The parabolic Anderson model in a dynamic random environment: basic properties of the quenched Lyapunov exponent

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Abstract

In this paper we study the parabolic Anderson equation \( \partial u(x,t)/\partial t = \kappa \Delta u(x,t) + \xi(x,t)u(x,t) \), \( x \in \mathbb{Z}^d, t \geq 0 \), where the \( u \)-field and the \( \xi \)-field are \( \mathbb{R} \)-valued, \( \kappa \in [0, \infty) \) is the diffusion constant, and \( \Delta \) is the discrete Laplacian. The \( \xi \)-field plays the role of a dynamic random environment that drives the equation. The initial condition \( u(x,0) = u_0(x), x \in \mathbb{Z}^d \), is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate \( 2^d \kappa \), split into two at rate \( \xi \lor 0 \), and die at rate \( (-\xi) \lor 0 \). Our goal is to prove a number of basic properties of the solution \( u \) under assumptions on \( \xi \) that are as weak as possible. These properties will serve as a jump board for later refinements.

Throughout the paper we assume that \( \xi \) is stationary and ergodic under translations in space and time, is not constant and satisfies \( \mathbb{E}(|\xi(0,0)|) < \infty \), where \( \mathbb{E} \) denotes expectation w.r.t. \( \xi \). Under a mild assumption on the tails of the distribution of \( \xi \), we show that the solution to the parabolic Anderson equation exists and is unique for all \( \kappa \in [0, \infty) \). Our main object of interest is the quenched Lyapunov exponent \( \lambda_0(\kappa) = \lim_{t \to \infty} \frac{1}{t} \log u(0,t) \). It was shown in Gärtner, den Hollander and Maillard [7] that this exponent exists and is constant \( \xi \)-a.s., satisfies \( \lambda_0(0) = \mathbb{E}(\xi(0,0)) \) and \( \lambda_0(\kappa) > \mathbb{E}(\xi(0,0)) \) for \( \kappa \in (0, \infty) \), and is such that \( \kappa \mapsto \lambda_0(\kappa) \) is globally Lipschitz on \( (0, \infty) \) outside any neighborhood of 0 where it is finite. Under certain weak space-time mixing assumptions on \( \xi \), we show the following properties: (1) \( \lambda_0(\kappa) \) does not depend on the initial condition \( u_0 \); (2) \( \lambda_0(\kappa) < \infty \) for all \( \kappa \in [0, \infty) \); (3) \( \kappa \mapsto \lambda_0(\kappa) \) is continuous on \( [0, \infty) \) but not Lipschitz at 0. We further conjecture: (4) \( \lim_{\kappa \to \infty} [\lambda_p(\kappa) - \lambda_p(0)] = 0 \) for all \( p \in \mathbb{N} \), where \( \lambda_p(\kappa) = \lim_{t \to \infty} \frac{1}{pt} \log \mathbb{E}([u(0,t)]^p) \) is the \( p \)-th annealed Lyapunov exponent. (In [7] properties (1), (2) and (4) were not addressed, while property (3) was shown under much more restrictive assumptions on \( \xi \).)

Finally, we prove that our weak space-time mixing conditions on \( \xi \) are satisfied for several classes of interacting particle systems.

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