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A CLASS OF ASYMPTOTICALLY SELF-SIMILAR STABLE PROCESSES WITH STATIONARY INCREMENTS

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Abstract. We generalize the BM-local time fractional symmetric α-stable motion introduced by Cohen and Samorodnitksy by replacing the local time with a general continuous additive functional (CAF). We show that the resulting process is symmetric α-stable and has stationary increments. Depending on the CAF considered, the process is either self-similar or asymptotically self-similar, lying in the domain of attraction of the BM-local time fractional symmetric α-stable motion. We also show that the process arises as a weak limit of a discrete "random rewards scheme" similar to the one described by Cohen and Samorodnitsky.

1. Introduction

α-stable self-similar processes with stationary increments (or α-stable sssi processes, for short) are attractive theoretical models for various natural phenomena exhibiting both heavy-tailed marginal distributions and invariant statistical behavior under suitable scaling. Recall that a stochastic process $\{X(t), t \geq 0\}$ is called α-stable if its finite-dimensional distributions are multivariate α-stable, and self-similar with index $H > 0$ if

$$\{X(ct), t \geq 0\} \stackrel{d}{=} \{c^H X(t), t \geq 0\}$$

for any $c > 0$. Stationary increments simply means that

$$\{X(t+c) - X(t), t \geq 0\} \stackrel{d}{=} \{X(t) - X(0), t \geq 0\}$$

for any $c > 0$. There is an extensive literature on α-stable sssi processes; we refer the reader to Samorodnitsky and Taqqu (1994), Chapter 7 and Embrechts and Maejima (2002), Chapter 3, for introductory expositions and references.

In the Gaussian case (α=2), fractional Brownian motions and their constant multiples are known to be the only non-trivial sssi processes; see, for example, §7.2 of Samorodnitsky and Taqqu (1994). More precisely, for any given index of self-similarity $H \in (0,1)$, there is a unique Gaussian H-sssi process (up to multiplicative constants), namely the fractional Brownian motion with index H. (There are no non-degenerate Gaussian H-sssi processes with $H \geq 1$.)

In contrast, when $0 < \alpha < 2$, there are typically many different α-stable H-sssi processes for any given feasible pair of indices (α, H). The feasible range of the pair (α, H) is given by

$$\begin{cases} 0 < H \leq 1/\alpha & \text{if } 0 < \alpha \leq 1, \\ 0 < H < 1 & \text{if } 1 < \alpha < 2. \end{cases}$$

Classification and understanding of α-stable sssi processes for $0 < \alpha < 2$ is an ongoing and fruitful project. Well-known examples of such processes include α-stable

Lévy motions, linear fractional stable motions introduced in Taqqu and Wolpert (1983) and Maejima (1983), and real harmonizable fractional stable motions introduced in Cambanis and Maejima (1989). The latter two classes of processes are defined for $0 < \alpha \leq 2, 0 < H < 1$, and both reduce to the fractional Brownian motion in the case $\alpha = 2$. The α-stable Lévy motion is defined for $0 < \alpha \leq 2, H = 1/\alpha$, and reduces to the Brownian motion when $\alpha = 2$.

In Cohen and Samorodnitsky (2006), the authors constructed a new class of symmetric α-stable (SαS) sssi processes. (The focus on *symmetric* α-stable distributions was for simplicity only, and we will adopt the same convention in this paper.) The construction is based on the local time process of a fractional Brownian motion with index of self-similarity H, so the authors called their model the *FBM-H-local time fractional stable motion*. They also showed that, in the case $H = 1/2$, this model arises naturally as a limiting process in a situation where many "users" perform independent symmetric random walks on distinct copies of the integer line and collect i.i.d. heavy-tailed random "rewards" associated with the integers that they visit. As the number of users increases, the properly normalized and time-scaled total reward process of all users converges weakly to the FBM-1/2-local time fractional stable motion, which can also be called the BM-local time fractional stable motion. The Brownian local time appearing in the limiting model can be regarded heuristically as a replacement for the local times of the random walks.

This paper extends the construction of Cohen and Samorodnitsky (2006) in the case $H = 1/2$, by considering a general *continuous additive functional* of Brownian motion instead of the Brownian local time. Following the authors' terminology, this model can be called the *BM-CAF fractional stable motion*, where CAF stands for continuous additive functional. CAFs of Brownian motion can be thought of as generalizations of the local time concept, since they include the local time as a special case. In fact, every Brownian CAF is a unique mixture of local times at different levels along $\mathbb{R}$, in a sense that will be made precise. This suggests that the BM-CAF fractional stable motion will be similar in structure to the BM-local time fractional motion, and in particular, it will be a natural approximating model for a generalized version of the random rewards scheme described in Cohen and Samorodnitsky (2006). Our aim is to show that this is indeed the case. We will formally introduce the BM-CAF fractional stable motion, explore its similarities and differences with the BM-local time stable motion, and prove that it is a limiting model in a situation where many independent users collect *moving averages* of i.i.d. heavy-tailed random rewards associated with the nodes around them.

In Section 2, we briefly discuss the construction of the FBM-H-local time fractional SαS motion and describe the random rewards scheme converging to it. Section 3 gives some preliminary information on Brownian continuous additive functionals, including a fundamental representation theorem which states that each Brownian continuous additive functional can be associated with a unique Radon measure on $\mathbb{R}$. The BM-CAF fractional SαS motion is formally defined in Section 4 for a large class of associated Radon measures; the conditions on the associated measures are stronger in the case $\alpha \in (0, 1]$ than in the case $\alpha \in (1, 2]$. In Section 5 we show that the BM-CAF fractional SαS motion has stationary increments, and in Section 6, we turn to the question of self-similarity. It turns out that, unlike the BM-local time fractional SαS motion, the BM-CAF fractional SαS motion is not always self-similar: under certain assumptions on the associated measure, the latter process will lie in the domain of attraction of the first one, in a sense that will be made precise. In Section 7 we study the smoothness of sample paths through their Hölder continuity properties. In Section 8 we state and prove a random rewards convergence result similar to the one presented in Cohen and Samorodnitsky (2006), once again under certain limitations on the associated measure. Finally, in Section 9 we study a special class of BM-CAF fractional SαS motions for which the associated measure takes on a particular form, and we use it to demonstrate that some of the sufficient conditions introduced in earlier results are merely sufficient and not necessary.

2. The FBM-H-Local Time Fractional Stable Motion

Let $\{B_H(t), t \geq 0\}$ be a fractional Brownian motion with index of self-similarity H, defined on a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, and let $\{l(x,t), x \in \mathbb{R}, t \geq 0\}$ be its jointly continuous local time process

$$l(x,t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}\left\{B_H(s) \in (x-\epsilon, x+\epsilon)\right\} ds,$$

which is known to exist as an almost sure limit (see, for example, Berman (1970)). Also let M be an independently scattered SαS random measure on the space $\Omega' \times \mathbb{R}$ with control measure $\mathbf{P}' \times$ Leb, where Leb denotes the Lebesgue measure on $\mathbb{R}$. M is assumed to be defined on another probability space $(\Omega, \mathcal{F}, \mathbf{P})$. (See Chapter 3 of Samorodnitsky and Taqqu (1994) for information on integrals with respect to stable random measures.) The FBM-H-local time fractional stable motion $\{\Gamma(t), t \geq 0\}$ is then defined as

$$\Gamma(t) = \int_{\Omega' \times \mathbb{R}} l(x,t) M(d\omega', dx), \quad t \geq 0. \tag{1}$$

This is a SαS H'-sssi process with index of self-similarity $H' = 1 - H + H/\alpha$. For $H = 1/2$, it arises as a weak limit of the following discrete scheme.

Let $\{W_k, k \in \mathbb{Z}\}$ be a sequence of i.i.d. symmetric random variables satisfying $P(W_0 > x) \sim cx^{-\alpha}$ as $x \to \infty$, for some $c > 0$ and $0 < \alpha < 2$. Also let $\{V_1, V_2, \ldots\}$ be a sequence of i.i.d. integer valued random variables having zero mean and unit variance, independent of $\{W_k, k \in \mathbb{Z}\}$. Consider the random walk $S_n = V_1 + \ldots + V_n, n \geq 1$. If one views S_n as describing the "position" of a "user" along the integer line at time n, and W_k as a "reward" associated with position k that will be collected whenever k is visited, then the total reward earned by time n will be

$$R_n = \sum_{j=1}^{n} W_{S_j}, \quad n \geq 1. \tag{2}$$

Assuming that there are many such users performing independent random walks and earning independent rewards, the properly normalized and time-scaled total reward process of all users will converge weakly to the BM-local time fractional stable motion as the number of users increases. A heuristic explanation for this result can be obtained by rewriting (2) as

$$R_n = \sum_{k=\infty}^{\infty} \varphi(k,n) W_k, \quad n \geq 1, \tag{3}$$

where $\varphi(k,n) = \sum_{j=1}^{n} \mathbf{1}\{S_k = j\}$ is the local time of the random walk $\{S_n, n \geq 1\}$. Comparing (1) and (3), one observes that the limiting procedure turns the sum into an integral, the local time of the random walk into that of a Brownian motion, and the heavy-tailed random rewards into a SαS random measure.

The construction of the FBM-H-local time fractional stable motion was likely motivated by a very similar process introduced in Kesten and Spitzer (1979), henceforth called the *Kesten-Spitzer process.* It is defined as

$$\Delta(t) = \int_{\mathbb{R}} l(x,t) M(dx), \quad t \geq 0, \tag{4}$$

where $\{l(x,t), x \in \mathbb{R}, t \geq 0\}$ is the local time of a Brownian motion as before, and M is a SαS random measure defined on $\mathbb{R}$ with Lebesgue control measure, assumed to be independent of the Brownian motion. This is a sssi process that can be seen as a mixture of stable processes; it is *not* a stable process. The random rewards scheme of Cohen and Samorodnitsky (2006) yields the Kesten-Spitzer process in the limit if one considers the total reward of a single user rather than that of many users. (Kesten and Spitzer called the random rewards scheme with a single user a "random walk in a random environment.") Once again, heuristic support for this convergence is provided by the similarity of (3) and (4).

Cohen and Dombry (2009) generalized the convergence result of Cohen and Samorodnitsky (2006) to $H \neq 1/2$, by considering random walks with dependent steps. More precisely, they assumed that each user performs a random walk $S_n = [V_1 + \ldots + V_n], n \geq 1$, where $[\cdot]$ denotes the usual "floor" function and the sequence of steps $\{V_1, V_2, \ldots\}$ forms a stationary Gaussian sequence satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{n} E(V_i V_j) \sim n^{2H} \text{ as } n \to \infty$$

for some $0 < H < 1$. The properly normalized and time-scaled cumulative reward process of all users then converges weakly to the FBM-H-local time fractional stable motion as the number of users increases.

Dombry and Guillotin-Plantard (2009) replaced the fractional Brownian local time $l(x,t)$ in (1) by the local time of a β-stable Lévy motion with $\beta \in (1,2]$, while still assuming M to be a SαS random measure, $0 < \alpha < 2$, independent of the Lévy motion. They showed that the resulting process is again α-stable sssi, and that the random rewards scheme of Cohen and Samorodnitsky (2006) yields their process in the limit if one allows the i.i.d. steps $\{V_1, V_2, \ldots\}$ to be in the domain of attraction of a β-stable law, rather than having unit variance. Following the terminology of Cohen and Samorodnitsky, they named their process the "β-stable Lévy motion local time fractional α-stable motion."

Our aim is to generalize the construction (1) in the case $H = 1/2$ by replacing the integrand local time $l(x,t)$ by a general continuous additive functional of Brownian motion, study the resulting process, and in particular construct a modified version of the random rewards scheme of Cohen and Samorodnitsky (2006) that yields the generalized process in the scaling limit. We start by reviewing some preliminaries on Brownian continuous additive functionals.

3. Preliminaries on Brownian Continuous Additive Functionals

Let $B = \{B(t), t \geq 0\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A *continuous additive functional* of B is a real-valued process $A = \{A(t), t \geq 0\}$ such that

(i) A is adapted to the natural filtration of B,

(ii) A is a.s. continuous, non-decreasing and vanishing at zero,

(iii) for each pair (s, t), $A(s+t) = A(t) + A(s) \circ \theta_t$ a.s.,

where $\theta_t : \Omega \to \Omega$ is the right-shift operator satisfying $B(s, \theta_t(\omega)) = B(t+s, \omega)$ for each $\omega \in \Omega$.

Clearly, the local time $\{l(x,t), t \geq 0\}$ at any point $x \in \mathbb{R}$ is a continuous additive functional, and so is the occupation time of any Borel set Γ,

$$A(t) = \int_0^t \mathbf{1}_\Gamma(B(s))ds.$$

For any continuous additive functional A of Brownian motion, there exists a unique Radon measure ν_A on $\mathbb{R}$ (called the *measure associated with* A) such that

$$\{A(t), t \geq 0\} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} l(y,t)\nu_A(dy), t \geq 0 \right\}. \tag{5}$$

Conversely, any Radon measure ν_A on $\mathbb{R}$ defines a continuous additive functional A through (5). In view of this result, we see that the local time at $x \in \mathbb{R}$ is the continuous additive functional with associated measure δ_x, the Dirac point measure of mass 1 concentrated at x. Similarly, the associated measure of the occupation time of a Borel set Γ is the restriction of the Lebesgue measure on Γ. These results and more on Brownian continuous additive functionals can be found in Chapter X of Revuz and Yor (1999).

In order to replace the local time l in (1) with a continuous additive functional A, we need to introduce dependence on a space variable x for A. We do that in the "obvious" way, by defining

$$A(x,t) = \int_{\mathbb{R}} l(x+y,t)\nu_A(dy), \quad x \in \mathbb{R}, t \geq 0, \tag{6}$$

i.e. we define $A(x,t)$ to be the value of $A(t)$ for the vertically shifted Brownian motion $\{B(t) - x, t \geq 0\}$.

4. The BM-CAF Fractional Stable Motion

Let $(\Omega', \mathcal{F}', \mathbf{P}')$ be a probability space supporting a Brownian motion $\{B(t), t \geq 0\}$ with local time process $\{l(x,t), x \in \mathbb{R}, t \geq 0\}$, and let $\{A(x,t), t \geq 0\}$ be an arbitrary continuous additive functional (CAF) of B with associated measure ν_A. Let M be a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $\mathbf{P}' \times$Leb. Suppose M itself lives on some other probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We define the BM-CAF fractional stable motion by

$$\begin{aligned} Y(t) &= \int_{\Omega' \times \mathbb{R}} A(x,t)M(d\omega', dx) \\ &= \int_{\Omega' \times \mathbb{R}} \int_{\mathbb{R}} l(x+y,t)\nu_A(dy)M(d\omega', dx), \quad t \geq 0. \end{aligned} \tag{7}$$

The first issue that needs to be addressed is that of well-definedness. The following two results identify sufficient conditions on the measure ν_A under which the process

(7) is a well-defined SαS process. The conditions are more restrictive in the case $0 < \alpha \le 1$ than in the case $1 < \alpha \le 2$.

Theorem 4.1. *Suppose* $0 < \alpha \le 1$ *and* ν_A *satisfies*

$$\sum_{i=0}^{\infty} \beta^{(1-\alpha)i} \nu_A\big([\beta^i, \beta^{i+1})\big)^\alpha + \sum_{i=0}^{\infty} \beta^{(1-\alpha)i} \nu_A\big([-\beta^{i+1}, -\beta^i)\big)^\alpha < \infty \tag{8}$$

for some constant $\beta > 1$*. Then,* $\{Y(t), t \ge 0\}$ *in (7) is a well-defined SαS process.*

Remark 4.2. The value of β in (8) does not matter, in the sense that (8) holds for all $\beta > 1$ if it holds for one. Indeed, given $\gamma = \beta^c$ for some $c > 0$,

$$\begin{aligned}
\sum_{i=0}^{\infty} \gamma^{(1-\alpha)i} \nu_A\big([\gamma^i, \gamma^{i+1})\big)^\alpha &= \sum_{i=0}^{\infty} \beta^{c(1-\alpha)i} \nu_A\big([\beta^{ci}, \beta^{ci+c})\big)^\alpha \\
&\le \sum_{i=0}^{\infty} \beta^{(1-\alpha)([ci]+1)} \nu_A\big([\beta^{[ci]}, \beta^{[ci]+[c]+2})\big)^\alpha \\
&\le \mathit{const} \sum_{i=0}^{\infty} \beta^{(1-\alpha)([ci]+1)} \sum_{j=0}^{[c]+1} \nu_A\big([\beta^{[ci]+j}, \beta^{[ci]+j+1})\big)^\alpha \\
&\le \mathit{const} \sum_{j=0}^{[c]+1} \sum_{i=0}^{\infty} \beta^{(1-\alpha)([ci]+j)} \nu_A\big([\beta^{[ci]+j}, \beta^{[ci]+j+1})\big)^\alpha \\
&\le \mathit{const} \sum_{i=0}^{\infty} \beta^{(1-\alpha)i} \nu_A\big([\beta^i, \beta^{i+1})\big)^\alpha.
\end{aligned}$$

Proof of Theorem 4.1. We need to check that, for any fixed $t \ge 0$,

$$\mathbf{E}'\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} l(x+y, t)\, \nu_A(dy)\right)^\alpha dx\right) < \infty\,. \tag{9}$$

It will suffice to prove

$$\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}\{|x+y| \le M(t)\}\, \nu_A(dy)\right)^\alpha dx\right) < \infty\,, \tag{10}$$

with

$$l_*(t) = \sup_{x \in \mathbb{R}} l(x, t), \tag{11}$$

$$M(t) = \sup_{0 \le s \le t} |B(s)|. \tag{12}$$

It is known that for any fixed $t \ge 0$, $l_*(t)$ has finite moments of all orders; see, for example, Theorem 1.7 of Borodin (1986). It is also known that $M(t)$ has Gaussian-like probability tails, or more precisely,

$$\mathbf{P}'\left(M(t) \ge x\right) \le \mathit{const} \int_{x/\sqrt{2t}}^{\infty} e^{-u^2} du \le \mathit{const}\, e^{-x^2/2t}$$

for $x > 0$; see, for example, §10.2 of Ross (2006). In particular, for any fixed $t \ge 0$, $M(t)$ has finite moments of all orders as well. In the following, we will make frequent use of these facts without explicitly mentioning them each time. Let us denote $I(x,y) = \mathbf{1}\{|x+y| \le M(t)\}$ for notational convenience. We first prove

$$\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_1^\infty I(x,y)\,\nu_A(dy)\right)^\alpha dx\right) < \infty\,. \tag{13}$$

The left hand side of (13) can be decomposed as

$$\begin{aligned} I_1 + I_2 := \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(x,y)\nu_A(dy)\right)^\alpha dx\right) \\ + \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(-x,y)\nu_A(dy)\right)^\alpha dx\right). \end{aligned}$$

Defining $B_i = [\beta^i, \beta^{i+1})$, we have

$$\begin{aligned} I_1 &= \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\sum_{i=0}^\infty \int_{B_i} I(x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &\le \mathbf{E}'\left(l_*(t)^\alpha \left(\sum_{i=0}^\infty \nu_A(B_i)\right)^\alpha \int_0^\infty \mathbf{1}\{x \le M(t)\}\,dx\right) \qquad (14) \\ &\le \sum_{i=0}^\infty \nu_A(B_i)^\alpha\, \mathbf{E}'\left(l_*(t)^\alpha\, M(t)\right) < \infty\,, \end{aligned}$$

where the finiteness follows from Cauchy-Schwarz inequality and (8). The term I_2 can be further decomposed as

$$\begin{aligned} I_2 &= \mathbf{E}'\left(l_*(t)^\alpha \int_0^{\beta^2} \left(\int_1^\infty I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &\quad + \mathbf{E}'\left(l_*(t)^\alpha \int_{\beta^2}^\infty \left(\int_1^\infty I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &:= I_{21} + I_{22}\,. \end{aligned}$$

Note that

$$\begin{aligned} I_{21} &= \mathbf{E}'\left(l_*(t)^\alpha \int_0^{\beta^2} \left(\sum_{i=0}^\infty \int_{B_i} I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &\le \mathbf{E}'\left(l_*(t)^\alpha \int_0^{\beta^2} \left(\sum_{i=0}^\infty \nu_A(B_i)\right)^\alpha dx\right) \qquad (15) \\ &= \beta^2 \left(\sum_{i=0}^\infty \nu_A(B_i)^\alpha\right) \mathbf{E}'(l_*(t)^\alpha) < \infty \end{aligned}$$

by (8), so it remains to prove that $I_{22} < \infty$. We write

$$I_{22} = \sum_{j=2}^\infty \mathbf{E}'\left(l_*(t)^\alpha \int_{B_j} \left(\sum_{i=0}^\infty \int_{B_i} I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) := \sum_{j=2}^\infty r_j\,.$$

For $j \ge 2$,

$$r_j \le \mathbf{E}'\left(l_*(t)^\alpha \int_{B_j} \left(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right)$$

$$
\begin{aligned}
&+\ \mathbf{E}'\Big(l_*(t)^\alpha \int_{B_j} \Big(\sum_{i=j-1}^{j+1} \int_{B_i} I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\
&+\ \mathbf{E}'\Big(l_*(t)^\alpha \int_{B_j} \Big(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\
:= {}& r_j^{(1)} + r_j^{(2)} + r_j^{(3)}\,.
\end{aligned}
$$

We will show that

$$
\sum_{j=2}^{\infty} r_j^{(k)} < \infty \quad \text{for } k = 1,2,3. \tag{16}
$$

Let $Z_j(t) = \mathbf{1}\{\beta^j - \beta^{j-1} \le M(t)\}$ and note that

$$
\begin{aligned}
r_j^{(1)} &= \mathbf{E}'\Big(Z_j(t)\, l_*(t)^\alpha \int_{B_j} \Big(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\
&\le \int_{B_j} \sum_{i=0}^{j-2} \nu_A(B_i)^\alpha\, dx\ \mathbf{E}'\left(Z_j(t)\, l_*(t)^\alpha\right) \\
&\le (\beta^{j+1} - \beta^j)\mathbf{E}'(Z_j(t)\, l_*(t)^\alpha) \sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \\
&\le c_1(\beta^{j+1} - \beta^j)\mathbf{E}'(l_*(t)^{2\alpha})^{1/2}\mathbf{P}'\left(\beta^j - \beta^{j-1} \le M(t)\right)^{1/2} \\
&= c_1\beta^j\, \mathbf{P}'\left(c_2\beta^{j-1} \le M(t)\right)^{1/2} \\
&\le c_1\beta^j \exp\left(-c_2\beta^{2(j-1)}\right),
\end{aligned}
$$

where c_1 and c_2 are positive constants that may change from instance to instance and may depend on t. Since the last expression is summable over j, (16) is true for $k = 1$. Next, note that

$$
\begin{aligned}
\sum_{j=2}^{\infty} r_j^{(2)} &\le \sum_{j=2}^{\infty} \mathbf{E}'\Big(l_*(t)^\alpha \int_{B_j} \sum_{i=j-1}^{j+1} \Big(\int_{B_i} I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\
&\le \sum_{j=2}^{\infty} (\beta^{j+1} - \beta^j)^{1-\alpha}\mathbf{E}'\Big(l_*(t)^\alpha \sum_{i=j-1}^{j+1} \Big(\int_{B_j}\int_{B_i} I(-x,y)\,\nu_A(dy)\,dx\Big)^\alpha\Big) \\
&\le \sum_{j=2}^{\infty} (\beta^{j+1} - \beta^j)^{1-\alpha}\mathbf{E}'\Big(l_*(t)^\alpha \sum_{i=j-1}^{j+1} \Big(\int_{B_i}\int_{\mathbb{R}} I(-x,y)\,dx\,\nu_A(dy)\Big)^\alpha\Big) \\
&\le \sum_{j=2}^{\infty} (\beta^{j+1} - \beta^j)^{1-\alpha}\mathbf{E}'(2l_*(t)^\alpha M(t)^\alpha) \sum_{i=j-1}^{j+1} \nu_A(B_i)^\alpha \\
&\le \mathit{const}\ \mathbf{E}'\left(2l_*(t)^\alpha M(t)^\alpha\right) \sum_{j=0}^{\infty} \beta^{j(1-\alpha)}\, \nu_A(B_j)^\alpha,
\end{aligned}
$$

where the expectation is finite by Cauchy-Schwarz inequality and the sum over j is finite by (8). Thus (16) is established for $k = 2$ as well. Finally,

$$\begin{aligned} r_j^{(3)} &= \mathbf{E}'\Big(Z_{j+2}(t)\, l_*(t)^\alpha \int_{B_j} \Big(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y)\, \nu_A(dy)\Big)^\alpha dx\Big) \\ &\le \int_{B_j} \sum_{i=j+2}^{\infty} \nu_A(B_i)^\alpha\, dx\, \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\right) \\ &\le (\beta^{j+1}-\beta^j)\mathbf{E}'(Z_{j+2}(t)\, l_*(t)^\alpha) \sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \\ &\le c_1(\beta^{j+1}-\beta^j)\mathbf{E}'(l_*(t)^{2\alpha})^{1/2}\mathbf{P}'\left(\beta^{j+2}-\beta^{j+1} \le M(t)\right)^{1/2} \\ &= c_1\beta^j\, \mathbf{P}'\left(c_2\beta^j \le M(t)\right)^{1/2} \\ &\le c_1\beta^j \exp\big(-c_2\beta^{2j}\big)\,, \end{aligned}$$

where c_1 and c_2 are positive constants that may change from instance to instance and may depend on t. Since the last expression is summable over j, (16) is true for $k = 3$. It now follows that $I_{22} < \infty$, and combined with (15), this yields $I_2 < \infty$. Thus we have shown (13).

Now, it can be shown by analogous arguments that

$$\mathbf{E}'\Big(l_*(t)^\alpha \int_{\mathbb{R}} \Big(\int_{-\infty}^{-1} I(x,y)\, \nu_A(dy)\Big)^\alpha dx\Big) < \infty\,, \tag{17}$$

and also note that

$$\begin{aligned} &\mathbf{E}'\Big(l_*(t)^\alpha \int_{\mathbb{R}} \Big(\int_{-1}^{1} I(x,y)\, \nu_A(dy)\Big)^\alpha dx\Big) \\ &\qquad \le \mathbf{E}'\Big(l_*(t)^\alpha \int_{-M(t)-1}^{M(t)+1} \nu_A\left([-1,1)\right)^\alpha dx\Big) \\ &\qquad = \nu_A\left([-1,1)\right)^\alpha \mathbf{E}'\left(2l_*(t)^\alpha(M(t)+1)\right) < \infty. \end{aligned} \tag{18}$$

Combining (13), (17) and (18) yields (10), and well-definedness follows. ☐

Theorem 4.3. *Suppose $1 < \alpha \le 2$ and ν_A satisfies*

$$\sum_{i=0}^{\infty} \nu_A\big([\beta^i, \beta^{i+1})\big)^\alpha + \sum_{i=0}^{\infty} \nu_A\big([-\beta^{i+1}, -\beta^i)\big)^\alpha < \infty \tag{19}$$

for some constant $\beta > 1$. Then, $(Y(t), t \ge 0)$ in (7) is a well-defined $S\alpha S$ process.

Remark 4.4. As in condition (8) of Theorem 4.1, condition (19) holds for all $\beta > 1$ if it holds for one. We omit the proof.

Proof of Theorem 4.3. The proof is similar to that of Theorem 4.1. Using the same notation as in that proof, it will suffice to prove that

$$\mathbf{E}'\Big(l_*(t)^\alpha \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} I(x,y)\, \nu_A(dy)\Big)^\alpha dx\Big) < \infty\,, \tag{20}$$

and the first step is to show that

$$\mathbf{E}'\Big(l_*(t)^\alpha \int_{\mathbb{R}} \Big(\int_{1}^{\infty} I(x,y)\, \nu_A(dy)\Big)^\alpha dx\Big) < \infty\,. \tag{21}$$

The left hand side of (21) can be decomposed as

$$I_1 + I_2 := \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(x,y)\nu_A(dy)\right)^\alpha dx\right) + \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(-x,y)\nu_A(dy)\right)^\alpha dx\right).$$

Defining $B_i = [\beta^i, \beta^{i+1})$ and $N(t) = \max\{i \geq 0 : \beta^i \leq M(t)\}$, we have

$$\begin{aligned} I_1 &= \mathbf{E}'\Big(l_*(t)^\alpha \int_0^\infty \Big(\sum_{i=0}^{N(t)} \int_{B_i} I(x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\ &\leq \mathbf{E}'\Big(l_*(t)^\alpha \Big(\sum_{i=0}^{N(t)} \nu_A(B_i)\Big)^\alpha \int_0^\infty \mathbf{1}\{x \leq M(t)\}\,dx\Big) \\ &\leq \mathbf{E}'\Big(l_*(t)^\alpha\, N(t)^{\alpha-1} \sum_{i=0}^{N(t)} \nu_A(B_i)^\alpha \int_0^\infty \mathbf{1}\{x \leq M(t)\}\,dx\Big) \\ &\leq \Big(\sum_{i=0}^{\infty} \nu_A(B_i)^\alpha\Big)\, \mathbf{E}'\left(l_*(t)^\alpha\, N(t)^{\alpha-1} M(t)\right) < \infty\,, \end{aligned} \tag{22}$$

since the sum over $i \geq 0$ is finite by (19) and the random variables $l_*(t), M(t)$ and $N(t)$ have all moments finite. (Finite moments for $N(t)$ are implied by the fact that $N(t) \leq \log M(t)/\log\beta$.)

The term I_2 can be further decomposed as

$$\begin{aligned} I_2 &= \mathbf{E}'\Big(l_*(t)^\alpha \int_0^{\beta^2} \Big(\int_1^\infty I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\ &\quad + \mathbf{E}'\Big(l_*(t)^\alpha \int_{\beta^2}^\infty \Big(\int_1^\infty I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\ &:= I_{21} + I_{22}\,. \end{aligned}$$

Defining $\widetilde{N}(t) = \max\{i \geq 0 : \beta^i \leq M(t) + \beta^2\}$, we see that

$$\begin{aligned} I_{21} &\leq \mathbf{E}'\Big(l_*(t)^\alpha \int_0^{\beta^2} \Big(\sum_{i=0}^{\widetilde{N}(t)} \int_{B_i} I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) \\ &\leq \mathbf{E}'\Big(l_*(t)^\alpha \Big(\sum_{i=0}^{\widetilde{N}(t)} \nu_A(B_i)\Big)^\alpha\Big) \int_0^{\beta^2} 1\,dx \\ &\leq \beta^2\, \mathbf{E}'\Big(l_*(t)^\alpha\, \widetilde{N}(t)^{\alpha-1} \sum_{i=0}^{\widetilde{N}(t)} \nu_A(B_i)^\alpha\Big) \\ &\leq \beta^2 \Big(\sum_{i=0}^{\infty} \nu_A(B_i)^\alpha\Big)\, \mathbf{E}'\big(l_*(t)^\alpha\, \widetilde{N}(t)^{\alpha-1}\big) < \infty \end{aligned} \tag{23}$$

as before, so it remains to prove that $I_{22} < \infty$. We write

$$I_{22} = \sum_{j=2}^{\infty} \mathbf{E}'\Big(l_*(t)^\alpha \int_{B_j} \Big(\sum_{i=0}^{\infty} \int_{B_i} I(-x,y)\,\nu_A(dy)\Big)^\alpha dx\Big) := \sum_{j=2}^{\infty} r_j\,.$$

For $j \geq 2$,

$$
\begin{aligned}
r_j \leq 3^{\alpha-1}\Bigg[&\mathbf{E}'\bigg(l_*(t)^\alpha \int_{B_j} \bigg(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\,\nu_A(dy)\bigg)^\alpha dx\bigg) \\
&+ \mathbf{E}'\bigg(l_*(t)^\alpha \int_{B_j} \bigg(\sum_{i=j-1}^{j+1} \int_{B_i} I(-x,y)\,\nu_A(dy)\bigg)^\alpha dx\bigg) \\
&+ \mathbf{E}'\bigg(l_*(t)^\alpha \int_{B_j} \bigg(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y)\,\nu_A(dy)\bigg)^\alpha dx\bigg)\Bigg] \\
:= 3^{\alpha-1}&\left(r_j^{(1)} + r_j^{(2)} + r_j^{(3)}\right).
\end{aligned}
$$

We will show that

$$
\sum_{j=2}^{\infty} r_j^{(k)} < \infty \quad \text{for } k = 1,2,3. \tag{24}
$$

Let $Z_j(t) = \mathbf{1}\{\beta^j - \beta^{j-1} \leq M(t)\}$ and note that

$$
\begin{aligned}
r_j^{(1)} &= \mathbf{E}'\bigg(Z_j(t)\,l_*(t)^\alpha \int_{B_j} \bigg(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\,\nu_A(dy)\bigg)^\alpha dx\bigg) \\
&\leq (j-1)^{\alpha-1}\mathbf{E}'\bigg(Z_j(t)\,l_*(t)^\alpha \int_{B_j} \sum_{i=0}^{j-2} \nu_A(B_i)^{\alpha-1} \int_{B_i} I(-x,y)^\alpha \nu_A(dy)\,dx\bigg) \\
&\leq (j-1)^{\alpha-1}\mathbf{E}'\left(Z_j(t)\,l_*(t)^\alpha\right) \sum_{i=0}^{j-2} \nu_A(B_i)^\alpha \int_{B_j} 1\,dx \\
&\leq (j-1)^{\alpha-1}\left(\beta^{j+1} - \beta^j\right) \mathbf{E}'(Z_j(t)\,l_*(t)^\alpha) \sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \\
&\leq c_1 (j-1)^{\alpha-1}\left(\beta^{j+1} - \beta^j\right) \mathbf{E}'\left(l_*(t)^{2\alpha}\right)^{1/2} \mathbf{P}'\left(\beta^j - \beta^{j-1} \leq M(t)\right)^{1/2} \\
&= c_1 (j-1)^{\alpha-1} \beta^j\,\mathbf{P}'\left(c_2 \beta^{j-1} \leq M(t)\right)^{1/2} \\
&\leq c_1 (j-1)^{\alpha-1} \beta^j \exp\left(-c_2 \beta^{2(j-1)}\right),
\end{aligned}
$$

where c_1 and c_2 are positive constants that may change from instance to instance and may depend on t. Since the last expression is summable over j, (24) is true for $k = 1$. Next, note that

$$
\begin{aligned}
\sum_{j=2}^{\infty} r_j^{(2)} &\leq 3^{\alpha-1} \sum_{j=2}^{\infty} \mathbf{E}'\bigg(l_*(t)^\alpha \int_{B_j} \sum_{i=j-1}^{j+1} \bigg(\int_{B_i} I(-x,y)\,\nu_A(dy)\bigg)^\alpha dx\bigg) \\
&\leq 3^{\alpha-1} \sum_{j=2}^{\infty} \mathbf{E}'\bigg(l_*(t)^\alpha \int_{B_j} \sum_{i=j-1}^{j+1} \bigg(\nu_A(B_i)^{\alpha-1} \int_{B_i} I(-x,y)\,\nu_A(dy)\bigg) dx\bigg) \\
&\leq 3^{\alpha-1} \sum_{j=2}^{\infty} \mathbf{E}'\bigg(l_*(t)^\alpha \sum_{i=j-1}^{j+1} \bigg(\nu_A(B_i)^{\alpha-1} \int_{B_i} \int_{\mathbb{R}} I(-x,y)\,dx\,\nu_A(dy)\bigg)\bigg)
\end{aligned}
$$

$$= 3^{\alpha-1}\sum_{j=2}^{\infty} 2\mathbf{E}'(l_*(t)^\alpha M(t)) \sum_{i=j-1}^{j+1} \nu_A(B_i)^\alpha$$
$$\leq const\ \mathbf{E}'(l_*(t)^\alpha M(t)) \sum_{j=0}^{\infty} \nu_A(B_j)^\alpha,$$

where the last expression is finite by Cauchy-Schwarz inequality and by the assumption (19), so that (24) is true for $k = 2$ as well. It remains to prove (24) for $k = 3$. Letting

$$K(t) = \min\left\{i \geq 0 : \beta^{i+1} - \beta^i > M(t)\right\},$$

we see that

$$\begin{aligned}
r_j^{(3)} &= \mathbf{E}'\bigg(Z_{j+2}(t)\, l_*(t)^\alpha \int_{B_j} \bigg(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y)\, \nu_A(dy)\bigg)^\alpha dx\bigg) \\
&\leq \mathbf{E}'\bigg(Z_{j+2}(t)\, l_*(t)^\alpha\, K(t)^{\alpha-1} \int_{B_j} \sum_{i=j+2}^{\infty} \bigg(\int_{B_i} I(-x,y)\, \nu_A(dy)\bigg)^\alpha dx\bigg) \\
&\leq \mathbf{E}'\bigg(Z_{j+2}(t)\, l_*(t)^\alpha\, K(t)^{\alpha-1} \int_{B_j} \sum_{i=j+2}^{\infty} \bigg(\nu_A(B_i)^{\alpha-1} \int_{B_i} I(-x,y)\, \nu_A(dy)\bigg) dx\bigg) \\
&\leq \left(\beta^{j+1} - \beta^j\right) \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\, K(t)^{\alpha-1}\right) \sum_{i=j+2}^{\infty} \nu_A(B_i)^\alpha \\
&\leq const\, \beta^j\, \mathbf{E}'\big(l_*(t)^{2\alpha} K(t)^{2(\alpha-1)}\big)^{1/2} \mathbf{P}'\left(\beta^{j+2} - \beta^{j+1} \leq M(t)\right)^{1/2} \\
&= const\, \beta^j\, \mathbf{P}'\left(\beta^{j+2} - \beta^{j+1} \leq M(t)\right)^{1/2},
\end{aligned}$$

where we use the fact that $l_*(t)$ and $K(t)$ have all moments finite. The last expression is summable over j as before, so we conclude that (24) holds for $k = 3$. It follows that $I_{22} < \infty$, and combined with (23), this yields $I_2 < \infty$. Thus we obtain (21).

Now, it can be shown by analogous arguments that

$$\mathbf{E}'\bigg(l_*(t)^\alpha \int_{\mathbb{R}} \bigg(\int_{-\infty}^{-1} I(x,y)\, \nu_A(dy)\bigg)^\alpha dx\bigg) < \infty. \tag{25}$$

Moreover,

$$\begin{aligned}
&\mathbf{E}'\bigg(l_*(t)^\alpha \int_{\mathbb{R}} \bigg(\int_{-1}^{1} I(x,y)\, \nu_A(dy)\bigg)^\alpha dx\bigg) \\
&\leq \mathbf{E}'\bigg(l_*(t)^\alpha \int_{-M(t)-1}^{M(t)+1} \nu_A\left([-1,1)\right)^\alpha dx\bigg) \\
&= \nu_A\left([-1,1)\right)^\alpha \mathbf{E}'\left(2l_*(t)^\alpha (M(t)+1)\right) < \infty.
\end{aligned} \tag{26}$$

Combining (21), (25) and (26) yields (20), and well-definedness follows. □

5. Stationary Increments

We will need the following lemma, which is proved for $d = 1$ in the Appendix of Samorodnitsky (2010). The proof for $d > 1$ is analogous.

Lemma 5.1. *Let $(B(t), t \geq 0)$ be a Brownian motion defined on $(\Omega, \mathcal{F}, P)$, with local time process $(l(x,t), x \in \mathbb{R}, t \geq 0)$. Then for any $y_1, \ldots, y_d \in \mathbb{R}$, the law of*

$$((l(x+y_i, t+s) - l(x+y_i, s), i = 1, \ldots, d), x \in \mathbb{R}, t \geq 0)$$

under Leb $\times P$ does not depend on $s \geq 0$.

The following is the main result of this section.

Theorem 5.2. *Let $(Y(t), t \geq 0)$ be a BM-CAF fractional stable motion as defined in (7), with α and ν_A satisfying the hypotheses of Theorem 4.1 or Theorem 4.3. Then, $(Y(t), t \geq 0)$ has stationary increments.*

Proof. Let $\nu_1, \nu_2, \ldots$ be a sequence of discrete measures defined as follows:

$$\nu_n = \sum_{i=-n^2}^{n^2} \nu_A\left(\left[\frac{i}{n}, \frac{i+1}{n}\right)\right) \delta_{\frac{i}{n}} \,,$$

with δ_x denoting the Dirac point measure of mass 1 concentrated at x. Also let $\theta_1, \ldots, \theta_k \in \mathbb{R}, 0 \leq t_1 < \ldots < t_k$ and $s \geq 0$. We have

$$\mathbf{E} \exp\left(i \sum_{j=1}^{k} \theta_j \big(Y(t_j+s) - Y(s)\big)\right)$$

$$= \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y, s)\big)\, \nu_A(dy) \right|^{\alpha} dx \right) \tag{27}$$

$$= \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \lim_{n\to\infty} \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y, s)\big)\, \nu_n(dy) \right|^{\alpha} dx \right).$$

Now note that for each $n \geq 1$,

$$\left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y, s)\big)\, \nu_n(dy) \right|^{\alpha}$$

$$\leq \mathit{const} \left(\int_{\mathbb{R}} \big(l(x+y, t_k+s) - l(x+y, s)\big)\, \nu_n(dy) \right)^{\alpha}$$

$$\leq \mathit{const}\; l_*(t_k+s)^{\alpha} \left(\int_{\mathbb{R}} \mathbf{1}\{|x+y| \leq M(t_k+s)\}\, \nu_n(dy) \right)^{\alpha}$$

$$\leq \mathit{const}\; l_*(t_k+s)^{\alpha} \left(\int_{\mathbb{R}} \mathbf{1}\{|x+y| \leq M(t_k+s)+1\}\, \nu_A(dy) \right)^{\alpha},$$

where M is as defined in (12). The last expression is integrable with respect to Leb$\times\mathbf{P'}$; the proof is analogous to that of (10) or (20), depending on the value of α. Therefore we can apply the dominated convergence theorem to the last expression in (27) and conclude that it is the same as

$$\lim_{n\to\infty} \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y, s)\big)\, \nu_n(dy) \right|^{\alpha} dx \right),$$

which is in turn equal to

$$\lim_{n\to\infty} \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} l(x+y, t_j)\, \nu_n(dy) \right|^{\alpha} dx \right), \tag{28}$$

by Lemma 5.1. Another application of the dominated convergence theorem now allows us to move the limit in (28) back under the expectation and conclude that

$$\begin{aligned}
&\mathbf{E}\exp\left(i\sum_{j=1}^{k}\theta_j\big(Y(t_j+s)-Y(s)\big)\right)\\
&\quad=\exp\left(-\int_{\mathbb{R}}\mathbf{E}'\left|\sum_{j=1}^{k}\theta_j\int_{\mathbb{R}}l(x+y,t_j)\,\nu_A(dy)\right|^{\alpha}dx\right)\\
&\quad=\mathbf{E}\exp\left(i\sum_{j=1}^{k}\theta_j Y(t_j)\right),
\end{aligned}$$

which proves the theorem. □

6. Asymptotic Self-Similarity

A natural question to ask is whether the self-similarity of the BM-local time fractional stable motion carries over to the BM-CAF fractional stable motion. The following result identifies a class of BM-CAF fractional stable motions that are in the domain of attraction of the BM-local time fractional stable motion, in the sense that they converge to it in finite dimensional distributions under proper scaling of time and space. Thus the processes in this class are generally not self-similar, but they can be considered *asymptotically* self-similar.

Theorem 6.1. *Suppose $0<\alpha\le 2$ and ν_A satisfies*

$$\sum_{i=0}^{\infty}\beta^i\nu_A\big([\beta^i,\beta^{i+1})\big)^{\alpha}+\sum_{i=0}^{\infty}\beta^i\nu_A\big([-\beta^{i+1},-\beta^i)\big)^{\alpha}<\infty \tag{29}$$

for some constant $\beta>1$. Then, for $H=\frac{1}{2}+\frac{1}{2\alpha}$,

$$\left(\frac{1}{c^H}\,Y(ct),t\ge 0\right)\xrightarrow{f.d.}(|\nu_A|\,\Gamma(t),t\ge 0)\quad as\quad c\to\infty\,,$$

where $\overset{f.d.}{\to}$ denotes convergence in finite-dimensional distributions, $|\nu_A|=\nu_A(\mathbb{R})$ and $(\Gamma(t),t\ge 0)$ is the BM-local time fractional $S\alpha S$ motion defined in (1).

Proof. We will take advantage of the following scaling property of the Brownian local time, which follows immediately from the self-similarity of Brownian motion. For any $c>0$,

$$\big(l(\sqrt{c}x,ct),x\in\mathbb{R},t\ge 0\big)\overset{d}{=}\big(\sqrt{c}\,l(x,t),x\in\mathbb{R},t\ge 0\big)\,. \tag{30}$$

Now, let $0 \le t_1 < t_2 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$. By (30), we have for any $c > 0$

$$
\begin{aligned}
&\mathbf{E} \exp\Big(i \sum_{j=1}^m \theta_j \,\frac{1}{c^H} Y(ct_j) \Big) \\
&\quad = \exp\Big(- \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \frac{\theta_j}{c^H} \int_{\mathbb{R}} l(x+y, ct_j)\, \nu_A(dy) \Big|^\alpha dx \Big) \\
&\quad = \exp\Big(- c^{\alpha\left(\frac{1}{2}-H\right)} \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_{\mathbb{R}} l\Big(\frac{x+y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big|^\alpha dx \Big) \\
&\quad = \exp\Big(- \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_{\mathbb{R}} l\Big(x + \frac{y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big|^\alpha dx \Big).
\end{aligned} \tag{31}
$$

We want to take the limit of this expression as $c \to \infty$. We claim that

$$
\begin{aligned}
&\lim_{c\to\infty} \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_{\mathbb{R}} l\Big(x + \frac{y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big|^\alpha dx \\
&\quad = \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_{\mathbb{R}} \lim_{c\to\infty} l\Big(x + \frac{y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big|^\alpha dx \\
&\quad = \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_{\mathbb{R}} l(x, t_j)\, \nu_A(dy) \Big|^\alpha dx \\
&\quad = \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \,|\nu_A|\, l(x, t_j) \Big|^\alpha dx,
\end{aligned} \tag{32}
$$

so that

$$
\begin{aligned}
\lim_{c\to\infty} \mathbf{E} \exp\Big(i \sum_{j=1}^m \theta_j \,\frac{1}{c^H} Y(ct_j) \Big) &= \exp\Big(- \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \,|\nu_A|\, l(x, t_j) \Big|^\alpha dx \Big) \\
&= \mathbf{E} \exp\Big(i \sum_{j=1}^m \theta_j \;|\nu_A|\, \Gamma(t_j) \Big),
\end{aligned}
$$

which proves the theorem. The only step that requires justification is the first equality in (32), and we now prove it using Lebesgue's Dominated Convergence Theorem.

For the rest of the proof, we will assume that $0 < \alpha < 1$. The arguments for the case $1 \le \alpha \le 2$ will be identical, up to different constants in some bounds. Note that

$$
\begin{aligned}
&\mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_{\mathbb{R}} l\Big(x + \frac{y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big|^\alpha \\
&\quad \le \mathbf{E}' \Big(\int_{\mathbb{R}} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big)^\alpha \\
&\quad \le \mathbf{E}' \Big(\int_{-\infty}^{-\sqrt{c}} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j \Big)\, \nu_A(dy) \Big)^\alpha
\end{aligned}
$$

$$+ \mathbf{E}'\Big(\int_{-\sqrt{c}}^{\sqrt{c}} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^\alpha$$

$$+ \mathbf{E}'\Big(\int_{\sqrt{c}}^{\infty} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^\alpha$$

$$:= h_1(c,x) + h_2(c,x) + h_3(c,x).$$

We will show that each of these components is bounded uniformly over c by an integrable function of x, which will justify passing the limit through the outer integral in (32). Let $M(t) = \sup_{0\le s\le t} |B(s)|$ and $l_*(t) = \sup_{x\in\mathbb{R}} l(x,t)$ as before. Then,

$$\begin{aligned}
&h_2(c,x)\\
&\le \max\{|\theta_1|,\dots,|\theta_m|\}\, \mathbf{E}'\Big(\int_{-\sqrt{c}}^{\sqrt{c}} \sum_{j=1}^m l_*(t_j) \mathbf{1}\Big\{\Big|x + \frac{y}{\sqrt{c}}\Big| \le M(t_j)\Big\} \nu_A(dy)\Big)^\alpha\\
&\le m \max\{|\theta_1|,\dots,|\theta_m|\}\, \mathbf{E}' l_*(t_m)^\alpha \Big(\int_{-\sqrt{c}}^{\sqrt{c}} \mathbf{1}\Big\{\Big|x + \frac{y}{\sqrt{c}}\Big| \le M(t_m)\Big\} \nu_A(dy)\Big)^\alpha\\
&\le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^\alpha \mathbf{1}\left\{|x| \le M(t_m)+1\right\}\big) \nu_A\left([-\sqrt{c},\sqrt{c})\right)^\alpha\\
&\le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^\alpha \mathbf{1}\left\{|x| \le M(t_m)+1\right\}\big),
\end{aligned}$$

and the last expression is integrable over x since both $M(t_m)$ and $l_*(t_m)$ have all moments finite.

Next, note that

$$h_3(c,x) = h_3(c,x)\mathbf{1}\{x > -\beta^2\} + h_3(c,x)\mathbf{1}\{x \le -\beta^2\}.$$

We have

$$\begin{aligned}
&h_3(c,x)\mathbf{1}\{x > -\beta^2\}\\
&\quad= \mathbf{E}'\Big(\int_{\sqrt{c}}^{\infty} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^\alpha \mathbf{1}\{-\beta^2 < x < M(t_m)\}\\
&\quad\le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^\alpha \nu_A\left([\sqrt{c},\infty)\right)^\alpha \mathbf{1}\{-\beta^2 < x < M(t_m)\}\big)\\
&\quad\le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^\alpha \mathbf{1}\{-\beta^2 < x < M(t_m)\}\big),
\end{aligned}$$

where the last expression is integrable as before. Now, let $B_k = [\beta^k, \beta^{k+1})$ and $\sqrt{c}B_k = \left[\sqrt{c}\beta^k, \sqrt{c}\beta^{k+1}\right)$. Then,

$$h_3(c,x)\mathbf{1}\{x \le -\beta^2\} = \sum_{k=2}^\infty h_3(c,x)\mathbf{1}\{-x \in B_k\} := \sum_{k=2}^\infty g_k(c,x),$$

with

$$\begin{aligned}
g_k(c,x) \le \mathbf{E}'\Big(\sum_{i=0}^{k-2} \int_{\sqrt{c}B_i} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^\alpha \mathbf{1}\{-x \in B_k\}\\
+\ \mathbf{E}'\Big(\sum_{i=k-1}^{k+1} \int_{\sqrt{c}B_i} \sum_{j=1}^m |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^\alpha \mathbf{1}\{-x \in B_k\}
\end{aligned}$$

$$+ \ \mathbf{E}'\Big(\sum_{i=k+2}^{\infty}\int_{\sqrt{c}B_i}\sum_{j=1}^{m}|\theta_j| l\Big(x+\frac{y}{\sqrt{c}},t_j\Big)\,\nu_A(dy)\Big)^{\alpha}\mathbf{1}\{-x\in B_k\}$$

$$:= g_k^{(1)}(c,x)+g_k^{(2)}(c,x)+g_k^{(3)}(c,x).$$

Letting $Z_k(t)=\mathbf{1}\{\beta^k-\beta^{k-1}\le M(t)\}$, we see that

$$\sum_{k=2}^{\infty} g_k^{(1)}(c,x) \le const\sum_{k=2}^{\infty}\mathbf{E}'\big(Z_k(t_m)\,l_*(t_m)^{\alpha}\big)\Big(\sum_{i=0}^{k-2}\int_{\sqrt{c}B_i}\nu_A(dy)\Big)^{\alpha}\mathbf{1}\{-x\in B_k\}$$

$$\le const\sum_{i=0}^{\infty}\nu_A(\sqrt{c}B_i)^{\alpha}\sum_{k=2}^{\infty}\mathbf{E}'\big(Z_k(t)\,l_*(t_m)^{\alpha}\big)\mathbf{1}\{-x\in B_k\} \tag{33}$$

$$\le const\sum_{i=0}^{\infty}\nu_A(B_i)^{\alpha}\sum_{k=2}^{\infty}\mathbf{E}'\big(Z_k(t)\,l_*(t_m)^{\alpha}\big)\mathbf{1}\{-x\in B_k\},$$

where the last inequality follows from the fact that, for $\beta^n\le\sqrt{c}<\beta^{n+1}$,

$$\nu_A(\sqrt{c}B_i)^{\alpha}\le \nu_A(B_{n+i})^{\alpha}+\nu_A(B_{n+i+1})^{\alpha}. \tag{34}$$

The last expression in (33) is integrable since

$$\int_{\mathbb{R}}\sum_{k=2}^{\infty}\mathbf{E}'\big(Z_k(t_m)\,l_*(t_m)^{\alpha}\big)\mathbf{1}\{-x\in B_k\}\,dx$$

$$=\sum_{k=2}^{\infty}\big(\beta^{k+1}-\beta^k\big)\,\mathbf{E}'\big(Z_k(t_m)\,l_*(t_m)^{\alpha}\big)$$

$$\le c_1\sum_{k=2}^{\infty}\beta^k\mathbf{E}'\left(l_*(t_m)^{2\alpha}\right)^{1/2}\mathbf{P}'\left(\beta^k-\beta^{k-1}\le M(t_m)\right)^{1/2}$$

$$=c_1\sum_{k=2}^{\infty}\beta^k\,\mathbf{P}'\left(c_2\beta^{k-1}\le M(t_m)\right)^{1/2}$$

$$=c_1\sum_{k=2}^{\infty}\beta^k\exp\big(-c_2\beta^{2(k-1)}\big)<\infty,$$

where c_1 and c_2 are positive constants that may change from instance to instance. Also,

$$\sum_{k=2}^{\infty} g_k^{(2)}(c,x)\le const\sum_{k=2}^{\infty}\mathbf{E}'\big(l_*(t_m)^{\alpha}\big)\Big(\sum_{i=k-1}^{k+1}\int_{\sqrt{c}B_i}\nu_A(dy)\Big)^{\alpha}\mathbf{1}\{-x\in B_k\}$$

$$\le const\sum_{k=2}^{\infty}\mathbf{1}\{-x\in B_k\}\sum_{i=k-1}^{\infty}\nu_A(\sqrt{c}B_i)^{\alpha}$$

$$\le const\sum_{k=2}^{\infty}\mathbf{1}\{-x\in B_k\}\sum_{i=k-1}^{\infty}\nu_A(B_i)^{\alpha},$$

where the last inequality follows from (34) as before. The last expression is integrable because

$$\int_{\mathbb{R}}\sum_{k=2}^{\infty}\mathbf{1}\{-x\in B_k\}\sum_{i=k-1}^{\infty}\nu_A(B_i)^{\alpha}\,dx=\sum_{k=2}^{\infty}\left(\beta^{k+1}-\beta^k\right)\sum_{i=k-1}^{\infty}\nu_A(B_i)^{\alpha}$$

$$
\begin{aligned}
&= const \sum_{k=1}^{\infty} \beta^k \sum_{i=k}^{\infty} \nu_A(B_i)^\alpha \\
&= const \sum_{i=1}^{\infty} \nu_A(B_i)^\alpha \sum_{k=1}^{i} \beta^k \\
&\le const \sum_{i=1}^{\infty} \beta^i \nu_A(B_i)^\alpha \sum_{k=0}^{\infty} \beta^{-k} \\
&= const \sum_{i=1}^{\infty} \beta^i \nu_A(B_i)^\alpha < \infty,
\end{aligned}
$$

by (29). Finally, we also have

$$
\begin{aligned}
\sum_{k=2}^{\infty} g_k^{(3)}(c,x) &\le \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_{k+2}(t_m)\, l_*(t_m)^\alpha\big) \Big(\sum_{i=k+2}^{\infty} \int_{\sqrt{c}B_i} \nu_A(dy) \Big)^\alpha \mathbf{1}\{-x \in B_k\} \\
&\le \sum_{i=0}^{\infty} \nu_A(\sqrt{c}B_i)^\alpha \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_{k+2}(t_m)\, l_*(t_m)^\alpha\big) \mathbf{1}\{-x \in B_k\} \\
&\le 2 \sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_{k+2}(t_m)\, l_*(t_m)^\alpha\big) \mathbf{1}\{-x \in B_k\},
\end{aligned}
$$

and the last expression is integrable as before. We have thus shown that $h_3(c,x)$ is bounded uniformly over c by an integrable function of x. It can be shown by analogous arguments that $h_1(c,x)$ is similarly bounded.

We conclude that we are justified in exchanging the limit with the outer integral in (32). But once we do that, we can also interchange the limit with the expectation since for any $x \in \mathbb{R}$, $c > 0$, and $\mathbf{P}'$-a.s.,

$$
\Big| \sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy) \Big|^\alpha \le \Big| \sum_{j=1}^{m} \theta_j \Big|^\alpha \nu_A(\mathbb{R})^\alpha l_*(t_m)^\alpha \,,
$$

where the right-hand side has finite expectation under $\mathbf{P}'$. Finally, the limit also goes through the inner integral since for any $y \in \mathbb{R}$, $x \in \mathbb{R}$, $c > 0$, $j \le m$ and $\mathbf{P}'$-a.s.,

$$
l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big) \le l_*(t_m)\,,
$$

and the right-hand side is integrable with respect to ν_A. □

Theorem 6.1 identifies a class of BM-CAF fractional stable motions that yield the BM-local time fractional stable motion (up to a multiplicative constant) in the large time scale limit, or under "shrinking" of the time scale. It turns out that a subclass of those processes yield the *same* limiting process, up to different multiplicative constants, in the small time scale limit as well. Being attracted to the same limiting process in both large and small time scale limits is an interesting behavior that, to our knowledge, has not been described in literature before.

Theorem 6.2. *Suppose* $0 < \alpha \le 2$ *and* ν_A *is of the form*

$$
\nu_A = \sum_{i=1}^{n} \mu_i \, \delta_{a_i} \,,
$$

where $\mu_1,\dots,\mu_n > 0$, $a_1,\dots,a_n \in \mathbb{R}$ and δ_{a_i} is the Dirac point measure of mass 1 concentrated at a_i. Then, for $H = \frac{1}{2} + \frac{1}{2\alpha}$,

$$\left(\frac{1}{c^H} Y(ct), t \geq 0\right) \overset{f.d.}{\longrightarrow} \left(\left(\sum_{i=1}^{n} \mu_i^\alpha\right)^{1/\alpha} \Gamma(t), t \geq 0\right) \quad as \quad c \downarrow 0\,, \tag{35}$$

where $\overset{f.d.}{\to}$ denotes convergence in finite-dimensional distributions and $(\Gamma(t), t \geq 0)$ is the BM-local time fractional $S\alpha S$ motion defined in (1).

Proof. Let $0 \leq t_1 < t_2 < \ldots < t_m$ and $\theta_1,\dots,\theta_m \in \mathbb{R}$. As in (31), we have for any $c > 0$

$$\begin{aligned}
&\mathbf{E}\exp\left(i\sum_{j=1}^{m}\theta_j\,\frac{1}{c^H}Y(ct_j)\right)\\
&= \exp\left(-\int_{\mathbb{R}} \mathbf{E}'\left|\sum_{j=1}^{m}\theta_j \int_{\mathbb{R}} l\Big(x+\frac{y}{\sqrt{c}}, t_j\Big)\,\nu_A(dy)\right|^\alpha dx\right)\\
&= \exp\left(-\int_{\mathbb{R}} \mathbf{E}'\left|\sum_{j=1}^{m}\theta_j \sum_{i=1}^{n} \mu_i l\Big(x+\frac{a_i}{\sqrt{c}}, t_j\Big)\right|^\alpha dx\right)\\
&:= \exp\left(-\int_{\mathbb{R}} \mathbf{E}'S(c,x)dx\right).
\end{aligned} \tag{36}$$

We want to take the limit of this expression as $c \downarrow 0$. We decompose $S(c,x)$ as

$$\begin{aligned}
S(c,x) &= \sum_{i=1}^{n}\mu_i^\alpha\left|\sum_{j=1}^{m}\theta_j l\Big(x+\frac{a_i}{\sqrt{c}}, t_j\Big)\right|^\alpha \mathbf{1}\Big\{l\Big(x+\frac{a_{i'}}{\sqrt{c}}, t_m\Big) = 0 \text{ for all } i' \neq i\Big\}\\
&\quad + \sum_{k=1}^{n-1}\left|\sum_{i=1}^{n}\mu_i\sum_{j=1}^{m}\theta_j l\Big(x+\frac{a_i}{\sqrt{c}}, t_j\Big)\right|^\alpha \mathbf{1}_{G_k(c,x)}\\
&:= S_1(c,x) + S_2(c,x)
\end{aligned} \tag{37}$$

where $G_k(c,x)$ denotes the event that

$$\begin{gathered}
l\Big(x+\frac{a_{k'}}{\sqrt{c}}, t_m\Big) = 0 \text{ for all } k' < k,\\
l\Big(x+\frac{a_k}{\sqrt{c}}, t_m\Big)\, l\Big(x+\frac{a_{k''}}{\sqrt{c}}, t_m\Big) \neq 0 \text{ for some } k'' > k.
\end{gathered}$$

We first show that

$$\int_{\mathbb{R}} \mathbf{E}'S_2(c,x)dx \to 0 \ \text{ as } c \downarrow 0. \tag{38}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{E}' S_2(c,x)dx \\
&\quad \le const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{E}' \Big(\sum_{i=1}^{n} \mu_i \sum_{j=1}^{m} |\theta_j| \, l\Big(x + \frac{a_i}{\sqrt{c}}, t_j\Big)\Big)^{\alpha} \mathbf{1}_{G_k(c,x)} \, dx \\
&\quad \le const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{E}' \big(l_*(t_m)^{\alpha} \mathbf{1}_{G_k(c,x)} \big) \, dx \\
&\quad \le const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \big(\mathbf{E}' l_*(t_m)^{2\alpha} \big)^{1/2} \, \mathbf{P}'(G_k(c,x))^{1/2} dx,
\end{aligned}$$

by Cauchy-Schwarz inequality. Since $l_*(t_m)$ has finite moments of all orders, we see that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{E}' S_2(c,x)dx \\
&\le const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{P}'(G_k(c,x))^{1/2} dx \\
&\le const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{P}' \Big\{ l\Big(x + \frac{a_k}{\sqrt{c}}, t_m\Big) l\Big(x + \frac{a_{k''}}{\sqrt{c}}, t_m\Big) \neq 0 \text{ for some } k'' > k \Big\}^{1/2} dx \\
&\le const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{P}' \Big\{ l(x,t_m) l\Big(x + \frac{a_{k''} - a_k}{\sqrt{c}}, t_m\Big) \neq 0 \text{ for some } k'' > k \Big\}^{1/2} dx \\
&:= const \sum_{k=1}^{n-1} \int_{\mathbb{R}} p_k(c,x)^{1/2} dx. \qquad (39)
\end{aligned}$$

Fix $k \in \{1, \ldots, n-1\}$. It is clear that for any $x \in \mathbb{R}, p_k(c,x) \to 0$ as $c \downarrow 0$. Also, for any $c > 0, p_k(c,x) \le \mathbf{P}' \{l(x,t_m) \neq 0\}$, with

$$\int_{\mathbb{R}} \mathbf{P}' \{l(x,t_m) \neq 0\}^{1/2} \, dx \le \int_{\mathbb{R}} \mathbf{P}' \{|x| \le M(t_m)\}^{1/2} \, dx < \infty,$$

since $M(t_m)$ has Gaussian-like probability tails. It now follows, by the Dominated Convergence Theorem, that the last expression in (39) vanishes as $c \downarrow 0$, and (38) is established.

Next, note that

$$\lim_{c\downarrow 0}\int_{\mathbb{R}}\mathbf{E}' S_1(c,x)dx$$

$$= \sum_{i=1}^{n}\mu_i^{\alpha}\lim_{c\downarrow 0}\int_{\mathbb{R}}\mathbf{E}'\left|\sum_{j=1}^{m}\theta_j l(x,t_j)\right|^{\alpha}\mathbf{1}\left\{l\left(x+\frac{a_{i'}-a_i}{\sqrt{c}},t_m\right)=0 \text{ for all } i'\neq i\right\}dx$$

$$= \sum_{i=1}^{n}\mu_i^{\alpha}\int_{\mathbb{R}}\mathbf{E}'\left|\sum_{j=1}^{m}\theta_j l(x,t_j)\right|^{\alpha}\lim_{c\downarrow 0}\mathbf{1}\left\{l\left(x+\frac{a_{i'}-a_i}{\sqrt{c}},t_m\right)=0 \text{ for all } i'\neq i\right\}dx$$

$$= \sum_{i=1}^{n}\mu_i^{\alpha}\int_{\mathbb{R}}\mathbf{E}'\left|\sum_{j=1}^{m}\theta_j l(x,t_j)\right|^{\alpha}dx, \tag{40}$$

where once again the Dominated Convergence Theorem provides justification for moving the limit:

$$\left|\sum_{j=1}^{m}\theta_j l(x,t_j)\right|^{\alpha}\mathbf{1}\left\{l\left(x+\frac{a_{i'}-a_i}{\sqrt{c}},t_m\right)=0 \text{ for all } i'\neq i\right\}\leq const\, l(x,t_m)^{\alpha},$$

and the right-hand side is integrable over $\mathbb{R}\times\Omega'$ with respect to Leb $\times\,\mathbf{P}'$.

By the decomposition (37) and the convergences (39), (40), we conclude that

$$\lim_{c\downarrow 0}\int_{\mathbb{R}}\mathbf{E}' S(c,x)dx = \sum_{i=1}^{n}\mu_i^{\alpha}\int_{\mathbb{R}}\mathbf{E}'\left|\sum_{j=1}^{m}\theta_j l(x,t_j)\right|^{\alpha}dx$$

$$= \mathbf{E}\exp\left(i\sum_{j=1}^{m}\theta_j\left(\sum_{i=1}^{n}\mu_i\right)^{1/\alpha}\Gamma(t_j)\right),$$

or, in view of (36),

$$\lim_{c\downarrow 0}\mathbf{E}\exp\left(i\sum_{j=1}^{m}\theta_j\frac{1}{c^H}Y(ct_j)\right)=\mathbf{E}\exp\left(i\sum_{j=1}^{m}\theta_j\left(\sum_{i=1}^{n}\mu_i\right)^{1/\alpha}\Gamma(t_j)\right),$$

which completes the proof. □

7. Hölder Continuity

Theorem 7.1. *Let $(Y(t),t\geq 0)$ be a BM-CAF fractional $S\alpha S$ motion as defined in (7), with α and ν_A satisfying the hypotheses of Theorem 4.1 or Theorem 4.3. Then, $(Y(t),t\geq 0)$ has a version with continuous sample paths satisfying*

$$\sup_{0\leq s<t\leq 1/2}\frac{|Y(t)-Y(s)|}{(t-s)^{1/2}\log\left(\frac{1}{t-s}\right)}<\infty \quad a.s. \tag{41}$$

Proof. We use the series representation

$$Y(t)\overset{d}{=}C_\alpha\sum_{j=1}^{\infty}G_j\Gamma_j^{-1/\alpha}e^{X_j^2/2\alpha}\int_{\mathbb{R}}l_j(X_j+y,t)\,dy\,, \tag{42}$$

where C_α is a constant determined by α, $(G_j),(\Gamma_j),(X_j),(l_j)$ are independent sequences, $(G_j),(X_j)$ are i.i.d. standard normal random variables, (Γ_j) are arrival times of a unit rate Poisson process, and (l_j) are i.i.d. copies of Brownian local time. We refer to §3.10 of Samorodnitsky and Taqqu (1994) for information on the series representation of stable stochastic integrals.

Assume that (G_j) are defined on some probability space $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$, while the other random variables on the right-hand side of (42) are defined on some other probability space $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, so that $(Y(t), t \geq 0)$ is defined on the product of these two spaces.

We let

$$K_j = \sup_{\substack{x \in \mathbb{R} \\ 0 \leq s < t \leq 1/2}} \frac{l_j(x,t) - l_j(x,s)}{(t-s)^{1/2} \left(\log\left(\frac{1}{t-s}\right)\right)^{1/2}}, \quad j = 1, 2, \dots$$

As mentioned in Cohen and Samorodnitsky (2006), K_j has finite moments of all orders. Note that, for fixed $\omega_2 \in \Omega_2$, $Y(t)$ is a centered Gaussian process with incremental variance

$$\begin{aligned}
&E_1(Y(t) - Y(s))^2 \\
&\quad = C_\alpha^2 \sum_{j=1}^\infty \Gamma_j^{-2/\alpha} e^{X_j^2/\alpha} \left(\int_{\mathbb{R}} (l_j(X_j + y, t) - l_j(X_j + y, s)) \, \nu_A(dy) \right)^2 \\
&\quad \leq C_\alpha^2 \sum_{j=1}^\infty \Gamma_j^{-2/\alpha} e^{X_j^2/\alpha} K_j^2 (t-s) \log\left(\frac{1}{t-s}\right) \\
&\qquad\qquad \times \left(\int_{\mathbb{R}} \mathbf{1}\{M_j(t) \geq |X_j + y|\} \, \nu_A(dy) \right)^2 \\
&\quad := J(\omega_2)(t-s) \log\left(\frac{1}{t-s}\right)
\end{aligned}$$

for all $0 \leq s < t \leq 1/2$, with $M_j(t) = \sup_{0 \leq r \leq t} |B_j(r)|$. We will prove that J is a $\mathbf{P}_2$-a.s. finite random variable on $(\Omega_2, \mathcal{F}_2, P_2)$. By Theorem 1.4.2 of Samorodnitsky and Taqqu (1994), it will suffice to show that

$$E_2 \, e^{X_j^2/2} K_j^\alpha \left(\int_{\mathbb{R}} \mathbf{1}\{M_j(t) \geq |X_j + y|\} \, \nu_A(dy) \right)^\alpha < \infty, \tag{43}$$

or equivalently,

$$E_2 \, K_j^\alpha \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}\{M_j(t) \geq |x + y|\} \, \nu_A(dy) \right)^\alpha dx < \infty. \tag{44}$$

But the proof of (44) is identical to that of (10) if $0 < \alpha < 1$, and that of (20) if $1 \leq \alpha \leq 2$, provided that one replaces $l_*(t)$ with K_j and $\mathbf{E}'$ with E_2.

We now conclude, by classical results on moduli of continuity of Gaussian processes (see, e.g., Corollary 2.3 of Dudley (1973)), that $(Y(t), t \geq 0)$ has a version with continuous paths satisfying

$$\sup_{\substack{0 \leq s < t \leq 1/2 \\ s,t \in \mathbb{Q}}} \frac{|Y(t) - Y(s)|}{(t-s)^{1/2} \log\left(\frac{1}{t-s}\right)} < \infty \quad \mathbf{P}_1\text{-a.s.}$$

For such a version, we also have, by Fubini's Theorem,

$$\sup_{\substack{0 \leq s < t \leq 1/2 \\ s,t \in \mathbb{Q}}} \frac{|Y(t) - Y(s)|}{(t-s)^{1/2} \log\left(\frac{1}{t-s}\right)} < \infty \quad \mathbf{P}_1 \times \mathbf{P}_2\text{-a.s.},$$

which is equivalent to the statement of the theorem. □

8. A Limit Theorem

Our aim in this section is to generalize the "random rewards" scheme presented in Cohen and Samorodnitsky (2006) and outlined in Section 2. We start by setting up the notation.

Let $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$ be an array of i.i.d. SαS random variables with scale parameter 1. Further, let $(V_k^{(i)}, k \geq 1, i \geq 1)$ be an array of i.i.d. mean zero and unit variance integer-valued random variables, independent of $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$. Let $S_n^{(i)} = V_1^{(i)} + \ldots + V_n^{(i)}, n \geq 0$ be the i^{th} random walk, $i = 1, 2, \ldots$, and define for $j \in \mathbb{Z}$ and $n \geq 1$

$$\varphi^{(i)}(j,n) = \sum_{k=1}^{n} \mathbf{1}\{S_k^{(i)} = j\},$$

the number of times the i^{th} random walk visits state j by time n. Define $\varphi^{(i)}(j,t)$ for noninteger values of t by linear interpolation, i.e. for $n < t < n+1$, let

$$\varphi^{(i)}(j,t) = (t-n)\varphi^{(i)}(j,n+1) + (1-t+n)\varphi^{(i)}(j,n).$$

We note here that the results presented below can likely be generalized to an array $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$ of i.i.d. infinite-variance random variables that are in the domain of attraction of a SαS distribution, but we do not pursue that goal in order to keep the technicalities at a minimum.

Theorem 8.1. *Let $(b_n, n \geq 1)$ be a sequence of positive integers with $b_n \to \infty$, let ν_A be a finite measure on $\mathbb{R}$ whose support is contained in $[-\kappa, \kappa)$ for some positive integer κ, and define, for $n \geq 1$ and $t \geq 0$,*

$$Y_n(t) = \frac{1}{\big(nb_n^{\alpha+1}\big)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \varphi^{(i)}(k, b_n^2 t) \sum_{j=-\infty}^{\infty} W_{k-j}^{(i)} \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right). \tag{45}$$

Then we have, as $n \to \infty$,

$$\big(Y_n(t), t \geq 0\big) \Longrightarrow \big(Y(t), t \geq 0\big)$$

weakly in $\mathbb{C}([0,\infty))$, where Y is the BM-CAF fractional stable motion with associated measure ν_A.

Remark 8.2. One way to interpret this result is the following. Suppose many independent "users," indexed by $i \geq 0$, are performing independent random walks $(S_n^{(i)}, n \geq 0)$ on distinct integer lines. The numbers (or "positions") along each integer line are assigned i.i.d. SαS random "rewards" $(W_k^{(i)}, n \geq 0)$. Whenever user i visits position k, she collects a weighted average

$$\sum_{j=-\infty}^{\infty} W_{k-j}^{(i)} \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right)$$

of the rewards around k, where the weighting is determined by the measure ν_A and does not depend on k. In other words, the collected amounts form a "moving average" of the i.i.d. rewards. If there are many such users earning rewards independently, their cumulative total reward process can be approximated by the BM-CAF fractional stable motion, up to proper scaling of time and space.

Proof of Theorem 8.1. Note that $Y_n(t)$ in (45) can also be written as

$$Y_n(t) = \frac{1}{\left(nb_n^{\alpha+1}\right)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2 t)\, \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right),$$

since all the sums involved have finitely many non-zero terms. In the following, we will work with this representation.

Let $(B^{(i)}(t), t \geq 0), i = 1, 2, \ldots$ be a sequence of i.i.d. Brownian motions with jointly continuous local time processes $(l^{(i)}(x,t), x \in \mathbb{R}, t \geq 0), i = 1, 2, \ldots$, such that for every $T > 0$,

$$\sup_{x\in\mathbb{R}, 0\leq t\leq nT} \left|\varphi^{(i)}([x], t) - \sqrt{n}\, l^{(i)}\left(\frac{x}{\sqrt{n}}, \frac{t}{n}\right)\right| \longrightarrow 0 \tag{46}$$

in probability as $n \to \infty, i = 1, 2, \ldots$. (Such a sequence of Brownian motions exists, by Borodin (1982).) Define, for $n \geq 1$ and $t \geq 0$,

$$X_n(t) = \frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_{\mathbb{R}} l^{(i)}\left(\frac{k}{b_n} + y, t\right) \nu_A(dy). \tag{47}$$

We first show that for any $t \geq 0$,

$$E_n(t) := Y_n(t) - X_n(t) \longrightarrow 0 \text{ in probability} \tag{48}$$

as $n \to \infty$. For notational simplicity, we take $t = 1$. We have

$$\begin{aligned}
&E_n(1)\\
&= \frac{1}{\left(nb_n^{\alpha+1}\right)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \Bigg(\sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2)\, \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right) \\
&\qquad\qquad - b_n \int_{\mathbb{R}} l^{(i)}\left(\frac{k}{b_n} + y, 1\right) \nu_A(dy)\Bigg) \\
&= \frac{1}{\left(nb_n^{\alpha+1}\right)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_{\mathbb{R}} \left(\varphi^{(i)}\left([k + yb_n], b_n^2\right) - b_n l^{(i)}\left(\frac{k}{b_n} + y, 1\right)\right) \nu_A(dy) \\
&:= \frac{1}{\left(nb_n^{\alpha+1}\right)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} D_{k,n}^{(i)}.
\end{aligned}$$

Since the last expression is equal in distribution to

$$\left(\frac{1}{nb_n^{\alpha+1}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(i)}\right|^{\alpha}\right) W_1^{(1)},$$

the convergence (48) will be proven if we can show that

$$\frac{1}{nb_n^{\alpha+1}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(i)}\right|^{\alpha} \longrightarrow 0 \text{ in probability} \tag{49}$$

as $n \to \infty$. The expectation of the left-hand side of (49) is

$$
\begin{aligned}
&\frac{1}{b_n^{\alpha+1}} E \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^{\alpha} \\
&= \frac{1}{b_n^{\alpha+1}} E \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^{\alpha} \mathbf{1}\left\{\left|D_{k,n}^{(1)}\right| \le 1\right\} + \frac{1}{b_n^{\alpha+1}} E \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^{\alpha} \mathbf{1}\left\{\left|D_{k,n}^{(1)}\right| > 1\right\} \\
&:= p_1 + p_2 .
\end{aligned}
$$

Now, letting

$$M^{(i)}(m) = \max\left\{\kappa m + \sup_{0 \le k \le m^2} \left|S_k^{(i)}\right|, m\Big(\sup_{0 \le t \le 1} \left|B^{(i)}(t)\right| + \kappa\Big)\right\}$$

for positive integers m, we see that

$$
\begin{aligned}
p_1 &= \frac{1}{b_n^{\alpha+1}} E \sum_{k=-M^{(1)}(b_n)}^{M^{(1)}(b_n)} \left|D_{k,n}^{(1)}\right|^{\alpha} \mathbf{1}\left\{\left|D_{k,n}^{(1)}\right| \le 1\right\} \\
&\le \frac{1}{b_n^{\alpha+1}} E\big(2M^{(1)}(b_n) + 1\big).
\end{aligned}
\tag{50}
$$

It is an easy consequence of Doob's martingale inequalities that

$$E\big(M^{(1)}(m)^r\big) \le const\ m^r \tag{51}$$

for integers $m \ge 1$ and real numbers $r \ge 1$, where the constant depends on r. Therefore, continuing from (50), we obtain

$$p_1 \le const\ \frac{1}{b_n^{\alpha+1}}\ b_n \longrightarrow 0$$

as $n \to \infty$.

Next, we consider p_2. By repeated use of Hölder's inequality,

$$
\begin{aligned}
p_2 &\le \frac{1}{b_n^{\alpha+1}} E\Big(\sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^2\Big)^{\frac{\alpha}{2}} \Big(\sum_{k=-\infty}^{\infty} \mathbf{1}\left\{\left|D_{k,n}^{(1)}\right| > 1\right\}\Big)^{1-\frac{\alpha}{2}} \\
&\le \frac{1}{b_n^{\alpha+1}} E\Big(\sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^2\Big)^{\frac{\alpha}{2}} \big(2M^{(1)}(b_n) + 1\big)^{1-\frac{\alpha}{2}} \mathbf{1}\Big\{\sup_k \left|D_{k,n}^{(1)}\right| > 1\Big\}^{1-\frac{\alpha}{2}} \\
&\le \frac{1}{b_n^{\alpha+1}} \Big(E \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^2\Big)^{\frac{\alpha}{2}} \Big(E\big(2M^{(1)}(b_n) + 1\big)\mathbf{1}\Big\{\sup_k \left|D_{k,n}^{(1)}\right| > 1\Big\}\Big)^{1-\frac{\alpha}{2}}.
\end{aligned}
$$

But note that

$$
\begin{aligned}
&E \sum_{k=-\infty}^{\infty} \left|D_{k,n}^{(1)}\right|^2 \\
&\qquad \le const \int_{\mathbb{R}} E \sum_{k=-\infty}^{\infty} \Big(\varphi^{(1)}\left([k + yb_n], b_n^2\right) - b_n l^{(1)}\Big(\frac{k}{b_n} + y, 1\Big)\Big)^2 \nu_A(dy) \\
&\qquad \le const \int_{\mathbb{R}} \Big(E \sum_{k=-\infty}^{\infty} \varphi^{(1)}\left(k, b_n^2\right)^2 + b_n^2\, E \sum_{k=-\infty}^{\infty} l^{(1)}\Big(\frac{k}{b_n} + y, 1\Big)^2\Big) \nu_A(dy) \\
&\qquad \le const\ b_n^3,
\end{aligned}
$$

where the last inequality follows from Lemma 1 of Kesten and Spitzer (1979) and from the fact that the largest value of a Brownian local time at time 1 has all moments finite. Furthermore,

$$\sup_k \big|D^{(1)}_{k,n}\big| \le |\nu_A| \sup_{x\in\mathbb{R}} \Big|\varphi^{(1)}([x], b_n^2) - b_n\, l^{(1)}\Big(\frac{x}{b_n}, 1\Big)\Big| := |\nu_A|\Delta^{(1)}(b_n),$$

where $|\nu_A| = \nu_A(\mathbb{R})$. Thus we obtain

$$\begin{aligned} p_2 &\le const\, \frac{1}{b_n^{\alpha+1}}\, b_n^{\frac{3\alpha}{2}} \Big(EM^{(1)}(b_n)^{3/2}\Big)^{\frac{2-\alpha}{3}} \Big(P\Big(\Delta^{(1)}(b_n) > |\nu_A|^{-1}\Big)\Big)^{(1-\frac{\alpha}{2})/3} \\ &\le const\, \frac{1}{b_n^{\alpha+1}}\, b_n^{\frac{3\alpha}{2}} \Big(b_n^{3/2}\Big)^{\frac{2-\alpha}{3}} \Big(P\Big(\Delta^{(1)}(b_n) > |\nu_A|^{-1}\Big)\Big)^{(1-\frac{\alpha}{2})/3} \\ &= const\, \Big(P\Big(\Delta^{(1)}(b_n) > |\nu_A|^{-1}\Big)\Big)^{(1-\frac{\alpha}{2})/3} \longrightarrow 0, \end{aligned}$$

by (46). Note that in the middle line we take advantage of the inequality (51). Thus (49) follows, and (48) is established.

The next step is to show that the finite-dimensional distributions of the process $(X_n(t), t \ge 0)$ in (47) converge to those of $(Y(t), t \ge 0)$. For this, it is enough to show that, for every $m \ge 1$, $0 < t_1 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$,

$$\sum_{j=1}^m \theta_j X_n(t_j) \xrightarrow{d} \sum_{j=1}^m \theta_j Y(t_j) \qquad \text{as } n \to \infty\,.$$

We will see that this is true for $m = 1$ and $t_1 = 1$; the general case is similar. So we will show that

$$\frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^n \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_{\mathbb{R}} l^{(i)}\Big(\frac{k}{b_n} + y, 1\Big)\nu_A(dy) \xrightarrow{d} Y(1). \tag{52}$$

Since both sides of (52) are conditionally SαS random variables, it will suffice to show the convergence in probability of the scale parameters. That is, it will suffice to show that

(53)

$$\frac{1}{nb_n} \sum_{i=1}^n \sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} l^{(i)}\Big(\frac{k}{b_n} + y, 1\Big)\nu_A(dy)\Big)^\alpha \longrightarrow E\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} l\,(x+y,1)\,\nu_A(dy)\Big)^\alpha dx$$

in probability. Let us denote the absolute difference

$$\Bigg|\frac{1}{nb_n} \sum_{i=1}^n \sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} l^{(i)}\Big(\frac{k}{b_n} + y, 1\Big)\nu_A(dy)\Big)^\alpha - \frac{1}{b_n}\, E \sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} l\Big(\frac{k}{b_n} + y, 1\Big)\nu_A(dy)\Big)^\alpha\Bigg|$$

by δ_n. By Chebyshev's inequality,

$$P(\delta_n > \epsilon) \le \frac{1}{\epsilon^2 n b_n^2}\, E\Big(\sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} l^{(1)}\Big(\frac{k}{b_n} + y, 1\Big)\nu_A(dy)\Big)^\alpha\Big)^2$$

$$\leq \frac{c}{\epsilon^2 n b_n^2} E\big(2M^{(1)}(b_n)+1\big)^2 l_*(1)^{2\alpha}$$
$$\leq \frac{c}{\epsilon^2 n} \longrightarrow 0.$$

Moreover,

$$\frac{1}{b_n} E \sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} l\Big(\frac{k}{b_n}+y,1\Big)\nu_A(dy)\Big)^{\alpha} \longrightarrow E\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} l\,(x+y,1)\,\nu_A(dy)\Big)^{\alpha} dx$$

by the Dominated Convergence Theorem. Hence the convergence (53) follows, and (52) is proven.

It remains to prove the tightness of the sequence $(Y_n(t), t \geq 0)$ in $\mathbb{C}([0,\infty))$. Given $K > 0$, we write

$$Y_n(t) = \frac{1}{\big(nb_n^{\alpha+1}\big)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \mathbf{1}\big\{|W_k^{(i)}| > K(nb_n)^{1/\alpha}\big\}$$
$$\times \sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2 t)\,\nu_A\Big(\Big[\frac{j}{b_n}, \frac{j+1}{b_n}\Big)\Big)$$
$$+ \frac{1}{\big(nb_n^{\alpha+1}\big)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \mathbf{1}\big\{|W_k^{(i)}| \leq K(nb_n)^{1/\alpha}\big\}$$
$$\times \sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2 t)\,\nu_A\Big(\Big[\frac{j}{b_n}, \frac{j+1}{b_n}\Big)\Big)$$
$$:= Y_{1,n}(t) + Y_{2,n}(t).$$

Note that

$$P\Big(\sup_{0\leq t\leq 1} \big|Y_{1,n}(t)\big| = 0\Big)^{1/n} \geq P\left(\text{for all } |k| \leq M^{(1)}(b_n), \big|W_k^{(1)}\big| \leq K(nb_n)^{1/\alpha}\right)$$
$$= E\,P\left(\big|W_1^{(1)}\big| \leq K(nb_n)^{1/\alpha}\right)^{2M^{(1)}(b_n)+1}$$
$$\geq E\left(1 - cK^{-\alpha}(nb_n)^{-1}\right)^{2M^{(1)}(b_n)+1}$$
$$\geq 1 + E\left(2M^{(1)}(b_n)+1\right)\log\left(1 - cK^{-\alpha}(nb_n)^{-1}\right)$$
$$\geq 1 + c_1 b_n \log\left(1 - c_2 K^{-\alpha}(nb_n)^{-1}\right),$$

where c, c_1, c_2 are, as usual, positive constants that may change from instance to instance. It now follows that

$$P\Big(\sup_{0\leq t\leq 1} \big|Y_{1,n}(t)\big| > 0\Big) \leq 1 - \left(1 + c_1 \log\left(1 - c_2 K^{-\alpha}(nb_n)^{-1}\right)^{b_n}\right)^n.$$

Letting n go to infinity, we obtain

$$\limsup_{n\to\infty} P\Big(\sup_{0\leq t\leq 1} \big|Y_{1,n}(t)\big| > 0\Big) \leq 1 - \exp\left(-c_2 K^{-\alpha}\right).$$

Since the right-hand side converges to zero as $K \to \infty$, it follows from the decomposition of $Y_n(t)$ above that it will suffice to prove the tightness of the processes $(Y_{2,n}(t), 0 \leq t \leq 1)$ for each fixed K.

Now, for any $0 \le s \le t \le 1$, we have

$$
\begin{aligned}
&E\left(Y_{2,n}(t) - Y_{2,n}(s)\right)^2 \\
&= \frac{1}{n^{\frac{2}{\alpha}-1} b_n^{\frac{2}{\alpha}+2}} E\left(\left(W_1^{(1)}\right)^2 \mathbf{1}\left\{\left|W_1^{(1)}\right| \le K(nb_n)^{1/\alpha}\right\}\right) \\
&\quad \times E \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} \left(\varphi^{(i)}(k+j, b_n^2 t) - \varphi^{(i)}(k+j, b_n^2 s)\right) \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right)\right)^2 .
\end{aligned}
$$

Since, for large x,

$$
E\left(W_1^{(1)}\right)^2 \mathbf{1}\left\{\left|W_1^{(1)}\right| \le x\right\} \le 4 \int_0^x y P\left(W_1^{(1)} > y\right) dy \le c x^{2-\alpha},
$$

we see that, for large n,

$$
\begin{aligned}
&E\left(Y_{2,n}(t) - Y_{2,n}(s)\right)^2 \\
&\le c\, b_n^{-3} E \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} \left(\varphi^{(i)}(k+j, b_n^2 t) - \varphi^{(i)}(k+j, b_n^2 s)\right) \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right)\right)^2 \\
&= c\, b_n^{-3} E \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} \left(\varphi^{(i)}\left([k+yb_n], b_n^2 t\right) - \varphi^{(i)}\left([k+yb_n], b_n^2 s\right)\right) \nu_A(dy)\right)^2 \\
&\le c\, b_n^{-3} \int_{\mathbb{R}} E \sum_{k=-\infty}^{\infty} \left(\varphi^{(i)}\left([k+yb_n], b_n^2 t\right) - \varphi^{(i)}\left([k+yb_n], b_n^2 s\right)\right)^2 \nu_A(dy) \\
&\le c\, b_n^{-3} \int_{\mathbb{R}} \left(b_n^2 (t-s)\right)^{3/2} \nu_A(dy) \\
&= c(t-s)^{3/2},
\end{aligned}
$$

as in the proof of Lemma 7 in Kesten and Spitzer (1979). We can now appeal to Theorem 12.3 in Billingsley (1968) to conclude the tightness of $(Y_{2,n}(t), 0 \le t \le 1)$ and, hence, complete the proof. □

9. A Special Case

In this section we study BM-CAF fractional SαS motions whose associated measures are of the form $\nu_A(dy) = y^{-\lambda} \mathbf{1}_{[0,\infty)}(y)\, dy$ for some $0 < \lambda < 1$. That is, we study processes of the form

$$
Y(t) = \int_{\Omega' \times \mathbb{R}} \int_0^\infty l(x+y, t)\, y^{-\lambda}\, dy\, M(d\omega', dx), \quad t \ge 0, \tag{54}
$$

where $(l(x,t), x \in \mathbb{R}, t \ge 0)$ is the local time of a Brownian motion $(B(t), t \ge 0)$ defined on $(\Omega', \mathcal{F}', \mathbf{P}')$, and M is a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $\mathbf{P}' \times$ Leb. M lives on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Theorem 9.1. *Suppose $1 < \alpha \le 2$ and $1/\alpha < \lambda < 1$. Then, the process $(Y(t), t \ge 0)$ in (54) is a well-defined SαS process. It is self-similar with exponent*

$$
H = 1 - \frac{\lambda}{2} + \frac{1}{2\alpha} = 1 - \frac{1}{2}\left(\lambda - \frac{1}{\alpha}\right). \tag{55}
$$

Remark 9.2. Well-definedness does *not* follow from Theorem 4.3 in the present case, since condition (19) is violated. Indeed, for any $\beta > 1$,

$$\sum_{i=0}^{\infty} \nu_A\left([\beta^i, \beta^{i+1})\right)^\alpha = \sum_{i=0}^{\infty} \left(\int_{\beta^i}^{\beta^{i+1}} y^{-\lambda}\, dy \right)^\alpha = const \sum_{i=0}^{\infty} \beta^{\alpha(1-\lambda)i} = \infty\,.$$

Hence Theorem 9.1 shows that in the case $1 < \alpha \le 2$, condition (19) is not necessary for well-definedness. Also, the self-similarity result proves that the BM-CAF fractional stable motion is not always in the domain of attraction of the BM-local time fractional stable motion.

Proof of Theorem 9.1. For well-definedness, we need to check that

$$\mathbf{E}' \int_{\mathbb{R}} \left(\int_0^\infty l(x+y,t)\, y^{-\lambda}\, dy \right)^\alpha dx < \infty\,. \tag{56}$$

It will suffice to prove

$$\mathbf{E}'\, l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_0^\infty \mathbf{1}\{|x+y| \le M(t)\}\, y^{-\lambda}\, dy \right)^\alpha dx < \infty\,, \tag{57}$$

with $l_*(t)$ and $M(t)$ as defined in (11) and (12). The left hand side of (57) can be decomposed as

$$\mathbf{E}'\, l_*(t)^\alpha \int_{-\infty}^{-M(t)} \left(\int_{-M(t)-x}^{M(t)-x} y^{-\lambda}\, dy \right)^\alpha dx + \mathbf{E}'\, l_*(t)^\alpha \int_{-M(t)}^{M(t)} \left(\int_0^{M(t)-x} y^{-\lambda}\, dy \right)^\alpha dx$$
$$:= I_1 + I_2.$$

Note that

$$\begin{aligned} I_1 &= \mathbf{E}'\, l_*(t)^\alpha \int_{M(t)}^\infty \left(\int_{x-M(t)}^{x+M(t)} y^{-\lambda}\, dy \right)^\alpha dx \\ &= const\ \mathbf{E}'\, l_*(t)^\alpha \int_{M(t)}^\infty \left((x+M(t))^{1-\lambda} - (x-M(t))^{1-\lambda} \right)^\alpha dx \\ &= const\ \mathbf{E}'\, l_*(t)^\alpha M(t)^{1+(1-\lambda)\alpha} \int_1^\infty \left((u+1)^{1-\lambda} - (u-1)^{1-\lambda} \right)^\alpha du \\ &= const\ \mathbf{E}'\, l_*(t)^\alpha M(t)^{1+(1-\lambda)\alpha} < \infty\,, \end{aligned}$$

since $\left((u+1)^{1-\lambda} - (u-1)^{1-\lambda}\right)^\alpha \sim u^{-\alpha\lambda}$ as $u \to \infty$, and $l_*(t)$ and $M(t)$ have finite moments of all orders. Also,

$$\begin{aligned} I_2 &= const\ \mathbf{E}'\, l_*(t)^\alpha \int_{-M(t)}^{M(t)} (M(t)-x)^{(1-\lambda)\alpha}\, dx \\ &= const\ \mathbf{E}'\, l_*(t)^\alpha M(t)^{1+(1-\lambda)\alpha} < \infty\,, \end{aligned}$$

and (57) follows.

For self-similarity, note that for any $c > 0, \theta_1, \dots, \theta_m \in \mathbb{R}$ and $t_1, \dots, t_m \ge 0$ we have, using (30),

$$\begin{aligned} &\mathbf{E} \exp\left(i \sum_{j=1}^m \theta_j Y(ct_j) \right) \\ &\quad = \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^m \theta_j \int_0^\infty l(x+y, ct_j)\, y^{-\lambda}\, dy \right|^\alpha dx \right) \end{aligned}$$

$$= \exp\Big(- \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j \int_0^\infty \sqrt{c}\, l\Big(\frac{x+y}{\sqrt{c}}, t_j\Big) y^{-\lambda}\, dy \Big|^\alpha dx \Big)$$

$$= \exp\Big(- \int_{\mathbb{R}} \mathbf{E}' \Big| \sum_{j=1}^m \theta_j c^{1-\frac{\lambda}{2}+\frac{1}{2\alpha}} \int_0^\infty l(u+v, t_j)\, v^{-\lambda}\, dv \Big|^\alpha du \Big)$$

$$= \mathbf{E} \exp\Big(i \sum_{j=1}^m \theta_j c^{1-\frac{\lambda}{2}+\frac{1}{2\alpha}} Y(t_j) \Big) .$$

Therefore, $(Y(t), t \geq 0)$ is H-self-similar, with H as defined in (55). □

Next, we prove a finite-dimensional analogue of Theorem 8.1 for the process $(Y(t), t \geq 0)$ defined in (54). Note that Theorem 8.1 does not apply in this case, since the measure $\nu_A(dy) = y^{-\lambda} \mathbf{1}_{[0,\infty)}(y)\, dy$ clearly does not satisfy its hypotheses.

As in Section 8.1, let $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$ be an array of i.i.d. SαS random variables with scale parameter 1. Further, let $(V_k^{(i)}, k \geq 1, i \geq 1)$ be an array of i.i.d. mean zero and unit variance integer-valued random variables, independent of $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$. Let $S_n^{(i)} = V_1^{(i)} + \ldots + V_n^{(i)}, n \geq 0$ be the i^{th} random walk, $i = 1, 2, \ldots$, and define for $j \in \mathbb{Z}$ and $n \geq 1$

$$\varphi^{(i)}(j,n) = \sum_{k=1}^n \mathbf{1}\{S_k^{(i)} = j\} ,$$

the number of times the i^{th} random walk visits state j by time n. Define $\varphi^{(i)}(j,t)$ for noninteger values of t by linear interpolation, i.e. for $n < t < n+1$, let

$$\varphi^{(i)}(j,t) = (t-n)\varphi^{(i)}(j,n+1) + (1-t+n)\varphi^{(i)}(j,n) .$$

Theorem 9.3. *Let $(b_n, n \geq 1)$ be a sequence of positive integers with $b_n \to \infty$, $1 < \alpha \leq 2$ and $1/\alpha < \lambda < 1$. Define, for $n \geq 1$ and $t \geq 0$,*

$$Y_n(t) = \frac{1}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^n \sum_{k=-\infty}^\infty \varphi^{(i)}(k, b_n^2 t) \sum_{j=0}^\infty W_{k-j}^{(i)} \left((j+1)^{1-\lambda} - j^{1-\lambda} \right) . \tag{58}$$

Then we have, as $n \to \infty$,

$$\big(Y_n(t), t \geq 0\big) \xrightarrow{f.d.} \big(Y(t), t \geq 0\big),$$

where $\xrightarrow{f.d.}$ denotes convergence in finite-dimensional distributions and $(Y(t), t \geq 0)$ is the process defined in (54).

Proof. The outline of the proof is the same as in Theorem 8.1. We work with the representation

$$Y_n(t) = \frac{1}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^n \sum_{k=-\infty}^\infty W_k^{(i)} \sum_{j=0}^\infty \varphi^{(i)}(k+j, b_n^2 t) \left((j+1)^{1-\lambda} - j^{1-\lambda} \right) .$$

Let $(B^{(i)}(t), t \geq 0), i = 1, 2, \ldots$ be a sequence of i.i.d. Brownian motions with jointly continuous local time processes $(l^{(i)}(x,t), x \in \mathbb{R}, t \geq 0), i = 1, 2, \ldots$, such that for every $T > 0$,

$$\sup_{x \in \mathbb{R}, 0 \leq t \leq nT} \Big| \varphi^{(i)}([x], t) - \sqrt{n}\, l^{(i)}\Big(\frac{x}{\sqrt{n}}, \frac{t}{n}\Big) \Big| \xrightarrow{L^2} 0 \tag{59}$$

as $n\to\infty, i=1,2,\dots$. Such a sequence of Brownian motions exists, by Kang and Wee (1997). Define, for $n\geq 1$ and $t\geq 0$,

$$X_n(t)=\frac{1}{(nb_n)^{1/\alpha}}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}W_k^{(i)}\int_0^\infty l^{(i)}\Big(\frac{k}{b_n}+y,t\Big)y^{-\lambda}\,dy. \tag{60}$$

We first show that for any $t\geq 0$,

$$E_n(t):=Y_n(t)-X_n(t)\longrightarrow 0\ \text{ in probability} \tag{61}$$

as $n\to\infty$. For notational simplicity, we take $t=1$. We have

$$\begin{aligned}
&E_n(1)\\
&=\frac{1}{\big(nb_n^{\alpha+1}\big)^{1/\alpha}}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}W_k^{(i)}\Big(\sum_{j=0}^{\infty}\varphi^{(i)}(k+j,b_n^2)\,b_n^{\lambda-1}\left((j+1)^{1-\lambda}-j^{1-\lambda}\right)\\
&\qquad\qquad -b_n\int_0^\infty l^{(i)}\Big(\frac{k}{b_n}+y,1\Big)y^{-\lambda}\,dy\Big)\\
&=\frac{1}{\big(nb_n^{\alpha+1}\big)^{1/\alpha}}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}W_k^{(i)}\int_0^\infty\Big(\varphi^{(i)}\left([k+yb_n],b_n^2\right)-b_nl^{(i)}\Big(\frac{k}{b_n}+y,1\Big)\Big)y^{-\lambda}\,dy\\
&=\frac{1}{n^{1/\alpha}b_n^{2-\lambda+1/\alpha}}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}W_k^{(i)}\int_k^\infty\Big(\varphi^{(i)}\left([u],b_n^2\right)-b_nl^{(i)}\Big(\frac{u}{b_n},1\Big)\Big)(u-k)^{-\lambda}\,du\\
&:=\frac{1}{n^{1/\alpha}b_n^{2-\lambda+1/\alpha}}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}W_k^{(i)}\int_k^\infty D^{(i)}(u,b_n)(u-k)^{-\lambda}\,du.
\end{aligned}$$

Thus, in order to prove (61), it will suffice to show that

$$\frac{1}{nb_n^{(2-\lambda)\alpha+1}}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}\left|\int_k^\infty D^{(i)}(u,b_n)(u-k)^{-\lambda}\,du\right|^\alpha\longrightarrow 0\ \text{ in probability} \tag{62}$$

as $n\to\infty$. For integers $m\geq 1$, we define

$$K^{(i)}(m)=\max\Big\{1+\sup_{0\leq k\leq m^2}\big|S_k^{(i)}\big|,m\Big(\sup_{0\leq t\leq 1}\big|B^{(i)}(t)\big|\Big)\Big\}.$$

Then, the expectation of the left-hand side of (62) is

$$\begin{aligned}
&\frac{1}{b_n^{(2-\lambda)\alpha+1}}E\Big(\sum_{k=-\infty}^{\infty}\left|\int_k^\infty D^{(1)}(u,b_n)(u-k)^{-\lambda}\,du\right|^\alpha\Big)\\
&=\frac{1}{b_n^{(2-\lambda)\alpha+1}}E\Big(\sum_{k=-\infty}^{-K^{(1)}(b_n)-1}\left|\int_{-K^{(1)}(b_n)}^{K^{(1)}(b_n)}D^{(1)}(u,b_n)(u-k)^{-\lambda}\,du\right|^\alpha\Big)\\
&\quad+\frac{1}{b_n^{(2-\lambda)\alpha+1}}E\Big(\sum_{k=-K^{(1)}(b_n)}^{K^{(1)}(b_n)}\left|\int_k^{K^{(1)}(b_n)}D^{(1)}(u,b_n)(u-k)^{-\lambda}\,du\right|^\alpha\Big)\\
&:=p_1+p_2\,.
\end{aligned} \tag{63}$$

Defining $D_*^{(i)}(m) = \sup_{u\in\mathbb{R}} \left|D^{(i)}(u,m)\right|$ and omitting the superscript "(1)" for notational convenience, we see that

$$p_1 \le \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\biggl(D_*(b_n)^\alpha \sum_{k=-\infty}^{-K(b_n)-1} \biggl(\int_{-K(b_n)}^{K(b_n)} (u-k)^{-\lambda}\, du \biggr)^\alpha \biggr),$$

with

$$\begin{aligned}
&\sum_{k=-\infty}^{-K(b_n)-1} \biggl(\int_{-K(b_n)}^{K(b_n)} (u-k)^{-\lambda}\, du \biggr)^\alpha \\
&= \sum_{k=K(b_n)+1}^{\infty} \bigl((k+K(b_n))^{1-\lambda} - (k-K(b_n))^{1-\lambda} \bigr)^\alpha \\
&= K(b_n)^{(1-\lambda)\alpha+1} \frac{1}{K(b_n)} \sum_{k=K(b_n)+1}^{\infty} \biggl(\Bigl(\frac{k}{K(b_n)} + 1 \Bigr)^{1-\lambda} - \Bigl(\frac{k}{K(b_n)} - 1 \Bigr)^{1-\lambda} \biggr)^\alpha \\
&\le K(b_n)^{(1-\lambda)\alpha+1} \int_1^\infty \bigl((u+1)^{1-\lambda} - (u-1)^{1-\lambda} \bigr)^\alpha \, du \\
&= const\; K(b_n)^{(1-\lambda)\alpha+1},
\end{aligned}$$

since $(u+1)^{1-\lambda} - (u-1)^{1-\lambda} \sim u^{-\lambda}$ as $u \to \infty$. Thus we obtain

$$\begin{aligned}
p_1 &\le const\, \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\Bigl(D_*(b_n)^\alpha K(b_n)^{(1-\lambda)\alpha+1} \Bigr) \\
&\le const\, \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, \bigl(E(D_*(b_n)^2) \bigr)^{\alpha/2} \Bigl(E\Bigl(K(b_n)^{\frac{2(1-\lambda)\alpha+2}{2-\alpha}} \Bigr) \Bigr)^{1-\alpha/2} \qquad (64)\\
&\le const\, \frac{b_n^{(1-\lambda)\alpha+1}}{b_n^{(2-\lambda)\alpha+1}}\, \bigl(E(D_*(b_n)^2) \bigr)^{\alpha/2} \longrightarrow 0 \quad \text{as } n \to \infty\,,
\end{aligned}$$

by (59).

The term p_2 can be bounded similarly:

$$p_2 \le \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\biggl(D_*(b_n)^\alpha \sum_{k=-K(b_n)}^{K(b_n)} \biggl(\int_{k}^{K(b_n)} (u-k)^{-\lambda}\, du \biggr)^\alpha \biggr),$$

with

$$\begin{aligned}
\sum_{k=-K(b_n)}^{K(b_n)} \biggl(\int_{k}^{K(b_n)} (u-k)^{-\lambda}\, du \biggr)^\alpha &\le \sum_{k=-K(b_n)}^{K(b_n)} \biggl(\int_{0}^{2K(b_n)} u^{-\lambda}\, du \biggr)^\alpha \\
&= const \sum_{k=-K(b_n)}^{K(b_n)} K(b_n)^{(1-\lambda)\alpha} \\
&= const\; K(b_n)^{(1-\lambda)\alpha+1}.
\end{aligned}$$

It follows that

$$p_2 \le const\, \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\Bigl(D_*(b_n)^\alpha K(b_n)^{(1-\lambda)\alpha+1} \Bigr) \longrightarrow 0 \quad \text{as } n \to \infty\,,$$

as in (64). Thus we have established (62).

The next step is to show that the finite-dimensional distributions of the process $(X_n(t), t \geq 0)$ in (60) converge to those of $(Y(t), t \geq 0)$. For this, it is enough to show that, for every $m \geq 1$, $0 < t_1 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$,

$$\sum_{j=1}^{m} \theta_j X_n(t_j) \xrightarrow{d} \sum_{j=1}^{m} \theta_j Y(t_j) \quad \text{as } n \to \infty .$$

We will see that this is true for $m = 1$ and $t_1 = 1$; the general case is similar. So we will show that

$$\frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_0^\infty l^{(i)}\Big(\frac{k}{b_n} + y, 1\Big) y^{-\lambda}\, dy \xrightarrow{d} Y(1). \tag{65}$$

Since both sides of (65) are conditionally SαS random variables, it will suffice to show the convergence in probability of the scale parameters. That is, it will suffice to show that

$$\frac{1}{nb_n} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \Big(\int_0^\infty l^{(i)}\Big(\frac{k}{b_n} + y, 1\Big) y^{-\lambda}\, dy \Big)^\alpha \longrightarrow E \int_{\mathbb{R}} \Big(\int_0^\infty l\,(x + y, 1)\, y^{-\lambda}\, dy \Big)^\alpha dx \tag{66}$$

in probability. Let us denote the absolute difference

$$\Bigg| \frac{1}{nb_n} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \Big(\int_0^\infty l^{(i)}\left(\frac{k}{b_n} + y, 1\right) y^{-\lambda}\, dy \Big)^\alpha - \frac{1}{b_n} E \sum_{k=-\infty}^{\infty} \Big(\int_0^\infty l\left(\frac{k}{b_n} + y, 1\right) y^{-\lambda}\, dy \Big)^\alpha \Bigg|$$

by δ_n. Now, by Chebyshev's inequality,

$$\begin{aligned}
P(\delta_n > \epsilon) &\leq \frac{1}{\epsilon^2 n b_n^2} E\Big(\sum_{k=-\infty}^{\infty} \Big(\int_0^\infty l\Big(\frac{k}{b_n} + y, 1\Big) y^{-\lambda}\, dy \Big)^\alpha \Big)^2 \\
&\leq \frac{1}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}} E\Big(\sum_{k=-\infty}^{\infty} \Big(\int_0^\infty l\Big(\frac{u}{b_n}, 1\Big) (u-k)^{-\lambda}\, du \Big)^\alpha \Big)^2 \\
&\leq \frac{const}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}} E\Big(\sum_{k=-\infty}^{-K(b_n)-1} \Big(\int_{-K(b_n)}^{K(b_n)} l\Big(\frac{u}{b_n}, 1\Big) (u-k)^{-\lambda}\, du \Big)^\alpha \Big)^2 \\
&\quad + \frac{const}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}} E\Big(\sum_{k=-K(b_n)}^{K(b_n)} \Big(\int_k^{K(b_n)} l\Big(\frac{u}{b_n}, 1\Big) (u-k)^{-\lambda}\, du \Big)^\alpha \Big)^2 \\
&:= p_1' + p_2' .
\end{aligned}$$

Note the similarity of p_1', p_2' to p_1, p_2 in (63). By arguments analogous to the ones used for p_1 and p_2, one can show that

$$p_1' + p_2' \leq const\ \frac{b_n^{2+2(1-\lambda)\alpha}}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}} \longrightarrow 0$$

as $n \to \infty$, hence $\delta_n \to 0$ in probability. Moreover, we have

$$\frac{1}{b_n} E \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} l\Big(\frac{k}{b_n} + y, 1\Big) y^{-\lambda}\, dy \right)^{\alpha} \longrightarrow E \int_{\mathbb{R}} \left(\int_{\mathbb{R}} l\,(x + y, 1)\, y^{-\lambda}\, dy \right)^{\alpha} dx$$

by the Dominated Convergence Theorem. The convergence (66) follows, hence (65) is proven, and so is the theorem. □

Corollary 9.4. *Let $(b_n, n \geq 1)$ be a sequence of positive integers with $b_n \to \infty$, $1 < \alpha \leq 2$ and $1/\alpha < \lambda < 1$. Define, for $n \geq 1$ and $t \geq 0$,*

$$Y_n(t) = \frac{1-\lambda}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \varphi^{(i)}(k, b_n^2 t) \sum_{j=0}^{\infty} j^{-\lambda} W_{k-j}^{(i)}\,. \tag{67}$$

Then we have, as $n \to \infty$,

$$\big(Y_n(t), t \geq 0\big) \overset{f.d.}{\longrightarrow} \big(Y(t), t \geq 0\big),$$

where $\overset{f.d.}{\longrightarrow}$ denotes convergence in finite-dimensional distributions and $(Y(t), t \geq 0)$ is the process defined in (54).

Proof. The proof is analogous to that of Theorem 9.3. □

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