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A Compound Poisson EOQ Model for Perishable Items with Intermittent High and Low Demand Periods

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Abstract

We consider a stochastic EOQ-type model, with demand operating in a two-state random environment. This environment alternates between exponentially distributed periods of high demand and generally distributed periods of low demand. The inventory level starts at some level q, and decreases according to different compound Poisson processes during the periods of high demand and of low demand. The inventory level is refilled to level q when level 0 is hit or when an expiration date is reached, whichever comes first. We determine various performance measures of interest, like the distribution of the time until refill, the expected amount of discarded material and of material held (inventory), and the expected values of various kinds of shortages. For a given cost/revenue structure, we can thus determine the long-run average profit.

Keywords: EOQ model; Perishable inventories; Outdatings; Unsatisfied demands; Regenerative process; Compound Poisson process

1 Introduction

Consider a stochastic EOQ (Economic Order Quantity) model in which demand occurs in a two-state random environment. This environment alternates between periods of high demand and periods of low demand according to a continuous-time semi-Markov process. The demand is represented by a compound Poisson process $\{Y_H(t),\ t \geq 0\}$ during high demand periods and another compound Poisson process $\{Y_L(t),\ t \geq 0\}$ during low demand

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periods. We assume that the mean increment per time unit during high demand periods is higher than during low demand periods, but this assumption is not necessary for the analysis. The high demand periods and the low demand periods alternate according to an alternating renewal process as follows: the high demand periods are independent and exponentially distributed random variables with rate μ and the low demand periods are *i.i.d.* random variables with some distribution G.

We denote by $\mathbf{Q} = \{Q(t) : t \geq 0\}$, with $Q(0) = 0$, the cumulative inventory demand process. $\mathbf{Q}$ initially increases according to $\mathbf{Y_H}$, subsequently according to $\mathbf{Y_L}$, etc. We assume that the goods under consideration are *perishable*, with a fixed expiration date. At some time t_0, the outdating or expiration time of the perishable goods, the remaining inventory is discarded and the buffer is instantaneously refilled with fresh perishable goods up to level q (all goods of the same batch have common shelf life). It is possible that the demand already reaches q before the expiration date t_0. In that case, an order of size q is placed (with negligible lead time), with the proviso that, if level zero is reached during a period of *low* demand, the buffer is refilled at the end of that low demand period. In either case, the buffer content process $X(t) := q - Q(t)$ now stochastically repeats itself, again starting at level q with a period of high demand. Thus the buffer content process is a *regenerative process* whose cycles start at moments of replenishments. In Remark 3 below we comment on the assumption to postpone the order until the begin of the next high demand period.

A controller might wish to maximize the long-run average profit, by controlling the order quantity q. Accordingly, he would wish to select the optimal q so as to properly balance revenues and costs. Revenues are earned by selling units. The costs are composed of setup costs, which are incurred each time an order is placed, holding costs, the costs for discarded units (outdating) and the costs for unsatisfied demand. We return to this in Section 8.

Remark 1. The two-state random environment model reflects situations in which the demand rate for a certain commodity undergoes periodically recurring changes. The demands for many goods change according to changes in the interest rate or due to fashion or other recurring seasonal/external effects. Models including a multi-state random environment can be suitable for such situations; the two-state case presented in this article could serve as a first approximation.

The rationale behind having exponentially distributed high demand periods is that low demand periods may represent some form of recession, whose starting times (in first approximation) may form a Poisson process. This yields exponentially distributed high demand periods. Recessions within recessions may give rise to extended low demand periods; it seems less natural to assume that low demand periods are exponentially distributed.

We determine various performance measures of interest, such as the distribution of the time until either shortage occurs or items are being discarded, the expected amount of discarded material and of material held (inventory), and the expectations of various kinds of shortage. That would allow one to analyze the following cost optimization problem: given the profit of selling one unit of stock, and given setup costs, holding costs, costs for discarding outdated units and costs for unsatisfied demand, choose q so as to maximize the long-run average profit. The study of this optimization problem does not fall in the scope of the present paper, but in Section 8 we indicate how it can be tackled numerically.

This is a companion paper of [4]. The latter paper considers a stochastic fluid EOQ model with a similar underlying semi-Markov process, the key difference to the present model being that demand there is constant, with a high demand rate β_H during high demand periods and a demand rate $\beta_L < \beta_H$ during low demand periods. We also refer to [4] for an extensive literature review. The case of compound Poisson demand was introduced in [2], but without a random environment. See also the surveys [6, 9, 13, 14, 17] on Perishable Inventory Systems (PIS). Our paper is closest to the subject area of [9], which focuses on the stochastic analysis of PIS that operate under certain heuristic control policies.

According to [9] continuous review inventory models can be classified into three categories: those without fixed ordering cost or lead times, those without fixed ordering cost having positive lead times, and those with fixed ordering cost (typically with zero lead times). The first category was originated by Graves [7], who assumed that items are continuously produced and perish after a deterministic time, and that demand follows a compound Poisson process with either a single-unit or an exponential demand at each arrival. The second category goes back to Pal [15], who investigated the performance of an $(S-1, S)$ control policy. The third category, originated by Weiss [18], is of relevance to our model; [5], [10], [11], [12] and [16] made significant contributions to models in this category. In particular, Lian, Liu and Neuts [11] consider discrete demand for items and perishability times that are either fixed (and known) or follow a phase-type distribution.

A large variety of inventory models is presented in detail in the monograph [19]. The stochastic models are based on point processes to represent the demand arrivals in a random environment. The fluid systems in [19] are deterministic EOQ models with the classical extensions such as planned backorders, limited capacity, quantity discounts, and imperfect quality. In the deterministic setting, time-varying demands are considered also, but without multiple order quantities.

Remark 2. If we let, in our compound Poisson model, the arrival rates tend to infinity, and take exponential demand sizes with rates that also go to infinity, such that the average increase rate equals β_H during high demand periods and β_L during low demand periods, then in the limit one arrives at the fluid EOQ model that was studied in [4]. Actually, both the fluid EOQ model of [4] and the compound Poisson EOQ model of the present paper are special cases of an EOQ model that alternates between two different non-decreasing Lévy demand processes. It would be interesting to provide an analysis for that, more general and more complicated, model.

The paper is organized as follows. Section 2 contains a detailed model description. We study the distribution of the total time in $(0,t)$ of high demand periods in Section 3. In Section 4 we use this to determine the distribution of $Q(t)$ and of the time until either complete stock depletion or stock discarding. In Section 5 we consider the expected amount of material discarded (because the deadline expires) and the expected amount of material held. Section 6 is devoted to determining the expected length of an inventory cycle; the mean shortages are derived in Section 7. Combining the various results from Sections 3-7 enables us to determine the expected profit per time unit. In Section 8 we present numerical examples, evaluating the formulae for cost functionals and exploring the effect of several parameters on the cost function.

2 The Model

Consider an inventory system filled with perishable items, which have to be discarded after t_0 time units. The demand follows intermittent periods of high demand (HD) followed by low demand (LD). Let $\{H_i, i = 1, 2, \ldots\}$ and $\{L_i, i \geq 1\}$ denote two independent sequences of i.i.d. positive random variables, representing the lengths of the HD-periods and of the LD-periods, respectively. The intermittent sequence of high and low demand periods are then represented by the alternating renewal process associated to $\{H_1, L_1, H_2, L_2, \ldots\}$. We denote by HD the union of all high demand periods. In the present paper we assume that $H_i \sim \text{Expo}(\mu)$ (exponential distribution with mean $1/\mu$) while the L_i have a general common absolutely continuous distribution G, with density g.

To describe the demand process we need *compound Poisson processes* (CPP(λ, F))

$$Y(t) = \sum_{n=0}^{N(t)} X_n, \quad X_0 \equiv 0, \tag{2.1}$$

where $\{N(t), t \geq 0\}$ is an ordinary Poisson process with intensity λ and $\{X_n, n \geq 1\}$ is a sequence of i.i.d. positive random variables with common distribution function F and independent of $\{N(t)\}$. We denote by $Y_H^{(i)}(t)$, $i \geq 1$, independent processes of the type CPP(λ_H, F_H) giving the total quantity demanded during the i-th HD-period (where t is the time elapsed since the beginning of that period). Similarly, let $Y_L^{(i)}(t)$, $i \geq 1$, be independent processes of type CPP(λ_L, F_L), also independent from the $Y_H^{(i)}(t)$, giving the total quantity demanded during the first time units of the i-th LD-period. Of course the pairs (λ_H, F_H) and (λ_L, F_L) of intensities and distribution functions can be different. Let $Q(t)$ denote the total demand up to time t. Formally, let $M(t)$ be the number of completed HD-periods in $(0, t)$. Then $Q(t) = Y_H^{(1)}(t)$ if $M(t) = 0$, while if $M(t) \geq 1$ and $t \in$ HD,

$$Q(t) = \sum_{i=1}^{M(t)} [Y_H^{(i)}(H_i) + Y_L^{(i)}(L_i)] + Y_H^{(M(t)+1)}\Big(t - \sum_{i=1}^{M(t)} [H_i + L_i]\Big)$$

and if $M(t) \geq 1$, $t \in (0, \infty) \backslash \text{HD}$,

$$\begin{aligned} Q(t) = & \sum_{i=1}^{M(t)-1} [Y_H^{(i)}(H_i) + Y_L^{(i)}(L_i)] + Y_H^{(M(t))}(H_{M(t)}) \\ & + Y_L^{(M(t))}\Big(t - \sum_{i=1}^{M(t)-1} [H_i + L_i] - H_{M(t)}\Big). \end{aligned}$$

In Figure 1 we display a possible sample path of $Q(t)$.

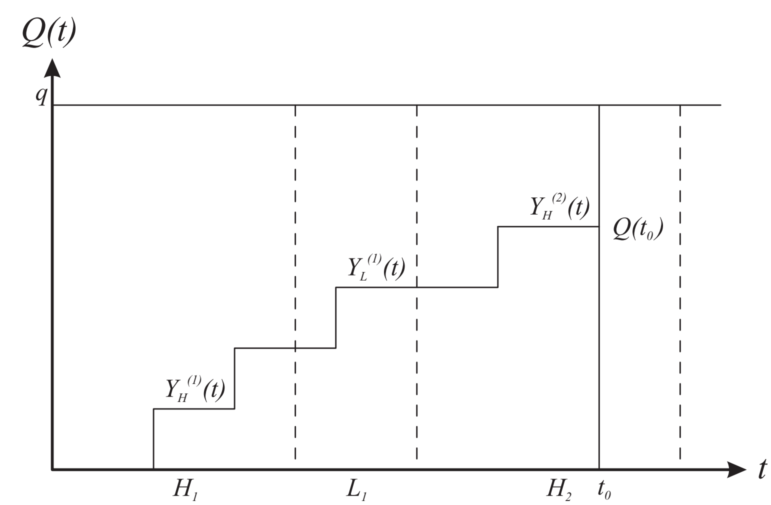

Figure 1. **A possible sample path of** $Q(t)$

In the present illustration the total demand at t_0, $Q(t_0)$, is smaller than q. The quantity *discarded* is $q - Q(t_0)$. q is the amount of material replenished; i.e. the stock level at any replenishment time.

Define the stopping times

$$\tau = \inf\{t > 0 : Q(t) = q\}, \tag{2.2}$$

and

$$\tau^* = \min\{\tau, t_0\}. \tag{2.3}$$

τ^* is the instant of complete stock depletion or stock discarding. Stock replenishment takes place at τ^*, if τ^* is in an HD-period. If τ^* falls in an LD-period, replenishment is deferred till the end of this LD-period. Accordingly, the inventory cycle has length

$$C = \begin{cases} \tau^*, & \text{if } \tau^* \in \text{HD} \\ \tau^* + R, & \text{if } \tau^* \notin \text{HD} \end{cases} \tag{2.4}$$

where R is the remaining time, after τ^*, in LD.

Remark 3. The above control policy guarantees that the cycle always starts with an HD-period. The following example shows that it is often natural to wait for the beginning of the next high demand period before ordering. HD-periods will typically be periods in which trade is relatively brisk. If the holding costs are relatively high compared to the costs of unsatisfied demand, entrepreneurs will not start new initiatives during LD-periods, since if they place orders during LD-periods the content level will stochastically increase and the holding cost component will cause the long-run average costs to go up. If they wait till the beginning of the next HD-period, the costs due to lost sales are low and the holding costs will also be low, since the content level will be depleted quickly. Hence in this case it may be worthwhile to pay the costs of unsatisfied demands by waiting with the content at level 0 rather than placing an order and for a long time paying considerable holding costs while the content level is near its maximum q.

Let $W(t)$ denote the total time in $(0, t)$ spent in HD-periods, i.e.,

$$W(t) = \int_0^t 1_{\{s \in \mathrm{HD}\}}(s) ds. \tag{2.5}$$

Since $Y_L^{(i)}(t)$ and $Y_H^{(i)}(t)$, $i = 1, 2, \ldots$, are independent stochastic copies of Lévy processes and independent of the process $W(t)$, the strong Markov property yields the distributional equality

$$Q(t) =_d Y_H(W(t)) + Y_L(t - W(t)), \tag{2.6}$$

where $Y_H(t) = Y_H^{(1)}(t)$ and $Y_L(t) = Y_L^{(1)}(t)$. In the following sections we present the distribution of $W(t)$, $Q(t)$, τ^*, R, the expected quantity of discarded material $ED(q; t_0) = q - EQ(\tau^*)$, and the expected total holding quantity.

3 The Distribution of $W(t)$

For $w > 0$, let $N^*(w) = \max\{n \geq 0 : \sum_{j=0}^{n} H_j \leq w\}$, $H_0 \equiv 0$. Since $\{H_j\}$ are Expo(μ), $\{N^*(w), w \geq 0\}$ is an ordinary Poisson process, with intensity μ. Consider the CPP(μ, G) given by

$$Y^*(w) = \sum_{n=0}^{N^*(w)} L_n, \quad L_0 \equiv 0. \tag{3.1}$$

The c.d.f. of $Y^*(w)$ is

$$H^*(y; w) = \sum_{n=0}^{\infty} p(n; \mu w) G^{(n)}(y), \tag{3.2}$$

where $p(n; \eta) = e^{-\eta} \eta^n / n!$ denotes the Poisson p.d.f. with mean η. Notice that $H^*(0; w) = e^{-\mu w}$ (atom of (3.2)). We denote by $h^*(y; w)$ the p.d.f. of $Y^*(w)$, for $0 < y < \infty$, namely:

$$h^*(y; w) = \sum_{n=1}^{\infty} p(n; \mu w) g^{(n)}(y). \tag{3.3}$$

Upon reflection (see Figure 2) we realize that $W(t)$ is the stopping time

$$W(t) = \inf\{w > 0 : Y^*(w) \geq t - w\}. \tag{3.4}$$

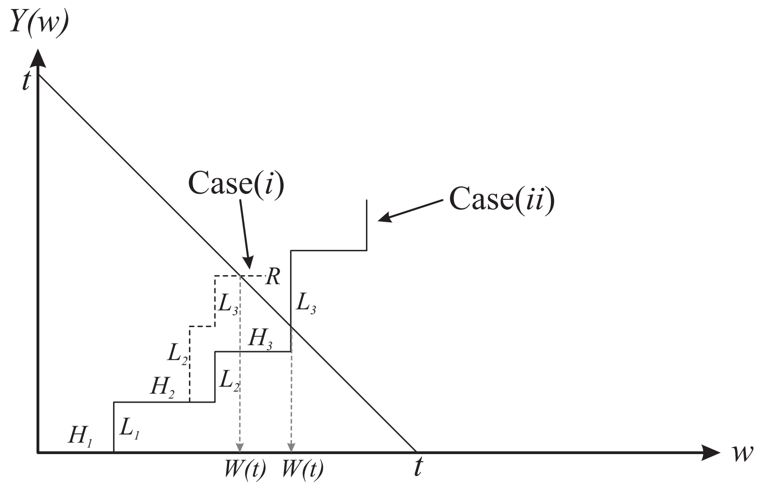

Figure 2. **The Process** $Y^*(w)$ **and** $W(t)$

Since $Y^*(w)$ is non-decreasing with probability 1, (3.4) yields, for $0 < w < t$,

$$P\{W(t) > w\} = H^*(t - w; w); \tag{3.5}$$

also

$$P\{W(t) = t\} = e^{-\mu t}. \tag{3.6}$$

As shown in Figure 2, if $W(t) = w$ and $Y^*(w) > t - w$, the alternating renewal process (ARP) is at an LD-period at t; and if $Y^*(w) = t - w$ the ARP is at an HD-period at t.

The p.d.f. of $W(t)$, on $(0, t)$, is $\psi_{W(t)}(w; t) = -\frac{d}{dw} H^*(t - w; w)$, which is

$$\begin{aligned} \psi_{W(t)}(w; t) &= \mu e^{-\mu w} + \mu \sum_{n=1}^{\infty} (p(n; \mu w) - p(n-1; \mu w)) G^{(n)}(t - w) \\ &+ \sum_{n=1}^{\infty} p(n; \mu w) g^{(n)}(t - w), \quad 0 < w < t. \end{aligned} \tag{3.7}$$

Thus the density of $W(t)$ can be written as a sum of two components, i.e. $\psi_{W(t)}(w; t) = \psi^{(\mathrm{LD})}_{W(t)}(w; t) + \psi^{(\mathrm{HD})}_{W(t)}(w; t)$ where

$$\psi^{(\mathrm{LD})}_{W(t)}(w; t) = \mu e^{-\mu w} + \mu \sum_{n=0}^{\infty} (p(n; \mu w) - p(n-1; \mu w)) G^{(n)}(t - w) \tag{3.8}$$

and

$$\psi^{(\mathrm{HD})}_{W(t)}(w; t) = h^*(t - w; w). \tag{3.9}$$

From the Bayes theorem we obtain

$$P\{t \in \mathrm{LD} \mid W(t) = w\} = \frac{\psi^{(\mathrm{LD})}_{W(t)}(w;t)}{\psi_{W(t)}(w;t)}, \quad 0 < w < t. \tag{3.10}$$

4 The Distribution of $Q(t)$ and τ^*

Let $H^{(i)}(y;t)$ and $h^{(i)}(y,t)$ be the c.d.f. and p.d.f. of $Y_i(t)$, $i \in \{L, H\}$. These are

$$H^{(i)}(y;t) = \sum_{n=0}^{\infty} p(n;\lambda_i t) F_i^{(n)}(y), \quad y \geq 0, \tag{4.1}$$

and

$$h^{(i)}(y;t) = \sum_{n=1}^{\infty} p(n;\lambda_i t) f_i^{(n)}(y), \quad y > 0. \tag{4.2}$$

The *conditional* c.d.f. of $Y_H(W(t))$, given $\{W(t) = w\}$, $0 \leq w \leq t$, is $H^{(H)}(y;w)$, and that of $Y_L(t - W(t))$ is $H^{(L)}(y; t-w)$.

4.1 Distribution of $Q(t)$

The conditional distribution of $Q(t)$, given $W(t) = w$ is, according to (2.6),

$$\begin{aligned} H_Q(y;w,t) &= I(y=0, w<t)e^{-\lambda_H w - \lambda_L(t-w)} \\ &\quad + I(0<y, 0<w<t)[e^{-\lambda_H w} H^{(L)}(y;t-w) \\ &\quad + \int_0^y h^{(H)}(x;w) H^{(L)}(y-x;t-w)dx] \\ &\quad + I(w=t)e^{-\mu t} H^{(H)}(y;t). \end{aligned} \tag{4.3}$$

Notice that the first term of (4.3) is $P\{Q(t) = 0 \mid W(t) = w\}$.

The corresponding conditional density of $Q(t)$, for $0 < y$, is for $W(t) = w$,

$$\begin{aligned} h_Q(y;w,t) &= e^{-\lambda_H w} h^{(L)}(y;t-w) + e^{-\lambda_L(t-w)} h^{(H)}(y;w) \\ &\quad + \int_0^y h^{(H)}(x;w) h^{(L)}(y-x;t-w)dx. \end{aligned} \tag{4.4}$$

We obtain

Theorem 4.1. *The c.d.f. of $Q(t)$ is*

$$H_Q^*(y;t) = e^{-\mu t} H^{(H)}(y;t) + \int_0^t \psi_{W(t)}(w;t) H_Q(y;w,t) dw. \tag{4.5}$$

The corresponding density is, for $y > 0$,

$$h_Q^*(y;t) = e^{-\mu t} h^{(H)}(y;t) + \int_0^t \psi_{W(t)}(w;t) h_Q(y;w,t) dw. \tag{4.6}$$

The m-th moment of the truncated cumulative demand $Q_q(t) = \min(q, Q(t))$ is

$$E\{Q_q^m(t)\} = q^m - m \int_0^q x^{m-1} H_Q^*(x;t) dx, \quad m \geq 1. \tag{4.7}$$

□

4.2 The Distribution of τ^*

Since $Q(t)$ is a non-decreasing process we have

$$P\{\tau^* > t\} = H_Q^*(q;t), \quad 0 < t < t_0 \tag{4.8}$$

and

$$P\{\tau^* = t_0\} = H_Q^*(q;t_0). \tag{4.9}$$

According to (4.3) and (4.5),

$$\begin{aligned} H_Q^*(q;t) &= e^{-\mu t} H^{(H)}(q;t) \\ &\quad + \int_0^t \psi_{W(t)}(w;t) \Big[e^{-\lambda_H w} H^{(L)}(q;t-w) \\ &\quad + \int_0^q h^{(H)}(x;w) H^{(L)}(q-x;t-w) dx \Big] dw. \end{aligned} \tag{4.10}$$

Let $p_{\tau^*}(t;q)$ denote the density of τ^*, for $0 < t < t_0$. Since $p_{\tau^*}(t;q) = -\frac{d}{dt} P\{\tau^* > t\}$, for $0 < t < t_0$, inserting (4.3) yields, for $0 < t < t_0$,

$$\begin{aligned} p_{\tau^*}(t;q) &= \mu e^{-\mu t} H^{(H)}(q;t) + e^{-\mu t} \left(-\frac{\partial}{\partial t} H^{(H)}(q;t) \right) \\ &\quad + \int_0^t \left(-\frac{\partial}{\partial t} \psi_{W(t)}(w;t) \right) H_Q(q;w,t) dw \\ &\quad + \int_0^t \psi_W(w;t) \left(-\frac{\partial}{\partial t} H_Q(q;w,t) \right) dw. \end{aligned} \tag{4.11}$$

Since $\frac{d}{dt}G^{(n)}(t-w) = g^{(n)}(t-w)$ and

$$\frac{d}{dt}g^{(n)}(t-w) = \int_0^{t-w} g^{(n-1)}(x)\frac{d}{dt}g(t-w-x)dx \tag{4.12}$$

we obtain

Lemma 4.1.

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{W(t)}(w;t) = \mu\sum_{n=1}^{\infty}(p(n;\mu w) - p(n-1;\mu w)) \,\cdot \\ \cdot\; g^{(n)}(t-w) + \sum_{n=1}^{\infty}p(n;\mu w)\left(\frac{d}{dt}g^{(n)}(t-w)\right).\end{aligned} \tag{4.13}$$

□

As an example, if $G(t) = 1 - e^{-\zeta t}$, $t \geq 0$ (exponential case), then $g^{(n)}(t) = \zeta p(n-1;\zeta t)$ and

$$\begin{aligned}-\frac{\partial}{\partial t}\psi_{W(t)}(w;t) = \mu\zeta\sum_{n=1}^{\infty}(p(n-1;\mu w) - p(n;\mu w))p(n-1;\zeta(t-w)) \\ - \zeta^2\sum_{n=1}^{\infty}p(n;\mu w)(p(n-2;\zeta(t-w)) - p(n-1;\zeta(t-w))),\end{aligned} \tag{4.14}$$

where $p(-1;\cdot) \equiv 0$.

We derive now a formula for $\frac{\partial}{\partial t}H_Q(y;w,t)$. We start with

Lemma 4.2. *For $y > 0$, if $H(y;t)$ is the c.d.f. of a CPP(λ, F), then*

$$-\frac{\partial}{\partial t}H(y;t) = \lambda e^{-\lambda t}\bar{F}(y) + \lambda\int_0^y h(x;t)\bar{F}(y-x)dx, \tag{4.15}$$

where $h(x;t)$ is the corresponding p.d.f. and $\bar{F}(\cdot) = 1 - F(\cdot)$.

Proof. For the CPP(λ, F),

$$H(y;t) = \sum_{n=0}^{\infty}p(n;\lambda t)F^{(n)}(y),$$

and

$$h(y;t) = \sum_{n=1}^{\infty}p(n;\lambda t)f^{(n)}(y).$$

Thus,

$$
(4.16)\quad
\begin{aligned}
-\frac{\partial}{\partial t}H(y;t) &= \sum_{n=0}^{\infty}\left(-\frac{\partial}{\partial t}p(n;\lambda t)\right)F^{(n)}(y) \\
&= \lambda e^{-\lambda t} + \lambda\sum_{n=1}^{\infty}(p(n;\lambda t) - p(n-1;\lambda t))F^{(n)}(y) \\
&= \lambda\sum_{n=0}^{\infty}p(n;\lambda t)F^{(n)}(y) - \lambda\sum_{n=0}^{\infty}p(n;\lambda t)F^{(n+1)}(y) \\
&= \lambda e^{-\lambda t}\bar{F}(y) + \lambda\sum_{n=1}^{\infty}p(n;\lambda t)[F^{(n)}(y) - F^{(n+1)}(y)].
\end{aligned}
$$

Moreover

$$
(4.17)\quad
\begin{aligned}
F^{(n)}(y) - F^{(n+1)}(y) &= \int_0^y f^{(n)}(x)(1 - F(y-x))dx \\
&= \int_0^y f^{(n)}(x)\bar{F}(y-x)dx.
\end{aligned}
$$

Substituting (4.17) in (4.16) we get (4.15). □

Remark 4. A probabilistic interpretation of (4.15) is readily obtained when one realizes the following. $H(y;t)$ not only is the probability that the CPP(λ, F) at t is below y, but also the probability that a crossing of level y has only taken place after t. Minus the derivative of the latter probability gives the density of the crossing time of level y, which is easily seen to be the righthand side of (4.15).

We use this lemma in the following derivation. First, according to (4.15)

$$
(4.18)\quad -\frac{\partial}{\partial t}H^{(H)}(q;t) = \lambda_H e^{-\lambda_H t}\bar{F}_H(q) + \lambda_H\int_0^q h^{(H)}(x;t)\bar{F}_H(q-x)dx.
$$

Moreover, by (4.10)

$$
(4.19)\quad -\frac{\partial}{\partial t}H^{(L)}(q;t-w) = \lambda_L e^{-\lambda_L(t-w)}\bar{F}_L(q) + \lambda_L\int_0^q h^{(L)}(u;t-w)\bar{F}_L(q-u)du.
$$

Collecting terms we get, for $w < t$,

Lemma 4.3. *For $0 < w < t < t_0$*

$$
\begin{aligned}
-\frac{\partial}{\partial t} H_Q(q; w, t) &= \lambda_L e^{-\lambda_H w - \lambda_L (t-w)} \bar{F}_L(q) \\
&\quad + \lambda_L e^{-\lambda_H w} \int_0^q h^{(L)}(x; t-w) \bar{F}_L(q-x) dx \\
&\quad + \lambda_L e^{-\lambda_L (t-w)} \int_0^q h^{(H)}(x; w) \bar{F}_L(q-x) dx \\
&\quad + \lambda_L \int_0^q h^{(H)}(x; w) \int_0^{q-x} h^{(L)}(u; t-w) \bar{F}_L(q-x-u) du\, dx.
\end{aligned}
\tag{4.20}
$$

□

An explicit formula for $p_{\tau^*}(t; q)$ is obtained by substituting (4.13), (4.18) and (4.20) into (4.11).

Theorem 4.2. *The m-th moment of τ^* is*

$$
E\{(\tau^*)^m\} = m \int_0^{t_0} t^{m-1} H_Q^*(q; t) dt, \quad m = 1, 2, \ldots \tag{4.21}
$$

Proof.

$$
E\{(\tau^*)^m\} = t_0^m P\{\tau^* = t_0\} + \int_0^{t_0^-} t^m dP\{\tau^* \le t\}. \tag{4.22}
$$

Substituting $t^m = m \int_0^t y^{m-1} dy$ in the second term of (4.22) we get, after changing the order of integration,

$$
E\{(\tau^*)^m\} = t_0^m P\{\tau^* = t_0\} + m \int_0^{t_0} y^{m-1} \int_y^{t_0} dP\{\tau^* \le t\} dy. \tag{4.23}
$$

Notice that

$$
\begin{aligned}
\int_y^{t_0} dP\{\tau^* \le t\} &= P\{\tau^* < t_0\} - P\{\tau^* \le y\} \\
&= (-(1 - P\{\tau^* < t_0\}) + 1 - P\{\tau^* \le y\} \\
&= -P\{\tau^* = t_0\} + H_Q^*(q; y).
\end{aligned}
\tag{4.24}
$$

Substituting (4.24) in (4.23) we get (4.21). □

From the above theorem we get, as a special case,

$$
E\{\tau^*\} = \int_0^{t_0} H_Q^*(q; t) dt. \tag{4.25}
$$

Notice that $E\{\tau^*\}$ is an increasing function of q with a limit t_0.

5 The Expected Amount of Discarded Material and The Expected Total Amount of Material Held

Material is discarded only if $Q(t_0) < q$, which is the case when $\tau^* = t_0$.

Theorem 5.1. *The expected amount of discarded material is*

$$E\{D(q)\} = \int_0^q H_Q^*(x; t_0)dx. \tag{5.1}$$

Proof.

$$E\{D(q)\} = \int_0^q P\{waste\ at\ t_0\ > q - x\}dx = \int_0^q H_Q^*(q - x; t_0)dx = \int_0^q H_Q^*(x; t_0)dx. \tag{5.2}$$

□

The expected total amount held during an inventory cycle is

$$\begin{aligned} E\{T(\tau^*)\} &= qt_0 P\{Q(t_0) = 0\} \\ &+ \int_0^{t_0} p_{\tau^*}(t; q) \int_0^t E\{q - Q(s) \mid \tau^* = t\}ds\, dt \\ &+ \int_0^q h_Q^*(y; t_0) \int_0^{t_0} E\{q - Q(s) \mid Q(t_0) = y\}dy\, ds. \end{aligned} \tag{5.3}$$

In order to compute (5.3) we have to develop formulae for $E\{Q(s) \mid \tau^* = t\}$ when $0 < t < t_0$, and for $E\{Q(s) \mid Q(t_0) = y\}$ for $0 < s < t_0$. Notice that

$$\int_0^{t_0} p_{\tau^*}(t; q) \int_0^t q\, ds\, dt + \int_0^q h_Q^*(y; t_0) \int_0^{t_0} q\, ds\, dy = qE\{\tau^*\}. \tag{5.4}$$

The conditional expectation of $Q(s)$, given $\tau^* = t$, for $s < t < t_0$ is

$$E\{Q(s) \mid \tau^* = t\} = \frac{\int_0^q y h_Q^*(y; s) h_Q^*(q - y; t - s)dy}{\int_0^q h_Q^*(y; s) h_Q^*(q - y; t - s)dy}. \tag{5.5}$$

Similarly, for $0 < s < t_0$

$$E\{Q(s) \mid Q(t_0) = y\} = \frac{\int_0^y x h_Q^*(x; s) h_Q^*(y - x; t_0 - s)dx}{\int_0^y h_Q^*(x; s) h_Q^*(y - x; t_0 - s)dx}. \tag{5.6}$$

These functions are substituted in (5.3) to obtain the expected total amount held, $E\{T(\tau^*)\}$.

6 The Expected Length of an Inventory Cycle

The inventory cycles are the periods between replenishing epochs. If $\tau^* \in \text{HD}$, replenishing is done at τ^*. On the other hand, if $\tau^* \in \text{LD}$ period, replenishing takes place at the beginning of the next HD-period. That is, if C denotes the length of an inventory cycle then

$$C = \begin{cases} \tau^*, & \text{if } \tau^* \in \text{HD} \\ \tau^* + R, & \text{if } \tau^* \in \text{LD}, \end{cases} \tag{6.1}$$

where R is the remaining length of the LD-period, after stopping. The expected length of an inventory cycle is thus

$$E\{C\} = E\{\tau^*\} + E\{R, \tau^* \in \text{LD}\}. \tag{6.2}$$

We have to derive a formula for $E\{R, \tau^* \in \text{LD}\}$.

Generally, for a renewal process $\{V_i\}_{i=1}^{\infty}$, if G is the c.d.f. of V, then the remaining length of the last renewal cycle, at time t, has the c.d.f. (see p. 109 of [8])

$$F_R(x;t) = G(t+x) - \int_0^t \bar{G}(t+x-y)m(y)dy, \tag{6.3}$$

where $\bar{G}(\cdot) = 1 - G(\cdot)$ and $m(y)$ is the renewal density

$$m(y) = \sum_{n=1}^{\infty} g^{(n)}(y), \quad y > 0. \tag{6.4}$$

Of course, if G is exponential with rate ζ, one has $m(y) \equiv \zeta$ and it is immediately verified that $F_R(x;t)$ is also exponential with rate ζ.

In the ARP, the conditional distribution of R, given $\{t \in \text{LD}\}$ and $\{W(t) = w\}$ is

$$P\{R \le x \mid t \in \text{LD}, W(t) = w\} = F_R(x; t-w). \tag{6.5}$$

Accordingly,

$$\begin{aligned} E\{R \mid t \in \text{LD}, W(t) = w\} &= \int_0^{\infty} \bar{F}_R(x; t-w)dx \\ &= \int_0^{\infty} \left[\bar{G}(t+x-w) + \int_0^{t-w} \bar{G}(t+x-w-y)m(y)dy \right] dx. \end{aligned} \tag{6.6}$$

Finally, we obtain from (4.7) and (6.6),

Theorem 6.1.

$$E\{R,\tau^* \in LD\} = \int_0^{t_0}\left(\int_0^t E\{R \mid t \in LD, W(t)=w\}\cdot \cdot\ \psi_{W(t)}^{(LD)}(w;t)\cdot\ h_Q(q;w,t)dw\right)dt + \int_0^{t_0} E\{R \mid t_0 \in LD, W(t_0)=w\}H_Q(q;w,t_0)\psi_{W(t_0)}^{(LD)}(w,t_0)dw. \tag{6.7}$$

7 Mean Shortages

There are three kinds of shortages in the present problem

(i) First kind of shortage, Sh_1, is the one when $\tau^* < t_0$ and $\tau^* \in \mathrm{HD}$. In this case

$$\mathrm{Sh}_1 = (Q(\tau^*) - q)I(\tau^* < t_0, \tau^* \in \mathrm{HD}). \tag{7.1}$$

(ii) Second kind of shortage occurs when $\tau^* < t_0$ and $\tau^* \in \mathrm{LD}$. In this case

$$\mathrm{Sh}_2 = (Q(\tau^*) - q + Y_L(R))I(\tau^* < t_0, \tau^* \in \mathrm{LD}). \tag{7.2}$$

(iii) The third kind of shortage occurs when $\tau^* = t_0$ and $\tau^* \in$ LD. In this case

$$\mathrm{Sh}_3 = Y_L(R)I(\tau^* = t_0 \in \mathrm{LD}). \tag{7.3}$$

Theorem 7.1. *The expected value of* Sh_3 *is*

$$E\{\mathrm{Sh}_3\} = \lambda_L\xi_L E\{R, \tau^* = t_0 \in LD\}, \tag{7.4}$$

where $E\{R,\tau^* = t_0 \in LD\}$ is the second term on the right hand side of (6.7), and $\xi_L = \int_0^\infty x dF_H(x)$.

Proof. First, since the process $Y_L(t)$ is conditionally independent of $W(t)$,

$$E\{Y_L(R) \mid R\} = \lambda_L\xi_L R. \tag{7.5}$$

Accordingly

$$E\{Y_L(R)I(\tau^* = t_0 \in \mathrm{LD})\} = \lambda_L\xi_L E\{R, \tau^* = t_0 \in \mathrm{LD}\} = \lambda_L\xi_L \int_0^{t_0} E\{R \mid t_0 \in \mathrm{LD}, W(t_0) = w\}H_Q(q;w,t_0)\psi_{W(t)}^{(\mathrm{LD})}(w;t_0)dw. \tag{7.6}$$

□

We develop now the formula for $E\{\mathrm{Sh}_1\}$ and $E\{\mathrm{Sh}_2\}$. Recall that $Q(\tau^*) - q$ when $W(\tau^*) = w$, is the overshoot of $Y_H(w)$ at $\{\tau^* \in \mathrm{HD}\}$. The joint density of τ^* and $S = Q(\tau^*) - q$, given $W(t) = w$ and $\{\tau^* \in \mathrm{HD}\}$ is given in (7.7).

Let $h_Q(y; w, t) = \dfrac{d}{dy} H_Q(y; w, t)$. Then, with f_H (f_L) denoting the density of F_H (F_L),

$$p_{\tau^*,S}(t, s \mid \tau^* \in \mathrm{HD}, W(t) = w) = \lambda_H e^{-\lambda_H w - \lambda_L (t-w)} f_H(q+s) + \lambda_H \int_0^q h_Q(y; w, t) f_H(q+s-y) dy. \tag{7.7}$$

Lemma 7.1.

$$\int_0^\infty s f_H(q+s) ds = \int_q^\infty \bar{F}_H(u) du. \tag{7.8}$$

Proof.

$$\int_0^\infty s f_H(q+s) ds = \int_q^\infty (u-q) f_H(u) du = \int_q^\infty f_H(u) \int_z^q dy du = \int_q^\infty \bar{F}_H(u) du. \tag{7.9}$$

□

Thus,

$$E\{S \mid \tau^* \in \mathrm{HD}, W(t) = w\} = \lambda_H e^{-\lambda_H w - \lambda_L (t-w)} \int_q^\infty \bar{F}_H(u) du + \lambda_H \int_0^q h_Q(y; w, t) \int_{q-y}^\infty \bar{F}_H(u) du dy. \tag{7.10}$$

Finally, integrating (7.10) with respect to $W(t)$ and t we get

Theorem 7.2.

$$\begin{aligned} E\{\mathrm{Sh}_1\} &= E\{SI\{\tau^* \in HD\}\} \\ &= \left[\int_q^\infty \bar{F}_H(u) du\right] \left[\frac{\lambda_H}{\lambda_H + \mu}(1 - e^{-(\lambda_H + \mu) t_0}) \right. \\ &\left. + \lambda_H \int_0^{t_0} \int_0^t e^{-\lambda_H w - \lambda_L (t-w)} \psi_{W(t)}^{(HD)}(w; t) dw dt \right] \\ &+ \lambda_H \int_0^q \left(\int_0^{t_0} \int_0^t h_Q(y; w, t) \psi_{W(t)}^{(HD)}(w; t) dw dt \right) \cdot \\ &\cdot \left(\int_{q-y}^\infty \bar{F}_H(u) du \right) dy. \end{aligned} \tag{7.11}$$

Similarly we get

Theorem 7.3.

$$
\begin{aligned}
E\{\mathrm{Sh}_2\} = \lambda_L & \left(\int_q^\infty \bar{F}_L(u)du\right) \cdot \\
& \cdot \int_0^{t_0}\int_0^t e^{-\lambda_H w - \lambda_L(t-w)} \psi_{W(t)}^{(LD)}(w,t)dwdt \\
& + \lambda_L \int_0^q \left(\int_0^{t_0}\int_0^t h_Q(y;w,t)\psi_{W(t)}^{(LD)}(w,t)dwdt\right) \cdot \\
& \cdot \left(\int_{q-y}^\infty \bar{F}_L(u)du\right) dy \\
& + \lambda_L \xi_L \int_0^{t_0}\int_0^t E\{R \mid t \in LD, W(t)=w\}\psi_{W(t)}^{(LD)}(w;t)dwdt.
\end{aligned} \tag{7.12}
$$

8 Cost Functionals for Exponential Distributions and Numerical Examples

In the present section we specialize the formulae for the various cost functionals assuming that

$$G(t) = 1 - e^{-\zeta t}, \quad t \geq 0$$

$$F_i(t) = 1 - e^{-\kappa_i t}, \quad t \geq 0,$$

where $\kappa_i = \dfrac{1}{\xi_i}$, $i \in \{L, H\}$. The main purposes of the section are to show how the formulae for the various key performance indicators simplify in this case, and how one may evaluate them numerically. At the end of the section we provide some tables with numerical values of cost functionals. In principle, one may thus perform optimization; e.g., one may determine the q value that maximizes profit. That is outside the scope of the present paper; we refer the interested reader to [4], where considerable attention has been given to the profit optimization problem in the fluid demand case.

The c.d.f. of $Y_i(t)$ is

$$H^{(i)}(y;t) = \sum_{j=0}^{\infty} p(j;\kappa_i y)P(j;\lambda_i t), \quad i = L, H \tag{8.1}$$

and

$$H^*(y;t) = \sum_{j=0}^{\infty} p(j;\zeta y) P(j;\mu t). \tag{8.2}$$

Lemma 8.1. *The p.d.f. of $H^*(y,t)$ is for $y > 0$*

$$h^*(y,t) = \zeta \sum_{j=0}^{\infty} p(j;\zeta y) p(j+1;\mu t). \tag{8.3}$$

Proof.

$$\begin{aligned} h^*(y;t) &= \frac{d}{dy} H^*(y;t) \\ &= \zeta \left[-\sum_{j=0}^{\infty} p(j;\zeta y) P(j;\mu t) + \sum_{j=1}^{\infty} p(j-1;\zeta y) P(j;\mu t) \right] \\ &= \zeta \left[\sum_{j=0}^{\infty} p(j;\zeta y)(P(j+1;\mu t) - P(j;\mu t)) \right]. \end{aligned}$$

This implies (8.3). □

Notice that the p.d.f. of $H^{(i)}(y;t)$ is similarly, for $y > 0$,

$$h^{(i)}(y;t) = \kappa_i \sum_{j=0}^{\infty} p(j;\kappa_i y) p(j+1;\lambda_i t), \quad i = L, H.$$

The densities of $W(t)$ at HD- and LD-periods are given in the next lemma.

Lemma 8.2.

$$\begin{aligned} \psi_{W(t)}^{(HD)}(w;t) &= h^*(t-w;w) \\ &= \zeta \sum_{j=0}^{\infty} p(j;\zeta(t-w)) p(j+1;\mu w). \end{aligned} \tag{8.4}$$

$$\psi_{W(t)}^{(LD)}(w;t) = \mu \sum_{n=0}^{\infty} p(n;\mu w) p(n;\zeta(t-w)). \tag{8.5}$$

Proof.

$$\psi_{W(t)}^{(\mathrm{LD})}(w;t) = \psi_{W(t)}(w;t) - \psi_{W(t)}^{(\mathrm{HD})}(w;t).$$

Moreover,

$$\begin{aligned}\psi_{W(t)}(w;t) &= -\frac{d}{dw}H^*(t-w;w)\\ &= -\frac{d}{dw}\sum_{j=0}^{\infty} p(j;\zeta(t-w))P(j;\mu w)\\ &= \zeta\sum_{j=0}^{\infty} p(j;\zeta(t-w))p(j+1;\mu w)\\ &+ \mu\sum_{j=0}^{\infty} p(j;\zeta(t-w))p(j;\mu w).\end{aligned}$$

□

An important function is $H_Q^*(y;t)$ given by (4.5). For numerical computations it is convenient to use numerical integration to evaluate $H_Q^*(y;t)$. For example

(8.6)
$$\begin{aligned}&\int_0^t e^{-\lambda_H w} H_L(y;t-w)\psi_{W(t)}(w;t)dw\\ &\qquad = t\int_0^1 e^{-\lambda_H tz} H_L(y;t(1-z))\psi_{W(t)}(tz;t)dz\\ &\qquad \cong t\sum_{i=1}^{8} e^{-\lambda_H tz_i} H_L(y;t(1-z_i))\psi_{W(t)}(tz_i;t)\cdot w_i,\end{aligned}$$

where z_i and w_i are the abscissas and weight factors of the Gaussian integration (see p. 921 of [1]).

In the following table we present the values of $H_Q^*(y;t)$ for various values of y, at $t = 10$, $\lambda_H = 1.5$, $\lambda_L = 1.0$, $\kappa_H = 0.5$, $\kappa_L = 1$, $\mu = 1$, $\zeta = 2$.

Table 8.1. Values of $H_Q^*(y;10)$

y	$H_Q^*(y,10)$	y	$H_Q^*(y;10)$
4	0.00267	24	0.56143
8	0.02677	28	0.70666
12	0.09970	32	0.81713
16	0.22877	36	0.89294
20	0.39346	40	0.94071

For a system with $q = 30$ and $t_0 = 20$, the survival function of τ^*, i.e., $P\{\tau^* > t\} = H_Q^*(30; t)$ is displayed in Table 8.2 for the above values of λ_H, λ_L etc.

Table 8.2. $P\{\tau^* > t\}$

t	5	10	12	15	20
$P(\tau^* > t)$	0.9837	0.7665	0.6036	0.3559	0.0968

The expected value of τ^*, $E(\tau^*) = \int_0^{20} H_Q^*(30; t)dt = 13.31$. Also $P\{\tau^* = t_0\} = 0.0968$. The median of τ^* is $\tau^*_{0.5} = 13.26$. The expected amount of discarded material is $E\{D(30)\} = \int_0^{30} H_Q^*(x; 20)dx = 9.3473$.

Since $G(t) = 1 - e^{-\zeta t}$, $t \geq 0$, $R \sim \text{Expo}(\zeta)$, independently of τ^*. Thus, for $E\{\text{Sh}_3\}$,

$$E\{R, \tau^* = t_0 \in \text{LD}\} = \frac{1}{\zeta} P\{\tau^* = t_0 \in \text{LD}\}, \tag{8.7}$$

and

$$P\{\tau^* = t_0 \in \text{LD}\} = \int_0^{t_0} H_Q(q; w, t_0)\psi^{(\text{LD})}_{W(t_0)}(w; t_0)dw. \tag{8.8}$$

For the above system, with $t_0 = 20$, $q = 30$, $\lambda_H = 1.5$, $\lambda_L = 1$, $\kappa_H = 0.5$, $\kappa_L = 1$, $\mu = 1$, $\zeta = 2$ we get $P\{\tau^* = 20 \in \text{LD}\} = 0.03431$, and $E\{\text{Sh}_3\} = 0.01716$.

For Sh_1 we have

$$\begin{aligned} E\{\text{Sh}_1 \mid \tau^* = t \in \text{HD}\} = {} & \frac{\lambda_H}{\kappa_H} e^{-\kappa_H q} \left(\frac{1}{\lambda_H + \mu} (1 - e^{-(\lambda_H + \mu)t_0}) \right. \\ & \left. + \int_0^{t_0} \int_0^t e^{-\lambda_H w - \lambda_L (t-w)} \psi^{(\text{HD})}_{W(t)}(w; t)dwdt \right) \\ & + \frac{\lambda_H}{\kappa_H} \int_0^q e^{-\kappa_H (q-y)} \left(\int_0^{t_0} \int_0^t h_Q(y; w, t)\psi^{(\text{HD})}_{W(t)}(w; t)dwdt \right) dy. \end{aligned} \tag{8.9}$$

Finally

$$E\{\text{Sh}_1\} = \int_0^{t_0} E\{\text{Sh}_1 \mid \tau^* = t \in \text{HD}\} P\{\tau^* = t \in \text{HD}\}dt. \tag{8.10}$$

For example, for a system with $t_0 = 20$, $q = 30$, $\lambda_H = 1.5$, $\lambda_L = 1$, $\kappa_H = 0.5$, $\kappa_L = 1$, $\mu = 1$, $\zeta = 2$ we get $E\{\text{Sh}_1\} = 0.37398$. In a similar manner we compute $E\{\text{Sh}_2\}$ using eq. (7.12).

We get $E\{\mathrm{Sh}_2\} = 0.28667$. Finally, according to (5.3), the expected total material held is $E\{T(\tau^*)\} = 245.037$.

In the following table we present a few values of $E\{Q(s) \mid \tau^* = t\}$ for the case of $\lambda_H = 5$, $\lambda_L = 2$, $\kappa_H = 1/6$, $\kappa_L = 1/5$, $\mu = 0.1$, $\zeta = 0.2$, $t = 15$ and $q = 350$, computed according to (5.5).

Table 8.3

s	2	4	6	8	10	12	14	15
$E\{Q(s) \mid \tau^* = t\}$	41.42	86.87	136.44	188.10	238.56	286.14	328.88	350.00

In the following table we present the values of $E\{\tau^*\}$, $E\{R, \tau^* \in \mathrm{LD}\}$, E $\{\mathrm{Sh}\}$ (which is defined as the sum of the three shortage terms $E\{\mathrm{Sh}_1\}$, $E\{\mathrm{Sh}_2\}$, and $E\{\mathrm{Sh}_3\}$), $E\{D(q)\}$ and $E\{T(\tau^*)\}$ for various q values. The parameters are those of Table 8.3.

Table 8.4

q	$E\{\tau^*\}$	$E\{R, \tau^* \in \mathrm{LD}\}$	$E\{\mathrm{Sh}\}$	$E\{D(q)\}$	$E\{T(\tau^*)\}$
300	12.66	0.1109	9.005	0.618	1955.51
350	14.72	0.2632	8.998	2.057	2664.45
400	16.66	0.5105	9.297	5.354	3457.77
450	18.47	0.8576	9.643	11.736	4342.10
500	20.05	1.2833	10.949	22.563	5424.51

A reasonable objective function is the long-run average net profit per time unit, which is given by

$$O(q) := \frac{c_p q - K - c_d E\{D(q)\} - c_{sh} E\{\mathrm{Sh}\} - c_h E\{T(\tau^*)\}}{E\{C\}}. \tag{8.11}$$

Here c_p are the earnings per item, K the set-up costs per cycle, and c_d, c_{sh} and c_h are the costs of a discarded unit, the costs for shortage of one unit, and the costs of holding the buffer content. It should be noted that, if not all items are sold, the profit from selling in one cycle is less than $c_p q$; however, that is taken into account by the $D(q)$ term. We compute the expected profit per cycle for the parameters of Table 8.3, when the cost values are $c_p = 5$,

$K = 10$, $c_d = 10$, $c_{sh} = 2$, and varying values of c_h. These are presented in the following table.

Table 8.5

q	$c_h = 0$	$c_h = 0.1$	$c_h = 0.2$	$c_h = 0.3$	$c_h = 0.4$	$c_h = 0.5$
300	114.78	99.47	84.15	68.84	53.52	38.22
350	113.56	95.77	77.99	60.21	42.42	24.64
400	112.86	92.72	72.58	52.44	32.31	12.17
450	108.83	86.36	63.89	41.43	18.96	-3.50
500	105.12	79.69	54.26	28.83	3.41	-22.00

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