^{-\alpha\gamma B_1}]}{1-(1-p)\mathbb{E}[e^{-\alpha\gamma B_1}]}, \end{aligned}$$

where $\mathbb{E}[e^{-\alpha\gamma B_1}]$ is obtained from (8) by substituting $\beta = 0$. Hence we have proven:

Theorem 2.

$$\mathbb{E}[e^{-\alpha D(\tau)}] = p_0 + (1-p_0)\frac{p\mathbb{E}[e^{-\alpha\gamma B_1}]}{1-(1-p)\mathbb{E}[e^{-\alpha\gamma B_1}]},$$

where the LST of B_1 is given in (8) and where p_0 and p are given in (14) and (15), respectively.

Mean of the ruin time and of the deficit at ruin – One could derive the mean ruin time from the transform results in Section 2. However, we prefer to derive it using a probabilistic argument, once again exploiting a relation between risk and queueing theory and using a queueing-theoretic level crossing argument. To compute the expectation of the ruin time τ and the expectation of the deficit at ruin Y, we construct the artificial regenerative process $\widehat{R}(t)$ with cycle length $\widehat{C} = \tau - \widehat{R}(\tau)$ as follows (recall that $\widehat{R}(\tau)$ is negative so that $\widehat{C}$ is the sum of the ruin time and the deficit $Y = -\widehat{R}(\tau)$). We define

$$\widehat{R}(t) = \begin{cases} R(t) & ; 0 \le t \le \tau, \\ R(\tau) + t - \tau & ; \tau < t \le \widehat{C}. \end{cases}$$

By the theory of level crossings we obtain the balance equations for the equilibrium density of $\widehat{R}(t)$ (the density of the weak limit $\widehat{R} = \lim_{t\to\infty} \widehat{R}(t)$):

$$(1 - \gamma \mathbb{1}(z > b))\widehat{f}(z) = \begin{cases} \mu \int_0^\infty \overline{G}(w - z)\widehat{f}(w)\, dw, & z < 0, \\ \mu \int_z^\infty \overline{G}(w - z)\widehat{f}(w)\, dw - \zeta, & 0 \le z < x, \\ \mu \int_z^\infty \overline{G}(w - z)\widehat{f}(w)\, dw, & x \le z \le b, \\ \mu \int_z^\infty \overline{G}(w - z)\widehat{f}(w)\, dw, & z > b, \end{cases} \tag{16}$$

where $\zeta = \widehat{f}(0-) = \mu \int_0^\infty \overline{G}(w)\widehat{f}(w)\, dw$. In order to get a better insight into these balance equations, recall that each cycle starts at level $\widehat{R}(0) = x$ and terminates at $\widehat{C}$ with $\widehat{R}(\widehat{C}) = 0$, where by construction of $\widehat{R}(t)$ the increase rate below level 0 is 1. By the level crossing approach, $(1 - \gamma \mathbb{1}(z > b))\widehat{f}(z)$ is interpreted as the long run average upcrossing rate of level z. The downcrossing rate is obviously represented by the integrals in the righthand side of (16), but the case $0 \le z < x$ deserves a remark. In this case, the number of upcrossings of level z per cycle is one less than the number of downcrossings. Hence we need to compensate the downcrossing rate by subtracting $1/\mathbb{E}[\widehat{C}]$, which equals $\zeta = \mu \int_0^\infty \overline{G}(w)\widehat{f}(w)dw$, the downcrossing rate of level 0.

The fourth equation of (16) is satisfied by $\widehat{f}(z) = Ce^{-\eta z}$ (with C a constant that has to be determined later via normalization); this is easily verified via substitution or by observing that, for $z > b$, $\widehat{f}(z)$ behaves like the workload in a $G/M/1$ queue. Now use this result to rewrite the third equation into: $\widehat{f}(z) = \mu \int_z^b \overline{G}(w-z)\widehat{f}(w)dw + H(z)$, with $H(z) = \mu \int_b^\infty \overline{G}(w-z)Ce^{-\eta w}dw$. Now $\widehat{f}(z)$ on $[x, b]$ can be obtained by successive substitutions and iterations: $\widehat{f}(z) = H(z) + \mu \int_z^b \overline{G}(w - z)H(w)dw + \ldots$. Subsequently, similar procedures are applied on $[0, x)$ and on $(-\infty, 0)$. Finally, C is determined by normalization.

The sequence of deficits associated with the regenerative process $\widehat{R}(t)$ is a sequence of i.i.d. random variables; let $K(\cdot)$ denote their common distribution function. A little thought will convince that $K_e(z) = \mathbb{P}(\widehat{R} \ge -z|\widehat{R} \le 0), \quad z \ge 0$, where $K_e(z)$ is the equilibrium distribution of K. Thus

$$K(z) = 1 - \mathbb{E}[Y]\tfrac{d}{dz}\mathbb{P}(\widehat{R} \ge -z|\widehat{R} \le 0).$$

To compute $\mathbb{E}[Y]$ and $\mathbb{E}[\tau]$ recall that (by level crossing) $\mathbb{E}[\widehat{C}] = 1/\zeta$ and use the theory of regenerative processes to conclude that

$$\mathbb{E}[Y] = \mathbb{E}[\widehat{C}]\mathbb{P}(\widehat{R} < 0) = \frac{\mathbb{P}(\widehat{R} < 0)}{\zeta}, \quad \mathbb{E}[\tau] = \mathbb{E}[\widehat{C}]\mathbb{P}(\widehat{R} \geq 0) = \frac{\mathbb{P}(\widehat{R} \geq 0)}{\zeta}.$$

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