EURANDOM PREPRINT SERIES
2011-041 October 23, 2011

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This report has been realized with support of NWO (grant 040.11.257)

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ISSN 1389-2355

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October 23, 2011

Abstract

We consider a generalized memoryless property which relates to Cantor's second functional equation, study its properties and demonstrate various examples.

Keywords: Generalized memoryless property, Markov kernel, Cantor's second functional equation.

AMS Subject Classification: Primary: 60K30 Secondary: 90B18, 60J25, 60J35,60G44, 68M20.

1 Introduction and setup

Let $\mathbb{R}$ denote the set of real numbers and $\mathcal{B}$ be it's Borel sets. Consider a Markov kernel $P(x, A)$ where for each $x \in \mathbb{R}$, $P(x, \cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. We say that the generalized memoryless property is satisfied if the following is satisfied for each nonnegative x, y and real z:

$$P(z,(x+y,\infty)) = P(z,(x,\infty))P(z+x,(y,\infty)) \tag{1}$$

We note that if $P(x, \cdot) = P(\cdot)$, that is, the Markov kernel is independent of the originating state, this is precisely the memoryless property of which the only solution is of the form $P((x,\infty)) = \rho^x$ for some $0 \le \rho \le 1$. That is, a random variable that has such a distribution is either almost surely (a.s.) zero, or a.s. infinite or has an exponential distribution.

To motivate the property given by (1) assume that in addition $P(\cdot, A)$ is a Borel function for each $A \in \mathcal{B}$ and let T_a (age) and T_r (remaining life) be a pair of random variables, where T_a has an arbitrary distribution and $\mathbb{P}(T_r \in A|T_a) = P(T_a, A)$. That is, one interprets $\mathbb{P}(T_r \in A|T_a = z) = P(z, A)$. Then (1) becomes

$$\mathbb{P}(T_r > x+y|T_a = z) = \mathbb{P}(T_r > x|T_a = z)\mathbb{P}(T_r > y|T_a = z+x) \ . \tag{2}$$

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†Supported in part by grant No. 434/09 from the Israel Science Foundation and the Vigevani Chair in Statistics.

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In other words, in order for the a component having age z to function at time $x + y$, it first has to function at time x. Then, independent of everything else, its age is modified to $z + x$ and, given that its age is now $z + x$, it has to function at time y. This model relates to *Model II* of [7]. In fact, given the condition that $P(0, (x, \infty)) > 0$ for each $x \geq 0$ this results in exactly the same model, even though we write it in more primitive terms. To see this, denote $1 - F(x) = P(0, (x, \infty))$ and observe that with $z = 0$, (2) becomes

$$1 - F(x + y) = (1 - F(x))\mathbb{P}(T_r > y | T_a = x), \tag{3}$$

so that indeed

$$\mathbb{P}(T_r > y | T_a = x) = \frac{1 - F(x + y)}{1 - F(x)}, \tag{4}$$

which is equation (1) of [7] with the roles of x and y reversed. However, if we only assume that $P(z, (x, \infty)) > 0$ for all $0 < z \leq x$ then we will see that it is possible that $\mathbb{P}(T_r > y | T_a = x) = s(x + y)/s(x)$ for some nonincreasing function s for which $s(x) \to \infty$ as $x \downarrow 0$. We will also see that other possibilities may occur as well. We also mention that for the case where $P(z, (x, \infty)) > 0$ a more general model was considered in [9]. However, in this paper the authors suffice in pointing out some examples of this property but do not characterize the general form. In this generality, a characterization might not be possible.

To continue, for real x and $y \geq x$ we denote $\mu(x, y) = P(x, (y - x, \infty))$ and observe that (1) becomes

$$\mu(z, y) = \mu(z, x)\mu(x, y) \tag{5}$$

for each $z \leq x \leq y$. If this was valid for all x, y, z then (5) is called *Cantor's second functional equation*. We note that in the latter case, if $\mu(z, y) \neq 0$ for some z, y then $\mu(z, x) \neq 0$ and $\mu(x, y) \neq 0$ for all x. Since $\mu(x, y) = \mu(x, u)\mu(u, y)$ then $\mu(x, u) \neq 0$ for all x, u and in particular if we denote $s(x) = \mu(z, x)$ then for any x, y we have that $\mu(x, y) = s(y)/s(x)$. Thus, as is well known (e.g., see [1]), the only solutions of Cantor's second functional equation are either $\mu(x, y) = 0$ for all x, y or $\mu(x, y) = s(y)/s(x)$ for some function $s(\cdot)$ that never vanishes. In the second case it is clear that for all x, $\mu(x, x) = 1$ and that for any nonvanishing function $s(\cdot)$, $s(y)/s(x)$ obeys Cantor's second functional equation.

When we assume that (5) is satisfied only if $z \leq x \leq y$, the solution requires a bit more care.

2 Main observations

In order to consider the most general setup, let us assume until further notice that

$$\mu : \{(x, y) |\ x \leq y\} \to \mathbb{R} \tag{6}$$

(note: $\mathbb{R}$ rather than just $[0, 1]$) and that (5) is satisfied for $z \leq x \leq y$. Denote

$$b(z) = \begin{cases} z & \text{if } \mu(z, x) = 0 \ \forall x > z \\ \sup\{x | \mu(z, x) \neq 0,\ x > z\} & \text{otherwise} \end{cases} \tag{7}$$

and

$$a(z) = \begin{cases} z & \text{if } \mu(x,z) = 0 \ \forall x < z \\ \inf\{x|\mu(x,z) \neq 0, \ x < z\} & \text{otherwise.} \end{cases} \tag{8}$$

Now, when $a(z) < b(z)$ we denote

$$I(z) = \begin{cases} (a(z), b(z)) & \text{if } \mu(a(z), z) = 0 = \mu(z, b(z)) \\ (a(z), b(z)] & \text{if } \mu(a(z), z) = 0 \neq \mu(z, b(z)) \\ [a(z), b(z)) & \text{if } \mu(a(z), z) \neq 0 = \mu(z, b(z)) \\ [a(z), b(z)] & \text{if } \mu(a(z), z) \neq 0 \neq \mu(z, b(z)) \end{cases} \tag{9}$$

and when $a(z) = b(z) = z$ we let $I(z) = \{z\}$, noting that from $\mu(z,z) = \mu(z,z)\mu(z,z)$ necessarily $\mu(z,z) \in \{0,1\}$. Moreover, since

$$\mu(u,v) = \mu(u,v)\mu(v,v) = \mu(u,u)\mu(u,v) \tag{10}$$

for $u \leq v$, it is easy to check that if $I(z)$ is not a singleton, then necessarily $\mu(z,z) = 1$.

Consider now the following.

Lemma 1. *Let $\mu : \{(x,y)| \ x \leq y\} \to \mathbb{R}$ satisfy (5) for all $z \leq x \leq y$. Then for $x < y$, $\mu(x,y) \neq 0$ if and only if $I(x) = I(y)$ and for every $u, v \in I(x)$ with $u \leq v$ we have that $\mu(u,v) \neq 0$.*

Proof. For $v \geq y$ we have that $\mu(x,v) = \mu(x,y)\mu(y,v)$, so that $\mu(x,v) \neq 0$ if and only if $\mu(y,v) \neq 0$. For $u \leq x$ we have by similar reasoning that $\mu(u,x) \neq 0$ if and only if $\mu(u,y) \neq 0$. For $w \in (x,y)$ we have that $\mu(x,y) = \mu(x,w)\mu(w,y)$ and thus $\mu(x,w) \neq 0$ and $\mu(w,y) \neq 0$. This implies that $I(x) = I(y)$. Now, for every $u \in I(x)$ we have that $I(u) = I(x)$ and thus for any $v \geq u$ with $v \in I(x)$ it follows that $v \in I(u)$ which implies that $\mu(u,v) \neq 0$. □

Theorem 1. *Under the conditions of Lemma 1 there exists a family $\{I_\theta| \ \theta \in \Theta\}$ of necessarily at most countably many disjoint intervals (open, half open or closed) and possibly uncountably many singletons with $\cup_{\theta\in\Theta} I_\theta = \mathbb{R}$ such that for each θ for which I_θ is not a singleton there exists a function $s_\theta : I_\theta \to \mathbb{R}$ which is nonvanishing such that for every $x, y \in I_\theta$ with $x \leq y$ we have that $\mu(x,y) = s_\theta(y)/s_\theta(x)$.*

Proof. For an arbitrary $z \in I_\theta$ define

$$s_\theta(x) = \begin{cases} \mu(z,x) & \text{if } z \leq x \in I_\theta \\ 1/\mu(x,z) & \text{if } z \geq x \in I_\theta \ . \end{cases} \tag{11}$$

Then, if $z \leq x \leq y$ and $x, y \in I_\theta$ then

$$s_\theta(y) = \mu(z,y) = \mu(z,x)\mu(x,y) = s_\theta(x)\mu(x,y) \ . \tag{12}$$

If $x \leq y \leq z$ and $x, y \in I_\theta$ then

$$\frac{1}{s_\theta(x)} = \mu(x,z) = \mu(x,y)\mu(y,z) = \mu(x,y)\frac{1}{s_\theta(y)} \ . \tag{13}$$

Finally, when $x \le z \le y$ with $x, y \in I_\theta$ then

$$\mu(x,y) = \mu(x,z)\mu(z,y) = \frac{1}{s_\theta(x)} s_\theta(y) \ . \tag{14}$$

□

In particular we observe that if $\mu(x,y) \ne 0$ for all $x \le y$ then $I(x) = \mathbb{R}$ and for some nonvanishing function $s : \mathbb{R} \to \mathbb{R}$ we have that $\mu(x,y) = s(y)/s(x)$ for all $x \le y$. It seems as if this is the same solution of Cantor's second functional equation until we recall that $\mu(x,y)$ is undefined for $x > y$.

Returning to the generalized memoryless property of (1) we recall that $\mu(x,y) = P(x,(y-x,\infty))$ and thus, for $y \ge 0$, $P(x,y) = \mu(x,x+y)$. Hence we conclude the following.

Theorem 2. *Assume that (1) is satisfied. Then there exists a family* $\{I_\theta |\ \theta \in \Theta\}$ *of disjoint intervals and singletons with* $\cup_{\theta\in\Theta} I_\theta = \mathbb{R}$ *such that for each* θ *for which* I_θ *is not a singleton there exists a nonincreasing right continuous strictly positive function* $s_\theta : I_\theta \to \mathbb{R}$ *such that for every* $x \in I_\theta$ *and* $y \ge 0$ *with* $x + y \in I_\theta$ *we have that*

$$P(x,(y,\infty)) = \frac{s_\theta(x+y)}{s_\theta(x)}.$$

Remark 1. We note that the same result holds, only with s_θ being left continuous, for $P(x,[y,\infty))$ (left closed interval) if we replace (1) by the left closed version

$$P(z,[x+y,\infty)) = P(z,[x,\infty))P(z+x,[y,\infty)) \tag{15}$$

which, by taking $y \downarrow u$, where $u \ge x$, is also equivalent to

$$P(z,(x+u,\infty)) = P(z,[x,\infty))P(z+x,(u,\infty)) \tag{16}$$

whenever $x, y \ge 0$.

Remark 2. For the case $\mu(x,y) = P(x,(y-x,\infty))$ clearly $\mu(x,y) \in [0,1]$ for all $x \le y$. Therefore, from $\mu(z,y) = \mu(z,x)\mu(x,y) \le \mu(x,y)$ for $z \le x \le y$ it follows that $\mu(\cdot,y)$ is a nondecreasing function on $(-\infty,y]$. Similarly $\mu(x,\cdot)$ is nonincreasing on $[x,\infty)$.

Remark 3. It is easy to check that these results remain unchanged if the domain of μ is $0 \le x \le y$ rather than $x \le y$ or $x \le y \le \infty$ or any other combination like this. Basically, whenever we have (5) for $z \le x \le y$ these results apply.

Remark 4. When $P(x,(y,\infty)) > 0$ for every x and every $y \ge 0$, then there is some necessarily nonincreasing and right continuous function $s : \mathbb{R} \to (0,\infty)$ for which $\mu(x,(y,\infty)) = s(x+y)/s(x)$ for all x and $y \ge 0$. Moreover the function s is uniquely determined up to a constant multiple. We note that unlike in [7], it is possible that for some θ, $I_\theta = (0,\infty)$ and $s_\theta(x) \to \infty$ as $x \downarrow 0$. If $[0,\infty) \subset I_\theta$ for some θ then one has the model considered in [7].

Remark 5. It is also evident that the same structure holds when one reverses (1) to

$$P(z,(-\infty,x+y)) = P(z,(-\infty,x))P(z+x,(-\infty,y)) \tag{17}$$

or to

$$P(z,(-\infty,x+y]) = P(z,(-\infty,x]))P(z+x,(-\infty,y]) \tag{18}$$

for all $x, y \le 0$. In particular when $P(z,(-\infty,0)) = 0$ and $-z \le x, y \le 0$. We will use this in the next section when modeling a certain growth collapse (additive increase multiplicative decrease) process with state dependent decrease ratios.

Consider now a possibly infinite interval I_θ which is not a singleton and its corresponding positive valued function s_θ. Then for each $z \in I_\theta$ and $x \ge 0$ such that $z + x \in I_\theta$ we have that $P(z,(x,\infty)) = s_\theta(z+x)/s_\theta(z)$. If $x + z \notin I_\theta$ then $P(z,(x,\infty)) = 0$ so that we may define $s_\theta(y) = 0$ for any $y \notin I_\theta$ which is on the right of I_θ (if any) where necessarily s_θ must be right continuous at $z^*(\theta) = \sup\{z|\ z \in I_\theta\}$. Now, for $z \in I_\theta$ and $0 < u < 1$ denote

$$\begin{aligned} t^z_\theta(u) &= \inf\left\{x|\ 1 - \frac{s_\theta(z+x)}{s_\theta(z)} \ge 1-u,\ x \ge 0\right\} \\ &= -z + \inf\left\{x|\ s_\theta(x) \le s_\theta(z)u,\ x \ge z\right\} \qquad (19) \\ &= -z + \inf\left\{x|\ s_\theta(x) \le s_\theta(z)u\right\} \end{aligned}$$

Where the last equatlity follows since $s_\theta(x) < s_\theta(z)u$ for every $x \le z$. It is standard that if $U \sim \text{Uniform}(0,1)$ then $t^z_\theta(1-U)$ and thus $t^z_\theta(U)$ have the distribution $P(z,\cdot)$. Thus, if we denote $t_\theta(v) = \inf\{x|\ s_\theta(x) \le v\}$ for $v > \inf\{z|\ z \in I_\theta\}$ then for every $z \in I_\theta$ we have that $t_\theta(s_\theta(z)U) - z$ has the distribution $P(z,\cdot)$. Recalling T_a and T_r from Section 1, this implies the following.

Theorem 3. *Assuming that T_a and $U \sim \text{Uniform}(0,1)$ are independent, then $(T_a, T_r)\mathbb{1}_{\{T_a \in I_\theta\}}$ and $(T_a, t_\theta(s_\theta(T_a)U) - T_a)\mathbb{1}_{\{T_a \in I_\theta\}}$ are identically distributed.*

Clearly, when Θ is countable then the immediate conclusion is that

$$(T_a, T_r) \sim \left(T_a, \sum_\theta (t_\theta(s_\theta(T_a)U) - T_a)\mathbb{1}_{\{T_a \in I_\theta\}}\right) . \tag{20}$$

It is interesting to check when for a given θ for which I_θ is not a singleton the value of $t_\theta(s_\theta(z)u) - z$ is independent of z. That is, it is only a function of u. The answer is not surprising.

Theorem 4. *When I_θ is not a singleton then $t_\theta(s_\theta(z)u) - z$ is independent of $z \in I_\theta$ if and only if for some $0 \le \lambda_\theta < \infty$ and $0 < c_\theta < \infty$, $s_\theta(z) = c_\theta e^{-\lambda_\theta z}$ for $z \in I_\theta$.*

Proof. Let $f(u) = t_\theta(s_\theta(z)u) - z$ (independent of z) for every $z \in I_\theta$ and $0 < u < 1$. Note that since the right side is left continuous in u, then so is f (as a function defined on

(0, 1)). In particular f is Borel. Denoting $X = f(U)$ we have that for every $z \in I_\theta$ and every $x, y \geq 0$, with $z + x + y \in I_\theta$,

$$\begin{aligned}\mathbb{P}(X > x + y) &= \frac{s_\theta(z + x + y)}{s_\theta(z)} \\ &= \frac{s_\theta(z + x)}{s_\theta(z)} \cdot \frac{s_\theta(z + x + y)}{s_\theta(z + x)} \\ &= \mathbb{P}(X > x)\mathbb{P}(X > y)\end{aligned}$$

The equation $g(x + y) = g(x)g(y)$ for $x, y \geq 0$ under minor regularity conditions on g implies that g is either identically zero, identically one or exponential. Monotonicity, right or left continuity or even Lebesgue measurability are sufficient conditions. The standard proof can be easily modified to the case where g is defined on and the equation is valid only when $x, y, x + y$ are in $[0, a)$ or $[0, a]$ for some $0 < a < \infty$, resulting in g being identically zero, identically one or exponential on $[0, a)$ or $[0, a]$. When we know that g is strictly positive and bounded above by one on $[0, a)$ or $[0, a]$, then zero is not an option and thus $g(x) = e^{-\lambda x}$ for some $0 \leq \lambda < \infty$. Thus, for every $z \in I_\theta$ and $x \geq 0$ such that $w = z + x \in \theta$ we have that for some $0 \leq \lambda_\theta < \infty$

$$\frac{s_\theta(z + x)}{s_\theta(z)} = e^{-\lambda_\theta x} = \frac{e^{-\lambda_\theta(z+x)}}{e^{-\lambda_\theta z}} \tag{21}$$

which implies that for every $z, w \in I_\theta$

$$s_\theta(w)e^{\lambda_\theta w} = s_\theta(z)e^{\lambda_\theta z} \equiv c_\theta \ , \tag{22}$$

as required. □

3 Maximum at a random time of a continuous time Markov process with no positive jumps

Consider a continuous time right continuous Markov process $\{X(t)\}_{t\geq 0}$ with convex state space $\mathfrak{X} \subset \mathbb{R}$, having no positive jumps and with generator $\mathcal{A}$. As is customary, we denote $\mathbb{P}_x$ and $\mathbb{E}_x$ the distribution measure and the expected value when the process is initiated at $x \in \mathfrak{X}$. Assuming its existence, let f be strictly positive and nondecreasing function in the extended domain, which is bounded on $(-\infty, x] \cap \mathfrak{X}$ for any $x \in \mathbb{R}$ and for which

$$M(t) = f(X(t)) \exp\Big(- \int_0^t \frac{\mathcal{A}f(X(s))}{f(X(s))} ds\Big) \tag{23}$$

is a martingale with respect to the right continuous augmented filtration $\{\mathcal{F}_t |\ t \geq 0\}$ generated by X. A sufficient condition for the latter is that f is bounded away from zero on $\mathfrak{X}$ (e.g. [4], p.175). Furthermore we assume that $\mathcal{A}f(x)$ is nonnegative for all $x \in \mathfrak{X}$. Now, denote $\tau(y) = \inf\{t |\ X(t) > y\}$ (infinite if X never exceeds y). Then $\tau(y)$ is right continuous in y and it is easy to check that $\sup_{0\le s\le t} X(s) \le y$ if and only if $\tau(y) \ge t$. With $a \wedge b = \min(a,b)$ it is well known that $M(\tau(y) \wedge t)$ is also a martingale and moreover, our assumptions assure that it is also bounded. Finally, denoting $\lambda(x) = \mathcal{A}f(x)/f(x)$, then by the bounded convergence theorem we have that for $y \ge x$ such that $y \in \mathfrak{X}$, if either $\tau(y) < \infty$ $\mathbb{P}_x$-a.s. (almost surely) or $\int_0^\infty \lambda(X(s))ds = \infty$ on $\{\tau(y) = \infty\}$ then

$$f(x) = \mathbb{E}_x M(0) = \mathbb{E}_x M(\tau(y)) = f(y)\mathbb{E}_x e^{-\int_0^{\tau(y)} \lambda(X(s))ds} . \tag{24}$$

In particular this means that it is impossible to find a positive f in the extended domain of $\mathcal{A}$ such that $\mathcal{A}f \ge 0$ and $\int_0^{\tau(y)} \lambda(X(s))ds = \infty$ $\mathbb{P}_x$-a.s. for some x.

Now, if we denote $s(x) = 1/f(x)$, we have that

$$\mathbb{E}_x e^{-\int_0^{\tau(y)} \lambda(X(s))ds} = \frac{s(y)}{s(x)} \tag{25}$$

for every y for which $\tau(y) < \infty$ $\mathbb{P}_x$-a.s. If Z is a random variable (possibly infinite) such that $\mathbb{P}_x\left(Z > t|\mathcal{F}_t\right) = e^{-\int_0^t \lambda(X(s))ds}$, then in fact

$$\mathbb{E}_x e^{-\int_0^{\tau(y)} \lambda(X(s))ds} = \mathbb{P}_x\left(Z > \tau(y)\right) = \mathbb{P}_x\left(\max_{0\le t\le Z} X(t) > y\right) . \tag{26}$$

Thus, we have that

$$P_x\left[\max_{0\le t\le Z} X(t) > y\right] = \frac{s(y)}{s(x)} \tag{27}$$

for $x \le y$. If one assumes that U is an independent Uniform$(0,1)$ random variable (if there isn't then it is easy to artificially modify our probability space so that there is), then taking $\mathcal{F}_t^U = \mathcal{F}_t \vee \sigma(U)$, we see that M is a martingale also with respect to this new filtration. Thus, if we let $F(X,t) = 1 - e^{-\int_0^t \lambda(X(s))ds}$ and $G(X,u) = \inf\{t|\ F(X,t) \ge u\}$ then $Z = G(X,U)$ has the correct conditional distribution.

3.1 Lévy processes

Sometimes, for various values of α, we may be lucky to find a function f satisfying the above conditions and for which $\lambda(x) = \alpha$. In this case we immediately obtain the Laplace transform

$$\mathbb{E}_x e^{-\alpha\tau(y)} = \frac{s(y)}{s(x)}. \tag{28}$$

For a Lévy process with no positive jumps (in particular a Brownian motion) and

$$\varphi(\alpha) = \log \mathbb{E}_0 e^{\alpha X(1)} = c\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(-\infty,0)} \left(e^{\alpha y} - 1 - \alpha y 1_{(-1,0)}(y)\right)\nu(dy) \tag{29}$$

then the equation that needs to be solved is the following

$$cf'(x) + \frac{\sigma^2}{2}f''(x) + \int_{(-\infty,0)} \left(f(x+y) - f(x) - yf'(x)1_{(-1,0)}(y)\right)\nu(dy) = \alpha f(x) . \tag{30}$$

Fortunately, when X is not nonincreasing (the negative of a subordinator or the zero function), then φ has an inverse on $[\beta, \infty)$ when $\beta = \inf\{\alpha | \ \varphi(\alpha) > 0, \alpha > 0\}$. It is well known that $\beta = 0$ if $\varphi'(0) \geq 0$ and $\beta > 0$ otherwise. In this case, for every $x \leq y$, $\tau(y)$ is $\mathbb{P}_x$-a.s. finite and $f(x) = e^{\varphi^{-1}(\alpha)x}$ for $\alpha \geq \beta$ satisfies all the needed requirements and in particular solves (30). So, as is well known,

$$\mathbb{E}_x e^{-\alpha\tau(y)} = \mathbb{E}_x e^{-\alpha\tau(y)} \mathbb{1}_{\{\tau(y)<\infty\}} = \frac{s(y)}{s(x)} = \frac{f(x)}{f(y)} = e^{-\varphi^{-1}(\alpha)(y-x)}, \tag{31}$$

even if $\tau(y)$ is not $\mathbb{P}_x$-a.s. finite (that is, when $\varphi'(0) < 0$). In this particular case $Z \sim \exp(\alpha)$ (independent of X) and it follows as is also well known that

$$\max_{0\leq t\leq Z} X(t) - X(0) \sim \exp(\varphi^{-1}(\alpha)),$$

so that this random variable obeys the standard (not-generalized) memoryless property.

3.2 Reflected Brownian motion

For the reflected Brownian motion on $\mathfrak{X} = [0, \infty)$ with general drift, the generator is the same as the one for Brownian motion, only that its domain is reduced to twice differentiable functions for which $f'(0) = 0$. In this case one needs to compute $\mu f'(x) + \frac{\sigma^2}{2} f''(x) = \alpha f(x)$ subject to $f'(0) = 0$ for $\alpha > 0$. It is easy to check that any positive constant multiple of the function

$$f(x) = \frac{e^{a^+x}}{a^+} + \frac{e^{-a^-x}}{a^-} \tag{32}$$

where $a^\pm = \frac{\sqrt{\mu^2+2\sigma^2\alpha} \ \pm \ \mu}{\sigma^2}$, would do the trick. In particular it is positive and increasing due to $0 < a^- < a^+$. In this case it is well known that $\tau(y) < \infty$ $\mathbb{P}_x$-a.s. for any $y > x$ regardless of the value of μ, but it can also be inferred from this without resorting to anything else, by letting $\alpha \to 0$. Of course, this particular result is quite standard (e.g., problem 4 on p. 95 of [5]).

More generally of course, given a positive function λ if it is possible to find some f satisfying our assumptions for which $\mathcal{A}f(x) = \lambda(x)f(x)$ for each $x \in \mathfrak{X}$, then if either $y > x$ is such that $\tau(y)$ is $\mathbb{P}_x$-a.s. finite or λ is bounded away from zero, then (27) is satisfied.

3.3 A growth collapse process with generalized memoryless jumps

In this section we consider a piecewise deterministic Markov process X_t with jumps that are governed by a jump measure with the generalized lack of memory property described above. See [2] and [8] for similar models.

Let $\{X(t)\}_{t\geq 0}$ be a Markov process on $\mathfrak{X} = [0, \infty)$ which is deterministically increasing with rate $r(x)$ between randomly occurring downward jumps. More specifically, we assume that inbetween jumps $dX_t = r(X_t)\,dt$, that $r(x)$ is positive and Lipschitz-continuous and that the time $t^*(x, y) = \int_x^y 1/r(u)\,du$ that is needed to reach the level y from x in the absence of any jumps is finite for all $x < y \in [0, \infty)$. Let $\kappa : \mathfrak{X} \to [0, \infty)$ denote the state-dependent jump rate, i.e. if the process is in the state $x \in \mathfrak{X}$, then a jump occurs during the next Δt time units with probability $\kappa(x)\Delta t + o(\Delta t)$ (and the probability to see more than one jump is $o(\Delta t)$). We assume κ to be bounded. Given that there is a jump at time t, the process jumps from state $x \in \mathfrak{X}$ into some measurable $A \subset [0, x)$ with probability $\nu(x, A)$. We assume that for $0 \leq y \leq x \leq z$ the kernel ν has the special property that

$$\nu(z, y) = \nu(z, x)\nu(x, y) \tag{33}$$

holds (compare with (5)). Here we write $\nu(x, y)$ for $\nu(x, [0, y])$. It is then easy to see that a similar situation as in Section 1 is present (let $P(z, A) = \nu(-z, -A)$ for $z \leq 0$ and $A \subseteq (x, 0]$). It follows that there exists a family $\{I_\theta | \ \theta \in \Theta\}$ of disjoint intervals and singletons with $\cup_{\theta \in \Theta} I_\theta = [0, \infty)$ such that for each θ for which I_θ is not a singleton there exists a function $s_\theta : I_\theta \to \mathbb{R}$ which is nonvanishing such that for every $x, y \in I_\theta$ with $x \leq y$ we have that

$$\nu(x, y) = \frac{s_\theta(y)}{s_\theta(x)}.$$

Note that $s_\theta(y) : I_\theta \to [0, \infty)$ is nondecreasing and is not necessarily bounded. The infinitesimal generator of the Markov process X_t is given by

$$\mathcal{A}f(x) = r(x)f'(x) + \kappa(x) \int_0^x \big(f(y) - f(x)\big)\nu(x, dy). \tag{34}$$

We assume that the domain $\mathcal{D}_\mathcal{A}$ of $\mathcal{A}$ consists of functions f that are absolutely continuous and for which the expectation of $\sum_{0<T_i\leq t} |f(X_{T_i-}) - f(X_{T_i})|$ is finite for every $t \geq 0$, where T_i denotes the ith jump time (see [3]).

The following Lemma generalizes formula (28) in [8].

Lemma 2. *Suppose that $r(x)$, $\kappa(x)$, $\lambda(x)$ and $s_\theta(x)$ are differentiable for $x \in I_\theta$. Define the functions $a(x) = r'(x) + r(x)\xi(x) - \lambda(x) - \kappa(x)$ and $b(x) = \lambda'(x) + \lambda(x)\xi(x)$, where $\xi(x) = \frac{s'_\theta(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)}$ if $\kappa(x) \neq 0$ and $\xi(x) = 0$ otherwise. Any twice differentiable solution f with $f'(x)s_\theta(x)$ being continuous of*

$$r(x)f''(x) + a(x)f'(x) - b(x)f(x) = 0, \tag{35}$$

fulfils $\mathcal{A}f(x) = \lambda(x)f(x)$.

Proof. The process X_t, if started in the state $x \in I_\theta$ will leave I_θ only at the moment when it passes through the upper boundary $z^*(\theta)$ and $\nu(x, y) = 0$ for $y < z_*(\theta)$. If $x \in I_\theta$ we may hence write

$$\mathcal{A}f(x) \quad = \quad r(x)f'(x) + \frac{\kappa(x)}{s_\theta(x)} \int_{z_*(\theta)}^x \int_x^y f'(u)\, du\, s_\theta(dy), \quad x \in I_\theta.$$

Applying Fubini's theorem we can write this as

$$\mathcal{A}f(x) \quad = \quad r(x)f'(x) - \frac{\kappa(x)}{s_\theta(x)} \int_{z_*(\theta)}^x f'(u)s_\theta(u)\, du, \quad x \in I_\theta. \tag{36}$$

Then $\mathcal{A}f(x) = \lambda(x)f(x)$ is equivalent to

$$\kappa(x)\int_{z_*(\theta)}^{x} f'(u)s_\theta(u)\,du = s_\theta(x)\Big(r(x)f'(x) - \lambda(x)f(x)\Big). \tag{37}$$

Differentiation yields

$$\begin{aligned}\frac{\kappa'(x)}{s_\theta(x)}\int_{z_*(\theta)}^{x} f'(u)s_\theta(u)\,du &= r(x)f''(x) + (r'(x) + r(x)\frac{s_\theta'(x)}{s_\theta(x)} \\ &\quad - \lambda(x) - \kappa(x))f'(x) - (\lambda'(x) + \lambda(x)\frac{s_\theta'(x)}{s_\theta(x)})f(x).\end{aligned}$$

If $\kappa(x) \neq 0$ then we divide (37) by $\kappa(x)$ and obtain (35) with $\xi(x) = \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)}$. If $\kappa(x) = 0$ then it follows from (37) that

$$r(x)f''(x) + (r'(x) - \lambda(x))f'(x) - \lambda'(x)f(x) = 0,$$

which is (35) with $\xi(x) = 0$. □

As is described earlier in the section via (27), the probability that the maximum process $\max_{0\le t\le Z} X(t)$ exceeds y, given $X(0) = x$, satisfies the generalized lack of memory property when Z is defined right before (26). More precisely,

Corollary 1. *Fix a $\theta \in \Theta$ and suppose that $f \in \mathcal{D}_\mathcal{A}$ is bounded away from zero (or is such that $M(t)$ in* (23) *is a martingale) and solves equation* (35) *in I_θ. Then*

$$P_x\Big[\max_{0\le t\le Z} X(t) > y\Big] = \frac{f(x)}{f(y)},$$

for all $x, y \in I_\theta$ with $x \le y$, where Z be a random variable, such that $\mathbb{P}_x(Z > t|\mathcal{F}_t) = e^{-\int_0^t \lambda(X(s))ds}$.

In general (35) is not easy to solve and closed form solutions may be obtained only in certain cases. We provide two examples, where the coefficients $a(x)$ and $b(x)$ are such that a solution can be given.

Example 1. Equation (35) reduces to a differential equation with contant coefficients if

$$\frac{r'(x)}{r(x)} + \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)} - \frac{\lambda(x)+\kappa(x)}{r(x)} \equiv C$$

$$\text{and} \qquad \frac{\lambda(x)}{r(x)}\Big(\frac{\lambda'(x)}{\lambda(x)} + \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)}\Big) \equiv D.$$

For example suppose that $\lambda(x) = c_1e^{\alpha x}$, $\kappa(x) = c_2e^{\alpha x}$, $r(x) = c_3e^{\alpha x}$, $s_\theta(x) = c_4e^{\beta x}$, with $c_1, c_2, c_3, c_4, \beta \ge 0$ and $\alpha \in \mathbb{R}$. Then (35) reads

$$f''(x) + \big(\beta - \frac{c_1+c_2}{c_3}\big)f'(x) - \frac{c_1\beta}{c_3}f(x) = 0,$$

which is solved by $f(x) = Ae^{a^- x} + Be^{a^+ x}$, where

$$a^\pm = \frac{1}{2}\Big(\beta - \frac{c_1 + c_2}{c_3} \pm \sqrt{\big(\beta - \frac{c_1 + c_2}{c_3}\big)^2 + 4\frac{c_1\beta}{c_3}}\Big).$$

If we set $f(z_*(\theta)) = 1$ (w.l.o.g.), then $f'(z_*(\theta)) = \lambda(z_*(\theta))/r(z_*(\theta)) = c_1/c_3$. This leads to the final solution

$$f(x) = \frac{a^+ - \frac{c_1}{c_3}}{a^+ - a^-} e^{a^-(x - z_*(\theta))} + \frac{\frac{c_1}{c_3} - a^-}{a^+ - a^-} e^{a^+(x - z_*(\theta))}.$$

Example 2. This example is a generalization of Example (A), Section 4.1 in [8]. Suppose that the jump measure $\nu(x, y) = s_\theta(y)/s_\theta(x)$ is defined such that for some $\alpha > 0$ $s_\theta(x)\lambda(x) = \alpha\kappa(x)$. Then $\xi(x) = -\lambda'(x)/\lambda(x)$ and as a consequence the second coefficient $b(x)$ is zero (while $a(x) = r'(x) - r(x)\lambda'(x)/\lambda(x) - \lambda(x) - \kappa(x)$). Hence (35) becomes

$$r(x)f''(x) + a(x)f'(x) = 0, \tag{38}$$

which is solved by

$$f(x) = f(z_*(\theta)) + f'(z_*(\theta))\frac{r(z_*(\theta))}{\lambda(z_*(\theta))}\int_{z_*(\theta)}^x \frac{\lambda(u)}{r(u)} e^{\int_{z_*(\theta)}^u \frac{\lambda(w)+\kappa(w)}{r(w)}\,dw}\,du.$$

Note that since $\mathcal{A}f(x) = \lambda(x)f(x)$ it follows that $\lambda(z_*(\theta))f(z_*(\theta)) = r(z_*(\theta))f'(z_*(\theta))$ and hence, choosing w.l.o.g. $f(z_*(\theta)) = 1$, we obtain the solution

$$f(x) = 1 + \int_{z_*(\theta)}^x \frac{\lambda(u)}{r(u)} e^{\int_{z_*(\theta)}^u \frac{\lambda(w)+\kappa(w)}{r(w)}\,dw}\,du.$$

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