The parabolic Anderson model in a dynamic random environment: basic properties of the quenched Lyapunov exponent

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Abstract

In this paper we study the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where the u-field and the ξ-field are $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, and Δ is the discrete Laplacian. The ξ-field plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$. Our goal is to prove a number of *basic properties* of the solution u under assumptions on ξ that are as weak as possible. These properties will serve as a jump board for later refinements.

Throughout the paper we assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Under a mild assumption on the tails of the distribution of ξ, we show that the solution to the parabolic Anderson equation exists and is unique for all $\kappa \in [0,\infty)$. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t}\log u(0,t)$. It was shown in Gärtner, den Hollander and Maillard [7] that this exponent exists and is constant ξ-a.s., satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$, and is such that $\kappa \mapsto \lambda_0(\kappa)$ is globally Lipschitz on $(0,\infty)$ outside any neighborhood of 0 where it is finite. Under certain weak space-time mixing assumptions on ξ, we show the following properties: (1) $\lambda_0(\kappa)$ does not depend on the initial condition u_0; (2) $\lambda_0(\kappa) < \infty$ for all $\kappa \in [0,\infty)$; (3) $\kappa \mapsto \lambda_0(\kappa)$ is continuous on $[0,\infty)$ but not Lipschitz at 0. We further conjecture: (4) $\lim_{\kappa\to\infty}[\lambda_p(\kappa) - \lambda_0(\kappa)] = 0$ for all $p \in \mathbb{N}$, where $\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt}\log \mathbb{E}([u(0,t)]^p)$ is the p-th *annealed Lyapunov exponent*. (In [7] properties (1), (2) and (4) were not addressed, while property (3) was shown under much more restrictive assumptions on ξ.) Finally, we prove that our weak space-time mixing conditions on ξ are satisfied for several classes of interacting particle systems.

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1 Introduction and main results

Section 1.1 defines the parabolic Anderson model and provides motivation, Section 1.2 describes our main targets and their relation to the literature, Section 1.3 contains our main results, while Section 1.4 discusses these results and state a conjecture.

1.1 The parabolic Anderson model (PAM)

The parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d\\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.2}$$

($\|\cdot\|$ is the l_1-norm), while

$$\xi = (\xi_t)_{t\geq 0} \text{ with } \xi_t = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d\} \tag{1.3}$$

is an $\mathbb{R}$-valued random field playing the role a of *dynamic random environment* that drives the equation. As initial condition for (1.1) we take

$$\blacktriangleright \quad u(x,0) = u_0(x),\ x \in \mathbb{Z}^d,\ \text{with } u_0 \text{ non-negative and bounded.} \tag{1.4}$$

One interpretation of (1.1) and (1.4) comes from *population dynamics*. Consider the special case where $\xi(x,t) = \gamma\bar{\xi}(x,t) - \delta$ with $\delta, \gamma \in (0,\infty)$ and $\bar{\xi}$ an $\mathbb{N}_0$-valued random field. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

- A-particles evolve autonomously according to a prescribed dynamics with $\bar{\xi}(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time;
- B-particles die at rate δ;
- the average number of B-particles at site x at time 0 is $u_0(x)$.

Then

$$u(x,t) \;=\; \text{the average number of } B\text{-particles at site } x \text{ at time } t \text{ conditioned on the evolution of the } A\text{-particles.} \tag{1.5}$$

The ξ-field is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper we assume that

- ► ξ is *stationary* and *ergodic* under translations in space and time.
- ► ξ is *not constant* and $\mathbb{E}(|\xi(0,0)|) < \infty$.

(1.6)

Without loss of generality we may assume that $\mathbb{E}(\xi(0,0)) = 0$.

1.2 Main targets and related literature

The goal of the present paper is to prove a number of *basic properties* about the Cauchy problem in (1.1) with initial condition (1.4). In this section we describe these properties informally. Precise results will be stated in Section 1.3.

• **Existence and uniqueness of the solution.** For *static* ξ, i.e.,

$$\xi = \{\xi(x)\colon\ x \in \mathbb{Z}^d\}, \tag{1.7}$$

existence and uniqueness of the solution to (1.1) with initial condition (1.4) were addressed by Gärtner and Molchanov [8]. Namely, for arbitrary $q\colon\ \mathbb{Z}^d \to \mathbb{R}$ and $u_0\colon\ \mathbb{Z}^d \to [0,\infty)$, they considered the deterministic equation

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa\Delta u(x,t) + q(x)u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{1.8}$$

with u_0 non-negative, and showed that there exists a non-negative solution if and only if the Feynman-Kac formula

$$v(x,t) = E_x\left(\exp\left\{\int_0^t q(X^\kappa(s))\,ds\right\}\,u_0(X^\kappa(t))\right) \tag{1.9}$$

is finite for all x and t. Here, $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is the continuous-time simple random walk jumping at rate $2d\kappa$ (i.e., the Markov process with generator $\kappa\Delta$) starting in x under the law P_x. Moreover, they showed that v in (1.9) is the minimal non-negative solution to (1.8). This result was later extended to *dynamic* ξ by Carmona and Molchanov [2], who proved the following.

Proposition 1.1. (Carmona and Molchanov [2]) *Suppose that $q\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is such that $q(x,\cdot)$ is locally integrable for every x. Then, for every non-negative initial condition u_0, the deterministic equation*

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa\Delta u(x,t) + q(x,t)u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{1.10}$$

has a non-negative solution if and only if the Feynman-Kac formula

$$v(x,t) = E_x\left(\exp\left\{\int_0^t q(X^\kappa(s), t-s)\,ds\right\}u_0(X^\kappa(t))\right) \tag{1.11}$$

is finite for all x and t. Moreover, v in (1.11) *is the minimal non-negative solution to* (1.10).

To complement Proposition 1.1, we need to find a condition on ξ that leads to uniqueness of (1.11). This will be the first of our targets. To answer the question of uniqueness for *static* ξ, Gärtner and Molchanov [8] introduced the following notion.

Definition 1.2. *A field $q = \{q(x)\colon\ x \in \mathbb{Z}^d\}$ is said to be percolating from below if for every $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{Z}^d\colon\ q(x) \le \alpha\}$ contains an infinite connected component. Otherwise q is said to be non-percolating from below.*

It was shown in [8] that if q is non-percolating from below, then (1.8) has at most one non-negative solution. We will show that a similar condition suffices for *dynamic* ξ, namely, (1.10) has a most one non-negative solution when there is a $T > 0$ such that

$$q^T = \{q^T(x)\colon\ x \in \mathbb{Z}^d\} \quad \text{with} \quad q^T(x) = \sup_{0 \le t \le T} q(x,t) \tag{1.12}$$

is non-percolating from below (Theorem 1.11 below), and that this solution is given by the Feynman-Kac formula (Theorem 1.12 below). The (surprisingly weak) condition in (1.12) is fulfilled for most choices of ξ.

• **Quenched Lyapunov exponent and initial condition.** The *quenched Lyapunov exponent* associated with (1.1) with initial condition u_0 is defined as

$$\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t). \tag{1.13}$$

Gärtner, den Hollander and Maillard [7] showed that if u_0 has finite support, then the limit exists ξ-a.s. and in $L^1(\mathbb{P})$, is ξ-a.s. constant, and does not depend on u_0. A natural question is whether the same is true for u_0 bounded with infinite support. This question was already addressed by Drewitz, Gärtner, Ramirez and Sun [5]. Define

$$\overline{\lambda}_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log E_0 \left(\exp\left\{ \int_0^t \xi(X^\kappa(s), s)\, ds \right\} u_0(X^\kappa(t)) \right). \tag{1.14}$$

Proposition 1.3. (Drewitz, Gärtner, Ramirez and Sun [5])
(I) *If ξ satisfies the first line of* (1.6) *and is bounded, then $\overline{\lambda}_0^{\mathbb{1}}(\kappa)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, and is ξ-a.s. constant.*
(II) *If, in addition, ξ is reversible in time or symmetric in space, then, for all u_0 subject to* (1.4)*, $\overline{\lambda}_0^{u_0}(\kappa)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, and coincides with $\overline{\lambda}_0^{\mathbb{1}}(\kappa)$.*

The time-reversal that distinguishes $\lambda_0^{\mathbb{1}}(\kappa)$ from $\overline{\lambda}_0^{\mathbb{1}}(\kappa)$ is non-trivial. Under appropriate space-time mixing conditions on ξ, we show how Proposition 1.3 can be used to settle the existence of $\lambda_0^{u_0}(\kappa)$ with the same limit for all u_0 subject to (1.4) (Theorem 1.14 below).

• **Finiteness of the quenched Lyapunov exponent.** Trivially, $\lambda_0^{u_0}(\kappa) \ge \mathbb{E}(\xi(0,0))$ for all κ, while if ξ is bounded from above, then also $\lambda_0^{u_0}(\kappa) < \infty$ for all κ. For unbounded ξ the same is expected to be true under a mild assumption on the positive tail of ξ. However, settling this issue seems far from easy. The only two choices of ξ for which finiteness has been established in the literature are an i.i.d. field of Brownian motions (Carmona and Molchanov [2]) and a Poisson random field of independent simple random walks (Kesten and Sidoravicius [10]). We will show that finiteness holds under an appropriate mixing condition on ξ (Theorem 1.13 below).

• **Dependence on κ.** In Gärtner, den Hollander and Maillard [7] it was shown that $\lambda_0^{\delta_0}(0) = \mathbb{E}(\xi(0,0))$, $\lambda_0^{\delta_0}(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$, and $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is globally Lipschitz outside any neighborhood of zero where it is finite. Under certain strong "noisiness" assumptions on ξ, it was further shown that continuity extends to zero while the Lipschitz property does not. It remained unclear, however, which characteristics of ξ are really necessary for the latter two properties to hold. We will show that if ξ is a Markov process, then in essence a weak condition on its Dirichlet form is enough to ensure continuity (Theorem 1.15 and Corollary 1.19 below), whereas the non Lipschitz property holds under a weak assumption on the fluctuations of ξ (Theorem 1.16). Finally, by the ergodicity of ξ in space, it is natural to expect (see Conjecture 1.20 below) that $\lim_{\kappa\to\infty}[\lambda_p^{\delta_0}(\kappa) - \lambda_0^{\delta_0}(\kappa)] = 0$ for all $p \in \mathbb{N}$, where

$$\lambda_p^{\delta_0}(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}([u(0,t)]^p) \tag{1.15}$$

is the p-th *annealed Lyapunov exponent* (provided this exists). It was proved for three special choices of ξ: (1) independent simple random walks; (2) the symmetric exclusion process; (3) the symmetric voter model, (for references, see [7]), that, when d is large enough, $\lim_{\kappa\to\infty} \lambda_p^{\delta_0}(\kappa) = \mathbb{E}(\xi(0,0))$, $p \in \mathbb{N}_0$. It is known from Carmona and Molchanov [2] that $\lim_{\kappa\to\infty} \lambda_p^{\delta_0}(\kappa) = \frac{1}{2} \neq \mathbb{E}(\xi(0,0))$ for all $p \in \mathbb{N}$ when ξ is an i.i.d. field of Brownian motions.

1.3 Main results

This section contains five definitions of space-time mixing assumptions on ξ, six theorems subject to these assumptions, as well as examples of ξ for which these assumptions are satisfied. The material is organized as Sections 1.3.1–1.3.4. The first theorem refers to the deterministic PAM, the other four theorems to the random PAM. Recall that the initial condition u_0 is assumed to be non-negative and bounded. Further recall that ξ satisfies (1.6).

1.3.1 Definitions: Space-time blocks, Gärtner-mixing, Gärtner-regularity and Gärtner-volatility

• **Good and bad space-time blocks.** For $A \geq 1$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $k,b,c \in \mathbb{N}_0$, define the space-time blocks

$$\tilde{B}_R^A(x,k;b,c) = \left(\prod_{j=1}^d \left[(x(j)-1-b)A^R, (x(j)+1+b)A^R\right) \cap \mathbb{Z}^d\right) \times [(k-c)A^R, (k+1)A^R), \tag{1.16}$$

abbreviate $B_R^A(x,k) = \tilde{B}_R^A(x,k;0,0)$, and define the space-blocks

$$Q_R^A(x) = x + [0, A^R)^d \cap \mathbb{Z}^d. \tag{1.17}$$

It is convenient to extend the ξ-process to negative times, to obtain a two-sided process $\overline{\xi} = (\overline{\xi}_t)_{t\in\mathbb{R}}$. Abbreviate $M = \operatorname{ess\,sup}[\xi(0,0)]$.

Definition 1.4. *For $A \geq 1$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $C \in [0,M]$ and $b,c \in \mathbb{N}_0$, the block $B_R^A(x,k)$ is called (C,b,c)-good when*

$$\sum_{z\in Q_R^A(y)} \xi(z,s) \leq C A^{Rd} \qquad \forall\, y \in \mathbb{Z}^d,\ s \geq 0\colon\ Q_R^A(y) \times \{s\} \subseteq \tilde{B}_R^A(x,k;b,c). \tag{1.18}$$

Otherwise it is called (C,b,c)-bad.

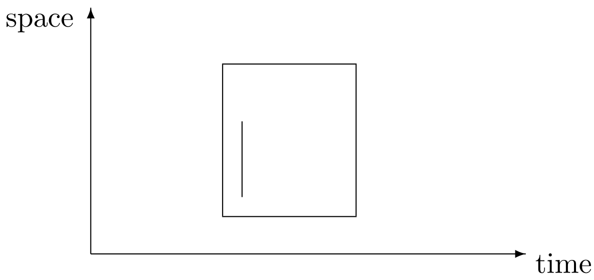

Figure 1: The box represents $B_R^A(x,k)$. The line is a possible realization of $Q_R^A(x)$.

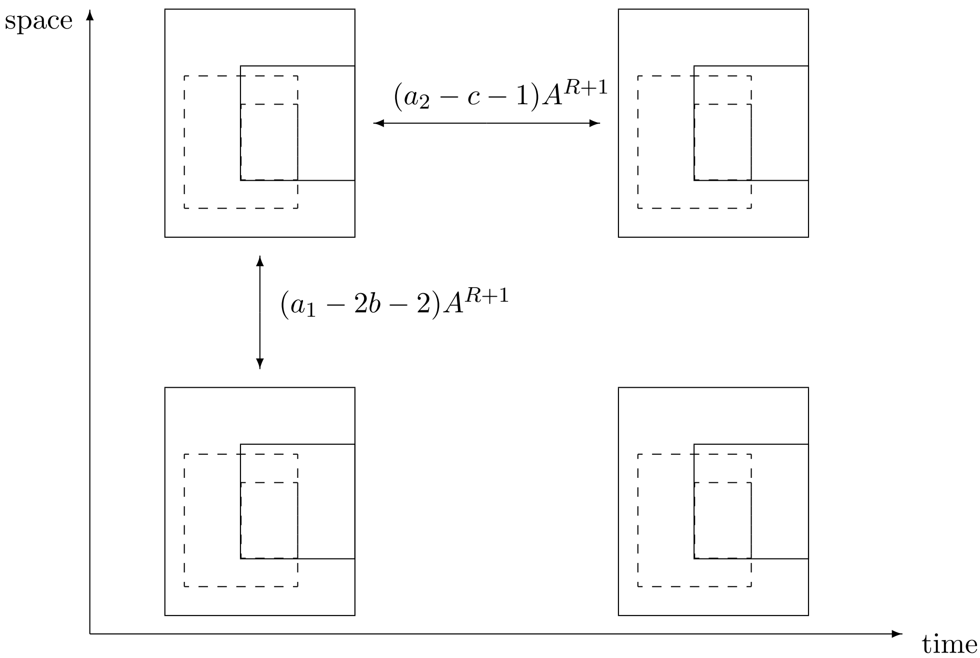

Figure 2: The dashed blocks are R-blocks, i.e., $B_R^A(x,k)$ (inner) and $\tilde{B}_R^A(x,k;b,c)$ (outer) for some choice of A,x,k,b,c. The solid blocks are $(R+1)$-blocks.

• **Gärtner-mixing.** For $A \geq 1$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $C \in [0, M]$ and $b, c \in \mathbb{N}_0$, let

$$\begin{aligned}
&\mathcal{A}_R^{A,C}(x,k;b,c)\\
&= \big\{B_{R+1}^A(x,k) \text{ is } (C,b,c)\text{-good, but contains an } R\text{-block that is } (C,b,c)\text{-bad}\big\}\\
&= \bigcup_{\substack{(x_i,k_i)\in\mathbb{Z}^d\times\mathbb{N},\\ B_R^A(x_i,k_i)\subseteq B_{R+1}^A(x,k)}} \Big(\big\{B_R^A(x_i,k_i) \text{ is } (C,b,c)\text{-bad}\big\} \cap \big\{B_{R+1}^A(x,k) \text{ is } (C,b,c)\text{-good}\big\} \Big).
\end{aligned} \tag{1.19}$$

In terms of these events we define the following *space-time mixing conditions* (see Fig. 2). For $D \subset \mathbb{Z}^d \times \mathbb{R}$, let $\sigma(D)$ be the σ-field generated by $\{\overline{\xi}(x,t)\colon\ (x,t) \in D\}$.

Definition 1.5. [Gärtner-mixing]
For $a_1, a_2 \in \mathbb{N}$, denote by $\Delta_n(a_1,a_2)$ the set of $\mathbb{Z}^d \times \mathbb{N}$-valued sequences $\{(x_i,k_i)\}_{i=0}^n$ that are increasing with respect to the lexicographic ordering of $\mathbb{Z}^d \times \mathbb{N}$ and are such that for all $0 \leq i < j \leq n$

$$x_j \equiv x_i \mod a_1 \quad \textit{and} \quad k_j \equiv k_i \mod a_2. \tag{1.20}$$

(a) *ξ is called (A,C,b,c)-type-I Gärtner-mixing when there are $a_1, a_2 \in \mathbb{N}$ such that there is an $R_0 \in \mathbb{N}$ such that, for all $R \in \mathbb{N}$ with $R \geq R_0$ and all $n \in \mathbb{N}$,*

$$\sup_{(x_i,k_i)_{i=0}^n \in \Delta_n(a_1,a_2)} \mathbb{P}\bigg(\bigcap_{i=0}^n \mathcal{A}_R^{A,C}(x_i,k_i;b,c) \bigg) \leq K \Big(A^{(1+2d)} \Big)^{-R(1+d)n}, \tag{1.21}$$

where $K > 0$ is a constant independent of R and n.
(b) *ξ is called (A,C,b,c)-type-II Gärtner-mixing when for each family of events*

$$\mathcal{A}_i^R \in \sigma(B_{R+1}^A(x_i,k_i)), \qquad (x_i,k_i)_{i=0}^n \in \Delta_n(a_1,a_2), \tag{1.22}$$

that are invariant under space-time shifts and satisfy

$$\lim_{R\to\infty} \mathbb{P}(\mathcal{A}_i^R) = 0, \tag{1.23}$$

there are $a_1, a_2 \in \mathbb{N}$ such that for each $\delta > 0$ there is an $R_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{P}\bigg(\bigcap_{i=0}^n \big\{ B_{R+1}^A(x_i,k_i) \textit{ is } (C,b,c)\textit{-good}, \mathcal{A}_i^R \big\} \bigg) \leq K\delta^n \qquad R \geq R_0, R \in \mathbb{N}. \tag{1.24}$$

Here $K > 0$ is a constant independent of R, n and δ.
(c) *ξ is called type-I, respectively type-II, Gärtner-mixing, if there are $A \geq 1$, $C \in [0,M]$, $R \in \mathbb{N}$, $b,c \in \mathbb{N}$ such that ξ is (A,C,b,c)-type-I, respectively, (A,C,b,c)-type-II, Gärtner-mixing.*

Definition 1.6. [Gärtner-hyper-mixing]
(a) *ξ is called Gärtner-positive-hyper-mixing when*
(a1) $\mathbb{E}\Big[e^{q \sup_{s\in[0,1]} \xi(0,s)}\Big] < \infty$ *for all $q \geq 0$.*
(a2) *There are $b, c \in \mathbb{N}$ and a constant C such that for each $A_0 > 1$ one can find $A \geq A_0$ such that ξ is (A,C,b,c)-type-I Gärtner-mixing.*
(a3) *There are $R_0, C_0 \geq 1$ such that*

$$\mathbb{P}\Bigg(\sup_{s\in[0,1]} \frac{1}{|B_R|} \sum_{y\in B_R} \xi(y,s) \geq C \Bigg) \leq |B_R|^{-\alpha} \quad \forall\, R \geq R_0, C \geq C_0, \tag{1.25}$$

for some $\alpha > (1+2d)(2+d)/d$, *where*

$$B_R = \Big([-R,R]^d \cap \mathbb{Z}^d\Big). \tag{1.26}$$

(b) ξ *is called Gärtner-negative-hyper-mixing, if* $-\xi$ *is Gärtner-positive-hyper-mixing.*

Remark 1.7. If ξ is bounded from above, then ξ is Gärtner-positive-hyper-mixing. For those examples where $\xi(x,t)$ represents "the number of particles at site x at time t", we may view Gärtner-mixing as a consequence of the fact that there are not enough particles in the blocks $\tilde{B}_R^A(x_i,k_i;b,c)$ that manage to travel to the blocks $\tilde{B}_R^A(x_j,k_j;b,c)$. Indeed, if there is a bad block on scale R that is contained in a good block on scale $R+1$, then in some neighborhood of this bad block the particle density cannot be too large. This also explains why we must work with the extended blocks $\tilde{B}_R^A(x,k;b,c)$ instead of with the original blocks $B_R^A(x,k;0,0)$. Indeed, the surroundings of a bad block on scale R can be bad when it is located near the boundary of a good block on scale $R+1$ (see Fig. 2).

• **Gärtner-regularity and Gärtner-volatility.** We say that $\Phi\colon [0,t] \to \mathbb{Z}^d$ is a path when

$$\|\Phi(s) - \Phi(s-)\| \le 1 \qquad \forall\, s \in [0,t]. \tag{1.27}$$

We write $\Phi \in B_r(0)$ (the ball in $\mathbb{R}^d$ of radius r centered at 0) when $\|\Phi(s)\| \le r$ for all $s \in [0,t]$ and $N(\Phi,t) \le M$, if Φ has at most M jumps up to time t.

Definition 1.8. [Gärtner-regularity]
ξ *is called Gärtner-regular when*
(a) ξ *is Gärtner-negative-hyper-mixing and Gärtner-positive-hyper-mixing.*
(b) *There are* $t_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every* $\delta_1 > 0$ *there is a* $\delta_2 = \delta_2(\delta_1) > 0$ *such that*

$$\mathbb{P}\left(\sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi\big(\Phi((j-1)t+1), s\big)\, ds \ge \delta_1 nt\right) \le e^{-\delta_2 nt} \qquad \forall\, t \ge t_0,\ n \ge n_0,\ \Phi \in B_{tn}(0). \tag{1.28}$$

Definition 1.9. [Gärtner-volatility]
ξ *is called Gärtner-volatile when*
(a) ξ *is Gärtner-negative-hyper-mixing.*
(b)

$$\lim_{t\to\infty} \frac{1}{\log t} \mathbb{E}\left(\Big|\int_0^t [\xi(0,s) - \xi(e,s)]\, ds\Big|\right) = \infty \qquad \textit{for some } e \in \mathbb{Z}^d \textit{ with } \|e\| = 1, \tag{1.29}$$

Remark 1.10. Corollary 1.19 below will show that condition (b) in Definition 1.8 is satisfied as soon as the Dirichlet form of ξ is non-degenerate, i.e., has a unique zero (see Section 7).

1.3.2 Theorems: Uniqueness, existence, finiteness and initial condition

Recall the definition of q^T (see (1.12)).

Theorem 1.11. [Uniqueness] *Consider a deterministic $q\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ such that:*
(1) *There is a $T > 0$ such that q^T is non-percolating from below.*
(2) *$q^T(x) < \infty$ for all $T > 0$ and $x \in \mathbb{Z}^d$.*
Then the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + q(x,t) u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{1.30}$$

has at most one non-negative solution.

Theorem 1.12. [Existence] *Suppose that:*
(1) *$s \mapsto \xi(x,s)$ is locally integrable for every x, ξ-a.s.*
(2) *$\mathbb{E}(e^{q\xi(0,0)}) < \infty$ for all $q \geq 0$.*
Then the function defined by the Feynman-Kac formula

$$u(x,t) = E_x\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\, ds\right\} u_0(X^\kappa(t))\right) \tag{1.31}$$

solves (1.1) *with initial condition u_0.*

From now on we assume that ξ satisfies the conditions of Theorems 1.11–1.12.

Theorem 1.13. [Finiteness] *If ξ is Gärtner-positive-hyper-mixing, then $\lambda_0^{\delta_0}(\kappa) < \infty$.*

From now on we also assume that ξ satisfies the conditions of Theorem 1.13. The following result extends Gärtner, den Hollander and Maillard [7], Theorem 1.1, in which it was shown that for the initial condition $u_0 = \delta_0$ the quenched Lyapunov exponent exists and is constant ξ-a.s.

Theorem 1.14. [Initial Condition] *If ξ is reversible in time or symmetric in space, type-II Gärtner-mixing and Gärtner-negative-hyper-mixing, then $\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, is constant ξ-a.s., and is independent of u_0.*

1.3.3 Theorems: Dependence on κ

Theorem 1.15. [Continuity at $\kappa = 0$] *If ξ is Gärtner-regular, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is continuous at zero.*

Theorem 1.16. [Not Lipschitz at $\kappa = 0$] *If ξ is Gärtner-volatile, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is not Lipschitz continuous in zero.*

Remark 1.17. Theorem 1.16 was already shown in [7], under the additional assumption that ξ is bounded from below.

1.3.4 Examples

We state two corollaries in which we give examples of classes of ξ for which the conditions in Theorems 1.13–1.15 are satisfied.

Corollary 1.18. [Examples for Theorems 1.13–1.14]
(1) *Let* $X = (X_t)_{t\geq 0}$ *be a stationary and ergodic* $\mathbb{R}$*-valued Markov process. Let* $(X.(x))_{x\in\mathbb{Z}^d}$ *be independent copies of* X*. Define* ξ *by* $\xi(x,t) = X_t(x)$*. If*

$$\mathbb{E}\left[e^{q\sup_{s\in[0,1]} X_s}\right] < \infty \qquad \forall\, q \geq 0, \tag{1.32}$$

then ξ *fulfills the conditions of Theorem* 1.13*. If, moreover, the left-hand side of (1.32) is finite for all* $q \leq 0$*, then* ξ *satisfies the conditions of Theorem* 1.14*.*
(2) *Let* ξ *be the zero-range process with rate function* $g\colon \mathbb{N}_0 \to (0,\infty)$*,* $g(k) = k^\beta$*,* $\beta \in (0,1]$*, and transition probabilities given by a simple random walk on* $\mathbb{Z}^d$*. If* ξ *starts from the product measure* π_ρ*,* $\rho \in (0,\infty)$*, with marginals*

$$\pi_\rho\big\{\eta \in \mathbb{N}_0^{\mathbb{Z}^d}\colon\ \eta(x) = k\big\} = \begin{cases} \gamma\, \frac{\rho^k}{g(1)\times\cdots\times g(k)}, & \text{if } k > 0 \\ \gamma, & \text{if } k = 0, \end{cases} \tag{1.33}$$

where $\gamma \in (0,\infty)$ *is a normalization constant, then* ξ *satisfies the conditions of Theorems* 1.13*–*1.14*.*

Corollary 1.19. [Examples for Theorem 1.15] (1) *If* ξ *is a bounded interacting particle system in the so-called* $M < \varepsilon$ *regime (see Liggett* [12]*), then the conditions of Theorem* 1.15 *are satisfied.*
(2) *If* ξ *is the exclusion process with an irreducible, symmetric and transient random walk transition kernel, then the conditions of Theorem* 1.15 *are satisfied.*
(3) *If* ξ *is the dynamics defined by*

$$\xi(x,t) = \sum_{y\in\mathbb{Z}^d} \sum_{j=1}^{N_y} \delta_{Y_j^y(t)}(x), \tag{1.34}$$

where $\{Y_j^y\colon\ y \in \mathbb{Z}^d, 1 \leq j \leq N_y\}$ *is a collection of independent continuous-time simple random walks jumping at rate one, and* $(N_y)_{y\in\mathbb{Z}^d}$ *is a Poisson random field with intensity* ν *for some* $\nu \in (0,\infty)$*. If* $d \geq 3$*, then the conditions of Theorem* 1.15 *are satisfied.*

Corollaries 1.18–1.19 list only a few examples that match the conditions. It is a separate problem to verify these conditions for as broad a class of interacting particle systems as possible.

1.4 Discussion and a conjecture

The proofs of Theorems 1.11–1.16 and Corollaries 1.18–1.19 are given in Sections 2–7. The content of Theorems 1.11–1.16 is summarized in Fig. 3.

The importance of $\lambda_0^{u_0}(\kappa)$ within the *population dynamics* interpretation of the parabolic Anderson model, as explained in Section 1.1, is the following. For $t > 0$, randomly draw an B-particle from the population of B-particles at the origin. Let L_t be the random time this B-particle and its ancestors have spent on top of A-particles. By appealing to the ergodic theorem, it may be shown that $\lim_{t\to\infty} L_t/t = \lambda_0^{u_0}(\kappa)$ a.s. Thus, $\lambda_0^{u_0}(\kappa)$ is the fraction of time the *best* B-particles spend on top of A-particles, where best means that they come from the fastest growing family ("survival of the fittest"). Fig. 3 shows that for all $\kappa \in (0,\infty)$ *clumping* occurs: the limiting fraction is strictly larger than the density of A-particles. In the limit as

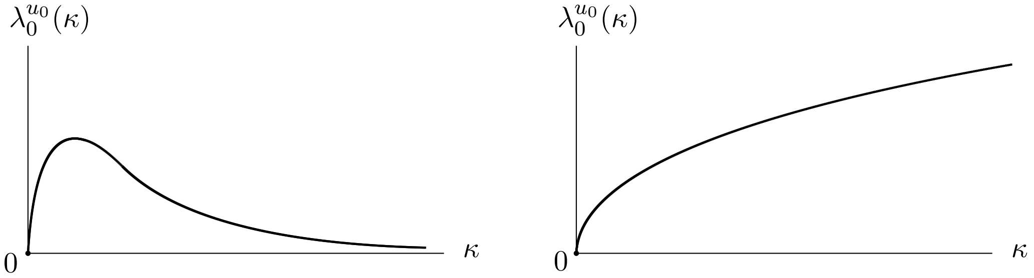

Figure 3: Qualitative picture of $\kappa \mapsto \lambda_0^{u_0}(\kappa)$ in the weakly, respectively, strongly catalytic regime.

$\kappa \downarrow 0$ the clumping vanishes because the motion of the A-particles is ergodic in time. The clumping is hard to suppress for $\kappa \downarrow 0$: even a tiny bit of mobility allows the best B-particles and their ancestors to successfully "hunt down" the A-particles.

In the limit as $\kappa \to \infty$ we expect the quenched Lyapunov exponent to merge with the *annealed Lyapunov exponents* defined in (1.15).

Conjecture 1.20. $\lim_{\kappa\to\infty}[\lambda_p^{u_0}(\kappa) - \lambda_0^{u_0}(\kappa)] = 0$ *for all* $p \in \mathbb{N}$.

The reason is that for large κ the B-particles can easily find the largest clumps of A-particles and spend most of their time there, so that it does not matter much whether the largest clumps are close to the origin or not.

It remains to identify the scaling behaviour of $\lambda_0^{u_0}(\kappa)$ for $\kappa \downarrow 0$ and $\kappa \to \infty$. Under strong noisiness conditions on ξ, it was shown in Gärtner, den Hollander and Maillard [7] that $\lambda^{u_0}(\kappa)$ tends to zero like $1/\log(1/\kappa)$ (in a rough sense), while it tends to $\mathbb{E}(\xi(0,0))$ as $\kappa \to \infty$. For the *annealed* Lyapunov exponents $\lambda_p^{u_0}(\kappa)$, $p \in \mathbb{N}$, there is no singular behavior as $\kappa \downarrow 0$, in particular, they are Lipschitz continuous at $\kappa = 0$ with $\lambda_p^{u_0}(0) > \mathbb{E}(\xi(0,0))$. For three specific choices of ξ it was shown that $\lambda_p^{u_0}(\kappa)$ with $u_0 \equiv 1$ decays like $1/\kappa$ as $\kappa \to \infty$ (see [7] and references therein). A distinction is needed between the *strongly catalytic regime* for which $\lambda_p^{u_0}(\kappa) = \infty$ for all $\kappa \in [0,\infty)$, and the *weakly catalytic regime* for which $\lambda_p^{u_0}(\kappa) < \infty$ for all $\kappa \in [0,\infty)$. (These regimes were introduced by Gärtner and den Hollander [6] for independent simple random walks.) We expect Conjecture 1.20 to be valid in both regimes.

2 Existence and uniqueness of the solution

In this section we prove Theorem 1.11 (uniqueness; Section 2.1) and Theorem 1.12 (existence; Section 2.2).

2.1 Uniqueness

The proof of Theorem 1.11 is based on the following lemma.

Lemma 2.1. *Let* $q_i\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, $i \in \{1,2\}$, *satisfy conditions* (1)–(2) *in Theorem* 1.11 *and be such that, for a given initial condition* u_0, *the two corresponding Cauchy problems*

$$\begin{cases} \frac{\partial}{\partial t}u_i(x,t) = \kappa\Delta u_i(x,t) + q_i(x,t)u_i(x,t), \\ u_i(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0,\ i \in \{1,2\}, \tag{2.1}$$

have a solution. If there exists a $T > 0$ such that $q_1(x,t) \geq q_2(x,t)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,T]$, then $u_1(x,t) \geq u_2(x,t)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,T]$, where u_1 and u_2 are any two solutions of (2.1).

We first prove Theorem 1.11 subject to Lemma 2.1.

Proof. Note from Definition 1.2 that whenever q^T is non-percolating from below for $T = T_0$ for some $T_0 > 0$, then the same is true for all $T \geq T_0$. Fix $T \geq T_0$, and let u be a non-negative solution of (1.30) with zero initial condition, i.e., $u_0(x) = 0$ for all $x \in \mathbb{Z}^d$. It is sufficient to prove that $u(x,t) = 0$ for all $x \in \mathbb{Z}^d$ and $t \in [0,T]$.

Let v be the solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) = \kappa \Delta v(x,t) + q^T(x) v(x,t), \\ v(x,0) = v_0(x) = 0, \end{cases} \qquad x \in \mathbb{Z}^d, t \in [0,T], \tag{2.2}$$

which exists because the corresponding Feynman-Kac representation is zero by Gärtner and Molchanov [8], Lemma 2.2. By Lemma 2.1 it follows that $0 \leq u \leq v$ on $\mathbb{Z}^d \times [0,T]$. Using that q^T is non-percolating from below, we may apply [8], Lemma 2.3, to conclude that (2.2) has at most one solution. Hence $u = v = 0$ on $\mathbb{Z}^d \times [0,T]$, which gives the claim. ∎

We next prove Lemma 2.1.

Proof. Fix $N \in \mathbb{N}$. Let $\bar{Q}_N = [-N,N]^d \cap \mathbb{Z}^d$, $Q_N = (-N,N)^d \cap \mathbb{Z}^d$, and $\partial Q_N = \bar{Q}_N \backslash Q_N$. If u_1 and u_2 are solutions of (2.1) on $\mathbb{Z}^d \times [0,\infty)$, then they are also solutions on $\bar{Q}_N \times [0,T]$. More precisely, for $i \in \{1,2\}$, u_i is a solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) = \kappa \Delta v(x,t) + q_i(x,t) v(x,t), & (x,t) \in Q_N \times [0,T], \\ v(x,0) = u_0(x), & x \in \bar{Q}_N, \\ v(x,t) = u_i(x,t), & (x,t) \in \partial Q_N \times [0,T]. \end{cases} \tag{2.3}$$

Recall that $q_1 \geq q_2$ on $\mathbb{Z}^d \times [0,T]$. Choose c_N^T such that

$$c_N^T > \max_{x \in \bar{Q}_N,\, t \in [0,T]} q_1(x,t) \geq \max_{x \in \bar{Q}_N,\, t \in [0,T]} q_2(x,t), \tag{2.4}$$

and abbreviate

$$\begin{cases} v(x,t) = e^{-c_N^T t} \big[u_1(x,t) - u_2(x,t) \big], & (x,t) \in \bar{Q}_N \times [0,T], \\ \bar{Q}_i = q_i - c_N^T, & i \in \{1,2\}. \end{cases} \tag{2.5}$$

Then, by (2.3), v satisfies

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) = \kappa \Delta v(x,t) \\ \qquad\qquad + e^{-c_N^T t} \bar{Q}_1(x,t) u_1(x,t) - e^{-c_N^T t} \bar{Q}_2(x,t) u_2(x,t), & (x,t) \in Q_N \times [0,T], \\ v(x,0) = 0, & x \in \bar{Q}_N, \\ v(x,t) = e^{-c_N^T t} \big[u_1(x,t) - u_2(x,t) \big], & (x,t) \in \partial Q_N \times [0,T]. \end{cases} \tag{2.6}$$

Now, suppose that there exists a $(x_*, t_*) \in Q_N \times [0,T]$ such that

$$v(x_*, t_*) = \min_{x \in Q_N,\, t \in [0,T]} v(x,t) < 0. \tag{2.7}$$

Then

$$\frac{\partial}{\partial t} v(x_*,t_*) \le 0 \tag{2.8}$$

and

$$\Delta v(x_*,t_*) = \sum_{\substack{y\in\mathbb{Z}^d\\ \|y-x_*\|=1}} \big[v(y,t_*)-v(x_*,t_*)\big] \ge 0. \tag{2.9}$$

Moreover, by (2.4–2.5) and (2.7),

$$\begin{aligned} &e^{-c_N^T t_*}\,\bar{Q}_1(x_*,t_*)u_1(x_*,t_*) - e^{-c_N^T t_*}\,\bar{Q}_2(x_*,t_*)u_2(x_*,t_*)\\ &\quad = \big[q_1(x_*,t_*)-c_N^T\big]\,v(x_*,t_*) + \big[q_1(x_*,t_*)-q_2(x_*,t_*)\big]\,e^{-c_N^T t_*}\,u_2(x_*,t_*) > 0. \end{aligned} \tag{2.10}$$

But (2.8–2.10) contradict the first line of (2.6) at $(x,t)=(x_*,t_*)$. Hence (2.7) fails, and so it follows from (2.5) that $u_1(x,t)\ge u_2(x,t)$ for all $x\in Q_N$ and $t\in[0,T]$. Since N can be chosen arbitrarily, the claim follows. ∎

2.2 Existence

In the sequel we use the abbreviations

$$\begin{aligned} \mathcal{I}^\kappa(a,b,c) &= \textstyle\int_a^b \xi(X^\kappa(s),c-s)ds, \qquad 0\le a\le b\le c, &(2.11)\\ \overline{\mathcal{I}}^\kappa(a,b,c) &= \textstyle\int_a^b \xi(X^\kappa(s),c+s)ds, \qquad 0\le a\le b\le c. &(2.12) \end{aligned}$$

Proof. To prove Theorem 1.12, by Proposition 1.1 it is enough to show that

$$E_x\left(e^{\mathcal{I}^\kappa(0,t,t)}u_0(X^\kappa(t))\right) < \infty \qquad \forall x\in\mathbb{Z}^d,\ t\ge 0. \tag{2.13}$$

Since u_0 is assumed to be non-negative and bounded (recall (1.4)), without loss of generality we may take $u_0\equiv 1$. We give the proof for $x=0$, the extension to $x\in\mathbb{Z}^d$ being straightforward. Fix $q\in\mathbb{Q}\cap[0,\infty)$. Using Jensen's inequality and the stationarity of ξ, we have (recall (1.6))

$$\begin{aligned} \mathbb{E}\left(E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\right)\right) &= E_0\left(\mathbb{E}\left(e^{\mathcal{I}^\kappa(0,q,q)}\right)\right)\\ &\le E_0\left(\mathbb{E}\left(\frac{1}{q}\int_0^q \exp\Big\{q\xi(X^\kappa(s),q-s)\Big\}\,ds\right)\right)\\ &= E_0\left(\frac{1}{q}\int_0^q \mathbb{E}\Big(\exp\Big\{q\xi(0,0)\Big\}\Big)\,ds\right)\\ &= \mathbb{E}\big(e^{q\xi(0,0)}\big) < \infty, \end{aligned} \tag{2.14}$$

where the finiteness follows by condition (2). Hence, for every $q\in\mathbb{Q}\cap[0,\infty)$ there exists a set A_q with $\mathbb{P}(A_q)=1$ such that

$$E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\right) < \infty \qquad \forall\,\xi\in A_q. \tag{2.15}$$

To extend (2.15) to $t\in[0,\infty)$, note that, by the Markov property of X^κ applied at time $q-t$, $q>t$, we have

$$\begin{aligned} E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\right) &\ge E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\mathbb{1}\Big\{X^\kappa(r)=0\ \forall\,r\in[0,q-t]\Big\}\right)\\ &= e^{\int_0^{q-t}\xi(0,q-s)\,ds}P_0\Big(X^\kappa(r)=0\ \forall\,r\in[0,q-t]\Big)E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\right). \end{aligned} \tag{2.16}$$

Because $s\mapsto\xi(0,s)$ is locally integrable ξ-a.s. by condition (1), we have $\int_0^{q-t}\xi(0,q-s)\,ds > -\infty$ ξ-a.s. The claim now follows from (2.15–2.16) by picking $q\in\mathbb{Q}\cap[0,\infty)$ and $t\in[0,\infty)$. ∎

3 Finiteness of the quenched Lyapunov exponent

In this section we prove Theorem 1.13. In Section 3.1 we sketch the strategy of the proof. In Sections 3.2–3.6 the details are worked out.

3.1 Strategy of the proof

The proof uses ideas from Kesten and Sidoravicius [10]. Fix C, b, c according to our assumptions on ξ. For $j \in \mathbb{N}$ and $t > 0$, define the set of random walk paths

$$\Pi(j,t) = \left\{ \Phi\colon\ [0,t] \to \mathbb{Z}^d\colon\ \Phi \text{ makes } j \text{ jumps, } \Phi([0,t]) \subseteq [-C_1 t\log t, C_1 t \log t]^d \cap \mathbb{Z}^d \right\}, \quad (3.1)$$

where C_1 will be determined later on. Abbreviate $[C_1]_t = [-C_1 t\log t, C_1 t\log t]^d \cap \mathbb{Z}^d$. For $A \geq 1$, $R \in \mathbb{N}$ and $\Phi \in \Pi(j,t)$, define

$$\begin{aligned}
\Psi_R^A(\Phi) &= \text{number of good } (R+1)\text{-blocks crossed by } \Phi \text{ containing a bad } R\text{-block,} && (3.2)\\
\Psi_R^{A,j} &= \sup_{\Phi\in\Pi(j,t)} \Psi_R^A(\Phi), && (3.3)\\
\Xi_R^A(\Phi) &= \text{number of bad } R\text{-blocks crossed by } \Phi, && (3.4)\\
\Xi_R^{A,j} &= \sup_{\Phi\in\Pi(j,t)} \Xi_R^A(\Phi). && (3.5)
\end{aligned}$$

The proof comes in 5 steps, organized as Sections 3.2–3.6: (1) the Feynman-Kac formula may be restricted to paths contained in $[C_1]_t$; (2) there are no bad R-blocks for sufficiently large R; (3) the Feynman-Kac formula can be estimated in terms of bad R-blocks; (4) bounds can be derived on the number of bad R-blocks; (5) completion of the proof.

3.2 Step 1: Restriction to $[C_1]_t$

Lemma 3.1. *Fix $C_1 > 0$. Suppose that $\mathbb{E}\left(e^{q\sup_{s\in[0,1]}\xi(0,s)}\right) < \infty$ for all $q > 0$. Then:*
(a) *ξ-a.s.*

$$\limsup_{t\to\infty} \left[\frac{1}{\log t} \sup\{\xi(x,s)\colon\ x \in [C_1]_t, 0 \leq s \leq t\}\right] \leq 1. \quad (3.6)$$

(b) *ξ-a.s. there exists a $t_0 \geq 0$ such that, for all $t \geq t_0$ and $x \notin [C_1]_t$,*

$$\sup_{s\in[0,t]} \xi(x,s) \leq \log\|x\|. \quad (3.7)$$

Proof. (a) For any $\theta > 0$ and $t \geq 1$, we may estimate

$$\begin{aligned}
&\mathbb{P}\left(\exists\, x \in [C_1]_t\colon\ \sup_{s\in[0,t]} \xi(x,s) \geq \log t\right)\\
&\leq \sum_{x\in[C_1]_t} \sum_{k=0}^{\lfloor t\rfloor} \mathbb{P}\left(\sup_{s\in[k,k+1]} \xi(x,s) \geq \log t\right) \qquad (3.8)\\
&\leq (2C_1 t\log t + 1)^d (\lfloor t\rfloor + 1) \exp\{-\theta\log t\} \mathbb{E}\left(\exp\left\{\theta \sup_{s\in[0,1]} \xi(0,s)\right\}\right).
\end{aligned}$$

Choosing $\theta > 2(d+1)+1$, we get that the right-hand side is summable over $t \in \mathbb{N}$. Hence, by the Borel-Cantelli Lemma, we get the claim.

(b) The proof is similar and is omitted. ∎

The main result of this section reads:

Lemma 3.2. *There exists a $C_0 > 0$ such that ξ-a.s. there exists a $t_0 > 0$ such that*

$$E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\mathbb{1}\{X^\kappa([0,t]) \not\subseteq [C_1]_t\}\right) \leq e^t \qquad \forall\, t \geq t_0, C_1 \geq C_0. \tag{3.9}$$

Proof. See Kesten and Sidoravicius [10], Eq. (2.38). We only sketch the main idea. Take a realization $\Phi\colon [0,t] \to \mathbb{Z}^d$ of a random walk path that leaves the box $[C_1]_t$. Then $\|\Phi\| = \max\{\|x\|\colon x \in \Phi([0,t])\} > C_1 t \log t$. By Lemma 3.1,

$$\sup_{s\in[0,t]}\ \sup_{\|x\|\leq\|\Phi\|} \xi(x,s) \leq \log\|\Phi\|, \tag{3.10}$$

and so we can estimate

$$\begin{aligned} &E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\mathbb{1}\{X^\kappa([0,t]) \not\subseteq [C_1]_t\}\right) \\ &\leq E_0\left(\exp\left\{t \sup_{s\in[0,t]} \log\|X^\kappa(s)\|\right\}\mathbb{1}\{X^\kappa([0,t]) \not\subseteq [C_1]_t\}\right). \end{aligned} \tag{3.11}$$

The rest of the proof consists of balancing the exponential growth of the term with the supremum against the superexponential decay of $P_0(X^\kappa([0,t]) \not\subseteq [C_1]_t)$. See [10] for details. ∎

3.3 Step 2: No bad R-blocks for large R

Lemma 3.3. *Fix $C_0 > 0$ according to Lemma* 3.2*, and suppose that ξ satisfies condition* (a3) *in the Gärtner-positive-hyper-mixing definition. Then for every $C_1 \geq C_0$ and $\varepsilon > 0$ there exists an $A = A(\varepsilon) > 2$ such that*

$$\mathbb{P}\Big(\Xi_R^{A,j} > 0 \textit{ for some } R \geq \varepsilon\log t \textit{ and some } j \in \mathbb{N}_0\Big) \tag{3.12}$$

is summable over $t \in \mathbb{N}$. (It suffices to choose $A = \lfloor e^{1/a(1+2d)\varepsilon}\rfloor$ for some $a > 1$.)

Proof. Fix $C_1 \geq C_0$, $A > 2$ and assume that $\Xi_R^{A,j} > 0$ for some $j \in \mathbb{N}_0$. Then there is a bad R-block $B_R^A(x,k)$ that intersects $[C_1]_t \times [0,t]$. Hence there is a pair $(y,s) \in \mathbb{Z}^d \times [0,\infty)$ such that $Q_R^A(y) \times \{s\} \subseteq \tilde{B}_R^A(x,k;b,c)$ and

$$\sum_{z\in Q_R^A(y)} \xi(z,s) > CA^{Rd}. \tag{3.13}$$

In particular, x and s satisfy $\mathrm{dist}(y,[C_1]_t)\le (b+2)A^R$ and $s\in[0,t+A^R]$. Hence, for $\varepsilon>0$,

$$
\begin{aligned}
&\mathbb{P}\Big(\Xi_R^{A,j}>0 \text{ for some } R\ge\varepsilon\log t, j\in\mathbb{N}_0\Big)\\
&\le \sum_{R\ge\varepsilon\log t}\mathbb{P}\Bigg(\sum_{z\in Q_R^A(y)}\xi(z,s)>CA^{Rd}\text{ for some }(y,s)\colon\\
&\qquad\qquad \mathrm{dist}(y,[C_1]_t)\le(b+2)A^R, s\in[0,t+A^R]\Bigg)\\
&\le \sum_{R\ge\varepsilon\log t}\ \sum_{y\colon \mathrm{dist}(y,[C_1]_t)\le(b+2)A^R}\ \sum_{k=0}^{\lfloor t+A^R\rfloor}\mathbb{P}\Big(\exists s\in[k,k+1)\colon \sum_{z\in Q_R^A(y)}\xi(z,s)>CA^{Rd}\Big).
\end{aligned}
\tag{3.14}
$$

By assumption (1.25), we may bound the two inner sums by

$$
(2C_1t\log t+1+(b+2)A^R)^d\times(\lfloor t+A^R\rfloor+1)\times(2A^R+1)^{-d\alpha}\stackrel{\mathrm{def}}{=}G(R,t). \tag{3.15}
$$

Recall the definition of α (see below 1.25)), to see that one can choose A as described in the formulation of Lemma 3.3 to get that

$$
\sum_{R\ge\varepsilon\log t}G(R,t) \tag{3.16}
$$

is summable over $t\in\mathbb{N}$. ∎

3.4 Step 3: Estimate of the Feynman-Kac formula in terms of bad blocks

Lemma 3.4. *Fix $\varepsilon>0$ and $A>2$. For all $C_1\ge C_0$ (where C_0 is determined by Lemma 3.2),*

$$
\begin{aligned}
&E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\mathbb{1}\{X^\kappa([0,t])\subseteq[C_1]_t\}\right)\\
&\le\sum_{j\in\mathbb{N}_0}\frac{(2dt\kappa)^j}{j!}\exp\left\{t(CA^d-2d\kappa)+\sum_{R=1}^{\infty}CA^{(R+1)d}A^R\Xi_R^{A,j}\right\}.
\end{aligned}
\tag{3.17}
$$

Proof. See [10], Lemma 9. We sketch the proof. Note that

$$
\begin{aligned}
E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\mathbb{1}\{X^\kappa([0,t])\subseteq[C_1]_t\}\right)&=\sum_{j\in\mathbb{N}_0}e^{-2dt\kappa}\frac{(2dt\kappa)^j}{j!}\\
\times\sum_{x_1,x_2,\dots,x_j\in\mathbb{Z}^d}\frac{1}{(2d)^j}E_0&\left(\exp\left\{\sum_{i=1}^j\int_{S_{i-1}}^{S_i}\xi(x_{i-1},t-u)\,du+\int_{S_j}^t\xi(x_j,t-u)\,du\right\}\right),
\end{aligned}
\tag{3.18}
$$

where j is the number of jumps, $0=x_0,x_1,\dots,x_j$, $x_i\in[C_1]_t, i\in\{0,1,\cdots,j\}$, are the nearest-neighbor sites visited, and $0=S_0<S_1<\cdots<S_j<t$ are the jump times. To analyze (3.18), fix $A>2$, $R\in\mathbb{N}$ as well as $0=s_0<s_1<\cdots<s_j$ and a path Φ with these jump times, and define

$$
\begin{aligned}
\Lambda_R(\Phi)=&\bigcup_{i=1}^j\left\{u\in[s_{i-1},s_i)\colon CA^{Rd}<\xi(x_{i-1},t-u)\le CA^{(R+1)d}\right\}\\
&\bigcup\left\{u\in[s_j,t)\colon CA^{Rd}<\xi(x_j,t-u)\le CA^{(R+1)d}\right\}.
\end{aligned}
\tag{3.19}
$$

The contribution of Φ to the exponential in (3.18) may be bounded from above by

$$tCA^d + \sum_{R=1}^{\infty} CA^{(R+1)d} |\Lambda_R(\Phi)|, \tag{3.20}$$

where the first term comes from the space-time points $(x_{i-1}, t-u)$ with $\xi(x_{i-1}, t-u) \leq CA^d$. If $CA^{Rd} < \xi(x_{i-1}, t-u) \leq CA^{(R+1)d}$, then $(x_{i-1}, t-u)$ belongs to a bad R-block. There are at most $\Xi_R^{A,j}$ such blocks, and any path spends at most a time A^R in each R-block. Hence

$$|\Lambda_R(\Phi)| \leq A^R \Xi_R^{A,j}. \tag{3.21}$$

The claim now follows from (3.18), (3.20–3.21) and the fact that there are at most $(2d)^j$ nearest-neighbor paths $(0 = x_0, x_1, x_2, \dots, x_j)$ that are contained in $[C_1]_t$. ∎

3.5 Step 4: Bound on the number of bad blocks

The goal of this section is to provide a bound on the number of bad blocks on all scales simultaneously (Lemma 3.5 below). In Section 3.6 we will combine Lemmas 3.2 and 3.4–3.22 to prove Theorem 1.13.

Lemma 3.5. *Fix $\varepsilon > 0$, pick A according to Lemma 3.3 and assume that $\Xi_R^{A,j} = 0$ for all $R \geq \lceil \varepsilon \log t \rceil$. Then, for some $C_2 > 0$,*

$$\mathbb{P}\Big(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R} \text{ for some } R \in \mathbb{N} \text{ and some } j \in \mathbb{N}_0\Big), \tag{3.22}$$

$$\mathbb{P}\Big(\Xi_R^{A,j} \geq C_2(t+j)(A^{(1+2d)})^{-R} \text{ for some } R \in \mathbb{N} \text{ and some } j \in \mathbb{N}_0\Big), \tag{3.23}$$

are summable on $t \in \mathbb{N}$.

The proof of Lemma 3.5 is based on Lemmas 3.6–3.7 below. The first estimates for fixed R the probability that there is a large number of good $(R+1)$-blocks containing a bad R-block, the second gives a recursion bound on the number of bad blocks in terms of $\Psi_R^{A,j}$.

Lemma 3.6. *Suppose that ξ satisfies condition* (a2) *in Definition* 1.6. *Then, for R large enough, $j \in \mathbb{N}_0$ and A chosen according to Lemma 3.3, for some constant $C_3 > 0$*

$$\mathbb{P}\left(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R}\right) \leq \exp\left\{-C_3(t+j)(A^{(1+2d)})^{-R}\right\}. \tag{3.24}$$

Lemma 3.7. *Fix $\varepsilon > 0$, and pick A according to Lemma* 3.3. *Assume that $\Xi_R^{A,j} = 0$ for all $R \geq \lceil \varepsilon \log t \rceil$. Then, with $N = \lceil \varepsilon \log t \rceil$,*

$$\Xi_R^{A,j} \leq 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} 2^{id} A^{i(1+d)} \Psi_{R+i}^{A,j}. \tag{3.25}$$

The proofs of Lemmas 3.6, 3.7 and 3.5 are given in Sections 3.5.1, 3.5.2 and 3.5.3, respectively.

3.5.1 Proof of Lemma 3.6

Proof. Throughout the proof, $x, x' \in \mathbb{Z}^d$ and $k, k' \in \mathbb{N}$. The idea of the proof is to divide space-time blocks into equivalence classes, such that in each equivalence class blocks are far enough away from each other so that they can be treated as being independent (see Fig. 2). The proof comes in four steps.

1. Fix $A > 2$ according to Lemma 3.3, fix $R \in \mathbb{N}$ and take $a_1, a_2, b, c \in \mathbb{N}_0$ according to condition (a2) in Definition 1.6. We say that (x, k) and (x', k') are equivalent if and only if

$$x \equiv x' \mod a_1 \quad \text{and} \quad k \equiv k' \mod a_2. \tag{3.26}$$

This equivalence relation divides $\mathbb{Z}^d \times \mathbb{N}$ into $a_1^d a_2$ equivalence classes. We write $\sum_{(x^*,k^*)}$ to denote the sum over all equivalence classes. Furthermore, we define

$$\chi^A(x,k) = 1\!\!1\{B^A_{R+1}(x,k) \text{ is good, but contains a bad } R\text{-block}\}. \tag{3.27}$$

We tacitly assume that all blocks under consideration intersect $[C_1]_t \times [0,t]$. Then

$$\begin{aligned}
&\mathbb{P}\left(\Psi^{A,j}_R \geq (t+j)(A^{(1+2d)})^{-R}\right) \\
&\leq \sum_{(x^*,k^*)} \mathbb{P}\Big(\exists \text{ a path with } j \text{ jumps that intersects at least} \\
&\qquad (t+j)(A^{(1+2d)})^{-R}/a_1^d a_2 \text{ blocks } B^A_{R+1}(x,k) \text{ with } \chi^A(x,k) = 1,\ (x,k) \equiv (x^*,k^*)\Big).
\end{aligned} \tag{3.28}$$

2. Define $\rho_R^{1/(1+d)} = (A^{(1+2d)})^{-R}$ and abbreviate

$$\begin{aligned}
\mathcal{A}^n_1 = \Big\{&\exists \text{ a path with } j \text{ jumps that intersects } n \text{ blocks } B^A_{R+1}(x,k) \\
&\text{with } \chi^A(x,k) = 1,\ (x,k) \equiv (x^*,k^*)\Big\}.
\end{aligned} \tag{3.29}$$

Then we may rewrite the right-hand side of (3.28) as

$$\sum_{(x^*,k^*)} \sum_{n=\frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}}^{L} \mathbb{P}(\mathcal{A}^n_1), \tag{3.30}$$

where L is the number of $(R+1)$-blocks that can be crossed by a path with j jumps contained in $[C_1]_t \times [0,t]$. To estimate the probability in (3.30), note that we can write $\mathcal{A}^n_1$ as

$$\mathcal{A}^n_1 = \bigcup_{(x_m^{a_1}, k_m^{a_2})} \mathcal{A}_1^{n,(x_m^{a_1}, k_m^{a_2})}, \tag{3.31}$$

where the union is taken over all possible choices of $(R+1)$-blocks such that exactly n of them are good, contain a bad R-block, and can be reached by a path with j jumps. Hence, (3.30) becomes

$$\sum_{(x^*,k^*)} \sum_{n=\frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}}^{L} \sum_{(x_m^{a_1}, k_m^{a_2})} \mathbb{P}\Big(\mathcal{A}_1^{n,(x_m^{a_1}, k_m^{a_2})}\Big). \tag{3.32}$$

By condition (a2) in Definition 1.6, the probability in (3.32) may be bounded from above by $K(A^{(1+2d)})^{-R(1+d)n}$. Note that there are at most $\binom{L}{n}$ elements in the union (3.31). Thus, the two inner sums in (3.32) are bounded from above by

$$(1-\rho_R)^{-L} K \mathbb{P}\left(T_L \geq \frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}\right), \tag{3.33}$$

where $T_L = \mathrm{BIN}(L, \rho_R)$.

3. To estimate the binomial random variable, we first bound L. As in [10], take $\nu = \lceil \rho_R^{-1/(1+d)} \rceil$ and define space-time blocks

$$\tilde{B}_R^A(x,k) = \left(\prod_{j=1}^d [\nu(x(j)-1)A^R, \nu(x(j)+1)A^R) \cap \mathbb{Z}^d\right) \times [\nu k A^R, \nu(k+1)A^R). \tag{3.34}$$

By the same reasoning as in [10], we see that at most

$$\mu(j) \stackrel{\text{def}}{=} 3^d \left(\frac{t+j}{\nu A^{R+1}} + 2\right) \tag{3.35}$$

blocks $\tilde{B}_{R+1}^A(x,k)$ can be crossed by a path Φ with j jumps. Hence $L \leq \nu^{(1+d)}\mu(j)$, and thus the probability in (3.33) is at most

$$\mathbb{P}\left(T \geq \frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}\right), \tag{3.36}$$

where $T = \mathrm{BIN}(\nu^{(1+d)}\mu(j), \rho_R)$. According to Bernstein's inequality (compare with [10], Lemma 11), there is a constant C' such that, for all $\lambda \geq 2\mathbb{E}(T)$,

$$\mathbb{P}(T \geq \lambda) \leq \exp\{-C'\lambda\}. \tag{3.37}$$

We may assume that $\rho_R^{-1/(1+d)} \in \mathbb{N}$, so that

$$\mathbb{E}(T) = \nu^{(1+d)}\mu(j)\rho_R = 3^d \left(\frac{t+j}{A^{R+1}}\,\rho_R^{1/(1+d)} + 2\right), \tag{3.38}$$

and hence, by Lemma 3.3 and the fact that $R \leq \varepsilon \log t$,

$$\frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2} \geq 2\mathbb{E}(T). \tag{3.39}$$

Since a_1, a_2 are independent of R, we may estimate, again using Bernstein's inequality,

$$\begin{aligned} \mathbb{P}\left(T \geq \frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}\right) &\leq \exp\left\{-C'(t+j)\rho_R^{1/(1+d)}\right\} \\ &= \exp\left\{-C'(t+j)(A^{(1+2d)})^{-R}\right\}. \end{aligned} \tag{3.40}$$

It rests to show that the first term in (3.33) does not contribute. Note that $1/(1-\rho_R) = 1 + \rho_R/(1-\rho_R)$, so that

$$\log\left(\frac{1}{1-\rho_R}\right) \leq \frac{\rho_R}{1-\rho_R}. \tag{3.41}$$

Thus, if we assume $\rho_r^{-1/(1+d)} \in \mathbb{N}$,

$$L \log\left(\frac{1}{1-\rho_R}\right) \le \frac{\mu(j)}{1-\rho_R}. \tag{3.42}$$

Inserting (3.42) into the first term of (3.33) and comparing it with the right hand side of (3.40), we see that the asymptotic of (3.33) is determined by the probability term.

4. Finally, we estimate (3.32). For that we insert (3.40) into (3.32), which yields

$$\begin{aligned} &\mathbb{P}\left(\Psi_R^{A,j} \ge (t+j)(A^{(1+2d)})^{-R}\right) \\ &\le K a_1^d a_2 \exp\left\{-C'(t+j)(A^{(1+2d)})^{-R}\right\}. \end{aligned} \tag{3.43}$$

This finishes the proof. ∎

3.5.2 Proof of Lemma 3.7

Proof. We first show that

$$\Xi_R^{A,j} \le 2^d A^{(1+d)} \Xi_{R+1}^{A,j} + 2^d A^{(1+d)} \Psi_R^{A,j}. \tag{3.44}$$

In order to see why (3.44) is true, take a bad R-block $B_R^A(x,k)$ that is crossed by a path with j jumps. Then there are two possibilities. Either $B_R^A(x,k)$ is contained in a bad $(R+1)$-block, or all $(R+1)$-blocks that contain $B_R^A(x,k)$ are good. Since an $(R+1)$-block contains $A^{(1+d)}$ R-blocks, and there are at most 2^d $(R+1)$-blocks, which may contain a given R-block, the first term in the above sum bounds the number of bad R-blocks contained in a bad $(R+1)$-block. In contrast, the second term bounds the number of bad R-blocks contained in a good $(R+1)$-block. Hence we obtain (3.44).

We can now prove the claim. Apply (3.44) iteratively to the terms in the sum, i.e., replace $\Xi_{R+i}^{A,j}$ by

$$2^d A^{(1+d)} \Xi_{R+i+1}^{A,j} + 2^d A^{(1+d)} \Psi_{R+i}^{A,j}. \tag{3.45}$$

This yields

$$\Xi_R^{A,j} \le 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} 2^{id} A^{i(1+d)} \Psi_{R+i}^{A,j}, \tag{3.46}$$

from which the claim follows. ∎

3.5.3 Proof of Lemma 3.5

Proof. Fix $\varepsilon > 0$ and $0 < R \le \varepsilon \log t$. Then, by Lemma 3.6,

$$\begin{aligned} &\mathbb{P}\left(\Psi_R^{A,j} \ge (t+j)(A^{(1+2d)})^{-R} \text{ for some } j \in \mathbb{N}_0\right) \\ &\le \sum_{j\in\mathbb{N}_0} \mathbb{P}\left(\Psi_R^{A,j} \ge (t+j)(A^{(1+2d)})^{-R}\right) \\ &\le \sum_{j\in\mathbb{N}_0} \exp\left\{-tC_3(t+j)(A^{(1+2d)})^{-R}\right\} \\ &\le \exp\left\{-C_3 t (A^{(1+2d)})^{-\varepsilon\log t}\right\} \sum_{j\in\mathbb{N}_0} \exp\left\{-C_3 j (A^{(1+2d)})^{-\varepsilon \log t}\right\}. \end{aligned} \tag{3.47}$$

Recall Lemma 3.3, which implies that $\delta \overset{\text{def}}{=} \log(A)\varepsilon(1+2d) < 1$. Consequently, the right-hand side of (3.47) is at most

$$\begin{aligned} &\exp\left\{-C_3 t^{(1-\delta)}\right\} \frac{1}{1-\exp\{-C_3 t^{-\delta}\}} \\ &\le \frac{1}{C_3} \exp\left\{-C_3 t^{(1-\delta)}\right\} t^{\delta} \exp\left\{C_3 t^{-\delta}\right\}. \end{aligned} \tag{3.48}$$

It therefore follows that

$$\begin{aligned} &\mathbb{P}\left(\Psi_R^{A,j} \ge (t+j)(A^{(1+2d)})^{-R} \text{ for some } j \in \mathbb{N}_0,\, R \in \mathbb{N}\right) \\ &\le \frac{1}{C_3} t^{\delta}\, \varepsilon \log t \, \exp\left\{-C_3 t^{(1-\delta)} + C_3 t^{-\delta}\right\}, \end{aligned} \tag{3.49}$$

which is summable over $t \in \mathbb{N}$. In order to prove the second statement, suppose that none of the events in (3.22) occurs. With Lemma 3.7 we may estimate

$$\begin{aligned} \Xi_R^{A,j} &\le 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} 2^{id} A^{i(1+d)} \Psi_{R+i}^{A,j} \\ &\le 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} (t+j) 2^{id} A^{i(1+d)} (A^{(1+2d)})^{-i-R} \\ &\le 2^d A^{(1+d)} (t+j) A^{-R(1+2d)} \sum_{i\in\mathbb{N}_0} 2^{id} A^{-id} \\ &\overset{\text{def}}{=} (t+j) A^{-R(1+2d)} C_2, \end{aligned} \tag{3.50}$$

where we use that $A > 2$ (see Lemma 3.3). ∎

3.6 Step 5: Proof of Theorem 1.13

Fix $\varepsilon > 0$ and A such that Lemma 3.5 applies. It follows from Lemma 3.3 that

$$\mathbb{P}\Big(\Xi_R^{A,j} > 0 \text{ for some } R \ge \varepsilon \log t, j \in \mathbb{N}_0\Big) \tag{3.51}$$

is summable over $t \in \mathbb{N}$. Hence, by the Borel-Cantelli Lemma, there is an $t_0 \in \mathbb{N}$ such that none of the events in the above probability occurs for integer $t \ge t_0$. Thus, by Lemma 3.4, for all integer $t \ge t_0$ we have, with $N = \lfloor \varepsilon \log t \rfloor$,

$$\begin{aligned} &E_0\left(e^{\mathcal{I}^\kappa(0,t,t)} \mathbb{1}\left\{X^\kappa([0,t]) \subseteq [C_1]_t\right\}\right) \\ &\le \sum_{j\in\mathbb{N}_0} \frac{(2dt\kappa)^j}{j!} \exp\left\{ t(CA^d - 2d\kappa) + \sum_{R=1}^{N} CA^{(R+1)d} A^R \Xi_R^{A,j} \right\}. \end{aligned} \tag{3.52}$$

Using the bound of Lemma 3.5, we have

$$\begin{aligned} \sum_{R=1}^{N} CA^{(R+1)d} A^R \Xi_R^{A,j} &\le \sum_{R=1}^{N} C(t+j) A^{(R+1)d} A^R A^{-R(1+2d)} C_2 \\ &\le (t+j) A^d C' \sum_{R\in\mathbb{N}} A^{-Rd} \le C_4(t+j). \end{aligned} \tag{3.53}$$

We can therefore estimate the last line of (3.52) by

$$\begin{aligned}&\sum_{j\in\mathbb{N}_0}\frac{(2dt\kappa)^j}{j!}\exp\Big\{t(CA^d-2d\kappa)+C_4(t+j)\Big\}\\&=\exp\Big\{t(CA_2^d-2d\kappa+C_4+2d\kappa e^{C_4})\Big\}.\end{aligned}\tag{3.54}$$

From (3.54) and Lemma 3.2, we obtain

$$\limsup_{\substack{t\to\infty\\ t\in\mathbb{N}}}\frac{1}{t}\log E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\right)<\infty.\tag{3.55}$$

To extend this to sequences along $\mathbb{R}$ instead of $\mathbb{N}$, note that

$$u(0,t)\leq u(0,n+1)\,e^{-\int_t^{n+1}\xi(0,s)\,ds}e^{2d\kappa(n+1-t)},\qquad t\in[n,n+1].\tag{3.56}$$

Since ξ is ergodic in time, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_t^{\lceil t\rceil}\xi(0,s)\,ds=0.\tag{3.57}$$

Theorem 1.13 follows from (3.56–3.57).

4 Initial condition

In this section we prove Theorem 1.14. Section 4.1 contains some preparations. Section 4.2 states three lemmas (Lemmas 4.5–4.7 below) that are needed for the proof of Theorem 1.14, which is given in Section 4.3. Section 4.4 provides the proof of these three lemmas.

4.1 Preparations

In this section we first state and prove a lemma (Lemma 4.1 below) that will be needed for the proof of Theorem 1.14. After that we introduce some further notation (Definitions 4.2–4.4 below).

Fix $R_0\in\mathbb{N}$ and take A,C according to our assumption (type-II Gärtner-mixing). Set $N=CA^{R_0d}$ and abbreviate $\xi_N=(\xi\wedge N)\vee(-N)$. Let u_N be the solution of (1.1) with ξ replaced by ξ_N. Abbreviate (recall (2.11–2.12))

$$\mathcal{I}_N^\kappa(a,b,c)\;=\int_a^b\xi_N(X^\kappa(s),c-s)\,ds,\qquad 0\leq a\leq b\leq c,\tag{4.1}$$

$$\overline{\mathcal{I}}_N^\kappa(a,b,c)\;=\int_a^b\xi_N(X^\kappa(s),c+s)\,ds,\qquad 0\leq a\leq b\leq c.\tag{4.2}$$

Lemma 4.1. *If, for all N of the form $N=CA^{R_0d}$ and for all $\varepsilon>0$ and some sequence $(t_r)_{r\in\mathbb{N}}$ of the form $t_r=rL$ with $L>0$,*

$$\mathbb{P}\left(E_0\left(e^{\overline{\mathcal{I}}_N^\kappa(0,t_r,0)}\right)>e^{(\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon)t_r}\right)\tag{4.3}$$

is summable on r, then Theorem 1.14 holds.

Proof. Fix $\varepsilon > 0$. Note that $u_N(0,t)$ has the same distribution as $E_0(e^{\overline{\mathcal{I}}_N^\kappa(0,t,0)})$, so that we can replace the latter by $u_N(0,t)$ in (4.3) without violating the summability condition. Thus, by the Borel-Cantelli Lemma, we have

$$\limsup_{r\to\infty} \frac{1}{t_r} \log E_0\left(e^{\mathcal{I}_N^\kappa(0,t_r,t_r)}\right) \le \overline{\lambda}_0^{1\!\!1}(\kappa) + \varepsilon \qquad \xi\text{-a.s.} \tag{4.4}$$

The extension to sequences along $\mathbb{R}$ may be done as in the proof of Theorem 1.13 (recall (3.56)). Standard arguments yield

$$\limsup_{t\to\infty} \frac{1}{t} \log u_N(0,t) \le \overline{\lambda}_0^{1\!\!1}(\kappa) \qquad \xi\text{-a.s.} \tag{4.5}$$

To extend this to the solution of (1.1) with initial condition $u_0 \equiv 1$, we estimate

$$\begin{aligned} &\int_0^t \xi(X^\kappa(s), t-s)ds \\ &\le \int_0^t \xi(X^\kappa(s),t-s) 1\!\!1\{\xi(X^\kappa(s),t-s) \ge N\}\, ds + \int_0^t \xi_N(X^\kappa(s),t-s)\, ds. \end{aligned} \tag{4.6}$$

Note that, by (3.53) and the arguments given in Lemma 3.4, we have, for $t \in \mathbb{N}$ sufficiently large,

$$\sup_{\Phi\in\Pi(j,t)} \int_0^t \xi(\Phi(s),t-s) 1\!\!1\{\xi(\Phi(s),t-s) \ge N\}\, ds \le (t+j)A^d C' \sum_{R=R_0}^\infty A^{-Rd}. \tag{4.7}$$

Next, choose $M > 1$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log E_0\left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{N(X^\kappa,t) > Mt\}\right) < \overline{\lambda}_0^{1\!\!1}(\kappa). \tag{4.8}$$

Then, by (4.6–4.7), for $t \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} &E_0\left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{N(X^\kappa,t) \le Mt\}\right) \\ &\le \exp\left\{(M+1)tA^dC' \sum_{R=R_0}^\infty A^{-Rd}\right\} E_0\left(e^{\mathcal{I}_N^\kappa(0,t,t)} 1\!\!1\{N(X^\kappa,t) \le Mt\}\right). \end{aligned} \tag{4.9}$$

We infer from (4.5) and (4.8–4.9) that

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{N}}} \frac{1}{t} u(0,t) \le (M+1)A^dC' \sum_{R=R_0}^\infty A^{-Rd} + \overline{\lambda}_0^{1\!\!1}(\kappa). \tag{4.10}$$

Taking the limit $R_0 \to \infty$, $R_0 \in \mathbb{N}$, we obtain

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{N}}} \frac{1}{t} \log u(0,t) \le \overline{\lambda}_0^{1\!\!1}(\kappa). \tag{4.11}$$

The extension to sequences along $\mathbb{R}$ may again be done as in the proof of Theorem 1.13 (recall (3.56)). Furthermore, Proposition 1.3 gives $\lambda_0^{\delta_0}(\kappa) = \overline{\lambda}_0^{\delta_0}(\kappa) = \overline{\lambda}_0^{1\!\!1}(\kappa)$, so that

$$\limsup_{t\to\infty} \frac{1}{t} \log u(0,t) \le \lambda_0^{\delta_0}(\kappa). \tag{4.12}$$

By monotonicity, the reverse inequality holds with the limsup replaced by the liminf. It follows that $\lambda_0^{\mathbb{1}}(\kappa)$ exists and equals $\lambda_0^{\delta_0}(\kappa)$. A further monotonicity argument shows that the same is true for $\lambda_0^{u_0}(\kappa)$ for any initial condition u_0 subject to (1.4). ∎

In view of Lemma 4.1, our target is to prove (4.3). We fix M subject to (4.8), N of the form $N = CA^{R_0 d}$, $\varepsilon > 0$ small, and write t as $t = rA^R$, $r, R \in \mathbb{N}$, $A > 2$. Note that the choice of M implies that it is enough to concentrate on path with at most Mt jumps.

We proceed by introducing space-time blocks and dividing them into good blocks and bad blocks, respectively, into N-sufficient blocks and N-insufficient blocks (compare with (1.18 and Fig. 2).

Definition 4.2. *For $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$ and $b, c \in \mathbb{N}_0$, define (see Fig. 4)*

$$\hat{B}_R^A(x, k; b, c) = \left(\prod_{j=1}^{d} [(x(j) - 1 - b)4MA^R, (x(j) + 1 + b)4MA^R) \cap \mathbb{Z}^d \right) \times [(k - c)A^R, (k + 1)A^R) \tag{4.13}$$

and abbreviate $\hat{B}_R^A(x, k) = \hat{B}_R^A(x, k; 0, 0)$.

For $S \subset \mathbb{Z}^d$, let ∂S denote the inner boundary of S. For $S \times S' \subset \mathbb{Z}^d \times \mathbb{R}$, let $\Pi_1(S \times S')$ denote the projection of $S \times S'$ onto the first d coordinates (the spatial coordinates).

Definition 4.3. *The subpedestal of $B_R^A(x, k)$ is defined as*

$$\hat{B}_R^{A,\mathrm{sub}}(x, k) = \Big\{ y \in \Pi_1(\hat{B}_R^A(x, k)) \colon |y(j) - z(j)| \geq 2MA^R,\ j \in \{1, 2, \ldots, d\}\ \forall\, z \in \partial \Pi_1(\hat{B}_R^A(x, k)) \Big\} \times \{kA^R\}. \tag{4.14}$$

Definition 4.4. *A block $\hat{B}_R^A(x, k)$ is called N-sufficient when, for every $y \in \Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x, k))$ (see Fig. 4),*

$$E_y \left(e^{\overline{\mathcal{I}}_N^{\kappa}(0, A^R, kA^R)} \mathbb{1}\{ N(X^\kappa, A^R) \leq MA^R \} \right) \leq e^{(\overline{\lambda}_0^{\mathbb{1}}(\kappa) + \varepsilon) A^R}. \tag{4.15}$$

Otherwise $\hat{B}_R^A(x, k)$ is called N-insufficient. A subpedestal is called N-sufficient/N-insufficient when its corresponding block is N-sufficient/N-insufficient.

The notion of good/bad is similar as in Definition 1.4 with the only difference that $B_R^A(x, k)$ is replaced by $\hat{B}_R^A(x, k)$ and $B_R^A(x, k; b, c)$ by $\hat{B}_R^A(x, k; b, c)$. Similarly as in (3.5), define $\hat{\Xi}_R^{A,j}$ to be the maximal number of bad R-blocks a path with j jumps can cross.

4.2 Three lemmas

For the proof of Theorem 1.14 we need Lemmas 4.5–4.7 below. The first says that each block is N-sufficient with a large probability (and is comparable with [4], Lemma 4.3), the second controls the number of bad blocks (and is comparable with Lemma 3.5), the third estimates the number of N-insufficient blocks that are good and are visited by a typical random walk path (see [4], Lemma 4.4 and [10], Lemma 11).

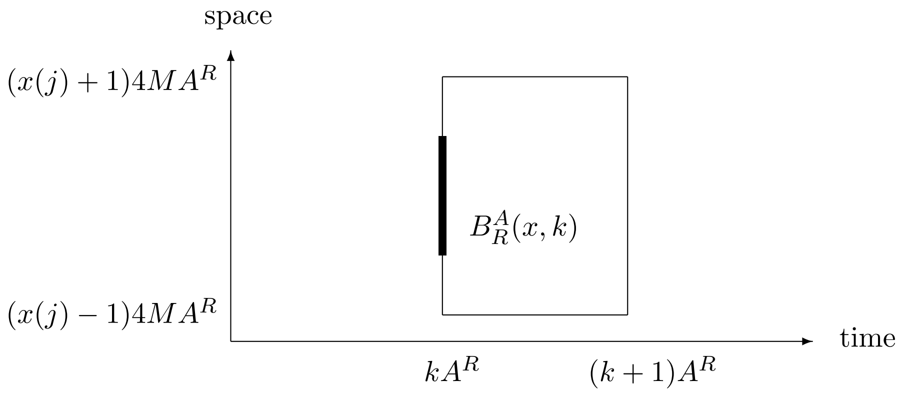

Figure 4: The thick line is the subpedestal.

Lemma 4.5. *Fix $A \in \mathbb{N}$. For every $\tilde\delta > 0$ there is an $R_0 = R_0(A, \tilde\delta) \in \mathbb{N}$ such that*

$$\mathbb{P}(\hat B_R^A(x,k) \text{ is } N\text{-sufficient }) \geq 1 - \tilde\delta \qquad \forall\, R \geq R_0,\ x \in \mathbb{Z}^d,\ k \in \mathbb{N}. \tag{4.16}$$

Lemma 4.6. *For every $A_0 > 2$ there is an $A \in \mathbb{N}$ with $A \geq A_0$ such that, for some $C > 0$ independent of A,*

$$\mathbb{P}\biggl(\hat\Xi_R^{A,j} \geq C(t+j)(A^{(1+2d)})^{-R} \text{ for some } j \in \mathbb{N}_0 \text{ and } R \in \mathbb{N}\biggr) \tag{4.17}$$

is summable on $t \in \mathbb{N}$.

Lemma 4.7. *Let $C(Mt, \eta t/A^R)$ be the event that there is a path Φ with $\Phi(0) = 0$ and $N(\Phi,t) \leq Mt$ that up to time t crosses more than $\eta t/A^R$ N-insufficient subpedestals of a good R-block. Then, under the Gärtner-mixing type-II condition, for every $\eta > 0$ there is an A (which can be chosen as in Lemma 4.6) and $R_0 \in \mathbb{N}$ such that, for some $c_1 > 0$,*

$$\mathbb{P}\left(C(Mt, \eta t/A^R)\right) \leq e^{-c_1 \eta t/A^R}, \qquad \forall R \geq R_0. \tag{4.18}$$

4.3 Proof of Theorem 1.14

Proof. The proof comes in three steps. Fix $0 < \eta < \varepsilon$, and choose $A, R \geq R_0$ according to Lemmas 4.5–4.7.

1. Consider all random walk path that start in zero, make $0 \leq j \leq Mt$ jumps, and attain values $\{x_1, x_2, \dots, x_{t/N_0-1}\}$ at times kA^R, $k \in \{1, 2, \dots, t/A^R - 1\}$. By the Markov property,

$$E_0\left(e^{\overline{\mathcal{I}}_N^\kappa(0,t,0)} \prod_{k=1}^{t/A^R-1} \mathbb{1}\{X^\kappa(kA^R) = x_k\}\right) \leq \prod_{k=0}^{t/A^R-1} E_{x_k}\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\right), \tag{4.19}$$

where $x_0 = 0$. Let I and S be the sets of indices k such that (x_k, kA^R) is in an N-insufficient, respectively, N-sufficient subpedestal. Then the right-hand side of (4.19) can be rewritten as

$$\prod_{k \in I} E_{x_k}\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\right) \prod_{k \in S} E_{x_k}\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\right). \tag{4.20}$$

Because of Lemmas 4.6 and 4.7, there is a measurable set, independent of j and of ξ-probability at least $1 - e^{-c_1\eta t/A^R}$, such that

$$|I| \leq \eta t/A^R + C(t+j)(A^{(1+2d)})^{-R}. \tag{4.21}$$

Since $\xi_N \leq N$, on a set of that probability the first term in (4.20) can be estimated from above by $e^{N\eta t}\exp\{NC(t+j)/A^{2Rd}\}$.

2. Pick a realization of ξ which satisfies (4.21). To bound the second term in (4.20), we split this term up as

$$\begin{aligned}\prod_{k\in S}\Big[&E_{x_k}\left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,kA^R)}1\!\!1\{N(X^\kappa,A^R)\leq MA^R\}\right)\\ &+E_{x_k}\left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,kA^R)}1\!\!1\{N(X^\kappa,A^R)> MA^R\}\right)\Big],\end{aligned} \tag{4.22}$$

which can be written as

$$\begin{aligned}\sum_{J\subset S}\Big[&\prod_{k\in J}E_{x_k}\left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,kA^R)}1\!\!1\{N(X^\kappa,A^R)\leq MA^R\}\right)\\ &\times\prod_{k\notin J}E_{x_k}\left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,kA^R)}1\!\!1\{N(X^\kappa,A^R)> MA^R\}\right)\Big].\end{aligned} \tag{4.23}$$

Take $c \gg 1$. Then, for M large enough, $P_{x_k}\left(N(X,A^R) > MA^R\right) \leq e^{-cA^R}$. Hence the second term in (4.23) can be bounded from above by $e^{A^R(-c+N)(t/A^R-|J|)}$. Recall the definition of a N-sufficient block, to bound the sum in (4.23) by

$$e^{t(-c+N)}\left(1+e^{A^R(\overline{\lambda}^{\mathbf{1}}_0(\kappa)+\varepsilon+c-N)}\right)^{t/A^R}. \tag{4.24}$$

Summing over all possible values $(x_1, x_2, \ldots, x_{t/A^R-1})$ compatible with a path Φ such that $\Phi(0) = 0$ and $N(\Phi, t) = j$, and fixing $\eta \leq \varepsilon$, we obtain

$$\begin{aligned}&E_0\left(e^{\overline{\mathcal{I}}^\kappa_N(0,t,0)}1\!\!1\{N(X^\kappa,t)=j\}\right)\\ &=\sum_{x_1,x_2,\ldots,x_{t/A^R-1}}E_0\left(e^{\overline{\mathcal{I}}^\kappa_N(0,t,0)}\prod_{k=1}^{t/A^R-1}1\!\!1\{X^\kappa(kA^R)=x_k\}\right)\\ &\qquad\times p_{A^R}(0,x_1)\times\cdots\times p_{A^R}(x_{t/A^R-2},x_{t/A^R-1})\\ &\leq\exp\left\{\frac{jNC}{A^{2Rd}}\right\}\exp\left\{t\left(N\eta+\frac{NC}{A^{2Rd}}-c+N\right)\right\}\left(1+e^{A^R(\overline{\lambda}^{\mathbf{1}}_0(\kappa)+\varepsilon+c-N)}\right)^{t/A^R}\\ &\stackrel{\text{def}}{=}\exp\left\{\frac{jNC}{A^{2Rd}}\right\}C^N_1(t,A^R,\varepsilon),\end{aligned} \tag{4.25}$$

where $(p_s(x,y))_{s\geq 0, x,y\in\mathbb{Z}^d}$ denote the transition probabilities of a continuous-time simple random walk jumping at rate κ.

3. We proceed by summing over the number of jumps, to obtain

$$\begin{aligned}
&E_0\left(e^{\overline{\mathcal{I}}_N^\kappa(0,t,0)}\mathbb{1}\{N(X^\kappa,t)\le Mt\}\right)\\
&\le \sum_{j=0}^{Mt} e^{-2d\kappa t}\frac{(2d\kappa t)^j}{j!}\exp\left\{\frac{jNC}{A^{2Rd}}\right\}C_1^N(t,A^R,\varepsilon)\\
&\le e^{-2d\kappa t}\exp\left\{2d\kappa t e^{NC/A^{2Rd}}\right\}C_1^N(t,A^R,\varepsilon)\overset{\text{def}}{=}C_2^N(t,A^R,\varepsilon).
\end{aligned}\tag{4.26}$$

Thus, we have shown that there is an $A>2$ such that for each $R\ge R_0$

$$\mathbb{P}\left[E_0\left(e^{\overline{\mathcal{I}}_N^\kappa(0,rA^R,0)}\mathbb{1}\big\{N(X^\kappa,rA^R)\le MrA^R\big\}\right)>C_2^N(rA^R,A^R,\varepsilon)\right]\le e^{-r\eta c_1},\tag{4.27}$$

which is summable on $r\in\mathbb{N}$. By the boundedness of ξ_N, the same is true without the indicator in the expectation (after a possible enlargement of C_2^N by ε). Further note that

$$\begin{aligned}
&\frac{1}{rA^R}\log C_2^N(rA^R,A^R,\varepsilon)\\
&=-2d\kappa+2d\kappa e^{NC/A^{2Rd}}+N\eta+\frac{NC}{A^{2Rd}}+\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon+\frac{1}{A^R}\log\left(e^{-A^R(\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon+c-N)}+1\right),
\end{aligned}\tag{4.28}$$

so that $C_2^N(rA^R,A^R,\varepsilon)$ is indeed of the form $\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon$. Thus, we have proved Lemma 4.1 and hence Theorem 1.14. ∎

4.4 Proof of Lemmas 4.5–4.7

4.4.1 Proof of Lemma 4.5

Proof. The proof comes in three steps and uses ideas from Cranston, Mountford and Shiga [4], Lemma 4.3, and Drewitz, Gärtner, Ramirez and Sun [5], Lemma 4.3.

1. Suppose that we already showed that, ξ-a.s. and independently of the realization of ξ, for all $\varepsilon>0$ there is an $\eta'>0$ such that, for all $0<\eta<\eta'$,

$$\begin{aligned}
\limsup_{R\to\infty}\sup_{\substack{x,y\in[-2MA^R,2MA^R]^d\cap\mathbb{Z}^d\\ \|x-y\|\le\eta A^R}}\frac{1}{A^R}\bigg|&\log E_x\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)}\mathbb{1}\Big\{N(X^\kappa,A^R)\le MA^R\Big\}\right)\\
&-\log E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)}\mathbb{1}\Big\{N(X^\kappa,A^R)\le MA^R\Big\}\right)\bigg|\le\varepsilon.
\end{aligned}\tag{4.29}$$

We show how (4.29) can be used to obtain the claim.

2. By Proposition 1.3 for fixed $\tilde\delta>0$ there is an $R_0=R_0(A,\tilde\delta)\in\mathbb{N}$ such that, for all $R\ge R_0$,

$$\mathbb{P}\left(E_0\left(e^{\mathcal{I}^\kappa(0,A^R,0)}\mathbb{1}\Big\{N(X^\kappa,A^R)\le MA^R\Big\}\right)\le e^{(\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon)A^R}\right)\ge 1-\tilde\delta.\tag{4.30}$$

To extend this to $\overline{\mathcal{I}}_N$, note that for each $t\in\mathbb{N}$

$$\int_0^t \xi_N(X^\kappa(s),s)\,ds\le\int_0^t\xi(X^\kappa(s),s)\,ds+\int_0^t-\xi(X^\kappa(s),s)\mathbb{1}\{\xi(X^\kappa(s),s)<-N\}\,ds.\tag{4.31}$$

Thus, given a realization of X^κ with no more than Mt jumps, by the fact that ξ is Gärtner-negative-hyper-mixing, (3.53) and the arguments given in the proof of Lemma 3.4, the second term in the right-hand side is at most

$$(M+1)tA^dC' \sum_{R=R_0}^{\infty} A^{-Rd}. \tag{4.32}$$

Hence (4.30) remains true when we replace $\overline{\mathcal{I}}$ by $\overline{\mathcal{I}}_N$. According to (4.29), this estimate also holds when we replace 0 by any x with $\|x\| \leq \eta A^R$ for η small enough, independently of the realization of ξ. Consequently, for any $\tilde{\delta} > 0$ there is an $R_0 \in \mathbb{N}$ such that

$$\mathbb{P}\left(\sup_{x\colon \|x\| \leq \eta A^R} E_x \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\Big\{ N(X^\kappa, A^R) \leq MA^R \Big\} \right) \leq e^{(\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon)A^R} \right) \geq 1-\tilde{\delta} \qquad R \geq R_0. \tag{4.33}$$

Next, note that $[-2MA^R, 2MA^R]^d$ can be divided into K boxes, with $K \sim 4^d M^d/\eta^d$, of the form $x_i(A^R) + B_{\eta A^R}$, where the $x_i(A^R)$'s are separated by ηA^R. By the stationarity of ξ in space, we have

$$\begin{aligned} &\mathbb{P}\left(E_0 \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\Big\{ N(X^\kappa, A^R) \leq MA^R \Big\} \right) \leq e^{(\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon)A^R} \right) \\ &= \mathbb{P}\left(E_{x_i(A^R)} \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\Big\{ N(X^\kappa, A^R) \leq MA^R \Big\} \right) \leq e^{(\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon)A^R} \right). \end{aligned} \tag{4.34}$$

Thus, for the same choice of R_0 as in (4.33),

$$\begin{aligned} &\mathbb{P}\left(\sup_{y\colon y \in x_i(A^R)+B_{\eta A^R}} E_y \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\Big\{ N(X^\kappa, A^R) \leq MA^R \Big\} \right) \leq e^{(\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon)A^R} \right) \\ &\qquad \geq 1-\tilde{\delta} \qquad R \geq R_0. \end{aligned} \tag{4.35}$$

Since K is independent of R_0, we may conclude that

$$\mathbb{P}\left(B_R^A(0,0) \text{ is } N\text{-sufficient} \right) \geq 1-\tilde{\delta}. \tag{4.36}$$

By the stationarity of ξ in space and time, the same statement holds for any block $B_R^A(x,k)$, which proves the claim. It therefore remains to prove (4.29).

3. Since for M large the event $\Big\{ N(X^\kappa, A^R) > MA^R \Big\}$ does not contribute on an exponential scale, in order to prove (4.29) it suffices to show that

$$\limsup_{R\to\infty} \sup_{\substack{x,y \in [-2MA^R, 2MA^R]^d \cap \mathbb{Z}^d \\ \|x-y\| \leq \eta A^R}} \frac{1}{A^R} \left| \log \frac{E_x \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} \right)}{E_y \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} \right)} \right| \leq \varepsilon. \tag{4.37}$$

To that end, we show that we can restrict ourself to contributions coming from random walk paths that stay within a certain distance of $[-2M, 2M]^d \cap \mathbb{Z}^d$. More precisely, ξ-a.s. there is a box $B_L = [-L, L]^d \cap \mathbb{Z}^d$, independent of $0 < \eta < 1$ and containing $[-2M, 2M]^d \cap \mathbb{Z}^d$, such that

$$\begin{aligned}
&\sup_{x \in [-2MA^R, 2MA^R]^d \cap \mathbb{Z}^d} \sum_{\substack{w \in \mathbb{Z}^d \\ w \notin A^R B_L}} \left[E_x \left(e^{\overline{\mathcal{I}}^\kappa_N(0,\eta A^R,0)} \delta_w(X^\kappa(\eta A^R)) \right) E_w \left(e^{\overline{\mathcal{I}}^\kappa_N(0,(1-\eta)A^R,\eta A^R)} \right) \right] \\
&\qquad \le e^{A^R} \sup_{x \in [-2MA^R, 2MA^R]^d \cap \mathbb{Z}^d} P_x \left(X^\kappa(\eta A^R) \notin A^R B_L \right) \\
&\qquad = e^{A^R} P_0 \Big(X^\kappa(\eta A^R) \notin A^R B_{L-2M} \Big) \\
&\qquad \le e^{A^R} \exp \left\{ -A^R (L-2M) \left(\log \left(\frac{L-2M}{\kappa d} \right) - 1 \right) \right\},
\end{aligned} \tag{4.38}$$

where the last inequality follows from Gärtner and Molchanov [8], Lemma 4.3. Consequently, we may concentrate in (4.37) on the contribution coming from paths that stay inside $A^R B_L \cap \mathbb{Z}^d$. Next, note that, ξ-a.s. and uniformly in $x, y \in [-2MA^R, 2MA^R]^d \cap \mathbb{Z}^d$,

$$\begin{aligned}
&\frac{\sum_{w \in A^R B_L} E_x \left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,0)} \delta_w(X^\kappa(\eta A^R)) \right)}{\sum_{w \in A^R B_L} E_y \left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,0)} \delta_w(X^\kappa(\eta A^R)) \right)} \\
&\le \sup_{w \in A^R B_L} \frac{E_x \left(e^{\overline{\mathcal{I}}^\kappa_N(0,\eta A^R,0)} \delta_w(X^\kappa(\eta A^R)) \right)}{E_y \left(e^{\overline{\mathcal{I}}^\kappa_N(0,\eta A^R,0)} \delta_w(X^\kappa(\eta A^R)) \right)} \\
&\le e^{2N\eta A^R} \sup_{w \in A^R B_L} \frac{P_0 \left(X^\kappa(\eta A^R) = w - x \right)}{P_0 \left(X^\kappa(\eta A^R) = w - y \right)}, \qquad 0 < \eta < 1.
\end{aligned} \tag{4.39}$$

To obtain (4.29), it remains to estimate the probabilities in the last line. This can be done by applying bounds on probabilities for simple random walks (see [5], Lemma 4.3 for details). ∎

4.4.2 Proof of Lemma 4.6

The only difference with the situation in the proof of Lemma 3.5 is that we replaced the R-blocks $B^A_R(x, k)$ by the R-blocks $\hat{B}^A_R(x, k)$. However, this does not affect the proof. Thus, the proof of Lemma 3.5 yields the claim.

4.4.3 Proof of Lemma 4.7

Proof. The proof comes in two steps and is essentially a copy of the proof of Lemma 3.6. Throughout the proof, $x, x' \in \mathbb{Z}^d$ and $k, k' \in \mathbb{N}$.

1. Pick $a_1, a_2 \in \mathbb{N}$ according to our main assumption. We say that (x, k) and (x', k') are equivalent if and only if

$$x \equiv x' \mod a_1 \quad \text{and} \quad k \equiv k' \mod a_2. \tag{4.40}$$

This equivalence relation divides $\mathbb{Z}^d \times \mathbb{N}$ into $a_1^d a_2$ equivalence classes. We write $\sum_{(x^*,k^*)}$ to denote the sum over all equivalence classes. Furthermore, we define

$$\hat{\chi}^A(x, k) = \mathbb{1}\Big\{ \hat{B}^A_R(x, k) \text{ is good, but has an } N\text{-insufficient subpedestal} \Big\}. \tag{4.41}$$

Henceforth we assume that all blocks under consideration intersect $[-Mt, Mt] \times [0, t]$. Then we have

$$\begin{aligned} &\mathbb{P}(C(Mt, \eta t/A^R)) \\ &\leq \sum_{(x^*,k^*)} \mathbb{P}\Big(\exists \text{ a path with no more than } Mt \text{ jumps that intersects at least} \\ &\qquad \eta t/A^R a_1^d a_2 \text{ blocks } \hat{B}_R^A(x,k) \text{ with } \hat{\chi}^A(x,k) = 1,\ (x,k) \equiv (x^*, k^*)\Big). \end{aligned} \tag{4.42}$$

We introduce the following abbreviation

$$\begin{aligned} \hat{\mathcal{A}}_1^n = \Big\{&\exists \text{ a path with no more than } Mt \text{ jumps that intersects } n \text{ blocks } \hat{B}_R^A(x,k) \\ &\text{with } \hat{\chi}^A(x,k) = 1,\ (x,k) \equiv (x^*, k^*)\Big\}. \end{aligned} \tag{4.43}$$

Consequently, we may rewrite the right-hand side of (4.42) as

$$\sum_{(x^*,k^*)} \sum_{n=\frac{\eta t}{A^R a_1^d a_2}}^{L} \mathbb{P}(\hat{\mathcal{A}}_1^n), \tag{4.44}$$

where L is the number of R-blocks that can be crossed by a path with no more than Mt jumps.

2. To estimate the probability in (4.44), note that we can write $\hat{\mathcal{A}}_1^n$ as

$$\hat{\mathcal{A}}_1^n = \bigcup_{(x_m^{a_1}, k_m^{a_2})} \hat{\mathcal{A}}_1^{n,(x_m^{a_1}, k_m^{a_2})}, \tag{4.45}$$

where the union is taken over all possible choices of good R-blocks $\hat{B}_R^A(x_m^{a_1}, k_m^{a_2})$ whose sub-pedestal is N-insufficient, can be reached by a path with at most Mt jumps and such that all points belonging to the sequence $(x_m^{a_1}, k_m^{a_2})$ are equivalent. Hence, (4.44) becomes

$$\sum_{(x^*,k^*)} \sum_{n=\frac{\eta t}{A^R a_1^d a_2}}^{L} \sum_{(x_m^{a_1}, k_m^{a_2})} \mathbb{P}\Big(\hat{\mathcal{A}}_1^{n,(x_m^{a_1}, k_m^{a_2})}\Big). \tag{4.46}$$

By our assumption, there is $R_0 \in \mathbb{N}$ such that the probability in (4.46) may be bounded from above by $K\delta^n$ for all $R \geq R_0$. Note that there are at most $\binom{L}{n}$ elements in the union (4.45), and so the two inner sums in (4.46) are bounded from above by

$$(1-\delta)^{-L} K \mathbb{P}\Big(T_L \geq \frac{\eta t}{A^R a_1^d a_2}\Big), \tag{4.47}$$

where $T_L = \mathrm{BIN}(L, \delta)$. As in Lemma 3.6, the probability in (4.47) is bounded from above by

$$\mathbb{P}\Big(T \geq \frac{\eta t}{A^R a_1^d a_2}\Big), \tag{4.48}$$

where $T = \mathrm{BIN}(\nu^{(1+d)}\mu(j), \delta)$, with $\nu = \lceil \delta^{-1/(1+d)} \rceil$ and $\mu(j) = 3^d \frac{t+j}{\nu A^R} + 3^d 2$. The same arguments as in Lemma 3.6 yield that this binomially distributed random variable may be bounded from above by

$$\exp\Big\{-C' \eta t / A^R a_1^d a_2\Big\} \tag{4.49}$$

for some $C' > 0$. Similarly as in Lemma 3.6, if $\delta^{-1/1+d} \in \mathbb{N}$, the first term in (4.47) may be bounded from above by

$$\frac{3^d(M+1)t\delta^{1/1+d}}{(1-\delta)A^R} + \frac{3^d 2}{1-\delta}. \tag{4.50}$$

Since δ tends to zero, if R tends to infinity, for t large enough the first term in (4.47) does not contribute. Finally, to estimate (4.46), insert (4.49) into (4.46), to obtain

$$\mathbb{P}\left(C(Mt, \eta t/A^R)\right) \leq K a_1^d a_2 M t \exp\left\{-C'\eta t/A^R\right\}, \tag{4.51}$$

which yields the claim. ∎

5 Continuity at $\kappa = 0$

The proof of Theorem 1.15 is given in Section 5.3. It is based on Lemmas 5.1–5.3 below, which are stated in Section 5.1 and are proved in Section 5.2.

5.1 Three lemmas

Fix $b \in (0, 1)$, and define the set of paths

$$\begin{aligned} A_{nt}^{\kappa} = \Big\{ &\Phi\colon [0, nt] \to \mathbb{Z}^d\colon N(\Phi, nt) \leq \frac{1}{\log(1/\kappa)^b}\, nt, \\ &\forall\, 1 \leq j \leq n\ \exists\, x_j \in \mathbb{Z}^d\colon \|x_j\| \leq \frac{1}{\log(1/\kappa)^b}\, nt,\ \Phi(s) = x_j\ \forall\, s \in [(j-1)t+1, jt) \Big\}, \end{aligned} \tag{5.1}$$

i.e., paths of length nt that do not jump in time intervals of length $t-1$ and whose number of jumps is bounded by $\frac{1}{\log(1/\kappa)^b}\, nt$. Note that $\kappa \mapsto A_{nt}^{\kappa}$ is non-decreasing.

Lemma 5.1. *Suppose that ξ satisfies condition* (b) *in Definition* 1.8*. Then, ξ-a.s., for any sequence of positive numbers $(a_m)_{m\in\mathbb{N}}$ tending to zero there exists a strictly positive and non-increasing sequence $(\kappa_m)_{m\in\mathbb{N}}$ such that, for all $m \in \mathbb{N}$ and $0 < \kappa \leq \kappa_m$, there exists a $t_m = t_m(\kappa_m)$ such that, for all $t \in \mathbb{Q} \cap [t_m, \infty)$, there exists an $n_m = n_m(\kappa_m, t)$ such that*

$$\sup_{\Phi \in A_{nt}^{\kappa}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi(j-1)t+1), s)\, ds \leq a_m nt \qquad \forall\, n \geq n_m. \tag{5.2}$$

We say that two paths Φ_1 and Φ_2 on $[0, nt]$ are equivalent, written $\Phi_1 \sim \Phi_2$, if and only if

$$\Phi_1|_{[(j-1)t+1, jt)} = \Phi_2|_{[(j-1)t+1, jt)} \qquad \forall\, 1 \leq j \leq n. \tag{5.3}$$

This defines an equivalence relation $\sim$, and we denote by $A_{nt}^{\kappa,\sim}$ the set of corresponding equivalent classes. The following lemma provides an estimation of the cardinality of $A_{nt}^{\kappa,\sim}$.

Lemma 5.2. $|A_{nt}^{\kappa,\sim}| \leq (nt/\log(1/\kappa)^b) 2^n (2d)^{nt/\log(1/\kappa)^b} + 1$.

Lemma 5.3. *Suppose that ξ is Gärtner-positive-hyper-mixing. Then there are $A, C > 0$ such that ξ-a.s. for $t \in \mathbb{Q}$ large enough and any choice of disjoint subintervals $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_k$, $k \in \mathbb{N}$, of $[0,t]$ such that $|\mathcal{I}_i| = |\mathcal{I}_l|$, $i, l \in \{1, 2, \cdots, k\}$, and each $R_0 \in \mathbb{N}$ and each path $\Phi \in B_t(0)$*

$$\sum_{i=1}^{k} \int_{\mathcal{I}_i} \xi(\Phi(s), s)\, ds \le k|\mathcal{I}_1| C A^{R_0 d} + (t + N(\Phi, t)) C' A^d \sum_{R=R_0}^{\infty} A^{-Rd}, \tag{5.4}$$

for some constant $C' > 0$.

5.2 Proof of Lemmas 5.1–5.3

5.2.1 Proof of Lemma 5.2

Proof. Fix an integer $k \le nt$. We start by estimating the number of possible arrangements of the jumps in paths with k jumps. Since we do not distinguish between two paths that coincide on the intervals $[(j-1)t+1, jt)$, $1 \le j \le n$, only the last jumps before the times $(j-1)t+1$, $1 \le j \le n$, need to be considered.

First, the number of arrangements with jumps in $0 \le l \le k$ different intervals is $\binom{n}{l}$. Since the number of different intervals cannot exceed n, the number of different arrangements is bounded from above by $\sum_{l=1}^{n} \binom{n}{l} \le 2^n$. Next, there are $(2d)^k$ different points in $\mathbb{Z}^d$ that can be visited by a path with k jumps. Therefore

$$|A_{nt}^{\kappa,\sim}| \le \sum_{k=1}^{nt/\log(1/\kappa)^b} 2^n (2d)^k + 1 \le (nt/\log(1/\kappa)^b) 2^n (2d)^{nt/\log(1/\kappa)^b} + 1, \tag{5.5}$$

which proves the claim. ∎

5.2.2 Proof of Lemma 5.1

Proof. Choose κ_1 such that

$$\frac{\log(2d)}{\log(1/\kappa_1)^b} < \delta_2(a_1), \tag{5.6}$$

and $t_1 = t_1(\kappa_1)$ such that

$$t_1 \left(-\delta_2 + \frac{\log(2d)}{\log(1/\kappa_1)^b} \right) < -\log 2. \tag{5.7}$$

Then, by condition (b) in Definition 1.8 and Lemma 5.2, for all $t \ge t_0 \vee t_1$ we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{\Phi \in A_{nt}^{\kappa_1}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \ge a_1 nt \right) \\ &\le \sum_{\Phi \in A_{nt}^{\kappa_1,\sim}} \mathbb{P}\left(\sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \ge a_1 nt \right) \\ &\le \left[(nt/\log(1/\kappa)^b) 2^n (2d)^{nt/\log(1/\kappa)^b} + 1 \right] e^{-\delta_2 nt}, \end{aligned} \tag{5.8}$$

which is summable on n. Hence, by the Borel-Cantelli lemma, there exists a set $B_{\kappa_1,t}$ with $\mathbb{P}(B_{\kappa_1,t}) = 1$ for which there exists an $n_0 = n_0(\xi, \kappa_1, t)$ such that

$$\sup_{\Phi \in A_{nt}^{\kappa_1}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \le a_1 nt \qquad \forall\, n \ge n_0. \tag{5.9}$$

Since $\kappa \mapsto A_{nt}^{\kappa}$ is non-decreasing, (5.9) is true for all $0 < \kappa \le \kappa_1$. Define

$$B_1 = \bigcap_{t \ge t_1, t \in \mathbb{Q}} B_{\kappa_1,t}, \tag{5.10}$$

for which still $\mathbb{P}(B_1) = 1$. Similarly, we can construct sets B_m, $m \in \mathbb{N} \setminus \{1\}$, with $\mathbb{P}(B_m) = 1$ such that on B_m there exist κ_m and $t_m = t_m(\kappa_m)$ such that for all $0 < \kappa \le \kappa_m$ and $t \ge t_m$ with $t \in \mathbb{Q}$ there exists an $n_0 = n_0(\xi, \kappa_m, t)$ such that

$$\sup_{\Phi \in A_{nt}^{\kappa}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \le a_m nt \qquad \forall\, n \ge n_0. \tag{5.11}$$

Hence $B = \cap_{m \in \mathbb{N}} B_m$ is the desired set. Note that we can control the value of t_m by choosing κ_m small enough. Indeed, with the right choice of κ_m, it follows that $t_{m-1}(\kappa_{m-1}) = t_m(\kappa_m)$ for all $m \in \mathbb{N}$. ∎

5.2.3 Proof of Lemma 5.3

Proof. Fix A, C as in Section 3, $R_0 \in \mathbb{N}$, a path $\Phi \in B_t(0)$ and disjoint subintervals $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_k$, $k \in \mathbb{N}$, of $[0, t]$ with equal length. Note that

$$\begin{aligned}\sum_{j=1}^{k} \int_{\mathcal{I}_j} \xi(\Phi(s), s)\, ds \le \sum_{j=1}^{k} \int_{\mathcal{I}_j} & \xi(\Phi(s), s) 1\!\!1\{\xi(\Phi(s), s) \le CA^{R_0 d}\}\, ds \\ &+ \int_0^t \xi(\Phi(s), s) 1\!\!1\{\xi(\Phi(s), s) > CA^{R_0 d}\}\, ds.\end{aligned} \tag{5.12}$$

By (3.53) and Lemma 3.4, for $t \in \mathbb{Q}$ sufficiently large, the second term on the right hand side in (5.12) may be bounded from above by

$$(t + N(\Phi, t)) C' A^d \sum_{R=R_0}^{\infty} A^{-Rd}. \tag{5.13}$$

Inserting (5.13) into (5.12) yields the claim. ∎

5.3 Proof of Theorem 1.15

In this section we prove Theorem 1.15 with the help of Lemmas 5.1–5.3. The proof comes in three steps, organized as Sections 5.3.1–5.3.3.

5.3.1 Estimation of the Feynman-Kac representation on A^κ_{nT}

Consider the case $u_0(x) = \delta_0(x)$, $x \in \mathbb{Z}^d$. Recall (1.13) and (1.31), and estimate

$$\lambda_0^{\delta_0}(\kappa) \le \lim_{n\to\infty} \frac{1}{nT} \log E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}\right) < \infty, \qquad T > 0, \tag{5.14}$$

where we reverse time, use that X^κ is a reversible dynamics, and remove the constraint $X^\kappa(nT) = 0$. Recalling (5.1) and (5.3), we have

$$\begin{aligned} E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)} 1\!\!1_{A^\kappa_{nT}}\right) &= \sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)} 1\!\!1_{A^\kappa_{nT}} 1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right) \\ &= \sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(\exp\left\{\sum_{j=1}^{n} \overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0)\right\}\right. \\ &\qquad \left.\times \exp\left\{\sum_{j=0}^{n-1} \overline{\mathcal{I}}^\kappa(jT, jT+1, 0)\right\} 1\!\!1_{A^\kappa_{nT}} 1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right). \end{aligned} \tag{5.15}$$

By the Hölder inequality with $p, q > 1$ and $1/p + 1/q = 1$, we have

$$\begin{aligned} &E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)} 1\!\!1_{\{A^\kappa_{nT}\}}\right) \\ &\le \sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(\exp\left\{p\sum_{j=1}^{n} \overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0)\, ds\right\} 1\!\!1_{A^\kappa_{nT}} 1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right)^{1/p} \\ &\quad \times E_0\left(\exp\left\{q\sum_{j=0}^{n-1} \overline{\mathcal{I}}^\kappa(jT, jT+1, 0)\right\} 1\!\!1_{A^\kappa_{nT}}\right)^{1/q}. \end{aligned} \tag{5.16}$$

Next, fix $(a_m)_{m\in\mathbb{N}}$, $(\kappa_m)_{m\in\mathbb{N}}$, t_m as in Lemma 5.1, choose $T > 0$ such that

$$t_m \le T = T(\kappa_m) = K\lfloor \log(1/\kappa_m)\rfloor, \qquad m \gg 1, \tag{5.17}$$

where K is a constant to be chosen later. For all $0 < \kappa \le \kappa_m$ and $n \ge n_m(\kappa_m, T(\kappa))$, by Lemma 5.1 we have

$$\begin{aligned} &\sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(\exp\left\{p\sum_{j=1}^{n} \overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0)\right\} 1\!\!1_{A^\kappa_{nT}} 1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right)^{1/p} \\ &\qquad \le \sum_{\Phi\in A^{\kappa,\sim}_{nT}} e^{a_m nT} P_0\left(A^\kappa_{nT}, X|_{[0,nT]} \sim \Phi\right)^{1/p} \le e^{a_m nT} |A^{\kappa,\sim}_{nT}|, \end{aligned} \tag{5.18}$$

while by Lemma 5.3 we have

$$\begin{aligned} &E_0\left(\exp\left\{q\sum_{j=0}^{n-1} \overline{\mathcal{I}}^\kappa(jT, jT+1, 0)\right\} 1\!\!1_{A^\kappa_{nT}}\right)^{1/q} \\ &\qquad \le \exp\left\{nCA^{R_0 d} + nT\left(1 + \frac{1}{\log(1/\kappa)^b}\right) C' A^d \sum_{R=R_0}^{\infty} A^{-Rd}\right\}. \end{aligned} \tag{5.19}$$

From Lemma 5.2 we know that $\frac{1}{nT}\log|A^{\kappa,\sim}_{nT}|$ tends to zero if we let first $n\to\infty$ and then $\kappa\downarrow 0$. Therefore, combining (5.16–5.19) and using that $\lim_{\kappa\downarrow 0} T=\lim_{\kappa\downarrow 0} T(\kappa)=0$, we get

$$\limsup_{\kappa\downarrow 0}\limsup_{n\to\infty}\frac{1}{nT}\log E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}\mathbb{1}_{A^{\kappa}_{nT}}\right)\le \max\left\{a_m, C'A^d\sum_{R=R_0}^{\infty}A^{-Rd}\right\}. \tag{5.20}$$

5.3.2 Estimation of the Feynman-Kac representation on $[A^{\kappa}_{nT}]^{c}$

The proof comes in three steps.

1. We start by estimating the corresponding Feynman-Kac term on $[A^{\kappa}_{nT}]^{c}$. Split

$$[A^{\kappa}_{nT}]^{c}=B^{\kappa}_{nT}\cup C^{\kappa}_{nT} \tag{5.21}$$

with

$$B^{\kappa}_{nT}=\left\{N(X^{\kappa},nT)>\frac{1}{\log(1/\kappa)^b}\,nT\right\} \tag{5.22}$$

and

$$\begin{aligned}C^{\kappa}_{nT}=\Big\{\exists\, 1\le j\le n\colon\ &\text{for all } x\in\mathbb{Z}^d \text{ with } \|x\|\le\frac{1}{\log(1/\kappa)^b}\,nT\\ &\text{there exists a } s_j\in[(j-1)T+1,jT) \text{ such that } X^{\kappa}(s_j)\neq x\Big\}.\end{aligned} \tag{5.23}$$

Then

$$P_0\big(B^{\kappa}_{nT}\big)\le\exp\left\{\big[-J_\kappa(1/\log(1/\kappa)^b)+o_n(1)\big]\,nT\right\}, \tag{5.24}$$

where

$$J_\kappa(x)=x\log(x/2d\kappa)-x+2d\kappa \tag{5.25}$$

is the large deviation rate function of the rate-$2d\kappa$ Poisson process. Thus, by the Hölder inequality with $p,q>1$ and $1/p+1/q=1$, we have

$$\begin{aligned}&E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}\mathbb{1}_{B^{\kappa}_{nT}}\right)\\ &\quad\le E_0\left(\exp\left\{p\overline{\mathcal{I}}\kappa(0,nT,0)\right\}\right)^{1/p}\exp\left\{1/q\big[-J_\kappa(1/\log(1/\kappa)^b)+o_n(1)\big]\,nT\right\}.\end{aligned} \tag{5.26}$$

Recalling Theorem 1.13 and using that $\lim_{\kappa\downarrow 0}J_\kappa(1/\log(1/\kappa)^b)=\infty$, we get

$$\lim_{\kappa\downarrow 0}\lim_{n\to\infty}\frac{1}{nT}\log E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}\mathbb{1}_{B^{\kappa}_{nT}}\right)=-\infty. \tag{5.27}$$

2. Note that

$$C^{\kappa}_{nT}\subseteq B^{\kappa}_{nT}\cup D^{\kappa}_{nT}\quad\text{with}\quad D^{\kappa}_{nT}=\left(C^{\kappa}_{nT}\cap[B^{\kappa}_{nT}]^{c}\right). \tag{5.28}$$

Since we have just proved that the Feynman-Kac representation on B^{κ}_{nT} is not contributing, we only have to look at the contribution coming from D^{κ}_{nT}, namely,

$$E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}\mathbb{1}_{D^{\kappa}_{nT}}\right). \tag{5.29}$$

On the event D^κ_{nT}, the random walk X^κ stays inside the box of radius $nT/\log(1/\kappa)^b$, and jumps during the time intervals $[(j-1)T+1, jT)$, $1 \le j \le n$, defined in (5.23). By the Hölder inequality with $p, q > 1$ and $1/p + 1/q = 1$, we have

$$E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}\mathbb{1}_{D^\kappa_{nT}}\right) \le I \times II, \tag{5.30}$$

where

$$\begin{aligned} I &= \left[E_0\left(\exp\left\{p\sum_{j=1}^{n}\overline{\mathcal{I}}\kappa((j-1)T+1, jT, 0)\right\}\mathbb{1}_{D^\kappa_{nT}}\right)\right]^{1/p}, \\ II &= \left[E_0\left(\exp\left\{q\sum_{j=0}^{n-1}\overline{\mathcal{I}}^\kappa(jT, jT+1, 0)\right\}\mathbb{1}_{D^\kappa_{nT}}\right)\right]^{1/q}. \end{aligned} \tag{5.31}$$

Define

$$J = \Big\{1 \le j \le n\colon\ N(X^\kappa, [(j-1)T+1, jT)) \ge 1\Big\}, \tag{5.32}$$

where $N(X^\kappa, \mathcal{I})$ is the number of jumps of the random walk X^κ during the time interval $\mathcal{I}$. Using that X^κ is not jumping in the time intervals $[(j-1)T+1, jT))$, $j \in J^{\mathrm{c}}$, we may write

$$\begin{aligned} &\sum_{j=1}^{n}\overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0) \\ &\quad = \sum_{j\in J}\overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0) + \sum_{j\in J^{\mathrm{c}}}\int_{(j-1)T+1}^{jT}\xi(X^\kappa((j-1)T+1), s)\,ds. \end{aligned} \tag{5.33}$$

3. To estimate the second term in the right-hand side of (5.33), pick any $\Phi^{X^\kappa} \in A^\kappa_{nT}$ such that $\Phi^{X^\kappa} = X^\kappa$ on $\cup_{j\in J^{\mathrm{c}}}[(j-1)T+1, jT)$ and apply Lemma 5.1, to get

$$\begin{aligned} &\sum_{j\in J^{\mathrm{c}}}\int_{(j-1)T+1}^{jT}\xi(X^\kappa((j-1)T+1), s)\,ds \\ &\quad \le a_m nT - \sum_{j\in J}\int_{(j-1)T+1}^{jT}\xi(\Phi^{X^\kappa}(s), s)\,ds \quad \xi\text{-a.s.} \end{aligned} \tag{5.34}$$

Note that ξ is Gärtner-negative-hyper-mixing, so that by Lemma 5.3 we may estimate the second term on the right hand side of (5.34) by

$$|J|(T-1)CA^{R_0 d} + nT\left(1 + \frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}^{\infty}A^{-Rd}. \tag{5.35}$$

To estimate the first term in the right-hand side of (5.33), apply Lemma 5.3, to get

$$\begin{aligned} &\sum_{k=1}^{n}\left[E_0\left(\exp\left\{p\sum_{j\in J}\overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0)\right\}\mathbb{1}_{D^\kappa_{nT}}\mathbb{1}_{\{|J|=k\}}\right)e^{pk(T-1)CA^{R_0 d}}\right] \\ &\quad \le \exp\left\{pnT\left(1 + \frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}^{\infty}A^{-Rd}\right\} \\ &\qquad \times \sum_{k=1}^{n}\exp\Big\{2pk(T-1)CA^{R_0 d}\Big\}P_0\left(D^\kappa_{nT}, |J| = k\right). \end{aligned} \tag{5.36}$$

The distribution of $|J|$ is $\mathrm{BIN}(n, 1-e^{-2d\kappa(T-1)})$. Hence the sum on the right hand side of (5.36) is bounded from above by

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n}{k}\left(1-e^{-2d\kappa(T-1)}\right)^{k}e^{-2d\kappa(T-1)(n-k)}e^{2pk(T-1)CA^{R_0d}}\\&\leq\left((1-e^{-2d\kappa(T-1)})e^{2p(T-1)CA^{R_0d}}+e^{-2d\kappa(T-1)}\right)^{n}.\end{aligned}\tag{5.37}$$

Combining (5.34–(5.37)), we arrive at

$$\begin{aligned}I\leq&e^{a_m nT}\exp\left\{2nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}\right\}\\&\times\left((1-e^{-2d\kappa(T-1)})e^{2p(T-1)CA^{R_0d}}+e^{-2d\kappa(T-1)}\right)^{n/p}.\end{aligned}\tag{5.38}$$

On the other hand, by Lemma 5.3, we have

$$II\leq e^{nCA^{R_0d}}\exp\left\{nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}\right\}.\tag{5.39}$$

Therefore, combining (5.38–5.39), we finally obtain

$$\begin{aligned}&E_0\left(e^{\overline{\mathcal{I}}^\kappa(0,nT,0)}1\!\!1_{D^\kappa_{nT}}\right)\\&\leq e^{a_m nT}e^{nCA^{R_0d}}\exp\left\{3nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}\right\}\\&\quad\times\left((1-e^{-2d\kappa(T-1)})e^{2p(T-1)CA^{R_0d}}+e^{-2d\kappa(T-1)}\right)^{n/p}.\end{aligned}\tag{5.40}$$

5.3.3 Final estimation

By (5.40), we have

$$\begin{aligned}&\frac{1}{nT}\log E_0\left(e^{\overline{\mathcal{I}}^\kappa(0,nT,0)}1\!\!1_{D^\kappa_{nT}}\right)\\&\leq a_m+\frac{2d\kappa(T-1)}{pT}e^{2p(T-1)CA^{R_0d}}+\frac{CA^{R_0d}}{T}+3\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}.\end{aligned}\tag{5.41}$$

Abbreviate $M_2=2pCA^{R_0d}$ and recall (5.17). Then the right-hand side of (5.41) is asymptotically equivalent to

$$a_m+\frac{2d\kappa}{p}(1/\kappa)^{M_2K}+3\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd},\qquad\kappa\downarrow 0.\tag{5.42}$$

Choosing $K\leq 1/2M_2$, $K\in\mathbb{Q}$, and recalling (5.20) we finally arrive at

$$\lambda_0^{\delta_0}(\kappa)\leq a_m+\frac{2d\sqrt{\kappa}}{p}+3\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd},\tag{5.43}$$

which tends to zero as $\kappa\downarrow 0$, $R_0\to\infty$ and $m\to\infty$.

6 No Lipschitz continuity at $\kappa = 0$

In this section we prove Theorem 1.16. The proof is very close to that of Gärtner, den Hollander and Maillard [7], Theorem 1.2(iii), where it is assumed that ξ is bounded from below. For completeness we will repeat the main steps in that proof.

Proof. Fix $C_1 > 0$, write (see 3.1),

$$\lambda_0^{\delta_0}(\kappa) = \lim_{n\to\infty} \frac{1}{nT+1} \log E_0 \left(e^{\overline{\mathcal{I}}^\kappa(0,nT+1,0)} \delta_0(X^\kappa(nT+1)) 1\!\!1\Big\{ X^\kappa([0,nT+1]) \subseteq [C_1]_{nT+1} \Big\} \right) \tag{6.1}$$

and abbreviate

$$I_j^\xi(x) = \int_{(j-1)T+1}^{jT} \xi(x,s)\, ds, \qquad Z_j^\xi = \operatorname*{argmax}_{x\in\{0,e\}} I_j^\xi(x), \qquad 1 \le j \le n. \tag{6.2}$$

Consider the event

$$A^\xi = \left[\bigcap_{j=1}^n \{ X^\kappa(t) = Z_j^\xi \ \forall\, t \in [(j-1)T+1, jT) \} \right] \cap \{ X^\kappa(nT+1) = 0 \}. \tag{6.3}$$

We have

$$\begin{aligned} &E_0 \left(\exp\left\{ \overline{\mathcal{I}}^\kappa(0,nT+1,0) \right\} \delta_0(X^\kappa(nT+1)) \right) \\ &\quad \ge E_0 \left(\exp\left\{ \sum_{j=1}^{n+1} \overline{\mathcal{I}}^\kappa((j-1)T,(j-1)T+1,0) + \sum_{j=1}^{n} \overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0) \right\} 1\!\!1_{A^\xi} \right). \end{aligned} \tag{6.4}$$

Using the reverse Hölder inequality with $q < 0 < p < 1$ and $1/q + 1/p = 1$, we have

$$\begin{aligned} &E_0 \left(\exp\left\{ \sum_{j=1}^{n+1} \overline{\mathcal{I}}^\kappa((j-1)T,(j-1)T+1,0) + \sum_{j=1}^{n} \overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0) \right\} 1\!\!1_{A^\xi} \right) \\ &\quad \ge \left[E_0 \left(\exp\left\{ q \sum_{j=1}^{n+1} \overline{\mathcal{I}}^\kappa((j-1)T,(j-1)T+1,0) \right\} 1\!\!1\Big\{ X^\kappa([0,nT+1]) \subseteq [C_1]_{nT+1} \Big\} \right) \right]^{1/q} \\ &\qquad \times \left[E_0 \left(\exp\left\{ p \sum_{j=1}^{n} \overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0) \right\} 1\!\!1_{A^\xi} 1\!\!1\Big\{ X^\kappa([0,nT+1]) \subseteq [C_1]_{nT+1} \Big\} \right) \right]^{1/p}. \end{aligned} \tag{6.5}$$

To estimate the first term on the right hand side of (6.5), fix $R_0 \in \mathbb{N}$ and choose $A, C > 0$ such that all results of Section 3 are satisfied for $q\xi$. Moreover, note that by a refinement of the arguments given in the proof of Lemma 3.4 and (3.53), one has ξ-a.s. for $nT+1 \in \mathbb{N}$ sufficiently large

$$\begin{aligned} &\sum_{j=1}^{n+1} \int_{(j-1)T}^{(j-1)T+1} q\xi(X^\kappa(s),s) 1\!\!1\Big\{ q\xi(X^\kappa(s),s) > -qCA^{R_0 d} \Big\}\, ds \\ &\quad \le -\frac{nT+1+N(X^\kappa,nT+1)}{T} qC'A^d \sum_{R=R_0}^{\infty} A^{-Rd}. \end{aligned} \tag{6.6}$$

Consequently,

$$\begin{aligned} &E_0\left(\exp\left\{q\sum_{j=1}^{n+1}\overline{\mathcal{I}}^\kappa((j-1)T,(j-1)T+1,0)\right\}1\!\!1\Big\{X^\kappa([0,nT+1])\subseteq[C_1]_{nT+1}\Big\}\right)\\ &\qquad\le e^{-q(n+1)CA^{R_0d}}E_0\left(\exp\left\{-\frac{nT+1+N(X^\kappa,nT+1)}{T}qC'A^d\sum_{R=R_0}^\infty A^{-Rd}\right\}\right), \end{aligned} \tag{6.7}$$

which equals

$$\begin{aligned} &e^{-q(n+1)CA^{R_0d}}\exp\left\{-\frac{nT+1}{T}qC'A^d\sum_{R=R_0}^\infty A^{-Rd}\right\}\\ &\qquad\times\exp\Big\{2d\kappa(nT+1)\Big(e^{-\frac{q}{T}C'A^d\sum_{R=R_0}^\infty A^{-Rd}}-1\Big)\Big\}. \end{aligned} \tag{6.8}$$

As in the proof of [7], Theorem 1.2(iii), we have

$$\begin{aligned} &\left[E_0\left(\exp\left\{p\sum_{j=1}^{n}\overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0)\right\}1\!\!1_{A^\xi}\right)\right]^{1/p}\\ &\qquad\ge\Big[\exp\Big\{[1+o_n(1)]\,np\,\mathbb{E}\Big(\max\{I_1^\xi(0),I_1^\xi(e)\}\Big)\Big\}\,[p_1^\kappa(e)]^{n+1}\,e^{-2d\kappa n(T-1)}\Big]^{1/p}. \end{aligned} \tag{6.9}$$

Combining (6.4–6.9), we arrive at

$$\begin{aligned} &\frac{1}{nT+1}\log E_0\Big(e^{\overline{\mathcal{I}}^\kappa(0,nT+1,0)}\,\delta_0(X^\kappa(nT+1))\Big)\\ &\qquad\ge-\frac{1}{(nT+1)}\,CA^{R_0d}(n+1)-\frac{1}{T}C'A^d\sum_{R=R_0}^\infty A^{-Rd}+\frac{2d\kappa}{q}\Big(e^{-\frac{q}{T}C'A^d\sum_{R=R_0}^\infty A^{-Rd}}-1\Big)\\ &\qquad\quad+\frac{1}{p(nT+1)}\Big[[1+o_n(1)]\,pn\,\mathbb{E}\Big(\max\Big\{I_1^\xi(0),I_1^\xi(e)\Big\}\Big)\Big]-\frac{2d\kappa n(T-1)}{p(nT+1)}\\ &\qquad\quad+\frac{n+1}{p(nT+1)}\,\log p_1^\kappa(e). \end{aligned} \tag{6.10}$$

Using that $p_1^\kappa(e)=\kappa[1+o_\kappa(1)]$ as $\kappa\downarrow0$ and letting $n\to\infty$, we get that

$$\begin{aligned} \lambda_0^{\delta_0}(\kappa)\ge&-\frac{1}{(nT+1)}\,CA^{R_0d}(n+1)-\frac{1}{T}C'A^d\sum_{R=R_0}^\infty A^{-Rd}+\frac{2d\kappa}{q}\Big(e^{-\frac{q}{T}C'A^d\sum_{R=R_0}^\infty A^{-Rd}}-1\Big)\\ &-\frac{2d\kappa(T-1)}{pT}+\frac{1}{T}\,[1+o_\kappa(1)]\,\Big(\tfrac{1}{2}\,\mathbb{E}(T-1)-\frac{1}{p}\,\log(1/\kappa)\Big). \end{aligned} \tag{6.11}$$

At this point we can copy the rest of the proof of [7], Theorem 1.2(iii), with a few minor adaptations of constants. ∎

7 Examples

In Section 7.1 we prove Corollary 1.18, in Section 7.2 we prove Corollary 1.19.

7.1 Proof of Corollary 1.18

In Section 7.1.1 we settle Part (1), in Section 7.1.2 we settle Part (2).

7.1.1 Proof of Corollary 1.18 (1)

1.1 The first condition in Definition 1.6 is satisfied by our assumption on ξ.

1.2 We show that ξ is type-I Gärtner-mixing. Fix $A > 1$, pick $b = c = 0$ and $a_1 = a_2 = 2$ (see (3.26)), and define

$$B_R^{A,\mathrm{sub}}(x,k) = \left(\prod_{j=1}^{d} [(x(j)-1)A^R, (x(j)+1)A^R) \cap \mathbb{Z}^d \right) \times \{kA^R\}. \tag{7.1}$$

We start by estimating the probability of the event

$$\mathrm{B}(x,k) \stackrel{\mathrm{def}}{=} \big\{ B_{R+1}^A(x,k) \text{ is good, but contains a bad } R\text{-block} \big\}. \tag{7.2}$$

Note that each $R+1$-block contains at most $2^d A^{(1+d)}$ R-blocks. For each such R-block $B_R^A(y,l)$ there are no more than A^{Rd} blocks

$$Q_R^A(z_1) = \prod_{j=1}^{d} [(z_1(j), z_1(j) + A^R) \tag{7.3}$$

contained in it. For any such block we may estimate, for $C_1 > 0$,

$$\begin{aligned} &\mathbb{P}\left(\exists s \in [lA^R, (l+1)A^R)\colon \sum_{z_2 \in Q_R^A(z_1)} \xi(z_2, s) > C_1 A^{Rd} \right) \\ &\le \sum_{k=0}^{\lfloor A^R \rfloor} \mathbb{P}\left(\sum_{z_2 \in Q_R^A(z_1)} \sup_{s \in [k,k+1)} X_s(z_2) > C_1 A^{Rd} \right) \\ &\le (A^R + 1) \exp\left\{ -C_1 A^{Rd} \right\} \mathbb{E}\left(\exp\left\{ \sup_{s\in[0,1)} X_s(0) \right\} \right)^{A^{Rd}}, \end{aligned} \tag{7.4}$$

where we use the time stationarity in the first inequality, and the time stationarity and the space independence in the second inequality. Thus, for C_1 sufficiently large, there is a $C_1' > 0$ such that

$$\mathbb{P}\big(\mathrm{B}(x,k)\big) \le e^{-C_1' A^{Rd}}. \tag{7.5}$$

Moreover, for space-time blocks that are disjoint in space, the corresponding events in (7.2) are independent. Hence we may assume that $B_{R+1}^A(x_1,k_1), \ldots, B_{R+1}^A(x_n,k_n)$ are equal in space but disjoint in time. Since

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \mathrm{B}(x_i,k_i) \right) = \mathbb{P}\left(\mathrm{B}(x_n,k_n) \;\Big|\; \bigcap_{i=1}^{n-1} \mathrm{B}(x_i,k_i) \right) \mathbb{P}\left(\bigcap_{i=1}^{n-1} \mathrm{B}(x_i,k_i) \right), \tag{7.6}$$

it is enough to show that there is a constant $K < \infty$, independent of R, such that the conditional probability in (7.6) may be estimated from above by $K\mathbb{P}(\mathrm{B}(x_n, k_n))$. To do this, we apply the Markov property to obtain

$$\begin{aligned}\mathbb{P}\Bigg(\mathrm{B}(x_n, k_n) \;\Bigg|\; \bigcap_{i=1}^{n-1} \mathrm{B}(x_i, k_i)\Bigg) &\le \mathbb{P}\Bigg(\mathrm{B}(x_n, k_n) \;\Bigg|\; \bigcap_{i=1}^{n-1} \mathrm{B}(x_i, k_i), B_{R+1}^{A,\mathrm{sub}}(x_n, k_n) \text{ is good}\Bigg) \\ &= \mathbb{P}\Bigg(\mathrm{B}(x_n, k_n) \;\Bigg|\; B_{R+1}^{A,\mathrm{sub}}(x_n, k_n) \text{ is good}\Bigg).\end{aligned} \tag{7.7}$$

Thus, the left-hand side of (7.7) is at most

$$\frac{\mathbb{P}(\mathrm{B}(x_n, k_n))}{\mathbb{P}(B_{R+1}^{A,\mathrm{sub}}(x_n, k_n) \text{ is good})}. \tag{7.8}$$

Since $\lim_{R\to\infty} \mathbb{P}(B_{R+1}^{A,\mathrm{sub}}(x_n, k_n) \text{ is good}) = 1$, we obtain that ξ is type-I Gärtner-mixing.

1.3 Condition (a3) in Definition 1.6 follows from the calculations in (7.4).

2. The same strategy as above works to show that ξ is type-II Gärtner-mixing. If X has exponential moments of all negative orders, then the same calculations as in the first part show that ξ is Gärtner-negative-hyper-mixing. All requirements of Theorems 1.13–1.14 are thus met.

7.1.2 Proof of Corollary 1.18(2)

Let ξ be the zero-range process as described in Corollary 1.18(2). We will use that each particle, independently of all the other particles, carries an exponential clock of parameter one. If there are k particles at a site x and one of these clocks rings, then the corresponding particle jumps to y with probability $\frac{g(k)}{2dk}$ and it stays at y with probability $1 - \frac{g(k)}{k}$.

1.1 By Andjel [1], Theorem 1.9, the product measures in (1.33) are extremal for ξ. Thus, $\mathbb{E}\left[e^{q\xi(0,0)}\right] < \infty$ for all $q \ge 0$. Consequently, to show that $\mathbb{E}\left[e^{q\sup_{s\in[0,1]}\xi(0,s)}\right] < \infty$, it suffices to prove that there is a constant $K > 0$ such that, for all $k \in \mathbb{N}$ sufficiently large,

$$\mathbb{P}\Bigg(\sup_{s\in[0,1]} \xi(0, s) \ge k\Bigg) \le K\mathbb{P}\Bigg(\xi(0, 1) \ge \frac{e^{-1}}{2} k\Bigg). \tag{7.9}$$

Write $\mathrm{NR}(\frac{e^{-1}}{2}k, \tau)$ for the event that there are at least $\frac{e^{-1}}{2}k$ exponential clocks of particles located at zero that do not ring in the time interval $[\tau, \tau + 1)$. Then we may estimate

$$\begin{aligned}&\mathbb{P}\Bigg(\xi(0, 1) \ge \frac{e^{-1}}{2} k \;\Bigg|\; \exists \tau \in [0, 1]\colon\; \xi(0, \tau) \ge k\Bigg) \\ &\quad\ge \mathbb{P}\Bigg(\mathrm{NR}\Big(\frac{e^{-1}}{2} k, \tau\Big) \;\Bigg|\; \exists \tau \in [0, 1]\colon\; \xi(0, \tau) \ge k\Bigg).\end{aligned} \tag{7.10}$$

Since the probability that a clock does not ring within a time interval of length one is equal to e^{-1}, and all clocks are independent, we may estimate the right-hand side of (7.10) from below by

$$\mathbb{P}\Big(T \ge \frac{e^{-1}}{2} k\Big), \qquad T = \mathrm{BIN}(k, e^{-1}). \tag{7.11}$$

Finally, note that the probability in (7.11) is bounded away from zero. Thus, inserting (7.10) and (7.11) into (7.9), we get the claim.

1.2. We show that ξ is type-I Gärtner-mixing. We use ideas from [10]. Fix $A > 3$, choose $b = 3$, $c = 1$, $a_1 = 13$, $a_2 = 2$ (see (3.26), and introduce additional space-time blocks

$$\begin{aligned}\tilde{B}_R^{A,+}(x,k) &= \left(\prod_{j=1}^{d}[(x(j)-2)A^R,(x(j)+2)A^R)\cap\mathbb{Z}^d\right)\times[kA^R-A^{R-1},(k+1)A^R)\\ \tilde{B}_R^{A,\mathrm{sub}}(x,k) &= \left(\prod_{j=1}^{d}[(x(j)-4)A^R,(x(j)+4)A^R)\cap\mathbb{Z}^d\right)\times\{(k-1)A^R\}.\end{aligned} \tag{7.12}$$

Given $S \subseteq \mathbb{Z}^d$ and $S' \subseteq \mathbb{N}_0$, write ∂S to denote the inner boundary of S and $\Pi_1(S \times S')$ to denote the projection onto the spacial coordinates. Furthermore, e_j, $j \in \{1, 2, \cdots, d\}$ denotes the j-th unit vector (and we agree that $\frac{0}{0} = 0$).

We call a space-time block $B_R^A(x,k)$ *contaminated* if there is a particle at some space-time point $(y,s) \in \tilde{B}_R^{A,+}(x,k)$ that has been outside $\Pi_1(\prod_{j=1}^d[(x(j)-4)A^R,(x(j)+4)A^R))$ at a time s' such that $(k-1)A^R \le s' < s$. The reason for introducing this notion is that events depending on non-contaminated blocks that are equal in time but disjoint in space are all independent.

Contaminated blocks. For $L > 0$, define

$$\chi(x,k) = \mathbb{1}\Big\{B_{R+1}^A(x,k) \text{ is good, but contaminated and intersects } [-L,L]^{d+1}\Big\} \tag{7.13}$$

and fix $(x^*,k^*) \in \mathbb{Z}^d \times \mathbb{N}$.

Claim 7.1. *There is a $C' > 0$ independent of L such that $(\chi(x,k))_{(x,k)\equiv(x^*,k^*)}$ is stochastically dominated by independent Bernoulli random variables $(Z(x,k))_{(x,k)=(x^*,k^*)}$ with success probability $e^{-C'A^{R+1}}$.*

Proof. We use a discretization scheme. More precisely, we construct a discrete-time version of the zero-range process where particles are allowed to jump at times k/n, $k \in \mathbb{N}_0$ only. Here, n is an integer that will later tend to infinity, and we will denote by $\xi^n(x,s)$ the number of particles at site x at time s. To construct this process, we take a family $X^n(x,s,q_1,q_2)$ of independent random variables with index set $\mathbb{Z}^d \times \frac{1}{n}\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ whose distribution is defined via

$$\begin{aligned}\mathbb{P}\Big(X^n(\cdot,\cdot,\cdot,q_2) = 0\Big) &= 1 - \frac{g(q_2)}{nq_2},\\ \mathbb{P}\Big(X^n(\cdot,\cdot,\cdot,q_2) = \pm e_j\Big) &= \frac{g(q_2)}{2dnq_2}, \qquad j \in \{1,2,\ldots,d\}.\end{aligned} \tag{7.14}$$

With this family in hand, we proceed as follows. At time zero start with an initial configuration that comes from the invariant measure π_ρ. Attach to each particle σ a uniform-$[0,1]$ random variable $\mathcal{U}(\sigma)$. Take all these random variables independent of each other and of $X^n(x,s,q_1,q_2)$ for all choices of $(x,s,q_1,q_2) \in \mathbb{Z}^d \times \frac{1}{n}\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$. For each site x, order all particles present at x at time zero so that their uniform random variables are increasing. To the q_1-th variable attach $X^n(x,0,q_1,\xi^n(x,0))$, i.e., the position of the q_1-th particle in this ordering at time $\frac{1}{n}$ is $x + X^n(x,0,q_1,\xi^n(x,0))$. In this way we obtain the configuration of the system at time $\frac{1}{n}$.

To construct the process $\frac{1}{n}$ time units further, repeat the first step, but let the particles jump according to $X^n(\cdot, \frac{1}{n}, \cdot, \xi^n(\cdot, \frac{1}{n}))$. Thus, our construction is such that each particle chooses at each step uniformly at random, but dependent on the number of particles at the same location, a new jump distribution. In what follows we will use the phrase "at level n" to emphasize that we refer to the discrete-time version of the process. For instance, we say that $B_R(x,k)$ is good at level n if

$$\sum_{z \in Q_R^A(y)} \xi^n(z,s) \le CA^{Rd} \qquad \forall\, y \in \mathbb{Z}^d, s \ge 0 \text{ s.t. } Q_R(y) \times \{s\} \subseteq \tilde{B}_R^A(x,k). \tag{7.15}$$

Next, we introduce

$$\begin{aligned} &\chi^n(x,k) \\ &= \mathbb{1}\Big\{ B_{R+1}^A(x,k) \text{ is good, but is contaminated at level } n \text{ and intersects } [-L,L]^{d+1} \Big\}. \end{aligned} \tag{7.16}$$

It is not hard to show that the joint distribution of χ^n converges weakly to the joint distribution of χ (use that only finitely many particles can enter a fixed region in space-time, so that the above family of random variables may be approximated by a function depending on finitely many particles only). Thus, to estimate the joint distribution of χ, it is enough to analyze the joint distribution of χ^n, as long as the estimates are uniform in n. In what follows, $s, s', s'' \in \frac{1}{n}\mathbb{N}_0$.

Let $B_{R+1}^A(x,k)$ be a good block that is contaminated at level n. Then there is a particle at a site $y \in \partial\Pi_1(\tilde{B}_{R+1}^A(x,k))$ at a time $s \in [(k-1)A^{R+1}, (k+1)A^{R+1})$ (see below 4.13)) that is at a site $y' \in \partial\Pi_1(\tilde{B}_{R+1}^{A,+}(x,k))$ at a time $s' \in [(k-1)A^{R+1}, (k+1)A^{R+1})$, $s < s'$. Furthermore, for all s'' such that $s < s'' < s'$, the particle is inside $\Pi_1(\tilde{B}_{R+1}^A(x,k))$. This implies, when $X^n(y,s,q_1,\xi^n(y,s))$ denotes the random variable attached to σ at time s, that $X^n(y,s,q_1,\xi^n(y,s)) \neq 0$. Pick any such particle. Since this particle travels over a distance larger than $2A^{R+1}$, there is at least one coordinate direction along which it makes at least $2A^{R+1}$ steps. We call this direction $e_j(\sigma)$, and say that each step in this direction is a success. Note the uniform estimate

$$\mathbb{P}\big(\sigma \text{ has a success at time } s''\big) \le \frac{1}{2dn} \qquad s'' \in [(k-1)A^{R+1}, (k+1)A^{R+1}). \tag{7.17}$$

Thus, if σ contaminates $B_{R+1}^A(x,k)$ at level n, then from time s' up to time $(k+1)A^{R+1}$ it has at least $2A^{R+1}$ successes in $\Pi_1(\tilde{B}_{R+1}^A(x,k))$. We write $\mathrm{S}(y,s,q_1,q_2)$ for the event just described, provided the particle was attached to $X^n(y,s,q_1,q_2)$ when it entered $\tilde{B}_{R+1}^A(x,k)$. Since $B_{R+1}^A(x,k)$ is good, at each space-time point $(y'',s'') \in \tilde{B}_{R+1}^A(x,k)$ there are at most $CA^{(R+1)d}$ particles that can contaminate $B_{R+1}^A(x,k)$. We therefore obtain

$$\begin{aligned} &\Big\{ B_{R+1}^A(x,k) \text{ is good, but contaminated at level } n \Big\} \\ &\subseteq \bigcup_{\substack{y \in \partial\Pi_1(\tilde{B}_{R+1}^A(x,k)) \\ s \in [(k-1)A^{R+1},(k+1)A^{R+1}),\, s \in \frac{1}{n}\mathbb{N}_0 \\ q_1 \le CA^{(R+1)d} \\ q_2 \le CA^{(R+1)d}}} \Big\{ X^n(y,s,q_1,q_2) \neq 0, \mathrm{S}(y,s,q_1,q_2) \Big\} \\ &\stackrel{\text{def}}{=} \mathrm{C}^n(x,k). \end{aligned} \tag{7.18}$$

Next, note that the event $\mathrm{C}^n(x,k)$ depends on the $X^n(y,s,q_1,q_2)$ with $(y,s) \in \tilde{B}^A_{R+1}(x,k)$ only. Hence, for $x \in \mathbb{Z}^d$, $k \in \mathbb{N}_0$, the family $(\mathcal{C}^n(x,k))_{(x,k)\equiv(x^*,k^*)}$ consists of independent events. We estimate $\mathbb{P}(C^n(x,k))$. By (7.17), the probability inside the union may be bounded from above by

$$\frac{1}{n}\mathbb{P}\big(T \geq 2A^{R+1}\big), \tag{7.19}$$

where $T = \mathrm{BIN}(2A^{R+1}n, \frac{1}{2dn})$. Note that the event in (7.19) is a large deviation event, so that Bernstein's inequality guarantees the existence of a constant $C' > 0$ such that (7.19) is at most

$$\frac{1}{n}\exp\left\{-C'A^{R+1}\right\}. \tag{7.20}$$

Recall the definition of $C^n(x,k)$ to see that, for a possibly different constant $C'' > 0$,

$$\mathbb{P}(C^n(x,k)) \leq \exp\left\{-C''A^{R+1}\right\}. \tag{7.21}$$

Hence there is a family of independent Bernoulli random variables $(Z^n(x,k))_{(x,k)\equiv(x^*,k^*)}$ that stochastically dominates $(\chi^n(x,k))_{(x,k)\equiv(x^*,k*)}$ and has success probability $e^{-C''A^{R+1}}$. Thus, if f is a positive and bounded function that is increasing in all of its arguments, then

$$\mathbb{E}\Big(f(\chi^n(x_1,k_1),\dots,\chi^n(x_n,k_n)\Big) \leq \mathbb{E}\Big(f(Z^n(x_1,k_1),\dots,Z^n(x_n,k_n)\Big). \tag{7.22}$$

As the right-hand side does not depend on n, we obtain, by letting $n \to \infty$,

$$\mathbb{E}\Big(f(\chi(x_1,k_1),\dots,\chi(x_n,k_n)\Big) \leq \mathbb{E}\Big(f(Z(x_1,k_1),\dots,Z(x_n,k_n)\Big), \tag{7.23}$$

which proves Claim 7.1. In particular, since all estimates are independent of L, we may even set $L = \infty$, to get that the whole field of good but contaminated $(R+1)$-blocks may be dominated by an independent family of Bernoulli random variables with the same success probability as above. This field will be denoted by $Z(x,k)$ as well. ∎

Non-contaminated blocks. We begin by estimating the probability of the event

$$\Big\{B^A_{R+1}(x,k) \text{ is good, but contains a bad } R\text{-block}\Big\}. \tag{7.24}$$

Let $B^A_R(y,l)$ be an R-block that is contained in $B^A_{R+1}(x,k)$. We bound the probability that this block is bad. For that we take $z_1 \in \mathbb{Z}^d$ such that

$$Q^A_R(z_1) = \prod_{j=1}^d [z_1(j), z_1(j) + A^R) \subseteq \tilde{B}^A_R(y,l). \tag{7.25}$$

Use the time stationarity of ξ and the fact that $[(l-1)A^R, (l+1)A^R)$ may be divided into at most $2A^R + 1$ time intervals of length one, to obtain

$$\begin{aligned} &\mathbb{P}\left(\exists s \in [(l-1)A^R, (l+1)A^R)\colon \sum_{z_2 \in Q^A_R(z_1)} \xi(z_2,s) > CA^{Rd}\right) \\ &\leq (2A^R+1)\mathbb{P}\left(\sup_{s\in[0,1]} \sum_{z_2 \in Q^A_R(z_1)} \xi(z_2,s) > CA^{Rd}\right). \end{aligned} \tag{7.26}$$

In the same way as in Step 1.1, we may now show that for some constant $K > 0$,

$$\mathbb{P}\Bigg(\sup_{s\in[0,1]} \sum_{z_2\in Q_R^A(z_1)} \xi(z_2,s) > CA^{Rd}\Bigg) \leq K\mathbb{P}\Bigg(\sum_{z_2\in Q_R^A(z_2)} \xi(z_2,1) \geq \frac{e^{-1}}{2}CA^{Rd}\Bigg). \tag{7.27}$$

Next, note that under the invariant measure π_ρ the sum in the right-hand side of (7.27) is a sum of i.i.d. random variables with finite exponential moments. Hence (7.27) is bounded from above by

$$K\exp\Big\{-\frac{e^{-1}}{2}CA^{Rd}\Big\}\mathbb{E}\left[e^{\xi(0,0)}\right]^{A^{Rd}}. \tag{7.28}$$

Now choose C large enough so that (7.26) decays superexponentially fast in R.

In order to estimate the joint distribution of non-contaminated blocks, we let $B_{R+1}^{A_1}(x_1,k_1)$, $\dots, B_{R+1}^{A_1}(x_n,k_n)$ be space-time blocks whose indices increase in the lexicographic order of $\mathbb{Z}^d\times\mathbb{N}$ and belong to the same equivalence class. We abbreviate

$$\begin{aligned} N^{\rm sub}(x_i,k_i) = \Big\{&\tilde{B}_{R+1}^{A,{\rm sub}}(x_i,k_i) \text{ is good, } B_{R+1}^{A}(x_i,k_i) \text{ contains a bad } R\text{-block},\\ &\text{but is not contaminated}\Big\}. \end{aligned} \tag{7.29}$$

Note that $N^{\rm sub}(x_i,k_i)$ and $N^{\rm sub}(x_j,k_j), i\neq j$, are independent when they depend on blocks that coincide in time but are disjoint in space. This observation, together with the Markov property applied, leads to

$$\begin{aligned} &\mathbb{P}\Bigg(N^{\rm sub}(x_n,k_n)\ \Bigg|\ \bigcap_{i=1}^{n-1} N^{\rm sub}(x_i,k_i)\Bigg)\\ &\leq \mathbb{P}\Bigg(N^{\rm sub}(x_n,k_n)\ \Bigg|\ \bigcap_{i=1}^{n-1} N^{\rm sub}(x_i,k_i), \tilde{B}_{R+1}^{A,{\rm sub}}(x_n,k_n) \text{ is good}\Bigg)\\ &= \mathbb{P}\Big(N^{\rm sub}(x_n,k_n)\ \Big|\ \tilde{B}_{R+1}^{A,{\rm sub}}(x_n,k_n) \text{ is good}\Big). \end{aligned} \tag{7.30}$$

Thus, the left-hand side of (7.30) is at most

$$\frac{\mathbb{P}(B_{R+1}^{A}(x,k) \text{ is good, but contains a bad } R\text{-block})}{\mathbb{P}(\tilde{B}_{R+1}^{A_1,{\rm sub}}(x_n,k_n) \text{ is good})}. \tag{7.31}$$

Note that the denominator tends to one as $R\to\infty$. This comes from the fact that, for all $t\geq 0$, $(\xi(x,t))_{x\in\mathbb{Z}^d}$ is an i.i.d. field of random variables distributed according to π_ρ. Thus, from (7.26) and the lines below, we infer that (7.31) decays superexponentially fast in R.

Finally, write

$$\Big\{B_{R+1}^{A}(x_i,k_i) \text{ is good, but contains a bad } R\text{-block}\ \Big\} \subseteq C(x_i,k_i)\cup N^{\rm sub}(x_i,k_i), \tag{7.32}$$

where we denote by $C(x_i,k_i)$ the event that $B_{R+1}^{A}(x_i,k_i)$ is good, contains a bad R-block, and is contaminated. Then

$$\begin{aligned} &\mathbb{P}\Bigg(\bigcap_{i=1}^{n}\Big\{B_{R+1}^{A_1}(x_i,k_i) \text{ is good, but contains a bad } R\text{-block}\Big\}\Bigg)\\ &\qquad\leq \mathbb{P}\Bigg(\bigcap_{i=1}^{n}\big(C(x_i,k_i)\cup N^{\rm sub}(x_i,k_i)\big)\Bigg). \end{aligned} \tag{7.33}$$

If we denote by C the subset of all $i \in \{1,2,\dots,n\}$ for which $C(x_i,k_i)$ occurs, then (7.33) may be rewritten as

$$\sum_{C \subseteq \{1,2,\dots,n\}} \mathbb{P}\Bigg(\bigcap_{i\in C} C(x_i,k_i) \cap \bigcap_{i \notin C} N^{\mathrm{sub}}(x_i,k_i) \Bigg). \tag{7.34}$$

Note that either $|C| \geq n/2$ or $|\{1,2,\dots,n\} \setminus C| \geq n/2$, so that by Claim 7.1 and (7.28) there is a $C'' > 0$ such that the expression in (7.34) is at most $2^n \exp\{-C'' A^{R+1} n/2\}$. A comparison with the right-hand side of (1.21) shows that ξ is type-I Gärtner-mixing.

1.3 From the previous calculations we infer that ξ satisfies condition (a3) of Definition 1.6.

2. Since ξ is bounded from below it is Gärtner-negative-hyper-mixing. Hence it remains to show that ξ is type-II Gärtner-mixing. To that end, fix the same constants as in the proof of the first part of the corollary. Furthermore, fix $\delta > 0$ and let $B^A_{R+1}(x_1,k_1),\dots,B^A_{R+1}(x_n,k_n)$ be space-time blocks whose indices increase in the lexicographic order of $\mathbb{Z}^d \times \mathbb{N}$, and belong to the same equivalence class. Take events $\mathcal{A}^R_i \in \sigma(B^A_{R+1}(x_n,k_n))$, $i \in \{1,2,\dots,n\}$, that are invariant under shifts in space and time, and satisfy

$$\lim_{R\to\infty} \mathbb{P}(\mathcal{A}^R_i) = 0. \tag{7.35}$$

As in part 1, we divide space-time blocks into contaminated and non-contaminated blocks. With the help of Claim 7.1, we may control contaminated blocks. To treat non-contaminated blocks, we introduce

$$N^{\mathrm{sub}}(x_i,k_i,\mathcal{A}^R_i) = \Big\{ \tilde{B}^{A,\mathrm{sub}}_{R+1}(x_i,k_i) \text{ is good, but contaminated, } \mathcal{A}^R_i \text{ occurs} \Big\} \tag{7.36}$$

and proceed as in the lines following (7.29), to finish the proof.

7.2 Proof of Corollary 1.19

In this section we prove Corollary 1.19. Suppose that

- ► ξ is Markov with initial distribution ν and generator L defined on a domain $D(L) \subset L^2(d\nu)$. (7.37)

Denote by

$$\bar{\varepsilon}(f,g) = \tfrac{1}{2}\big[\langle -Lf,g\rangle + \langle -Lg,f\rangle\big], \qquad f,g \in D(L), \tag{7.38}$$

its symmetrized Dirichlet form, assume that $(\bar{\varepsilon}, D(L))$ is closable, and denote its closure by $(\bar{\varepsilon}, D(\bar{\varepsilon}))$. Furthermore, for $V\colon\ \Omega \to \mathbb{R}$, define

$$J_V(r) = \inf\left\{ \bar{\varepsilon}(f,f)\colon\ f \in D(\bar{\varepsilon}) \cap L^2(|V|d\nu),\ \int f^2\,d\nu = 1,\ \int V f^2\,d\nu = r \right\}, \qquad r \in \mathbb{R}, \tag{7.39}$$

note that $r \mapsto J_V(r)$ is convex, and let I_V be its lower semi-continuous regularization. For $W\colon\ \Omega \times [0,\infty) \to \mathbb{R}$, define

$$\Gamma^L_t = \sup_{\substack{f \in D(L)\cap L^2(|W(\cdot,t)|d\nu) \\ \|f\|_2 = 1}} \left\{ \langle Lf,f\rangle + \langle W(\cdot,t), f^2\rangle \right\}, \qquad t \geq 0. \tag{7.40}$$

In the particular case of a static $W_0\colon\ \Omega \to \mathbb{R}$, we denote the corresponding variational expression by Γ^L_0.

Lemma 7.2. *Suppose that W is bounded from below, piecewise continuous in the time-coordinate, and ν-integrable in the space-coordinate. Suppose further that:*
(i) $\Gamma_t^L < \infty$ for all $t > 0$.
(ii) ξ is reversible in time and càdlàg.
Then

$$E_\nu\left(\exp\left\{\int_0^t W(\xi(s),s)\,ds\right\}\right) \le \exp\left\{\int_0^t \Gamma_s^L\,ds\right\} \qquad \forall\, t \ge 0. \tag{7.41}$$

Proof. The proof is based on ideas in Kipnis and Landim [9], Appendix 1.7, and comes in three steps.

1. Suppose that W is piecewise continuous, not necessarily bounded from above, and define $W^n = W \wedge n$. Then, by an argument similar to that in [9], Appendix 1, Lemma 7.2, we have

$$\mathbb{E}_\nu\left(\exp\left\{\int_0^t W^n(\xi(s),s)\,ds\right\}\right) \le \exp\left\{\int_0^t \Gamma_s^{L,n}\,ds\right\}, \tag{7.42}$$

where $\Gamma^{L,n}$ is defined as in (7.40) but with W replaced by W^n. It is here that we use that ξ is reversible and càdlàg, since under this condition we have a Feynman-Kac representation for the parabolic Anderson equation with potential W^n and with Δ replaced by L. Because W^n is bounded, we have that $L^2(|W^n(\cdot,t)|d\nu) = L^2(d\nu)$. Hence

$$\Gamma_s^{L,n} = \sup_{\substack{f\in D(L)\\ \|f\|_2=1}} \left\{\langle Lf,f\rangle + \langle W^n(\cdot,s),f^2\rangle\right\}. \tag{7.43}$$

Since, by monotone convergence,

$$\lim_{n\to\infty} \mathbb{E}_\nu\left(\exp\left\{\int_0^t W^n(\xi(s),s)\,ds\right\}\right) = \mathbb{E}_\nu\left(\exp\left\{\int_0^t W(\xi(s),s)\,ds\right\}\right), \tag{7.44}$$

it suffices to show that

$$\lim_{n\to\infty} \Gamma_s^{L,n} = \Gamma_s^L, \qquad s \ge 0. \tag{7.45}$$

2. To show that the left-hand side in (7.45) is an upper bound, fix $\varepsilon > 0$ and pick $f \in D(L) \cap L^2(|W(\cdot,t)|d\nu)$ such that

$$\langle Lf,f\rangle + \langle W(\cdot,t),f^2\rangle + \varepsilon \ge \Gamma_s^L. \tag{7.46}$$

Then, by monotone convergence,

$$\lim_{n\to\infty} \left\{\langle Lf,f\rangle + \langle W^n(\cdot,t),f^2\rangle\right\} = \langle Lf,f\rangle + \langle W(\cdot,t),f^2\rangle. \tag{7.47}$$

3. To prove that the left-hand side in (7.45) is also a lower bound, we need to assume that L is self-adjoint. Then, by Wu [14], Remark on p. 209, we have

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}_\nu\left(\exp\left\{\int_0^t W^n(\xi(s),0)\,ds\right\}\right) = \Gamma_0^{L,n}, \tag{7.48}$$

and the same is true when W^n is replaced by W. But, obviously,

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}_\nu\left(\exp\left\{\int_0^t W^n(\xi(s),0)\,ds\right\}\right) \le \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}_\nu\left(\exp\left\{\int_0^t W(\xi(s),0)\,ds\right\}\right), \tag{7.49}$$

which shows that $\Gamma_0^{L,n} \le \Gamma_0^L$. Note that the time point 0 does not play any special role. Hence we obtain (7.45). ∎

Proposition 7.3. *Suppose that ξ satisfies* (7.37) *and condition (ii) in Lemma 7.2. Then, for all paths Φ (recall* (1.27)*),*

$$\mathbb{P}_\nu\left(\frac{1}{n(T-1)}\sum_{j=1}^{n}\int_{(j-1)T+1}^{jT}\big[\xi(\Phi(s),s)\,ds-\rho\big]>\delta\right)\leq\exp\big\{-n(T-1)I_{V_0}(\rho+\delta)\big\},\quad(7.50)$$

where $V_0(\eta)=\eta(0)$ and $\rho=\mathbb{E}(\xi(0,0))$.

Proof. First note that by Lemma 7.2,

$$\mathbb{E}_\nu\left(\exp\left\{\sum_{j=1}^{n}\int_{(j-1)T+1}^{jT}\xi(\Phi(s),s)\,ds\right\}\right)\leq\exp\big\{n(T-1)\Gamma_0^L\big\},\tag{7.51}$$

where Γ_0^L is defined with $W(\cdot,0)=V_0$. To see that, take a path Φ which starts in zero and define $W\colon\,\mathbb{R}^{\mathbb{Z}^d}\times[0,\infty)\to\mathbb{R}$ by

$$W(\eta,t)=\begin{cases}\eta(\Phi(t)), & \text{if } t\in[(j-1)T+1,jT) \text{ for some } 1\leq j\leq n,\\ 0, & \text{otherwise.}\end{cases}\tag{7.52}$$

Let $x\in\mathbb{Z}^d$ be such that $\Phi(t)=x$, and denote by τ_x the space-shift over x. Then, using the fact that ν is shift-ergodic, we get

$$\begin{aligned}\langle W(\cdot,t),f^2\rangle&=\int_{\mathbb{R}^{\mathbb{Z}^d}}\eta(x)f^2(\eta)\,d\nu(\eta)\\&=\int_{\mathbb{R}^{\mathbb{Z}^d}}\tau_x\eta(0)f^2(\eta)\,d\nu(\eta)\\&=\int_{\mathbb{R}^{\mathbb{Z}^d}}\eta(0)(\tau_{-x}f)^2(\eta)\,d\nu(\eta)=\langle W(\cdot,0),(\tau_{-x}f)^2\rangle,\end{aligned}\tag{7.53}$$

which yields $\Gamma_t^L=\Gamma_0^L$. Since the space point 0 does not play any special role, Lemma 7.2 leads to (7.51) for any path Φ. Next apply the Chebyshev inequality to the left-hand side of (7.50). After that it remains to solve an optimization problem. See Wu [15] for details. ∎

Note that, for a reversible dynamics, I_{V_0} is the large deviation rate function for the occupation time

$$T_t=\int_0^t\xi(0,s)\,ds,\qquad t\geq0.\tag{7.54}$$

We are now ready to give the proof of Corollary 1.19.

Proof. All three dynamics in (1)–(3) satisfy condition (a) in Definition 1.8. The proof of (b) below consists of an application of Proposition 7.3, combined with a suitable analysis of I_{V_0}.

(1) Redig and Völlering [13], Theorem 4.1, shows that for all $\delta_1>0$ there is a $\delta_2=\delta_2(\delta_1)$ such that

$$\mathbb{P}_\nu\left(\int_0^{nt}\xi(0,s)\,ds\geq\delta_1 nt\right)\leq e^{-\delta_2 nt}.\tag{7.55}$$

A straightforward extension of this result implies that condition (b) in Definition 1.8 is satisfied. All requirements in Theorem 1.15 are thus met.

(2) By Landim [11], Theorem 4.2, the rate function of the simple exclusion process is non-degenerate (i.e., it has a unique zero at ρ). Hence condition (b) in Definition 1.8 is satisfied. Thus, all requirements of Theorem 1.15 are met.

(3) By Cox and Griffeath [3], Theorem 1, the rate function for independent simple random walks is non-degenerate. Hence condition (b) in Definition 1.8 is satisfied. All requirements in Theorem 1.15 are thus met. ∎

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