

EURANDOM PREPRINT SERIES

2012-023

November 27, 2012

**Limit theorems for renewal shot noise processes
with decreasing response function**

Alexander Iksanov, Alexander Marynych, Matthias Meiners

ISSN 1389-2355

Limit theorems for renewal shot noise processes with decreasing response function

Alexander Iksanov*, Alexander Marynych[†]
and Matthias Meiners[‡]

November 27, 2012

Abstract

We consider shot noise processes $(X(t))_{t \geq 0}$ with deterministic response function h and the shots occurring at the renewal epochs $0 = S_0 < S_1 < S_2 \dots$ of a zero-delayed renewal process. We prove convergence of the finite-dimensional distributions of $(X(ut))_{u \geq 0}$ as $t \rightarrow \infty$ in different regimes. If the response function h is directly Riemann integrable, then the finite-dimensional distributions of $(X(ut))_{u \geq 0}$ converge weakly as $t \rightarrow \infty$. Neither scaling nor centering are needed in this case. If the response function is non-negative, eventually decreasing but non-integrable, then, after suitable shifting, the finite-dimensional distributions of the process converge. Again, no scaling is needed. In both cases, the limit is identified as a collection of independent shot noise processes in equilibrium. If (the distribution of) S_1 is in the domain of attraction of an α -stable law and the response function is regularly varying at ∞ with index β (with $\beta < 1/\alpha$ or $\beta \leq 1/\alpha$, depending on whether $\mathbb{E} S_1 < \infty$ or $\mathbb{E} S_1 = \infty$), then scaling is needed to obtain weak convergence of the finite-dimensional distributions of $(X(ut))_{u \geq 0}$. The limiting processes are fractionally integrated stable processes if $\mathbb{E} S_1 < \infty$ and fractionally integrated inverse stable subordinators if $\mathbb{E} S_1 = \infty$.

2010 Mathematics Subject Classification: Primary: 60F05, 60K05

Secondary: 60G55

Keywords: continuous mapping theorem; finite-dimensional convergence; fractionally integrated stable processes; fractionally integrated inverse stable subordinators; shot noise processes; stationary renewal processes

*Faculty of Cybernetics, National T. Shevchenko University of Kiev, 01033 Kiev, Ukraine, e-mail: iksan@univ.kiev.ua

[†]Eindhoven University of Technology, The Netherlands, e-mail: O.Marynych@tue.nl. The research of A. Marynych was financially supported by a Free Competition Grant of the Netherlands Organisation for Scientific Research (NWO)

[‡]Institut für Mathematische Statistik, Westfälische Wilhelms-Universität Münster, 48149 Münster, Germany, e-mail: mmeiners@math.uni-muenster.de. Research supported by DFG SFB 878 “Geometry, Groups and Actions”

A part of this work was carried out during visits of A. Iksanov and A. Marynych to Münster and a visit of M Meiners to Kiev. Grateful acknowledgment is made for financial support and hospitality.

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 2 |
| 2 | Main results | 3 |
| 2.1 | Limit theorems without scaling | 4 |
| 2.2 | Limit theorems with scaling | 7 |
| 2.3 | Properties of the limiting processes in Theorems 2.7 and 2.9 . . . | 10 |
| 3 | Preliminaries | 16 |
| 3.1 | Stationary renewal processes and coupling | 16 |
| 3.2 | Stable distributions and domains of attraction | 18 |
| 3.3 | Weak convergence of the renewal counting process | 19 |
| 3.4 | Moment convergence | 20 |
| 4 | Proofs of the limit theorems without scaling | 23 |
| 4.1 | Preparatory results | 23 |
| 4.2 | One-dimensional convergence | 31 |
| 4.3 | Finite-dimensional convergence | 34 |
| 5 | Proofs of the limit theorems with scaling | 40 |
| 5.1 | Reduction to continuous and decreasing h | 40 |
| 5.2 | Proofs of Theorems 2.7 and 2.9 | 42 |
| 5.3 | Convergence of moments in the case $\mu = \infty$ and the proof of Proposition 2.11 | 47 |
| A | Appendix: Auxiliary results | 52 |

1 Introduction

Continuing the line of research initiated in [18], in the present paper, we are investigating the convergence of the finite-dimensional distributions of renewal shot noise processes. Some work on convergence of renewal shot noise processes has already been done by other authors [17, 32, 34]. However, their results do not intersect with those obtained here. The special case of Poisson shot noise has received more attention, see *e.g.* [15, 23, 25, 26, 36].

Initially, shot noise processes were introduced to model the current induced by a stream of electrons arriving at the anode of a vacuum tube [39]. Since their first appearance in the literature, shot noise processes have been used to model rainfall [35, 43], stream- and riverflows [27, 44], earthquake occurrences [42], computer failures [28], traffic noise [30], delay in claim settlement in insurance [23, 24], and several processes in finance [37], to name but a few. The recent paper [1] offers a list of further references.

We now start with the mathematical setup. Let ξ_1, ξ_2, \dots be a sequence of independent copies of a positive random variable ξ . The distribution of ξ is denoted by F . By $(S_k)_{k \in \mathbb{N}_0}$ we denote the random walk with initial position

$S_0 := 0$ and increments $S_k - S_{k-1} = \xi_k$, $k \in \mathbb{N}$. The corresponding renewal counting measure is denoted by N , that is,

$$N = \sum_{k \geq 0} \delta_{S_k},$$

where δ_x denotes the Dirac distribution concentrated at x . We write $N(t)$ for $N[0, t]$, $t \geq 0$. By U we denote the intensity measure corresponding to N . Hence, $U(B) := \mathbb{E} N(B)$ for Borel sets $B \subseteq \mathbb{R}$. We write $U(t)$ for $U[0, t]$, $t \geq 0$. Throughout the paper, we denote by h a real-valued, measurable and locally bounded function on the positive half-line $\mathbb{R}_+ = [0, \infty)$. Further, let

$$X(t) := \sum_{k=0}^{N(t)-1} h(t - S_k) = \int_{[0,t]} h(t-y) N(dy), \quad t \geq 0. \quad (1.1)$$

While the stochastic process $(X(t))_{t \geq 0}$ is called *renewal shot noise process*, h is called *response function*.

In the recent paper [18], functional limit theorems for $(X(ut))_{u \geq 0}$ are derived in the case that the response function is eventually increasing¹. The motivation behind the present work in general and the use of the specific time scaling in particular is the following. First, we intend to obtain counterparts of the results derived in [18] for functions h that are eventually decreasing. Second, in a forthcoming publication [20] some results of this paper will be used to prove the finite-dimensional convergence of the number of empty boxes in the Bernoulli sieve (see [12] for the definition and properties of the Bernoulli sieve). Of course, transformations of time other than ut may also lead to useful limit theorems. For instance, convergence of $(X(t+u))_{u \geq 0}$ may be worth investigating. Yet another transformation of time has proved important [17], where one only rescales the time of the underlying renewal process, whereas the deterministic component runs in its original time scale.

Unlike in [18], where functional limit theorems are derived, in the present paper, we investigate convergence of finite-dimensional distributions only. In some cases, the limiting processes do not take values in the Skorokhod space of right-continuous functions with left limits which excludes the possibility that a classical functional limit theorem holds. However, in some cases, if h belongs to the Skorokhod space, so does the limit. Whether in these cases there is actually convergence in a functional space remains open for future research.

2 Main results

As mentioned in the introduction, we centre our attention on the case of eventually decreasing response functions. Let us remark right away that the situations where h is eventually decreasing and either $\lim_{t \rightarrow \infty} h(t) = c \in (-\infty, 0)$ or $\lim_{t \rightarrow \infty} h(t) = -\infty$ and $-h(t)$ is regularly varying at ∞ with some index $\beta \geq 0$

¹Notice that we call a function h increasing if for all $s < t$ we have $h(s) \leq h(t)$. We call h *strictly* increasing if $s < t$ we have $h(s) < h(t)$. Analogously, h is said to be decreasing if $s < t$ implies $h(s) \geq h(t)$ and it is said to be *strictly* decreasing if $s < t$ implies $h(s) > h(t)$.

are covered by Theorem 1.1 in [18]. See Remark 2.10 for more details. Keeping this in mind our main results mainly treat eventually decreasing functions with non-negative limit at infinity.

Our results fall into two fundamentally different categories. The first type of results considers finite-dimensional convergence of the process $(X(ut))_{u \geq 0}$ as $t \rightarrow \infty$ when no scaling (normalization) is needed. In this case, all randomness in the limiting process can be described in terms of (copies of) the stationary renewal counting process N^* to be introduced below. In the second type of results scaling is needed. As an effect of the scaling, some of the fine features of the process $(X(ut))_{u \geq 0}$ vanish in the limit. Therefore, in the second case, robust limit theorems are obtained in the sense that the limiting behavior only depends on the asymptotic behavior of h and the tails of S_1 . The limiting processes are stochastic integrals with integrators being certain Lévy processes. For instance, if ξ is square-integrable, the integrator is Brownian motion.

For the formulation of our main results, we need to introduce further notation. First, let $\mu := \mathbb{E}\xi$. Since $\xi > 0$ a.s., μ is well-defined but may equal $+\infty$. Whenever $\mu < \infty$ and the law of ξ is non-lattice, we denote by S_0^* a positive random variable which is independent of the sequence $(\xi_k)_{k \in \mathbb{N}}$ and which has distribution function

$$F^*(t) := \mathbb{P}\{S_0^* \leq t\} := \frac{1}{\mu} \int_0^t \mathbb{P}\{\xi > x\} dx, \quad t \geq 0. \quad (2.1)$$

Moreover, we set $S_k^* := S_0^* + S_k$, $k \in \mathbb{N}_0$. The associated renewal counting process $N^* := \sum_{k \geq 0} \delta_{S_k^*}$ has stationary increments. Equivalently, the corresponding intensity measure $U^*(\cdot) := \mathbb{E}N^*(\cdot)$ satisfies $U^*(dx) = \mu^{-1} dx$, see Subsection 3.1 and [29, Section III.1.2] or [41, Section II.9] for further background information and details.

For stochastic processes $(Z_t(u))_{u \geq 0}$, $t \geq 0$ and $(Z(u))_{u \geq 0}$, we write $Z_t(u) \xrightarrow{\text{f.d.}} Z(u)$ as $t \rightarrow \infty$ to denote weak convergence of finite-dimensional distributions, *i.e.*, for any $n \in \mathbb{N}$ and any selection $0 \leq u_1 < \dots < u_n < \infty$

$$(Z_t(u_1), \dots, Z_t(u_n)) \xrightarrow{\text{d}} (Z(u_1), \dots, Z(u_n)) \quad \text{as } t \rightarrow \infty.$$

The record $Z_t(u) \xrightarrow{\text{f.d.}} Z(u)$ as $t \rightarrow \infty$ means that the above relation holds for all choices $0 < u_1 < \dots < u_n < \infty$, whereas $Z_t(0)$ does not (necessarily) converge to $Z(0)$.

2.1 Limit theorems without scaling

If $\mu < \infty$ and F is non-lattice, define

$$X^* := \sum_{k \geq 0} h(S_k^*) \quad (2.2)$$

whenever the sum converges in probability. In this case, denote by $(X^*(u))_{u \geq 0}$ a family of i.i.d. copies of X^* .

Our first result states that if h is directly Riemann integrable (d.R.i.), then the finite-dimensional distributions of $(X(ut))_{u > 0}$ converge weakly to the finite-dimensional distributions of the process $(X^*(u))_{u > 0}$.

Theorem 2.1. *Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be directly Riemann integrable and F be non-lattice.*

(a) *If $\mu = \mathbb{E}\xi < \infty$, then the random series X^* converges a.s. and*

$$X(ut) \xrightarrow{\text{f.d.}} X^*(u) \quad \text{as } t \rightarrow \infty. \quad (2.3)$$

(b) *If $\mu = \infty$, then*

$$X(t) \xrightarrow{\mathbb{P}} 0 \quad \text{as } t \rightarrow \infty.$$

If F is spread-out, then assertions (a) and (b) also hold for functions h that are bounded, a.e. continuous and Lebesgue integrable.

Remark 2.2. Since the focus of this paper is on eventually decreasing response functions it is worth mentioning that Theorem 2.1 covers the case when h is eventually decreasing and improperly Riemann integrable since any such function is necessarily directly Riemann integrable.

Example 2.3. Assume that $\mu < \infty$ and that F is non-lattice. For fixed $0 \leq a < b$, choose $h(t) := \mathbb{1}_{[a,b)}(t)$, $t \geq 0$ as response function. Then Theorem 2.1 implies that

$$N(t-a) - N(t-b) = \sum_{k \geq 0} h(t - S_k) \xrightarrow{\text{d}} N^*(b-a) \quad \text{as } t \rightarrow \infty. \quad (2.4)$$

Though the one-dimensional convergence in Theorem 2.1 is quite expected, a rigorous proof is necessary. It is tempting to conclude this from Theorem 6.1 in [32]. However, the cited theorem does not hold in the generality stated there. Regularity assumptions on the function h appearing in the theorem above or the function g in the cited result, respectively, cannot be avoided. This will be demonstrated in Example 2.6 at the end of this subsection.

Our second result is an extension of Theorem 2.1 to the situation where h is not integrable. In this case, X^* is not well-defined and in order to still obtain non-trivial finite-dimensional convergence of the process $(X(ut))_{u \geq 0}$ as $t \rightarrow \infty$ centering is needed. Let

$$X_{\circ}(t) := X(t) - \mu^{-1} \int_0^t h(y) dy, \quad t \geq 0. \quad (2.5)$$

Under suitable assumptions, we obtain finite-dimensional convergence of the process $(X_{\circ}(ut))_{u > 0}$ as $t \rightarrow \infty$. The limiting process is a close relative of $(X^*(u))_{u \geq 0}$ and is to be introduced next. When $\mu < \infty$ and F is non-lattice, define

$$X_{\circ}^* := \lim_{t \rightarrow \infty} \left(\sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{S_k^* \leq t\}} - \frac{1}{\mu} \int_0^t h(y) dy \right). \quad (2.6)$$

Whenever X_{\circ}^* exists as the limit in probability and is a.s. finite, denote by $(X_{\circ}^*(u))_{u \geq 0}$ a family of i.i.d. copies of X_{\circ}^* .

Theorem 2.4. *Assume that F is non-lattice. Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally bounded, eventually decreasing and non-integrable.*

(C1) Suppose $\sigma^2 := \text{Var } \xi < \infty$ and

$$\int_0^\infty h(y)^2 dy < \infty. \quad (2.7)$$

Then X_\circ^* exists as the limit in \mathcal{L}^2 in (2.6) and

$$X_\circ(ut) \xrightarrow{\text{f.d.}} X_\circ^*(u) \quad \text{as } t \rightarrow \infty. \quad (2.8)$$

(2.8) also holds when $(X_\circ(ut))_{u \geq 0}$ is replaced by $(X(ut) - \mathbb{E} X(ut))_{u \geq 0}$.

For the rest of the theorem, assume that h is eventually twice differentiable and that h'' is eventually nonnegative.

(C2) Suppose $\mathbb{E} \xi^r < \infty$ for some $1 < r < 2$. If

$$\int_0^\infty h(y)^r dy < \infty, \quad (2.9)$$

and²

$$h''(t) = O(t^{-2-1/r}) \quad \text{as } t \rightarrow \infty, \quad (2.10)$$

then X_\circ^* is well-defined and the a.s. limit in (2.6). Further, (2.8) holds.

(C3) Suppose $\mathbb{P}\{\xi > x\} \sim x^{-\alpha} \ell(x)$ as $x \rightarrow \infty$ for some $1 < \alpha < 2$ and some ℓ slowly varying at ∞ . If

$$\int_1^\infty h(y)^\alpha \ell(1/h(y)) dy < \infty, \quad (2.11)$$

and

$$h''(t) = O(t^{-2} c(t)^{-1}) \quad \text{as } t \rightarrow \infty \quad (2.12)$$

where $c(t)$ is any positive function such that

$$\lim_{t \rightarrow \infty} \frac{t \ell(c(t))}{c(t)^\alpha} = 1, \quad (2.13)$$

then X_\circ^* exists as the limit in probability in (2.6) and (2.8) holds.

Remark 2.5. The cases (C2) and (C3) of Theorem 2.4 impose, besides conditions on the law of ξ , smoothness and integrability conditions on h . The smoothness conditions may seem rather restrictive but are an essential ingredient of our proof which is based on an idea we have learned in [22]. We believe that in each assertion (C1)–(C3), given the respective assumption on the law of ξ , the corresponding integrability condition is close to optimal. In a sense, the extra smoothness conditions in (C2) and (C3) are the price one has to pay for this precision (we do not claim, however, that the smoothness conditions are indeed necessary using them seems to be a restriction caused by the method). For comparison, we mention the following. Assuming nothing

²If h'' is eventually monotone, then (2.10) and (2.12) are consequences of (2.9) and (2.11), respectively.

beyond the standing conditions of the theorem (in particular, not requiring h to be differentiable) we can prove that (2.8) holds under more restrictive integrability conditions:

$$\mathbb{E} \xi^r < \infty \quad \text{and} \quad \int_{[0, \infty)} y^{1/r} d(-h(y)) < \infty$$

for some $1 < r < 2$, and

$$\mathbb{P}\{\xi > x\} \sim x^{-\alpha} \ell(x) \quad \text{as } x \rightarrow \infty \quad \text{and} \quad \int_{[1, \infty)} c(y) d(-h(y)) < \infty$$

for some $1 < \alpha < 2$ and some ℓ slowly varying at ∞ , respectively. Without going into the details, we mention that the conditions $\int_{[0, \infty)} y^{1/r} d(-h(y)) < \infty$ and $\int_{[1, \infty)} c(y) d(-h(y)) < \infty$ are sufficient for $(N^*(y) - y/\mu)$ being a.s. absolutely integrable w.r.t. $d(-h(y))$ whereas the conditions (2.9) and (2.11) are sufficient for the a.s. (improper) integrability of $(N^*(y) - y/\mu)$ w.r.t. $d(-h(y))$.

Example 2.6. Let $h(t) := (1 \wedge 1/t^2) \mathbb{1}_{\mathbb{Q}}(t)$, $t \geq 0$. Further, let the distribution of ξ be such that $\mathbb{P}\{\xi \in \mathbb{Q} \cap (0, 1]\} = 1$ and $\mathbb{P}\{\xi = r\} > 0$ for any $r \in \mathbb{Q} \cap (0, 1]$. Then the distribution of ξ is non-lattice. From (2.1) we conclude that the distribution of S_0^* is continuous w.r.t. Lebesgue measure and concentrated on $[0, 1]$. Therefore, with probability 1, S_0^* takes values in $[0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})$. Since the ξ_k take rational values a.s., all S_k^* take irrational values on a set of probability 1. Consequently, the random variable $X^* = \sum_{k \geq 0} h(S_k^*)$ equals 0 a.s. But in the given situation, $X(t)$ does not converge to 0 in distribution when t approaches $+\infty$ along a sequence of rationals. In fact, for $t \in \mathbb{Q}$, $X(t) = Y(t)$ a.s. where $Y(t) = \sum_{k \geq 0} f(t - S_k) \mathbb{1}_{\{S_k \leq t\}}$ with $f(t) = 1 \wedge 1/t^2$ for $t \geq 0$. Therefore, from Theorem 2.1 we conclude that

$$X(t) = Y(t) \xrightarrow{d} \sum_{k \geq 0} f(S_k^*) \quad \text{as } t \rightarrow \infty, t \in \mathbb{Q}.$$

Plainly, the latter random variable is positive a.s.

Example 2.6 does not only demonstrate that Theorem 6.1 in [32] fails when assuming only that $\lim_{t \rightarrow \infty} h(t) = 0$. It moreover shows that also Lebesgue integrability of h is not enough to ensure (2.3) to hold. A stronger assumption such as the direct Riemann integrability of h is needed.

2.2 Limit theorems with scaling

In the case when scaling is needed our main assumption on the response function h is regular variation at ∞ :

$$h(t) \sim t^{-\beta} \ell_h(t) \quad \text{as } t \rightarrow \infty \tag{2.14}$$

for some $\beta \geq 0$ and some ℓ_h slowly varying at ∞ . Recall that $\ell_h(t) > 0$ for all $t \geq 0$ by the definition of slow variation, see [5]. Note further that the functions h with $\lim_{t \rightarrow \infty} h(t) = b \in (0, \infty)$ are covered by condition (2.14) with $\beta = 0$ and $\lim_{t \rightarrow \infty} \ell_h(t) = b$.

Theorem 2.7. Assume that $\mu < \infty$ and that F is non-lattice. Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally bounded, measurable and eventually decreasing.

(A1) Suppose $\sigma^2 := \text{Var } \xi < \infty$ and let $(W(u))_{u \geq 0}$ denote a standard Brownian motion. If (2.14) holds for some $\beta \in (0, 1/2)$, then

$$\frac{X(ut) - \mu^{-1} \int_0^{ut} h(y) dy}{\sqrt{\sigma^2 \mu^{-3} t h(t)}} \xrightarrow{\text{f.d.}} \int_{[0, u]} (u - y)^{-\beta} dW(y) \quad \text{as } t \rightarrow \infty,$$

whereas if condition (2.14) holds with $\beta = 0$, the limiting process is $(W(u))_{u \geq 0}$.

(A2) Suppose $\sigma^2 = \infty$ and

$$\int_{[0, t]} y^2 \mathbb{P}\{\xi \in dy\} \sim \ell(t) \quad \text{as } t \rightarrow \infty$$

for some ℓ slowly varying at ∞ . Let $c(t)$ be any positive continuous function such that $\lim_{t \rightarrow \infty} \frac{t\ell(c(t))}{c(t)^2} = 1$ and let $(W(u))_{u \geq 0}$ denote a standard Brownian motion. If condition (2.14) holds with $\beta \in (0, 1/2)$, then

$$\frac{X(ut) - \mu^{-1} \int_0^{ut} h(y) dy}{\mu^{-3/2} c(t) h(t)} \xrightarrow{\text{f.d.}} \int_{[0, u]} (u - y)^{-\beta} dW(y) \quad \text{as } t \rightarrow \infty,$$

whereas if condition (2.14) holds with $\beta = 0$, the limiting process is $(W(u))_{u \geq 0}$.

(A3) Suppose

$$\mathbb{P}\{\xi > t\} \sim t^{-\alpha} \ell(t) \quad \text{as } t \rightarrow \infty$$

for some $1 < \alpha < 2$ and some ℓ slowly varying at ∞ . Let $c(t)$ be any positive continuous function such that $\lim_{t \rightarrow \infty} \frac{t\ell(c(t))}{c^\alpha(t)} = 1$ and let $(W(u))_{u \geq 0}$ denote a spectrally negative α -stable Lévy process such that $W(1)$ has the characteristic function

$$z \mapsto \exp \left\{ -|z|^\alpha \Gamma(1-\alpha) (\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \text{sign}(z)) \right\}, \quad z \in \mathbb{R} \quad (2.15)$$

where $\Gamma(\cdot)$ denotes the gamma function. If condition (2.14) holds with $\beta \in (0, 1/\alpha)$, then

$$\frac{X(ut) - \mu^{-1} \int_0^{ut} h(y) dy}{\mu^{-1-1/\alpha} c(t) h(t)} \xrightarrow{\text{f.d.}} \int_{[0, u]} (u - y)^{-\beta} dW(y) \quad \text{as } t \rightarrow \infty,$$

whereas if condition (2.14) holds with $\beta = 0$, the limiting process is $(W(u))_{u \geq 0}$.

Remark 2.8. In Theorem 2.7, we only consider limit theorems with regularly varying scaling. However, there are cases in which the scaling function is slowly varying. The treatment of these requires different techniques and is left for future research.

Our second result in this subsection is concerned with the case of infinite μ . Here the assumptions on the response function h are less restrictive.

Theorem 2.9. *Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally bounded and measurable. Suppose that $\mathbb{P}\{\xi > t\} \sim t^{-\alpha}\ell(t)$ as $t \rightarrow \infty$ for some $0 < \alpha < 1$ and some ℓ slowly varying at ∞ , and that h satisfies (2.14) for some $\beta \in [0, \alpha]$. If $\alpha = \beta$, assume additionally that*

$$\lim_{t \rightarrow \infty} \frac{h(t)}{\mathbb{P}\{\xi > t\}} = \lim_{t \rightarrow \infty} \frac{\ell_h(t)}{\ell(t)} = c \in (0, \infty]$$

and if $c = \infty$ that there exists an increasing function $u(t)$ such that

$$\lim_{t \rightarrow \infty} \frac{\ell_h(t)}{\ell(t)u(t)} = 1.$$

Let $(W(u))_{u \geq 0}$ denote an inverse α -stable subordinator defined by

$$W(u) := \inf\{t \geq 0 : D(t) > u\}$$

where $(D(t))_{t \geq 0}$ is an α -stable subordinator with $-\log \mathbb{E} e^{-tD(1)} = \Gamma(1 - \alpha)t^\alpha$ for $t \geq 0$. If $\alpha > \beta$ or $\alpha = \beta$ and $c = \infty$, then

$$\frac{\mathbb{P}\{\xi > t\}}{h(t)} X(ut) \xrightarrow{\text{f.d.}} \int_{[0, u]} (u - y)^{-\beta} dW(y) \quad \text{as } t \rightarrow \infty.$$

If $\alpha = \beta$ and $c < \infty$, then the same convergence holds with $\xrightarrow{\text{f.d.}}$ replaced by $\xrightarrow{\text{f.d.}}$. Furthermore, in all cases above there is convergence of moments:

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\frac{\mathbb{P}\{\xi > t\}}{h(t)} \right)^k \mathbb{E} X(ut)^k &= \mathbb{E} \left(\int_{[0, u]} (u - y)^{-\beta} dW(y) \right)^k \\ &= u^{k(\alpha - \beta)} \frac{k!}{\Gamma(1 - \alpha)^k} \prod_{j=1}^k \frac{\Gamma(1 - \beta + (j - 1)(\alpha - \beta))}{\Gamma(j(\alpha - \beta) + 1)}, \quad k \in \mathbb{N}, \end{aligned} \quad (2.16)$$

where $\Gamma(\cdot)$ denotes the gamma function.

Remark 2.10. Let the assumptions concerning ξ in Theorem 2.7 or Theorem 2.9 be in force with eventually decreasing h and with condition (2.14) replaced by

$$-h(t) \sim t^\beta \ell_h(t) \quad \text{as } t \rightarrow \infty$$

for some $\beta \geq 0$ and some ℓ_h slowly varying at ∞ . No further restrictions on β like those appearing in Theorem 2.7 are needed. Then the limit relations of the theorems remain valid when the limiting processes are replaced by $\int_{[0, u]} (u - y)^\beta dW(y)$, cf. Theorem 1.1 in [18].

From Theorem 2.9 it follows that if $\alpha = \beta$ and

$$\lim_{t \rightarrow \infty} \frac{h(t)}{\mathbb{P}\{\xi > t\}} = c \in (0, \infty), \quad (2.17)$$

then $X(t) \xrightarrow{\text{d}} \text{Exp}(c^{-1})$ as $t \rightarrow \infty$ where $\text{Exp}(c^{-1})$ denotes an exponentially distributed random variable with mean c . In fact, the one-dimensional convergence takes place under the sole assumption (2.17). In particular, the regular variation of neither $h(t)$, nor $\mathbb{P}\{\xi > t\}$ is needed.

Proposition 2.11. *Assume that $\mu = \infty$ and let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable and locally bounded function which satisfies condition (2.17). Then*

$$\lim_{t \rightarrow \infty} \mathbb{E} X(t)^k = c^k k!, \quad k \in \mathbb{N},$$

which entails $X(t) \xrightarrow{d} \text{Exp}(c^{-1})$ as $t \rightarrow \infty$.

2.3 Properties of the limiting processes in Theorems 2.7 and 2.9

In this section we find it more transparent to add subscripts in the notation of the processes to bring out the dependence on the parameters α and β . Throughout this subsection, we use $\Gamma(\cdot)$ to denote the gamma function.

Limits in Theorem 2.7

Assuming in what follows that $\alpha = 2$ corresponds to the cases (A1) and (A2), in particular, that $(W_2(u))_{u \geq 0}$ is a Brownian motion, we define the limiting stochastic integral

$$Y_{\alpha, \beta}(u) := \int_{[0, u]} (u - y)^{-\beta} dW_\alpha(y)$$

via the formula

$$\int_{[0, u]} (u - y)^{-\beta} dW_\alpha(y) := u^{-\beta} W_\alpha(u) + \beta \int_0^u (W_\alpha(u) - W_\alpha(y))(u - y)^{-\beta-1} dy. \quad (2.18)$$

This definition is consistent with the usual definition of a stochastic integral with a deterministic integrand and the integrator being a semimartingale. However, since $\lim_{y \uparrow u} (u - y)^{-\beta-1} = \infty$, it is necessary to check the existence of the Lebesgue integral $\int_0^u (W_\alpha(u) - W_\alpha(y))(u - y)^{-\beta-1} dy$. Indeed, in view of the inequality

$$\begin{aligned} & \mathbb{E} \left| \int_0^u (W_\alpha(u) - W_\alpha(y))(u - y)^{-\beta-1} dy \right| \\ & \leq \int_0^u \mathbb{E} |W_\alpha(u) - W_\alpha(y)| (u - y)^{-\beta-1} dy \\ & = \int_0^u \mathbb{E} |W_\alpha(u - y)| (u - y)^{-\beta-1} dy \\ & = \mathbb{E} |W_\alpha(1)| \int_0^u (u - y)^{1/\alpha - \beta - 1} dy. \end{aligned}$$

the integral exists in the a.s. sense if $\beta < 1/\alpha$, which explains the restrictions imposed on β in the theorem. Further, if $1 < \alpha < 2$, it can be checked that,

for $z \in \mathbb{R}$,

$$\begin{aligned} & \log \mathbb{E} \exp \left(iz \int_{[0, u]} (u - y)^{-\beta} dW_\alpha(y) \right) \\ &= \int_0^u \log \mathbb{E} \exp(iz(u - y)^{-\beta} W_\alpha(1)) dy \\ &= -\Gamma(1 - \alpha) (\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \operatorname{sign}(z)) |z|^\alpha \int_0^u (u - y)^{-\alpha\beta} dy, \end{aligned}$$

where the last equality is a consequence of (2.15). Note that the last integral converges iff $\beta \in [0, 1/\alpha)$. Also we infer that, with u fixed,

$$Y_{\alpha, \beta}(u) \stackrel{d}{=} \frac{u^{1/\alpha - \beta}}{(1 - \alpha\beta)^{1/\alpha}} W_\alpha(1), \quad (2.19)$$

which means that $Y_{\alpha, \beta}(u)$ has a spectrally negative α -stable law in the case (A3). It is easy to check that (2.19) carries over to the case $\alpha = 2$, in other words, $Y_{\alpha, \beta}(u)$ has a normal law in the cases (A1) and (A2).

Assuming that $\beta \in (0, 1/\alpha)$, it follows from the defining equation (2.18) that $(Y_{\alpha, \beta}(u))_{u \geq 0}$ has a.s. continuous paths with $Y_{\alpha, \beta}(0) = 0$. Arguing along the lines of Subsection 2.2 in [18] one further concludes that $(Y_{\alpha, \beta}(u))_{u \geq 0}$ is self-similar with Hurst index $1/\alpha - \beta$, i.e., for every $c > 0$,

$$(Y_{\alpha, \beta}(cu))_{u \geq 0} \stackrel{\text{f.d.}}{=} (c^{1/\alpha - \beta} Y_{\alpha, \beta}(u))_{u \geq 0};$$

its increments are neither independent, nor stationary.

Limits in Theorem 2.9

In this case, the $(W_\alpha(u))_{u \geq 0}$ denotes an inverse α -stable subordinator as defined in Theorem 2.9 and the limiting integral is $Y_{\alpha, \beta}(u) := \int_{[0, u]} (u - y)^{-\beta} dW_\alpha(y)$. This integral can be thought of as a pathwise Lebesgue-Stieltjes integral since the integrator W_α has increasing paths. However, the finiteness of the integral should be verified. This is done in the following lemma:

Lemma 2.12. *For every $0 < \beta \leq \alpha < 1$, $\mathbb{E} Y_{\alpha, \beta}(u) < \infty$. In particular, $Y_{\alpha, \beta}(u) < \infty$ a.s. and*

$$\int_{(\rho u, u]} (u - y)^{-\beta} dW_\alpha(y) \rightarrow 0 \quad \text{a.s. as } \rho \uparrow 1.$$

Proof. It is well known that $W_\alpha(1)$ has a Mittag-Leffler law with $\mathbb{E} W_\alpha(1) = (\Gamma(1 - \alpha)\Gamma(1 + \alpha))^{-1} =: c_\alpha$. Since the α -stable subordinator is self-similar with Hurst index $1/\alpha$, $(W_\alpha(u))_{u \geq 0}$ is self-similar with Hurst index α . In particular, $\mathbb{E} W_\alpha(u) = c_\alpha u^\alpha$. Therefore,

$$\begin{aligned} \mathbb{E} Y_{\alpha, \beta}(u) &= \mathbb{E} \left(\int_{[0, u]} \left(u^{-\beta} + \beta \int_0^y (u - x)^{-\beta-1} dx \right) dW_\alpha(y) \right) \\ &= u^{-\beta} \mathbb{E} W_\alpha(u) + \beta \int_0^u \mathbb{E} (W_\alpha(u) - W_\alpha(x)) (u - x)^{-\beta-1} dx \\ &= c_\alpha u^{\alpha - \beta} + \beta c_\alpha u^{\alpha - \beta} \int_0^1 (1 - x^\alpha) (1 - x)^{-\beta-1} dx < \infty. \end{aligned}$$

□

It is implicit in the proof of Lemma 2.12 that (2.18) holds also in the given situation. From this representation, recalling the standing assumption $0 < \beta \leq \alpha < 1$, it follows that $(Y_{\alpha,\beta}(u))_{u \geq 0}$ has a.s. continuous paths. Arguing in the same way as in Section 3 of [18] one can check that $(Y_{\alpha,\beta}(u))_{u \geq 0}$ is self-similar with Hurst index $\alpha - \beta$. The latter implies that its increments are not stationary.

Further we infer from [19] that the law of $Y_{\alpha,\beta}(u)$ is uniquely determined by its moments

$$\mathbb{E} Y_{\alpha,\beta}(u)^k = u^{k(\alpha-\beta)} \frac{k!}{\Gamma(1-\alpha)^k} \prod_{j=1}^k \frac{\Gamma(1-\beta+(j-1)(\alpha-\beta))}{\Gamma(j(\alpha-\beta)+1)}, \quad k \in \mathbb{N}. \quad (2.20)$$

In particular,

$$Y_{\alpha,\beta}(u) \stackrel{d}{=} u^{\alpha-\beta} \int_0^R e^{-cZ_\alpha(t)} dt, \quad (2.21)$$

where R is a random variable with the standard exponential law which is independent of $(Z_\alpha(u))_{u \geq 0}$, a drift-free subordinator with no killing and Lévy measure

$$\nu_\alpha(dt) = \frac{e^{-t/\alpha}}{(1 - e^{-t/\alpha})^{\alpha+1}} \mathbb{1}_{(0,\infty)}(t) dt,$$

and $c := (\alpha - \beta)/\alpha$.

Now we want to investigate the covariance structure of $(Y_{\alpha,\beta}(u))_{u \geq 0}$. One can check that formula (2.20) with $k = 1$ remains valid whenever $\alpha \in (0, 1)$ and $\beta \in (-\infty, 1)$. Hence the process $(Y_{\alpha,\beta}(u))_{u \geq 0}$ is well-defined for such α and β .

Lemma 2.13. *For any $\alpha \in (0, 1)$, $\beta \in (-\infty, 1)$ and $0 < t_1 \leq t_2$,*

$$\begin{aligned} \mathbb{E} Y_{\alpha,\beta}(t_1) Y_{\alpha,\beta}(t_2) &= \frac{\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(1-\alpha)^2\Gamma(1+\alpha-\beta)} \\ &\times \int_0^{t_1} (t_1-y)^{-\beta} (t_2-y)^{-\beta} y^{\alpha-1} ((t_1-y)^\alpha + (t_2-y)^\alpha) dy. \end{aligned} \quad (2.22)$$

Proof. If $t_1 = t_2$, (2.22) coincides with (2.20) in the case $k = 2$ as it must be. Now fix $t_1 < t_2$ and set

$$\begin{aligned} H_1(y) &:= \int_{[0,y]} (t_1-x)^{-\beta} dW_\alpha(x), \quad y \in [0, t_1], \\ H_2(y) &:= \int_{[0,t_2-t_1+y]} (t_2-x)^{-\beta} dW_\alpha(x), \quad y \in [0, t_1]. \end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
Y_{\alpha,\beta}(t_1)Y_{\alpha,\beta}(t_2) &= H_1(t_1)H_2(t_1) \\
&= \int_{[0,t_1]} H_1(x) dH_2(x) + \int_{[0,t_1]} H_2(x) dH_1(x) \\
&= \int_{[0,t_1]} \int_{[0,x]} (t_1 - y)^{-\beta} dW_\alpha(y) (t_1 - x)^{-\beta} dW_\alpha(t_2 - t_1 + x) \\
&\quad + \int_{[0,t_1]} \int_{[0,x]} (t_2 - y)^{-\beta} dW_\alpha(y) (t_1 - x)^{-\beta} dW_\alpha(x) \\
&\quad + \int_{[0,t_1]} \int_{[x,t_2-t_1+x]} (t_2 - y)^{-\beta} dW_\alpha(y) (t_1 - x)^{-\beta} dW_\alpha(x) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

According to Proposition 1(a) in [4]³,

$$\mathbb{E} (dW_\alpha(x)dW_\alpha(y)) = \frac{x^{\alpha-1}(y-x)^{\alpha-1}}{\Gamma^2(\alpha)\Gamma^2(1-\alpha)} dx dy, \quad 0 < x < y < \infty. \quad (2.23)$$

Below we make a repeated use of the formula (see Lemma A.5)

$$\mathbb{E} \int_A f(x)g(y)dW_\alpha(x)dW_\alpha(y) = \int_A f(x)g(y) \mathbb{E} (dW_\alpha(x)dW_\alpha(y)),$$

where f, g are arbitrary non-negative measurable functions and $A \subset \mathbb{R}_+^2$ Borel. Using (2.23) and the change of variable yields

$$\mathbb{E} I_3 = b_\alpha \int_0^{t_1} (t_1 - x)^{-\beta} x^{\alpha-1} \int_0^{t_2-t_1} (t_2 - x - y)^{-\beta} y^{\alpha-1} dy dx$$

where $b_\alpha := \Gamma(\alpha)^{-2}\Gamma(1-\alpha)^{-2}$. Using (2.23) and changing the order of integration followed by a change of variable ($z = t_2 - t_1 + x - y$) gives

$$\begin{aligned}
\mathbb{E} I_1 &= b_\alpha \int_0^{t_1} (t_1 - x)^{-\beta} \int_0^x (t_1 - y)^{-\beta} (t_2 - t_1 + x - y)^{\alpha-1} y^{\alpha-1} dy dx \\
&= b_\alpha \int_0^{t_1} (t_1 - y)^{-\beta} y^{\alpha-1} \int_y^{t_1} (t_1 - x)^{-\beta} (t_2 - t_1 + x - y)^{\alpha-1} dx dy \\
&= b_\alpha \int_0^{t_1} (t_1 - y)^{-\beta} y^{\alpha-1} \int_0^{t_2-y} (t_2 - y - x)^{-\beta} x^{\alpha-1} dx dy \\
&\quad - b_\alpha \int_0^{t_1} (t_1 - y)^{-\beta} y^{\alpha-1} \int_0^{t_2-t_1} (t_2 - y - x)^{-\beta} x^{\alpha-1} dx dy \\
&= B_{\alpha,\beta} \int_0^{t_1} (t_1 - y)^{-\beta} (t_2 - y)^{\alpha-\beta} y^{\alpha-1} dy - \mathbb{E} I_3,
\end{aligned}$$

³Keep in mind that Bingham uses a different scaling.

where $B_{\alpha,\beta} := \frac{\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma^2(1-\alpha)\Gamma(1+\alpha-\beta)}$. Analogously

$$\begin{aligned}
\mathbb{E} I_2 &= b_\alpha \int_0^{t_1} (t_1 - x)^{-\beta} \int_0^x (t_2 - y)^{-\beta} (x - y)^{\alpha-1} y^{\alpha-1} dy dx \\
&= b_\alpha \int_0^{t_1} (t_2 - y)^{-\beta} y^{\alpha-1} \int_y^{t_1} (t_1 - x)^{-\beta} (x - y)^{\alpha-1} dx dy \\
&= b_\alpha \int_0^{t_1} (t_2 - y)^{-\beta} y^{\alpha-1} \int_0^{t_1-y} (t_1 - y - x)^{-\beta} x^{\alpha-1} dx dy \\
&= B_{\alpha,\beta} \int_0^{t_1} (t_1 - y)^{\alpha-\beta} (t_2 - y)^{-\beta} y^{\alpha-1} dy.
\end{aligned}$$

It remains to sum up these expectations. \square

Similar to the preceding lemma our next result treats both positive and negative β thereby solving a problem which has remained open in [18].

Proposition 2.14. *For $\alpha \in (0, 1)$ and $\beta \in (-\infty, 1)$, the process $(Y_{\alpha,\beta}(u))_{u \geq 0}$ does not have independent increments.*

Proof. We use the idea of the proof of Theorem 3.1 in [31]. Assume that the increments are independent. Then, with $0 < t_1 < t_2 < t_3 < \infty$,

$$\begin{aligned}
&\mathbb{E}(Y_{\alpha,\beta}(t_2) - Y_{\alpha,\beta}(t_1))(Y_{\alpha,\beta}(t_3) - Y_{\alpha,\beta}(t_2)) \\
&= \mathbb{E}(Y_{\alpha,\beta}(t_2) - Y_{\alpha,\beta}(t_1)) \mathbb{E}(Y_{\alpha,\beta}(t_3) - Y_{\alpha,\beta}(t_2)) \\
&= \frac{\Gamma(1-\beta)^2}{\Gamma(1-\alpha)^2 \Gamma(1+\alpha-\beta)^2} (t_2^{\alpha-\beta} - t_1^{\alpha-\beta})(t_3^{\alpha-\beta} - t_2^{\alpha-\beta}) =: A(t_1, t_2, t_3).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}(Y_{\alpha,\beta}(t_2) - Y_{\alpha,\beta}(t_1))(Y_{\alpha,\beta}(t_3) - Y_{\alpha,\beta}(t_2)) \\
&= \mathbb{E} Y_{\alpha,\beta}(t_2) Y_{\alpha,\beta}(t_3) - \mathbb{E} Y_{\alpha,\beta}(t_2)^2 - \mathbb{E} Y_{\alpha,\beta}(t_1) Y_{\alpha,\beta}(t_3) + \mathbb{E} Y_{\alpha,\beta}(t_1) Y_{\alpha,\beta}(t_2) \\
&= \frac{\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma^2(1-\alpha)\Gamma(1+\alpha-\beta)} \\
&\times \left(\int_0^{t_2} (t_2 - y)^{-\beta} (t_3 - y)^{-\beta} y^{\alpha-1} ((t_2 - y)^\alpha + (t_3 - y)^\alpha) dy \right. \\
&\quad - \int_0^{t_1} (t_1 - y)^{-\beta} (t_3 - y)^{-\beta} y^{\alpha-1} ((t_1 - y)^\alpha + (t_3 - y)^\alpha) dy \\
&\quad \left. + \int_0^{t_1} (t_1 - y)^{-\beta} (t_2 - y)^{-\beta} y^{\alpha-1} ((t_1 - y)^\alpha + (t_2 - y)^\alpha) dy \right) \\
&= \frac{2\Gamma(1-\beta)\Gamma(1+\alpha-2\beta)}{\Gamma^2(1-\alpha)\Gamma(1+\alpha-\beta)\Gamma(1+2\alpha-2\beta)} t_2^{2\alpha-2\beta} =: B(t_1, t_2, t_3).
\end{aligned}$$

By the assumption $A(t_1, t_2, t_3) = B(t_1, t_2, t_3)$ for all $t_1 < t_2 < t_3$. But

$$\frac{\partial^2 A(t_1, t_2, t_3)}{\partial t_1 \partial t_3} = - \frac{\Gamma^2(1-\beta)}{\Gamma^2(1-\alpha)\Gamma^2(1+\alpha-\beta)} (\alpha-\beta)^2 (t_1 t_3)^{\alpha-\beta-1},$$

and

$$\begin{aligned}
& \frac{\partial^2 B(t_1, t_2, t_3)}{\partial t_1 \partial t_3} \\
&= - \frac{\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma^2(1 - \alpha)\Gamma(1 + \alpha - \beta)} \\
& \quad \frac{\partial^2}{\partial t_1 \partial t_3} \left(\int_0^{t_1} (t_1 - y)^{-\beta} (t_3 - y)^{-\beta} y^{\alpha-1} ((t_1 - y)^\alpha + (t_3 - y)^\alpha) dy \right) \\
&= - \frac{\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma^2(1 - \alpha)\Gamma(1 + \alpha - \beta)} \\
& \quad \times \frac{\partial^2}{\partial t_1 \partial t_3} \left(t_1^{\alpha-\beta} \int_0^1 (1 - y)^{-\beta} (t_3 - t_1 y)^{-\beta} y^{\alpha-1} (t_1^\alpha (1 - y)^\alpha + (t_3 - t_1 y)^\alpha) dy \right) \\
&= - \frac{\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma^2(1 - \alpha)\Gamma(1 + \alpha - \beta)} \\
& \quad \times \left[\frac{\partial}{\partial t_1} \left(-\beta t_1^{2\alpha-\beta} \int_0^1 (1 - y)^{\alpha-\beta} y^{\alpha-1} (t_3 - t_1 y)^{-\beta-1} dy \right) \right. \\
& \quad \left. + \frac{\partial}{\partial t_1} \left((\alpha - \beta) t_1^{\alpha-\beta} \int_0^1 (1 - y)^{-\beta} y^{\alpha-1} (t_3 - t_1 y)^{\alpha-\beta-1} dy \right) \right] \\
&= - \frac{\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma^2(1 - \alpha)\Gamma(1 + \alpha - \beta)} \\
& \quad \times \left[-\beta(2\alpha - \beta) t_1^{2\alpha-\beta-1} \int_0^1 (1 - y)^{\alpha-\beta} y^{\alpha-1} (t_3 - t_1 y)^{-\beta-1} dy \right. \\
& \quad - \beta(\beta + 1) t_1^{2\alpha-\beta} \int_0^1 (1 - y)^{\alpha-\beta} y^\alpha (t_3 - t_1 y)^{-\beta-2} dy \\
& \quad + (\alpha - \beta)^2 t_1^{\alpha-\beta-1} \int_0^1 (1 - y)^{-\beta} y^{\alpha-1} (t_3 - t_1 y)^{\alpha-\beta-1} dy \\
& \quad \left. - (\alpha - \beta)(\alpha - \beta - 1) t_1^{\alpha-\beta} \int_0^1 (1 - y)^{-\beta} y^\alpha (t_3 - t_1 y)^{\alpha-\beta-2} dy \right].
\end{aligned}$$

To show that these expressions are not equal, assume that $0 < t_1 < 1$ and set $t_3 = t_1^{-1}$, $z := t_1^2$. Then the first one does not depend on z . The second, after some manipulations, becomes

$$\begin{aligned}
D(z) &:= - \frac{\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma^2(1 - \alpha)\Gamma(1 + \alpha - \beta)} \\
& \quad \times \left[-\beta(2\alpha - \beta) z^\alpha \int_0^1 (1 - y)^{\alpha-\beta} y^{\alpha-1} (1 - zy)^{-\beta-1} dy \right. \\
& \quad - \beta(\beta + 1) z^{\alpha+1} \int_0^1 (1 - y)^{\alpha-\beta} y^\alpha (1 - zy)^{-\beta-2} dy \\
& \quad + (\alpha - \beta)^2 \int_0^1 (1 - y)^{-\beta} y^{\alpha-1} (1 - zy)^{\alpha-\beta-1} dy \\
& \quad \left. - (\alpha - \beta)(\alpha - \beta - 1) z \int_0^1 (1 - y)^{-\beta} y^\alpha (1 - zy)^{\alpha-\beta-2} dy \right].
\end{aligned}$$

Using the asymptotic expansion $(1 - z)^\alpha = 1 - \alpha z + O(z^2)$ which holds for $\alpha \in \mathbb{R}$ as $z \rightarrow 0$ yields

$$D(z) = -\frac{\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma^2(1 - \alpha)\Gamma(1 + \alpha - \beta)} \times \\ \times \left[(\alpha - \beta)^2 \int_0^1 (1 - y)^{-\beta} y^{\alpha-1} dy - \beta(2\alpha - \beta)z^\alpha \int_0^1 (1 - y)^{\alpha-\beta} y^{\alpha-1} dy \right. \\ \left. + O(z) \right],$$

as $z \rightarrow 0$. From this expansion it is clear that $D(z)$ depends on z if $\beta(2\alpha - \beta) \neq 0$ since $\alpha < 1$. If $\beta = 0$ then $Y_{\alpha,\beta}(u) = W_\alpha(u)$ and this process does not have independent increments as was shown in Theorem 3.1 in [31]. If $2\alpha = \beta$ using the same idea one can show

$$D(z) = c_1 + c_2 z + O(z^{\alpha+1})$$

where $c_1 c_2 \neq 0$, we omit the details. \square

Formula (2.21) entails that

$$Y_{\alpha,\alpha}(u) \stackrel{d}{=} R,$$

i.e., all one-dimensional distributions of $(Y_{\alpha,\alpha}(u))_{u \geq 0}$ are standard exponential. This leads to a conjecture that the process $(Y_{\alpha,\alpha}(u))_{u \geq 0}$ may bear some kind of stationarity.

Lemma 2.15. *The process $(Y_{\alpha,\alpha}(e^u))_{u \in \mathbb{R}}$ is strictly stationary with covariance function $R(s) := \mathbb{E}(Y_{\alpha,\alpha}(e^u) - 1)(Y_{\alpha,\alpha}(e^{u+s}) - 1)$, $s \in \mathbb{R}$ given by*

$$R(s) = \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)} \int_{|s|}^{\infty} (1 - e^{-y})^{-\alpha} e^{-\alpha y} dy, \quad s \in \mathbb{R}. \quad (2.24)$$

Proof. The strict stationarity follows from the case $\alpha = \beta$ and $c < \infty$ of Theorem 2.9. Indeed, by that theorem, as $t \rightarrow \infty$, $(X(u_1 t), \dots, X(u_n t)) \xrightarrow{d} c(Y_{\alpha,\alpha}(u_1), \dots, Y_{\alpha,\alpha}(u_n))$ for any $n \in \mathbb{N}$ and any $0 < u_1 < \dots < u_n$, and, for any $h > 0$, the weak limit of $(X(u_1 h t), \dots, X(u_n h t))$ is obviously the same. To prove (2.24), it suffices to show that, for $0 < t_1 < t_2 < \infty$,

$$\mathbb{E} Y_{\alpha,\alpha}(t_1) Y_{\alpha,\alpha}(t_2) = 1 + \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)} \int_0^{t_1/t_2} (1 - y)^{-\alpha} y^{\alpha-1} dy.$$

The last equality follows from (2.22) with $\alpha = \beta$. \square

3 Preliminaries

3.1 Stationary renewal processes and coupling

Our first result in this section shows that the finite-dimensional distributions of the increments of the stationary renewal counting process is invariant under time reversal. The proof is based on a simple coupling argument and an application of the point-at-zero duality (Theorem 4.1 in Chapter 8 of [41]).

Proposition 3.1. *Let $\mu < \infty$ and F be non-lattice. Then, for every $t > 0$,*

$$(N^*(t) - N^*((t-s)-) : 0 \leq s \leq t) \stackrel{d}{=} (N^*(s) : 0 \leq s \leq t).$$

Proof. For the proof of this proposition, it is convenient to embed the zero-delayed random walk $(S_k)_{k \in \mathbb{N}_0}$ into a two-sided random walk $(S_k)_{k \in \mathbb{Z}}$. To this end, assume that on the basic probability space, there is an independent copy $(\xi_{-k})_{k \in \mathbb{N}}$ of the sequence $(\xi_k)_{k \in \mathbb{N}}$. Let $S_{-k} := -(\xi_{-1} + \dots + \xi_{-k})$ for $k \in \mathbb{N}$. Further, assume there is a random variable ξ_0 which is independent of the ξ_k , $k \in \mathbb{Z} \setminus \{0\}$ with size-biased distribution

$$\mathbb{P}\{\xi_0 \in B\} = \mu^{-1} \mathbb{E} \xi \mathbb{1}_{\{\xi \in B\}} \quad (B \subseteq [0, \infty) \text{ Borel}).$$

Finally, let U have the uniform distribution on $(0, 1)$ and assume that U is independent of the sequence $(\xi_k)_{k \in \mathbb{Z}}$. Define $S_0^* := U\xi_0$ and $S_{-1}^* := -(1 - U)\xi_0$. Then S_0^* and $-S_{-1}^*$ have distribution function F^* (see *e.g.* [41, p. 261] for a quick proof). Now recall that $S_k^* = S_0^* + S_k$ for $k \in \mathbb{N}$ and define analogously $S_k^* := S_{-1}^* + S_{k+1}$ for $k < -1$. By the first Palm duality (point-at-zero duality, Theorem 4.1 in Chapter 8 of [41]), the process $N_{\mathbb{Z}}^* := \sum_{k \in \mathbb{Z}} \delta_{S_k^*}$ is distributionally invariant under shifts. Using this and the fact that $(-S_{-k}^*)_{k \in \mathbb{N}}$ has the same distribution as $(S_{k-1}^*)_{k \in \mathbb{N}}$, we infer for given $t > 0$

$$(N_{\mathbb{Z}}^*[0, s] : 0 \leq s \leq t) \stackrel{d}{=} (N_{\mathbb{Z}}^*[-t, -(t-s)] : 0 \leq s \leq t) \stackrel{d}{=} (N_{\mathbb{Z}}^*[t-s, t] : 0 \leq s \leq t).$$

This implies the assertion in view of the fact that $N^*(\cdot) = N_{\mathbb{Z}}^*(\cdot \cap [0, \infty))$. \square

Next, we briefly introduce a classical coupling that will prove useful in several proofs in this paper and which works in the case when $\mu < \infty$ and F is non-lattice. Let $(\hat{\xi}_k)_{k \in \mathbb{N}}$ be an independent copy of the sequence $(\xi_k)_{k \in \mathbb{N}}$. Let $\hat{\xi}_0^*$ denote a random variable that is independent of all previously introduced random variables and has distribution function F^* (recall the definition of F^* from (2.1)). Put

$$\hat{S}_0 := 0 \quad \text{and} \quad \hat{S}_k := \hat{\xi}_1 + \dots + \hat{\xi}_k, \quad k \in \mathbb{N}.$$

Let \hat{N} be the renewal counting process associated with the process $(\hat{S}_k)_{k \in \mathbb{N}_0}$. In particular, $\hat{N}(t) := \hat{N}[0, t] = \#\{k \in \mathbb{N}_0 : \hat{S}_k \leq t\}$, $t \geq 0$. Further, define $\hat{S}_k^* := \hat{\xi}_0^* + \hat{S}_k$, $k \in \mathbb{N}_0$ and let $\hat{N}^* := \sum_{k \geq 0} \delta_{\hat{S}_k^*}$ denote the associated renewal counting process. As usual, put $\hat{N}^*(t) := \hat{N}^*[0, t]$, $t \geq 0$. By construction, $(\hat{N}^*(t))_{t \geq 0}$ is a stationary renewal process.

It is known (see *e.g.* [29, p. 74]) that, for any fixed $\varepsilon > 0$, the stopping time

$$\tau := \tau(\varepsilon) = \begin{cases} \inf\{k \in \mathbb{N}_0 : |S_k - \hat{S}_k^*| \leq \varepsilon\}, & \text{if } \inf_{k \in \mathbb{N}_0} |S_k - \hat{S}_k^*| \leq \varepsilon, \\ +\infty, & \text{otherwise,} \end{cases}$$

called the *time of ε -coupling*, is a.s. finite. Define the coupled random walk

$$\tilde{S}_k^* := \begin{cases} \hat{S}_k^*, & \text{for } k \leq \tau, \\ S_k - (S_\tau - \hat{S}_\tau^*) = \hat{S}_\tau^* + \sum_{j=\tau+1}^k \xi_j, & \text{for } k \geq \tau + 1, \end{cases}$$

$k \in \mathbb{N}_0$. Then $(\tilde{S}_k^*)_{k \in \mathbb{N}_0} \stackrel{d}{=} (\hat{S}_k^*)_{k \in \mathbb{N}_0} \stackrel{d}{=} (S_k^*)_{k \in \mathbb{N}_0}$. In particular, the random process $(\tilde{N}^*(t))_{t \geq 0}$ defined by

$$\tilde{N}^*(t) := \#\{k \in \mathbb{N}_0 : \tilde{S}_k^* \leq t\},$$

is a stationary renewal process. Further, the construction of the process $(\tilde{S}_k^*)_{k \in \mathbb{N}_0}$ guarantees that

$$\tilde{S}_k^* - \varepsilon \leq S_k \leq \tilde{S}_k^* + \varepsilon \quad (3.1)$$

for $k \geq \tau + 1$.

Further, one can check that, for any fixed $\varepsilon > 0$ and arbitrary $0 \leq y \leq t$, on $\{\tau < \infty\}$ (hence, with probability one),

$$\begin{aligned} \sum_{k \geq \tau+1} \mathbb{1}_{\{t-y < S_k \leq t\}} &\leq \sum_{k \geq \tau+1} \mathbb{1}_{\{t-y-\varepsilon < \tilde{S}_k^* \leq t+\varepsilon\}} \\ &\leq \tilde{N}^*(t+\varepsilon) - \tilde{N}^*(t-y-\varepsilon) \end{aligned} \quad (3.2)$$

where $\tilde{N}^*(x) := 0$ for $x < 0$ is stipulated. Similarly, for any fixed $\varepsilon > 0$ and $0 \leq y \leq t$, again on $\{\tau < \infty\}$ (and thus with probability one),

$$\begin{aligned} \sum_{k \geq \tau+1} \mathbb{1}_{\{t-y < S_k \leq t\}} &\geq \sum_{k \geq \tau+1} \mathbb{1}_{\{t-y+\varepsilon < \tilde{S}_k^* \leq t-\varepsilon\}} \\ &= \tilde{N}^*(t-\varepsilon) - \tilde{N}^*(t-y+\varepsilon) - \sum_{k=0}^{\tau} \mathbb{1}_{\{t-y+\varepsilon < \tilde{S}_k^* \leq t-\varepsilon\}}. \end{aligned} \quad (3.3)$$

3.2 Stable distributions and domains of attraction

Naturally, the asymptotics of the shot noise process $(X(ut))_{u \geq 0}$ as $t \rightarrow \infty$ is connected to the limiting behavior of S_k as $k \rightarrow \infty$. Under the assumptions of the limit theorems with scaling, F (the distribution of ξ) is in the domain of attraction of a stable law. To be more precise, for appropriate constants $c_k > 0$ and $b_k \in \mathbb{R}$,

$$\frac{S_k - b_k}{c_k} \xrightarrow{d} W \quad \text{as } k \rightarrow \infty \quad (3.4)$$

where W has a stable law $S_\alpha(\sigma', \beta', \mu')$ that is characterized by four parameters, the *index of stability* α and the *scale*, *skewness*, and *shift* parameters, σ' , β' , and μ' , respectively. We refer to [38] for the precise definition of the law $S_\alpha(\sigma', \beta', \mu')$ as well as for a general introduction to stable random variables. Due to the assumption that $\xi > 0$ a.s., $S_\alpha(\sigma', \beta', \mu')$ will automatically be totally skewed to the right, *i.e.*, it will have skewness parameter $\beta' = 1$.

By changing b_k and c_k if necessary, it can be arranged that the shift parameter μ' equals 0 and the scale parameter σ' is as we wish. Particularly, the limit can be arranged to be standard normal in the case $\alpha = 2$ and to have characteristic function

$$z \mapsto \exp \left\{ -|z|^\alpha \Gamma(1-\alpha) (\cos(\pi\alpha/2) - i \sin(\pi\alpha/2) \operatorname{sign}(z)) \right\}, \quad z \in \mathbb{R} \quad (3.5)$$

in case $0 < \alpha < 2$, $\alpha \neq 1$. The constants b_k and c_k that produce this limit in (3.4) can be described in terms of the distribution of ξ .

- (D1) The case $\sigma^2 < \infty$:
 If $\sigma^2 := \text{Var } \xi < \infty$, then (3.4) holds with $b_k = k\mu$, $c_k = \sigma\sqrt{k}$, $k \in \mathbb{N}$. W then has the standard normal law.
- (D2) The case when $\sigma^2 = \infty$, yet there is attraction to a normal law:
 If $\int_{[0,t]} y^2 \mathbb{P}\{\xi \in dy\} \sim \ell(t)$ for some function ℓ that is slowly varying at $+\infty$, then (3.4) holds with $b_k = k\mu$ and the c_k , $k \geq 1$ being such that $\lim_{k \rightarrow \infty} \frac{k\ell(c_k)}{c_k^2} = 1$. Again, W has the standard normal law.
- (D3) The case $1 < \alpha < 2$:
 If $\mathbb{P}\{\xi > t\} \sim t^{-\alpha}\ell(t)$ for some $1 < \alpha < 2$ and some ℓ slowly varying at ∞ , then (3.4) holds with $b_k = k\mu$ and the c_k , $k \geq 1$ being such that $k\mathbb{P}\{\xi > c_k\} \sim k\ell(c_k)/c_k^\alpha \rightarrow 1$. W then has a spectrally positive stable law with characteristic function given by (3.5).
- (D4) The case $0 < \alpha < 1$:
 If $\mathbb{P}\{\xi > t\} \sim t^{-\alpha}\ell(t)$ as $t \rightarrow \infty$ for some $0 < \alpha < 1$ and some ℓ slowly varying at ∞ , (3.4) holds with $b_k = 0$ and c_k , $k \geq 1$ being such that $k\mathbb{P}\{\xi > c_k\} \sim k\ell(c_k)/c_k^\alpha \rightarrow 1$. Again, W has a spectrally positive stable law with characteristic function given by (3.5). Further, $W > 0$ a.s. in this case and its Laplace exponent is given by

$$-\log \varphi(s) = \Gamma(1 - \alpha)s^\alpha, \quad s \geq 0.$$

Clearly, (D1) is the classical central limit theorem. (D2) follows from [10, Theorem IX.8.1 and Eq. (IX.8.12)]. (D3) and (D4) follow from the lemma on p. 107 of [9]. (D4) is also Theorem XIII.7.2 in [10] (where Laplace transform methods are used rather than characteristic functions).

3.3 Weak convergence of the renewal counting process

When F is in the domain of attraction of a stable law of index $1 < \alpha \leq 2$, then $N(t)$, centered and suitably scaled, converges in law to a stable random variable as $t \rightarrow \infty$. This convergence in law carries over to a functional convergence which forms the core of our analysis. Here, we give a brief summary of the relevant results and the necessary notation.

Denote by $D := D[0, \infty)$ the space of right-continuous real-valued functions on $[0, \infty)$ with finite limits from the left. It is well known (see, for instance, Theorem 5.3.1 and Theorem 5.3.2 in [14] or Section 7.3.1 in [45]) that the following functional limit theorems hold:

$$W_t(u) := \frac{N(ut) - \mu^{-1}ut}{g(t)} \Rightarrow W(u) \quad \text{as } t \rightarrow \infty \quad (3.6)$$

where

- in the case (D1) $(W(u))_{u \geq 0}$ is a Brownian motion; $g(t) = \sqrt{\sigma^2 \mu^{-3}t}$ and the convergence takes place in J_1 topology on D ;

- in the case (D2) $(W(u))_{u \geq 0}$ is a Brownian motion; $g(t) = \mu^{-3/2}c(t)$ where $c(t)$ is any positive continuous function such that $\lim_{t \rightarrow \infty} t\ell(c(t))c(t)^{-2} = 1$; the convergence takes place in J_1 topology on D ;
- in the case (D3) $(W(u))_{u \geq 0}$ is a spectrally negative α -stable Lévy process such that $W(1)$ has the characteristic function given in (2.15); $g(t) = \mu^{-1-1/\alpha}c(t)$ where $c(t)$ is any positive continuous function with $\lim_{t \rightarrow \infty} t\ell(c(t))c^{-\alpha}(t) = 1$; the convergence takes place in M_1 topology on D .

We refer to [45] for extensive information concerning both the J_1 and M_1 convergence in D .

For later use we note the following lemma.

Lemma 3.2. *$c(t)$ and $g(t)$ are regularly varying with index $1/2$ in the cases (D1) and (D2), and with index $1/\alpha$ in the case (D3). As a consequence, given $A > 1$ and $\delta \in (0, 1/\alpha)$ (here, $\alpha = 2$ in the cases (D1) and (D2)) there exists $t_0 > 0$ such that*

$$\frac{g(tv)}{g(t)} \leq Av^{1/\alpha-\delta}, \quad (3.7)$$

for all $v \leq 1$ and $tv \geq t_0$.

Proof. We only check this for the case (D2), the case (D3) being similar, and the case (D1) being trivial.

The function $c(t)$ is an asymptotic inverse of

$$t^2 \left(\int_{[0,t]} y^2 \mathbb{P}\{\xi \in dy\} \right)^{-1} \sim t^2/\ell(t).$$

Hence, by Proposition 1.5.15 in [5], $c(t) \sim t^{1/2}(L^\#(t))^{1/2}$ where $L^\#(t)$ is the de Bruijn conjugate of $L(t) = 1/\ell(t^{1/2})$. The de Bruijn conjugate is slowly varying and hence c regularly varying of index $1/2$.

(3.7) is Potter's bound for g (see Theorem 1.5.6 in [5]). \square

There is also an analog of (3.6) in the case (D4). The functional convergence

$$W_t(u) := \frac{N(ut)}{g(t)} \Rightarrow W(u) \quad \text{as } t \rightarrow \infty \quad (3.8)$$

under the J_1 topology in D where $(W(u))_{u \geq 0}$ is an inverse α -stable subordinator and $g(t) := 1/\mathbb{P}\{\xi > t\}$ was proved in Corollary 3.4 in [31]. (Notice that the function $\tilde{b}(t)$ in the cited article can be chosen as $g(t) = \mathbb{P}\{\xi > t\}^{-1}$, $t \geq 0$.)

3.4 Moment convergence

The next result is on the asymptotic behavior of $\mathbb{E}|N(t) - t/\mu|$ in the case when $\mu = \mathbb{E}\xi < \infty$. We do not claim the result stated below is new. However, we have been unable to locate it in the literature.

Theorem 3.3. *Let $\mu := \mathbb{E} \xi < \infty$. Then the following assertions hold:*

(A1) *If $\sigma^2 := \text{Var} \xi < \infty$, then*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} |N(t) - \mu^{-1}t|}{\sqrt{t}} = \frac{\sigma}{\mu^{3/2}} \mathbb{E} |W| = \sigma \sqrt{\frac{2}{\pi \mu^3}}$$

where W is a random variable with standard normal law.

(A2) *Suppose $\sigma^2 = \infty$ and*

$$\int_{[0, t]} y^2 \mathbb{P}\{\xi \in dy\} \sim \ell(t) \quad \text{as } t \rightarrow \infty$$

for some ℓ slowly varying at ∞ . Then

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} |N(t) - \mu^{-1}t|}{c(t)} = \frac{1}{\mu^{3/2}} \mathbb{E} |W| = \sqrt{\frac{2}{\pi \mu^3}}$$

where $c(t)$ is a positive function satisfying $\lim_{t \rightarrow \infty} t\ell(c(t))/c(t)^2 = 1$, and W is a random variable with standard normal law.

(A3) *Suppose $\mathbb{P}\{\xi > t\} \sim t^{-\alpha}\ell(t)$ as $t \rightarrow \infty$ for some $\alpha \in (1, 2)$ and some ℓ slowly varying at ∞ . Then*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} |N(t) - \mu^{-1}t|}{c(t)} = \frac{\mathbb{E} |W|}{\mu^{1+1/\alpha}} = \frac{2\Gamma(1 - \frac{1}{\alpha})|\Gamma(1 - \alpha)|^{1/\alpha} \sin(\frac{\pi}{\alpha})}{\pi \mu^{1+1/\alpha}}$$

where $c(t)$ is a positive function such that $\lim_{t \rightarrow \infty} t\ell(c(t))c(t)^{-\alpha} = 1$, and W is a random variable with characteristic function given by (2.15).

In any of the three cases (A1)-(A3), $\mathbb{E} |N^*(t) - \mu^{-1}t| \sim (\mathbb{E} |N(t) - \mu^{-1}t|)$ as $t \rightarrow \infty$.

Assertion (A1) is well known and can be found, for instance, in [14, Theorem 3.8.4(i)]. The proof of assertion (A1) is included here, for it requires no extra work in the given framework.

Knowing already the weak convergence relation (3.6) (evaluated at $u = 1$), uniform integrability of the family $|N(t) - t/\mu|/c(t)$, $t \geq 1$ would suffice to conclude convergence of the first absolute moments. This approach has been used in [14]. However, the relevant results on uniform integrability presented in the cited book (such as Theorem 3.7.3 there) are not strong enough for our purposes. This is why we follow a different approach in the proof.

Proof of Theorem 3.3. Our purpose is to show that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} |N(t) - \mu^{-1}t|}{g(t)} = \mathbb{E} |W|. \quad (3.9)$$

where g is defined as in the context of (3.6). We start with the representation

$$\begin{aligned} \mathbb{E} |S_{N(\mu n)} - S_n| &= \mathbb{E} (S_{N(\mu n) \vee n} - S_{N(\mu n) \wedge n}) \\ &= \mu \mathbb{E} ((N(\mu n) \vee n) - (N(\mu n) \wedge n)) = \mu \mathbb{E} |N(\mu n) - n|, \end{aligned}$$

where the second equality follows from Wald's identity. We thus infer

$$\begin{aligned}\mathbb{E}(|S_n - \mu n| - (S_{N(\mu n)} - \mu n)) &\leq \mu \mathbb{E}|N(\mu n) - n| \\ &\leq \mathbb{E}(|S_n - \mu n| + (S_{N(\mu n)} - \mu n)).\end{aligned}\quad (3.10)$$

From the discussion in (3.2) we know that $c(n)^{-1}(S_n - n\mu) \xrightarrow{d} -W$. Furthermore, according to Lemma 5.2.2 in [16],

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left(\frac{|S_n - \mu n|}{c(n)} \right)^{1+\delta} < \infty$$

for some $\delta > 0$. Consequently,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}|S_n - \mu n|}{c(n)} = \mathbb{E}|W|. \quad (3.11)$$

If F is non-lattice, then from [33] it is known that

$$\mathbb{E}(S_{N(t)} - t) \sim \begin{cases} \text{const} & \text{in the case (A1),} \\ \text{const} \cdot \ell(t) & \text{in the case (A2),} \\ \text{const} \cdot t^{2-\alpha} \ell(t) & \text{in the case (A3),} \end{cases} \quad (3.12)$$

as $t \rightarrow \infty$. Similar asymptotics hold in the lattice case. In fact, when F is lattice with span $d > 0$, then, in the case (A1),

$$\lim_{n \rightarrow \infty} \mathbb{E}(S_{N(nd)} - nd) \rightarrow \text{const}$$

by Theorem 9 in [9]. Hence

$$\mathbb{E}(S_{N(t)} - t) = O(1) \quad \text{as } t \rightarrow \infty.$$

In the cases (A2) and (A3), according to Theorem 6 in [40], $\mathbb{E}(S_{N(s)} - s)$ again exhibits the same asymptotics as in the non-lattice case.

Recalling that $c(t)$ is regularly varying at ∞ with index $1/\alpha$ (where $\alpha = 2$ in the Cases (A1) and (A2)), we conclude that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} S_{N(\mu n)} - \mu n}{c(n)} = 0.$$

Applying this and (3.11) to (3.10) we infer

$$\lim_{n \rightarrow \infty} \mu \frac{\mathbb{E}|N(\mu n) - n|}{c(n)} = \mathbb{E}|W|.$$

Now we have to check that this relation implies (3.9). For any $t > 0$ there exists $n = n(t) \in \mathbb{N}_0$ such that $t \in (\mu n, \mu(n+1)]$. Hence, by subadditivity,

$$\mathbb{E}(N(t) - N(\mu n)) \leq \mathbb{E}(N(\mu(n+1)) - N(\mu n)) \leq \mathbb{E}N(\mu).$$

It remains to observe that, as a consequence of the regular variation of $c(t)$, we have $\lim_{t \rightarrow \infty} c(\mu n(t)\mu^{-1})/c(t) = \mu^{-1/\alpha}$, hence (3.9).

The formula for $\mathbb{E}|W|$ in the case (A3) is proved in Lemma A.1.

It remains to check that $\mathbb{E}|N^*(t) - \mu^{-1}t|$ exhibits the same asymptotics as $\mathbb{E}|N(t) - \mu^{-1}t|$. But this is a consequence of the chain of equalities

$$\mathbb{E}|N^*(t) - N(t)| = \mathbb{E}(N(t) - N^*(t)) = \mu^{-1}(\mathbb{E}S_{N(t)} - t) = o(c(t)) \text{ as } t \rightarrow \infty$$

where the second equality follows from Wald's equation and the third is a consequence of (3.12). The proof is complete. \square

4 Proofs of the limit theorems without scaling

4.1 Preparatory results

A couple of preliminary results are needed before Theorems 2.1 and 2.4 can be proved.

Proposition 4.1. *Assume that $\mu < \infty$ and that F is non-lattice. Then, for any $k \in \mathbb{N}_0$,*

$$(t - S_{N(t)-1}, \xi_{N(t)-1}, \dots, \xi_{N(t)-k}, \mathbb{1}_{\{N(t) > k\}}) \xrightarrow{d} (S_0^*, \xi_1, \dots, \xi_k, 1) \text{ as } t \rightarrow \infty.$$

Notice that in Proposition 4.1, there is a small change in perspective regarding the number of visits $N(t)$ to the interval $[0, t]$. $N(t)$ can also be viewed as the first time at which the walk exceeds the level t . Therefore, we always have $t - S_{N(t)-1} \geq 0$.

Proof. We prove the assertion for $k = 1$. Since $\mathbb{P}\{N(t) \geq 2\} \rightarrow 1$ we can centre our attention on events contained in $\{N(t) \geq 2\}$. Fix $x, y \geq 0$. Then

$$\begin{aligned} & \mathbb{P}\{t - S_{N(t)-1} \leq x, \xi_{N(t)-1} \leq y, N(t) \geq 2\} \\ &= \sum_{k \geq 2} \mathbb{P}\{S_{k-1} \leq t, S_k > t, t - S_{k-1} \leq x, \xi_{k-1} \leq y\} \\ &= \sum_{k \geq 2} \mathbb{E} \mathbb{1}_{\{t - S_{k-2} - x \leq \xi_{k-1} \leq (t - S_{k-2}) \wedge y\}} (1 - F(t - S_{k-2} - \xi_{k-1})) \\ &= \int_{[0, t]} \int_{[t-s-x, (t-s) \wedge y]} (1 - F(t - s - z)) F(dz) U(ds) \\ &= \int_{[0, t]} f_{x,y}(t - s) U(ds) \end{aligned} \tag{4.1}$$

where $f_{x,y}(t) = \int_{[t-x, t \wedge y]} (1 - F(t - z)) F(dz)$. We claim that $f_{x,y}$ is d.R.i. To see this, first observe that

$$\begin{aligned} 0 \leq f_{x,y}(t) &= \mathbb{P}\{t - x \leq \xi_1 \leq t \wedge y, \xi_1 + \xi_2 > t\} \\ &\leq \mathbb{P}\{\xi_1 + \xi_2 > t\} =: f(t). \end{aligned}$$

f is decreasing and integrable since $\mathbb{E}(\xi_1 + \xi_2) = 2\mu < \infty$. Thus, f is d.R.i. and, therefore, $f_{x,y}$ is sandwiched between two d.R.i. functions and has only

countably many discontinuities. Consequently, $f_{x,y}$ is d.R.i. This justifies the application of the key renewal theorem in (4.1) which gives

$$\lim_{t \rightarrow \infty} \mathbb{P}\{t - S_{N(t)-1} \leq x, \xi_{N(t)-1} \leq y, N(t) \geq 2\} = \frac{1}{\mu} \int_{[0, \infty)} f_{x,y}(s) \, ds.$$

The latter integral can be rewritten as follows:

$$\begin{aligned} \int_0^\infty f_{x,y}(s) \, ds &= \int_0^\infty \int_{[s-x, s \wedge y]} (1 - F(s-z)) F(dz) \, ds \\ &= \int_{[0, \infty)} \int_0^\infty \mathbb{1}_{\{s-x \leq z \leq s \wedge y\}} (1 - F(s-z)) \, ds \, F(dz) \\ &= \int_{[0, y]} \int_z^{(z+x) \wedge y} (1 - F(s-z)) \, ds \, F(dz) \\ &\quad + \int_{[0, y]} \int_y^{z+x} (1 - F(s-z)) \, ds \, F(dz) \\ &= \int_{[0, y]} \int_0^{x \wedge (y-z)} (1 - F(s)) \, ds \, F(dz) \\ &\quad + \int_{[0, y]} \int_{y-z}^x (1 - F(s)) \, ds \, F(dz) \\ &= \int_{[0, y]} \int_0^x (1 - F(s)) \, ds \, F(dz) \\ &= F(y) \mu F^*(x) = \mu \mathbb{P}\{S_0^* \leq x, \xi_1 \leq y\}. \end{aligned}$$

The case $k = 0$ is a particular case of the case $k = 1$. The case $k > 1$ can be treated similarly but the calculations become more involved. \square

Now we show that under the assumptions of Theorems 2.1 and 2.4, the limiting variables X^* and X_\circ^* , respectively, are well-defined. It turns out that in the case of X_\circ^* , this is more than halfway to proving one-dimensional convergence in Theorem 2.4.

Proposition 4.2. *Assume that $\mu < \infty$ and that F is non-lattice. If h is d.R.i., then X^* exists as the a.s. limit and the limit in \mathcal{L}^1 in (2.2). In particular, X^* is integrable, a fortiori a.s. finite.*

Proof. Since $\mu < \infty$, the stationary walk $(S_k^*)_{k \in \mathbb{N}_0}$ exists and we can define $X_n^* := \sum_{k=0}^n h(S_k^*)$, $n \in \mathbb{N}_0$. Recall that $U^*(dx) = \sum_{k \geq 0} \mathbb{P}\{S_k^* \in dx\}$ equals $\mu^{-1} dx$. Hence, since $|h|$ is assumed to be d.R.i., we have

$$\mathbb{E} \sum_{k \geq 0} |h(S_k^*)| = \int_0^\infty |h(x)| \, dx < \infty.$$

Consequently, $X_n^* \rightarrow X^*$ a.s. and in \mathcal{L}^1 . \square

Proposition 4.3. *Assume that $\mu < \infty$ and that F is non-lattice. Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally bounded, eventually decreasing and non-integrable and recall that*

$$X_\circ^* := \lim_{t \rightarrow \infty} \left(\sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{S_k^* \leq t\}} - \frac{1}{\mu} \int_0^t h(y) \, dy \right).$$

Under the assumptions of Theorem 2.4, X_\circ^ exists as the limit in \mathcal{L}^2 in the case (C1), as the a.s. limit in the case (C2) and as the limit in probability in the case (C3), and in all three cases, it is a.s. finite.*

Proof. Define

$$X_t^* := \sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{S_k^* \leq t\}} - \frac{1}{\mu} \int_0^t h(y) \, dy, \quad t \geq 0.$$

Our aim is to show that X_t^* converges as $t \rightarrow \infty$ in the asserted sense.

We start with the case (C1) and first prove the result assuming that h is decreasing on \mathbb{R}_+ . We then have to show that X_t^* converges in \mathcal{L}^2 as $t \rightarrow \infty$, equivalently,

$$\limsup_{s \rightarrow \infty} \mathbb{E}(X_t^* - X_s^*)^2 = 0.$$

Since

$$X_t^* - X_s^* = \sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{s < S_k^* \leq t\}} - \mathbb{E} \sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{s < S_k^* \leq t\}}$$

for $t > s$, we conclude that

$$\begin{aligned} \mathbb{E}(X_t^* - X_s^*)^2 &= \mathbb{E} \left(\sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{s < S_k^* \leq t\}} \right)^2 - \left(\mathbb{E} \sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{s < S_k^* \leq t\}} \right)^2 \\ &= \mathbb{E} \left(\sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{s < S_k^* \leq t\}} \right)^2 - \left(\frac{1}{\mu} \int_s^t h(y) \, dy \right)^2 \\ &= \mathbb{E} \left(\sum_{k \geq 0} h(t - S_k^*) \mathbb{1}_{\{S_k^* < t - s\}} \right)^2 - \left(\frac{1}{\mu} \int_s^t h(y) \, dy \right)^2, \end{aligned}$$

where the last equality follows from Proposition 3.1. The first term on the right-hand side equals

$$\begin{aligned} &\mathbb{E} \sum_{k \geq 0} h(t - S_k^*)^2 \mathbb{1}_{\{S_k^* < t - s\}} + 2 \mathbb{E} \sum_{0 \leq i < j} h(t - S_i^*) \mathbb{1}_{\{S_i^* < t - s\}} h(t - S_j^*) \mathbb{1}_{\{S_j^* < t - s\}} \\ &= \frac{1}{\mu} \int_0^{t-s} h(t-y)^2 \, dy + \frac{2}{\mu} \int_0^{t-s} h(t-y) \int_{(0, t-s-y)} h(t-y-x) U(dx) \, dy \\ &= \frac{1}{\mu} \int_s^t h(y)^2 \, dy + \frac{2}{\mu} \int_s^t h(y) \int_{(0, y-s)} h(y-x) U(dx) \, dy. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E}(X_t^* - X_s^*)^2 \\ &= \frac{1}{\mu} \int_s^t h(y)^2 \, dy + \frac{2}{\mu} \int_s^t h(y) \int_{(0, y-s)} h(y-x) \, d(U(x) - \mu^{-1}x) \, dy. \end{aligned}$$

Since h^2 is assumed to be integrable, $\lim_{s \rightarrow \infty} \sup_{t > s} \int_s^t h(y)^2 dy = 0$ and it remains to check that

$$\lim_{t \rightarrow \infty} \sup_{t > s} \int_s^t h(y) \int_{(0, y-s)} h(y-x) d(U(x) - \mu^{-1}x) dy = 0. \quad (4.2)$$

Put $H_{s,t}(x) := \int_s^{t-x} h(x+y)h(y) dy$ for $x \in [0, t-s)$ and $H_{s,t}(x) := 0$ for all other x . Note that $H_{s,t}(x)$ is right-continuous and decreasing on $[0, \infty)$. Changing the order of integration followed by integration by parts gives

$$\begin{aligned} & \int_s^t h(y) \int_{(0, y-s)} h(y-x) d(U(x) - \mu^{-1}x) dy \\ &= \int_{(0, t-s)} \int_s^{t-x} h(x+y)h(y) dy d(U(x) - \mu^{-1}x) \\ &\leq \int_{(0, t-s)} (U(x) - \mu^{-1}x) d(-H_{s,t}(x)) \\ &\leq \sup_{x \geq 0} |U(x) - \mu^{-1}x| H_{s,t}(0) \\ &= \sup_{x \geq 0} |U(x) - \mu^{-1}x| \int_s^t h(y)^2 dy. \end{aligned}$$

It is known (see Theorem XI.3.1 in [10]) that $\lim_{t \rightarrow \infty} (U(t) - \mu^{-1}t) = \mu^{-2}(\sigma^2 + \mu^2) < \infty$, hence $\sup_{x \geq 0} |U(x) - \mu^{-1}x| < \infty$, and (4.2) follows.

Next we assume that h is only eventually decreasing (rather than decreasing everywhere). Then we can pick some $t_0 > 0$ such that h is decreasing on $[t_0, \infty)$. Define $\bar{h}(t) := h(t_0 + t)$, $t \geq 0$. Then \bar{h} is decreasing on \mathbb{R}_+ . Further, the post- t_0 walk $(\bar{S}_k^*)_{k \in \mathbb{N}_0} := (S_{N^*(t_0)+k}^* - t_0)_{k \in \mathbb{N}_0}$ is a distributional copy of $(S_k^*)_{k \in \mathbb{N}_0}$. Therefore, by what we have already shown,

$$\bar{X}_\circ^* := \lim_{t \rightarrow \infty} \left(\sum_{k \geq 0} \bar{h}(\bar{S}_k^*) \mathbb{1}_{\{\bar{S}_k^* \leq t\}} - \frac{1}{\mu} \int_0^t \bar{h}(y) dy \right)$$

exists in the \mathcal{L}^2 -sense. Therefore, also

$$\begin{aligned} X_\circ^* &= \lim_{t \rightarrow \infty} \left(\sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{S_k^* \leq t_0+t\}} - \frac{1}{\mu} \int_0^{t_0+t} h(y) dy \right) \\ &= X_{t_0}^* + \lim_{t \rightarrow \infty} \left(\sum_{k \geq 0} \bar{h}(\bar{S}_k^*) \mathbb{1}_{\{\bar{S}_k^* \leq t\}} - \frac{1}{\mu} \int_0^t \bar{h}(y) dy \right) \end{aligned}$$

exists in the \mathcal{L}^2 -sense.

Now we turn to the cases (C2) and (C3). Again, we begin by assuming that h satisfies the assumptions of the theorem and is decreasing and twice differentiable on \mathbb{R}_+ with $h'' \geq 0$. The proof is divided into three steps.

STEP 1: Prove that if, as $n \rightarrow \infty$, $U_n := \sum_{k=0}^n (h(S_k^*) - h(\mu k))$ converges a.s. in the case (C2) or converges in probability in the case (C3), then, as $t \rightarrow \infty$, $\sum_{k \geq 0} h(S_k^*) \mathbb{1}_{\{S_k^* \leq t\}} - \mu^{-1} \int_0^t h(y) dy$ converges in the same sense.

STEP 2: Prove that if the series $\sum_{j \geq 0} (\xi_j^* - \mu) \sum_{k \geq j} h'(\mu k)$ converges a.s., then, as $n \rightarrow \infty$, U_n converges a.s. in the case (C2) and converges in probability in the case (C3).

STEP 3. Use the three series theorem to check that, under the conditions stated, the series $\sum_{j \geq 0} (\xi_j^* - \mu) \sum_{k \geq j} h'(\mu k)$ converges a.s.

STEP 1.

CASE (C2). Assume that U_n converges a.s. Then, by Lemma A.6, the sequence $\sum_{k=0}^n h(S_k^*) - \mu^{-1} \int_0^{\mu n} h(y) dy$ converges a.s., too. Since $\lim_{t \rightarrow \infty} N^*(t) = \infty$ a.s., we further have that $\sum_{k=0}^{N^*(t)-1} h(S_k^*) - \mu^{-1} \int_0^{\mu(N^*(t)-1)} h(y) dy$ converges a.s. as $t \rightarrow \infty$. To complete this step, it remains to prove that

$$\lim_{t \rightarrow \infty} \left| \int_0^{\mu(N^*(t)-1)} h(y) dy - \int_0^t h(y) dy \right| = 0 \quad \text{a.s.} \quad (4.3)$$

To this end, write

$$\begin{aligned} & \left| \int_0^{\mu(N^*(t)-1)} h(y) dy - \int_0^t h(y) dy \right| \\ &= \int_{\mu(N^*(t)-1) \wedge t}^{\mu(N^*(t)-1) \vee t} h(y) dy \\ &\leq |\mu(N^*(t) - 1) - t| h(\mu(N^*(t) - 1) \wedge t), \end{aligned} \quad (4.4)$$

where the inequality follows from the monotonicity of h . By Theorem 3.4.4 in [14], $\mathbb{E} \xi^r < \infty$ implies that

$$N(t) - \mu^{-1}t = o(t^{1/r}) \quad \text{a.s. as } t \rightarrow \infty, \quad (4.5)$$

where it should be recalled that

$$N(t) := \inf\{k \in \mathbb{N} : S_k > t\} = \inf\{k \in \mathbb{N} : S_k^* - S_0^* > t\}.$$

Since

$$N^*(t) = \mathbb{1}_{\{S_0^* \leq t\}} + N(t - S_0^*) \mathbb{1}_{\{S_0^* \leq t\}} \quad \text{a.s.}$$

and S_0^* is a.s. finite, we infer

$$N^*(t) - \mu^{-1}t = o(t^{1/r}) \quad \text{a.s. as } t \rightarrow \infty.$$

This relation implies that the first factor in (4.4) is $o(t^{1/r})$, whereas the second factor is $o(t^{-1/r})$ as $t \rightarrow \infty$. The latter relation can be derived as follows. First, in view of (2.9), we have

$$h(t) = o(t^{-1/r}) \quad \text{as } t \rightarrow \infty. \quad (4.6)$$

Second, by the strong law of large numbers for $N^*(t)$, we have

$$[\mu(N^*(t) - 1)] \wedge t \sim t \quad \text{a.s. as } t \rightarrow \infty.$$

Altogether, (4.3) has been proved.

CASE (C3). Assume that U_n converges in probability. In view of Lemma A.6 we conclude that, as $t \rightarrow \infty$, $\sum_{k=0}^{\lfloor t/\mu \rfloor} h(S_k^*) - \mu^{-1} \int_0^{\mu \lfloor t/\mu \rfloor} h(y) dy$ converges in probability, too.

From (2.11) it follows that $h(t)^\alpha \ell(h(t)^{-1}) = o(t^{-1})$. This and (2.13) imply that

$$0 = \lim_{t \rightarrow \infty} t h(t)^\alpha \ell(h(t)^{-1}) = \lim_{t \rightarrow \infty} c(t)^\alpha h(t)^\alpha \frac{\ell(h(t)^{-1})}{\ell(c(t))} = \lim_{t \rightarrow \infty} \frac{\mathbb{P}\{\xi > h(t)^{-1}\}}{\mathbb{P}\{\xi > c(t)\}}.$$

From this, using the monotonicity and regular variation of $x \mapsto \mathbb{P}\{\xi > x\}$, we conclude that

$$\lim_{t \rightarrow \infty} c(t)h(t) = 0 \quad (4.7)$$

The latter relation implies that $\lim_{t \rightarrow \infty} h(t) = 0$. Hence

$$\lim_{t \rightarrow \infty} \left(\int_0^{\mu \lfloor t/\mu \rfloor} h(y) dy - \int_0^t h(y) dy \right) = 0.$$

Further,

$$\lim_{t \rightarrow \infty} \frac{N^*(t) \wedge (\lfloor t/\mu \rfloor + 1)}{t} = \mu^{-1} \text{ a.s.} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{S_{N^*(t) \wedge (\lfloor t/\mu \rfloor + 1)}}{N^*(t) \wedge (\lfloor t/\mu \rfloor + 1)} = \mu \text{ a.s.}$$

by the strong laws of large numbers for renewal processes and random walks, respectively. Hence

$$\frac{S_{N^*(t) \wedge (\lfloor t/\mu \rfloor + 1)}}{t} = 1 \quad \text{a.s.}$$

Using this and (4.6) we obtain that

$$\begin{aligned} \left| \sum_{k=0}^{\lfloor t/\mu \rfloor} h(S_k^*) - \sum_{k=0}^{N^*(t)-1} h(S_k^*) \right| &= \sum_{k=N^*(t) \wedge (\lfloor t/\mu \rfloor + 1)}^{(N^*(t)-1) \vee \lfloor t/\mu \rfloor} h(S_k^*) \\ &\leq |N^*(t) - 1 - \lfloor t/\mu \rfloor| h(S_{N^*(t) \wedge (\lfloor t/\mu \rfloor + 1)}) \\ &= |N^*(t) - 1 - \lfloor t/\mu \rfloor| o(1/c(t)). \end{aligned}$$

From (3.6) for $u = 1$ in the case (D3) in Subsection 3.2 we infer that $\frac{|\mu(N^*(t)-1)-t|}{c(t)}$ converges in distribution to an α -stable law with characteristic function given by (2.15). This entails

$$\sum_{k=0}^{\lfloor t/\mu \rfloor} h(S_k^*) - \sum_{k=0}^{N^*(t)-1} h(S_k^*) \xrightarrow{\mathbb{P}} 0 \quad \text{as } t \rightarrow \infty.$$

Combining pieces together gives the needed conclusion for this step.

STEP 2. For each $k \in \mathbb{N}$, by Taylor's formula, there exists a $\theta_k \in [S_k^* \wedge \mu k, S_k^* \vee \mu k]$ such that

$$h(S_k^*) - h(\mu k) = h'(\mu k)(S_k^* - \mu k) + \frac{1}{2} h''(\theta_k)(S_k^* - \mu k)^2.$$

Set

$$I_n := \frac{1}{2} \sum_{k=1}^n h''(\theta_k)(S_k^* - \mu k)^2$$

and write

$$\begin{aligned} U_n - h(S_0^*) + h(0) &= \sum_{k=1}^n h'(\mu k)(S_k^* - \mu k) + I_n \\ &= S_0^* \sum_{k=1}^n h'(\mu k) + \sum_{k=1}^n (\xi_k - \mu) \sum_{j=k}^n h'(\mu j) + I_n \\ &= S_0^* \sum_{k=1}^n h'(\mu k) + \sum_{k=1}^n (\xi_k - \mu) \sum_{j \geq k} h'(\mu j) \\ &\quad - (S_n - \mu n) \sum_{k \geq n+1} h'(\mu k) + I_n. \end{aligned} \quad (4.8)$$

Since $-h'$ is decreasing and nonnegative we have

$$\sum_{k \geq n+1} -h'(\mu k) \leq \int_n^\infty -h'(\mu y) dy = \mu^{-1} h(\mu n) \leq \sum_{k \geq n} -h'(\mu k). \quad (4.9)$$

for all n . Using the first inequality in (4.9) and the fact that $\lim_{y \rightarrow \infty} h(y) = 0$, one immediately infers that the first summand in the penultimate line of (4.8) converges as $n \rightarrow \infty$. The a.s. convergence of the second (principal) term is assumed to hold here. As to the third and fourth terms, we have to consider the cases (C2) and (C3) separately.

CASE (C2). By the Marcinkiewicz-Zygmund law of large numbers [7, Theorem 2 on p. 125],

$$S_n - \mu n = o(n^{1/r}) \quad \text{as } n \rightarrow \infty \text{ a.s.} \quad (4.10)$$

Therefore, in view of (4.6) and (4.9), the third term converges to zero a.s. Further, $\lim_{k \rightarrow \infty} k^{-1} \theta_k = \mu$ a.s. by the strong law of large numbers. Hence, in view of (2.10),

$$h''(\theta_k) = O(\theta_k^{-2-1/r}) = O(k^{-2-1/r}) \quad \text{as } k \rightarrow \infty.$$

From (4.10) we infer

$$h''(\theta_k)(S_k^* - \mu k)^2 = o(k^{-(2-1/r)}) \quad \text{a.s. as } k \rightarrow \infty,$$

which implies that I_n converges a.s. as $n \rightarrow \infty$, for $2 - 1/r > 1$. Hence the a.s. convergence of $\sum_{k \geq 1} (\xi_k^* - \mu) \sum_{j \geq k} h'(\mu j)$ entails that of U_n .

CASE (C3). By the discussion in Subsection 3.2 $\frac{S_n - \mu n}{c(n)}$ converges in distribution to an α -stable law. Hence, in view of (4.7) and (4.9), the third term converges to zero in probability.

Now pick some $0 < \varepsilon < \alpha - 1$. Since $\mathbb{E} \xi^{\alpha-\varepsilon} < \infty$, we conclude (again from the Marcinkiewicz-Zygmund law of large numbers, [7, Theorem 2 on p. 125]) that

(4.10) holds with $r = \alpha - \varepsilon$. Using (2.12) and the facts that $\theta_k \sim \mu k$ a.s. and that $c(t) \sim t^{1/\alpha} L(t)$ for some slowly varying L (see Lemma 3.2), we conclude:

$$h''(\theta_k) = O(\theta_k^{-2} c(\theta_k)^{-1}) = O(k^{-2-1/\alpha} L(k)^{-1}) \quad \text{a.s. as } k \rightarrow \infty.$$

Therefore,

$$h''(\theta_k)(S_k^* - \mu k)^2 = o(k^{-(2-\frac{\alpha+\varepsilon}{\alpha(\alpha-\varepsilon)})} L(k)^{-1}) \quad \text{a.s., } k \rightarrow \infty,$$

which implies that the fourth term I_n converges a.s., as for sufficiently small ε , $2 - \frac{\alpha+\varepsilon}{\alpha(\alpha-\varepsilon)} > 1$. Hence we arrive at the conclusion that the a.s. convergence of $\sum_{k \geq 1} (\xi_k - \mu) \sum_{j \geq k} h'(\mu j)$ entails convergence in probability of U_n .

STEP 3. Set

$$c_k := \sum_{j \geq k} -h'(\mu j) \quad \text{and} \quad \zeta_k := -c_k(\xi_k - \mu), \quad k \in \mathbb{N}.$$

CASE (C2). Condition (2.9) ensures that $\sum_{k \geq 1} h(\mu k)^r < \infty$. In view of (4.9),

$$\begin{aligned} \sum_{k \geq 1} \mathbb{E} |\zeta_k|^r &= \mathbb{E} |\xi - \mu|^r \sum_{k \geq 1} \left(\sum_{j \geq k} (-h'(\mu j)) \right)^r \\ &\leq \mu^{-r} \mathbb{E} |\xi - \mu|^r \sum_{k \geq 1} h(\mu(k-1))^r < \infty. \end{aligned}$$

Hence the series $\sum_{k \geq 1} \zeta_k$ converges a.s. by Corollary 3 on p. 117 in [7].

CASE (C3). By the three series theorem [7, Theorem 2 on p. 117], it suffices to show that the following series converge

$$\sum_{k \geq 1} \mathbb{P}\{|\zeta_k| > 1\}, \quad \sum_{k \geq 1} \mathbb{E}(\zeta_k \mathbb{1}_{\{|\zeta_k| \leq 1\}}) \quad \text{and} \quad \sum_{k \geq 1} \text{Var}(\zeta_k \mathbb{1}_{\{|\zeta_k| \leq 1\}}).$$

By Markov's inequality, the first series converges if $\sum_{k \geq 1} \mathbb{E}(|\zeta_k| \mathbb{1}_{\{|\zeta_k| > 1\}})$ converges. Since $\mathbb{E} \zeta_j = 0$ for all $j \geq 1$, the second series converges if and only if the series $\sum_{k \geq 1} \mathbb{E} \zeta_k \mathbb{1}_{\{|\zeta_k| > 1\}}$ converges. By Theorem 1.6.5 in [5],

$$\mathbb{E}(|\zeta_k| \mathbb{1}_{\{|\zeta_k| > 1\}}) = c_k \int_{[c_k^{-1}, \infty)} x \mathbb{P}\{|\xi - \mu| \in dx\} \sim \frac{\alpha}{\alpha - 1} c_k^\alpha \ell(c_k^{-1}) \quad \text{as } k \rightarrow \infty.$$

Hence, recalling (4.9) and (2.11),

$$\sum_{k \geq 1} \mathbb{E}(|\zeta_k| \mathbb{1}_{\{|\zeta_k| > 1\}}) < \infty.$$

Further, by Theorem 1.6.4 in [5],

$$\mathbb{E}(\zeta_k^2 \mathbb{1}_{\{|\zeta_k| \leq 1\}}) = c_k^2 \int_{[0, c_k^{-1}]} x^2 \mathbb{P}\{|\xi - \mu| \in dx\} \sim \frac{\alpha}{2 - \alpha} c_k^\alpha \ell(c_k^{-1}) \quad \text{as } k \rightarrow \infty.$$

Again using (4.9) and (2.11), this entails

$$\sum_{k \geq 1} \text{Var}(\zeta_k \mathbb{1}_{\{|\zeta_k| \leq 1\}}) \leq \sum_{k \geq 1} \mathbb{E}(\zeta_k^2 \mathbb{1}_{\{|\zeta_k| \leq 1\}}) < \infty.$$

Finally, we need to prove that the assertion also holds for h that are only eventually decreasing and eventually twice differentiable with $h'' \geq 0$ eventually. Indeed, for any such h , there is some $t_0 > 0$ such that h is decreasing and twice differentiable on $[t_0, \infty)$ with $h'' \geq 0$ on $[t_0, \infty)$. Using this t_0 , define \bar{h} and $(\bar{S}_k^*)_{k \in \mathbb{N}_0}$ as in the proof in the case (C1). Notice that with h , also \bar{h} satisfies the assumptions of the theorem, for instance, in case (C3), $\bar{h}''(t) = h''(t_0+t) = O((t_0+t)^{-2}c(t_0+t)^{-1}) = O(t^{-2}c(t)^{-1})$ as $t \rightarrow \infty$. Now one can argue as in the corresponding part of the proof in the case (C1) to conclude that X_\circ^* exists as the a.s. limit or the limit in probability, respectively. \square

4.2 One-dimensional convergence

The proofs of Theorems 2.1 and 2.4 are preceded by the corresponding statements on one-dimensional convergence and their proofs.

Proposition 4.4. *Assume that F is non-lattice and let h be directly Riemann integrable.*

(a) *If $\mu < \infty$, then the random series X^* converges a.s. and*

$$X(t) \xrightarrow{d} X^* \quad \text{as } t \rightarrow \infty.$$

(b) *If $\mu = \infty$, then*

$$X(t) \xrightarrow{\mathcal{L}^1} 0 \quad \text{as } t \rightarrow \infty.$$

Proof. We first deal with assertion (a) and assume $\mu < \infty$. Recall the definition $X_n^* := \sum_{k=0}^n h(S_k^*)$, $n \geq 0$ from the proof of Proposition 4.2. The latter proposition implies that $X_n^* \rightarrow X^*$ a.s. and in \mathcal{L}^1 . We now claim that

$$X_n(t) := \sum_{k=N(t)-n-1}^{N(t)-1} h(t - S_k) \mathbb{1}_{\{N(t) \geq n+1\}} \xrightarrow{d} X_n^* \quad \text{as } t \rightarrow \infty. \quad (4.11)$$

This follows from Proposition 4.1 and the continuous mapping theorem once we have shown that the random vector (S_0^*, \dots, S_n^*) takes values in the set of discontinuities of the mapping $s \circ g$ with probability 0 where $s \circ g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is defined as follows. $s : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ maps (x_0, \dots, x_n) to the sum $x_0 + \dots + x_n$. $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ maps (x_0, \dots, x_n) to $(h(x_0), \dots, h(x_n))$. Since s is continuous, attention can centre on g . Let D_h denote the set of discontinuities of h on \mathbb{R} and let D_g denote the set of discontinuities of g in \mathbb{R}^{n+1} . It is readily checked that $D_g = \bigcup_{k=0}^n (\mathbb{R}^k \times D_h \times \mathbb{R}^{n-k})$. For h is d.R.i., D_h has Lebesgue measure 0. Consequently,

$$\mathbb{P}\{(S_0^*, \dots, S_n^*) \in D_g\} \leq \sum_{k=0}^n \mathbb{P}\{S_k^* \in D_h\} = 0,$$

where we have used that S_0^* and therefore also S_k^* for $k \in \mathbb{N}$ have absolutely continuous distributions. From (4.11) and the fact that $X_n^* \rightarrow X^*$ a.s. (and, in

particular, in distribution), we want to conclude that $X(t) \xrightarrow{d} X^*$ as $t \rightarrow \infty$. This follows from Theorem 3.2 in [3] if we can show that for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}\{|X_{n-1}(t) - X(t)| > \varepsilon\} \\ &= \lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}\left\{\left|\sum_{k=0}^{N(t)-n-1} h(t - S_k) \mathbb{1}_{\{N(t) \geq n\}} + X(t) \mathbb{1}_{\{N(t) < n\}}\right| > \varepsilon\right\} = 0. \end{aligned}$$

Since $X(t) \mathbb{1}_{\{N(t) < n\}} = \sum_{k=0}^{N(t)-1} h(t - S_k) \mathbb{1}_{\{N(t) < n\}} \rightarrow 0$ a.s. as $t \rightarrow \infty$, this reduces to showing that

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}\left\{\left|\sum_{k=0}^{N(t)-n-1} h(t - S_k) \mathbb{1}_{\{N(t) \geq n\}}\right| > \varepsilon\right\} = 0.$$

By Markov's inequality, it suffices to verify that

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{E} \sum_{k=0}^{N(t)-n-1} |h(t - S_k)| \mathbb{1}_{\{N(t) \geq n\}} = 0. \quad (4.12)$$

To see this, we define $p_n(x) := \mathbb{P}\{S_n \leq x\}$. Then, for $n \geq 2$,

$$\begin{aligned} \mathbb{E} \sum_{k=0}^{N(t)-n-1} |h(t - S_k)| \mathbb{1}_{\{N(t) \geq n\}} &= \mathbb{E} \sum_{k \geq 0} |h(t - S_k)| \mathbb{1}_{\{S_{k+n} \leq t\}} \\ &= \mathbb{E} \sum_{k \geq 0} |h(t - S_k)| p_n(t - S_k) \mathbb{1}_{\{S_k \leq t\}} \\ &\rightarrow \frac{1}{\mu} \int_0^\infty |h(x)| p_n(x) dx \quad \text{as } t \rightarrow \infty, \end{aligned} \quad (4.13)$$

where in the last step we have used the key renewal theorem and the fact that if h is d.R.i., then so is $|h|p_n$. Using the dominated convergence theorem and $\lim_{n \rightarrow \infty} p_n(x) = 0$, we infer (4.12) by taking the limit $n \rightarrow \infty$ in (4.13).

Assertion (b) is a straightforward consequence of the key renewal theorem. \square

Proposition 4.5. *Assume that $\mu < \infty$ and that F is non-lattice. Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally bounded, a.e. continuous, eventually decreasing and non-integrable with $\lim_{t \rightarrow \infty} h(t) = 0$. If X_\circ^* exists as a limit in distribution, then*

$$X_\circ(t) \xrightarrow{d} X_\circ^* \quad \text{as } t \rightarrow \infty. \quad (4.14)$$

In particular, (4.14) holds under the assumptions of Theorem 2.4. Further, $X(t) - \mathbb{E} X(t) \xrightarrow{d} X_\circ^$ as $t \rightarrow \infty$ in the case (C1) of Theorem 2.4.*

Proof. First assume that h is decreasing everywhere. Let $\varepsilon > 0$ and $\tau = \tau(\varepsilon)$ be defined as in Section 3.1. Then

$$X(t) = \sum_{k=0}^{\tau} h(t - S_k) \mathbb{1}_{\{S_k \leq t\}} + \sum_{k \geq \tau+1} h(t - S_k) \mathbb{1}_{\{S_k \leq t\}}.$$

By assumption, $\lim_{t \rightarrow \infty} h(t) = 0$. Hence $\lim_{t \rightarrow \infty} \sum_{k=0}^{\tau} h(t - S_k) \mathbb{1}_{\{S_k \leq t\}} = 0$ a.s. Using (3.1) and the monotonicity of h , we infer for any $t > \varepsilon$:

$$\begin{aligned} & \sum_{k \geq \tau+1} h(t - S_k) \mathbb{1}_{\{S_k \leq t\}} \\ & \leq \sum_{k \geq \tau+1} h(t - \varepsilon - \tilde{S}_k^*) \mathbb{1}_{\{\tilde{S}_k^* \leq t - \varepsilon\}} + \sum_{k \geq \tau+1} h(0) \mathbb{1}_{\{t - \varepsilon < \tilde{S}_k^* \leq t + \varepsilon\}} \\ & \leq \sum_{k \geq 0} h(t - \varepsilon - \tilde{S}_k^*) \mathbb{1}_{\{\tilde{S}_k^* \leq t - \varepsilon\}} + h(0)(\tilde{N}^*(t + \varepsilon) - \tilde{N}^*(t - \varepsilon)). \end{aligned}$$

In view of Proposition 3.1,

$$\sum_{k \geq 0} h(t - \tilde{S}_k^*) \mathbb{1}_{\{\tilde{S}_k^* \leq t\}} \stackrel{d}{=} \sum_{k \geq 0} h(\tilde{S}_k^*) \mathbb{1}_{\{\tilde{S}_k^* \leq t\}}.$$

Hence, since X_\circ^* exists as a limit in distribution,

$$\sum_{k \geq 0} h(t - \varepsilon - \tilde{S}_k^*) \mathbb{1}_{\{\tilde{S}_k^* \leq t - \varepsilon\}} - \frac{1}{\mu} \int_{[0, t - \varepsilon]} h(y) \, dy \xrightarrow{d} X_\circ^*.$$

From $\lim_{t \rightarrow \infty} \int_{t - \varepsilon}^t h(y) \, dy = 0$, $\tilde{N}^*(t + \varepsilon) - \tilde{N}^*(t - \varepsilon) \stackrel{d}{=} \tilde{N}^*(2\varepsilon)$, and $\tilde{N}^*(2\varepsilon) \downarrow 0$ a.s. as $\varepsilon \downarrow 0$, we conclude

$$\limsup_{t \rightarrow \infty} \mathbb{P}\{X_\circ(t) > x\} \leq \mathbb{P}\{X_\circ^* > x\}$$

for every continuity point x of the law of X_\circ^* . To be more precise, let x and $x - \delta$ ($\delta > 0$) be continuity points of the law of X_\circ^* . Then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \mathbb{P}\{X_\circ(t) > x\} \\ & \leq \limsup_{t \rightarrow \infty} \mathbb{P}\left\{ \sum_{k \geq 0} h(t - \varepsilon - \tilde{S}_k^*) \mathbb{1}_{\{\tilde{S}_k^* \leq t - \varepsilon\}} - \frac{1}{\mu} \int_0^{t - \varepsilon} h(y) \, dy > x - \delta \right\} \\ & \quad + \limsup_{t \rightarrow \infty} \mathbb{P}\{h(0)(\tilde{N}^*(t + \varepsilon) - \tilde{N}^*(t - \varepsilon)) > \delta/2\} \\ & \quad + \limsup_{t \rightarrow \infty} \mathbb{P}\left\{ \sum_{k=0}^{\tau} h(t - S_k) \mathbb{1}_{\{S_k \leq t\}} > \delta/2 \right\} \\ & = \mathbb{P}\{X_\circ^* > x - \delta\} + \mathbb{P}\{h(0)\tilde{N}^*(2\varepsilon) > \delta/2\}. \end{aligned}$$

As $\varepsilon \downarrow 0$, the second probability goes to zero. Sending now $\delta \downarrow 0$ along a sequence such that each $x - \delta$ is a continuity point of the law of X_\circ^* , we arrive at the desired conclusion. The lower bound can be obtained similarly, we omit the details.

We next extend the assertion to response functions h that are only eventually decreasing. Indeed, when h is assumed to be eventually decreasing, there exists a $t_0 > 0$ such that $h(t)$ is decreasing on $[t_0, \infty)$. Define $h_1 := h(t_0) \mathbb{1}_{[0, t_0]} + h(t) \mathbb{1}_{(t_0, \infty)}$ and $h_2 := h - h_1$. Let $(X_1(t))_{t \geq 0}$ and $(X_2(t))_{t \geq 0}$ be the shot noise processes with response functions h_1 and h_2 , respectively. Then

$X(t) = X_1(t) + X_2(t)$, $t \geq 0$. Now h_1 is d.R.i. Therefore, Proposition 4.4 implies that $X_1(t) \xrightarrow{d} X_1^*$ as $t \rightarrow \infty$. What is more, $X_{1,\circ}(t) \xrightarrow{d} X_{1,\circ}^*$ as $t \rightarrow \infty$, where these quantities are defined in the obvious way. Further, from what we have already shown, $X_{2,\circ}(t)$, which is again defined in the obvious way, converges to $X_{2,\circ}^*$ in distribution as $t \rightarrow \infty$.

It now follows from Proposition 4.3 that the assumptions of Theorem 2.4 are sufficient for (4.14) to hold.

In the case (C1) of Theorem 2.4, $\lim_{t \rightarrow \infty} |\mathbb{E} X(t) - \mu^{-1} \int_0^t h(y) dy| = 0$ by Corollary 3.1 in [21]. Hence, the limiting distribution of $X(t) - \mathbb{E} X(t)$ as $t \rightarrow \infty$ is the same as that of $X_\circ^*(t)$. \square

4.3 Finite-dimensional convergence

It remains to extend one-dimensional convergence to finite-dimensional convergence. This is done in this subsection.

Proof of Theorem 2.1. We have to show that for all $0 < u_1 < \dots < u_n < \infty$, the random vector $(X(u_1 t), \dots, X(u_n t))$ converges to $(X^*(u_1), \dots, X^*(u_n))$ in distribution as $t \rightarrow \infty$. For ease of notation, we will prove this in the case $n = 2$; the case $n > 2$ can be treated analogously. Without loss of generality we may assume that $u_1 = 1$. We write u for $u_2 > 1$. Set $m_1 = (1 + u)/2$ and let $m_1 < m_2 < u$. For $t > 0$ set also

$$X_1(ut) := \sum_{k=0}^{N(m_1 t)-1} h(ut - S_k) \mathbb{1}_{\{S_k \leq ut\}} \quad \text{and} \quad X_2(ut) := \sum_{k=N(m_1 t)}^{\infty} h(ut - S_k) \mathbb{1}_{\{S_k \leq ut\}}.$$

Clearly, $X(ut) = X_1(ut) + X_2(ut)$ for all $t > 0$. We claim that

$$X_1(ut) \xrightarrow{\mathbb{P}} 0 \quad \text{as } t \rightarrow \infty. \quad (4.15)$$

Hence, by Slutsky's lemma it is enough to show that

$$\mathbb{P}\{X(t) \leq a, X_2(ut) \leq b\} \rightarrow \mathbb{P}\{X^* \leq a\} \mathbb{P}\{X^* \leq b\} \quad (4.16)$$

as $t \rightarrow \infty$ for continuity points $a, b \in \mathbb{R}$ of the law of X^* .

We first prove (4.16) and then (4.15). Write the probability on the left-hand side of (4.16) as follows:

$$\begin{aligned} & \mathbb{P}\{X(t) \leq a, X_2(ut) \leq b\} \\ &= \int_{(m_1 t, \infty)} \mathbb{P}\{X(t) \leq a, X_2(ut) \leq b, S_{N(m_1 t)} \in dy\} \\ &= \left(\int_{(m_1 t, m_2 t]} \dots + \int_{(m_2 t, \infty)} \dots \right) =: J_1(t) + J_2(t). \end{aligned}$$

Clearly, $0 \leq J_2(t) \leq \mathbb{P}\{S_{N(m_1 t)} > m_2 t\} = \mathbb{P}\{S_{N(m_1 t)} - m_1 t > (m_2 - m_1)t\} \rightarrow 0$ as $t \rightarrow \infty$ since $S_{N(m_1 t)} - m_1 t \xrightarrow{d} S_0^*$ and $(m_2 - m_1)t \rightarrow +\infty$ as $t \rightarrow \infty$. For $J_1(t)$ we may write:

$$J_1(t) = \int_{(m_1 t, m_2 t]} \mathbb{P}\{X(ut - y) \leq b\} \mathbb{P}\{X(t) \leq a, S_{N(m_1 t)} \in dy\}$$

where we have used that $(S_{k+N(m_1t)} - S_{N(m_1t)})_{k \in \mathbb{N}_0}$ has the same distribution as $(S_k)_{k \in \mathbb{N}_0}$ and is independent of $(S_k)_{0 \leq k \leq N(m_1t)}$. Further,

$$\begin{aligned} J_1(t) &= \mathbb{P}\{X^* \leq b\} \int_{(m_1t, m_2t]} \mathbb{P}\{X(t) \leq a, S_{N(m_1t)} \in dy\} \\ &\quad + \int_{(m_1t, m_2t]} (\mathbb{P}\{X(ut-y) \leq b\} - \mathbb{P}\{X^* \leq b\}) \mathbb{P}\{X(t) \leq a, S_{N(m_1t)} \in dy\} \\ &=: J_{11}(t) + J_{12}(t). \end{aligned}$$

The integral in the first summand converges to $\mathbb{P}\{X^* \leq a\}$ by Proposition 4.4 since a is a continuity point of the law of X^* and $\mathbb{P}\{m_1t \leq S_{N(m_1t)} \leq m_2t\} \rightarrow 1$ as $t \rightarrow \infty$. To show that $J_{12}(t)$ converges to zero, write

$$\begin{aligned} |J_{12}(t)| &= \sup_{y \in [m_1t, m_2t]} \left| \mathbb{P}\{X(ut-y) \leq b\} - \mathbb{P}\{X^* \leq b\} \right| \int_{(m_1t, m_2t]} \mathbb{P}\{S_{N(m_1t)} \in dy\} \\ &\leq \sup_{y \geq (u-m_2)t} \left| \mathbb{P}\{X(y) \leq b\} - \mathbb{P}\{X^* \leq b\} \right|, \end{aligned}$$

which goes to zero since $(u-m_2)t \rightarrow \infty$, as $t \rightarrow \infty$, in view of Proposition 4.4. The proof of (4.16) is complete. It remains to show (4.15). By the key renewal theorem,

$$\begin{aligned} \mathbb{E}|X_1(ut)| &\leq \mathbb{E} \sum_{k=0}^{N(m_1t)-1} |h(ut - S_k)| = \int_{(0, m_1t]} |h(ut-y)| U(dy) \\ &= \int_{(0, ut]} |h(ut-y)| U(dy) - \int_{(m_1t, ut]} |h(ut-y)| U(dy) \\ &= \frac{1}{\mu} \int_0^\infty |h(y)| dy - \int_{(m_1t, ut]} |h(ut-y)| U(dy) + o(1) \end{aligned}$$

as $t \rightarrow \infty$. So it is sufficient to show that

$$\liminf_{t \rightarrow \infty} \int_{(m_1t, ut]} |h(ut-y)| U(dy) \geq \frac{1}{\mu} \int_0^\infty |h(y)| dy.$$

For fixed $\delta > 0$ set $I_k^\delta = [k\delta, (k+1)\delta)$ and $m_k^\delta = \inf_{x \in I_k^\delta} |h(x)|$. Then, for large enough $t > 0$,

$$\int_{(m_1t, ut]} |h(ut-y)| U(dy) \geq \sum_{k=0}^{\lfloor ((u-m_1)t)/\delta \rfloor - 1} m_k^\delta U(ut - I_k^\delta)$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$. Given $0 < \delta < 1$ there is $N = N(\delta)$ such that $\sum_{k > N} m_k^\delta < \delta$, for $|h|$ is dRi. Let t be so large that $\lfloor (u-m_1)t/\delta \rfloor - 1 >$

N . Then

$$\begin{aligned}
\int_{(m_1 t, ut]} |h(ut - y)| U(dy) &\geq \sum_{k=0}^{\lfloor (u-m_1)t/\delta \rfloor - 1} m_k^\delta U(ut - I_k^\delta) \\
&\geq \sum_{k=0}^N m_k^\delta U(ut - I_k^\delta) \xrightarrow{t \rightarrow \infty} \frac{\delta}{\mu} \sum_{k=0}^N m_k^\delta \\
&\geq \frac{\delta}{\mu} \sum_{k=0}^{\infty} m_k^\delta - \frac{\delta^2}{\mu} \xrightarrow{\delta \downarrow 0} \frac{1}{\mu} \int_0^\infty |h(y)| dy,
\end{aligned}$$

which finishes both the proof of (4.15) and of the whole theorem for $n = 2$.

The case of general $n \in \mathbb{N}$ can be treated in a similar manner by conditioning the probability $\mathbb{P}\{X(u_1 t) \leq a_1, \dots, X(u_n t) \leq a_n\}$ on the values of $(S_{N(m_1 t)}, \dots, S_{N(m_{n-1} t)})$ at appropriately chosen middle points $u_i < m_i < u_{i+1}$. \square

The scheme of the proof of Theorem 2.4 is the same as that of the proof of Theorem 2.1 above. On the other hand, it differs in many details which is why we decided to include it in the paper.

Proof of Theorem 2.4. As in the case of Theorem 2.1, we will prove this theorem only for $n = 2$ and assume that $u_1 = 1$ and $u := u_2 > 1$. Let $p(t) := \mu^{-1} \int_0^t h(y) dy$ and set $m_1 := (1 + u)/2$, $z_2(t) := m_1 t + r(t)$ where $r(t)$ is some function to be specified below. Decompose $X_\circ(ut) := X(ut) - p(ut)$ as follows

$$\begin{aligned}
X_\circ(ut) &= \left(\sum_{k=0}^{N(m_1 t) - 1} h(ut - S_k) - \frac{1}{\mu} \int_{(u-m_1)t}^{ut} h(y) dy \right) \\
&\quad + \left(\sum_{k=N(m_1 t)}^{\infty} h(ut - S_k) \mathbb{1}_{\{S_k \leq ut\}} - p((u - m_1)t) \right) \\
&=: Y_1(t) + Y_2(t).
\end{aligned}$$

Similar to the proof of Theorem 2.1, we conclude that it is enough to show that

$$Y_1(t) \xrightarrow{\mathbb{P}} 0, \quad (4.17)$$

and

$$\mathbb{P}\{X(t) \leq a + p(t), Y_2(t) \leq b\} \rightarrow \mathbb{P}\{X_\circ^* \leq a\} \mathbb{P}\{X_\circ^* \leq b\}, \quad (4.18)$$

as $t \rightarrow \infty$ for all $a, b \in \mathbb{R}$ that are continuity points of the law of X_\circ^* .

We begin by proving (4.17). Using integration by parts (see Lemma A.3),

we infer

$$\begin{aligned}
Y_1(t) &= \left(\sum_{k=0}^{N(m_1 t)-1} h(ut - S_k) \mathbb{1}_{\{S_k \leq ut\}} - \frac{1}{\mu} \int_{(u-m_1)t}^{ut} h(y) dy \right) \\
&= \int_{[0, m_1 t]} h(ut - y) d\left(N(y) - \frac{y}{\mu}\right) \\
&= h(ut) + h((u - m_1)t-) \left((N(m_1 t) - \frac{m_1 t}{\mu}) - h(ut-) \right) \\
&\quad - \int_{(0, m_1 t]} \left(N(y) - \frac{y}{\mu} \right) dh(ut - y) \\
&= h(ut) - h(ut-) + h((u - m_1)t-) \left(N(m_1 t) - \frac{m_1 t}{\mu} \right) \\
&\quad + \int_{[(u-m_1)t, ut]} \left(N(ut - y) - \frac{ut - y}{\mu} \right) d(-h(y)). \tag{4.19}
\end{aligned}$$

We now distinguish between two cases:

CASES (C1) AND (C3): We invoke the estimate $\mathbb{E} |N(t) - \frac{t}{\mu}| \leq K_1 + K_2 c(t)$ which holds for all $t \geq 0$ and some fixed $K_1, K_2 > 0$, see Theorem 3.3. Here $c(t)$ is chosen as \sqrt{t} in case (C1) and in case (C3) it is as stated there. Then

$$\begin{aligned}
\mathbb{E} |Y_1(t)| &\leq h((u - m_1)t) \mathbb{E} \left| N(m_1 t) - \frac{m_1 t}{\mu} \right| + o(1) \\
&\quad + \int_{((u-m_1)t, ut]} \mathbb{E} \left| N(ut - y) - \frac{ut - y}{\mu} \right| d(-h(y)) \\
&\leq h((u - m_1)t) \left(K_1 + K_2 c(m_1 t) \right) + o(1) \\
&\quad + \int_{((u-m_1)t, ut]} (K_1 + K_2 c(ut - y)) d(-h(y)).
\end{aligned}$$

Note that since c is regularly varying with positive index we may assume, without loss of generality, that $c(t)$ is eventually increasing (see Theorem 1.5.3 in [5]). This observation implies that, for arbitrary $\kappa, \lambda > 0$,

$$\lim_{t \rightarrow \infty} c(\kappa t) h(\lambda t) = \lim_{t \rightarrow \infty} c(t) h(t) = 0, \tag{4.20}$$

under each assumption (i), (ii) and (iii). Recalling that $m_1 = (1 + u)/2 < u$ and using (4.20) we infer that the first summand in the estimate for $\mathbb{E} |Y_1(t)|$ above converges to zero as $t \rightarrow \infty$. Further, again using (4.20),

$$\begin{aligned}
&\int_{((u-m_1)t, ut]} (K_1 + K_2 c(ut - y)) d(-h(y)) \\
&= K_2 \int_{((u-m_1)t, ut]} c(ut - y) d(-h(y)) + o(1) \\
&\leq c(m_1 t) (h((u - m_1)t) - h(ut)) + o(1) \rightarrow 0
\end{aligned}$$

as $t \rightarrow \infty$.

CASE (C2): In this case $\mathbb{E}\xi^r < \infty$. Thus, by the Marcinkiewicz-Zygmund law of large numbers for $N(t)$, see (4.5),

$$\lim_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} |N(s) - \frac{s}{\mu}|}{t^{1/r}} = 0 \quad \text{a.s.}$$

From the monotonicity of h and (2.9) it follows that $h(t) = O(t^{-1/r})$ as $t \rightarrow \infty$. Hence

$$\lim_{t \rightarrow \infty} h(\kappa t) \sup_{0 \leq s \leq \lambda t} \left| N(s) - \frac{s}{\mu} \right| = 0 \quad \text{a.s.}$$

for arbitrary $\kappa, \lambda > 0$. Using this in (4.19) implies that $Y_1(t) \rightarrow 0$ a.s., in particular also in probability.

We now turn to the the proof of (4.18). Write the probability on the left-hand side of (4.18) as follows:

$$\begin{aligned} & \mathbb{P}\{X(t) \leq a + p(t), Y_2(t) \leq b\} \\ &= \int_{(m_1 t, \infty)} \mathbb{P}\{X(t) \leq a + p(t), Y_2(t) \leq b, S_{N(m_1 t)} \in dy\} \\ &= \left(\int_{(m_1 t, z_2(t)]} \cdots + \int_{(z_2(t), \infty)} \cdots \right) =: J_1(t) + J_2(t). \end{aligned}$$

Clearly, $0 \leq J_2(t) \leq \mathbb{P}\{S_{N(m_1 t)} \geq z_2(t)\} = \mathbb{P}\{S_{N(m_1 t)} - m_1 t \geq r(t)\}$. The latter probability tends to 0 as $t \rightarrow \infty$ whenever

$$r(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty \quad (4.21)$$

since $S_{N(m_1 t)} - m_1 t \xrightarrow{d} S_0^*$.

For $J_1(t)$ we may write:

$$\begin{aligned} & J_1(t) \\ &= \int_{(m_1 t, z_2(t)]} \mathbb{P}\{X(t) \leq a + p(t), S_{N(m_1 t)} \in dy, \\ & \quad \sum_{k=N(m_1 t)}^{\infty} h(ut - y - (S_k - S_{N(m_1 t)})) \mathbb{1}_{\{S_k - S_{N(m_1 t)} \leq ut - y\}} \leq b + p((u - m_1)t)\} \\ &= \int_{(m_1 t, z_2(t)]} \mathbb{P}\{X(ut - y) \leq b + p((u - m_1)t)\} \mathbb{P}\{X(t) \leq a + p(t), S_{N(m_1 t)} \in dy\}, \end{aligned}$$

where the last equality follows from the independence of $(S_j)_{j=1, \dots, N(m_1 t)}$ and $(S_{k+N(m_1 t)} - S_{N(m_1 t)})_{k \in \mathbb{N}}$, the distributional identity $(S_{N(m_1 t)+k} - S_{N(m_1 t)})_{k \in \mathbb{N}_0} \stackrel{d}{=} (S_k)_{k \in \mathbb{N}_0}$ and the inequality $t < m_1 t$. Further,

$$\begin{aligned} J_1(t) &= \mathbb{P}\{X_\circ^* \leq b\} \int_{(m_1 t, z_2(t)]} \mathbb{P}\{X(t) \leq a + p(t), S_{N(m_1 t)} \in dy\} \\ & \quad + \int_{(m_1 t, z_2(t)]} (\mathbb{P}\{X(ut - y) \leq b + p((u - m_1)t)\} - \mathbb{P}\{X_\circ^* \leq b\}) \\ & \quad \quad \quad \times \mathbb{P}\{X(t) \leq a + p(t), S_{N(m_1 t)} \in dy\} \\ &=: J_{11}(t) + J_{12}(t). \end{aligned}$$

When (4.21) holds, then $\mathbb{P}\{m_1 t < S_{N(m_1 t)} \leq z_2(t)\} \rightarrow 1$ as $t \rightarrow \infty$. Consequently, an application of Proposition 4.5 shows that the integral in the first summand converges to $\mathbb{P}\{X_\circ^* \leq a\}$. To show that $J_{12}(t)$ converges to zero, write

$$\begin{aligned} |J_{12}(t)| &\leq \sup_{m_1 t < y \leq z_2(t)} \left| \mathbb{P}\{X(ut-y) \leq b + p(ut-m_1 t)\} - \mathbb{P}\{X_\circ^* \leq b\} \right| \\ &\quad \times \int_{(m_1 t, z_2(t)]} \mathbb{P}\{S_{N(m_1 t)} \in dy\} \\ &\leq \sup_{m_1 t < y \leq z_2(t)} \left| \mathbb{P}\{X_\circ(ut-y) \leq b + p(ut-m_1 t) - p(ut-y)\} - \mathbb{P}\{X_\circ^* \leq b\} \right| \\ &= \sup_{m_1 t < y \leq z_2(t)} \left| \mathbb{P}\{X_\circ(ut-y) \leq b + \mu^{-1} \int_{ut-y}^{(u-m_1)t} h(x) dx - \mathbb{P}\{X_\circ^* \leq b\} \right|. \end{aligned}$$

Using the fact that h is non-negative, we proceed as follows:

$$\begin{aligned} |J_{12}(t)| &\leq \sup_{m_1 t < y \leq z_2(t)} \left| \mathbb{P}\{X_\circ(ut-y) \leq b\} - \mathbb{P}\{X_\circ^* \leq b\} \right| \\ &\quad + \sup_{m_1 t < y \leq z_2(t)} \mathbb{P} \left\{ b < X_\circ(ut-y) \leq b + \mu^{-1} \int_{ut-y}^{ut-m_1 t} h(y) dy \right\} \\ &\leq \sup_{y \geq ut-z_2(t)} \left| \mathbb{P}\{X_\circ(y) \leq b\} - \mathbb{P}\{X_\circ^* \leq b\} \right| \\ &\quad + \sup_{m_1 t < y \leq z_2(t)} \mathbb{P} \left\{ b < X_\circ(ut-y) \leq b + \mu^{-1} \int_{ut-y}^{ut-m_1 t} h(y) dy \right\}. \end{aligned}$$

Due to Proposition 4.5, the first summand converges to zero whenever

$$ut - z_2(t) = (u-1)t/2 - r(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (4.22)$$

Assume that r also satisfies

$$\lim_{t \rightarrow \infty} h((u-1)t/2 - r(t))r(t) = 0. \quad (4.23)$$

Then, for arbitrary $\varepsilon > 0$, there exists t_0 such that for all $t > t_0$

$$0 \leq \mu^{-1} \int_{ut-z_2(t)}^{ut-m_1 t} h(x) dx \leq \mu^{-1} h((u-1)t/2 - r(t))r(t) < \varepsilon.$$

Thus, when (4.22) and (4.23) hold and $\varepsilon > 0$ is chosen such that $b + \varepsilon$ is a continuity point of the law of X_\circ^*

$$\begin{aligned} &\sup_{m_1 t < y \leq z_2(t)} \mathbb{P} \left\{ b < X_\circ(ut-y) \leq b + \mu^{-1} \int_{ut-z_2(t)}^{(u-m_1)t} h(x) dx \right\} \\ &\leq \sup_{m_1 t < y \leq z_2(t)} \mathbb{P}\{X_\circ(ut-y) \in (b, b + \varepsilon]\} \\ &\xrightarrow[t \rightarrow \infty]{} \mathbb{P}\{X_\circ^* \in (b, b + \varepsilon]\} \xrightarrow[\varepsilon \rightarrow 0]{} 0, \end{aligned}$$

since b and $b + \varepsilon$ are continuity points of the law of X_\circ^* . We thus have proved that $\lim_{t \rightarrow \infty} J_{12}(t) = 0$ if r satisfies the conditions (4.21), (4.22) and (4.23). A possible choice of r such that the above conditions are satisfied is the following. Let $\delta = (u - 1)/4$. Then choose r as $r(t) = h(\delta t)^{-1/2} \wedge \delta t$, $t \geq 0$. Then (4.21) holds since $h(t) \rightarrow 0$ as $t \rightarrow \infty$ and (4.22) holds since $r(t) \leq \delta t$. Finally, (4.23) holds since $h(t(u - 1)/2 - r(t))r(t) \leq h(\delta t)^{1/2} \rightarrow 0$ as $t \rightarrow \infty$. The proof of (4.18) is complete. \square

5 Proofs of the limit theorems with scaling

Whenever possible we intend to treat the all cases simultaneously. To this end, for $t > 0$, put

$$X_t(u) := \frac{X(ut) - \int_0^{ut} h(y) dy}{g(t)h(t)}, \quad u \geq 0.$$

5.1 Reduction to continuous and decreasing h

Regarding the proof of Theorem 2.7, we make our life easier by showing that without loss of generality we can replace h by a decreasing and continuous function h^* on \mathbb{R}_+ satisfying $h^*(t) \sim h(t)$ as $t \rightarrow \infty$. This follows an idea in [18].

Suppose we have already proved that

$$X_t^*(u) := \frac{\int_{[0, ut]} h^*(ut - y) dN(y) - \mu^{-1} \int_0^{ut} h^*(y) dy}{g(t)h^*(t)} \xrightarrow{\text{f.d.}} Y(u), \quad t \rightarrow \infty,$$

where $Y(u) := W(u)$, $u \geq 0$ if $\beta = 0$, and

$$Y(u) := W(u)u^{-\beta} + \beta \int_0^u (W(u) - W(y))(u - y)^{-\beta-1} dy, \quad u \geq 0$$

if $\beta > 0$. To ensure the convergence $X_t(u) \xrightarrow{\text{f.d.}} Y(u)$ as $t \rightarrow \infty$, it suffices to check that, for any $u > 0$,

$$\left| \frac{\int_{[0, ut]} (h(ut - y) - h^*(ut - y)) dN(y)}{g(t)h(t)} \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } t \rightarrow \infty, \quad (5.1)$$

and

$$\left| \frac{\int_0^{ut} (h(y) - h^*(y)) dy}{g(t)h(t)} \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (5.2)$$

By assumption, h is eventually decreasing. Hence, there exists an $a > 0$ such that h is decreasing on $[a, \infty)$. Let \widehat{h} be a bounded, right-continuous and decreasing function such that $\widehat{h}(t) = h(t)$ for $t \geq a$. Note that the so defined \widehat{h} is non-negative. The first observation is that replacing h by \widehat{h} in the definition of $X(t)$ will not change the asymptotics. Indeed, if \widehat{X} denotes the shot noise

process with the shots occurring at times S_0, S_1, \dots and response function \widehat{h} instead of h , then for given $u > 0$ and all large enough t ,

$$\begin{aligned} |X(ut) - \widehat{X}(ut)| &= \left| \sum_{k=0}^{N(ut)} h(ut - S_k) - \widehat{h}(ut - S_k) \right| \\ &\leq \sup_{y \in [0, a]} |h(y) - \widehat{h}(y)| (N(ut) - N(ut - a)) \\ &\stackrel{d}{\leq} \sup_{y \in [0, a]} |h(y) - \widehat{h}(y)| N(a) \end{aligned}$$

by the well-known distributional subadditivity of N . The boundedness of h and \widehat{h} ensures the finiteness of the last supremum. Since $\beta < 1/\alpha$, in all cases we have $\lim_{t \rightarrow \infty} g(t)h(t) = \infty$. Consequently,

$$\left| \frac{X(ut) - \widehat{X}(ut)}{g(t)h(t)} \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } t \rightarrow \infty. \quad (5.3)$$

Further, for $ut \geq a$,

$$\begin{aligned} \left| \frac{\int_0^{ut} (h(y) - \widehat{h}(y)) \, dy}{g(t)h(t)} \right| &\leq \frac{\int_0^{ut} |h(y) - \widehat{h}(y)| \, dy}{g(t)h(t)} \\ &= \frac{\int_{[0, a]} |h(y) - \widehat{h}(y)| \, dy}{g(t)h(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (5.4)$$

Thus, in what follows, we can replace h by \widehat{h} . We will now construct h^* from \widehat{h} . To this end, let θ be a random variable with standard exponential distribution. Set

$$h^*(t) := \mathbb{E} \widehat{h}((t - \theta)^+) = e^{-t} \left(\widehat{h}(0) + \int_0^t \widehat{h}(y) e^y \, dy \right), \quad t \geq 0. \quad (5.5)$$

It is clear that $\widehat{h}(t) \leq h^*(t)$, $t \geq 0$ and that h^* is continuous and decreasing on \mathbb{R}_+ with $h^*(0) = \widehat{h}(0) < \infty$. Furthermore, $h^*(t) \sim \widehat{h}(t) \sim h(t)$, $t \rightarrow \infty$. While this is trivial if $\lim_{t \rightarrow \infty} h(t) \neq 0$, in the opposite case ($\lim_{t \rightarrow \infty} h(t) = 0$) the first equivalence does require a proof. We use the second equality in (5.5). Being a regularly varying function $1/\widehat{h}$ grows subexponentially fast. Using this and the regular variation of \widehat{h} at infinity, we infer for any $\varepsilon \in (0, 1)$:

$$\begin{aligned} \frac{h^*(t)}{\widehat{h}(t)} &= \mathbb{E} \left[\frac{\widehat{h}((t - \theta)^+)}{\widehat{h}(t)} \mathbb{1}_{\{\theta > \varepsilon t\}} \right] + \mathbb{E} \left[\frac{\widehat{h}((t - \theta)^+)}{\widehat{h}(t)} \mathbb{1}_{\{\theta \leq \varepsilon t\}} \right] \\ &\leq \frac{\widehat{h}(0)}{\widehat{h}(t)} e^{-\varepsilon t} + \frac{\widehat{h}((1 - \varepsilon)t)}{\widehat{h}(t)} (1 - e^{-\varepsilon t}) \xrightarrow{t \rightarrow \infty} (1 - \varepsilon)^{-\beta} \xrightarrow{\varepsilon \rightarrow 0} 1. \end{aligned}$$

Since $\widehat{h}(t) \leq h^*(t)$ for all $t \geq 0$, this implies $h^*(t) \sim \widehat{h}(t)$ as $t \rightarrow \infty$. Let us now prove that

$$\lim_{t \rightarrow \infty} \int_0^t (h^*(y) - \widehat{h}(y)) \, dy = \widehat{h}(0). \quad (5.6)$$

We use the representation

$$\begin{aligned} & \int_0^t (h^*(y) - \widehat{h}(y)) \, dy \\ &= \widehat{h}(0)(1 - e^{-t}) - \mathbb{E} \int_{t-\theta}^t \widehat{h}(y) \, dy \mathbb{1}_{\{\theta \leq t\}} - \int_0^t \widehat{h}(y) \, dy e^{-t}. \end{aligned}$$

Since \widehat{h} grows subexponentially, the last term vanishes as $t \rightarrow \infty$, and we are left with investigating the second term. By the monotonicity of \widehat{h} and the dominated convergence theorem,

$$\mathbb{E} \int_{[t-\theta, t]} \widehat{h}(y) \, dy \mathbb{1}_{\{\theta \leq t\}} \leq \mathbb{E} \theta \widehat{h}(t - \theta) \mathbb{1}_{\{\theta \leq t\}} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

which proves (5.6). In particular,

$$\left| \frac{\int_0^{ut} (\widehat{h}(y) - h^*(y)) \, dy}{g(t)h(t)} \right| = \frac{\int_0^{ut} (h^*(y) - \widehat{h}(y)) \, dy}{g(t)h(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

because $\lim_{t \rightarrow \infty} g(t)h(t) = \infty$. In combination with (5.4) the latter proves (5.2). Recalling (5.6) and the fact that in all cases $\lim_{t \rightarrow \infty} g(t)h(t) = \infty$, we conclude from Lemma A.7 (with $f_1 = h^*$ and $f_2 = \widehat{h}$)

$$\begin{aligned} & \left| \frac{\int_{[0, ut]} (\widehat{h}(ut - y) - h^*(ut - y)) \, dN(y)}{g(t)h(t)} \right| \\ &= \frac{\int_{[0, ut]} (h^*(ut - y) - \widehat{h}(ut - y)) \, dN(y)}{g(t)h(t)} \xrightarrow{\mathcal{L}^1} 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This together with (5.3) leads to (5.1).

5.2 Proofs of Theorems 2.7 and 2.9

Proof of Theorem 2.7. By the Cramér-Wold device and the discussion in Subsection 5.1, in order to show finite-dimensional convergence of $X_t(u)$, it suffices to prove that for any $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ and $0 \leq u_1 < \dots < u_n$ we have that

$$\sum_{k=1}^n \gamma_k X_t^*(u_k) \xrightarrow{d} \sum_{k=1}^n \gamma_k Y(u_k) \quad \text{as } t \rightarrow \infty.$$

Since

$$X_t^*(0) = \frac{h^*(0)}{g(t)h^*(t)} \rightarrow 0 = W(0) \quad \text{as } t \rightarrow \infty,$$

in what follows we assume that $u_1 > 0$. Integrating by parts we obtain that

$$\begin{aligned}
& \sum_{k=1}^n \gamma_k X_t^*(u_k) \\
&= \sum_{k=1}^n \frac{\gamma_k}{g(t)h^*(t)} \left(h^*(u_k t) + \int_{(0, u_k t]} h^*(u_k t - y) d\left(N(y) - \frac{y}{\mu}\right) \right) \\
&= \sum_{k=1}^n \frac{\gamma_k}{g(t)h^*(t)} \left(h^*(0) \left(N(u_k t) - \frac{u_k t}{\mu} \right) \right. \\
&\quad \left. - \int_{(0, u_k t]} (N(y) - \mu^{-1}y) d(h^*(u_k t - y)) \right) \\
&= \sum_{k=1}^n \gamma_k W_t(u_k) \frac{h^*(u_k t)}{h^*(t)} \\
&\quad + \sum_{k=1}^n \frac{\gamma_k}{g(t)h^*(t)} \left((h^*(0) - h^*(u_k t)) \left(N(u_k t) - \frac{u_k t}{\mu} \right) \right. \\
&\quad \left. - \int_{[0, u_k t)} \left(N(u_k t - y) - \frac{u_k t - y}{\mu} \right) d(-h^*(y)) \right) \\
&= \sum_{k=1}^n \gamma_k W_t(u_k) \frac{h^*(u_k t)}{h^*(t)} \\
&\quad + \sum_{k=1}^n \gamma_k \frac{\int_{[0, u_k t)} (N(u_k t) - N(u_k t - y) - \mu^{-1}y) d(-h^*(y))}{g(t)h^*(t)} \tag{5.7}
\end{aligned}$$

where the definition of $W_t(u)$ should be recalled from (3.6).

CASE $\beta = 0$. Our aim is to show that each summand of the second term in (5.7) converges to zero in probability, for the convergence

$$\sum_{k=1}^n \gamma_k X_t^*(u_k) \xrightarrow{d} \sum_{k=1}^n \gamma_k Y(u_k) = \sum_{k=1}^n \gamma_k W(u_k)$$

is then an immediate consequence of $\lim_{t \rightarrow \infty} h^*(u_k t)/h^*(t) = 1$, (3.6) and Slutsky's theorem.

For given $0 < \varepsilon < 1$, recall the construction of the coupled stationary random walk $(\tilde{S}_k^*)_{k \in \mathbb{N}_0}$ which satisfies (3.1), that is, $\tilde{S}_k^* - \varepsilon \leq S_k \leq \tilde{S}_k^* + \varepsilon$ for $k \geq \tau + 1$ where τ is an almost surely finite stopping time (the time of ε -coupling). For the renewal counting process \tilde{N}^* corresponding to $(\tilde{S}_k^*)_{k \in \mathbb{N}_0}$, we then have

$$\sum_{k \geq \tau+1} \mathbb{1}_{\{t-y < S_k \leq t\}} \leq \tilde{N}^*(t+\varepsilon) - \tilde{N}^*(t-y-\varepsilon)$$

and

$$\sum_{k \geq \tau+1} \mathbb{1}_{\{t-y < S_k \leq t\}} \geq \tilde{N}^*(t-\varepsilon) - \tilde{N}^*(t-y+\varepsilon) - \sum_{k=0}^{\tau} \mathbb{1}_{\{t-y+\varepsilon < \tilde{S}_k^* \leq t-\varepsilon\}}$$

by (3.2) and (3.3), respectively. For each summand in (5.7), we can use these two inequalities with t replaced by $u_k t$ to obtain that

$$\begin{aligned} & \left| \frac{\int_{[0, u_k t)} (N(u_k t) - N(u_k t - y) - \mu^{-1} y) d(-h^*(y))}{g(t)h^*(t)} \right| \\ & \stackrel{d}{\leq} \left| \frac{\int_{[0, u_k t)} (\tilde{N}^*(u_k t) - \tilde{N}^*(u_k t - y) - \mu^{-1} y) d(-h^*(y))}{g(t)h^*(t)} \right| \\ & \quad + \frac{(\tilde{N}^*(2\varepsilon) + \tau + 1)h^*(0)}{g(t)h^*(t)} \end{aligned} \quad (5.8)$$

where the distributional identity $\tilde{N}^*(x + c) - \tilde{N}^*(x) \stackrel{d}{=} \tilde{N}^*(c)$ ($x, c \geq 0$) has been utilized. Since $g(t)h^*(t) \rightarrow \infty$ as $t \rightarrow \infty$, an application of Markov's inequality shows that it is sufficient to check that

$$\lim_{t \rightarrow \infty} \frac{\int_{[0, ut]} \mathbb{E} |N^*(y) - \mu^{-1} y| d(-h^*(y))}{g(t)h^*(t)} = 0.$$

By Theorem 3.3, $\mathbb{E} |N^*(y) - \mu^{-1} y| = O(g(y))$ as $y \rightarrow \infty$. Consequently, it is enough to show that

$$\lim_{t \rightarrow \infty} \frac{\int_{[0, ut]} g(y) d(-h^*(y))}{g(t)h^*(t)} = 0.$$

Since the function $g(t)h^*(t)$ is regularly varying, the latter is equivalent to

$$\lim_{t \rightarrow \infty} \frac{\int_{[0, t]} g(y) d(-h^*(y))}{g(t)h^*(t)} = 0. \quad (5.9)$$

Using (3.7) gives

$$\frac{\int_{[t_0, t]} g(y) d(-h^*(y))}{g(t)h^*(t)} \leq A \frac{\int_{[t_0, t]} y^{1/\alpha - \delta} d(-h^*(y))}{t^{1/\alpha - \delta} h^*(t)}$$

for $t \geq t_0$, and, as $t \rightarrow \infty$, the last ratio tends to zero by Theorem 1.6.4 in [5]. Further, since $g(t)h^*(t) \rightarrow 0$, also

$$\lim_{t \rightarrow \infty} \frac{\int_{[0, t_0]} g(y) d(-h^*(y))}{g(t)h^*(t)} = 0.$$

Thus, (5.9) follows.

CASE $\beta > 0$. For any $\rho \in (0, 1)$, one can write

$$\begin{aligned} X_t^*(u_k) &= W_t(u_k) \frac{h^*(u_k t)}{h^*(t)} + \int_{(0, u_k]} (W_t(u_k) - W_t(v)) \nu_{t, k}^*(dv) \\ &= W_t(u_k) \frac{h^*(u_k t)}{h^*(t)} + \int_{(0, \rho u_k]} \cdots + \int_{(\rho u_k, u_k]} \cdots, \end{aligned}$$

where $\nu_{t,k}^*$ is the finite measure on $[0, u_k]$ defined by

$$\nu_{t,k}^*(a, b] := \frac{h^*(t(u_k - b)) - h^*(t(u_k - a))}{h^*(t)}, \quad 0 \leq a < b \leq u_k.$$

In view of (3.6) and the continuous mapping theorem,

$$W_t(u_k) - W_t(v) \Rightarrow W(u_k) - W(v) \quad \text{as } t \rightarrow \infty.$$

Further, by the regular variation of h^* , the finite measures $\nu_{t,k}^*$ converge weakly on $[0, \rho u_k]$ to a finite measure ν_k^* on $[0, \rho u_k]$ which is defined by $\nu_k^*(a, b] = (u_k - b)^{-\beta} - (u_k - a)^{-\beta}$. Clearly, the limiting measure is absolutely continuous with density $x \mapsto \beta(u_k - x)^{-\beta-1}$, $x \in [0, \rho u_k]$. Hence, by Lemma A.2,

$$\begin{aligned} W_t(u_k) \frac{h(u_k t)}{h(t)} + \int_{(0, \rho u_k]} (W_t(u_k) - W_t(v)) \nu_{t,k}^*(dv) \\ \xrightarrow{d} W(u_k) u_k^{-\beta} + \beta \int_0^{\rho u_k} (W(u_k) - W(v))(u_k - v)^{-\beta-1} dv. \end{aligned}$$

Likewise, again by the continuous mapping theorem,

$$\begin{aligned} \sum_{k=1}^n \gamma_k W_t(u_k) \frac{h(u_k t)}{h(t)} + \sum_{k=1}^n \gamma_k \int_{(0, \rho u_k]} (W_t(u_k) - W_t(v)) \nu_{t,k}^*(dv) \\ \xrightarrow{d} \sum_{k=1}^n \gamma_k W(u_k) u_k^{-\beta} + \sum_{k=1}^n \gamma_k \beta \int_0^{\rho u_k} (W(u_k) - W(v))(u_k - v)^{-\beta-1} dv. \end{aligned}$$

According to Theorem 3.2 in [3], it remains to check that, as $\rho \uparrow 1$,

$$\begin{aligned} \sum_{k=1}^n \gamma_k W(u_k) u_k^{-\beta} + \sum_{k=1}^n \gamma_k \beta \int_0^{\rho u_k} (W(u_k) - W(v))(u_k - v)^{-\beta-1} dv \\ \xrightarrow{d} \sum_{k=1}^n \gamma_k W(u_k) u_k^{-\beta} + \sum_{k=1}^n \gamma_k \beta \int_0^{u_k} (W(u_k) - W(v))(u_k - v)^{-\beta-1} dv \end{aligned}$$

and that, for any $c > 0$,

$$\lim_{\rho \uparrow 1} \limsup_{t \rightarrow \infty} \mathbb{P} \left\{ \left| \sum_{k=1}^n \gamma_k \int_{[\rho u_k, u_k]} (W_t(u_k) - W_t(v)) \nu_{t,k}^*(dv) \right| > c \right\} = 0. \quad (5.10)$$

The first relation is equivalent to

$$\sum_{k=1}^n \gamma_k \beta \int_{\rho u_k}^{u_k} (W(u_k) - W(v))(u_k - v)^{-\beta-1} dv \xrightarrow{\mathbb{P}} 0 \quad \text{as } \rho \uparrow 1. \quad (5.11)$$

To prove (5.11) it suffices to verify that each summand converges to zero in probability. But each summand actually tends to zero a.s. by the discussion in Subsection 2.3.

The sum in (5.10) equals

$$\sum_{k=1}^n \gamma_k \frac{\int_{[0, (1-\rho)u_k t]} (N(u_k t) - N(u_k t - y) - \mu^{-1}y) d(-h^*(y))}{g(t)h^*(t)}.$$

In view of (5.8) and Markov's inequality it suffices to check that, for $k = 1, \dots, n$,

$$\lim_{\rho \uparrow 1} \limsup_{t \rightarrow \infty} \frac{\int_{[0, (1-\rho)u_k t]} g(y) d(-h^*(y))}{g(t)h^*(t)} = 0$$

or just

$$\lim_{\rho \uparrow 1} \limsup_{t \rightarrow \infty} \frac{\int_{[0, (1-\rho)t]} g(y) d(-h^*(y))}{g(t)h^*(t)} = 0, \quad (5.12)$$

for $g(t)h^*(t)$ is regularly varying. Since $g(t)h^*(t) \rightarrow \infty$ as $t \rightarrow \infty$ by the assumptions of the theorem, we have

$$\lim_{t \rightarrow \infty} \frac{\int_{[0, t_0]} g(y) d(-h^*(y))}{g(t)h^*(t)} = 0$$

and, further,

$$\begin{aligned} \frac{\int_{[t_0, (1-\rho)t]} g(y) d(-h^*(y))}{g(t)h^*(t)} &\stackrel{(3.7)}{\leq} A \frac{\int_{[t_0, (1-\rho)t]} y^{1/\alpha-\delta} d(-h^*(y))}{t^{1/\alpha-\delta}h^*(t)} \\ &\sim \frac{\beta}{1/\alpha - \beta - \delta} (1-\rho)^{1/\alpha-\beta-\delta}, \end{aligned}$$

where the last relation is justified by Theorem 1.6.4 in [5], (5.12) follows. The proof is complete. \square

Proof of Theorem 2.9. For $t > 0$, put

$$X_t(u) := \frac{X(ut)}{g(t)h(t)} = \frac{\int_{[0, ut]} h(ut-x) dN(x)}{g(t)h(t)}, \quad u \geq 0.$$

For any $n \in \mathbb{N}$, fix $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ and $0 < u_1 < \dots < u_n$. We have to show that

$$\sum_{k=1}^n \gamma_k X_t(u_k) \xrightarrow{d} \sum_{k=1}^n \gamma_k Y(u_k) \quad \text{as } t \rightarrow \infty$$

in all cases, where $Y(u) := \int_{[0, u]} (u-y)^{-\beta} dW(y)$, $u \geq 0$ and that

$$X_t(0) = \frac{h(0)}{g(t)h(t)} \rightarrow 0 = Y(0) \quad \text{as } t \rightarrow \infty$$

in the cases $\alpha > \beta$, and $\alpha = \beta$ and $c = \infty$, the latter relation being trivial. We want to stress that in the remaining case $\lim_{t \rightarrow \infty} X_t(0) = h(0)/c$, whereas $Y(0) = 0$.

Since the convergence $\lim_{t \rightarrow \infty} h(ut)/h(t) = u^{-\beta}$ is uniform on compact sets not containing zero, and $(W(u))_{u \geq 0}$ has a.s. continuous paths, the relation (3.8) and Lemma A.2 entail

$$\int_{[0, \rho u_k]} \frac{h(t(u_k - y))}{h(t)} d \frac{N(ty)}{g(t)} \xrightarrow{d} \int_{[0, \rho u_k]} (u_k - y)^{-\beta} dW(y) \quad \text{as } t \rightarrow \infty$$

for any $\rho \in (0, 1)$ where here and in the rest of the proof integration w.r.t. $d \frac{N(ty)}{g(t)}$ means integration w.r.t. $\nu_t(dy)$ where the finite measure ν_t is defined by $\nu_t(A) = g(t)^{-1} N(tA)$, $A \subset \mathbb{R}$ Borel. By the continuous mapping theorem,

$$\sum_{k=1}^n \gamma_k \int_{[0, \rho u_k]} \frac{h(t(u_k - y))}{h(t)} d \frac{N(ty)}{g(t)} \xrightarrow{d} \sum_{k=1}^n \gamma_k \int_{[0, \rho u_k]} (u_k - y)^{-\beta} dW(y)$$

as $t \rightarrow \infty$. According to Theorem 3.2 in [3], it remains to check that, as $\rho \uparrow 1$,

$$\sum_{k=1}^n \gamma_k \int_{[0, \rho u_k]} (u_k - y)^{-\beta} dW(y) \xrightarrow{d} \sum_{k=1}^n \gamma_k \int_{[0, u_k]} (u_k - y)^{-\beta} dW(y)$$

and that, for any $c > 0$,

$$\lim_{\rho \uparrow 1} \limsup_{t \rightarrow \infty} \mathbb{P} \left\{ \left| \sum_{k=1}^n \gamma_k \int_{[\rho u_k, u_k]} \frac{h(t(u_k - y))}{h(t)} d \frac{N(ty)}{g(t)} \right| > c \right\} = 0. \quad (5.13)$$

The first relation is equivalent to

$$\sum_{k=1}^n \gamma_k \int_{[\rho u_k, u_k]} (u_k - y)^{-\beta} dW(y) \xrightarrow{\mathbb{P}} 0 \quad \text{as } \rho \uparrow 1. \quad (5.14)$$

To prove (5.14) it suffices to verify that each summand converges to zero in probability. Hence (5.14) is a direct consequence of Lemma 2.12.

For (5.13), in view of Markov's inequality it suffices to check that

$$\lim_{\rho \uparrow 1} \limsup_{t \rightarrow \infty} \frac{\int_{[\rho u_k t, u_k t]} h(u_k t - y) d \mathbb{E} N(y)}{h(t)g(t)} = 0.$$

Recalling that $h(t)g(t)$ is regularly varying the latter holds true by Lemma 5.2 below. This completes the proof of finite-dimensional convergence.

Convergence of moments follows from Lemma 5.3 below. The proof is complete. \square

5.3 Convergence of moments in the case $\mu = \infty$ and the proof of Proposition 2.11

Lemma 5.1. *Assume that $\mu = \infty$ and let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable and locally bounded function such that*

$$\lim_{t \rightarrow \infty} \frac{h(t)}{\mathbb{P}\{\xi > t\}} = c \in [0, \infty].$$

Then

$$\lim_{t \rightarrow \infty} \mathbb{E} X(t) = c.$$

In particular, $X(t) \xrightarrow{\mathbb{P}} 0$ as $t \rightarrow \infty$ if $c = 0$.

Proof. Denote by $Z(t) := t - S_{N(t)-1}$ the undershoot at time t and put

$$f(t) := \frac{h(t)}{\mathbb{P}\{\xi > t\}}, \quad t \geq 0.$$

The proof is based on the representation

$$\mathbb{E} X(t) = \int_{[0, t]} h(t-x) U(dx) = \mathbb{E} f(Z(t)), \quad t \geq 0$$

where $U(\cdot) := \mathbb{E} N(\cdot)$ denotes the renewal measure associated with $(S_k)_{k \in \mathbb{N}_0}$. Under the sole assumption $\mu = \infty$, the renewal theorem gives $Z(t) \rightarrow \infty$ in probability. Hence $f(Z(t)) \rightarrow c$ in probability. If $c < \infty$, the function f is bounded, and $\lim_{t \rightarrow \infty} \mathbb{E} f(Z(t)) = c$ by the dominated convergence theorem. If $c = \infty$, we obtain $\lim_{t \rightarrow \infty} \mathbb{E} f(Z(t)) = c = \infty$ by Fatou's lemma.

The last assertion of the lemma follows from Markov's inequality. \square

Now we are ready to prove Proposition 2.11:

Proof of Proposition 2.11. Since the exponential law is uniquely determined by its moments, the second assertion of the proposition is an immediate consequence of the first.

To prove convergence of moments, we use induction on k , the order of the moments. The case $k = 0$ is trivial, the case $k = 1$ follows from Lemma 5.1. Assuming that

$$\lim_{t \rightarrow \infty} \mathbb{E} X^j(t) = c^j j! \quad \text{for } j = 0, \dots, k-1,$$

we will prove that

$$\lim_{t \rightarrow \infty} \mathbb{E} X^k(t) = c^k k!. \quad (5.15)$$

To this end, we use the representation

$$X(t) = h(t) + X_*(t - \xi_1) \mathbb{1}_{\{\xi_1 \leq t\}}, \quad (5.16)$$

where

$$X_*(t) := \sum_{j \geq 1} h(t - S_j + S_1) \mathbb{1}_{\{S_j - S_1 \leq t\}} \stackrel{d}{=} X(t).$$

The latter implies

$$X(t)^k = X_*(t - \xi_1)^k \mathbb{1}_{\{\xi_1 \leq t\}} + \sum_{j=0}^{k-1} \binom{k}{j} h(t)^{k-j} X_*(t - \xi_1)^j \mathbb{1}_{\{\xi_1 \leq t\}}.$$

We have $\mathbb{E} X(t)^k = \mathbb{E} f_k(Z(t))$ where

$$f_k(t) := \frac{\sum_{j=0}^{k-1} \binom{k}{j} h(t)^{k-j} \mathbb{E} X_*(t - \xi_1)^j \mathbb{1}_{\{\xi_1 \leq t\}}}{\mathbb{P}\{\xi > t\}}.$$

Arguing as in Lemma 5.1, it suffices to show that

$$\lim_{t \rightarrow \infty} f_k(t) = c^k k!,$$

as $f_k(t)$ is then bounded and relation (5.15) follows by the dominated convergence theorem.

Since $\lim_{t \rightarrow \infty} h(t) = 0$ by the assumption of the proposition, and the expectations $\mathbb{E} X_*^j(t - \xi_1)^j \mathbb{1}_{\{\xi_1 \leq t\}}$, $j = 0, \dots, k-1$ are bounded by the induction hypothesis, we conclude

$$\frac{\sum_{j=0}^{k-2} \binom{k}{j} h^{k-j}(t) \mathbb{E} X_*^j(t - \xi_1) \mathbb{1}_{\{\xi_1 \leq t\}}}{\mathbb{P}\{\xi > t\}} = 0.$$

Hence

$$\begin{aligned} \lim_{t \rightarrow \infty} f_k(t) &= k \lim_{t \rightarrow \infty} \frac{h(t) \mathbb{E} X_*(t - \xi_1)^{k-1} \mathbb{1}_{\{\xi_1 \leq t\}}}{\mathbb{P}\{\xi > t\}} \\ &= ck \lim_{t \rightarrow \infty} \mathbb{E} (X(t) - h(t))^{k-1} \\ &= ck \lim_{t \rightarrow \infty} \left(\mathbb{E} X(t)^{k-1} + \sum_{j=0}^{k-2} \binom{k-1}{j} h(t)^{k-1-j} \mathbb{E} X(t)^j \right) \\ &= ck \lim_{t \rightarrow \infty} \mathbb{E} X(t)^{k-1} = c^k k!, \end{aligned}$$

where the penultimate equality follows from the induction hypothesis and $\lim_{t \rightarrow \infty} h(t) = 0$. \square

Lemma 5.2. *Assume that the assumptions of Theorem 2.9 hold. Then*

$$\lim_{\rho \uparrow 1} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}\{\xi > t\}}{h(t)} \int_{[\rho t, t]} h(t-y) U(dy) = 0 \quad (5.17)$$

where $U(\cdot) := \mathbb{E} N(\cdot)$. In particular,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}\{\xi > t\}}{h(t)} \mathbb{E} X(t) = \mathbb{E} \int_{[0,1]} (1-y)^{-\beta} dW(y) = \frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)\Gamma(1+\alpha-\beta)}.$$

Proof. We use the notation of Lemma 5.1, that is, $Z(t) := t - S_{N(t)-1}$ denotes the undershoot at time t and $f(t) := h(t)/\mathbb{P}\{\xi > t\}$, $t \geq 0$. Then, the expression under the double limit in (5.17) equals $\mathbb{E} f(Z(t)) \mathbb{1}_{\{Z(t) \leq (1-\rho)t\}} / f(t)$.

CASE 1: We first consider the case when $\alpha > \beta$ or $\alpha = \beta$ and $c = \infty$.

If $\alpha > \beta$ then, by Theorem 1.5.3 in [5] there exists an *increasing* function u such that $u(t) \sim f(t)$ as $t \rightarrow \infty$. If $\alpha = \beta$ and $c = \infty$ such a function u exists by assumption. Now fix $\varepsilon > 0$ and let $t_0 > 0$ be such that $(1-\varepsilon)u(t) \leq f(t) \leq (1+\varepsilon)u(t)$ for all $t \geq t_0$. Then

$$\frac{\mathbb{E} f(Z(t)) \mathbb{1}_{\{Z(t) \leq t_0\}}}{f(t)} \leq \frac{\sup_{0 \leq y \leq \varepsilon t_0} f(y)}{f(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

by the local boundedness of f . Further, for t such that $(1 - \rho)t > t_0$,

$$\begin{aligned} \frac{\mathbb{E} f(Z(t)) \mathbb{1}_{\{t_0 < Z(t) \leq (1-\rho)t\}}}{f(t)} &\leq \frac{1 + \varepsilon}{1 - \varepsilon} \frac{\mathbb{E} u(Z(t)) \mathbb{1}_{\{Z(t) \leq (1-\rho)t\}}}{u(t)} \\ &\leq \frac{1 + \varepsilon}{1 - \varepsilon} \mathbb{P}\{Z(t) \leq (1 - \rho)t\}. \end{aligned}$$

By a well-known result due to Dynkin (see, for instance, Theorem 8.6.3 in [5])

$$\lim_{t \rightarrow \infty} \mathbb{P}\{Z(t) \leq (1 - \rho)t\} = \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha)} \int_0^{1-\rho} y^{-\alpha}(1 - y)^{\alpha-1} dy$$

When $\rho \uparrow 1$ the last integral goes to zero, which proves (5.17).

CASE 2: Now consider the case when $\alpha = \beta$ and $c < \infty$. Then f is bounded. Hence $\mathbb{E} f(Z(t)) \mathbb{1}_{\{Z(t) \leq (1-\rho)t\}} \leq \text{const} \cdot \mathbb{P}\{Z(t) \leq (1 - \rho)t\}$, $t \geq 0$. The rest of the proof is the same as in the previous case.

We now turn to the second assertion of the lemma. Define $U(t) := \mathbb{E} N(t)$. Then, by the transformation formula,

$$\frac{\mathbb{P}\{\xi > t\}}{h(t)} \int_{[0, \rho t]} h(t - y) U(dy) = \mathbb{P}\{\xi > t\} U(t) \int_{[0, \rho]} \frac{h(t(1 - y))}{h(t)} U_t(dy)$$

where $U_t([0, x]) = U([0, tx])/U(t)$, $0 \leq x \leq 1$. Formula (8.6.4) on p. 361 in [5] says that $\lim_{t \rightarrow \infty} \mathbb{P}\{\xi > t\} U(t) = \Gamma(1 - \alpha)^{-1} \Gamma(1 + \alpha)^{-1}$. Further, by [5, Formula (8.6.3) on p. 361], $U_t(dx)$ converges weakly to $\alpha x^{\alpha-1} dx$ as $t \rightarrow \infty$. This in combination with the uniform convergence theorem [5, Theorem 1.2.1] yields:

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{P}\{\xi > t\} U(t) \int_{[0, \rho]} \frac{h(t(1 - y))}{h(t)} U_t(dy) \\ = \frac{\alpha}{\Gamma(1 - \alpha)\Gamma(1 + \alpha)} \int_0^\rho (1 - y)^{-\beta} y^{\alpha-1} dy \\ \xrightarrow{\rho \rightarrow 1} \frac{\Gamma(1 - \beta)}{\Gamma(1 + \alpha - \beta)\Gamma(1 - \alpha)} \end{aligned}$$

where we have used the expression of the Beta function in terms of the Gamma function and the functional equation of the Gamma function. (5.17) now gives the second assertion of the lemma. \square

Lemma 5.3. *Under the assumptions of Theorem 2.9, (2.16) holds.*

Proof. CASE 1: First assume that $\alpha > \beta$ and case $\alpha = \beta$ and $c = \infty$. Set $g(t) = 1/\mathbb{P}\{\xi > t\}$ and observe that $\lim_{t \rightarrow \infty} g(t)h(t) = \infty$ in the present situation. We prove the result only for $u = 1$. Define

$$d_k := \frac{k!}{\Gamma(1 - \alpha)^k} \prod_{j=1}^k \frac{\Gamma(1 - \beta + (j - 1)(\alpha - \beta))}{\Gamma(j(\alpha - \beta) + 1)}, \quad k \in \mathbb{N}.$$

We then have to prove that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} X(t)^j}{g(t)^j h(t)^j} = d_j \tag{5.18}$$

for all $j \in \mathbb{N}$. We will use induction on j . The case $j = 1$ follows from Lemma 5.2. Assuming that (5.18) holds for $j = 1, \dots, k-1$, we will prove (5.18) for $j = k$.

From the decomposition (5.16), one can derive the following representation for $\mathbb{E} X(t)^k$:

$$\mathbb{E} X(t)^k = \int_{[0,t]} r_k(t-y) U(dy) \quad (5.19)$$

where $U(\cdot) := \mathbb{E} N(\cdot)$ and

$$r_k(t) = \sum_{j=0}^{k-1} \binom{k}{j} h(t)^{k-j} \mathbb{E} X_*(t - \xi_1)^j \mathbb{1}_{\{\xi_1 \leq t\}} = \sum_{j=0}^{k-1} v_j h(t)^{k-j} \mathbb{E} X(t)^j$$

for some real constants v_j , $0 \leq j \leq k-1$ with $v_{k-1} = k$.

If we can prove that

$$\lim_{t \rightarrow \infty} \frac{r_k(t)}{g(t)^{k-1} h(t)^k} = kd_{k-1}, \quad (5.20)$$

which, among other things, means that $r_k(t)$ is regularly varying at ∞ with index $(k-1)\alpha - k\beta$, then (5.19) in combination with the argument given in the proof of Lemma 5.2 shows that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathbb{E} X(t)^k}{g(t)^k h(t)^k} &= kd_{k-1} \lim_{t \rightarrow \infty} \frac{U(t) \int_{[0,t]} r_k(t-y) U(dy)}{g(t) r_k(t) U(t)} \\ &= \frac{\alpha kd_{k-1}}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^1 (1-y)^{(k-1)\alpha - k\beta} y^{\alpha-1} dy \\ &= \frac{\Gamma(1-\beta + (k-1)(\alpha - \beta))}{\Gamma(1-\alpha)\Gamma(k(\alpha - \beta) + 1)} kd_{k-1} = d_k. \end{aligned}$$

We now verify (5.20). By the induction hypothesis, for $j = 0, \dots, k-1$, $\mathbb{E} X(t)^j$ is regularly varying with index $j(\alpha - \beta)$. Hence, for such a j , $h(t)^{k-j} \mathbb{E} X(t)^j$ is regularly varying with index $j\alpha - k\beta$. Since $g(t)^{k-1} h(t)^k$ is regularly varying with index $(k-1)\alpha - k\beta$, we conclude that

$$\lim_{t \rightarrow \infty} \frac{\sum_{j=0}^{k-2} v_j h(t)^{k-j} \mathbb{E} X(t)^j}{g^{k-1}(t) h^k(t)} = 0.$$

Hence

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} X(t)^k}{g(t)^{k-1} h(t)^k} = \lim_{t \rightarrow \infty} \frac{k \mathbb{E} X(t)^{k-1}}{g(t)^{k-1} h(t)^{k-1}} = kd_{k-1},$$

which proves (5.20).

CASE 2: $\alpha = \beta$ and $c < \infty$. (2.16) has been proved in Proposition 2.11 under weaker assumptions. \square

A Appendix: Auxiliary results

Lemma A.1. *Let W be a random variable with characteristic function given by (2.15). Then, for $r < \alpha$,*

$$\mathbb{E}|W|^r = \frac{2\Gamma(r+1)}{\pi r} \sin(r\pi/2)\Gamma(1-r/\alpha)|\Gamma(1-\alpha)|^{r/\alpha} \cos(\pi r/2 - \pi r/\alpha)$$

where $\Gamma(\cdot)$ denotes the gamma function. In particular,

$$\mathbb{E}|W| = 2\pi^{-1}\Gamma(1-1/\alpha)|\Gamma(1-\alpha)|^{1/\alpha} \sin(\pi/\alpha).$$

Proof. We use the integral representation for the r th absolute moment (see Lemma 2 in [2])

$$m_r := \mathbb{E}|W|^r = \frac{\Gamma(r+1)}{\pi} \sin\left(\frac{r\pi}{2}\right) \int_{\mathbb{R}} \frac{1 - \operatorname{Re} \mathbb{E} e^{itW}}{|t|^{r+1}} dt. \quad (\text{A.1})$$

Set $A := \pi^{-1}\Gamma(r+1) \sin(r\pi/2)$, $B := \Gamma(1-\alpha) \cos(\pi\alpha/2)$ and $C := \Gamma(1-\alpha) \sin(\pi\alpha/2)$. Using Euler's identity $e^{ix} = \cos x + i \sin x$ in (2.15), we obtain

$$\operatorname{Re} \mathbb{E} e^{itW} = \exp(-B|t|^\alpha) \cos(-C|t|^\alpha \operatorname{sgn}(t)).$$

Substituting this into formula (A.1) yields

$$m_r = 2A \int_0^\infty \frac{1 - \exp(-Bt^\alpha) \cos(Ct^\alpha)}{t^{r+1}} dt.$$

A change of variables ($u := t^\alpha$) gives

$$\begin{aligned} m_r &= \frac{2A}{\alpha} \int_0^\infty (1 - \exp(-Bu) \cos(Cu)) u^{-1-r/\alpha} du \\ &= \frac{2A}{\alpha} \int_0^\infty (1 - \exp(-Bu)) u^{-1-r/\alpha} du \\ &\quad + \frac{2A}{\alpha} \int_0^\infty (\exp(-Bu) - \exp(-Bu) \cos(Cu)) u^{-1-r/\alpha} du \\ &=: I_1 + I_2. \end{aligned} \quad (\text{A.2})$$

According to formula (3.945(2)) in [13], we have

$$I_2 = \frac{2A}{\alpha} \Gamma(-r/\alpha) \left(B^{r/\alpha} - \Gamma(1-\alpha)^{r/\alpha} \cos(\pi r/2 - \pi r/\alpha) \right).$$

To calculate I_1 we use integration by parts:

$$\begin{aligned} I_1 &= \frac{2A}{\alpha} \int_0^\infty (1 - \exp(-Bu)) u^{-1-r/\alpha} du \\ &= \frac{2AB}{r} \int_0^\infty u^{-r/\alpha} \exp(-Bu) du \\ &= \frac{2AB^{r/\alpha}}{r} \Gamma(1-r/\alpha) = -\frac{2AB^{r/\alpha}}{\alpha} \Gamma(-r/\alpha). \end{aligned}$$

Now plugging in the values of I_1 and I_2 in (A.2) gives

$$\begin{aligned} m_r &= -\frac{2A}{\alpha} \Gamma(-r/\alpha) \Gamma(1-\alpha)^{r/\alpha} \cos(\pi r/2 - \pi r/\alpha) \\ &= \frac{2A}{r} \Gamma(1-r/\alpha) \Gamma(1-\alpha)^{r/\alpha} \cos(\pi r/2 - \pi r/\alpha). \end{aligned}$$

The proof is complete. \square

Lemma A.2. *Let $0 \leq a < b < \infty$. Assume that $X_t(\cdot) \Rightarrow X(\cdot)$ as $t \rightarrow \infty$ in $D[a, b]$ in the M_1 topology. Further, assume that ν_t , $t \geq 0$ are finite measures such that $\nu_t \rightarrow \nu$ weakly as $t \rightarrow \infty$ where ν is a finite measure on $[a, b]$, which is continuous w.r.t. Lebesgue measure. Then*

$$\int_{[a, b]} X_t(y) \nu_t(dy) \xrightarrow{d} \int_{[a, b]} X(y) \nu(dy) \quad \text{as } t \rightarrow \infty,$$

and, for any fixed $c \in [a, b]$,

$$X_t(c) + \int_{[a, b]} X_t(y) \nu_t(dy) \xrightarrow{d} X(c) + \int_{[a, b]} X(y) \nu(dy) \quad \text{as } t \rightarrow \infty.$$

Proof. The assertion of the lemma follows from Lemma 6.5 in [18] in combination with Skorokhod's representation theorem. \square

Lemma A.3. *Let $F : [a, b] \rightarrow \mathbb{R}$ and $G : [a, b] \rightarrow \mathbb{R}$ be left- and right-continuous functions of bounded variation, respectively. Then*

$$\int_{(a, b]} F(y) dG(y) = F(b+)G(b) - F(a+)G(a) - \int_{(a, b]} G(y) dF(y). \quad (\text{A.3})$$

Proof. The result follows from Theorem T2 (Product Formula) of [6] with $f(x) = G(a+x)$ and $g(x) := F((a+x)+)$. The so-defined g is right-continuous and the theorem applies. Since F is assumed to be left continuous, $g(x-) = F(a+x)$. The asserted formula therefore follows from (2.1) in the cited book. \square

Lemma A.4. *Let (X_1, \mathcal{A}_1) and (X_2, \mathcal{A}_2) be measurable spaces. Let $\mu_1(\cdot)$ be a finite measure on (X_1, \mathcal{A}_1) . Assume that $\mu_2(x, \cdot)$ is a measure on (X_2, \mathcal{A}_2) for every $x \in X_1$, and that for every $B \in \mathcal{A}_2$ the function $X_1 \ni x \mapsto \mu_2(x, B)$ is measurable with respect to \mathcal{A}_1 . Then $\nu(B) := \int_{X_1} \mu_2(x, B) d\mu_1(x)$ is a measure on (X_2, \mathcal{A}_2) . Furthermore, for every non-negative measurable function f on (X_2, \mathcal{A}_2) , the function $g(x) := \int_{X_2} f(y) \mu_2(x, dy)$ is measurable with respect to \mathcal{A}_1 and*

$$\int_{X_2} f d\nu = \int_{X_1} g d\mu_1. \quad (\text{A.4})$$

The proof of this lemma is a standard approximation argument: check (A.4) for indicators of sets in \mathcal{A}_2 , then for the finite linear combinations of such indicators with positive coefficients and finally for arbitrary non-negative measurable functions.

Lemma A.5. Let $(X(\omega, u))_{u \in \mathbb{R}}$ be an arbitrary increasing random process defined on probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For fixed $k \in \mathbb{N}$ let $h : \mathbb{R}^k \rightarrow \mathbb{R}_+$ be a positive Borel function. Then

$$\mathbb{E} \int_{\mathbb{R}^k} h(s_1, \dots, s_k) dX(s_1) \dots dX(s_k) = \int_{\mathbb{R}^k} h(s_1, \dots, s_k) \mathbb{E} (dX(s_1) \dots dX(s_k)).$$

Proof. Lemma A.4 can be applied as follows. Define $(X_1, \mathcal{A}_1) := (\Omega, \mathcal{A})$ and $(X_2, \mathcal{A}_2) := (\mathbb{R}^k, \mathcal{B})$ where \mathcal{B} is a standard Borel σ -algebra of subsets of \mathbb{R}^k . Set also $\mu_1 := \mathbb{P}$. For every $\omega \in \Omega$ put

$$\mu_2(\omega, (a_1, b_1] \times \dots \times (a_k, b_k]) := (X(\omega, b_1) - X(\omega, a_1)) \dots (X(\omega, b_k) - X(\omega, a_k))$$

and extend this to a measure on (X_2, \mathcal{A}_2) . This is possible since $X(\omega, u)$ is increasing for almost all ω . It is clear that $\omega \mapsto \mu_2(\omega, A)$ is measurable for every $A \in \mathcal{B}$. By Lemma A.4, $\nu(A) = \mathbb{E} \mu_2(\omega, A)$ is a measure and for any given positive Borel function $h : \mathbb{R}^k \rightarrow \mathbb{R}_+$, the function $g : \Omega \rightarrow \mathbb{R}_+$ defined by $g(\omega) := \int_{\mathbb{R}^k} h(s_1, \dots, s_k) \mu_2(\omega, ds_1, \dots, ds_k)$ is measurable and

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^k} h(s_1, \dots, s_k) dX(s_1) \dots dX(s_k) \\ &= \mathbb{E} \int_{\mathbb{R}^k} h(s_1, \dots, s_k) \mu_2(\omega, ds_1 \times \dots \times ds_k) \\ &= \int_{\Omega} g(\omega) \mathbb{P}(d\omega) \stackrel{(A.4)}{=} \int_{\mathbb{R}^k} h(s_1, \dots, s_k) \nu(ds_1 \times \dots \times ds_k) \\ &= \int_{\mathbb{R}^k} h(s_1, \dots, s_k) \mathbb{E} (dX(s_1) \dots dX(s_k)). \end{aligned}$$

□

Lemma A.6. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a decreasing function with $\lim_{t \rightarrow \infty} f(t) \geq 0$. Then, for every $\theta > 0$,

$$\int_0^n f(\theta y) dy = \sum_{k=0}^n f(\theta k) + \delta_n(\theta), \quad n \in \mathbb{N},$$

where $\delta_n(\theta)$ converges as $n \rightarrow \infty$ to some $\delta(\theta) \leq 0$.

Proof. We assume w.l.o.g. that $\theta = 1$. For each $n \geq 1$,

$$\sum_{k=0}^n f(k) - \int_0^n f(y) dy = \sum_{k=0}^{n-1} \left(f(k) - \int_k^{k+1} f(y) dy \right) + f(n).$$

Since f is decreasing, each summand in the sum is non-negative. Hence, the sum is increasing in n . On the other hand, it can be bounded from above by

$$\sum_{k=0}^{n-1} \left(f(k) - \int_k^{k+1} f(y) dy \right) \leq \sum_{k=0}^{n-1} (f(k) - f(k+1)) \leq f(0) < \infty.$$

Consequently, the series $\sum_{k \geq 0} (f(k) - \int_k^{k+1} f(y) dy)$ converges. Recalling that $\lim_{n \rightarrow \infty} f(n)$ exists completes the proof. □

Lemma A.7. Let $f_1, f_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be bounded decreasing functions such that $f_1(t) \geq f_2(t)$ for all $t \in \mathbb{R}_+$ and such that $\int_0^{t_0} (f_1(y) - f_2(y)) dy > 0$ for some $t_0 > 0$. Then

$$\sup_{t \geq t_0} \frac{\mathbb{E} \int_{[0, t]} (f_1(t-y) - f_2(t-y)) dN(y)}{\int_0^t (f_1(y) - f_2(y)) dy} < \infty. \quad (\text{A.5})$$

Proof. Decompose the integral in the numerator of the left-hand side of (A.5) as follows:

$$\int_{[0, [t]]} (f_1(t-y) - f_2(t-y)) dN(y) + \int_{([t], t]} (f_1(t-y) - f_2(t-y)) dN(y) =: I_1(t) + I_2(t).$$

By the distributional subadditivity of N , we get for $I_2(t)$:

$$\begin{aligned} I_2(t) &\leq \int_{([t], t]} f_1(t-y) dN(y) \leq f_1(t - [t])(N(t) - N([t])) \\ &\leq f_1(0)(N(t) - N(t-1)) \stackrel{d}{\leq} f_1(0)N(1), \end{aligned}$$

hence $\mathbb{E} I_2(t) < f_1(0) \mathbb{E} N(1) < \infty$ for all $t \geq 0$. It remains to consider $I_1(t)$:

$$\begin{aligned} \mathbb{E} I_1(t) &= f_1(t) - f_2(t) + \mathbb{E} \sum_{j=0}^{[t]-1} \int_{(j, j+1]} (f_1(t-y) - f_2(t-y)) dN(y) \\ &\leq f_1(t) - f_2(t) + \sum_{j=0}^{[t]-1} (f_1(t-j-1) - f_2(t-j)) \mathbb{E}(N(j+1) - N(j)) \\ &\leq f_1(0) + \sum_{j=0}^{[t]-1} (f_1(t-j-1) - f_2(t-j)) \mathbb{E} N(1) \\ &= \mathbb{E} N(1) \left(\int_0^{[t]} (f_1(y) - f_2(y)) dy + O(1) \right). \end{aligned}$$

This proves the lemma. □

References

- [1] ALSMEYER, G., IKSANOV, A. AND MEINERS, M. (2012+). Power and exponential moments of the number of visits and related quantities for perturbed random walks. *Submitted*. Preprint available at <http://arxiv.org/abs/1111.4159>.
- [2] VON BAHR, B. AND ESSEEN, C.-G. (1965). Inequalities for the r th absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist.* **36**, 299–303.
- [3] BILLINGSLEY, P. (1999). *Convergence of probability measures*, 2nd Ed., Wiley Series in Probability and Statistics: New York.
- [4] BINGHAM, N. H. (1971). Limit theorems for occupation times of Markov processes. *Z. Wahrsch. verw. Gebiete.* **17**, 1–22.

- [5] BINGHAM N. H., GOLDIE C. M., AND TEUGELS, J. L. (1989). *Regular variation*. Cambridge University Press: Cambridge.
- [6] BRÉMAUD, P. (1981). *Point Processes and Queues. Martingale dynamics*. Springer-Verlag: New York.
- [7] CHOW, Y. S. AND TEICHER, H. (1997). *Probability theory: independence, interchangeability, martingales*. 3rd Ed., Springer-Verlag: New York etc.
- [8] DALEY, D. J. AND VERE-JONES, D. (2003). *An introduction to the theory of point processes. Volume I: Elementary theory and methods*. Springer: New York.
- [9] FELLER, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67**, 98–119.
- [10] FELLER, W. (1971). *An introduction to probability theory and its applications*. Vol. II. John Wiley & Sons: New York.
- [11] GNEDIN, A. AND IKSANOV, A. (2012). Regenerative compositions in the case of slow variation: A renewal theory approach. *Electron. J. Probab.* **17**, no. 77, 1–19.
- [12] GNEDIN, A., IKSANOV, A. AND MARYNYCH, A. (2010). The Bernoulli sieve: an overview. *Discr. Math. Theoret. Comput. Sci.* Proceedings Series, **AM**, 329–342.
- [13] GRADSTEIN, I. S. AND RYZHIK, I. M. (2000). *Tables of integrals, series and products*. Translated from the Russian. Sixth edition, Academic Press Inc.: San Diego.
- [14] GUT, A. (2009). *Stopped random walks: Limit theorems and applications*, 2nd edition. Springer: New York.
- [15] HEINRICH, L. AND SCHMIDT, V. (1985). Normal convergence of multidimensional shot noise and rates of this convergence. *Adv. Appl. Probab.* **17**, 709–730.
- [16] IBRAGIMOV, I. A. AND LINNIK, YU. V. (1971). *Independent and stationary sequences of random variables*. Wolters-Noordhoff Publishing: Groningen.
- [17] IGLEHART, D. L. (1973). Weak convergence of compound stochastic process I. *Stoch. Proc. Appl.* **1**, 11–31.
- [18] IKSANOV, A. (2012+). Functional limit theorems for explosive renewal shot noise processes. Preprint available at <http://arxiv.org/abs/1202.1950v2>.
- [19] IKSANOV, A. (2012+). On the number of empty boxes in the Bernoulli sieve I. To appear in *Stochastics*. DOI:10.1080/17442508.2012.688974
- [20] IKSANOV, A., MARYNYCH, A. AND VATUTIN, V. (2013+). On the number of empty boxes in the Bernoulli sieve: convergence of finite-dimensional distributions. *In preparation*.
- [21] KAPLAN, N. (1975). Limit theorems for a GI/G/ ∞ queue. *Ann. Probab.* **3**, 780–789.
- [22] KERSTING, G. (2012). The asymptotic distribution of the length of beta-coalescent trees. *Ann. Appl. Probab.* **22**, 2086–2107.
- [23] KLÜPPELBERG, C. AND MIKOSCH, T. (1995). Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli* **1**, 125–147.
- [24] KLÜPPELBERG, C. AND MIKOSCH, T. (1995). Delay in claim settlement and ruin probability approximations. *Scand. Actuar. J.* 1995(2), 154–168.

- [25] KLÜPPELBERG, C., MIKOSCH, T. AND SCHÄRF, A. (2003). Regular variation in the mean and stable limits for Poisson shot noise. *Bernoulli* **9**, 467–496.
- [26] LANE J. A. (1984). The central limit theorem for the Poisson shot-noise process. *J. Appl. Probab.* **21**, 287–301.
- [27] LAWRENCE, A. J. AND KOTTEGODA, N. T.(1977). Stochastic modelling of riverflow time series. *J. R. Statist. Soc. A* **140**, 1–47.
- [28] LEWIS, P. A. W. (1964). A branching Poisson process model for the analysis of computer failure patterns. *J. R. Statist. Soc. B* **26**, 398–456.
- [29] LINDVALL, T. (1992). *Lectures on the coupling method*, Wiley Series in Probability and Mathematical Statistics: New York.
- [30] MARCUS, A. H. (1975). Some exact distributions in traffic noise theory. *Adv. Appl. Probab.* **7**, 593–606.
- [31] MEERSCHAERT, M. M. AND SCHEFFLER, H. P. (2004). Limit theorems for continuous time random walks with infinite mean waiting times. *J. Appl. Probab.* **41**, 623–638.
- [32] MILLER. D. R. (1974). Limit theorems for path-functionals of regenerative processes, *Stoch. Proc. Appl.* **2**, 141–161.
- [33] MOHAN, N. R. (1976). Teugels’ renewal theorem and stable laws. *Ann. Probab.* **4**, 863–868.
- [34] RICE, J. (1977). On generalized shot noise. *Adv. Appl. Probab.* **9**, 553–565.
- [35] RODRIGUEZ-ITURBE I., COX D. R., ISHAM, V. (1987). Some models on rainfall based on stochastic processes. *Proc. R. Soc. Lond. A* **410**, 269–288.
- [36] ROSIŃSKI, J. (2001). Series representations of Lvy processes from the perspective of point processes. Barndorff-Nielsen, Ole E. (ed.) et al., Lvy processes. Theory and applications. Birkhäuser: Boston. 401–415.
- [37] SAMORODNITSKY, G. (1996). A class of shot noise models for financial applications. Athens conference on applied probability and time series analysis, Athens, Greece, March 2226, 1995. Vol. I: Applied probability. In honor of J. M. Gani. Berlin: Springer. Lect. Notes Stat., Springer-Verlag. 114, 332–353.
- [38] SAMORODNITSKY, G., AND TAQQU, M. S. (1994). *Stable non-Gaussian Random Processes*. Stochastic Modeling. Chapman & Hall: New York. Stochastic Models with Infinite Variance.
- [39] W. SCHOTTKY (1918). Spontaneous current fluctuations in electron streams. *Ann. Phys.* **57**, 541–567.
- [40] SGIBNEV, M. S. (1981). Renewal theorem in the case of an infinite variance. *Sibirskii Matematicheskii Zhurnal* **22**, no. 5, 178–189.
- [41] THORISSON, H. (2000). *Coupling, Stationarity, and Regeneration*, Springer: New York.
- [42] VERE-JONES, D. (1970). Stochastic models for earthquake occurrence. *J. R. Statist. Soc. B* **32**, 1–62.
- [43] WAYMIRE, E., GUPTA, V. K. (1981). The mathematical structure of rainfall representations: 1. A review of the stochastic rainfall models. *Water Resour. Res.* **17**, 1261–1272.

- [44] WEISS, G. (1977). Shot noise models for the generation of synthetic streamflow data. *Water Resour. Res.* **13**, 101–108.
- [45] WHITT, W. (2002). *Stochastic-process limits: an introduction to stochastic-process limits and their application to queues*. Springer-Verlag: New York.