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Abstract

The partition function of the random energy model at inverse temperature β is defined by $\mathcal{Z}_N(\beta) = \sum_{k=1}^N \exp(\beta\sqrt{n}X_k)$, where $X_1, X_2, \dots$ are independent real standard normal random variables, and $n = \log N$. We identify the asymptotic structure of complex zeros of $\mathcal{Z}_N$, as $N \to \infty$, confirming predictions made in the theoretical physics literature. Also, we describe the limiting complex fluctuations for a model generalizing $\mathcal{Z}_N(\beta)$.

1. Introduction and statement of results

1.1. **Introduction.** Let $X, X_1, X_2, \dots$ be independent real standard normal random variables. The partition function of Derrida's random energy model (REM) [6] at inverse temperature β is defined by

$$\mathcal{Z}_N(\beta) = \sum_{k=1}^N \mathrm{e}^{\beta\sqrt{n}X_k}. \tag{1.1}$$

Here, N is a large integer, and we use the notation $n = \log N$. Under the assumption $\beta > 0$, the limiting fluctuations of (1.1), as $N \to \infty$ (or equivalently, as $n \to \infty$), have been extensively studied in the literature; see [4, 1]. In particular, it was shown in these works that the asymptotic behavior of the fluctuations of (1.1) depends strongly on the parameter β and displays phase transitions. Specifically, upon suitable rescaling, for $\beta < 1/\sqrt{2}$, the fluctuations of (1.1) become Gaussian, whereas, for $\beta > 1/\sqrt{2}$, they become stable non-Gaussian, as $N \to \infty$.

Using heuristic arguments, Derrida [7] studied the REM at *complex* inverse temperature $\beta = \sigma + i\tau$. The motivation here is to identify the mechanisms causing

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phase transitions in the asymptotic behavior of (1.1). These transitions manifest themselves in the analyticity breaking of the logarithm of the partition function (1.1) in the limit $N \to \infty$. In this respect, complex inverse temperatures allow for a cleaner identification of analyticity breaking. Using non-rigorous methods, Derrida [7] derived the following logarithmic asymptotics of (1.1):

$$\lim_{N\to\infty} \frac{1}{n} \log |\mathcal{Z}_N(\beta)| = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3, \end{cases} \tag{1.2}$$

where B_1, B_2, B_3 are three subsets of the complex plane defined by

$$B_1 = \mathbb{C} \backslash \overline{B_2 \cup B_3}, \tag{1.3}$$

$$B_2 = \{\beta \in \mathbb{R}^2 : 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\}, \tag{1.4}$$

$$B_3 = \{\beta \in \mathbb{R}^2 : 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}. \tag{1.5}$$

Here, $\overline{A}$ denotes the closure of the set A. It has long been realized that analyticity breaking of the log-partition function (1.2) is closely related to the *complex zeros* of $\mathcal{Z}_N$. Based on (1.2) and on the numerical simulations of Moukarzel and Parga [16], Derrida made predictions concerning the distribution of zeros of $\mathcal{Z}_N$ on the complex plane. To cite Derrida [7], "there should be no zeros (or at least the density of zeros vanishes) in phases B_1 and B_2" and "the density of zeros is uniform in phase B_3". For the REM in an exterior magnetic field, similar non-rigorous analysis has been done by Moukarzel and Parga [17, 18]. For directed polymers with complex weights on a tree, which is another related model, the logarithmic asymptotics (1.2) has been derived in [8]; see also [2].

Our contribution is two-fold. First, we confirm the predictions of [7, 16] rigorously. Moreover, we derive further results on the asymptotic distribution of complex zeros of $\mathcal{Z}_N$, as $N \to \infty$. We relate the zeros of $\mathcal{Z}_N$ to the zeros of two random analytic functions: $\mathbb{G}$, a Gaussian analytic function, and ζ_P, a zeta-function associated to the Poisson process. Second, we identify the limiting fluctuations for a complex-valued generalization of $\mathcal{Z}_N$. Both questions are, in fact, closely related to each other.

The paper is organized as follows. In Sections 1.2 and 1.3, we state our results on zeros and fluctuations, respectively. Proofs can be found in Sections 2 and 3. In Section 1.4, we discuss possible extensions and open problems related to our results.

Notation. We will write the complex inverse temperature β in the form $\beta = \sigma + i\tau$, where $\sigma, \tau \in \mathbb{R}$. We use the notation $n = \log N$, where N is a large integer and the logarithm is natural. Note that in the physics literature on the REM, it is customary to take the logarithm at basis 2. Replacing β by $\beta/\sqrt{\log 2}$ in our results we can easily switch to the physics notation. We denote by $N_{\mathbb{R}}(0, s^2)$ the real Gaussian distribution with variance $s^2 > 0$ and by $N_{\mathbb{C}}(0, s^2)$ the complex Gaussian distribution with density $1/(\pi s^2)\mathrm{e}^{-|z/s|^2}$ w.r.t. the Lebesgue measure on $\mathbb{C}$. Note that $Z \sim N_{\mathbb{C}}(0, s^2)$ iff $Z = X + iY$, where $X, Y \sim N_{\mathbb{R}}(0, s^2/2)$ are independent. The distribution is referred to as standard if $s = 1$.

1.2. **Results on zeros.** Our first result describes the global structure of complex zeros of $\mathcal{Z}_N$, as $N \to \infty$. Confirming the predictions of [7], we show that the zeros of $\mathcal{Z}_N$ are distributed approximately uniformly with density n/π in the set B_3, whereas outside B_3 the density of zeros is of order $o(n)$. Let λ be the Lebesgue measure on $\mathbb{C}$.

Theorem 1.1. *For every continuous function $f : \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\frac{\pi}{n} \sum_{\beta \in \mathbb{C} : \mathcal{Z}_N(\beta)=0} f(\beta) \underset{N\to\infty}{\overset{P}{\longrightarrow}} \int_{B_3} f(\beta)\lambda(\mathrm{d}\beta).$$

Remark 1.2. As a consequence, the random measure assigning weight π/n to each zero of $\mathcal{Z}_N$ converges weakly to the Lebesgue measure restricted to B_3.

In the next theorem, we look more closely at the zeros of $\mathcal{Z}_N$ located in B_3. We describe the local structure of zeros of $\mathcal{Z}_N$ in a neighborhood of area $1/n$ of a fixed point $\beta_0 \in B_3$. Let $\{\mathbb{G}(t), t \in \mathbb{C}\}$ be a random analytic function given by

$$\mathbb{G}(t) = \sum_{k=0}^{\infty} N_k \frac{t^k}{\sqrt{k!}}, \tag{1.6}$$

where $N_1, N_2, \ldots$ are independent standard complex Gaussian random variables. The complex zeros of $\mathbb{G}$ form a remarkable point process which has intensity $1/\pi$ and is translation invariant. Up to rescaling, this is the only translation invariant zero set of a Gaussian analytic function; see [11, Section 2.5]. This and related zero sets have been much studied; see the monograph [11].

Theorem 1.3. *Let $\beta_0 \in B_3$ be fixed. For every continuous function $f : \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\sum_{\beta \in \mathbb{C} : \mathcal{Z}_N(\beta)=0} f(\sqrt{n}(\beta - \beta_0)) \underset{N\to\infty}{\overset{d}{\longrightarrow}} \sum_{\beta \in \mathbb{C} : \mathbb{G}(\beta)=0} f(\beta).$$

Remark 1.4. Equivalently, the point process consisting of the points $\sqrt{n}(\beta - \beta_0)$, where β is a zero of $\mathcal{Z}_N$, converges weakly to the point process of zeros of $\mathbb{G}$.

Derrida [7] predicted that the set B_1 should be free of zeros. The next result makes this statement rigorous.

Theorem 1.5. *Let K be a compact subset of B_1. Then, there exists $\varepsilon > 0$ depending on K such that*

$$\mathbb{P}[\mathcal{Z}_N(\beta) = 0, \text{ for some } \beta \in K] = O(N^{-\varepsilon}), \quad N \to \infty.$$

Consider now the zeros of $\mathcal{Z}_N$ in the set B_2. We will show that in the limit as $N \to \infty$ the zeros of $\mathcal{Z}_N$ in B_2 look like the zeros of certain random analytic function ζ_P. This function may be viewed as a zeta-function associated to the Poisson process. It is defined as follows. Let $P_1 < P_2 < \ldots$ be the arrival times of a unit intensity homogeneous Poisson process on the positive half-line. That is, $P_k = \varepsilon_1 + \ldots + \varepsilon_k$, where $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. standard exponential random variables, i.e., $\mathbb{P}[\varepsilon_k > t] = \mathrm{e}^{-t}$, $t \geq 0$. For $T > 1$, define the random process

$$\tilde{\zeta}_P(\beta; T) = \sum_{k=1}^{\infty} \frac{1}{P_k^{\beta}} \mathbb{1}_{P_k \in [0,T]} - \int_1^T t^{-\beta} \mathrm{d}t, \quad \beta \in \mathbb{C}. \tag{1.7}$$

Theorem 1.6. *With probability 1, the sequence $\tilde{\zeta}_P(\beta; T)$ converges as $T \to \infty$ to a limit function denoted by $\tilde{\zeta}_P(\beta)$. The convergence is uniform on compact subsets of the half-plane $\{\beta \in \mathbb{C} : \operatorname{Re} \beta > 1/2\}$.*

Corollary 1.7. *With probability* 1*, the Poisson process zeta-function*

$$\zeta_P(\beta) = \sum_{k=1}^{\infty} \frac{1}{P_k^{\beta}} \tag{1.8}$$

defined originally for $\operatorname{Re}\beta > 1$*, admits a meromorphic continuation to the domain* $\operatorname{Re}\beta > 1/2$*. The function* $\tilde{\zeta}_P(\beta) = \zeta_P(\beta) - \frac{1}{\beta-1}$ *is a.s. analytic in this domain.*

The next theorem describes the limiting structure of zeros of $\mathcal{Z}_N$ in B_2.

Theorem 1.8. *Let* $f : B_2 \to \mathbb{R}$ *be a continuous function with compact support. Let* $\zeta_P^{(1)}$ *and* $\zeta_P^{(2)}$ *be two independent copies of* ζ_P*. Then,*

$$\sum_{\beta \in B_2 : \mathcal{Z}_N(\beta)=0} f(\beta) \overset{d}{\underset{N\to\infty}{\longrightarrow}} \sum_{\substack{\beta \in B_2: \\ \zeta_P^{(1)}(\beta/\sqrt{2})=0}} f(\beta) + \sum_{\substack{\beta \in B_2: \\ \zeta_P^{(2)}(\beta/\sqrt{2})=0}} f(-\beta).$$

Remark 1.9. The theorem tells that the zeros of $\mathcal{Z}_N$ in the domain $\sigma > 1/\sqrt{2}, |\sigma| + |\tau| > \sqrt{2}$ (which constitutes one half of B_2) have approximately the same law as the zeros of ζ_P, as $N \to \infty$. Let us stress that the approximation breaks down in the triangle $\sigma > 1/\sqrt{2}, |\sigma|+|\tau| < \sqrt{2}$. Although the function ζ_P is well-defined and may have zeros there, the function $\mathcal{Z}_N$ has, with high probability, no zeros in any compact subset of the triangle by Theorem 1.5.

Next we state some properties of the function ζ_P. Let $\beta > 1/2$ be real. For $\beta \neq 1$, the random variable $\zeta_P(\beta)$ is stable with index $1/\beta$ and skewness parameter 1. In fact, (1.7) is just the series representation of this random variable; see [22, Theorem 1.4.5]. For $\beta = 1$, the random variable $\tilde{\zeta}_P(1)$ (which is the residuum of ζ_P at 1) is 1-stable with skewness 1. For general complex β, we have the following stability property.

Proposition 1.10. *If* $\zeta_P^{(1)}, \ldots, \zeta_P^{(k)}$ *are independent copies of* ζ_P*, then we have the following distributional equality of stochastic processes:*

$$\zeta_P^{(1)} + \ldots + \zeta_P^{(k)} \overset{d}{=} k^{\beta} \zeta_P.$$

To see this, observe that the union of k independent unit intensity Poisson processes has the same law as a single unit intensity Poisson process scaled by the factor $1/k$. As a corollary, the distribution of the random vector $(\operatorname{Re}\zeta_P(\beta), \operatorname{Im}\zeta_P(\beta))$ belongs to the family of operator stable laws; see [15].

Proposition 1.11. *Fix* $\tau \in \mathbb{R}$*. As* $\sigma \downarrow 1/2$*, we have*

$$\sqrt{2\sigma - 1}\, \zeta_P(\sigma + i\tau) \overset{d}{\longrightarrow} \begin{cases} N_{\mathbb{C}}(0,1), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0,1), & \text{if } \tau = 0. \end{cases}$$

As a corollary, there is a.s. no meromorphic continuation of ζ_P beyond the line $\sigma = 1/2$. Using the same method of proof it can be shown that for every different $\tau_1, \tau_2 > 0$ the random variables $\sqrt{2\sigma-1}\, \zeta_P(\sigma + i\tau_j)$, $j = 1, 2$, become asymptotically independent as $\sigma \downarrow 1/2$. Thus, the function ζ_P looks like a naïve white noise near the line $\sigma = 1/2$.

1.3. **Results on fluctuations.** We state our results on fluctuations for a generalization of (1.1) which we call complex random energy model. This model involves complex phases and allows for a dependence between the energies and the phases. Let $(X, Y), (X_1, Y_1), \dots$ be i.i.d. zero-mean bivariate Gaussian random vectors with

$$\operatorname{Var} X_k = 1, \quad \operatorname{Var} Y_k = 1, \quad \operatorname{Corr}(X_k, Y_k) = \rho.$$

Here, $-1 \le \rho \le 1$ is fixed. Recall that $n = \log N$. We consider the following partition function:

$$\mathcal{Z}_N(\beta) = \sum_{k=1}^{N} \mathrm{e}^{\sqrt{n}(\sigma X_k + i\tau Y_k)}, \qquad \beta = (\sigma, \tau) \in \mathbb{R}^2. \tag{1.9}$$

For $\tau = 0$, this is the REM of Derrida [6] at real inverse temperature σ. For $\rho = 1$, we obtain the REM at the complex inverse temperature $\beta = \sigma + i\tau$ considered above. For $\rho = 0$, the model is a REM with independent complex phases considered in [8].

Define the log-partition function as

$$p_N(\beta) = \frac{1}{n} \log |\mathcal{Z}_N(\beta)|, \qquad \beta = (\sigma, \tau) \in \mathbb{R}^2. \tag{1.10}$$

Theorem 1.12. *The limit*

$$p(\beta) := \lim_{N \to \infty} p_N(\beta) \tag{1.11}$$

exists in probability and in L^q, $q \ge 1$, and is explicitly given as

$$p(\beta) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3. \end{cases} \tag{1.12}$$

The next theorem shows that $\mathcal{Z}_N(\beta)$ satisfies the central limit theorem in the domain $\sigma^2 < 1/2$.

Theorem 1.13. *If $\sigma^2 < 1/2$ and $\tau \neq 0$, then*

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2 - \tau^2) + i\sigma\tau\rho}}{N^{\frac{1}{2}+\sigma^2}} \xrightarrow[N \to \infty]{d} N_{\mathbb{C}}(0, 1). \tag{1.13}$$

Remark 1.14. If $\sigma^2 < 1/2$ and $\tau = 0$, then the limiting distribution is real normal, as was shown in [4].

Remark 1.15. If in addition to $\sigma^2 < 1/2$ we have $\sigma^2 + \tau^2 > 1$, then $N^{1+\frac{1}{2}(\sigma^2-\tau^2)} = o(N^{\frac{1}{2}+\sigma^2})$ and, hence, the theorem simplifies to

$$\frac{\mathcal{Z}_N(\beta)}{N^{\frac{1}{2}+\sigma^2}} \xrightarrow[N \to \infty]{d} N_{\mathbb{C}}(0, 1). \tag{1.14}$$

Eq. (1.14) explains the difference between phases B_1 and B_3: in phase B_1 the expectation of $\mathcal{Z}_N(\beta)$ is of larger order than the mean square deviation, in phase B_3 otherwise.

In the boundary case $\sigma^2 = 1/2$, the limiting distribution is normal, but it has truncated variance.

Theorem 1.16. *If $\sigma^2 = 1/2$ and $\tau \neq 0$, then*

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\frac{1}{2} - \tau^2) + i\sigma\tau\rho}}{N} \xrightarrow[N \to \infty]{d} N_{\mathbb{C}}(0, 1/2).$$

Next, we describe the fluctuations of $\mathcal{Z}_N(\beta)$ in the domain $\sigma^2 > 1/2$. Since $\mathcal{Z}_N(\beta)$ has the same law as $\mathcal{Z}_N(-\beta)$, it is not a restriction of generality to assume that $\sigma > 0$. Let b_N be a sequence such that $\sqrt{2\pi}b_N e^{b_N^2/2} \sim N$ as $N \to \infty$. We can take

$$b_N = \sqrt{2n} - \frac{\log(4\pi n)}{2\sqrt{2n}}. \tag{1.15}$$

Theorem 1.17. *Let $\sigma > 1/\sqrt{2}$, $\tau \neq 0$, and $|\rho| < 1$. Then,*

$$\frac{\mathcal{Z}_N(\beta) - N\mathbb{E}[e^{\sqrt{n}(\sigma X + i\tau Y)} \mathbb{1}_{X<b_N}]}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{d} S_{\sqrt{2}/\sigma}, \tag{1.16}$$

where S_α denotes a complex isotropic α-stable random variable with a characteristic function of the form $\mathbb{E}[e^{i\,\mathrm{Re}(S_\alpha \bar{z})}] = e^{-\mathrm{const}\cdot|z|^\alpha}$, $z \in \mathbb{C}$.

Remark 1.18. If $\sigma > 1/\sqrt{2}$ and $\tau = 0$, then the limiting distribution is real totally skewed α-stable; see [4]. If $\sigma > 1/\sqrt{2}$ and $\rho = 1$ (resp., $\rho = -1$), then it follows from Theorem 3.7 below that

$$\frac{\mathcal{Z}_N(\beta) - N\mathbb{E}[e^{\beta\sqrt{n}X} \mathbb{1}_{X<b_N}]}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{d} \tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}\right) \quad \left(\text{resp., } \tilde{\zeta}_P\left(\frac{\bar{\beta}}{\sqrt{2}}\right)\right). \tag{1.17}$$

Remark 1.19. We will compute asymptotically the truncated expectation on the left-hand side of (1.16) in Section 2.2 below. We will obtain that under the assumptions of Theorem 1.17,

$$\frac{\mathcal{Z}_N(\beta)}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{d} S_{\sqrt{2}/\sigma}, \qquad \text{if } \sigma + |\tau| > \sqrt{2}, \tag{1.18}$$

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau\rho}}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{d} S_{\sqrt{2}/\sigma}, \qquad \text{if } \sigma + |\tau| \leq \sqrt{2}. \tag{1.19}$$

Similarly, if $\sigma > 1/\sqrt{2}$, but $\rho = 1$, then we have

$$\frac{\mathcal{Z}_N(\beta)}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{d} \zeta_P\left(\frac{\beta}{\sqrt{2}}\right), \qquad \text{if } \sigma + |\tau| > \sqrt{2}, \tag{1.20}$$

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau}}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{d} \tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}\right), \qquad \text{if } \sigma + |\tau| \leq \sqrt{2}. \tag{1.21}$$

For $\rho = -1$, we have to replace β by $\bar{\beta}$.

1.4. **Discussion, extensions and open questions.** The results on fluctuations are closely related, at least on the heuristic level, to the results on the zeros of $\mathcal{Z}_N$. In Section 1.3, we claimed that regardless of the value of $\beta \neq 0$ we can find normalizing constants $m_N(\beta) \in \mathbb{C}$, $v_N(\beta) > 0$ such that $(\mathcal{Z}_N(\beta) - m_N(\beta))/v_N(\beta)$ converges in distribution to a non-degenerate random variable $Z(\beta)$. It turns out that in phase B_1 the sequence $m_N(\beta)$ is of larger order than $v_N(\beta)$, which suggests that there should be no zeros in this phase. In phases B_2 and B_3, the sequence $v_N(\beta)$ dominates $m_N(\beta)$, which suggests that there should be zeros in these phases. The main difference between the phases B_2 and B_3 is the density of zeros. The density of zeros is essentially determined by the correlation structure of the process $\mathcal{Z}_N$. In phase B_3, it can be seen from Theorem 3.5 below that $\mathcal{Z}_N(\beta_1)$ and $\mathcal{Z}_N(\beta_2)$ become asymptotically decorrelated if the distance between β_1 and β_2 is of order larger than $1/\sqrt{n}$. This suggests that the distances between the close zeros in phase B_3 should be of order $1/\sqrt{n}$ and hence, the density of zeros should be of order n. Similarly, in phase B_2 the variables $\mathcal{Z}_N(\beta_1)$ and $\mathcal{Z}_N(\beta_2)$ remain non-trivially correlated at distances of order 1 by Theorem 3.7 below, which suggests that the density of zeros in this phase should be of order 1. All these heuristics are rigorously confirmed by our results.

Our results suggest the following approximate picture of zeros of $\mathcal{Z}_N$ for large N. Generate independently three objects: the zero set of the Gaussian analytic function $\mathbb{G}$ scaled down by the factor $1/\sqrt{n}$, and two copies of the zero set of the Poisson process zeta-function ζ_P. Use the zeros of $\mathbb{G}$ to fill the phase B_3 and the two copies of the zero set of ζ_P to fill the phase B_2. Leave the phase B_1 empty. Given this description, it is natural to ask about the behavior of zeros on the boundaries between the phases. Derrida [7] stated that "the boundaries between phases B_1 and B_2, and between phases B_1 and B_3 are lines of zeros whereas the separation between phases B_2 and B_3 is not". We were unable to find a satisfactory interpretation of these statements. On the rigorous side, it is possible to prove the following results. Take some point β on the boundary between B_1 and B_3. This means that $\sigma^2 < 1/2$ and $\sigma^2 + \tau^2 = 1$. In an infinitesimal neighborhood of β with linear size of order $1/\sqrt{n}$, the boundary (which is a circular arc) looks like a straight line dividing the plane into two half-planes. It can be shown that in one of the half-planes (located in B_3) the zeros of $\mathcal{Z}_N$ converge to the zeros of the Gaussian analytic function $\mathbb{G}$, whereas in the other half-plane (located in B_1) the zeros converge to the empty point process. Our results suggest that the probability that there is a zero in the ε-neighborhood of the boundary between B_1 and B_2 (which consists of 4 straight line segments) converges to 0 as $N \to \infty$ and then $\varepsilon \downarrow 0$. If β is on the boundary between B_3 and B_2 meaning that $\sigma^2 = 1/2$ and $\tau^2 > 1/2$, then it can be shown by combining the proofs of Theorems 1.3 and 1.16 that the zeros of $\mathcal{Z}_N$ in an infinitesimal neighborhood of β converge to the zeros of $\mathbb{G}$. The behavior of zeros is thus the same as inside B_2.

The intensity of complex zeros of the function ζ_P at β can be computed by the formula $g(\beta) = \frac{1}{2\pi}\mathbb{E}\Delta \log|\zeta_P(\beta)|$, where Δ is the Laplace operator; see [11, Section 2.4]. Proposition 1.11 suggests that $g(\sigma + i\tau) \sim \frac{1}{\pi}\frac{1}{(2\sigma-1)^2}$ as $\sigma \downarrow 1/2$. In particular, every point of the line $\sigma = 1/2$ should be an accumulation point for the zeros of ζ_P with probability 1.

It is possible to extend or strengthen our results in several directions. The statements of Theorem 1.1 and Theorem 1.12 should hold almost surely, although it seems difficult to prove this. It seems also that Theorem 1.5 can be strengthened as follows: the probability that $\mathcal{Z}_N$ has zeros in B_1 goes to 0 as $N \to \infty$. Several authors considered models involving sums of random exponentials generalizing the REM; see [1, 3, 13, 5]. They analyze the case of real β only. We believe that our results (both on zeros and on fluctuations) hold after appropriate modifications for these models. In particular, the assumption of Gaussianity can be relaxed. It is plausible that, for our results to hold, it suffices to require that e^X is in the Gumbel max-domain of attraction, and that a minor technical assumption as in [13, Corollary 5.1] holds.

2. Proofs of the results on fluctuations

2.1. **Truncated exponential moments.** We will often need estimates for the truncated exponential moments of the normal distribution. In the next lemmas, we denote by X a real standard normal random variable and by Φ the distribution function of X. It is well-known that

$$\Phi(t) \sim \frac{1}{\sqrt{2\pi}|t|} e^{-\frac{t^2}{2}}, \quad t \to -\infty \tag{2.1}$$

Lemma 2.1. *Let $w, a \in \mathbb{R}$. The following estimates hold.*

(1) *If $w > a$, then $\mathbb{E}[e^{wX} \mathbb{1}_{X<a}] < e^{aw - \frac{a^2}{2}}$.*
(2) *If $w < a$, then $\mathbb{E}[e^{wX} \mathbb{1}_{X>a}] < e^{aw - \frac{a^2}{2}}$.*

Proof. Consider only the case $w > a$, case (2) being similar. We have
(2.2)

$$\mathbb{E}[e^{wX} \mathbb{1}_{X<a}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{wz - \frac{z^2}{2}} dz = \frac{e^{\frac{w^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{(z-w)^2}{2}} dz = e^{\frac{w^2}{2}} \Phi(a-w).$$

Using the inequality $\Phi(t) < e^{-t^2/2}$ valid for $t \leq 0$, we obtain the statement. □

Lemma 2.2. *Let $F(\lambda) = \mathbb{E}[e^{\lambda w X} \mathbb{1}_{X<\lambda a(\lambda)}]$, where $w = u + iv \in \mathbb{C}$ and $a(\lambda)$ is a real-valued function with $\lim_{\lambda\to+\infty} a(\lambda) = a$. The following holds, as $\lambda \to +\infty$:*

$$F(\lambda) \sim \begin{cases} \frac{1}{\sqrt{2\pi}(w-a)\lambda} \exp\left\{\lambda^2 \left(a(\lambda)w - \frac{1}{2}a^2(\lambda)\right)\right\}, & \textit{if } a < u + v \\ \exp\left\{\frac{1}{2}\lambda^2 w^2\right\}, & \textit{if } a > u + v. \end{cases} \tag{2.3}$$

Remark 2.3. If $w \in \mathbb{R}$ and $a(\lambda) = w + \frac{c}{\lambda} + o(\frac{1}{\lambda})$ for some $c \in \mathbb{R}$, then

$$F(\lambda) \sim \Phi(c) \exp\left\{\frac{1}{2}\lambda^2 w^2\right\}. \tag{2.4}$$

Proof. Let first $w \in \mathbb{R}$. By the identity (2.2), we have $F(\lambda) = e^{\frac{\lambda^2 w^2}{2}} \Phi(\lambda a(\lambda) - \lambda w)$. The lemma and the remark follow readily. In the case $a < w$, we apply the formula (2.1).

Therefore, we may restrict ourselves to the case $w \in \mathbb{C} \backslash \mathbb{R}$. With $S(z) = wz - z^2/2$ write

$$F(\lambda) = \frac{\lambda}{\sqrt{2\pi}} \int_{-\infty}^{a(\lambda)} e^{\lambda^2 S(z)} dz.$$

If $a(\lambda) \leq u$, then the maximum of $\operatorname{Re} S(z)$ on the interval $z \leq a(\lambda)$ is attained at the boundary point $z = a(\lambda)$. A standard application of the stationary phase method, see [9, Chapter III, Theorem 1.1], yields the first formula of (2.3). Assume that $a(\lambda) > u$. Then, the maximum is attained at the interior point $z = u$. However, we have

$$F(\lambda) - e^{\lambda^2 w^2/2} = -\frac{\lambda}{\sqrt{2\pi}} \int_{a(\lambda)}^{\infty} e^{\lambda^2 S(z)} dz \sim \frac{1}{\sqrt{2\pi}(w-a)\lambda} e^{\lambda^2 (a(\lambda)w - \frac{1}{2}a^2(\lambda))}, \tag{2.5}$$

where the last step is by the same stationary phase argument. If $a < u + v$, the right-hand side dominates the term $e^{\lambda^2 w^2/2}$ on the left-hand side and we have the first case in (2.3). If $a > u + v$, then the term $e^{\lambda^2 w^2/2}$ becomes dominating and we arrive at the second formula in (2.3). □

Lemma 2.4. *If (X, Y) is a real Gaussian vector with standard margins and correlation ρ, then, for $s, a \in \mathbb{R}$, it holds that*

$$\mathbb{E}[e^{s(\sigma X + i\tau Y)} \mathbb{1}_{X<a}] = e^{-s^2\tau^2(1-\rho^2)/2} \mathbb{E}[e^{s(\sigma + i\tau\rho)X} \mathbb{1}_{X<a}].$$

In particular, $\mathbb{E}[e^{s(\sigma X + i\tau Y)}] = e^{s^2(\sigma^2 - \tau^2 + 2i\sigma\tau\rho)/2}$.

Proof. We have a distributional equality $(X,Y) = (X, \rho X + \sqrt{1-\rho^2} W)$, where (X,W) are independent standard normal real random variables. It follows that

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{s(\sigma X + i\tau Y)} \mathbb{1}_{X<a}] &= \mathbb{E}[\mathrm{e}^{s(\sigma + i\tau\rho)X + is\tau\sqrt{1-\rho^2}W} \mathbb{1}_{X<a}] \\ &= \mathrm{e}^{-s^2\tau^2(1-\rho^2)/2} \mathbb{E}[\mathrm{e}^{s(\sigma+i\tau\rho)X} \mathbb{1}_{X<a}],\end{aligned}$$

where we have used that $\mathbb{E}[\mathrm{e}^{tW}] = \mathrm{e}^{t^2/2}$ and that W and X are independent. □

2.2. **Proof of Theorems 1.13, 1.16, 1.17.** The main tool to prove the results on the fluctuations is the summation theory of triangular arrays of random vectors; see [10] and [15]. The following theorem can be found in [10, § 25] in the one-dimensional setting and in [21], [15, Theorem 3.2.2] in the d-dimensional setting. Denote by $|\cdot|$ the Euclidean norm and by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product.

Theorem 2.5. *For every $N \in \mathbb{N}$, let a series $W_{1,N}, \dots, W_{N,N}$ of independent random d-dimensional vectors be given. Assume that, for some locally finite measure ν on $\mathbb{R}^d \backslash \{0\}$, and some positive semidefinite matrix Σ, the following conditions hold:*

1. $\lim_{N\to\infty} \sum_{k=1}^N \mathbb{P}[W_{k,N} \in B] = \nu(B)$, *for every Borel set $B \subset \mathbb{R}^d \backslash \{0\}$ such that $\nu(\partial B) = 0$.*
2. *The following limits exist:*

$$\Sigma = \lim_{\varepsilon \downarrow 0} \limsup_{N\to\infty} \sum_{k=1}^N \mathrm{Var}[W_{k,N} \mathbb{1}_{|W_{k,N}|<\varepsilon}] = \lim_{\varepsilon \downarrow 0} \liminf_{N\to\infty} \sum_{k=1}^N \mathrm{Var}[W_{k,N} \mathbb{1}_{|W_{k,N}|<\varepsilon}].$$

Then, the random vector $S_N := \sum_{k=1}^N (W_{k,N} - \mathbb{E}[W_{k,N} \mathbb{1}_{|W_{k,N}|<R}])$ converges, as $N \to \infty$, to an infinitely divisible random vector S whose characteristic function is given by the Lévy–Khintchine representation

$$\log \mathbb{E}[\mathrm{e}^{i\langle t, S\rangle}] = -\frac{1}{2}\langle t, \Sigma t\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle t,s\rangle} - 1 - i\langle t, s\rangle \mathbb{1}_{|s|<R}) \nu(\mathrm{d}s), \qquad t \in \mathbb{R}^d.$$

Here, $R > 0$ is any number such that ν does not charge the set $\{s \in \mathbb{R}^d : |s| = R\}$.

Proof of Theorem 1.13. For $k = 1, \dots, N$, define

$$W_{k,N} = N^{-\frac{1}{2}-\sigma^2} \mathrm{e}^{\sqrt{n}(\sigma X_k + i\tau Y_k)}.$$

Let W_N be a random variable having the same law as the $W_{k,N}$'s. Note that $N\mathbb{E}[W_N] = N^{(1-\sigma^2-\tau^2+2i\sigma\tau\rho)/2}$ by Lemma 2.4. To prove the theorem, we need to show that

$$\sum_{k=1}^N (W_{k,N} - \mathbb{E}W_{k,N}) \xrightarrow[N\to\infty]{d} N_{\mathbb{C}}(0,1).$$

The proof is based on the two-dimensional Lindeberg central limit theorem. We consider $W_{k,N}$ as an $\mathbb{R}^2$-valued random vector $(\operatorname{Re} W_{k,N}, \operatorname{Im} W_{k,N})$. Let Σ_N be the covariance matrix of this vector. First, we show that

$$\lim_{N\to\infty} N\Sigma_N = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{2.6}$$

We have

$$N\mathbb{E}[(\operatorname{Re} W_N)^2 + (\operatorname{Im} W_N)^2] = N\mathbb{E}[|W_N|^2] = N^{-2\sigma^2} \mathbb{E}[\mathrm{e}^{2\sigma\sqrt{n}X}] = 1. \tag{2.7}$$

Also, we have $N\mathbb{E}[W_N^2] = N^{-2\tau^2+4i\sigma\tau\rho}$ by Lemma 2.4. Since we assume that $\tau \neq 0$, this implies that $\lim_{N\to\infty} N\mathbb{E}[W_N^2] = 0$. By taking real and imaginary parts, we obtain that

$$\lim_{N\to\infty} N\mathbb{E}[(\operatorname{Re} W_N)^2 - (\operatorname{Im} W_N)^2] = \lim_{N\to\infty} N\mathbb{E}[(\operatorname{Re} W_N)(\operatorname{Im} W_N)] = 0. \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$\lim_{N\to\infty} N\mathbb{E}[(\operatorname{Re} W_N)^2] = \lim_{N\to\infty} N\mathbb{E}[(\operatorname{Im} W_N)^2] = 1/2. \tag{2.9}$$

Also, by Lemma 2.4, we have

$$\lim_{N\to\infty} \sqrt{N}\mathbb{E}[W_N] = \lim_{N\to\infty} N^{-(\sigma^2+\tau^2-2i\sigma\tau\rho)/2} = 0. \tag{2.10}$$

It follows from (2.8), (2.9), (2.10) that (2.6) holds. Fix an arbitrary $\varepsilon > 0$. We complete the proof of the theorem by verifying the Lindeberg condition

$$\lim_{N\to\infty} N\mathbb{E}[|W_N - \mathbb{E}W_N|^2 \mathbb{1}_{|W_N - \mathbb{E}W_N|>\varepsilon}] = 0. \tag{2.11}$$

Assume first that $\sigma \neq 0$, say $\sigma > 0$. Write $a_N = \sigma + \frac{1}{2\sigma} + \frac{\log\varepsilon}{\sigma n}$. Then, $\lim_{N\to\infty} a_N = \sigma + \frac{1}{2\sigma} > 2\sigma$ by the assumption $\sigma^2 < 1/2$. Hence, by Part 2 of Lemma 2.1, we have

$$\lim_{N\to\infty} N\mathbb{E}[|W_N|^2 \mathbb{1}_{|W_N|>\varepsilon}] = \lim_{N\to\infty} \mathrm{e}^{-2\sigma^2 n}\mathbb{E}[\mathrm{e}^{2\sigma\sqrt{n}X}\mathbb{1}_{X>\sqrt{n}a_N}] = 0. \tag{2.12}$$

This also trivially holds for $\sigma = 0$. Together with (2.10), (2.12) implies (2.11). $\square$

Proof of Theorem 1.16. Without loss of generality let $\sigma = 1/\sqrt{2}$. For $k = 1, \ldots, N$, define

$$W_{k,N} = N^{-1}\mathrm{e}^{\sqrt{n}(\sigma X_k + i\tau Y_k)}.$$

Let W_N be a random variable with the same distribution as the $W_{k,N}$'s. To prove the theorem, we need to verify that

$$\sum_{k=1}^{N}(W_{k,N} - \mathbb{E}W_{k,N}) \xrightarrow[N\to\infty]{d} N_{\mathbb{C}}(0, 1/2).$$

As we will see in Eqn. (2.13) below, the Lindeberg condition (2.11) is not satisfied. We are going to apply Theorem 2.5 instead. Fix $\varepsilon > 0$ and let $a_N = \sqrt{2} + \frac{\sqrt{2}\log\varepsilon}{n}$. By Remark 2.3, we have

$$\lim_{N\to\infty} N\mathbb{E}[|W_N|^2 \mathbb{1}_{|W_N|<\varepsilon}] = \lim_{N\to\infty} N^{-1}\mathbb{E}[\mathrm{e}^{\sqrt{2n}X}\mathbb{1}_{X<\sqrt{n}a_N}] = 1/2. \tag{2.13}$$

Also, by Lemma 2.4 and Lemma 2.2 (first part of (2.3)),

$$\lim_{N\to\infty} N\mathbb{E}[W_N^2 \mathbb{1}_{|W_N|<\varepsilon}] = \lim_{N\to\infty} \mathrm{e}^{-2n(1-\rho^2)\tau^2} N^{-1}\mathbb{E}[\mathrm{e}^{2\sqrt{n}(\sigma+i\tau\rho)X}\mathbb{1}_{X<\sqrt{n}a_N}] = 0. \tag{2.14}$$

By Part 2 of Lemma 2.1,

$$\lim_{N\to\infty} N\mathbb{E}[|W_N| \mathbb{1}_{|W_N|>\varepsilon}] = \lim_{N\to\infty} \mathbb{E}[\mathrm{e}^{\sqrt{n}\sigma X}\mathbb{1}_{X>\sqrt{n}a_N}] = 0. \tag{2.15}$$

We consider W_N as an $\mathbb{R}^2$-valued random vector $(\operatorname{Re} W_N, \operatorname{Im} W_N)$. It follows from (2.13), (2.14), (2.15) that the covariance matrix $\Sigma_N := \operatorname{Var}[W_N \mathbb{1}_{|W_N|<\varepsilon}]$ satisfies

$$\lim_{N\to\infty} N\Sigma_N = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}. \tag{2.16}$$

It follows from (2.15) that $\lim_{N\to\infty} N\mathbb{P}[|W_N| > \varepsilon] = 0$. Therefore, the conditions of Theorem 2.5 are satisfied with $\nu = 0$ and Σ given by the right-hand side of (2.16). Applying Theorem 2.5, we obtain the required statement. $\square$

Proof of Theorem 1.17. Recall that $\alpha = \sqrt{2}/\sigma \in (0,2)$. For $k = 1,\dots,N$, define random variables

$$W_{k,N} = \mathrm{e}^{\sqrt{n}(\sigma X_k + i\tau Y_k - \sigma b_N)}.$$

Let W_N be a random variable having the same law as the $W_{k,N}$'s. We will verify the conditions of Theorem 2.5. To verify the first condition, fix $0 < r_1 < r_2$, $0 < \theta_1 < \theta_2 < 2\pi$ and consider the set

$$B = \{z \in \mathbb{C} : r_1 < |z| < r_2, \theta_1 < \arg z < \theta_2\}.$$

We will show that

$$\lim_{N\to\infty} N\mathbb{P}[W_N \in B] = \left(\frac{1}{r_1^\alpha} - \frac{1}{r_2^\alpha}\right) \cdot \frac{\theta_2 - \theta_1}{2\pi}. \tag{2.17}$$

Define a set

$$A_N = \bigcup_{j\in\mathbb{Z}} \left(\frac{2\pi j + \theta_1}{\tau\sqrt{n}}, \frac{2\pi j + \theta_2}{\tau\sqrt{n}}\right) \subset \mathbb{R}.$$

We have

$$\begin{aligned}\mathbb{P}[W_N \in B] &= \mathbb{P}[\mathrm{e}^{\sigma\sqrt{n}(X-b_N)} \in (r_1,r_2), Y \in A_N]\\ &= \int_{r_1}^{r_2} \mathbb{P}[Y \in A_N | \sigma\sqrt{n}(X - b_N) = \log r] f_N(r)\mathrm{d}r.\end{aligned}$$

Here, $f_N(r)$ is the density of the log-normal random variable $\mathrm{e}^{\sqrt{n}\sigma(X-b_N)}$:

$$f_N(r) = \frac{1}{\sqrt{2\pi n}\sigma r} \exp\left\{-\frac{1}{2}\left(\frac{\log r}{\sigma\sqrt{n}} + b_N\right)^2\right\} \sim \frac{1}{N}\alpha r^{-(1+\alpha)}, \quad N \to \infty, \tag{2.18}$$

where the asymptotic equivalence holds uniformly in $r \in [r_1, r_2]$. To prove (2.18), recall that $\sqrt{2\pi} b_N \mathrm{e}^{b_N^2/2} \sim N$ and $b_N \sim \sqrt{2n}$. Conditionally on $\sigma\sqrt{n}(X - b_N) = \log r$, the random variable Y is normal with mean $\mu_N = \rho(\frac{\log r}{\sigma\sqrt{n}} + b_N)$ and variance $\sqrt{1-\rho^2}$. The variance is strictly positive by the assumption $|\rho| \neq 1$. It follows easily that

$$\lim_{N\to\infty} \mathbb{P}[Y \in A_N | \sigma\sqrt{n}(X - b_N) = \log r] = \frac{\theta_2 - \theta_1}{2\pi}.$$

Bringing everything together, we arrive at (2.17). So, the first condition of Theorem 2.5 holds with

$$\nu(\mathrm{d}x\mathrm{d}y) = \frac{\alpha}{2\pi} \cdot \frac{\mathrm{d}x\mathrm{d}y}{r^{2+\alpha}}, \quad r = \sqrt{x^2 + y^2}.$$

To verify the second condition of Theorem 2.5 with $\Sigma = 0$, it suffices to show that

$$\lim_{\varepsilon\downarrow 0} \limsup_{N\to\infty} N\mathbb{E}[|W_N|^2 \mathbb{1}_{|W_N|\le\varepsilon}] = 0. \tag{2.19}$$

Condition $|W_N| \le \varepsilon$ is equivalent to $X < a_N$, where $a_N = b_N + \frac{\log\varepsilon}{\sigma\sqrt{n}} \sim \sqrt{2n}$. By Lemma 2.2 (first case of (2.3)) with $\lambda = \sqrt{n}$, $w = 2\sigma$, we have

$$\mathbb{E}[\mathrm{e}^{2\sigma\sqrt{n}X} \mathbb{1}_{X<a_N}] \sim Cn^{-1/2}\mathrm{e}^{2\sigma\sqrt{n}a_N - a_N^2/2} \sim CN^{-1}\mathrm{e}^{2\sigma\sqrt{n}b_N}\varepsilon^{2-\sqrt{2}/\sigma}, \quad N \to \infty,$$

where we have again used that $\sqrt{2\pi}b_N e^{b_N^2/2} \sim N$. We obtain that

$$\lim_{N\to\infty} N\mathbb{E}[|W_N|^2 \mathbb{1}_{|W_N|\le\varepsilon}] = \lim_{N\to\infty} N e^{-2\sigma\sqrt{n}b_N}\mathbb{E}[e^{2\sigma\sqrt{n}X}\mathbb{1}_{X<a_N}] = C\varepsilon^{2-\sqrt{2}/\sigma}.$$

Recalling that $2 > \sqrt{2}/\sigma$, we arrive at (2.19). By Theorem 2.5,

$$\sum_{k=1}^{N}(W_{k,N} - \mathbb{E}[W_N \mathbb{1}_{|W_N|<1}]) \xrightarrow[N\to\infty]{d} S_\alpha,$$

where the limiting random vector S_α is infinitely divisible with a characteristic function given by

$$\psi(z) := \log \mathbb{E}[e^{i\langle S_\alpha, z\rangle}] = \frac{\alpha}{2\pi}\int_{\mathbb{R}^2}(e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle \mathbb{1}_{|u|<1})\frac{dxdy}{|u|^{2+\alpha}}, \quad z\in\mathbb{C}.$$

Here, $u = x+iy$ and $\langle u, z\rangle = \operatorname{Re}(u\bar{z})$. Clearly, $\psi(z)$ depends on $|z|$ only and satisfies $\psi(\lambda z) = \lambda^\alpha\psi(z)$ for every $\lambda > 0$. It follows that $\psi(z) = \text{const}\cdot|z|^\alpha$. □

Proof of Remark 1.19. We assume that $\tau \neq 0$. By Lemma 2.4, we have

$$m_N := N\mathbb{E}[e^{\sqrt{n}(\sigma X + i\tau Y)}\mathbb{1}_{X<b_N}] = N^{1-\tau^2(1-\rho^2)/2}\mathbb{E}[e^{\sqrt{n}(\sigma+i\tau\rho)X}\mathbb{1}_{X<b_N}].$$

Write $w = \sigma + i\tau\rho$. Applying Lemma 2.2, we have

$$\mathbb{E}[e^{\sqrt{n}(\sigma+i\tau\rho)X}\mathbb{1}_{X<b_N}] \sim \begin{cases} \frac{1}{(w/\sqrt{2})-1}N^{-1}e^{b_N\sqrt{n}w}, & \sigma+\tau\rho > \sqrt{2}, \\ e^{w^2 n/2}, & \sigma+\tau\rho \le \sqrt{2}. \end{cases}$$

Strictly speaking, the case $\sigma+\tau\rho = \sqrt{2}$ is not contained in Lemma 2.2, but an easy computation shows that that the term $e^{w^2n/2}$ is dominating in (2.5).

□

2.3. **Proof of Theorem 1.12.** We will deduce the stochastic convergence of the log-partition function $p_N(\beta) = \frac{1}{n}\log|\mathcal{Z}_N(\beta)|$ from the weak convergence of $\mathcal{Z}_N(\beta)$. This will be done via the following lemma.

Lemma 2.6. *Let $Z, Z_1, Z_2, \dots$ be random variables with values in $\mathbb{C}$ and let $m_N \in \mathbb{C}$, $v_N \in \mathbb{C}\backslash\{0\}$ be sequences of normalizing constants such that*

$$\frac{Z_N - m_N}{v_N} \xrightarrow[N\to\infty]{d} Z. \tag{2.20}$$

The following two statements hold:

1. *If $|v_N| = o(|m_N|)$ and $|m_N| \to \infty$ as $N\to\infty$, then $\frac{\log|Z_N|}{\log|m_N|} \xrightarrow[N\to\infty]{P} 1$.*
2. *If $|m_N| = O(|v_N|)$, $|v_N| \to \infty$ as $N\to\infty$ and Z has no atoms, then $\frac{\log|Z_N|}{\log|v_N|} \xrightarrow[N\to\infty]{P} 1$.*

Proof of (1). Fix $\varepsilon > 0$. For sufficiently large N, we have $|m_N| > 1$ and, hence,

$$\begin{aligned}\mathbb{P}\left[1-\varepsilon < \frac{\log|Z_N|}{\log|m_N|} < 1+\varepsilon\right] &= \mathbb{P}[|m_N|^{1-\varepsilon} < |Z_N| < |m_N|^{1+\varepsilon}] \\ &\ge \mathbb{P}\left[\left|\frac{Z_N - m_N}{v_N}\right| < \frac{1}{2}\frac{|m_N|}{|v_N|}\right].\end{aligned}$$

The right-hand side converges to 1 by our assumptions. □

Proof of (2). Fix $\varepsilon > 0$. For sufficiently large N,

$$\mathbb{P}\left[\frac{\log|Z_N|}{\log|v_N|} > 1+\varepsilon\right] = \mathbb{P}\left[\frac{|Z_N|}{|v_N|} > |v_N|^{\varepsilon}\right] \le \mathbb{P}\left[\left|\frac{Z_N - m_N}{v_N}\right| > \frac{1}{2}|v_N|^{\varepsilon}\right].$$

The right hand-side converges to 0 by our assumptions. Consider now

$$\mathbb{P}\left[\frac{\log|Z_N|}{\log|v_N|} < 1-\varepsilon\right] = \mathbb{P}\left[\frac{|Z_N|}{|v_N|} < |v_N|^{-\varepsilon}\right] = \mathbb{P}\left[\left|\frac{Z_N - m_N}{v_N} + \frac{m_N}{v_N}\right| < |v_N|^{-\varepsilon}\right].$$

Assume that there is $\delta > 0$ such that the right-hand side is $> \delta$ for infinitely many N's. Recall that m_N/v_N is bounded. Taking a subsequence, we may assume that $-m_N/v_N$ converges to some $a \in \mathbb{C}$. Recall that $|v_N| \to \infty$. But then, for every $\eta > 0$,

$$\mathbb{P}[|Z-a| < \eta] \ge \limsup_{N\to\infty} \mathbb{P}\left[\left|\frac{Z_N - m_N}{v_N} - a\right| < \frac{\eta}{2}\right] > \delta.$$

This contradicts the assumption that Z has no atoms. □

Proof of Theorem 1.12. We may assume that $\tau \neq 0$, since otherwise the result is known [19]. Let $p(\beta)$ be defined by (1.12). First, we show that $\lim_{N\to\infty} p_N(\beta) = p(\beta)$ in probability. It follows from Theorems 1.13, 1.17, 1.16 and Remark 1.18 that condition (2.20) is satisfied with $Z_N = \mathcal{Z}_N(\beta)$ and an appropriate choice of m_N, v_N. Straightforward calculation (see in particular Remarks 1.15 and 1.19) shows that the normalizing constants m_N and v_N satisfy the first condition of Lemma 2.6 if $\beta \in B_1$ and the second condition if $\beta \in \overline{B_2 \cup B_3}$. Applying Lemma 2.6, we obtain that $p_N(\beta) \to p(\beta)$ in probability.

Let us show that $p_N(\beta) \to p(\beta)$ in L^q, where $q \ge 1$ is fixed. From the fact that $p_N(\beta) \to p(\beta)$ in probability, and since $p(\beta) > 0$, for every $\beta \in \mathbb{C}$, we conclude that, for every $C > p(\beta)$,

$$\lim_{N\to\infty} p_N(\beta) \mathbb{1}_{0 \le p_N(\beta) \le C+1} = p(\beta) \text{ in } L^q.$$

For every $u \in \mathbb{R}$, we have

$$\mathbb{P}[p_N(\beta) \ge u] \le \mathrm{e}^{-nu} \mathbb{E}|\mathcal{Z}_N(\beta)| \le \mathrm{e}^{-nu} N \mathbb{E}[\mathrm{e}^{\sigma\sqrt{n}X}] = \mathrm{e}^{n(C-u)},$$

where $C = 1 + \sigma^2/2$. From this, we conclude that

$$\begin{aligned}\mathbb{E}\left[|p_N(\beta)|^q \mathbb{1}_{p_N(\beta) > C+1}\right] &= \sum_{k=1}^{\infty} \mathbb{E}[|p_N(\beta)|^q \mathbb{1}_{C+k < p_N(\beta) \le C+k+1}] \\ &\le \sum_{k=1}^{\infty} \mathrm{e}^{-nk}(C+k+1)^q,\end{aligned}$$

which converges to 0, as $N \to \infty$. To complete the proof, we need to show that

$$\lim_{N\to\infty} \mathbb{E}\left[|p_N(\beta)|^q \mathbb{1}_{p_N(\beta) < 0}\right] = 0. \tag{2.21}$$

The problem is to bound the probability of small values of $\mathcal{Z}_N(\beta)$, where the logarithm has a singularity and $|p_N(\beta)|$ becomes large. Fix a small $\varepsilon > 0$. Clearly,

$$\mathbb{E}\left[|p_N(\beta)|^q \mathbb{1}_{-\varepsilon\sigma^2 \le p_N(\beta) \le 0}\right] \le (\varepsilon\sigma^2)^q. \tag{2.22}$$

To prove (2.21), we would like to estimate from above the probability $\mathbb{P}[|\mathcal{Z}_N(\beta)| \le r]$ for $0 < r < \mathrm{e}^{-\varepsilon\sigma^2 n}$. Recall that $\mathcal{Z}_N(\beta)$ is a sum of N independent copies of the random variable $\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)}$. Unfortunately, the distribution of the latter random variable does not possess nice regularity properties. For example, in the most interesting case $\rho = 1$ it has no density. This is why we need a smoothing argument. Denote by $B_r(t)$ the disc of radius r centered at $t \in \mathbb{C}$. Fix a large $A > 1$. We will show that uniformly in $t \in \mathbb{C}$, $1/A < |\beta| < A$, $n > (2A)^2$, and $0 < r < \mathrm{e}^{-\varepsilon\sigma^2 n}$,

$$\mathbb{P}[\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} \in B_r(t)] < C r^{\frac{\varepsilon}{20}}. \tag{2.23}$$

This inequality is stated in a form which will be needed later in the proof of Theorem 1.1.

Let $|t| \geq \sqrt{r}$ and $\tau \geq 1/(2A)$. The argument $\arg t$ of a complex number t is considered to have values in the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let $P : \mathbb{R} \to \mathbb{T}$ be the canonical projection. Denote by $I_r(t)$ be the sector $\{z \in \mathbb{C} : |\arg z - \arg t| \leq 2\sqrt{r}\}$, where we take the geodesic distance between the arguments. A simple geometric argument shows that the disc $B_r(t)$ is contained in the sector $I_r(t)$. The density of the random variable $P(\tau\sqrt{n}Y)$ at $\theta \in \mathbb{T}$ is given by

$$\mathbb{P}[P(\tau\sqrt{n}Y) \in \mathrm{d}\theta] = \frac{1}{\sqrt{2\pi n}\tau} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-(\theta+2\pi k)^2/(2\tau^2 n)} \mathrm{d}\theta.$$

By considering the right-hand side as a Riemann sum and recalling that $\tau \geq 1/(2A)$, we see that the density converges to $1/(2\pi)$ uniformly in $\theta \in \mathbb{T}$ as $N \to \infty$. We have

$$\mathbb{P}[\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} \in B_r(t)] \leq \mathbb{P}[\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} \in I_r(t)] < C\sqrt{r},$$

which implies (2.23).

Let now $|t| < \sqrt{r}$. Then, recalling that $\log r < -\varepsilon\sigma^2 n$, we obtain

$$\mathbb{P}[\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} \in B_r(t)] \leq \mathbb{P}[\mathrm{e}^{\sigma\sqrt{n}X} < r^{1/4}] = \mathbb{P}\left[X < \frac{\log r}{4\sigma\sqrt{n}}\right] < \mathrm{e}^{-\frac{(\log r)^2}{16\sigma^2 n}} < r^{\frac{\varepsilon}{16}}.$$

It remains to consider the case $t \geq \sqrt{r}$, $|\sigma| \geq 1/(2A)$. The density of the random variable $\mathrm{e}^{\sigma\sqrt{n}X}$ is given by

$$g(x) = \frac{1}{\sqrt{2\pi n}\sigma x} \mathrm{e}^{-\frac{(\log x)^2}{2\sigma^2 n}}, \qquad x > 0.$$

It attains its maximum at $x_0 = \mathrm{e}^{-\sigma^2 n}$. The maximum is equal to $g(x_0) = \frac{1}{\sqrt{2\pi n}\sigma} \mathrm{e}^{\sigma^2 n/2}$. Let $r \leq (2\pi n)\sigma^2 \mathrm{e}^{-\sigma^2 n}$. Then,

$$\mathbb{P}[\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} \in B_r(t)] \leq \mathbb{P}[t - r \leq \mathrm{e}^{\sigma\sqrt{n}X} \leq t + r] \leq \frac{Cr}{\sqrt{n}\sigma} \mathrm{e}^{\sigma^2 n/2} \leq C r^{1/2}.$$

Let $r \geq (2\pi n)\sigma^2 \mathrm{e}^{-\sigma^2 n}$, which, together with $|\sigma| > 1/(2A)$, implies that $r > \mathrm{e}^{-\sigma^2 n}$. Using the unimodality of the density g and the inequality $t - r > r$, we have

$$\mathbb{P}[\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} \in B_r(t)] \leq \mathbb{P}[t - r \leq \mathrm{e}^{\sigma\sqrt{n}X} \leq t + r] < 2rg(r) < \mathrm{e}^{-\frac{(\log r)^2}{2\sigma^2 n}} < r^{\frac{\varepsilon}{2}}.$$

The last inequality follows from $r < \mathrm{e}^{-\sigma^2 n}$. This completes the proof of (2.23).

Now we are in position to complete the proof of (2.21). Let U_r be a random variable distributed uniformly on the disc $B_r(0)$. It follows from (2.23) that the density of the random variable $\mathrm{e}^{\sqrt{n}(\sigma X + i\tau Y)} + U_r$ is bounded above by $C r^{-2+(\varepsilon/20)}$. Hence, the density of $\mathcal{Z}_N(\beta) + U_r$ is bounded by the same term $C r^{-2+(\varepsilon/20)}$. With the notation $r = \mathrm{e}^{-kn}$ it follows that, for every $k \geq \varepsilon\sigma^2$,

$$\mathbb{P}[p_N(\beta) \leq -k] = \mathbb{P}[|\mathcal{Z}_N(\beta)| \leq \mathrm{e}^{-kn}] \leq \mathbb{P}[|\mathcal{Z}_N(\beta) + U_r| \leq 2r] \leq C r^{\frac{\varepsilon}{20}}.$$

From this, we obtain that

$$\mathbb{E}[|p_N(\beta)|^q \mathbb{1}_{p_N(\beta)\in[-k-1,-k]}] \le C(k+1)^q \mathrm{e}^{-\frac{kn}{20}}.$$

Taking the sum over all $k = \varepsilon\sigma^2 + l$, $l = 0, 1, \dots$, we get

$$\mathbb{E}[|p_N(\beta)|^q \mathbb{1}_{p_N(\beta)<-\varepsilon\sigma^2}] \le C\mathrm{e}^{-\frac{\varepsilon\sigma^2 n}{20}} \sum_{l=1}^{\infty} l^q \mathrm{e}^{-\frac{ln}{20}} \le C\mathrm{e}^{-\frac{\varepsilon\sigma^2 n}{20}}.$$

Recalling (2.22), we arrive at (2.21). □

Remark 2.7. As a byproduct of the proof, we have the following statement. For every $A > 0$, there is a constant $C = C(A)$ such that $\mathbb{E}|p_N(\beta)| < C$, for all $1/A < |\beta| < A$ and sufficiently large N.

3. Proofs of the results on zeros

3.1. **Convergence of random analytic functions.** In this section, we collect some lemmas on weak convergence of stochastic processes whose sample paths are analytic functions. As we will see, the analyticity assumption simplifies the things considerably. For a metric space M denote by $C(M)$ the space of complex-valued continuous functions on M endowed with the topology of uniform convergence on compact sets. Let $D \subset \mathbb{C}$ be a simply connected domain.

Lemma 3.1. *Let $\{U(t), t \in D\}$ be a random analytic function defined on D. Let $\Gamma \subset D$ be a closed differentiable contour and let K be a compact subset located strictly inside Γ. Then, for every $p \in \mathbb{N}_0$, there is a constant $C = C_p(K, \Gamma)$ such that*

$$\mathbb{E}\left[\sup_{t\in K} |U^{(p)}(t)|\right] \le C \oint_\Gamma \mathbb{E}|U(w)||\mathrm{d}w|.$$

Proof. By the Cauchy formula, $|U^{(p)}(t)| \le C \oint_\Gamma |U(w)||\mathrm{d}w|$, for all $t \in K$. Take the supremum over $t \in K$ and then the expectation. □

It is easy to check that a sequence of stochastic processes with paths in $C(D)$ is tight (resp., weakly convergent) if and only if it is tight (resp., weakly convergent) in $C(K)$, for every compact set $K \subset D$.

Lemma 3.2. *Let $U_1, U_2, \dots$ be random analytic functions on D. Assume that there is a continuous function $f : D \to \mathbb{R}$ such that $\mathbb{E}|U_N(t)| < f(t)$, for all $t \in D$, and all $N \in \mathbb{N}$. Then, the sequence U_N is tight on $C(D)$.*

Proof. Let $K \subset D$ be a compact set. Let Γ be a contour enclosing K and located inside D. By Lemma 3.1,

$$\mathbb{E}\left[\sup_{t\in K} |U_N(t)|\right] \le C \oint_\Gamma f(w)|\mathrm{d}w|, \quad \mathbb{E}\left[\sup_{t\in K} |U_N'(t)|\right] \le C \oint_\Gamma f(w)|\mathrm{d}w|.$$

By standard arguments, this implies that the sequence U_N is tight on $C(K)$. □

Lemma 3.3. *Let $U, U_1, U_2, \dots$ be random analytic functions on D such that U_N converges as $N \to \infty$ to U weakly on $C(D)$ and $\mathbb{P}[U \equiv 0] = 0$. Then, for every continuous function $f : D \to \mathbb{R}$ with compact support, we have*

$$\sum_{z\in\mathbb{C}:U_N(z)=0} f(z) \xrightarrow[N\to\infty]{d} \sum_{z\in\mathbb{C}:U(z)=0} f(z).$$

Remark 3.4. Equivalently, the zero set of U_N, considered as a point process on D, converges weakly to the zero set of U.

Proof. Let H be a closed linear subspace of $C(D)$ consisting of all analytic functions. Consider a functional $\Psi : H \to \mathbb{R}$ mapping an analytic function φ which is not identically 0 to $\sum_z f(z)$, where the sum is over all zeros of φ. Define also $\Psi(0) = 0$. It is an easy consequence of Rouché's theorem that Ψ is continuous on $H \backslash \{0\}$. Note that $H\backslash\{0\}$ is a set of full measure with respect to the law of U. Recall that $U_N \to U$ weakly on H. By the continuous mapping theorem [20, § 3.5], $\Psi(U_N)$ converges in distribution to $\Psi(U)$. This proves the lemma. □

3.2. **Proof of Theorem 1.1.** Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function with compact support. We need to show that

$$\frac{\pi}{n} \sum_{\beta \in \mathbb{C} : \mathcal{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{P} \int_{B_3} f(\beta)\lambda(\mathrm{d}\beta). \tag{3.1}$$

We need to restrict somewhat the class of f under consideration. A standard approximation argument shows that we can assume that f is smooth. We may represent f as a sum of two functions, the first one vanishing on $|\beta| < 1/4$ and the second one vanishing outside $|\beta| < 1/2$. The second function makes no contribution to (3.1) by Theorem 1.5. So, we may assume that f vanishes on $|\beta| < 1/4$.

Denote by Δ the Laplace operator (interpreted in the distributional sense) and by $\delta(\beta)$ the unit point mass at $\beta \in \mathbb{C}$. It is well known, see [11, Section 2.4], that

$$\Delta \log |\mathcal{Z}_N| = 2\pi \sum_{\beta \in \mathbb{C} : \mathcal{Z}_N(\beta)=0} \delta(\beta). \tag{3.2}$$

Recall that $p_N(\beta) = \frac{1}{n}\log|\mathcal{Z}_N(\beta)|$ and $p(\beta)$ have been defined in Theorem 1.12. An easy computation shows that $\Delta p = 2 \cdot \mathbb{1}_{B_3}$. Using (3.2) and the self-adjointness of the Laplacian, we conclude that (3.1) is equivalent to

$$\int_{\mathbb{C}} p_N(\beta)\Delta f(\beta)\lambda(\mathrm{d}\beta) \xrightarrow[N\to\infty]{P} \int_{\mathbb{C}} p(\beta)\Delta f(\beta)\lambda(\mathrm{d}\beta).$$

We will show that this holds even in L^1. By Fubini's theorem, it suffices to show that

$$\lim_{N\to\infty} \int_{\mathbb{C}} \mathbb{E}|p_N(\beta) - p(\beta)||\Delta f(\beta)|\lambda(\mathrm{d}\beta) = 0. \tag{3.3}$$

We know from Theorem 1.12 that $\lim_{N\to\infty} \mathbb{E}|p_N(\beta) - p(\beta)| = 0$, for every $\beta \in \mathbb{C}$. To complete the proof of (3.3), we use the dominated convergence theorem, which is justified by Remark 2.7.

3.3. **Proof of Theorem 1.5.** Let Γ be a differentiable contour enclosing the set K and located inside B_1. We have

$$\begin{aligned}
\mathbb{P}[\mathcal{Z}_N(\beta) = 0, \text{ for some } \beta \in K] &\le \mathbb{P}\left[\sup_{\beta\in K}\left|\frac{\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)}{\mathbb{E}\mathcal{Z}_N(\beta)}\right| \ge 1\right] \\
&\le \mathbb{E}\sup_{\beta\in K}\left|\frac{\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)}{\mathbb{E}\mathcal{Z}_N(\beta)}\right| \\
&\le C \oint_{\Gamma} \mathbb{E}\left|\frac{\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)}{\mathbb{E}\mathcal{Z}_N(\beta)}\right| |\mathrm{d}w|,
\end{aligned}$$

where the last step is by Lemma 3.1. Note that $|\mathbb{E}\mathcal{Z}_N(\beta)| = N^{1+\frac{1}{2}(\sigma^2-\tau^2)}$. To complete the proof, we need to show that there exist $\varepsilon > 0$ and $C > 0$ depending on Γ such that, for every $\beta \in \Gamma$, $N \in \mathbb{N}$,

$$\mathbb{E}\,|\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)| < CN^{1-\varepsilon+\frac{1}{2}(\sigma^2-\tau^2)}. \tag{3.4}$$

Since $\Gamma \subset B_1$, we can choose $\varepsilon > 0$ so small that $\Gamma \subset B_1'(\varepsilon) \cup B_1''(\varepsilon)$, where

$$B_1'(\varepsilon) = \{\beta \in \mathbb{C} : \sigma^2 + \tau^2 < 1 - 2\varepsilon\},$$
$$B_1''(\varepsilon) = \{\beta \in \mathbb{C} : (|\sigma| - \sqrt{2})^2 - \tau^2 > 2\varepsilon, 1/2 < \sigma^2 < 2\}.$$

We have

$$\mathbb{E}|\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)|^2 = N\mathbb{E}|\mathrm{e}^{\beta\sqrt{n}X} - \mathbb{E}\mathrm{e}^{\beta\sqrt{n}X}|^2 \le N\mathbb{E}\mathrm{e}^{2\sigma\sqrt{n}X} = N^{1+2\sigma^2}.$$

If $\beta \in B_1'(\varepsilon)$, then it follows that

$$\mathbb{E}\,|\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)| \le N^{\frac{1}{2}+\sigma^2} \le N^{1-\varepsilon+\frac{1}{2}(\sigma^2-\tau^2)}.$$

This implies (3.4). Assume now that $\beta \in B_1''(\varepsilon)$ and $\sigma > 0$. For $k = 1, \ldots, N$, define random variables

$$U_{k,N} = \mathrm{e}^{\beta\sqrt{n}X_k - \sigma\sqrt{2}n}\mathbb{1}_{X_k \le \sqrt{2n}}, \quad V_{k,N} = \mathrm{e}^{\beta\sqrt{n}X_k - \sigma\sqrt{2}n}\mathbb{1}_{X_k > \sqrt{2n}},$$

By Part 1 of Lemma 2.1, we have

$$\mathbb{E}\left|\sum_{k=1}^{N}(U_{k,N} - \mathbb{E}U_{k,N})\right|^2 \le N\mathbb{E}|U_{1,N}|^2 = N\mathrm{e}^{-2\sqrt{2}\sigma n}\mathbb{E}[\mathrm{e}^{2\sigma\sqrt{n}X}\mathbb{1}_{X<\sqrt{2n}}] < 1. \tag{3.5}$$

Similarly, By Part 2 of Lemma 2.1,

$$\mathbb{E}\left|\sum_{k=1}^{N}(V_{k,N} - \mathbb{E}V_{k,N})\right| \le 2N\mathbb{E}|V_{k,N}| = 2N\mathrm{e}^{-\sigma\sqrt{2}n}\mathbb{E}[\mathrm{e}^{\sigma\sqrt{n}X}\mathbb{1}_{X>\sqrt{2n}}] < 2. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain $\mathbb{E}\,|\mathcal{Z}_N(\beta) - \mathbb{E}\mathcal{Z}_N(\beta)| \le 3\mathrm{e}^{\sigma\sqrt{2}n}$. Since $\beta \in B_1''(\varepsilon)$ this implies the required estimate (3.4).

3.4. **Proof of Theorem 1.3.** Recall that $\mathbb{G}$ is the Gaussian analytic function defined in (1.6). Theorem 1.3 will be deduced from the following result.

Theorem 3.5. *Fix some $\beta_0 = \sigma_0 + i\tau_0$ with $\sigma_0^2 < 1/2$ and $\tau_0 \neq 0$. Define a random process $\{G_N(t) : t \in \mathbb{C}\}$ by*

$$G_N(t) := \frac{\mathcal{Z}_N\left(\beta_0 + \frac{t}{\sqrt{n}}\right) - N^{1+\frac{1}{2}(\beta_0+\frac{t}{\sqrt{n}})^2}}{N^{\frac{1}{2}+(\sigma_0+\frac{t}{\sqrt{n}})^2}}.$$

Then, the process G_N converges weakly, as $N \to \infty$, to the process $\mathrm{e}^{-t^2/2}\mathbb{G}(t)$ on $C(\mathbb{C})$.

Proof. For $k = 1, \ldots, N$, define a random process $\{W_{k,N}(t) : t \in \mathbb{C}\}$ by

$$W_{k,N}(t) = N^{-1/2}\mathrm{e}^{(\beta_0\sqrt{n}+t)X_k - (\sigma_0\sqrt{n}+t)^2}.$$

Then, $G_N(t) = \sum_{k=1}^N (W_{k,N}(t) - \mathbb{E}W_{k,N}(t))$. First, we show that the convergence stated in Theorem 3.5 holds in the sense of finite-dimensional distributions. Take $t_1, \ldots, t_d \in \mathbb{C}$. Write $\mathbf{W}_{k,N} = (W_{k,N}(t_1), \ldots, W_{k,N}(t_d))$. We need to prove that

$$\sum_{k=1}^{N}(\mathbf{W}_{k,N} - \mathbb{E}\mathbf{W}_{k,N}) \xrightarrow[N\to\infty]{d} (\mathrm{e}^{-t_1^2/2}\mathbb{G}(t_1), \ldots, \mathrm{e}^{-t_d^2/2}\mathbb{G}(t_d)). \tag{3.7}$$

Let W_N be a process having the same law as the $W_{k,N}$'s and define $\mathbf{W}_N = (W_N(t_1), \dots, W_N(t_d))$. A straightforward computation shows that for all $t, s \in \mathbb{C}$,

$$N\mathbb{E}[W_N(t)\overline{W_N(s)}] = \mathrm{e}^{-(t-\overline{s})^2/2}, \tag{3.8}$$

$$\lim_{N\to\infty} N|\mathbb{E}[W_N(t)W_N(s)]| = 0. \tag{3.9}$$

Also, we have

$$\lim_{N\to\infty} \sqrt{N}|\mathbb{E}[W_N(t)]| = \mathrm{e}^{-t^2/2} \lim_{N\to\infty} \mathrm{e}^{-\tau^2 n} = 0. \tag{3.10}$$

Note that by (1.6),

$$\mathbb{E}[\mathrm{e}^{-t^2/2}\mathbb{G}(t)\overline{\mathrm{e}^{-s^2/2}\mathbb{G}(s)}] = \mathrm{e}^{-(t-\overline{s})^2/2}, \qquad \mathbb{E}[\mathrm{e}^{-t^2/2}\mathbb{G}(t)\mathrm{e}^{-s^2/2}\mathbb{G}(s)] = 0.$$

We see that the covariance matrix of the left-hand side of (3.7) converges to the covariance matrix of the right-hand side of (3.7) if we view both sides as $2d$-dimensional real random vectors. To complete the proof of (3.7), we need to verify the Lindeberg condition: for every $\varepsilon > 0$,

$$\lim_{N\to\infty} N\mathbb{E}[|\mathbf{W}_N|^2 \mathbb{1}_{|\mathbf{W}_N|>\varepsilon}] = 0. \tag{3.11}$$

For $l = 1, \dots, d$, let A_l be the random event $|W_N(t_l)| \geq |W_N(t_j)|$ for all $j = 1, \dots, d$. On A_l, we have $|\mathbf{W}_N|^2 \leq d|W_N(t_l)|^2$. It follows that

$$N\mathbb{E}[|\mathbf{W}_N|^2 \mathbb{1}_{|\mathbf{W}_N|>\varepsilon}] \leq d \sum_{l=1}^{d} N\mathbb{E}\left[|W_N(t_l)|^2 \mathbb{1}_{|W_N(t_l)|>\frac{\varepsilon}{\sqrt{l}}}\right] \to 0,$$

where the last step is by the same argument as in (2.12). This completes the proof of the finite-dimensional convergence stated in (3.7). The tightness follows from Lemma 3.2 which can be applied since

$$\mathbb{E}|G_N(t)| \leq \sqrt{\mathbb{E}[|G_N(t)|^2]} \leq \sqrt{N\mathbb{E}[|W_N(t)|^2]} = \mathrm{e}^{(\operatorname{Im} t)^2}.$$

The last equality follows from (3.8). □

Proof of Theorem 1.3. If $\beta_0 \in B_3$, then the expectation term in the definition of G_N can be ignored: we have $\lim_{N\to\infty} |G_N(t) - U_N(t)| = 0$ uniformly on compact sets, where

$$U_N(t) = N^{-\frac{1}{2}-(\sigma_0+\frac{t}{\sqrt{n}})^2} \mathcal{Z}_N\left(\beta_0 + \frac{t}{\sqrt{n}}\right).$$

It follows from Theorem 3.5 that U_N converges to $\mathbb{G}$ weakly on $C(\mathbb{C})$. Applying Lemma 3.3, we obtain the statement of Theorem 1.3. □

3.5. **Proof of Theorems 1.6 and 1.8.**

Proof of Theorem 1.6. Fix a compact set K contained in the half-plane $\sigma > 1/2$. Define random $C(K)$-valued elements $S_k(\beta) = s_1(\beta) + \ldots + s_k(\beta)$, where

$$s_k(\beta) = \sum_{j=1}^{\infty} P_j^{-\beta} \mathbb{1}_{k \leq P_j < k+1} - \int_k^{k+1} t^{-\beta} \mathrm{d}t, \quad \beta \in K.$$

Note that $s_1, s_2, \dots$ are independent. By the properties of the Poisson process,

$$\mathbb{E}[s_k(\beta)] = 0, \quad \sum_{k=1}^{\infty} \mathbb{E}[|s_k(\beta)|^2] = \int_1^{\infty} t^{-2\sigma} \mathrm{d}t < \infty. \tag{3.12}$$

Thus, as long as $\sigma > 1/2$, the sequence $S_k(\beta)$, $k \in \mathbb{N}$, is an L^2-bounded martingale. Hence, $S_k(\beta)$ converges a.s. to a limiting random variable denoted by $S(\beta)$. We need to show that the convergence is uniform a.s. It follows from (3.12) and Lemma 3.2 that the sequence S_k, $k \in \mathbb{N}$, is tight on $C(K)$. Hence, S_k converges weakly on $C(K)$ to the process S. By the Itô–Nisio theorem [12], this implies that S_k converges to S a.s. as a random element of $C(K)$. This proves the theorem. □

Proof of Theorem 1.8. Let us first describe the idea. Consider the case $\sigma > 1/\sqrt{2}$. Arrange the values $X_1, \dots, X_N$ in an increasing order, obtaining the order statistics $X_{1:N} \leq \dots \leq X_{N:N}$. It turns out that the main contribution to the sum $\mathcal{Z}_N(\beta) = \sum_{k=1}^N \mathrm{e}^{\beta\sqrt{n}X_k}$ comes from the upper order statistics $X_{N-k:N}$, where $k = 0, 1, \dots$. Their joint limiting distribution is well-known in the extreme-value theory, see [20, Corollary 4.19(i)], and will be recalled now. Denote by $\mathbb{M}$ the space of locally finite counting measures on $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We endow $\mathbb{M}$ with the (Polish) topology of vague convergence. A point process on $\bar{\mathbb{R}}$ is a random element with values in $\mathbb{M}$. Let $P_1, P_2, \dots$ be the arrivals of the unit intensity Poisson process on the positive half-line. Define the sequence b_N as in (1.15).

Proposition 3.6. *The point process $\pi_N := \sum_{k=1}^N \delta(\sqrt{n}(X_k - b_N))$ converges as $N \to \infty$ to the point process $\pi_\infty = \sum_{k=1}^\infty \delta(-(\log P_k)/\sqrt{2})$ weakly on $\mathbb{M}$.*

Utilizing this result, we will show that it is possible to approximate $\mathcal{Z}_N(\beta)$ (after appropriate normalization) by $\tilde{\zeta}_P(\beta/\sqrt{2})$ in the half-plane $\sigma > 1/\sqrt{2}$. Consider now the case $\sigma < -1/\sqrt{2}$. This time, the main contribution to the sum $\mathcal{Z}_N(\beta)$ comes from the lower order statistics $X_{k:N}$, $k = 1, 2, \dots$. Their joint limiting distribution is the same as for the upper order statistics, only the sign should be reversed. Moreover, it is known that the upper and the lower order statistics become asymptotically independent as $N \to \infty$. Thus, in the half-plane $\sigma < -1/\sqrt{2}$ it is possible to approximate $\mathcal{Z}_N(\beta)$ by an independent copy of $\zeta_P(-\beta/\sqrt{2})$. In the rest of the proof we make this idea rigorous. For simplicity of notation we restrict ourselves to the half-plane $D = \{\beta \in \mathbb{C} : \sigma > 1/\sqrt{2}\}$.

Theorem 3.7. *The following convergence holds weakly on $C(D)$:*

$$\xi_N(\beta) := \frac{\mathcal{Z}_N(\beta) - N\mathbb{E}[\mathrm{e}^{\beta\sqrt{n}X} 1_{X<b_N}]}{\mathrm{e}^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} \tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}\right).$$

The proof consists of two lemmas. Take $A > 0$ and write $\xi_N(\beta) = \xi_N^A(\beta) - e_N^A(\beta) + \Delta_N^A(\beta)$, where

$$\xi_N^A(\beta) = \sum_{k=1}^N \mathrm{e}^{\beta\sqrt{n}(X_k - b_N)} \mathbb{1}_{b_N - \frac{A}{\sqrt{n}} < X_k},$$

$$e_N^A(\beta) = N\mathbb{E}\left[\mathrm{e}^{\beta\sqrt{n}(X_k - b_N)} \mathbb{1}_{b_N - \frac{A}{\sqrt{n}} \leq X_k < b_N}\right],$$

$$\Delta_N^A(\beta) = \sum_{k=1}^N \left(\mathrm{e}^{\beta\sqrt{n}(X_k - b_N)} \mathbb{1}_{X_k \leq b_N - \frac{A}{\sqrt{n}}} - \mathbb{E}\left[\mathrm{e}^{\beta\sqrt{n}(X_k - b_N)} \mathbb{1}_{X_k \leq b_N - \frac{A}{\sqrt{n}}}\right]\right).$$

Lemma 3.8. *Let $\tilde{\zeta}_P(\cdot;\cdot)$ be defined as in (1.7). Then, the following convergence holds weakly on $C(D)$:*

$$\xi_N^A(\beta) - e_N^A(\beta) \xrightarrow[N\to\infty]{w} \tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}; \mathrm{e}^{\sqrt{2}A}\right).$$

Proof. Recall from Proposition 3.6 that the point process π_N converges to the point process π_∞ weakly on $\mathbb{M}$. Consider a functional $\Psi : \mathbb{M} \to C(D)$ which maps a locally finite counting measure $\rho = \sum_{i\in I} \delta(y_i) \in \mathbb{M}$ to the function $\Psi(\rho)(\beta) = \sum_{i\in I} \mathrm{e}^{\beta y_i} \mathbb{1}_{y_i > -A}$, where $\beta \in D$. Here, I is at most countable index set. If ρ charges the point $+\infty$, define $\Psi(\rho)$, say, as 0. The functional Ψ is continuous on the set of all $\rho \in \mathbb{M}$ not charging the points $-A$ and $+\infty$, which is a set of full measure with respect to the law of π_∞. It follows from the continuous mapping theorem [20, § 3.5] that $\xi_N^A = \Psi(\pi_N)$ converges weakly on $C(D)$ to $\Psi(\pi_\infty)$. Note that

$$\Psi(\pi_\infty)(\beta) = \sum_{k=1}^{\infty} P_k^{-\beta/\sqrt{2}} \mathbb{1}_{P_k < \mathrm{e}^{\sqrt{2}A}}.$$

We prove the convergence of $e_N^A(\beta)$. Using the change of variables $\sqrt{n}(x - b_N) = y$, we obtain

$$e_N^A(\beta) = \frac{N}{\sqrt{2\pi}} \int_{b_N - \frac{A}{\sqrt{n}}}^{b_N} \mathrm{e}^{\beta\sqrt{n}(x-b_N)} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \frac{N}{\sqrt{2\pi n}} \int_{-A}^{0} \mathrm{e}^{\beta y} \mathrm{e}^{-\frac{1}{2}(b_N + \frac{y}{\sqrt{n}})^2} \mathrm{d}y.$$

Recalling that $\sqrt{2\pi} b_N \mathrm{e}^{b_N^2/2} \sim N$ and $b_N \sim \sqrt{2n}$ as $N \to \infty$, we obtain that $\lim_{N\to\infty} e_N^A(\beta) = \int_1^{\mathrm{e}^{\sqrt{2}A}} t^{-\beta/\sqrt{2}} \mathrm{d}t$, as required. □

Lemma 3.9. *For every compact set $K \subset D$ there is $C > 0$ such that, for all sufficiently large N,*

$$\mathbb{E}\left[\sup_{\beta\in K} |\Delta_N^A(\beta)|\right] \le C\mathrm{e}^{(1-\sqrt{2}\sigma)A/2}.$$

Proof. Let Γ be a contour enclosing K and located inside D. First, $\mathbb{E}[\Delta_N^A(\beta)] = 0$ by definition. Second, uniformly in $\beta \in \Gamma$ it holds that

$$\begin{aligned}
\mathbb{E}[|\Delta_N^A(\beta)|^2] &\le N\mathbb{E}\left[\mathrm{e}^{2\sigma\sqrt{n}(X-b_N)} \mathbb{1}_{X < b_N - \frac{A}{b_N}}\right] \\
&= N\mathrm{e}^{-2\sigma\sqrt{n}b_N} \mathrm{e}^{2\sigma^2 n} \Phi\left(b_N - \frac{A}{b_N} - 2\sigma\sqrt{n}\right) \\
&\le C\mathrm{e}^{(1-\sqrt{2}\sigma)A},
\end{aligned}$$

where the second step follows from (2.2) and the last step follows from (2.1). By Lemma 3.1, we have

$$\mathbb{E}\left[\sup_{\beta\in K} |\Delta_N^A(\beta)|\right] \le C \oint_\Gamma \mathbb{E}|\Delta_N^A(\beta)||\mathrm{d}\beta| \le C\mathrm{e}^{(1-\sqrt{2}\sigma)A/2}.$$

The proof is complete. □

Proof of Theorem 3.7. By Theorem 1.6, we have the weak convergence

$$\tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}; \mathrm{e}^{\sqrt{2}A}\right) \xrightarrow[A\to\infty]{d} \tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}\right).$$

Together with Lemmas 3.8 and 3.9, this implies Theorem 3.7 by a standard argument; see for example [14, Lemma 6.7]. □

The proof of Theorem 1.8 can be completed as follows. Let $\sigma > 1/\sqrt{2}$. By Lemma 2.2, we have

$$\lim_{N\to\infty} N\mathrm{e}^{-\beta\sqrt{n}b_N}\mathbb{E}[\mathrm{e}^{\beta\sqrt{n}X}1_{X<b_N}] = \begin{cases} \frac{\sqrt{2}}{\beta-\sqrt{2}}, & \text{if } |\sigma|+|\tau|>\sqrt{2},\\ \infty, & \text{if } |\sigma|+|\tau|\le\sqrt{2}.\end{cases}$$

The first equality holds uniformly on compact subsets of B_2. By Theorem 3.7, the process $\mathrm{e}^{-\beta\sqrt{n}b_N}\mathcal{Z}_N(\beta)$ converges to $\zeta_P(\beta/\sqrt{2})$ weakly on the space of continuous functions on the set $B_2\cap\{\sigma>1/\sqrt{2}\}$. By Lemma 3.3, this implies Theorem 1.8. □

3.6. **Proof of Proposition 1.11.** Let $\tau\neq 0$ be fixed. Let $S(\beta)$ be a random variable defined as in the proof of Theorem 1.6. Take $a,b\in\mathbb{R}$. For $\sigma>1/2$ consider a random variable

$$Y(\sigma) = a\operatorname{Re} S(\beta) + b\operatorname{Im} S(\beta) = \lim_{k\to\infty}\left(\sum_{j=1}^{\infty} f(P_j;\sigma)\mathbb{1}_{1\le P_j<k} - \int_1^k f(t;\sigma)\mathrm{d}t\right),$$

where $f(t;\sigma)=\sqrt{a^2+b^2}\,t^{-\sigma}\cos(\tau\log t-\theta)$ and $\theta\in\mathbb{R}$ is such that $\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$.

We need to show that $\sqrt{2\sigma-1}\,Y(\sigma)$ converges, as $\sigma\downarrow 1/2$, to a centered real Gaussian distribution with variance $(a^2+b^2)/2$. By the properties of the Poisson process, the log-characteristic function of $Y(\sigma)$ is given by

$$\log\mathbb{E}\mathrm{e}^{izY(\sigma)} = \int_1^\infty\left(\mathrm{e}^{izf(t;\sigma)}-1-izf(t;\sigma)+\frac{z^2}{2}f^2(t;\sigma)\right)\mathrm{d}t - \frac{z^2}{2}\int_1^\infty f^2(t;\sigma)\mathrm{d}t.$$

We will compute the second term and show that the first term is negligible. By elementary integration we have

$$\int_1^\infty f^2(t;\sigma)\mathrm{d}t = \frac{a^2+b^2}{2}\int_1^\infty \frac{1+\cos(2\tau\log t-2\theta)}{t^{2\sigma}}\mathrm{d}t = \frac{a^2+b^2}{2}\left(\frac{1}{2\sigma-1}-\operatorname{Re}\frac{\mathrm{e}^{-2\theta i}}{(1-2\sigma)+2i\tau}\right). \tag{3.13}$$

Using the inequalities $|\mathrm{e}^{ix}-1-ix+\frac{x^2}{2}|\le|x|^3$ and $|f(t;\sigma)|<Ct^{-\sigma}$ we obtain

$$\left|\int_1^\infty\left(\mathrm{e}^{izf(t;\sigma)}-1-izf(t;\sigma)+\frac{z^2}{2}f^2(t;\sigma)\right)\mathrm{d}t\right| \le \frac{C}{3\sigma-1}|z|^3. \tag{3.14}$$

Bringing (3.13) and (3.14) together and recalling that $\tau\neq 0$ we arrive at

$$\lim_{\sigma\downarrow 1/2}\log\mathbb{E}\mathrm{e}^{i\sqrt{2\sigma-1}\,zY(\sigma)} = -\frac{1}{4}(a^2+b^2)z^2. \tag{3.15}$$

This proves the result for $\tau\neq 0$. For $\tau=0$, the limit is (3.15) is $-a^2z^2/2$.

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