EURANDOM PREPRINT SERIES
2011-046

December 18, 2011

Useful martingales for stochastic storage processes with Lévy-input and decomposition results

Offer Kella, Onno Boxma

ISSN 1389-2355

Useful martingales for stochastic storage processes with Lévy-type input and decomposition results

Offer Kella[*†] Onno Boxma[‡]

December 18, 2011

Abstract

In this paper we generalize the martingale of Kella and Whitt to the setting of Lévy-type processes and show that under some quite minimal conditions the local martingales are actually L^2 martingales which upon dividing by the time index converge to zero a.s. and in L^2. We apply these results to generalize known decomposition results for Lévy queues with secondary jump inputs and queues with server vacations or service interruptions. Special cases are polling systems with either compound Poisson or more general Lévy inputs.

Keywords: Lévy-type processes, Lévy storage systems, Kella-Whitt martingale, decomposition results, queues with server vacations

AMS 2000 Subject Classification: 60K25, 60K37, 60K30, 60H30, 90B05, 90B22

1 Introduction

Consider a process that can be either in an *on* state or an *off* state. When it is in the *on* state it behaves like some Lévy process with no negative jumps and a negative drift. When it is in an *off* state it behaves like a subordinator, that is, a nondecreasing Lévy process. It is

[*]Department of Statistics; The Hebrew University of Jerusalem; Mount Scopus, Jerusalem 91905; Israel (offer.kella@huji.ac.il)

[†]Supported in part by grant 434/09 from the Israel Science Foundation, the Vigevani Chair in Statistics and visitor grant No. 040.11.257 from The Netherlands Organisation for Scientific Research.

[‡]EURANDOM and Department of Mathematics and Computer Science; Eindhoven University of Technology; P.O. Box 513; 5600 MB Eindhoven; The Netherlands (boxma@win.tue.nl)

well known in queueing theory (e.g., [13]) that in a stable M/G/1 queue with server down times (vacations, interruptions, etc.) the steady state waiting time distribution (properly defined) is a convolution of two or more distributions, one of which is always the steady state waiting time distribution of an ordinary M/G/1 queue. As Poisson arrivals see time averages, this result also holds for the workload process. [16] studies a general model of a Lévy process with no negative jumps and additional jumps that occur at stopping epochs and the size of which is measurable with respect to the current information. This model is interesting in its own right but can also be viewed as a weak limit of queues with off times where during these off times workload can only accumulate as the server is idle. The interesting outcome of [16] was that the same (and even more general) decomposition results that were known for queues also turned out to hold for these Lévy processes with additional jumps. The question that comes to mind is whether the on/off process that we sketched in the beginning of this section (and for which we give a precise definition later) obeys a similar decomposition property. This would immediately imply a decomposition in certain polling systems as described in Section 6. It is a simple observation that if one cuts and pastes the on/off process such that only the on times are visible, then the resulting process is the one that was considered in [16]. As it seemed that the results of [16] could not be used in our setting, we found it necessary to develop a more general theory, in particular a certain martingale theory that would streamline our work and could be useful in other applications as well. We describe this direction in the next paragraph.

In [17] a certain (local) martingale associated with Lévy processes and its various applications is discussed (see also Section IX.3 of [2] and Section 4.4 of [19]). This has become a standard tool for studying various storage systems with Lévy inputs and other problems associated with Lévy process modeling. In [3] a generalization to a multidimensional (local) martingale associated with Markov additive processes with finite state space Markov modulation is considered, and in [4] a special case of the martingale of [17] for a reflected and a nonreflected Lévy process with no negative jumps and applications to certain hitting times associated with these processes. A generalization to martingales associated with more general functions (than exponential) is given in [20]. The focus is on reflected and nonreflected processes but the main results seem to hold for the more general structure considered in [17]. There are many papers which apply this and related martingales. As these particular applications are not the scope of this study, we will not attempt to list them here.

The first goal of this paper is to extend the results of [17] to the case where the driving process is a Lévy-type process. That is, it is a sum of stochastic integrals of some locally bounded left continuous right limit process with respect to coordinate processes associated with some multidimensional Lévy process. Such processes with an even more general (predictable) integrand are discussed in [1]. As a second goal, we want to learn when our local martingale is in fact an L^2 martingale and moreover when upon dividing by the time parameter t it converges to zero almost surely and in L^2 as $t \to \infty$. The third goal is to apply these martingale results to establish decomposition results for the on/off model introduced in the beginning of this section.

This article is organized as follows. In Section 2 we develop the main (local) martingale results of this paper. In Section 3 we show that under some further conditions the local martingale must actually be an L^2 martingale and moreover, that its rate (defined appropriately) is zero almost surely and in L^2. In Section 4 we apply our results to establish decomposition results for the on/off model described in the beginning, thereby considerably generalizing the results of [16]. In Section 5 we identify the non-standard component in the decomposition associated with off times. Finally in Section 6 a discussion of polling systems, the motivation for this study, is given and the contribution of our results to this area is emphasized.

2 A more general martingale

For what follows given a càdlàg (right continuous left limit) function $g : \mathbb{R}_+ \to \mathbb{R}$ we denote $g(t-) = \lim_{s \uparrow t} g(s)$, $\Delta g(t) = g(t) - g(t-)$ with the convention that $\Delta g(0) = g(0)$ and if g is VF (finite variation on finite intervals), then $g^d(t) = \sum_{0 \le s \le t} \Delta g(s)$ and $g^c(t) = g(t) - g^d(t)$. Also, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$ and *a.s.* abbreviates *almost surely*.

Let $X = (X_1, \ldots, X_K)$ be a càdlàg K-dimensional Lévy process with respect to some standard filtration $\{\mathcal{F}_t |\ t \ge 0\}$ with exponent

$$\psi(\alpha) = ic^T\alpha - \frac{\alpha^T \Sigma \alpha}{2} + \int_{\mathbb{R}^K} \left(e^{i\alpha^T x} - 1 - i\alpha^T x 1_{\{\|x\| \le 1\}} \right) \nu(dx) \quad (1)$$

where ^{T} denotes transposition, Σ is positive semidefinite and $\|x\| = \sqrt{x^T x}$. When X_k have no negative jumps ($\nu_k(-\infty, 0) = 0$), then for vector $\alpha \ge 0$ the Laplace-Stieltjes exponent is

$$\begin{aligned} \varphi(\alpha) &= \log E e^{-\alpha^T X(1)} = \psi(i\alpha) \qquad (2) \\ &= -c^T\alpha + \frac{\alpha^T \Sigma \alpha}{2} + \int_{\mathbb{R}_+^K} \left(e^{-\alpha^T x} - 1 + \alpha^T x 1_{\{\|x\| \le 1\}} \right) \nu(dx) \ . \end{aligned}$$

It is well known that in this case $\varphi(\alpha)$ is finite for each $\alpha \ge 0$, that it is convex (thus continuous) with $\varphi(0) = 0$ and infinitely differentiable in the interior of $\mathbb{R}_+$ and that for every $\alpha \ge 0$ for which $\alpha^T X$

is not a subordinator (not nondecreasing), $\varphi(t\alpha) \to \infty$ as $t \to \infty$. Furthermore, $EX_k(t) = -t\frac{\partial \varphi}{\partial \alpha_k}(0+)$ (finite or $+\infty$, but can never be $-\infty$) and when the first two right derivatives at zero are finite, then $\mathrm{Cov}(X_k(t), X_\ell(t)) = t\frac{\partial^2 \varphi}{\partial \alpha_k \partial \alpha_\ell}\varphi(0+)$.

Theorem 1 *Let $I = (I_1, \ldots, I_K)$ be a bounded K-dimensional adapted càdlàg process. Then*

$$e^{i\sum_{k=1}^K \int_{(0,t]} I_k(s-)dX_k(s) - \int_0^t \psi(I(s))ds} \tag{3}$$

is a (complex valued) martingale. When in addition X_k have no negative jumps and I_k are nonnegative then

$$e^{-\sum_{k=1}^K \int_{(0,t]} I_k(s-)dX_k(s) - \int_0^t \varphi(I(s))ds} \tag{4}$$

is a real valued martingale.

Proof: Follows, for example, by applying a multidimensional generalization of Corollary 5.2.2 and Theorem 5.2.4 on pages 253-254 of [1] to the process

$$\begin{aligned} dY(t) &= \left(\sum_{k=1}^K c_k I_k(s) - \varphi(I(s))\right) dt \\ &\quad + \sum_{k=1}^K \Big(I_k(t) dB_k(t) + I_k(t-) x \tilde{N}_k(dt, dx) \\ &\qquad + I_k(t-) x N_k(dt, dx) \Big) \end{aligned} \tag{5}$$

where Y, B_k, N_k and $\tilde{N}_k$ are the notations from [1] with the obvious additional index k. Since we will not use these notations in this paper we only mention them briefly here. Moreover, Y will soon be used for something else, in line with [17] and [3]. ■

Setting $Z(t) = \sum_{k=1}^K \int_{(0,t]} I_k(s-)dX_k(s) + Y(t)$ the exact same proof from [17] can be employed to prove the following, where $a \wedge b = \min(a, b)$. We recall here that in [17] the driving process was some one dimensional Lévy process X rather than $\sum_{k=1}^K \int_{(0,t]} I_k(s-)dX_k(s)$.

Theorem 2 *Let $X = (X_1, \ldots, X_K)$ be a Lévy process with exponent ψ and, when it has no negative jumps, Laplace-Stieltjes exponent φ. Let $I = (I_1, \ldots, I_K)$ be bounded càdlàg and adapted. Assume that Y is* *càdlàg, VF (a.s.) and adapted. Then*

$$
\begin{aligned}
M(t) &= \int_0^t \psi(I(s))e^{iZ(s)}ds + e^{iZ(0)} - e^{iZ(t)} + i\int_0^t e^{iZ(s)}dY^c(s) \\
&\quad + \sum_{0<s\le t} e^{iZ(s)}\left(1-e^{-i\Delta Y(s)}\right)
\end{aligned} \tag{6}
$$

is a local martingale and if the expected variation of Y^c is finite and

$$E\sum_{0<s\le t}|\Delta Y(s)|\wedge 1 < \infty \ ,$$

then it is a zero mean martingale.
When X_k have no negative jumps and I_k are nonnegative, then

$$
\begin{aligned}
M(t) &= \int_0^t \varphi(I(s))e^{-Z(s)}ds + e^{-Z(0)} - e^{-Z(t)} - \int_0^t e^{-Z(s)}dY^c(s) \\
&\quad + \sum_{0<s\le t} e^{-Z(s)}\left(1-e^{\Delta Y(s)}\right)
\end{aligned} \tag{7}
$$

is a local martingale and when the expected variation of Y^c is finite,

$$E\sum_{0<s\le t}|\Delta Y(s)|\wedge 1 < \infty$$

and Z is bounded below (in particular non-negative), then it is a zero mean martingale.

Remark 1 We note that in [17] it was assumed that the expected number of jumps of Y on finite intervals is finite in order for the local martingale to be a martingale. It is easy to show that with the same proof the weaker condition

$$E\sum_{0<s\le t}|\Delta Y(s)|\wedge 1 < \infty \ ,$$

is sufficient. For example, if Y is a subordinator (a nondecreasing Lévy process) then it satisfies this condition.

Remark 2 It may seem more general to consider the multidimensional process defined via $Z_\ell(t) = \sum_{k=1}^K \int_{(0,t]} I_{\ell k}(s-)dX_k(s) + Y_\ell$, but we immediately see that the one dimensional process

$$\sum_{\ell=1}^L \alpha_\ell Z_\ell(t) = \sum_{k=1}^K \int_{(0,t]} \sum_{\ell=1}^L \alpha_\ell I_{\ell k}(s-)dX_k(s) + \sum_{\ell=1}^K \alpha_\ell Y_\ell(t) \tag{8}$$

has the same structure, resulting in the following (local) martingales

$$\begin{aligned} M(t) &= \int_0^t \psi(\alpha^T I(s)) e^{i\alpha^T Z(s)} ds + e^{i\alpha^T Z(0)} - e^{i\alpha^T Z(t)} \\ &\quad + i \sum_{\ell=1}^{L} \alpha_\ell \int_0^t e^{i\alpha^T Z(s)} dY_\ell^c(s) + \sum_{0<s\le t} e^{i\alpha^T Z(s)} \left(1 - e^{-i\alpha^T \Delta Y(s)}\right) \end{aligned} \tag{9}$$

and

$$\begin{aligned} M(t) &= \int_0^t \varphi(\alpha^T I(s)) e^{-\alpha^T Z(s)} ds + e^{-\alpha^T Z(0)} - e^{-\alpha^T Z(t)} \\ &\quad - \sum_{\ell=1}^{L} \alpha_\ell \int_0^t e^{-\alpha^T Z(s)} dY_\ell^c(s) + \sum_{0<s\le t} e^{-\alpha^T Z(s)} \left(1 - e^{\alpha^T \Delta Y(s)}\right) \end{aligned} \tag{10}$$

where I is an $L \times K$-matrix valued function.

Remark 3 We note that when J is a (right continuous) continuous time Markov chain with states $1, \ldots, K$, then with $I_k(t) = 1_{\{J(t)=k\}}$ one has that $\sum_{k=1}^K \int_{(0,t]} I_k(s-) dX_k(s)$ is a Markov additive process. Adding additional jumps at state change epochs can be modeled by the process Y, which is obviously VF. For the case where Y is continuous, this kind of a process and associated martingales were considered in [3]. The one dimensional martingales considered here are not the same as the multidimensional ones considered there. However, the sum of the components of the latter does agree with the former.

Remark 4 We recall that when $X_1, \ldots, X_K$ are independent, then $\psi(\alpha) = \sum_{k=1}^K \psi_k(\alpha_k)$, where

$$\psi_k(\alpha_k) = ic_k\alpha_k - \frac{\sigma_k^2 \alpha_k^2}{2} + \int_{\mathbb{R}} \left(e^{i\alpha_k} - 1 - i\alpha_k 1_{\{|x_k| \le 1\}}\right) \nu_k(dx_k) \tag{11}$$

and when there are no negative jumps, $\varphi(\alpha) = \sum_{k=1}^K \varphi_k(\alpha_k)$, where $\varphi_k(\alpha_k) = \psi_k(i\alpha_k)$ for $\alpha_k \ge 0$.

Remark 5 We conclude this section with the following observation. Assume that J is a càdlàg adapted process taking values in some finite set $1, \ldots, K$ (not necessarily Markovian). Let $I_k(t) = \alpha_k 1_{\{J(t)=k\}}$. Then

$$\psi(I(t)) = \sum_{k=1}^K \psi_k(\alpha_k) 1_{\{J(t)=k\}} \ , \tag{12}$$

where $\psi_k(\alpha_k) = \psi(0, ...0, \alpha_k, 0, \ldots, 0)$ with α_k in the kth coordinate, is defined in the previous remark (and similarly with φ when there are no negative jumps). Thus, in this case

$$\int_0^t \psi(I(s))e^{-Z(s)}ds = \sum_{k=1}^K \psi_k(\alpha_k) \int_0^t e^{-Z(s)} 1_{\{J(s)=k\}} ds \ . \tag{13}$$

If in addition we replace Y by βY for some $\beta \geq 0$ and denote $\tilde{X}_k(t) = \int_{(0,t]} 1_{\{J(s)=k\}} dX_k(s)$ then

$$Z(t) = \alpha^T \tilde{X}(t) + \beta Y(t) \tag{14}$$

and the (local) martingale becomes

$$\begin{aligned} M(t) &= \sum_{k=1}^K \psi_k(\alpha_k) \int_0^t e^{iZ(s)} 1_{\{J(s)=k\}} ds + e^{iZ(0)} - e^{iZ(t)} \\ &\quad + i\beta \int_0^t e^{iZ(s)} dY^c(s) + \sum_{0<s\leq t} e^{iZ(s)} \left(1 - e^{-i\beta \Delta Y(s)}\right) \end{aligned} \tag{15}$$

and similarly

$$\begin{aligned} M(t) &= \sum_{k=1}^K \varphi_k(\alpha_k) \int_0^t e^{-Z(s)} 1_{\{J(s)=k\}} ds + e^{-Z(0)} - e^{-Z(t)} \\ &\quad - \beta \int_0^t e^{-Z(s)} dY^c(s) + \sum_{0<s\leq t} e^{-Z(s)} \left(1 - e^{\beta \Delta Y(s)}\right) \end{aligned} \tag{16}$$

when there are no negative jumps.

It seems that the joint structure of X is not important here. This is partly true in the sense that the evolution of the Lévy part of the process during times when J is at a given state is that of a one dimensional Lévy process. However both J and Y may also depend on the joint structure.

3 $M(t)/t \to 0$ when Y is continuous

In this section we will show that when Y is continuous then M is an L^2 martingale and moreover $M(t)/t \to 0$ almost surely and in L^2. This is interesting in its own right, is something that was overlooked in [17] and we find it extremely useful in the following Section 4 regarding decomposition results for the on/off processes which were described in the introduction.

Let us begin with the following.

Lemma 1 *Let X be a semimartingale and $f \in \mathcal{C}^2$ (twice continuously differentiable). Denote by $[\cdot,\cdot]$ the quadratic variation process associated with a semimartingale. Then $f(X)$ is also a semimartingale with the following quadratic variation:*

$$[f(X), f(X)](t) = \int_0^t \left(f'(X(s))\right)^2 d[X,X]^c(s) + \sum_{0 \le s \le t} \left(\Delta f(X(s))\right)^2 . \tag{17}$$

Proof: Although this should have been a standard result in a book (such as [21]) we did not find a direct reference. For its proof we apply the extended Itô's Lemma (Th. 32 on p. 78 of [21]) to conclude that

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_{(0,t]} f'(X(s-))dX(s) \\ &\quad + \text{continuous VF part} + \text{discrete VF part}. \end{aligned} \tag{18}$$

As in the displayed equation following the definition of $[X,X]^c$ on p. 70 of [21] we have that

$$[f(X), f(X)](t) = [f(X), f(X)]^c(t) + \sum_{0 \le s \le t} \left(\Delta f(X(s))\right)^2 . \tag{19}$$

Finally we note that the only term that can contribute to the continuous part of the quadratic variation associated with $f(X)$ is the stochastic integral part. Thus with the notation $f(X_-) \cdot X(t) = \int_{(0,t]} f(X(s-))dX(s)$ we now have via Th. 29 on p. 75 of [21] that

$$\begin{aligned} [f(X), f(X)]^c &= [f(X_-) \cdot X, f(X_-) \cdot X]^c = \left((f'(X_-))^2 \cdot [X,X] \right)^c \\ &= ((f'(X_-))^2 \cdot [X,X]^c \end{aligned} \tag{20}$$

and the proof is complete. ■

Corollary 1 *Assume that X is a semimartingale, Y is continuous, VF, adapted and $Z = X + Y$. Then*

$$[e^{-Z}, e^{-Z}](t) = \int_0^t e^{-2Z(s)} d[X,X]^c(s) + \sum_{0 \le s \le t} e^{-2Z(s-)} \left(1 - e^{-\Delta X(s)}\right)^2 . \tag{21}$$

Proof: When Y is continuous, VF and adapted then $[Z,Z] = [X,X]$ and $\Delta Z = \Delta X$. The rest is by substitution and some obvious manipulations. ■

Remark 6 Given the above, it is now an easy exercise to show that in fact for X a semimartingale and $f, g \in \mathcal{C}^2$ we have that

$$[f(X), g(X)] = \int_0^t f(X(s))g(X(s))d[X,X]^c(s) + \sum_{0 \le s \le t} \Delta f(X(s))\Delta g(X(s)) \tag{22}$$

and to conclude from this that, under the assumptions of Corollary 1,

$$[e^{iZ}, e^{iZ}](t) = \int_0^t e^{i2Z(s)}d[X,X]^c(s) + \sum_{0 \le s \le t} e^{i2Z(s-)}\left(1 - e^{i\Delta X(s)}\right)^2 \tag{23}$$

by treating the real and imaginary parts separately. We will not need this in what follows.

Remark 7 We note that when Y is not continuous but the jump points of X and Y are distinct, that is, when $\Delta X(t)\Delta Y(t) = 0$ for all $t \ge 0$ (a.s.), then one needs to add the term $\sum_{0 \le s \le t} e^{-2Z(s-)}\left(1 - e^{-\Delta Y(s)}\right)^2$ to (21) and similarly $\sum_{0 \le s \le t} e^{i2Z(s-)}\left(1 - e^{i\Delta Y(s)}\right)^2$ to (23).

Theorem 3 *With the notations and under the assumptions of Theorem 2 and with the added assumption that Y is continuous, I_k are nonnegative and that X_k have no negative jumps,*

$$[M,M](t) \quad = \quad \int_0^t e^{-2Z(s)}A(s)ds + \tilde{M}(t) \tag{24}$$

where

$$A(s) = \varphi(2I(s)) - 2\varphi(I(s)), \tag{25}$$

is nonnegative and $\tilde{M}$ is a martingale having bounded jumps.

Proof: We first observe that since Y is continuous, the only part of $M(t)$ which might have a nonzero quadratic variation is $-e^{-Z}$ and thus $[M,M] = [e^{-Z}, e^{-Z}]$. From Corollary 1 we have with $\tilde{X}(t) = \sum_{k=1}^K \int_{(0,t]} I_k(s-)dX_k(s)$ that

$$[e^{-Z}, e^{-Z}](t) = \int_0^t e^{-2Z(s)}d[\tilde{X},\tilde{X}]^c(s) + \sum_{0 \le s \le t} e^{-2Z(s-)}\left(1 - e^{-\Delta\tilde{X}(s)}\right)^2 \tag{26}$$

and from Th. 29 on p. 75 of [21] we have that

$$[\tilde{X}, \tilde{X}] = \sum_{k=1}^K \sum_{\ell=1}^K [I_k \cdot X_k, I_\ell \cdot X_\ell] = \sum_{k=1}^K \sum_{\ell=1}^K I_k I_\ell \cdot [X_k, X_\ell] \tag{27}$$

and thus also that

$$[\tilde{X}, \tilde{X}]^c = \sum_{k=1}^{K} \sum_{\ell=1}^{K} I_k I_\ell \cdot [X_k, X_\ell]^c \ . \tag{28}$$

Now since we can write $X = B + C$, where B is a Brownian motion and C is a quadratic pure jump Lévy process (e.g. see top of p. 71 of [21] for the one dimensional case), then $[X_k, X_\ell]^c(t) = [B_k, B_\ell](t) = \sigma_{k\ell} t$ which implies that

$$[\tilde{X}, \tilde{X}]^c(t) = \int_0^t I(s)^T \Sigma I(s) ds = \int_0^t \left[\frac{(2I(s)^T)\Sigma(2I(s))}{2} - 2\frac{I(s)^T \Sigma I(s)}{2} \right] ds. \tag{29}$$

Next, we observe that since X and thus its pure quadratic jump part is a Lévy process then

$$\tilde{N}(t) = \sum_{0 \le s \le t} \left(1 - e^{-I(s-)^T \Delta X(s)}\right)^2 - \int_0^t \left[\int_{(0,\infty)} \left(1 - e^{-I(s)^T x}\right)^2 \nu(dx) \right] ds \tag{30}$$

is a martingale having bounded jumps. To show this, one, e.g., first shows it for (multi-dimensional) compound Poisson processes and then takes appropriate limits. Thus also

$$\tilde{M}(t) = \int_{[0,t]} e^{-2Z(s-)} d\tilde{N}(s) \tag{31}$$

is a martingale (see Th. 51 on p. 38 and Th. 29 on p. 128 of [21]). Finally we observe that for any $a, x \in \mathbb{R}_+$

$$\begin{aligned} \left(1 - e^{-a^T x}\right)^2 &= \left(e^{-(2a)^T x} - 1 + (2a)^T x 1_{\{\|x\| \le 1\}}\right) \\ &\quad -2\left(e^{-a^T x} - 1 + a^T x 1_{\{\|x\| \le 1\}}\right) \end{aligned} \tag{32}$$

and upon replacing a by $I(s-)$ and integrating with respect to $\nu(dx)$, then together with (29) the result is obtained. ∎

Corollary 2 *Under the conditions of Theorem 3, $M(t)/t \to 0$ as $t \to \infty$ a.s. and in L^2.*

Proof: Since I is bounded and φ is continuous, then so is $\varphi(I)$. Thus there exists a constant C such that $\varphi(2I(s)) - 2\varphi(I(s)) \le C$ so that also $e^{-2Z(s)}(\varphi(2I(s)) - 2\varphi(I(s))) \le C$. Since $\int_{(0,t]} (1+s)^{-2} d\tilde{M}(s)$ is a zero mean martingale then

$$E \int_0^t (1+s)^{-2} d[M,M](s) \le C \int_0^t (1+s)^{-2} ds = C\left(1 - \frac{1}{1+t}\right) \le C. \tag{33}$$

Letting $t \to \infty$ and applying monotone convergence on the left hand side, together with Cor. 3 on p. 73 of [21], implies that $\int_0^t (1+s)^{-1} dM(s)$ is an L^2 martingale with second moment given by the left side of (33), that $\int_0^\infty (1+s)^{-1} dM(s)$ converges a.s. and thus, Ex. 14 on p. 95 of [21] implies that $M(t)/(1+t) \to 0$, hence also $M(t)/t \to 0$ a.s. Since by the same arguments $EM^2(t) = E[M, M](t) \le Ct$ then $E(M(t)/t)^2 \to 0$, thus L^2 convergence also holds. ■

Finally, we can state the following

Theorem 4 *Let $X = (X_1, \ldots, X_K)$ be a Lévy process with no negative jumps and Laplace-Stieltjes exponent φ given by (2). Let $I = (I_1, \ldots, I_K)$ be bounded and nonnegative càdlàg and adapted. Assume that Y is continuous, VF (a.s.) and adapted. Finally assume that*

$$Z(t) = \sum_{k=1}^{K} \int_{(0,t]} I_k(s-) dX_k(s) + Y(t) \tag{34}$$

is nonnegative (in particular $Y(0) = Z(0) \ge 0$). Then

$$\frac{1}{t} \int_0^t \varphi(I(s)) e^{-Z(s)} ds - \frac{1}{t} \int_0^t e^{-Z(s)} dY(s) \to 0 \tag{35}$$

a.s. and in L^2.

Remark 8 We note that we do not need to explicitly assume that Y has expected finite variation, but only that it is continuous, VF and adapted. For example

$$L(t) = -\inf_{0 \le s \le t} (Y(0) + \tilde{X}(s))^- \tag{36}$$

is such a process when there are no negative jumps for which Z is a nonnegative process. In this particular case $Z(t) = 0$ at points of increase of L and thus, as $t \to \infty$,

$$\frac{1}{t} \int_0^t \varphi(I(s)) e^{-Z(s)} ds - \frac{1}{t} L(t) \to 0 \tag{37}$$

a.s. and in L^2. Also, we recall from Theorem 1 of [18] that if $\tilde{X}(t)/t \to \xi \le 0$ then $Z(t)/t \to 0$ and thus, as $L = Z - Z(0) - X$, we have that $L(t)/t \to -\xi$ and (37) becomes

$$\frac{1}{t} \int_0^t \varphi(I(s)) e^{-Z(s)} ds \to -\xi. \tag{38}$$

A result related to Theorem 4 which will be useful in the next section is the following.

Lemma 2 *Let X be a one dimensional Lévy process with Lévy measure ν satisfying*

$$\int_{|x|>1} |x|\nu(dx) < \infty$$

(equivalently $E|X(1)| < \infty$). Then for any bounded càdlàg adapted process A,

$$\frac{\int_{(0,t]} A(s-)dX(s) - EX(1)\int_0^t A(s)ds}{t} \to 0 \tag{39}$$

a.s.

Proof: Assume that $|A(t)| \le B < \infty$. Set, for $M > 0$,

$$\begin{aligned} X_M(t) &= \sum_{0<s\le t} \Delta X(s) 1_{\{\Delta X(s)>M\}}, \\ X_{-M}(t) &= \sum_{0<s\le t} \Delta X(s) 1_{\{\Delta X(s)<-M\}}, \\ X_0(t) &= X(t) - X_M(t) - X_{-M}(t). \end{aligned} \tag{40}$$

Also, denote $\xi_i = EX_i(1)$ for $i = M, -M, 0$. Then X_M, X_{-M}, X_0 are independent Lévy processes. X_M is nondecreasing and X_{-M} is nonincreasing. Now,

$$\left| \frac{1}{t} \int_{(0,t]} A(s-)dX_M(s) \right| \le B \frac{X_M(t)}{t} , \tag{41}$$

and by the strong law of large numbers for Lévy processes we have that a.s.

$$\limsup_{t\to\infty} \left| \frac{1}{t} \int_{(0,t]} A(s-)dX_M(s) \right| \le B\xi_M = B \int_{(M,\infty)} x\nu(dx). \tag{42}$$

Clearly, we also have that

$$\left| \frac{1}{t} \int_0^t A(s)ds \right| \le B \tag{43}$$

and thus

$$\limsup_{t\to\infty} \left| \frac{\int_{(0,t]} A(s-)dX_M(s) - \xi_M \int_0^t A(s)ds}{t} \right| \le 2B \int_{(M,\infty)} x\nu(dx) . \tag{44}$$

Similarly

$$\limsup_{t\to\infty} \left| \frac{\int_{(0,t]} A(s-)dX_{-M}(s) - \xi_{-M} \int_0^t A(s)ds}{t} \right| \le 2B \int_{(-\infty,-M)} |x|\nu(dx) . \tag{45}$$

Next, we observe that the martingale $M_0(t) = X_0(t) - \xi_0 t$ is a Lévy process with bounded jumps and thus its quadratic variation is a nondecreasing Lévy process with bounded jumps which can also be compensated by a linear function to create a martingale. Thus, like in the proof of Corollary 2, this implies that

$$\frac{\int_{(0,t]} A(s-)dX_0(s) - \xi_0 \int_0^t A(s)ds}{t} \to 0 \tag{46}$$

a.s. (and also in L^2, but this is not needed here). To conclude, denoting $\xi = \xi_0 + \xi_M + \xi_{-M} = EX(1)$, we now clearly have that, a.s.,

$$\limsup_{t\to\infty} \left| \frac{\int_{(0,t]} A(s-)dX(s) - \xi \int_0^t A(s)ds}{t} \right| \leq 2B \int_{(-\infty,-M)\cup(M,\infty)} |x|\nu(dx) \tag{47}$$

and letting $M \to \infty$, recalling that $\int_{|x|>1} |x|\nu(dx) < \infty$, the proof is complete. ■

Remark 9 We note that if $EX(1) \neq 0$ then since $X(t)/t \to EX(1)$ a.s., (39) is equivalent to

$$\frac{1}{X(t)} \int_{(0,t]} A(s-)dX(s) - \frac{1}{t} \int_0^t A(s)ds \to 0, \tag{48}$$

and thus $\frac{1}{X(t)} \int_{(0,t]} A(s-)dX(s)$ converges a.s. if and only if $\frac{1}{t} \int_0^t A(s)ds$ does and the limits coincide. When X is a Poisson process, this is no less than an equivalent statement of the famous and often cited PASTA (Poisson Arrivals See Time Averages) property.

4 Application to decomposition results for Lévy storage processes

In this section we complement the results of [16] as follows. Let $0 = T_0 \leq S_1 \leq T_1 \leq S_2 \leq T_2 \ldots$ be an increasing sequence of a.s. finite stopping times with respect to the standard filtration $\{\mathcal{F}_t | \ t \geq 0\}$ satisfying $T_{n-1} < T_n$ and $T_n \to \infty$ a.s. Let $X_n = S_n - T_{n-1}$ and $Y_n = T_n - S_n$. The model here is that $(T_{n-1}, S_n]$ with lengths X_n are *down* times, where there is no output (the "server" is not working) and therefore the buffer content can only accumulate. $(S_n, T_n]$ with length Y_n are *up* times where there is both input and output, which is modeled as usual by a reflected (Skorohod map of the) process.

Remark 10 We note that in some models it is possible that there is no reflection. For example, whenever the server is shut off as soon as the system empties, which may be modeled via the stopping times.

Remark 11 Throughout this and the following section we will focus on almost sure convergence for the sake of convenience. However, throughout, most "almost sure" statements could be trivially replaced by "in probability" without changing anything else (simply by looking at subsequences that converge a.s.). We are not aware of related applications where the convergence is in probability but not almost surely and thus did not see a point in making this issue more precise.

Remark 12 In [16] the focus is on convergence in distribution rather than long run a.s. convergence. As in the previous remark, we could follow the same ideas with similar proofs (but with more restrictive assumptions). We chose to leave this out as, given what follows, and what is already available in [16], it may be considered an exercise.

Let X_u be a one-dimensional càdlàg Lévy process with no negative jumps which is not a subordinator (not nondecreasing), and with Laplace-Stieltjes exponent

$$\varphi(\alpha) = -c_u\alpha + \frac{\sigma_u^2\alpha^2}{2} + \int_{(0,\infty)} \left(e^{-\alpha x} - 1 + \alpha x 1_{\{x\le 1\}}\right)\nu_u(dx) \quad (49)$$

with $EX_u(1) = -\varphi'(0) < 0$ (necessarily well defined and finite). This models the net input process (input minus potential output) during up times. Let X_d be a one-dimensional right continuous subordinator (nondecreasing Lévy process) with Laplace-Stieltjes exponent $-\eta$ where

$$\eta(\alpha) = c_d\alpha + \int_{(0,\infty]} \left(1 - e^{-\alpha x}\right)\nu_d(dx) \quad (50)$$

with $EX_d(1) = \eta'(0) < \infty$. The latter models the process according to which work accumulates during down times.

Now, set $N(t) = \sup\{n|\ T_n \le t\}$ and let $J(t) = 1_{\{S_{N(t)+1}>t\}}$ and thus $J(t) = 1_{\{J(t)=1\}}$ and $1 - J(t) = 1_{\{J(t)=0\}}$. Therefore, $J(t) = 1$ during down times and $J(t) = 0$ during up times. Finally, for $W(0) \in \mathcal{F}_0$ let

$$\begin{aligned}
\tilde{X}_d(t) &= \int_{(0,t]} J(s-)dX_d(s) \\
\tilde{X}_u(t) &= \int_{(0,t]} (1 - J(s-))dX_u(s) \\
\tilde{X}(t) &= \tilde{X}_u(t) + \tilde{X}_d(t) \\
L(t) &= -\inf_{0\le s\le t}(W(0) + \tilde{X}(s))^- \\
W(t) &= W(0) + \tilde{X}(t) + L(t)
\end{aligned} \quad (51)$$

where $a^- = \min(a, 0)$. As in Remark 8, since there are no negative jumps, L is continuous. It is also monotone (thus VF) and satisfies that $W(t)=0$ whenever $L(t)<L(s)$ for every $s>t$ (e.g. [14]). It is also not difficult to check that in fact $EL(t)<\infty$ for every $t\geq 0$ (e.g. [17]). Remarks 5 and 8 and the fact that (since X_d is nondecreasing) $\int_0^t J(s)dL(s)=0$ imply that the following is a zero mean martingale:

$$-\eta(\alpha)\int_0^t e^{-\alpha W(s)}J(s)ds+\varphi(\alpha)\int_0^t e^{-\alpha W(s)}(1-J(s))ds$$
$$+e^{-\alpha W(0)}-e^{-\alpha W(t)}-\alpha L(t). \tag{52}$$

Dividing by $\varphi(\alpha)$ and collecting terms, the following is also a martingale:

$$\begin{aligned} M(t) &= \int_0^t e^{-\alpha W(s)}ds-\left(1+\frac{\eta(\alpha)}{\varphi(\alpha)}\right)\int_0^t e^{-\alpha W(s)}J(s)ds \\ &\quad +\frac{e^{-\alpha W(0)}-e^{-\alpha W(t)}}{\varphi(\alpha)}-\frac{\alpha}{\varphi(\alpha)}L(t). \end{aligned}\tag{53}$$

By Theorem 4, $M(t)/t\to 0$ a.s. and in L^2. From Lemma 2, if $\frac{1}{t}\int_0^t J(s)ds\to p_d$ a.s. then a.s.

$$\begin{aligned} \lim_{t\to\infty}\frac{\tilde{X}(t)}{t} &= \lim_{t\to\infty}\frac{EX_d(1)\int_0^t J(s)ds+EX_u(1)\int_0^t(1-J(s))ds}{t} \\ &= \eta'(0)p_d-\varphi'(0)(1-p_d) \end{aligned}\tag{54}$$

and if in addition $\eta'(0)p_d-\varphi'(0)(1-p_d)\leq 0$ then by Remark 8 we automatically have that a.s.

$$\frac{L(t)}{t}\to -\eta'(0)p_d+\varphi'(0)(1-p_d) \tag{55}$$

as $t\to\infty$.

We note that when $\eta'(0)p_d-\varphi'(0)(1-p_d)>0$ then $\tilde{X}(t)\to\infty$ a.s. and thus $L(t)$ is a.s. bounded. Hence, in this case $L(t)/t\to 0$ and $W(t)/t\to\eta'(0)p_d-\varphi'(0)(1-p_d)$ (thus $W(t)\to\infty$) and there cannot be any ("reasonably" defined) form of a steady state distribution for W.

We also note that if $W(t)/t\to 0$ and $L(t)/t\to\ell$ (necessarily $\ell\geq 0$), then $\tilde{X}(t)/t\to -\ell\leq 0$ and by Lemma 2 we also have that

$$\frac{\eta'(0)\int_0^t J(s)ds-\varphi'(0)\int_0^t(1-J(s))ds}{t}=(\eta'(0)+\varphi'(0))\frac{1}{t}\int_0^t J(s)ds-\varphi'(0) \tag{56}$$

converges to $-\ell$, so that necessarily $\frac{1}{t}\int_0^t J(s)ds\to p_d=\frac{\varphi'(0)-\ell}{\varphi'(0)+\eta'(0)}$. Thus we have

Lemma 3 *$L(t)/t \to \ell$ and $W(t)/t \to 0$ a.s. as $t \to \infty$ if and only if $\frac{1}{t}\int_0^t J(s)ds \to p_d \le \frac{\varphi'(0)}{\eta'(0)+\varphi'(0)}$, where equivalently*

$$\ell = -\eta'(0)p_d + \varphi'(0)(1-p_d) \tag{57}$$

and

$$p_d = \frac{\varphi'(0)-\ell}{\varphi'(0)+\eta'(0)} \ . \tag{58}$$

When this holds then necessarily $\ell \le \varphi'(0)$.

Next, observe that for each $t \ge 0$ (and each ω in the sample space) for which $\int_0^t J(s)ds > 0$ we have that

$$\frac{\int_0^t e^{-\alpha W(s)}J(s)ds}{\int_0^t J(s)ds} \tag{59}$$

is the Laplace-Stieltjes transform of an a.s. nonnegative and finite random variable and thus if this ratio converges to some constant $g(\alpha)$ for each α then g must be a Laplace-Stieltjes transform of some nonnegative (not necessarily a.s. finite) random variable. If in addition $g(\alpha) \to 1$ as $\alpha \downarrow 0$ then necessarily g is the Laplace-Stieltjes transform of a proper distribution on $\mathbb{R}_+$.

From these observations, the following is now evident.

Theorem 5 *If a.s. as $t \to \infty$,*

$$\frac{1}{t}\int_0^t e^{-\alpha W(s)}ds \to Ee^{-\alpha W(\infty)} \tag{60}$$

(ergodic convergence) for some finite random variable $W(\infty)$ and

$$\frac{1}{t}\int_0^t J(s)ds \to p_d \le \frac{\varphi'(0)}{\eta'(0)+\varphi'(0)} \tag{61}$$

(equivalently $W(t)/t \to 0$ and $L(t)/t \to \ell$ where necessarily $\ell \le \varphi'(0)$), then there exists a nonnegative random variable W_d such that if $p_d > 0$ then a.s.

$$\frac{\int_0^t e^{-\alpha W(s)}J(s)ds}{\int_0^t J(s)ds} \to Ee^{-\alpha W_d} \tag{62}$$

for every $\alpha \ge 0$. Moreover, with

$$\pi_\ell = \frac{\ell}{\varphi'(0)} = 1 - \left(1 + \frac{\eta'(0)}{\varphi'(0)}\right)p_d \tag{63}$$

and

$$\pi = \frac{\eta'(0)}{\eta'(0)+\varphi'(0)} \tag{64}$$

we have that

$$Ee^{-\alpha W(\infty)} = \pi_\ell \frac{\varphi'(0)\alpha}{\varphi(\alpha)} + (1-\pi_\ell)\left(1-\pi+\pi\frac{\eta(\alpha)}{\eta'(0)\alpha}\frac{\varphi'(0)\alpha}{\varphi(\alpha)}\right)Ee^{-\alpha W_d}. \tag{65}$$

Let us now interpret (65). First we note that, since $\varphi'(0) > 0$, $\frac{\alpha\varphi'(0)}{\varphi(\alpha)}$ is the Laplace-Stieltjes transform of the stationary, limit and ergodic distribution associated with the process $Z_u(t) = X_u(t) + L_u(t)$ where $L_u(t) = -\inf_{0\le s\le t} X_u(s)$, as well as the Laplace-Stieltjes transform of the random variable $\sup_{s\ge 0} X(s)$. This is well known and there are quite a few proofs of this generalized Pollaczek-Khinchin formula in the literature, one of which is in [17].

Next we observe that from [15], $\frac{\eta(\alpha)}{\alpha\eta'(0)}$ is the Laplace-Stieltjes transform of the stationary excess lifetime distribution associated with the jumps of the subordinator X_d. For ease of reference simply observe that from

$$\eta(\alpha) - c_d\alpha = \int_{(0,\infty)} (1-e^{-\alpha x})\nu_d(dx) = \alpha\int_0^\infty e^{-\alpha x}\nu(x,\infty)dx \tag{66}$$

and $\eta'(0) = c_d + \bar{\nu}_d$, where $\int_{(0,\infty)} x\nu_d(dx) = \int_0^\infty \nu(x,\infty)dx$, we have that

$$\frac{\eta(\alpha)}{\alpha\eta'(0)} = \frac{c_d}{c_d+\bar{\nu}_d} + \frac{\bar{\nu}_d}{c_d+\bar{\nu}_d}\int_0^\infty e^{-\alpha x}\frac{\nu(x,\infty)}{\bar{\nu}_d}dx \tag{67}$$

which is the Laplace-Stieltjes transform of the following distribution function:

$$F_e(y) = \frac{c_d}{c_d+\bar{\nu}_d} + \frac{\bar{\nu}_d}{c_d+\bar{\nu}_d}\int_0^y \frac{\nu(x,\infty)}{\bar{\nu}_d}dx \tag{68}$$

for $y \ge 0$ and $F_e(y) = 0$ for $y < 0$. This is a somewhat generalized stationary excess lifetime distribution associated with the jumps of X_u.

Now assume that W_u, Y_e, I_l, I, W_d are independent random variables where $Ee^{-\alpha W_u} = \frac{\alpha\varphi'(0}{\varphi(\alpha)}$, $Y_e \sim F_e$, $P(I_\ell = 1) = 1 - P(I_\ell = 0) = \pi_\ell$, $P(I = 1) = 1 - P(I = 0) = \pi$ and W_d as well as $W(\infty)$ are as in Theorem 5, then

Corollary 3 *Under the conditions of Theorem 5*

$$W(\infty) \sim I_\ell W_u + (1-I_\ell)(I(W_u+Y_e) + W_d)\ . \tag{69}$$

One important special case of this model is when during on times, whenever there is a positive content, the input has the same law as during down times and the output is at a fixed rate $r > 0$. That is, $\varphi(\alpha) = \alpha r - \eta(\alpha)$. A special case of this model was studied in [15]. In this particular case it is easy to check that (as in equation (4.12) of [15])

$$1-\pi+\pi\frac{\eta(\alpha)}{\eta'(0)\alpha}\frac{\varphi'(0)\alpha}{\varphi(\alpha)} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}, \tag{70}$$

that is, that $I(W_u + Y_e) \sim W_u$. So in this case we have the following.

Corollary 4 *When $\varphi(\alpha) = \alpha r - \eta(\alpha)$ then*

$$W(\infty) \sim W_u + (1 - I_\ell) W_d \tag{71}$$

and when in addition $\ell = 0$ (equivalently $\tilde{X}(t)/t \to 0$ or $p_d = 1 - \pi = \frac{\varphi'(0)}{\eta'(0)+\varphi'(0)} = 1 - \frac{\eta'(0)}{r}$), then

$$W(\infty) \sim W_u + W_d \ . \tag{72}$$

We note that in Corollary 4 the term $\pi = \frac{\eta'(0)}{r}$ may be referred to as the *traffic intensity* and is consistent with queueing theory.

5 What about W_d?

We recall that under the assumptions of Theorem 5,

$$Ee^{-\alpha W_d} = \lim_{t\to\infty} \frac{\int_0^t e^{-\alpha W(s)} J(s) ds}{\int_0^t J(s) ds} \tag{73}$$

and since for every nonnegative random variable U we have that $e^{-\alpha U} = \alpha \int_0^\infty e^{-\alpha x} 1_{\{U \le x\}} dx$, then also here

$$\frac{\int_0^t e^{-\alpha W(s)} J(s) ds}{\int_0^t J(s) ds} = \alpha \int_0^\infty e^{-\alpha x} \frac{\int_0^t 1_{\{W(s)\le x\}} J(s) ds}{\int_0^t J(s) ds} dx \tag{74}$$

and thus, a.s., $\frac{\int_0^t 1_{\{W(s)\in\cdot\}} J(s) ds}{\int_0^t J(s) ds}$ (probability distribution valued process) converges in distribution to W_d. This holds in particular if we replace t by S_n. In this case $\int_0^{S_n} J(s) ds = \sum_{k=1}^n X_k$ and thus we have that

$$\frac{\int_0^{S_n} 1_{\{W(s)\in\cdot\}} J(s) ds}{\int_0^{S_n} J(s) ds} = \frac{\sum_{k=1}^n \int_0^{X_k} 1_{\{W(T_{k-1}+s)\in\cdot\}} ds}{\sum_{k=1}^n X_k} \tag{75}$$

where for $s \in [0, X_n)$ we have that

$$W(T_{n-1} + s) = W(T_{n-1}) + X_d(T_{n-1} + s) - X_d(T_{n-1}) \tag{76}$$

and thus

$$\int_0^{X_n} e^{-\alpha W(T_{n-1}+s)} ds = e^{-\alpha W(T_{n-1})} \int_0^{X_n} e^{-\alpha (X_d(T_{n-1}+s) - X_d(T_{n-1}))} ds \ . \tag{77}$$

Now, since T_{n-1}, S_n are stopping times with respect to $\{\mathcal{F}_t |\ t \ge 0\}$, X_n is a stopping time with respect to $\left\{\mathcal{F}_{T_{n-1}+t} |\ t \ge 0\right\}$ (of course, not with respect to the original filtration in general). Moreover, $W(T_{n-1}) \in \mathcal{F}_{T_{n-1}}$ and by the strong Markov property $X_d^{T_{n-1}} \equiv \{X_d(T_{n-1}+t) - X_d(T_{n-1})|\ t \ge 0\}$ is a subordinator with respect to $\left\{\mathcal{F}_{T_{n-1}+t}|\ t \ge 0\right\}$ with exponent η (that is, distributed like X_d) and is independent of $F_{T_{n-1}}$ (thus, of $W(T_{n-1})$). Thus from [17] we have that

$$-\eta(\alpha)\int_0^t e^{-\alpha X_d^{T_{n-1}}(s)}ds + 1 - e^{-\alpha X_d^{T_{n-1}}(t)} \tag{78}$$

is a zero mean martingale with respect to $\left\{\mathcal{F}_{T_{n-1}+t}|\ t \ge 0\right\}$ and thus by the optional stopping theorem together with monotone and bounded convergence where appropriate we have with

$$\Delta_n = -\eta(\alpha)\int_0^{X_n} e^{-\alpha X_d^{T_{n-1}}(s)}ds + 1 - e^{-\alpha X_d^{T_{n-1}}(X_n)}, \tag{79}$$

that $E[\Delta_n|\mathcal{F}_{T_{n-1}}] = 0$. Moreover, from Theorem 3, and the fact that $M(t)^2 - [M,M](t)$ is a (zero mean) martingale, we can conclude that when X_n is a.s. finite then

$$E[\Delta_n^2|\mathcal{F}_{T_{n-1}}] = (2\eta(\alpha) - \eta(2\alpha))E\left[\int_0^{X_n} e^{-2\alpha X_d^{T_{n-1}}(s)}ds \middle| \mathcal{F}_{T_{n-1}}\right] \tag{80}$$

and in the same way that led to $E[\Delta_n|\mathcal{F}_{T_{n-1}}] = 0$, by substituting 2α instead of α, we have that

$$\eta(2\alpha)E\left[\int_0^{X_n} e^{-2\alpha X_d^{T_{n-1}}(s)}ds \middle| \mathcal{F}_{T_{n-1}}\right] = 1 - E\left[e^{-2\alpha X_d^{T_{n-1}}(X_n)} \middle| \mathcal{F}_{T_{n-1}}\right] \tag{81}$$

and we conclude that

$$E[\Delta_n^2|\mathcal{F}_{T_{n-1}}] = \left(\frac{2\eta(\alpha)}{\eta(2\alpha)} - 1\right)\left(1 - E\left[e^{-2\alpha X_d^{T_{n-1}}(X_n)} \middle| \mathcal{F}_{T_{n-1}}\right]\right) . \tag{82}$$

In particular, upon multiplying by $e^{-\alpha W(T_{n-1})} \in \mathcal{F}_{T_{n-1}}$, we have that

$$\sum_{k=1}^n e^{-\alpha W(T_{k-1})}\Delta_k \tag{83}$$

is a zero mean martingale, where

$$E\left[\left(e^{-\alpha W(T_{k-1})}\Delta_k\right)^2 \middle| \mathcal{F}_{T_{k-1}}\right] \le \frac{2\eta(\alpha)}{\eta(2\alpha)} - 1 < \infty . \tag{84}$$

It is well known (cf. Theorem 3 on p. 243 of [12]) that an L^2 martingale M_n satisfying

$$\sum_{k=1}^\infty \frac{E(M_k - M_{k-1})^2}{k^2} < \infty \tag{85}$$

also satisfies $M_n/n \to 0$ a.s. and in L^2 and thus

$$\frac{1}{n}\sum_{k=1}^{n} e^{-\alpha W(T_{k-1})}\Delta_k \to 0 \tag{86}$$

a.s. and in L^2 and we finally have the following.

Theorem 6 *Under the assumptions of Theorem 5,*

$$\frac{1}{n}\left(-\eta(\alpha)\int_0^{S_n} e^{-\alpha W(s)}J(s)ds + \sum_{k=1}^{n}\left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)}\right)\right) \to 0 \tag{87}$$

a.s. and in L^2 and if in addition $p_d > 0$ then

$$\frac{-\eta(\alpha)\int_0^{S_n} e^{-\alpha W(s)}J(s)ds + \sum_{k=1}^{n}\left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)}\right)}{\int_0^{S_n} J(s)ds} \to 0 \tag{88}$$

and thus

$$\frac{\sum_{k=1}^{n}\left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)}\right)}{\sum_{k=1}^{n} X_k} \to \eta(\alpha)Ee^{-\alpha W_d}. \tag{89}$$

Now, note that from $\frac{1}{t}\int_0^t J(s)ds \to \pi_d > 0$, if also $T_n/n \to \mu > 0$ a.s. (and thus also $S_n/n \to \mu$) then

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \frac{S_n}{n}\frac{1}{S_n}\int_0^{S_n} 1_{\{J(s)\}}ds \to \mu p_d > 0 \tag{90}$$

and thus

$$\frac{1}{n}\sum_{k=1}^{n}\left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)}\right) \tag{91}$$

converges a.s. In particular, we have that

Theorem 7 *Under the assumptions of Theorem 5, if $p_d > 0$ and $T_n/n \to \mu > 0$ a.s., then $\frac{1}{n}\sum_{k=1}^{n} e^{-\alpha W(S_k)} \to Ee^{-\alpha W^+}$ a.s. for some nonnegative random variable W^+ if and only if $\frac{1}{n}\sum_{k=1}^{n} e^{-\alpha W(T_{k-1})} \to Ee^{-\alpha W^-}$ a.s. for some random variable W^- and we have that*

$$\frac{Ee^{-\alpha W^-} - Ee^{-\alpha W^+}}{\alpha\eta'(0)\mu p_d} = \frac{\eta(\alpha)}{\alpha\eta'(0)}Ee^{-\alpha W_d}\ . \tag{92}$$

Moreover if any two of EW_d, EW^-, EW^+ are finite, then so is the third and we have that

$$\frac{Ee^{-\alpha W^-} - Ee^{-\alpha W^+}}{\alpha(EW^+ - EW^-)} = \frac{\eta(\alpha)}{\alpha\eta'(0)}Ee^{-\alpha W_d}\ . \tag{93}$$

For more details regarding the left side of (93), see Theorems 5.1 and 5.2. of [16]. In particular, it is a Laplace-Stieltjes transform of a bona fide distribution if and only if W^- is stochastically smaller than W^+. This form was also observed and discussed in the M/G/1 queue setting in [22]. Finally, if there are enough assumptions to assure that W^- and $U = W^+ - W^-$ are independent then the left side of (93) becomes

$$Ee^{-\alpha W^-} \frac{1 - Ee^{-\alpha U}}{\alpha EU} . \tag{94}$$

That is, it is the transform of a sum of two independent random variables, the first is W^- and the second has the stationary residual lifetime distribution of U. If we denote this variable by U_e then we have the following decomposition

$$W_- + U_e \sim W_d + Y_e , \tag{95}$$

where we recall that Y_e has the transform $\frac{\eta(\alpha)}{\alpha\eta'(0)}$ and the variables on either side are assumed independent. The special case where this kind of independence (between W^- and U) occurs is discussed in the M/G/1 queue setting in [13]. We also refer the reader to Theorem 4.1 and its proof in [15] for the special case considered there.

We recall that by Corollary 3,

$$W(\infty) \sim I_\ell W_u + (1 - I_\ell)(I(W_u + Y_e) + W_d) . \tag{96}$$

Thus, replacing Y_e by an independent $Y_e^1 \sim Y_e$ and adding Y_e on both sides we have that

$$W(\infty) + Y_e \sim I_\ell(W_u + Y_e) + (1 - I_\ell)(I(W_u + Y_e^1) + (W_d + Y_e)) . \tag{97}$$

With $W_\pm \sim W_d + Y_e$ (a random variable with LST given by the left side of (93)) this implies that

$$W(\infty) + Y_e \sim I_\ell(W_u + Y_e) + (1 - I_\ell)(I(W_u + Y_e^1) + W_\pm) , \tag{98}$$

where the expressions either side of the equation are independent. Finally, replacing Y_e^1 on the right by Y_e does not change the distribution (due to the indicator I_ℓ) so that

$$W(\infty) + Y_e \sim I_\ell(W_u + Y_e) + (1 - I_\ell)(I(W_u + Y_e) + W_\pm) , \tag{99}$$

where again all variables appearing on the expressions on either side of the equation are assumed independent so that only their marginal distribution matters. In the special case of $\varphi(\alpha) = \alpha r - \eta(\alpha)$ we replace $I(W_u + Y_e)$ on the right by W_u (see Corollary 4) and obtain

$$\begin{aligned} W(\infty) + Y_e &\sim I_\ell(W_u + Y_e) + (1 - I_\ell)(W_u + W_\pm) \\ &= W_u + I_\ell Y_e + (1 - I_\ell)W_\pm , \end{aligned} \tag{100}$$

and in particular when $\ell = 0$

$$W(\infty) + Y_e \sim W_u + W_\pm \ , \tag{101}$$

where, again, throughout all random variables appearing in the expressions on either sides of the equations are assumed independent.

6 Applications to polling systems

In this section we relate our decomposition results to decomposition results for so-called polling systems. A polling system is a single-server multi-queue system, in which the server visits the queues one at a time, typically in a cyclic order. The service discipline at each queue specifies the duration of a visit. E.g., under the exhaustive service discipline, the server visits a queue until it has become empty; under the 1-limited discipline, it serves exactly one customer during a visit. In many applications (e.g., in production systems, where the server is a machine and the customers of a queue are orders of a particular type) it is natural to have nonnegligible switchover times from one queue to the next. Stimulated by a wide variety of applications (not only production systems, but also computer- and communication systems, traffic lights, repair systems), polling models have been extensively studied. It is almost always assumed that the input processes to the queues are independent Poisson processes. For such a situation, it was proven in [5] that the steady-state total workload in the polling system *with* switchover times can be decomposed into two independent quantities, viz. (i) the workload in the corresponding polling system *without* switchover times, and (ii) the steady-state total amount of work at an epoch the server is not working. Item (i) is the workload in an $M/G/1$ queueing system; the distribution of item (ii) was determined for a few service disciplines in [10]. In [9] the joint steady-state workload distribution at arbitrary epochs was expressed in the joint queue length distribution at visit beginning and visit completion epochs. The latter distributions are known for certain polling models, in particular, for polling models in which the service discipline at all queues is of so-called branching type.

The cyclic polling model of [5] was generalized in [7] to the case of a fixed non-cyclic visit order of the queues; again a work decomposition result was derived. A further generalization is contained in [6]. That paper considers a single-server multi-class system with a work-conserving scheduling discipline as long as the server *is* serving and with a service interruption process (which could correspond to switchover times in a polling system) that does not affect the amount of service time given to a customer or the arrival time of any customer. Furthermore, the arrival process is a batch Poisson process that allows correlations between the numbers of arrivals of the various customer types in a batch. Again a decomposition result was proven: the steady-state workload in the model with interruptions is in distribution equal to the sum of two independent quantities, viz. (i) the steady-state workload in the corresponding model without interruptions, and (ii) the steady-state amount of work at an epoch in which the server is not serving.

Another extension of the cyclic polling model with independent Poisson arrivals was recently studied in [8]. It considers a cyclic polling system with N queues, extending the Poisson arrival process to an N-dimensional Lévy subordinator (so the sample paths are non-decreasing in all coordinates). If a particular queue is being served, then the workload level at that queue behaves as a spectrally positive Lévy process with a negative drift. Another special feature of the model is that the Lévy input process changes at polling and switching instants. A restrictive assumption is that the service discipline at each queue is of branching type. That assumption implies that the N-dimensional workload process at successive instants that the server arrives at the first queue is a *Jirina* process, which is a multi-type continuous-state branching process. The joint steady-state workload distribution at such epochs, and subsequently also at arbitrary epochs, is determined in [8]; no workload decomposition is derived. A special case (constant fluid input at all queues) had been studied by Czerniak and Yechiali [11], who also obtained the joint workload distribution at arbitrary epochs. In Section 4 of their paper they point out that, if there is a workload decomposition, the term (i) without switchover times is zero because the outflow is larger than the inflow during visit times.

In Section 4 of the present paper, we derive workload transforms and workload decompositions in a system that alternates between up and down times. The input process is one Lévy process X_u during up times and another Lévy process X_d during down times. Our Theorem 5 generalizes exact workload transform results in [9] and [10], where the input process is a sum of independent compound Poisson processes, to the case of a Lévy input process. It complements the exact workload transform result of [8] in the sense that it only gives total workload and does not give a joint transform, but that it does allow more general visit disciplines. Our assumption on the *up* and *down* times (visit times and interruptions), viz., the assumption that $0 = T_0 \leq S_1 \leq T_1 \leq S_2 \leq T_2 \dots$ is an increasing sequence of a.s. finite stopping times, in particular includes non-branching service disciplines. Our Corollary 4 generalizes/complements decomposition results for total workload in [5, 6] for polling systems and, more generally, single-server multi-class systems with interruptions. Our Lévy input process generalizes the (batch) Poisson processes of those and other polling papers.

If fact, due to our general setup it seems that under appropriate stability conditions, decomposition results would hold for quite general polling mechanisms. Some examples are cases were the lengths of the switching times depend on the state of the system in various ways (e.g., shorter switching when certain queues are large), or when the decision of when to leave a certain queue may depend on the overall information of the system rather than following a fixed mechanism.

References

[1] Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus.* Cambridge University Press.

[2] Asmussen, S. (2003). *Applied Probability and Queues*, 2nd Ed., Springer.

[3] Asmussen, S. and O. Kella. (2000). A multi-dimensional martingale for Markov additive processes and its applications. *Adv. Appl. Probab.* **32**, 376-393.

[4] Asmussen S. and O. Kella. (2001). On optional stopping of some exponential martingales for Lévy processes with or without reflection. *Stoch. Proc. Appl.* **91**, 47-55.

[5] Boxma, O.J. and W.P. Groenendijk (1987). Pseudo-conservation laws in cyclic-service systems. *J. Appl. Probab.* **24**, 949-964.

[6] Boxma, O.J. (1989). Workloads and waiting times in single-server systems with multiple customer classes. *Queueing Systems* **5**, 185-214.

[7] Boxma, O.J., Groenendijk, W.P. and J.A. Weststrate (1990). A pseudoconservation law for service systems with a polling table. *IEEE Trans. Comm.* **38**, 1865-1870.

[8] Boxma, O.J., Ivanovs, J., Kosiński, K.M. and M.R.H. Mandjes (2011). Lévy-driven polling systems and continuous-state branching processes. *Stochastic Systems* **1**, 411-436.

[9] Boxma, O.J., Kella, O. and K.M. Kosiński (2011). Queue lengths and workloads in polling systems. *Oper. Res. Letters* **39**, 401-405.

[10] Boxma, O.J., Takagi, H. and T. Takine (1992). Distribution of the workload in multiclass queueing systems with server vacations. *Naval Research Logistics* **39**, 41-52.

[11] Czerniak, O. and U. Yechiali (2009). Fluid polling systems. *Queueing Systems* **63**, 401-435.

[12] Feller, W. (1971). *An Introduction to Probability Theory and its Applications, Vol. II.* Wiley.

[13] Fuhrmann, S. W. and R. B. Cooper. (1985). Stochastic decomposition in the M/G/1 queue with generalized vacations. *Oper. Res.* **33**, 1117-1129.

[14] Kella, O. (2006). Reflecting thoughts. *Stat. Probab. Letters* **76**, 1808-1811.

[15] Kella, O. (1998). An exhaustive Lévy storage process with intermittent output. *Stoch. Models* **14**, 979-992.

[16] Kella, O. and W. Whitt. (1991). Queues with server vacations and Lévy processes with secondary jump input. *Ann. Appl. Probab.* **1**, 104-117.

[17] Kella, O. and W. Whitt. (1992). Useful martingales for stochastic storage processes with Lévy input. *J. Appl. Probab.* **29**, 396-403.

[18] Kella, O. and W. Whitt. (1996). Stability and structural properties of stochastic fluid networks. *J. Appl. Probab.* **33**, 1169-1180.

[19] Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer.

[20] Nguyen-Ngoc, L. and M. Yor. (2005). Some martingales associated to reflected Lévy processes. *Séminaire de Probabilités* **XXXVIII**, 42-69.

[21] Protter, P. E. (2004). *Stochastic Integration and Differential Equations.* 2nd Edition. Springer.

[22] Shanthikumar, J. G. (1988). On stochastic decomposition in M/G/1 type queues with generalized server vacations. *Oper. Res.* **36**, 566-569.