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Abstract.
We consider retrial queues with slow retrials for $M/M/s/s$ loss systems in the Halfin-Whitt regime. In this regime the service capacity s is large and the arrival rate λ is related to s according to $\lambda = s - \gamma\sqrt{s}$ with $\gamma < \sqrt{s}$. We present s-uniform asymptotics for the inverse Poisson-Mills ratio $f_s(\delta)$ as $\delta \to -\infty$. This f_s occurs in Cohen's equation $a = f_s(\gamma - a)$, $\gamma > 0$, for the retrial factor $a = a_s(\gamma)$ that determines the retrial rate Ω according to $\Omega = a\sqrt{s}$. We thus obtain approximation results for $a_s(\gamma)$ with small $\gamma > 0$ (critical range) with uniform error assessment. Furthermore, we present uniform asymptotics of $f_s(\delta)$ in the form of an asymptotic series in powers of $1/\sqrt{s}$ with coefficients expressed in terms of the inverse Gaussian Mills ratio $f_\infty(\delta)$ in a range $\delta \leq s^{1/6}$. This allows explicit corrections, in powers of $1/\sqrt{s}$, of the limiting retrial factor $a = a_\infty(\gamma)$ that are valid uniformly in the range $0 < \gamma \leq s^{1/6}$.

1 Introduction

Loss systems with slow retrials in the Halfin-Whitt regime were considered recently in [1]. Thus, there is considered the $M/M/s/s$ queue (multi-server loss model) with s servers under a heavy-traffic scaling with arrival rate λ of the form $\lambda = s - \gamma\sqrt{s}$ with γ a constant, often of order unity, less than $\sqrt{s}$. Without retrials, system stability is guaranteed for any $\gamma < \sqrt{s}$, but in the presence of (slow) retrials, one must assume γ to be positive. Under the assumption of slow retrials, the additional load Ω on the system due to retrials was shown by Cohen [2] to satisfy the equation

$$\Omega = (\lambda + \Omega)\, B(s, \lambda + \Omega) \ . \tag{1}$$

Here, $B(s, \lambda)$ is the Erlang B expression

$$B(s, \lambda) = \frac{\lambda^s / s!}{\sum\limits_{k=0}^{s} \lambda^k / k!} = \frac{e^{-\lambda} (\lambda/s)^2}{\int\limits_{\lambda}^{\infty} e^{-\lambda'} (\lambda'/s)^s \, d\lambda'} \tag{2}$$

that represents the steady-state blocking probability in the Erlang loss system. In the Halfin-Whitt regime, we write

$$\lambda = s - \gamma\sqrt{s} \ , \qquad \Omega = a\sqrt{s} \ , \tag{3}$$

with a the retrial factor, and then Cohen's equation (1) takes the form

$$a = f_s(\gamma - a) \ , \tag{4}$$

where f_s is expressed in terms of B as

$$f_s(\delta) = \sqrt{s}\left(1 - \frac{\delta}{\sqrt{s}}\right) B(s, s - \delta\sqrt{s}) \ , \tag{5}$$

with $\delta < \sqrt{s}$, and negative values of δ allowed. The function f_s is called the inverse Poisson-Mills ratio.

Cohen's equation (4) in the Halfin-Whitt regime was studied in considerable detail in [1]. Key results are

- existence, uniqueness and monotonicity properties of the solutions $a = a_s(\gamma)$ for $s \geq 1$ and $0 < \gamma < \sqrt{s}$,

- asymptotics of $a_s(\gamma)$ for fixed s according to

$$a_s(\gamma) = \frac{1}{\gamma} - \frac{2}{\sqrt{s}} - \left(1 - \frac{2}{s}\right)\gamma + O(\gamma^2) \ , \qquad \gamma \downarrow 0 \ , \tag{6}$$

– convergence as $s \to \infty$ of $a_s(\gamma)$ to $a_\infty(\gamma)$ according to

$$a_s(\gamma) - a_\infty(\gamma) = O\Big(\frac{1}{\sqrt{s}}\Big) \ , \tag{7}$$

uniformly in γ on any compact interval $[\gamma_0, \gamma_1]$ with $0 < \gamma_0 < \gamma_1 < \infty$,

– analysis of $a_\infty(\gamma)$ occurring in (7) as the unique solution a of the limiting form of Cohen's equation

$$a = f_\infty(\gamma - a) \ , \tag{8}$$

with $\gamma > 0$ and f_∞ the inverse of Mills ratio for the Gaussian distribution

$$f_\infty(\delta) = \frac{e^{-\frac{1}{2}\delta^2}}{\int\limits_{-\infty}^{\delta} e^{-\frac{1}{2}(\delta')^2}\, d\delta'} \tag{9}$$

– asymptotics of $a_\infty(\gamma)$ according to

$$a_\infty(\gamma) = \frac{1}{\gamma} - \gamma + 2\gamma^3 - 20\gamma^5 + 82\gamma^7 + O(\gamma^9) \ , \qquad \gamma \downarrow 0 \ . \tag{10}$$

The results in [1] on the retrial factors a_s and a_∞ were obtained by a thorough study of the Mills ratios f_s and f_∞. In this study the following facts were noted for f_∞ and established for f_s

– with $f = f_s$ or f_∞, there holds

$$f(\delta) > -\delta \ , \qquad -1 < f'(\delta) < 0 \ , \qquad f''(\delta) > 0 \ , \tag{11}$$

for $\delta < \sqrt{s}$ or $\delta < \infty$ in the respective cases,

– for fixed $s \geq 1$, there holds

$$f_s(\delta) = -\delta - \frac{1}{\delta} - \frac{2}{\delta^2\sqrt{s}} + \Big(2 - \frac{6}{s}\Big)\frac{1}{\delta^3} + O\Big(\frac{1}{\delta^4}\Big) \ , \qquad \delta < 0 \ , \tag{12}$$

and for the limiting case f_∞, there holds

$$f_\infty(\delta) = -\delta - \frac{1}{\delta} + \frac{2}{\delta^3} - \frac{10}{\delta^5} + \frac{74}{\delta^7} + O\Big(\frac{1}{\delta^9}\Big) \ , \qquad \delta < 0 \ , \tag{13}$$

– convergence as $s \to \infty$ of $f_s(\delta)$ to $f_\infty(\delta)$ according to

$$f_s(\delta) - f_\infty(\delta) = O\Big(\frac{1}{\sqrt{s}}\Big) \tag{14}$$

uniformly in δ on any compact interval $[\delta_0, \delta_1]$ with $-\infty < \delta_1 < \delta_2 < \infty$.

Cohen's equation (4) is ill-posed, and this manifests itself by divergence to $+\infty$ of the solutions $a_s(\gamma)$ and $a_\infty(\gamma)$, see (6) and (10), as $\gamma \downarrow 0$. This ill-posedness stems from (11–13) showing that the graphs of $f_s(\delta)$ and $f_\infty(\delta)$ lie above and are tangent to the straight line $\{(\delta, -\delta) \,|\, \delta < 0\}$. Hence, there is no solution a of the equation $a = f(\gamma - a)$ when $\gamma \le 0$ and $f = f_s$ or f_∞, and there is a unique solution a of this equation when $0 < \gamma < \sqrt{s}$ or $\gamma > 0$ in the respective cases. Moreover, the solution a increases to ∞ as γ tends to 0. From the leading asymptotics of $a_s(\gamma)$ and $a_\infty(\gamma)$ as $\gamma \downarrow 0$, one is led to expect that $a_s(\gamma) - a_\infty(\gamma) = O(1/\sqrt{s})$, also, see (7), when $\gamma \downarrow 0$. Such information is interesting from the system point of view, since it means that the Gaussian approximation is valid also in the critical regime $\gamma \downarrow 0$ with vanishing small overcapacity. However, the $O(\gamma^2)$ and $O(\gamma^9)$ terms at the right-hand sides of (6) and (10) do not allow such a conclusion.

A second concern is the fact that convergence of $a_s(\gamma)$ to $a_\infty(\gamma)$ is rather slow. One would be interested in being more precise about the $O(1/\sqrt{s})$ at the right-hand side of (7), for instance, in the form of a series

$$a_s(\gamma) = a_\infty(\gamma) + \frac{1}{\sqrt{s}}\, b_\infty(\gamma) + \frac{1}{s}\, c_\infty(\gamma) + \dots\,, \tag{15}$$

with validity in a set $\gamma \in (0, \gamma_s)$ with $\gamma_s \to \infty$ as $s \to \infty$. Evidently, an expansion as in (15) can only be expected to exist when a corresponding expansion

$$f_s(\delta) = f_\infty(\delta) + \frac{1}{\sqrt{s}}\, g_\infty(\delta) + \frac{1}{s}\, h_\infty(\delta) + \dots \tag{16}$$

would be available on a range $\delta \in (-\delta, \delta_s)$ with $\delta_s \to \infty$ as $s \to \infty$. An expansion of the type (16) for $B^{-1}(s, s - \delta\sqrt{s})$ has been developed by Jagerman, see [3], Theorem 14. However, there is no statement in [3], Theorem 14, about uniform validity of the expansion with respect to δ, which is required in our retrial setting where $\gamma \downarrow 0$ implies that $\delta = \gamma - a_s(\gamma) \to -\infty$ in Cohen's equation (4).

Overview of results

In Section 2 we consider, what is called in [1], Subsec. 4.1, the quasi-Gaussian representation of $f_s(\delta)$ in which $f_s(\delta)$ is displayed in a form reminiscent of the expression for f_∞ in (9). By using a substitution involving Lambert's W-function, see [4], Sec. 2, the integral contained in this quasi-Gaussian representation is brought into a form that is appropriate for repeated partial integrations, just as this is done when deriving the asymptotic series

$$l(x) \ := \ \frac{1}{f_\infty(-x)} \ = \ e^{\frac{1}{2}x^2} \int\limits_x^\infty e^{-\frac{1}{2}w^2}\, dw \ \sim \ \sum_{m=0}^\infty \frac{(-1)^m (2m-1)!!}{x^{2m+1}} \ = \ = \ \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \ldots \ , \qquad x \to \infty \ , \tag{17}$$

see [5], 7.2.14, case $n = 0$, on p. 300. There results the asymptotic series

$$\frac{1}{f_s(\delta)} \sim \frac{-1}{\delta} + \sum_{l=0}^\infty \Big\{ \sum_{j=0}^l \frac{(l+j+1)\, c_{lj}}{\delta^{l+3+j}\, s^{\frac{1}{2}(l-j)}} \Big\} \ , \qquad \delta \to -\infty \ , \tag{18}$$

in which the error after truncating the series after the term with $l = L$ has the same sign as and smaller modulus than the term with $l = L + 1$. The coefficients c_{jl} are given by means of a recursion and they are related to Ward polynomials, see [6]. When in (18) the limit $s \to \infty$ is taken, so that all terms with $j < l$ vanish, the asymptotic expansion (17) for $1/f_\infty(-x)$ with $x = -\delta$ is obtained. More generally, by formally summing the triangular array (l, j), $0 \le j \le l$, along diagonals with constant value of $l - j$, one obtains a formal expansion of $1/f_s(\delta)$ in powers of $1/\sqrt{s}$ with δ-dependent coefficients given in the form of asymptotic series.

Alternatively, by collecting the finite number of terms l, j with common value of $l + j$, one obtains an asymptotic expansion of $1/f_s(\delta)$ in powers of $1/\delta$. This expansion is not just formal: a truncation error assessment appropriate in the contact of asymptotic expansions can be made. In this form, approximations for $f_s(\delta)$ as $\delta \to -\infty$, and, subsequently, for $a_s(\gamma)$ as $\gamma \downarrow 0$ with uniform truncation error assessment can be found.

Interestingly, by employing the recursion satisfied by the c_{lj}, it can be shown, see [5], that

$$\frac{1}{B(s, s - \delta\sqrt{s})} = \sqrt{s}\, \frac{1 - \delta/\sqrt{s}}{f_s(\delta)} \sim 1 - \sum_{l=0}^\infty \Big\{ \sum_{j=0}^l \frac{c_{lj}}{\delta^{l+1+j}\, s^{\frac{1}{2}(l-j-1)}} \Big\} \ , \qquad \delta \to -\infty \ , \tag{19}$$

with a similar truncation error assessment as in (18).

In Section 3 we consider a representation of $1/f_s(\delta)$ in the form of a Laplace transform that can be derived from an integral representation of $B^{-1}(s, \lambda)$ as advocated by Jagers and Van Doorn in [7]. From this representation of $1/f_s(\delta)$, the asymptotic series for $1/f_s(\delta)$ in powers of $1/\delta$ is readily reestablished, and we use a method developed by Van Veen in [8] to handle truncation errors. The same approach can be used to show rigourously that $1/f_s(\delta) - 1/f_\infty(\delta)$ has an asymptotic series in powers of $1/\delta$ as $\delta \to -\infty$, obtained from the one for $1/f_s(\delta)$ by deleting the s-independent parts of the coefficients. In particular, we get $1/f_s(\delta) - 1/f_\infty(\delta) = O(\delta^{-4}/\sqrt{s})$, uniformly in $\delta < 0$ and $s \geq 1$. The latter result can be used to show that $a_s(\gamma) - a_\infty(\gamma) = O(1/\sqrt{s})$ as $s \to \infty$ uniformly in $\gamma \in (0, 1/3)$.

While both $\sqrt{s}\, B(s, s - \delta\sqrt{s})$ and $f_s(\delta)$ converge to $f_\infty(\delta)$ for any δ as $s \to \infty$, it is striking to see how much smaller the error $f_s(\delta) - f_\infty(\delta)$ is than the error $\sqrt{s}\, B(s, s - \delta\sqrt{s}) - f_\infty(\delta)$ for negative δ : $O(\delta^{-2}/\sqrt{s})$ compared to $O(\delta^2/(-\delta + \sqrt{s}))$ when $\delta \leq -1$. It is apparently inclusion of the factor $1 - \delta/\sqrt{s}$ at the right-hand side of (5) that brings about this accuracy improvement, a thing one would perhaps never have thought of to do when one were unaware of the connection with Cohen's equation in the Halfin-Whitt regime.

In Section 4 we combine the representation of $1/f_s(\delta)$ as a Laplace transform with Van Veen's method in a form slightly different from the form that was used in Section 3. This results into an asymptotic series

$$\frac{1}{f_s(\delta)} \sim \sum_{m=0}^{\infty} s^{-\frac{1}{2}m}\, K_m(s)\, I_m(\delta) \;, \qquad -\infty < \delta \leq s^{1/6} \;, \tag{20}$$

with term $m = 0$ given by $1/f_\infty(\delta)$. In (20) we have that $K_0(s) = 1$, $K_1(s) = K_2(s) = 0$, and $K_m(s)$, $m = 3, 4, \ldots$, are given recursively as polynomials in s with no constant term and of degree $\lfloor m/3 \rfloor$. Furthermore, $I_m(\delta)$ is a function of the form

$$I_m(\delta) = \frac{P_m(-\delta)}{f_\infty(\delta)} - Q_m(-\delta) \;, \tag{21}$$

with P_m and Q_m polynomials of degree m and $m - 1$, respectively, that occur as denominators and numerators of the continued fraction expansion of Laplace for the function $l(x)$, $x > 0$, in (17). For the series in (20), a truncation error analysis, based on Van Veen's method, can be made that is appropriate in the context of asymptotic series. In particular, when we truncate the summation in (20) after the term with $m = 3$, we get

$$\frac{1}{f_\infty(\delta)} + \frac{1}{3 f_\infty(\delta)\sqrt{s}}\,(\delta^3 + 3\delta + (\delta^2 + 2)\, f_\infty(\delta)) \tag{22}$$

as an approximation of $1/f_s(\delta)$ on a range $-\infty < \delta \leq s^{1/6}$ with an error $O(E(\delta)/s)$ with $E(\delta)$ expressible in the first few deleted terms of the series in (20). As a consequence, approximations of $f_s(\delta)$ in terms of $f_\infty(\delta)$ on a range $-\infty < \delta \leq s^{1/6}$, and subsequently, approximations of $a_s(\gamma)$ in terms of $a_\infty(\gamma)$ on a range $0 < \gamma < s^{1/6}$ can be given. Here the fact that the I_m can be expressed according to (21) into $f_\infty(\delta)$ is convenient when using $\delta = \gamma - a_\infty(\gamma)$ in the Gaussian limit form $a = f_\infty(\gamma - a)$ of Cohen's equation. Thus we find explicit forms for the first two correction terms $s^{-1/2}b_\infty(\gamma)$ and $s^{-1}c_\infty(\gamma)$ in the expansion of $a_s(\gamma)$ in (15).

2 Uniform asymptotics of $f_s(\delta)$, $\delta \to -\infty$, from quasi-Gaussian representation

We let for $s \geq 1$ and $\delta < \sqrt{s}$

$$\alpha(\delta) = \alpha_s(\delta) = \left(-2s\left(\frac{\delta}{\sqrt{s}} + \ln\left(1 - \frac{\delta}{\sqrt{s}}\right)\right)\right)^{1/2}, \tag{23}$$

where the square-root is taken such that $\mathrm{sgn}(\alpha_s(\delta)) = \mathrm{sgn}(\delta)$. Using this in the integral form (2) of B with $\lambda = s - \delta\sqrt{s}$, remembering the definition of f_s in (5) and substituting $\lambda' = s - \delta'\sqrt{s}$, it follows that

$$f_s(\delta) = \frac{(1 - \delta/\sqrt{s})\, e^{-\frac{1}{2}\alpha_s^2(\delta)}}{\int\limits_{-\infty}^{\delta} e^{-\frac{1}{2}\alpha_s^2(\delta')}\, d\delta'}, \qquad \delta < \sqrt{s}\,. \tag{24}$$

From Taylor expansion in (23), one has for $\delta \in \mathbb{R}$

$$\alpha_s(\delta) = \delta\left(1 + \frac{2\delta}{3\sqrt{s}} + \frac{\delta^2}{2s} + ...\right)^{1/2} = \delta + O\Big(\frac{1}{\sqrt{s}}\Big), \qquad s \to \infty\,. \tag{25}$$

This has been used in [1], Subsec. 5.9 to show that $f_\infty(\delta) - f_s(\delta) = O(1/\sqrt{s})$ uniformly in any compact set of $\delta \in \mathbb{R}$.

We have, see (17), when $\delta \to -\infty$,

$$\frac{1}{f_\infty(\delta)} = e^{\frac{1}{2}\delta^2} \int\limits_{-\infty}^{\delta} e^{-\frac{1}{2}(\delta')^2}\, d\delta' \sim -\sum_{m=0}^{\infty} \frac{(-1)^m(2m-1)!!}{\delta^{2m+1}}, \tag{26}$$

in which the error after truncating the series after the term with $m = M$ has the same sign as and smaller modulus than the term with $m = M + 1$. This is obtained from the integral representation of $1/f_\infty(\delta)$ in (26) by repeated partial integration. We shall derive a similar result for $1/f_s(\delta)$.

We denote for $x \in \mathbb{R}$ the solution $y \in (-\infty, 1)$ of

$$-y - \ln(1-y) = \tfrac{1}{2}\,x^2 \tag{27}$$

by $y(x)$, where we choose signs according to $\mathrm{sgn}(y(x)) = \mathrm{sgn}(x)$. This y is closely related to Lambert's W-function and is considered in detail in [4] and in [9], Appendix. In particular, y is an increasing and concave function of $x \in \mathbb{R}$ with $y(0) = 0$, $y'(0) = 1$. Noting from (23) that

$$-\frac{\delta}{\sqrt{s}} - \ln\Big(1 - \frac{\delta}{\sqrt{s}}\Big) = \frac{1}{2}\,(\alpha_s(\delta)/\sqrt{s})^2 \ , \tag{28}$$

it is seen that

$$y(\alpha_s(\delta)/\sqrt{s}) = \delta/\sqrt{s} \ , \qquad \delta < \sqrt{s} \ . \tag{29}$$

We consider the integral

$$I_s(\delta) := \int_{-\infty}^{\delta} e^{-\frac{1}{2}\alpha_s^2(\delta')}\,d\delta' \tag{30}$$

that occurs in the quasi-Gaussian representation (24) of $f_s(\delta)$, and we substitute $x = \alpha(\delta')$. By (29) we have

$$y'(\alpha(\delta)/\sqrt{s})\,\alpha'(\delta) = 1 \ , \qquad \delta < \sqrt{s} \ , \tag{31}$$

from differentiation with respect to δ, and so

$$I_s(\delta) = \int_{-\infty}^{\alpha(\delta)} e^{-\frac{1}{2}x^2}\,y'(x/\sqrt{s})\,dx \ , \qquad \delta < \sqrt{s} \ . \tag{32}$$

The form (32) of $I_s(\delta)$ lends itself for repeated partial integrations in which (29) and

$$\frac{1}{x}\,y'(x) = \frac{1}{y(x)} - 1 \ , \qquad x \in \mathbb{R} \ , \tag{33}$$

are instrumental in keeping the resulting expressions manageable. We first need some definitions. For $l = 0, 1, ...$ and integer j, we define c_{lj} recursively by

$$c_{00} = 1\,; \quad c_{0j} = 0 \ , \qquad j < 0 \text{ or } j > 0 \ , \tag{34}$$

$$\begin{aligned} c_{l+1,j} &= (l+1+j)\,c_{lj} - (l+j)\,c_{l,j-1} \ , \qquad j = 0, 1, ..., l+1 \ ; \\ c_{l+1,j} &= 0 \ , \qquad j < 0 \text{ or } j > l+1 \ . \end{aligned} \tag{35}$$

The following table gives $(-1)^j\,c_{lj}$ for $j = 0, 1, ..., l$ and $l = 0, 1, ..., 5$.

l \ j	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	2	5	3	0	0	0
3	6	26	35	15	0	0
4	24	154	340	315	105	0
5	120	1044	3304	4900	3465	945

Table 1. $(-1)^j\, c_{lj}$ for $j = 0, 1, ..., l$ and $l = 0, 1, 2, 3, 4, 5$.

It is not hard to show by induction that for $l = 1, 2, ...$

$$0 \le (-1)^j\, c_{lj} \le \frac{(l+j)!}{j!\, 2^j} \begin{pmatrix} l \\ j \end{pmatrix}, \qquad j = 0, 1, ..., l\ , \tag{36}$$

$$c_{l0} = l!\,, \quad (-1)^l\, c_{ll} = \frac{(2l)!}{l!\, 2^l} = (2l-1)!!\ . \tag{37}$$

Hence, (37) shows that there is equality in the second inequality in (36) for $j = 0$ and $j = l$. It follows also by induction that for $k \ge 1$

$$|c_{l,k-l}| \le \tfrac{1}{3}\,(k+2)\,k!\ , \qquad 0 \le l \le \tfrac{1}{2}\,k\ . \tag{38}$$

The inequality (38) sharpens the second inequality in (36) for large l and relatively small $j \ne 0$.

The relevance of the c_{lj} stems from the following result. We have for $l = 1, 2, ...$ that

$$\begin{aligned} \Big(\frac{1}{x}\,\frac{d}{dx}\Big)^l \Big(\frac{1}{x}\,y'\Big(\frac{x}{\sqrt{s}}\Big)\Big) \quad &= \quad -\,\frac{1-y(x/\sqrt{s})}{s^{l+1/2}} \sum_{j=0}^{l-1} \frac{(l+j)\,c_{l-1,j}}{y^{l+2+j}(x/\sqrt{s})} \quad = \\ &= \quad \frac{1}{s^{l+1/2}} \sum_{j=0}^{l} \frac{c_{lj}}{y^{l+1+j}(x/\sqrt{s})}\ . \end{aligned} \tag{39}$$

Indeed, from (33) we get

$$\begin{aligned} &\frac{1}{x}\,\frac{d}{dx}\Big(\frac{1}{x}\,y'\Big(\frac{x}{\sqrt{s}}\Big)\Big) = \frac{-1}{xs}\,\frac{y'(s/\sqrt{s})}{y^2(x/\sqrt{s})} \ = \\ &= -\,\frac{1-y(x/\sqrt{s})}{s^{3/2}}\,\frac{1}{y^3(x/\sqrt{s})} = \frac{1}{s^{3/2}}\Big(\frac{1}{y^2(x/\sqrt{s})} - \frac{1}{y^3(x/\sqrt{s})}\Big)\ , \end{aligned} \tag{40}$$

which shows (39) for $l = 1$, see Table 1. By induction, using (33) and the definition of the c_{lj} in (34–35), the validity of (39) is established for general $l = 1, 2, \ldots$. This leads to the following result.

Lemma 2.1. For $k = 0, 1, \ldots$ and $\alpha < 0$, we have

$$\int_{-\infty}^{\alpha} e^{-\frac{1}{2}x^2}\, y'(x/\sqrt{s})\, dx = \sum_{l=0}^{k} E_l(\alpha) + R_k(\alpha)\ , \tag{41}$$

where

$$E_0(\alpha) = -\,\frac{1 - y(\alpha/\sqrt{s})}{s^{1/2}}\ \frac{1}{y(\alpha/\sqrt{s})}\, e^{-\frac{1}{2}\alpha^2}\ , \tag{42}$$

and, for $l = 1, 2, \ldots$,

$$E_l(\alpha) = \frac{1 - y(\alpha/\sqrt{s})}{s^{l+1/2}} \sum_{j=0}^{l-1} \frac{(l+j)\, c_{l-1,j}}{y^{l+2+j}(\alpha/\sqrt{s})}\, e^{-\frac{1}{2}\alpha^2}\ . \tag{43}$$

Furthermore, $R_k(\alpha)$ is given by

$$R_k(\alpha) = \int_{-\infty}^{\alpha} \frac{d}{dx}\Big[\Big(\frac{1}{x}\ \frac{d}{dx}\Big)^k \Big(\frac{1}{x}\, y'(x/\sqrt{s})\Big)\Big]\, e^{-\frac{1}{2}x^2}\, dx\ , \tag{44}$$

and $R_k(\alpha)$ has the same sign as and smaller modulus than $E_{k+1}(\alpha)$.

Proof. We have by repeated partial integrations

$$\int_{-\infty}^{\alpha} e^{-\frac{1}{2}x^2}\, y'(x/\sqrt{s})\, dx\ =$$

$$=\ -\,\frac{1}{\alpha}\, y'(\alpha/\sqrt{s})\, e^{-\frac{1}{2}\alpha^2} + \int_{-\infty}^{\alpha} \frac{d}{dx}\Big(\frac{1}{x}\, y'(x/\sqrt{s})\Big)\, e^{-\frac{1}{2}x^2}\, dx\ =\ \ldots\ =$$

$$=\ -\sum_{l=0}^{k} \Big[\Big(\frac{1}{x}\ \frac{d}{dx}\Big)^l \Big(\frac{1}{x}\, y'(x/\sqrt{s})\Big)\, e^{-\frac{1}{2}x^2}\Big]_{x=\alpha} + R_k(\alpha)\ =$$

$$=\ \sum_{l=0}^{k} E_l(\alpha) + R_k(\alpha) \tag{45}$$

by (39), and this establishes (41) with $R_k(\alpha)$ as in (44).

Next, we have from (36), the last member of (39) and the fact that $y(x) < 0$ when $x < 0$ that

$$\operatorname{sgn}\Big[\Big(\frac{1}{x}\,\frac{d}{dx}\Big)^l\,\Big(\frac{1}{x}\,y'(x/\sqrt{s})\Big)\Big] = (-1)^{l+1}\ . \tag{46}$$

Therefore, $\operatorname{sgn}(E_l(\alpha)) = (-1)^l$. By differentiating the last member of (39) and using $y'(x) > 0$, it is also seen that

$$\operatorname{sgn}\Big[\frac{d}{dx}\Big[\Big(\frac{1}{x}\,\frac{d}{dx}\Big)^l\,\Big(\frac{1}{x}\,y'(x/\sqrt{s})\Big)\Big]\Big] = (-1)^{l+1}\ ,\qquad x < 0\ . \tag{47}$$

Therefore, $\operatorname{sgn}(R_k(\alpha)) = (-1)^{k+1}$. Then from

$$R_k(\alpha) = E_{k+1}(\alpha) + R_{k+1}(\alpha) \tag{48}$$

and

$$\operatorname{sgn}(R_k(\alpha)) = \operatorname{sgn}(E_{k+1}(\alpha)) = -\operatorname{sgn}(R_{k+1}(\alpha))\ , \tag{49}$$

it follows that $|R_k(\alpha)| < |E_{k+1}(\alpha)|$, as required.

Theorem 2.2. We have

$$\frac{1}{f_s(\delta)} \sim -\frac{1}{\delta} + \sum_{l=0}^{\infty}\Big\{\sum_{j=0}^{l}\frac{(l+j+1)\,c_{lj}}{\delta^{l+3+j}\,s^{\frac{1}{2}(l-j)}}\Big\}\ ,\qquad \delta\to-\infty\ . \tag{50}$$

The error

$$\frac{1}{f_s(\delta)} + \frac{1}{\delta} - \sum_{l=0}^{k}\Big\{\sum_{j=0}^{l}\frac{(l+j+1)\,c_{lj}}{\delta^{l+3+j}\,s^{\frac{1}{2}(l-j)}}\Big\} \tag{51}$$

that occurs when the series in (50) is truncated after the term with $l = k$ has for any $\delta < 0$ the same sign as and smaller modulus than the term in the series over l with $l = k+1$.

Proof. We use Lemma 2.1 in which we take $\alpha = \alpha_s(\delta/\sqrt{s})$ with $\delta < 0$. Then we recall the definition (30) of $I_s(\delta)$ and the formula (32) and use that in the expression (24) for $f_s(\delta)$. Finally, the term with $l = 0$ in (41) is set apart, we take $k+1$ instead of k in (41), and we write $l+1$ with $l = 0, 1, ..., k$ in the terms in the series instead of l with $l = 1, ..., k+1$.

Notes.

1. For any $l = 0, 1, \ldots$, all terms in the series over j between { } in (50) have sign $(-1)^{l+1}$. Hence, the modulus of term with $l = k+1$ equals

$$\sum_{j=0}^{k+1}\frac{(k+j+2)\,|c_{k+1,j}|}{|\delta|^{k+4+j}\,s^{\frac{1}{2}(k+1-j)}} = O\Big[\frac{1}{|\delta|^{k+4}\,s^{\frac{1}{2}(k+1)}} + \frac{1}{|\delta|^{2k+5}}\Big] \tag{52}$$

in which the constant implied bij O only depends on k when $\delta \le -1$ and $s \ge 1$. The right-hand side of (52) accounts for the two extreme terms with $j = 0$ and $j = k + 1$ in the series at the left-hand side.

2. Denote the terms in the double series in (50) by

$$G_{lj} = \frac{(l+j+1)\, c_{lj}}{\delta^{l+3+j}\, s^{\frac{1}{2}(l-j)}} \;, \qquad j = 0, 1, ..., j \;, \quad l = 0, 1, ... \;. \tag{53}$$

We have

$$G_{ll} = \frac{(2l+1)!!\,(-1)^l}{\delta^{2l+3}} \;, \qquad l = 0, 1, ... \;, \tag{54}$$

and this G_{ll} coincides with the term with index $m = l+1$ in the asymptotic series in (26) for $1/f_\infty(\delta)$, $\delta \to -\infty$. In Section 4 we shall identify the terms $G_{l,l-1}$, $l = 1, 2, ...$, in a similar manner.

3. Summing the terms G_{lj} in (50) by grouping the terms with constant value k of $l + j$ yields the expansion

$$\frac{1}{f_s(\delta)} \sim -\frac{1}{\delta} + \sum_{k=0}^{\infty} \frac{k+1}{\delta^{k+3}} \left\{ \sum_{l=\lceil \frac{k}{2} \rceil}^{k} \frac{c_{l,k-l}}{s^{l-k/2}} \right\} \tag{55}$$

as $\delta \to -\infty$. The truncation error when we truncate the series over k at the right-hand side of (55) at $k = K$ can be estimated by Theorem 2.2 by

$$\sum_{\substack{l+j>K,\\ 0\le j\le l\le K+1}} |G_{lj}| \;. \tag{56}$$

The quantity in (56) is $O(\delta^{-K-4})$ when K is odd and $O(\delta^{-K-4}\, s^{-1/2}) + O(\delta^{-K-5})$ when K is even and the O's implicit constants only depend on K when $\delta \le -1$ and $s \ge 1$. The asymptotic series (55) and its truncation analysis shall be considered in more detail in Section 3, using a different method to derive (55).

4. There holds

$$\frac{1}{B(s, s - \delta\sqrt{s})} \sim 1 - \sum_{l=0}^{\infty} \left\{ \sum_{j=0}^{l} \frac{c_{lj}}{\delta^{l+1+j}\, s^{\frac{1}{2}(l-1-j)}} \right\} \tag{57}$$

as $\delta \to -\infty$, and the truncation error when truncating the series over l after the term with $l = k$ has the same sign as and smaller modulus than $-$ the expression in $\{\ \}$ with $l = K + 1$. This follows from

$$\frac{1}{\sqrt{s}}\, \frac{1}{B(s, s - \delta\sqrt{s})} = \frac{1 - \delta/\sqrt{s}}{f_s(\delta)} = e^{\frac{1}{2}\alpha^2(\delta)} \int_{-\infty}^{\alpha(\delta)} e^{-\frac{1}{2}x^2}\, y'(x/\sqrt{s})\, dx \tag{58}$$

and Lemma 2.1 where the recursion (34–35) for the c_{lj} is used to write

$$\sum_{l=1}^{k} \frac{1-\delta/\sqrt{s}}{s^{l+1/2}} \sum_{j=0}^{l-1} \frac{(l+j)\, c_{l-1,j}}{(\delta/\sqrt{s})^{l+2+j}} = -\sum_{l=1}^{k} \frac{1}{s^{l+1/2}} \sum_{j=0}^{l} \frac{c_{lj}}{(\delta/\sqrt{s})^{l+1+j}} \ . \quad (59)$$

5. As in 3, see (55), we have from (57) the asymptotic expansion, as $\delta \to -\infty$,

$$\frac{1}{B(s, s-\delta\sqrt{s})} \sim 1 - \sum_{k=0}^{\infty} \frac{1}{\delta^{k+1}} \sum_{l=\lceil \frac{k}{2} \rceil}^{k} \frac{c_{l,k-l}}{s^{l-\frac{1}{2}(k+1)}} \ . \quad (60)$$

   For integer s, we have that $B^{-1}(s, s-\delta\sqrt{s})$ is a rational function of δ and thus possesses an absolutely convergent expansion in powers of $1/\delta$ when $|\delta|$ is large. The coefficients in this expansion coincide with what the expansion in (60) gives. Hence, in that case the expansion (60) actually converges for large values of $|\delta|$.

6. There is the following connection with Ward polynomials (that occur in the theory of Stirling numbers, see [6]). We have for $l = 1, 2, ...$

$$\sum_{j=0}^{l} (-1)^j\, c_{lj}\, t^j = \sum_{r=0}^{l-1} (-1)^r\, H_l^r (1+t)^{l-r} \ , \quad (61)$$

   where the positive integers H_l^r for $l = 1, 2, ...$ and $r = 0, 1, ..., l-1$ are the coefficients that occur in the expression in [6], (3.31) for the Stirling polynomials. These H_l^r are given in the table in [6] on p. 92 for $0 \le r \le l-1 \le 9$. Thus a series $\sum_{j=0}^{l} c_{lj} (\delta/\sqrt{s})^{-l-1-j}$, as occurs at the right-hand side of (59), can be expressed directly in terms of the Ward polynomial $\sum_{r=0}^{l-1} H_l^r\, x^{l-r}$.

Approximation of $f_s(\delta)$ as $\delta \to -\infty$ and of $a_s(\gamma)$ as $\gamma \downarrow 0$

As an example, we consider the case that we truncate in (55) at $k = 4$. Using Table 1, we thus have

$$\begin{aligned} \frac{1}{f_s(\delta)} \quad &= \quad -\frac{1}{\delta} + \frac{1}{\delta^3} + \frac{2}{\delta^4\sqrt{s}} - \frac{1}{\delta^5}\left(3 - \frac{6}{s}\right) - \frac{1}{\delta^6\sqrt{s}}\left(20 - \frac{24}{s}\right) + \\ &\quad + \frac{1}{\delta^7}\left(15 - \frac{130}{s} + \frac{124}{s^2}\right) + \varepsilon \ , \end{aligned} \quad (62)$$

in which $\varepsilon = O(\delta^{-8}\, s^{-1/2}) + O(\delta^{-9})$ for $\delta \leq -1$ and $s \geq 1$. Using $(1-x)^{-1} = 1 + x + x^2 + x^3 + O(x^4)$, $|x| \leq 1/2$, we then compute the approximation

$$
\begin{aligned}
f_s(\delta) &= -\delta\Big(1 - \frac{1}{\delta^2} - \frac{2}{\delta^3\sqrt{s}} + \frac{1}{\delta^4}\Big(3 - \frac{6}{s}\Big) + \frac{1}{\delta^5\sqrt{s}}\Big(20 - \frac{24}{s}\Big) + \\
&\quad - \frac{1}{\delta^6}\Big(15 - \frac{130}{s} + \frac{124}{s^2}\Big) - \varepsilon\Big)^{-1} = \\
&= -\,\delta - \frac{1}{\delta} - \frac{2}{\delta^2\sqrt{s}} + \Big(2 - \frac{6}{s}\Big)\frac{1}{\delta^3} + \Big(16 - \frac{24}{s}\Big)\frac{1}{\delta^4\sqrt{s}} + \\
&\quad - \Big(10 - \frac{114}{s^2} + \frac{120}{s^2}\Big)\frac{1}{\delta^5} + \varepsilon_f\ ,
\end{aligned}
\tag{63}
$$

where $\varepsilon_f = O(\delta^{-6}\, s^{-1/2}) + O(\delta^{-7})$ for $\delta \leq -1$ and $s \geq 1$.

We use this approximation of $f_s(\delta)$ for $\delta \leq -1$ and $s \geq 1$ to compute an approximation of $a_s(\gamma)$ as $\gamma \downarrow 0$, using the fact that $a_s(\gamma) \to +\infty$ as $1/\gamma$ when $\gamma \downarrow 0$. From Cohen's equation $a_s(\gamma) = f_s(\gamma - a_s(\gamma))$, we need to know when $\gamma - a_s(\gamma) \leq -1$ when we want to use (63). Now, by [1], Theorem 7, $a_s(\gamma)$ increases in $s \geq 1$ and $a_1(\gamma) = (1-\gamma^2)\,\gamma^{-1}$. Hence, $\gamma - a_s(\gamma) \leq -1$ for all $s \geq 1$ when $\gamma - (1-\gamma^2)\,\gamma^{-1} \leq -1$, i.e., when $0 < \gamma \leq 1/3$. Thus, in this region of γ, we approximate (ignoring terms like ε_f)

$$
\begin{aligned}
a &= f_s(\gamma - a) = \\
&= a - \gamma + \frac{1}{a-\gamma} - \frac{2}{(a-\gamma)^2\sqrt{s}} - \Big(2 - \frac{6}{s}\Big)\frac{1}{(a-\gamma)^3} + \\
&\quad + \Big(16 - \frac{24}{s}\Big)\frac{1}{(a-\gamma)^4\sqrt{s}} + \Big(10 - \frac{114}{s} + \frac{120}{s^2}\Big)\frac{1}{(a-\gamma)^5}\ .
\end{aligned}
\tag{64}
$$

With $b = \gamma(a-\gamma)$, equation (64) can be written, after multiplication by $a - \gamma$, as

$$
b = 1 - \frac{2\gamma}{b\sqrt{s}} - \frac{\Big(2 - \frac{6}{s}\Big)\gamma^2}{b^2} + \frac{\Big(16 - \frac{24}{s}\Big)\gamma^3}{b^3\sqrt{s}} + \Big(10 - \frac{114}{s} + \frac{120}{s^2}\Big)\frac{\gamma^4}{b^4}\ . \tag{65}
$$

This form is appropriate for iteration and we find, by also keeping track of terms like ε_f in (63), that

$$\gamma\, a_s(\gamma) \;=\; 1 - \frac{2\gamma}{\sqrt{s}} - \Big(1 - \frac{2}{s}\Big)\gamma^2 \;+ \\ + 4\Big(1-\frac{1}{s}\Big)\frac{\gamma^3}{\sqrt{s}} + 2\Big(1-\frac{1}{s}\Big)\Big(1-\frac{8}{s}\Big)\gamma^4 \;+ \\ + O(\gamma^5/\sqrt{s}) + O(\gamma^6)\;, \qquad (66)$$

where the O's hold uniformly in $s \geq 1$ and $0 < \gamma \leq 1/3$.

3 Asymptotic expansion of $1/f_s(\delta)$ and $1/f_s(\delta) - 1/f_\infty(\delta)$ as $\delta \to -\infty$ from the Jagers-Van Doorn representation of Erlang's B using Van Veen's approach

With the substitution $t = (\lambda'/\lambda) - 1$ in (2), we have for $s \geq 1$ and $\lambda > 0$

$$\lambda\, B(s,\lambda) = \left(\int_0^\infty e^{-\lambda t}(1+t)^{-s}\, dt\right)^{-1}\;, \qquad (67)$$

see [7]. Therefore, from (5) and (67) with $\lambda = s - \delta\sqrt{s}$, we have for $\delta < \sqrt{s}$

$$\frac{1}{f_s(\delta)} \;=\; \sqrt{s}\int_0^\infty e^{-(s-\delta\sqrt{s})t}(1+t)^{-s}\, dt \;= \\ = \int_0^\infty e^{\delta v}\, e^{-s[\frac{v}{\sqrt{s}} - \ln(1+\frac{v}{\sqrt{s}})]}\, dv\;, \qquad (68)$$

where the substitution $v = t\sqrt{s}$ has been used to obtain the last integral. The representation of $1/f_s(\delta)$ in the form (68) as a Laplace transform gives a direct way to get an asymptotic result in powers of $1/\delta$ as $\delta \to -\infty$. We develop

$$C(z) = C_s(z) := (1+z)^s\, e^{-sz} = e^{-s[z-\ln(1+z)]} = \sum_{k=0}^{\infty} C_k(s)\, z^k\;. \qquad (69)$$

Using this expansion in (68) with $z = v/\sqrt{s}$, we get at once the asymptotic expansion

$$\frac{1}{f_s(\delta)} \sim \sum_{k=0}^{\infty} \frac{C_k(s)}{s^{k/2}} \int_0^{\infty} e^{\delta v}\, v^k \, dv =$$

$$= \sum_{k=0}^{\infty} \frac{k!\, C_k(s)}{s^{k/2}} \left(\frac{-1}{\delta}\right)^{k+1} , \qquad \delta \to -\infty \ . \tag{70}$$

By identification of the terms in the asymptotic expansions in (55) and (70), it is seen that $C_0(s) = 1$, $C_1(s) = 0$, and that for $k = 0, 1, ...$

$$C_{k+2}(s) = (-1)^{k+1} \sum_{l=\lceil \frac{k}{2} \rceil}^{k} \frac{c_{l,k-l}}{(k+2)\, k!}\, s^{k+1-l} \ . \tag{71}$$

In particular, the coefficients in $C_{k+2}(s)$ of s^j with $j = k+1, ..., k+2-\lceil \frac{1}{2}\, k \rceil$ vanish, and the C_k's are closely related to Ward polynomials. Also recall the bound (38).

We note that for fixed δ

$$\lim_{s\to\infty} \frac{1}{f_s(\delta)} = \int_0^{\infty} e^{\delta v - \frac{1}{2} v^2}\, dv = e^{\frac{1}{2}\delta^2} \int_{-\infty}^{\delta} e^{-\frac{1}{2} w^2}\, dw = \frac{1}{f_\infty(\delta)} \ , \tag{72}$$

see (9). We aim at relating the asymptotic expansions of $1/f_s(\delta)$ and of $1/f_\infty(\delta)$, see (26), and to that end a careful analysis of the truncation errors in (70) has to be carried out. We follow for this a method developed by Van Veen [8] in the 1930's for obtaining uniform asymptotic expansions for the Hermite polynomials. The actual case considered by Van Veen led to power series expansion of $(1+z)^s \exp(-s\,[z - \frac{1}{2}\, z^2])$, with $s = -n-1 = -1, -2, \ldots$, and, in fact we shall use his results for this case in Section 4. Also, see [10], Ch. 19 and in particular Sec. 19.7 for expansions involving the exponential function.

We have as in [8], §4,with C given in (69), for $z \geq 0$

$$(1+z)\, C'(z) + sz\, C(z) = 0 \ , \qquad C(0) = C_0 = 1 \ . \tag{73}$$

From this it is readily seen that

$$C_1 = 0 \ ; \qquad (m+1)\, C_{m+1} + m\, C_m + s\, C_{m-1} = 0 \, , \quad m = 1, 2, \ldots \ . \tag{74}$$

Setting $B_m = (-1)^m \, m \, C_m$, we then find

$$B_0 = 0 \, , \quad B_1 = 0 \, , \quad B_2 = -s \; ; \qquad B_{m+1} = B_m - \frac{s}{m-1} \, B_{m-1} \, , \quad m = 2, 3, \ldots \, . \tag{75}$$

The first 9 B's are given by (75) and by

$$B_3 = -s \, , \quad B_4 = -s + \tfrac{1}{2} \, s^2 \, , \quad B_5 = -s + \tfrac{5}{6} \, s^2 \, , \quad B_6 = -s + \tfrac{13}{12} \, s^2 - \tfrac{1}{8} \, s^3 \, ,$$
$$B_7 = -s + \tfrac{77}{60} \, s^2 - \tfrac{7}{24} \, s^3 \, , \quad B_8 = -s + \tfrac{29}{20} \, s^2 - \tfrac{17}{36} \, s^3 + \tfrac{1}{48} \, s^4 \; . \tag{76}$$

From (71) and $B_m = (-1)^m \, m \, C_m$, we have

$$B_{k+2} = - \sum_{l=\lceil \frac{k}{2} \rceil}^{k} \frac{1}{k!} \, c_{l,k-l} \, s^{k+1-l} \; , \tag{77}$$

and one can check (76) with Table 1 for the c's.

We next set for $z \geq 0$ and $k = 0, 1, \ldots$

$$C(z) = \sum_{m=0}^{k} C_m \, z^m + R_k(z) \; . \tag{78}$$

Inserting this into (73) and using (74), we get

$$(1+z) \, R'_k(z) + sz \, R_k(z) = -k \, C_k \, z^k - s(C_{k-1} \, z^k + C_k \, z^{k+1}) \; . \tag{79}$$

Again using (74) and $B_m = (-1)^m \, m \, C_m$, we finally get

$$(1+z) \, R'_k(z) + sz \, R_k(z) = (-1)^{k+1} \left\{ B_{k+1} \, z^k (1+z) - B_{k+2} \, z^{k+1} \right\} \; , \tag{80}$$

i.e., we have for $z \geq 0$

$$R'_k(z) + \frac{sz}{1+z} \, R_k(z) = (-1)^{k+1} \left\{ B_{k+1} \, z^k - B_{k+2} \, \frac{z^{k+1}}{1+z} \right\} \; . \tag{81}$$

Solving this first order ODE for R_k, using

$$C(0) = 1 \, , \quad R_k(0) = 0 \; ; \qquad \frac{sz}{1+z} = - \frac{C'(z)}{C(z)} \, , \quad z \geq 0 \; , \tag{82}$$

we then get for $z \geq 0$

$$R_k(z) = (-1)^{k+1} \int_0^z \frac{C(z)}{C(\bar{z})} \left\{ B_{k+1} \, \bar{z}^k - B_{k+2} \, \frac{\bar{z}^{k+1}}{1+\bar{z}} \right\} d\bar{z} \; . \tag{83}$$

From

$$\frac{C(z)}{C(\bar{z})} = \exp\left[-s\int\limits_{\bar{z}}^{z} \frac{x}{1+x}\,dx\right] \leq 1\ , \qquad 0 \leq \bar{z} \leq z\ , \tag{84}$$

it is then see that for $z \geq 0$

$$|R_k(z)| \leq |B_{k+1}|\,\frac{z^{k+1}}{k+1} + |B_{k+2}|\,\frac{z^{k+2}}{k+2}\ . \tag{85}$$

We now use (85) in (68) with $z = v/\sqrt{s}$, so that for $v \geq 0$

$$\exp(-s\,[v/\sqrt{s} - \ln(1+v/\sqrt{s})]) = \sum_{m=0}^{k} \frac{C_m}{s^{m/2}}\,v^m + R_k(v/\sqrt{s})\ , \tag{86}$$

with R_k bounded according to (85). We thus obtain

$$\frac{1}{f_s(\delta)} = \sum_{m=0}^{k} \frac{m!\,C_m}{s^{m/2}} \left(\frac{-1}{\delta}\right)^{m+1} + E_k\ , \tag{87}$$

where for $\delta < 0$ and $s \geq 1$

$$\begin{aligned} |E_k| \ &\leq\ \frac{|B_{k+1}|}{k+1}\,\frac{1}{s^{\frac{1}{2}(k+1)}} \int\limits_0^v v^{k+1}\,e^{\delta v}\,dv\ + \\ &\quad + \frac{|B_{k+2}|}{k+2}\,\frac{1}{s^{\frac{1}{2}(k+2)}} \int\limits_0^v v^{k+2}\,e^{\delta v}\,dv\ = \\ &=\ \frac{(k+1)!\,|C_{k+1}|}{s^{\frac{1}{2}(k+1)}(-\delta)^{k+2}} + \frac{(k+2)!\,|C_{k+2}|}{s^{\frac{1}{2}(k+2)}(-\delta)^{k+3}}\ . \end{aligned} \tag{88}$$

Thus we have reestablished the result (55) in Section 2 with a better and simpler estimate for the remainder than what is provided by (56).

We now proceed with applying Van Veen's approach to estimating truncation errors in the asymptotic expansion of $1/f_s(\delta) - 1/f_\infty(\delta)$ as $\delta \to -\infty$. We let

$$D_m(s) = \frac{C_m(s)}{s^{m/2}}\ , \qquad D_m = \lim_{s\to\infty} D_m(s)\ , \tag{89}$$

so that, see (86),

$$C(v/\sqrt{s}) = \sum_{m=0}^{k} D_m(s)\,v^m + R_k(v/\sqrt{s})\ , \tag{90}$$

and we write

$$H(v) := \lim_{s\to\infty} C(v/\sqrt{s}) = e^{-\frac{1}{2}v^2} = \sum_{m=0}^{k} D_m\, v^m + S_k(v)\ . \tag{91}$$

In Table 2 we display $D_m(s)$ and D_m, noting that $D_m(s)$ is given by (89) and (71) and that

$$D_{2j} = (-1/2)^j/j!\ ,\quad D_{2j+1} = 0\ ,\qquad j = 0, 1, \ldots\ . \tag{92}$$

m	$D_m(s)$	D_m
0	1	1
1	0	0
2	$-1/2$	$-1/2$
3	$\frac{1}{3\sqrt{s}}$	0
4	$1/8 - \frac{1}{4s}$	$1/8$
5	$\frac{-1}{6\sqrt{s}} + \frac{1}{5s\sqrt{s}}$	0
6	$-1/48 + \frac{13}{72s} - \frac{1}{6s^2}$	$-1/48$

Table 2. $D_m(s)$ and D_m for $m = 0, 1, ..., 6$.

From (83) with $z = v/\sqrt{s}$ and substituting $\bar{z} = \bar{v}/\sqrt{s}$ we have

$$R_k(v/\sqrt{s}) = \frac{1}{\sqrt{s}} \int_0^v Q_k\Big(\frac{\bar{v}}{\sqrt{s}}\Big) \frac{C(v/\sqrt{s})}{C(\bar{v}/\sqrt{s})}\, d\bar{v}\ , \tag{93}$$

where

$$\begin{aligned} Q_k\Big(\frac{\bar{v}}{\sqrt{s}}\Big) &= (-1)^{k+1}\Big\{B_{k+1}(s)\Big(\frac{\bar{v}}{\sqrt{s}}\Big)^k - B_{k+2}(s)\,\frac{(\bar{v}/\sqrt{s})^{k+1}}{1+\bar{v}/\sqrt{s}}\Big\} = \\ &= \sqrt{s}\Big\{(k+1)\,D_{k+1}(s)\,\bar{v}^k + (k+2)\,D_{k+2}(s)\,\frac{\bar{v}^{k+1}}{1+\bar{v}/\sqrt{s}}\Big\}\,. \end{aligned} \tag{94}$$

Furthermore, by letting $s \to \infty$, we have

$$S_k(v) := \lim_{s\to\infty} R_k(v/\sqrt{s}) = \int_0^v T(\bar{v})\,\frac{H(v)}{H(\bar{v})}\, d\bar{v}\ , \tag{95}$$

where

$$T(\bar{v}) = \lim_{s\to\infty} \frac{1}{\sqrt{s}}\, Q_k\Big(\frac{\bar{v}}{\sqrt{s}}\Big) = (k+1)\, D_{k+1}\, \bar{v}^k + (k+2)\, \bar{v}^{k+1} \ . \qquad (96)$$

Therefore, for $\delta < 0$,

$$\frac{1}{f_s(\delta)} - \frac{1}{f_\infty(\delta)} \;=\; \sum_{m=0}^{k} m!(D_m(s) - D_m)\Big(\frac{-1}{\delta}\Big)^{m+1} \;+$$

$$+ \int_0^\infty e^{\delta v}(R_k(v/\sqrt{s}) - S_k(v))\, dv \ , \qquad (97)$$

with S_k given by (95) and R_k given by (93–94). We write for $v \geq 0$, using (94) and (96)

$$R_k(v/\sqrt{s}) - S_k(v) \;=$$

$$= \int_0^v \Big(\frac{C(v/\sqrt{s})}{C(\bar{v}/\sqrt{s})} - \frac{H(v)}{H(\bar{v})}\Big) \bar{v}^k \left\{(k{+}1)\, D_{k+1}(s) {-} (k+2)\, D_{k+2}(s)\, \frac{\bar{v}}{1{+}\bar{v}/\sqrt{s}}\right\} d\bar{v} \;+$$

$$+ \int_0^v \frac{H(v)}{H(\bar{v})}\, \bar{v}^k \Big\{(k+1)(D_{k+1}(s) - D_{k+1}) + (k+2)\, \bar{v}(D_{k+2}(s) - D_{k+2}) \;+$$

$$- \frac{1}{\sqrt{s}}\, (k+2)\, \bar{v}^2\, \frac{D_{k+2}(s)}{1 + \bar{v}/\sqrt{s}}\Big\}\, d\bar{v} \ . \qquad (98)$$

The second term at the right-hand side of (98), T_2, can be estimated using

$$\int_0^v \frac{H(v)}{H(\bar{v})}\, \bar{v}^l\, d\bar{v} = \int_0^v \bar{v}^l\, e^{-\frac{1}{2}(v^2 - \bar{v}^2)}\, d\bar{v} \leq \frac{v^{l+1}}{l+\ 1} \ . \qquad (99)$$

Hence

$$|T_2| \;\leq\; |D_{k+1}(s) - D_{k+1}|\, v^{k+1} + |D_{k+2}(s) - D_{k+2}|\, v^{k+2} \;+$$

$$+ \frac{1}{\sqrt{s}}\, \frac{k+2}{k+3}\, |D_{k+2}(s)|\, v^{k+3} \ . \qquad (100)$$

As to the first term at the right-hand side of (98), T_1, we use

$$\frac{C(v/\sqrt{s})}{C(\bar{v}/\sqrt{s})} - \frac{H(v)}{H(\bar{v})} =$$

$$= \exp\left[-\int_{\bar{v}}^{v} \frac{w}{1+w/\sqrt{s}}\,dv\right] - \exp\left[-\int_{\bar{v}}^{v} w\,dw\right] =$$

$$= \exp\left[-\int_{\bar{v}}^{v} \frac{w}{1+w/\sqrt{s}}\,dw\right]\left(1-\exp\left[-\frac{1}{\sqrt{s}}\int_{\bar{v}}^{v} \frac{w^2}{1+w/\sqrt{s}}\,dw\right]\right) \leq$$

$$\leq \exp\left[-\int_{\bar{v}}^{v} \frac{w}{1+w/\sqrt{s}}\,dw\right] \frac{1}{\sqrt{s}}\int_{\bar{v}}^{v} w^2\,dw\ . \tag{101}$$

By concavity of the function $w \geq 0 \mapsto w/(1+w/\sqrt{s})$, we have

$$\int_{\bar{v}}^{v} \frac{w}{1+w/\sqrt{s}}\,dw \geq \tfrac{1}{2}(v-\bar{v})\left(\frac{v}{1+v/\sqrt{s}}+\frac{\bar{v}}{1+\bar{v}/\sqrt{s}}\right) \geq$$

$$\geq \tfrac{1}{2}(v-\bar{v})\,\frac{v}{1+v/\sqrt{s}}\ . \tag{102}$$

Also,

$$\int_{\bar{v}}^{v} w^2\,dw \leq v^2(v-\bar{v})\ . \tag{103}$$

Therefore,

$$\frac{C(v/\sqrt{s})}{C(\bar{v}/\sqrt{s})} - \frac{H(v)}{H(\bar{v})} \leq \frac{1}{\sqrt{s}}\,(v-\bar{v})\,v^2 \exp\Big(-\frac{1}{2}\,\frac{v}{1+v/\sqrt{s}}\,(v-\bar{v})\Big)\ . \tag{104}$$

It follows that the first term T_1 can be estimated as

$$|T_1| \leq \frac{1}{\sqrt{s}}\,v^2((k+1)\,|D_{k+1}(s)|\,I_k + (k+2)\,|D_{k+2}(s)|\,I_{k+1})\ , \tag{105}$$

where, with $\alpha = v/2(1+v/\sqrt{s})$,

$$I_j \;=\; \int\limits_0^v (v-\bar{v})\,\bar{v}^j\, e^{-\alpha(v-\bar{v})}\, d\bar{v} \;\le$$

$$\le \int\limits_0^v (v-w)^j\, w\, dw = \frac{v^{j+2}}{(j+1)(j+2)}\ . \tag{106}$$

It follows that

$$|T_1| \le \frac{1}{\sqrt{s}}\ \frac{|D_{k+1}(s)|}{k+2}\, v^{k+3} + \frac{1}{\sqrt{s}}\ \frac{|D_{k+2}(s)|}{k+3}\, v^{k+4}\ . \tag{107}$$

We then finally find from (98) and (100), (107) that

$$|R_k(v/\sqrt{s}) - S_k(v)| \le \frac{1}{\sqrt{s}}\ \frac{|D_{k+1}(s)|}{k+2}\, v^{k+3} + \frac{1}{\sqrt{s}}\ \frac{|D_{k+2}(s)|}{k+3}\, v^{k+4}\ +$$

$$+\ |D_{k+1}(s) - D_{k+1}|\, v^{k+1} + |D_{k+2}(s) - D_{k+2}|\, v^{k+2} + \frac{1}{\sqrt{s}}\, |D_{k+2}(s)|\, v^{k+3}\ . \tag{108}$$

This estimate should now be inserted in the integral at the right-hand side of (97). We observe now that, see (89) and (71) and Table 2.

$$|D_{k+1}(s) - D_{k+1}|, |D_{k+2}(s) - D_{k+2}| = O\Big(\frac{1}{\sqrt{s}}\Big)\ . \tag{109}$$

Furthermore, from

$$\int\limits_0^\infty v^j\, e^{\delta v}\, dv = j!(-1/\delta)^{j+1}\ , \tag{110}$$

it is seen that the contributions of the terms at the right-hand side of (108) to the integral in (97) are all $O(\delta^{-k-2}\, s^{-1/2})$ in which the constant implied by the O only depends on k when $s \ge 1$ and $\delta \le -1$. Therefore,

$$\frac{1}{f_s(\delta)} - \frac{1}{f_\infty(\delta)} = \sum_{m=0}^{k} m!(D_m(s) - D_m)\Big(\frac{-1}{\delta}\Big)^{m+1} + O\Big(\frac{1}{\delta^{k+2}\sqrt{s}}\Big) \tag{111}$$

uniformly in $s \ge 1$ and $\delta \le -1$. That is, the asymptotic expansion of $1/f_s(\delta) - 1/f_\infty(\delta)$ as $\delta \to -\infty$ is obtained from the expansions of $1/f_s(\delta)$ and

$1/f_\infty(\delta)$ by subtracting them termwise, and the remainder after truncation at the k^{th} term is of the order of the term with index $k+1$, uniformly in $s \geq 1$ and $\delta \leq -1$.

We use this result to estimate $f_\infty(\delta) - f_s(\delta)$ when $\delta \leq -1$ and $s \geq 1$. With $k = 3$ in (111), using Table 2, we get for $s \geq 1$ and $\delta \leq -1$

$$\frac{1}{f_s(\delta)} - \frac{1}{f_\infty(\delta)} = \frac{2}{\delta^4\sqrt{s}} + O\Big(\frac{1}{\delta^5\sqrt{s}}\Big) \ . \tag{112}$$

Therefore,

$$\begin{aligned} f_s(\delta) &= \Big(\frac{1}{f_\infty(\delta)} + \frac{2}{\delta^4\sqrt{s}} + O\Big(\frac{1}{\delta^5\sqrt{s}}\Big)\Big)^{-1} = \\ &= f_\infty(\delta) - \frac{2f_\infty^2(\delta)}{\delta^4\sqrt{s}} + O\Big(\frac{f_\infty^2(\delta)}{\delta^5\sqrt{s}}\Big) \ , \end{aligned} \tag{113}$$

where it has been used that $f_\infty(\delta) = -\delta + O(1/\delta)$, $\delta < 0$.

We can use (113) to approximate $a_s(\gamma)$ as $\gamma \downarrow 0$. From

$$\begin{aligned} &a_s(\gamma) = f_s(\gamma - a_s(\gamma)) \ , \quad a_\infty(\gamma) = f_\infty(\gamma - a_\infty(\gamma)) \ ; \\ &a_s(\gamma), a_\infty(\gamma) = \frac{1}{\gamma} + O(\gamma) \ , \qquad \gamma > 0 \ , \end{aligned} \tag{114}$$

we get

$$\begin{aligned} a_s(\gamma) &= f_s(\gamma - a_s(\gamma)) \ = \\ &= f_\infty(\gamma - a_s(\gamma)) - \frac{2}{(\gamma - a_s(\gamma))^2\sqrt{s}} + O\Big(\frac{1}{(\gamma - a_s(\gamma))^3\sqrt{s}}\Big) \ = \\ &= f_\infty(\gamma - a_\infty(\gamma)) + (a_\infty(\gamma) - a_s(\gamma))\, f'_\infty(\gamma - a_\infty(\gamma)) \ + \\ &\quad - \frac{2\gamma^2}{\sqrt{s}} + O\Big(\frac{\gamma^3}{\sqrt{s}}\Big) \ = \\ &= a_\infty(\gamma) + (a_\infty(\gamma) - a_s(\gamma))\, f'_\infty(\gamma - a_\infty(\gamma)) - \frac{2\gamma^2}{\sqrt{s}} + O\Big(\frac{\gamma^3}{\sqrt{s}}\Big) \ . \end{aligned} \tag{115}$$

Hence,

$$a_\infty(\gamma) - a_s(\gamma) = \Big(\frac{2\gamma^2}{\sqrt{s}} + O\Big(\frac{\gamma^3}{\sqrt{s}}\Big)\Big)\, \frac{1}{1 + f'_\infty(\gamma - a_\infty(\gamma))} \ . \tag{116}$$

Now

$$f'_\infty(\delta) = -f_\infty(\delta)(\delta + f_\infty(\delta)) = -1 + \frac{1}{\delta^2} + O\Big(\frac{1}{\delta^4}\Big) \ , \qquad \delta < 0 \ , \tag{117}$$

where (13) has been used. With $\delta = \gamma - a_\infty(\gamma) = -1/\gamma + O(\gamma)$, this gives

$$\frac{1}{1 + f'_\infty(\gamma - a_\infty(\gamma))} = \frac{1}{\gamma^2} + O(1) \ , \qquad \gamma > 0 \ . \tag{118}$$

Thus finally

$$a_\infty(\gamma) - a_s(\gamma) = \frac{2}{\sqrt{s}} + O\Big(\frac{\gamma}{\sqrt{s}}\Big) \ , \qquad \gamma > 0 \ . \tag{119}$$

We conclude this section by showing the following consequence of (112).

Proposition 3.1. $f_\infty(\delta) - f_s(\delta) = O(1/\sqrt{s})$ uniformly in $\delta \in \mathbb{R}$ (where we set $f_s(\delta) = 0$ for $\delta \geq \sqrt{s}$).

Proof. It has been shown in [1], Subsec. 5.9 that $f_\infty(\delta) - f_s(\delta) = O(1/\sqrt{s})$ uniformly in any compact set of $\delta \in \mathbb{R}$. From (112) we have that $f_\infty(\delta) = f_s(\delta) = O(1/\sqrt{s})$ uniformly in $\delta \leq -1$, and so it is sufficient to consider the range $\delta \geq 0$. For this range we consider the quasi-Gaussian representation (24) of $f_s(\delta)$. It was shown in [1], Subsec. 5.9 that

$$\int_{-\infty}^{\delta} e^{-\frac{1}{2}(\delta')^2}\, d\delta' - \int_{-\infty}^{\delta} e^{-\frac{1}{2}\alpha_s^2(\delta')}\, d\delta' = O\Big(\frac{1}{\sqrt{s}}\Big) \tag{120}$$

uniformly in $\delta < \sqrt{s}$ while either integral at the left-hand side of (120) is bounded away from 0 when $0 \leq \delta < \sqrt{s}$. Hence, it is sufficient to show that

$$e^{-\frac{1}{2}\delta^2} - \Big(1 - \frac{\delta}{\sqrt{s}}\Big)\, e^{-\frac{1}{2}\alpha_s^2(\delta)} = O\Big(\frac{1}{\sqrt{s}}\Big) \ , \qquad 0 \leq \delta < \sqrt{s} \ . \tag{121}$$

The left-hand side of (121) is positive when $0 < \delta < \sqrt{s}$ by [1], (5.64), and $e^{-\frac{1}{2}\delta^2} \leq 18/s$ when $\delta \geq \frac{1}{3}\,\delta s$. So, it is sufficient to consider the range $0 \leq \delta \leq \frac{1}{3}\sqrt{s}$.

We have, when $0 \le \delta \le \frac{1}{3}\sqrt{s}$,

$$-\tfrac{1}{2}\,\alpha_s^2(\delta) + \ln\Big(1 - \frac{\delta}{\sqrt{s}}\Big) = s\Big(\frac{\delta}{\sqrt{s}} + \ln\Big(1 - \frac{\delta}{\sqrt{s}}\Big)\Big) + \ln\Big(1 - \frac{\delta}{\sqrt{s}}\Big) =$$

$$= -\tfrac{1}{2}\,\delta^2 - \frac{\delta^3}{3\sqrt{s}} - \frac{\delta^4}{4s} - ... - \frac{\delta}{\sqrt{s}} - \frac{\delta^2}{2s} - ... \ge$$

$$\ge -\tfrac{1}{2}\,\delta^2 - \frac{\delta^3}{3\sqrt{s}}\Big(1 - \frac{\delta}{\sqrt{s}}\Big)^{-1} - \frac{\delta}{\sqrt{s}}\Big(1 - \frac{\delta}{\sqrt{s}}\Big)^{-1} \ge -\tfrac{1}{2}\,\delta^2 - \frac{\delta^3 + 3\delta}{2\sqrt{s}} \; . \tag{122}$$

Hence, when $0 \le \delta \le \frac{1}{3}\sqrt{s}$,

$$e^{-\frac{1}{2}\delta^2} - \Big(1 - \frac{\delta}{\sqrt{s}}\Big)\, e^{-\frac{1}{2}\alpha_s^2(\delta)} \le e^{-\frac{1}{2}\delta^2}\,\frac{\delta^3 + 3\delta}{2\sqrt{s}} \le \frac{M}{2\sqrt{s}} \; , \tag{123}$$

with $M = \max_{\delta \ge 0}(\delta^3 + 3\delta)\, e^{-\frac{1}{2}\delta^2}$, which is assumed at $\delta = 3^{1/4}$. This completes the proof.

4 Approximation of $1/f_s(\delta)$ in terms of $1/f_\infty(\delta)$ on a range $\delta \le s^{1/6}$ using Van Veen's approach

We write now for $\delta < \sqrt{s}$, compare (68),

$$\frac{1}{f_s(\delta)} = \int_0^\infty e^{\delta v - \frac{1}{2}v^2}\, e^{-s[v/\sqrt{s} - \frac{1}{2}(v/\sqrt{s})^2 - \ln(1 + v/\sqrt{s})]}\, dv \; . \tag{124}$$

Consider with $z = v/\sqrt{s}$ the expansion

$$K(z) \;:=\; (1+z)^s\, e^{\frac{1}{2}s(z^2 - 2z)} = e^{-s[z - \frac{1}{2}z^2 - \ln(1+z)]} =$$

$$= e^{-s[\frac{1}{3}z^3 - \frac{1}{4}z^4 + ...]} = \sum_{m=0}^\infty K_m(s)\, z^m = 1 + \sum_{m=1}^\infty \frac{(-1)^m}{m}\, L_m(s)\, z^m \; , \tag{125}$$

just as Van Veen does in [8], §4. The $L_m(s)$ are given recursively by

$$L_0 = L_1 = L_2 = 0 \; , \qquad L_3 = -s \; , \tag{126}$$

$$L_{m+1} = L_m - \frac{s}{m-2}\, L_{m-2}\ , \qquad m = 3, 4, \dots\ . \tag{127}$$

Van Veen shows the following result in [8], §5. There holds for $z \geq 0$ and $k = 0, 1, \dots$

$$K(z) = \sum_{m=0}^{k} K_m(s)\, z^m + T_k(z)\ , \tag{128}$$

where

$$T_k(z) = \int_0^z \frac{K(z)}{K(\bar{z})}\, Q_k(\bar{z})\, d\bar{z}\ , \tag{129}$$

and

$$Q_k(z) = (-1)^{k+1}\, z^k \left\{ L_{k+1} - L_{k+2}\, z + L_{k+3}\, \frac{z^2}{1+z} \right\}\ . \tag{130}$$

Now

$$K(z)/K(\bar{z}) = \exp\left[s \int_{\bar{z}}^{z} \frac{w^2}{1+w}\, dw \right] \leq K(z)\ , \qquad 0 \leq \bar{z} \leq z\ , \tag{131}$$

and so

$$\begin{aligned} |T_k(z)| \ &\leq\ K(z) \int_0^z \bar{z}^k (|L_{k+1}| + |L_{k+2}|\, \bar{z} + |L_{k+3}|\, \bar{z}^2)\ = \\ &=\ K(z)(|K_{k+1}| + |K_{k+2}|\, z + |K_{k+3}|\, z^2)\, z^{k+1}\ , \qquad z \geq 0\, . \end{aligned} \tag{132}$$

We shall restrict attention to the range $\frac{1}{3}\, sz^2 \leq 1$ which ensures that $1 \leq K(z) \leq e$ when $z \geq 0$, see (131). Recalling that $z = v/\sqrt{s}$, this means that v is restricted by

$$0 \leq v \leq v(s) := 3^{1/3}\, s^{1/6}\ . \tag{133}$$

We next find out for which $\delta < \sqrt{s}$ truncation of the integral in (68) and (124) to the range $0 \leq v \leq v(s)$ leads to exponentially small errors. Thus we consider

$$\varphi(v) := \delta v - s\, [v/\sqrt{s} - \ln(1 + v/\sqrt{s})]\ , \qquad v \geq 0\ . \tag{134}$$

When $\delta \leq 0$, we have that $\varphi(v)$ is maximal 0 at $v = 0$, and when $0 \leq \delta < \sqrt{s}$, the maximum of $\varphi(v)$ is assumed at

$$\hat{v} = \hat{v}(\delta, s) = \frac{\delta}{1 - \delta/\sqrt{s}}\ . \tag{135}$$

We have

$$\varphi(\hat{v}) = -\delta\sqrt{s} - s\ln(1-\delta/\sqrt{s}) \geq \tfrac{1}{2}\,\delta^2 \ , \tag{136}$$

while for $v \geq 0$

$$\varphi(v) = \delta v - s \int\limits_0^{v/\sqrt{s}} \frac{z}{1+z}\, dz \leq \delta v - \frac{\frac{1}{2}\, v^2}{1+v/\sqrt{s}} \ . \tag{137}$$

Therefore,

$$\varphi(\hat{v}) - \varphi(v(s)) \geq \tfrac{1}{2}\,\delta^2 - \delta v(s) + \frac{\frac{1}{2}\, v^2(s)}{1+v(s)/\sqrt{s}} \ =$$

$$= \ \tfrac{1}{2}\,(v(s)-\delta)^2 - \frac{v^3(s)/2\sqrt{s}}{1+v(s)/\sqrt{s}} \geq \tfrac{1}{2}\,(v(s)-\delta)^2 - 3/2 \ . \tag{138}$$

Then choosing $\delta \leq \frac{1}{2}\, v(s) = 2^{-1}\, 3^{1/3}\, s^{1/6}$ produces relative errors in truncating $\int\limits_0^{\infty} \exp(\varphi(v))\, dv$ as $\int\limits_0^{v(s)} \exp(\varphi(v))\, dv$ of the order $\exp(-\frac{1}{8}\, 3^{2/3}\, s^{1/3})$. Here it has also been used that $\varphi(v)$ is a concave function of $v \geq 0$ and that for $\delta \leq \frac{1}{2}\, v(s)$, we have

$$\varphi'(v(s)) \leq -\tfrac{1}{2}\, v(s)\, \frac{1-v(s)/\sqrt{s}}{1+v(s)/\sqrt{s}} = -\tfrac{1}{2}\, 3^{1/3}\, s^{1/6}\, \frac{1-(3/s)^{1/2}}{1+(3/s)^{1/2}} \tag{139}$$

which is negative when $s > 3$.

We therefore have, with uniformly exponentially small error when $\delta \leq \frac{1}{2}\, v(s)$, that

$$\frac{1}{f_s(\delta)} \ = \ \int\limits_0^{v(s)} e^{\delta v - \frac{1}{2}v^2}\, K(v/\sqrt{s})\, dv \ =$$

$$= \ \sum_{m=0}^{k} s^{-\frac{1}{2}m}\, K_m(s) \int\limits_0^{v(s)} v^m\, e^{\delta v - \frac{1}{2}v^2}\, dv + \int\limits_0^{v(s)} T_k(z)\, e^{\delta v - \frac{1}{2}v^2}\, dv \ , \tag{140}$$

with T_k bounded as in (132) and $K_0 = 1$, $K_m = m^{-1}(-1)^m\, L_m$ where L_m is given through (126–127). The asymptotic analysis is now completed by replacing the integrals $\int\limits_0^{v(s)} v^m \exp\,[\delta v - \frac{1}{2}\, v^2]\, dv$ by

$$I_m(\delta) := \int\limits_0^{\infty} v^m\, e^{\delta v - \frac{1}{2}v^2}\, dv \ , \tag{141}$$

at the expense of exponentially small errors, and by bounding $|T_k(z)|$ by

$$\frac{e}{s^{\frac{1}{2}(k+1)}}\left(|K_{k+1}|\,I_{k+1}(\delta)+\frac{1}{\sqrt{s}}\,|K_{k+2}|\,I_{k+2}(\delta)+\frac{1}{s}\,|K_{k+3}|\,I_{k+3}\right)\ . \tag{142}$$

We thus arrive at an asymptotic series

$$\frac{1}{f_s(\delta)}\sim\sum_{m=0}^{\infty}s^{-\frac{1}{2}m}\,K_m(s)\,I_m(\delta)\ , \tag{143}$$

with uniform relative truncation error assessment in the range $-\infty<\delta\leq 2^{-1}\,3^{1/3}\,s^{1/6}=(9s/8)^{1/6}$.

We next consider the quantities $K_m(s)$ and $I_m(\delta)$ in (143) in more detail. For the first few $K_m(s)$ we have Table 3, and as to the $I_m(\delta)$ we shall be concerned in expressing them in terms of $1/f_\infty(\delta)$. We have

$$I_0(\delta)=e^{\frac{1}{2}\delta^2}\int_0^\infty e^{-\frac{1}{2}(v-\delta)^2}\,dv=e^{\frac{1}{2}\delta^2}\int_{-\delta}^\infty e^{-\frac{1}{2}w^2}\,dw=\frac{1}{f_\infty(\delta)}\ . \tag{144}$$

m	$s^{-\frac{1}{2}m}\,K_m(s)$	m	$s^{-\frac{1}{2}m}\,K_m(s)$
1	0	5	$\frac{1}{5s^{3/2}}$
2	0	6	$\frac{-1}{6s^2}+\frac{1}{18s}$
3	$\frac{1}{3s^{1/2}}$	7	$\frac{1}{7s^{5/2}}-\frac{1}{12s^{3/2}}$
4	$\frac{-1}{4s}$	8	$\frac{-1}{8s^3}+\frac{37}{480s^2}$

Table 3. $s^{-\frac{1}{2}m}\,K_m(s)$ with $K_0(s)=1$ and $K_m(s)=(-1)^m\,m^{-1}\,L_m(s)$, $m=1,2,...$ where $L_m(s)$ given by (126–127).

Furthermore, $I_m(\delta)=I_0^{(m)}(\delta)$. With $x=-\delta$, we let

$$l(x)=\frac{1}{f_\infty(-x)}=e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}w^2}\,dw\ . \tag{145}$$

Then

$$I_m(\delta)=(-1)^m\,l^{(m)}(-\delta)=m!\,e^{\frac{1}{4}\delta^2}\,U(m+\tfrac{1}{2},-\delta) \tag{146}$$

with $U(a,x)$ the parabolic cylinder function, see [5], Ch. 19. Now there holds for $m = 0, 1, ...$

$$l^{(m)}(x) = P_m(x)\, l(x) - Q_m(x) \ , \tag{147}$$

where P_m and Q_m are polynomials defined recursively by

$$P_0(x) = 1 \ , \qquad Q_0(x) = 0 \ , \tag{148}$$

and for $m = 0, 1, ...,$

$$P_{m+1}(x) = x\, P_m(x) + P_m'(x) \ , \qquad Q_{m+1}(x) = P_m(x) + Q_m'(x) \ . \tag{149}$$

In Table 4 we display the first few polynomials P, Q. The polynomials P_m can be expressed in terms of the Hermite polynomials He_m, see [5], Ch. 22, as

$$P_m(x) = \frac{1}{i^m}\,(He_m)(ix) \ , \qquad m = 0, 1, ... \ , \tag{150}$$

but for the Q_m such a simple closed form does not seem to exist. Alternatively, the P_m and Q_m occur in the m^{th} convergent $l_m(x) = Q_m(x)/P_m(x)$ of the continued fraction expansion of Laplace for Mills ratio,

$$l(x) = \frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+} \cdots \frac{m}{x+} \cdots \ , \tag{151}$$

see [11], Sec. 3 and Appendix, [12], pp. 83–84.

m	$P_m(x)$	$Q_m(x)$
0	1	0
1	x	1
2	x^2+1	x
3	x^3+3x	x^2+2
4	x^4+6x^2+3	x^3+5x
5	x^5+10x^3+15x	x^4+9x^2+8
6	$x^6+15x^4+45x^2+15$	x^5+14x^3+33x
7	$x^7+21x^5+105x^3+105x$	$x^6+20x^4+87x^2+48$
8	$x^8+28x^6+210x^4+420x^2+105$	$x^7+27x^5+185x^3+279x$

Table 4. $P_m(x)$ and $Q_m(x)$ for $m = 0, 1, ..., 8$.

We use (143) to find approximations of $f_s(\delta)$ in terms of $f_\infty(\delta)$. Truncating (143) at $m = 3$, we get

$$\frac{1}{f_s(\delta)} = K_0(s)\, I_0(\delta) + \frac{1}{s^{3/2}}\, K_3(s)\, I_3(\delta) + \varepsilon \ , \tag{152}$$

where ε accounts for a relative order of $O(\exp(-s^{1/3}))$ uniformly in the range $\delta \leq s^{1/6}$ together with T_4, bounded as in (142). Hence, $1/f_s(\delta)$ is approximated by

$$\frac{1}{f_\infty(\delta)} + \frac{1}{3\sqrt{s}}\,(-1)^3\, l^{(3)}(-\delta) = \frac{1}{f_\infty(\delta)} - \frac{1}{3\sqrt{s}}\,(P_3(-\delta)\, l(-\delta) - Q_3(-\delta)) \;=$$

$$= \;\frac{1}{f_\infty(\delta)} + \frac{1}{3 f_\infty(\delta)\sqrt{s}}\,(\delta^3 + 3\delta + (\delta^2+2)\, f_\infty(\delta))\;, \tag{153}$$

where the Table 4 and $l(-\delta) = 1/f_\infty(\delta)$ has been used.

The $\frac{1}{\sqrt{s}}$-correction in (153) has the asymptotic series

$$\frac{1}{3}\left(\delta^2 + 2 + \frac{\delta^3 + 3\delta}{f_\infty(\delta)}\right) \;\sim$$

$$\sim\; \frac{1}{3}\left(\delta^2 + 2 - (\delta^3 + 3\delta)\sum_{m=0}^{\infty}(-1)^m\,(2m-1)!!\,\delta^{-2m-1}\right) \;=$$

$$= \sum_{m=2}^{\infty}(-1)^m\,\tfrac{2}{3}\,(m-1)(2m-1)!!\,\delta^{-2m}\;, \tag{154}$$

where (17) has been used. On the other hand, when we collect in (50) all terms containing a factor $1/\sqrt{s}$, i.e., all terms (l,j) with $j = l-1$ and $l = 1, 2, \ldots$, we get for the coefficients of $1/\sqrt{s}$ in $1/f_s(\delta)$ the formal expression

$$\sum_{l=1}^{\infty}\frac{2l\, c_{l,l-1}}{\delta^{2l+2}}\;. \tag{155}$$

It can be shown from the recursion (34–35) and from (37) that

$$c_{l-1,l-2} = \tfrac{1}{3}\,(-1)^l\,(2l-1)!!\;, \qquad l = 2, 3, \ldots\;, \tag{156}$$

confirming formal equality of the respective $\frac{1}{\sqrt{s}}$-corrections.

From (153), we find a $\frac{1}{\sqrt{s}}$-correction on the approximation $f_\infty(\delta)$ of $f_s(\delta)$ as

$$f_\infty(\delta) - \frac{1}{3\sqrt{s}}\, f_\infty(\delta)\,(\delta^3 + 3\delta + (\delta^2+2)\, f_\infty(\delta))\;. \tag{157}$$

We shall use this to find a $\frac{1}{\sqrt{s}}$-correction

$$a_\infty(\gamma) + \frac{1}{\sqrt{s}}\, b_\infty(\gamma) \tag{158}$$

to the Gaussian limit approximation $a_\infty(\gamma)$ of $a_s(\gamma)$. With

$$g_\infty(\delta) := -\tfrac{1}{3}\, f_\infty(\delta)(\delta^3 + 3\delta + (\delta^2+2)\, f_\infty(\delta)) \ , \tag{159}$$

we thus consider the approximated version

$$a = f_\infty(\gamma - a) + \frac{1}{\sqrt{s}}\, g_\infty(\gamma - a) \tag{160}$$

of the finite-s Cohen equation (4). Ignoring errors $O(1/s)$, we get (γ suppressed in $a_\infty(\gamma)$ and $b_\infty(\gamma)$)

$$\begin{aligned} a_\infty + \frac{1}{\sqrt{s}}\, b_\infty &= f_\infty\Big(\gamma - a_\infty - \frac{1}{\sqrt{s}}\, b_\infty\Big) + \frac{1}{\sqrt{s}}\, g_\infty\Big(\gamma - a_\infty - \frac{1}{\sqrt{s}}\, b_\infty\Big) = \\ &= f_\infty(\gamma - a_\infty) - \frac{1}{\sqrt{s}}\, b_\infty\, f'_\infty(\gamma - a_\infty) + \frac{1}{\sqrt{s}}\, g_\infty(\gamma - a_\infty) \ . \end{aligned} \tag{161}$$

Using $a_\infty = f_\infty(\gamma - a_\infty)$, we then see that

$$b_\infty = \frac{g_\infty(\gamma - a_\infty)}{1 + f'_\infty(\gamma - a_\infty)} \ . \tag{162}$$

This can be elaborated further, using

$$f'_\infty(\delta) = -f_\infty(\delta)(\delta + f_\infty(\delta)) \tag{163}$$

and $f_\infty(\gamma - a_\infty) = a_\infty$. This yields

$$f'_\infty(\gamma - a_\infty) = -\gamma\, a_\infty \ . \tag{164}$$

Furthermore, from (159) and $f_\infty(\gamma - a_\infty) = a_\infty$, we compute

$$g_\infty(\gamma - a_\infty) = -\tfrac{1}{3}\, a_\infty(3\gamma - a_\infty + \gamma(\gamma - a_\infty)^2) \ .$$

Therefore, we get as our final result

$$b_\infty(\gamma) = b_\infty = \frac{-\tfrac{1}{3}\, a_\infty(3\gamma - a_\infty + \gamma(\gamma - a_\infty)^2)}{1 - \gamma\, a_\infty} \ . \tag{165}$$

Using that, see (10),

$$\gamma\, a_\infty(\gamma) = 1 - \gamma^2 + 2\gamma^4 + O(\gamma^6) \ , \qquad \gamma > 0 \ ,$$

it can be shown that $b_\infty(\gamma) \to -2$ as $\gamma \downarrow 0$, compare (6).

Further corrections can be considered as well. Thus, with

$$f_\infty(\delta) + \frac{1}{\sqrt{s}}\,g_\infty(\delta) + \frac{1}{s}\,h_\infty(\delta)\,, \quad a_\infty(\gamma) + \frac{1}{\sqrt{s}}\,b_\infty(\gamma) + \frac{1}{s}\,c_\infty(\gamma) \tag{166}$$

being the $\frac{1}{s}$-correction of $f_s(\delta)$ and $a_s(\gamma)$, respectively, we get upon ignoring errors $1/s^{3/2}$ and suppressing γ in $a_\infty(\gamma)$, $b_\infty(\gamma)$, $c_\infty(\gamma)$,

$$\begin{aligned}
a_\infty + \frac{1}{\sqrt{s}}\,b_\infty + \frac{1}{s}\,c_\infty &= f_\infty\Big(\gamma - a_\infty - \frac{1}{\sqrt{s}}\,b_\infty - \frac{1}{s}\,c_\infty\Big) \; + \\
&+ \frac{1}{\sqrt{s}}\,g_\infty\Big(\gamma{-}a_\infty - \frac{1}{\sqrt{s}}\,b_\infty - \frac{1}{s}\,c_\infty\Big) + \frac{1}{s}\,h_\infty\Big(\gamma{-}a_\infty - \frac{1}{\sqrt{s}}\,b_\infty - \frac{1}{s}\,c_\infty\Big) \; = \\
= \; f_\infty(\gamma - a_\infty) &- \Big(\frac{1}{\sqrt{s}}\,b_\infty + \frac{1}{s}\,c_\infty\Big)\,f'_\infty(\gamma - a_\infty) + \frac{1}{2s}\,b_\infty^2\,f''_\infty(\gamma - a_\infty) \; + \\
&+ \frac{1}{\sqrt{s}}\,\Big(g_\infty(\gamma - a_\infty) - \frac{1}{\sqrt{s}}\,b_\infty\,g'_\infty(\gamma - a_\infty)\Big) + \frac{1}{s}\,h_\infty(\gamma - a_\infty)\;.
\end{aligned} \tag{167}$$

Using $a_\infty = f_\infty(\gamma - a_\infty)$ and equating terms with factors $1/\sqrt{s}$ and $1/s$, we again get (162) for b_∞ and

$$c_\infty = \frac{\frac{1}{2}\,b_\infty^2\,f''_\infty - b_\infty\,g'_\infty + h_\infty}{1 + f'_\infty} \tag{168}$$

with f'_∞, f''_∞, g'_∞, h_∞ evaluated at $\gamma - a_\infty$ and b_∞ evaluated at γ.

Again, this can be elaborated. Using $f_\infty(\gamma - a_\infty) = a_\infty$ and (163–164), we find

$$f''_\infty(\gamma - a_\infty) = \gamma\,a_\infty(\gamma + a_\infty) - a_\infty\;. \tag{169}$$

Furthermore, b_∞ was already found in (165), and we can compute $g'_\infty(\gamma - a_\infty)$ from (159) and (163–164) using $f_\infty(\gamma - a_\infty) = a_\infty$. This is still feasible, with the final result

$$g'_\infty(\gamma - a_\infty) = \tfrac{1}{3}\,a\,[\gamma(\gamma - a)^2\,(\gamma + a) + 5\gamma a - a^3 - 3]\;, \tag{170}$$

where we have written $a = a_\infty$ at the right-hand side. Finally, we should compute $h_\infty(\gamma - a_\infty)$. For this, we write down all terms at the right-hand side of (143) comprising factors s^0, $s^{-1/2}$, s^{-1}. Using Table 3, it is seen that we should include the terms with $m = 0, 3, 4$ and 6. The $1/s$-correction to

$1/f_s(\delta)$ is then obtained by adding to (153)

$$-\frac{1}{4s}\,I_4(\delta)+\frac{1}{18s}\,I_6(\delta)=-\frac{1}{4s}\,l^{(4)}(-\delta)+\frac{1}{18s}\,l^{(6)}(-\delta)\;=$$

$$=\;-\frac{1}{4s}\,[P_4(-\delta)\,l(-\delta)-Q_4(-\delta)]+\frac{1}{18s}\,[P_6(-\delta)\,l(-\delta)-Q_6(-\delta)]\;=$$

$$=\;\frac{1}{36s}\,[(2\delta^6+21\delta^4+36\delta^2+3)\,l(-\delta)+2\delta^5+19\delta^3+21\delta]\;, \tag{171}$$

where Table 4 has been used. Recalling that $l(-\delta)=1/f_\infty(\delta)$, we then get

$$-\frac{1}{4s}\,I_4(\delta)+\frac{1}{18s}\,I_6(\delta)\;=$$

$$=\;\frac{1}{36s\,f_\infty(\delta)}\,[2\delta^6+21\delta^4+36\delta^2+3+(2\delta^5+19\delta^3+21\delta)\,f_\infty(\delta)]\;. \tag{172}$$

The expression (172) should be added to the right-hand side of (153) to obtain the $1/s$-correction of $1/f_s(\delta)$ in terms of $f_\infty(\delta)$ and δ. The two correcting terms being small compared to $1/f_\infty(\delta)$, the $1/s$-correction of $f_s(\delta)$ is obtained by expanding $(\frac{1}{f}+\varepsilon)^{-1}=f-\varepsilon f^2+\varepsilon^2 f^3$ with ε small compared to f. The coefficient $h_\infty(\delta)$ of $1/s$ in this $1/s$-correction of $f_s(\delta)$ is then given as

$$\begin{aligned}h_\infty(\delta)\;&=\;f_\infty(\delta)\,\Big[\tfrac{1}{9}\,(\delta^3+3\delta+(\delta^2+2)\,f_\infty(\delta))^2\;+\\&\qquad-\;\tfrac{1}{36}\,(2\delta^6+21\delta^4+36\delta^2+3+(2\delta^5+19\delta^3+21\delta)\,f_\infty(\delta))\Big]\;.\end{aligned} \tag{173}$$

This works out to

$$h_\infty(\delta)\;=$$

$$=\;\frac{f_\infty(\delta)}{36}\,[2\delta^6+3\delta^4-3+(6\delta^5+21\delta^3+27\delta)\,f_\infty(\delta)+4(\delta^2+2)^2\,f_\infty^2(\delta)]\;, \tag{174}$$

and in this expression we should set $\delta=\gamma-a_\infty$ and use that $f_\infty(\gamma-a_\infty)=a_\infty$. Therefore, the computation of the term $\frac{1}{s}\,c_\infty(\gamma)$ in the $1/s$-correction in (166) of $a_s(\gamma)$ according to (168) is feasible but quite involved.

The corrections based on (165) and on (168–170), (174) have been checked for the case that $\gamma=1$ and $s=10,20,50,100,1000$, see Table 5.

s	$a_s(1)$	$a_\infty(1)+\frac{1}{\sqrt{s}}\,b_\infty(1)$	$a_\infty(1)+\frac{1}{\sqrt{s}}\,b_\infty(1)+\frac{1}{s}\,c_\infty(1)$
10	0.2563620	0.2105055	0.2575139
20	0.3239228	0.3008443	0.3243485
50	0.3902951	0.3810048	0.3904065
100	0.4260664	0.4214055	0.4261064
1000	0.4885668	0.4880980	0.4885680

Table 5. $\frac{1}{\sqrt{s}}$-correction and $\frac{1}{s}$-correction of $a_\infty(1) = 0.5189416$ as an approximation of $a_s(1)$, $s = 10, 20, 50, 100, 1000$, using $b_\infty(1) = -0.9753608$ from (165) and $c_\infty(1) = 0.4700844$ from (168–170), (174). J.S.H. van Leeuwaarden is acknowledged for providing the numerical values of $a_s(1)$.

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