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# RANDOM WALK IN A HIGH DENSITY DYNAMIC RANDOM ENVIRONMENT

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ABSTRACT. The goal of this note is to prove a law of large numbers for the empirical speed of a green particle that performs a random walk on top of a field of red particles which themselves perform independent simple random walks on  $\mathbb{Z}^d$ ,  $d \geq 1$ . The red particles jump at rate 1 and are in a Poisson equilibrium with density  $\mu$ . The green particle also jumps at rate 1, but uses different transition kernels  $p'$  and  $p''$  depending on whether it sees a red particle or not. It is shown that, in the limit as  $\mu \rightarrow \infty$ , the speed of the green particle tends to the average jump under  $p'$ . This result is far from surprising, but it is non-trivial to prove. The proof that is given in this note is based on techniques that were developed in [10] to deal with spread-of-infection models. The main difficulty is that, due to particle conservation, space-time correlations in the field of red particles decay slowly. This places the problem in a class of random walks in dynamic random environments for which scaling laws are hard to obtain.

## 1. INTRODUCTION AND BACKGROUND

**1.1. Model and main theorem.** We consider a *green* particle that performs a continuous-time random walk on  $\mathbb{Z}^d$ ,  $d \geq 1$ , under the influence of a field of *red* particles which themselves perform independent continuous-time simple random walks jumping at rate 1, constituting a dynamic random environment. The latter is denoted by

$$(1.1) \quad N = (N(t))_{t \geq 0} \quad \text{with} \quad N(t) = \{N(x, t) : x \in \mathbb{Z}^d\},$$

where  $N(x, t) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  is the number of red particles at site  $x$  at time  $t$ . As initial state we take  $N(0) = \{N(x, 0) : x \in \mathbb{Z}^d\}$  to be i.i.d. Poisson random variables with mean  $\mu$ . As is well known, this makes  $N$  invariant under translations in space and time.

Also the green particle jumps at rate 1, however, our assumption is that the jump is drawn from two different random walk transition kernels  $p'$  and  $p''$  on  $\mathbb{Z}^d$  depending on whether the space-time point of the jump is occupied by a red particle or not. We assume that  $p'$  and  $p''$  have finite range, and write

$$(1.2) \quad v' = \sum_{x \in \mathbb{Z}^d} x p'(0, x), \quad v'' = \sum_{x \in \mathbb{Z}^d} x p''(0, x),$$

to denote their mean. We write

$$(1.3) \quad \mathcal{G} = (\mathcal{G}(t))_{t \geq 0}$$

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to denote the path of the green particle with  $\mathcal{G}(0) = 0$ , and write  $P^\mu$  to denote the joint law of  $N$  and  $\mathcal{G}$ . Our main result is the following asymptotic weak law of large numbers ( $\|\cdot\|$  is the Euclidean norm on  $\mathbb{R}^d$ ).

**Theorem 1.1.** *For every  $\varepsilon > 0$ ,*

$$(1.4) \quad \lim_{\mu \rightarrow \infty} \limsup_{t \rightarrow \infty} P^\mu \{ \|t^{-1}\mathcal{G}(t) - v'\| > \varepsilon \} = 0.$$

**1.2. Discussion.** The result in Theorem 1.1 is far from surprising. As  $\mu \rightarrow \infty$ , at any given time the fraction of *sites* occupied by red particles tends to 1. Therefore we may expect that the fraction of *time* the green particle sees a red particle tends to 1 also. Consequently, we may expect the green particle to almost satisfy a weak law of large numbers corresponding to the transition kernel  $p'$ , as if it were seeing a red particle always. Despite this simple intuition, the result in Theorem 1.1 seems non-trivial to prove. The proof in the present note relies on techniques developed in [10] to deal with spread-of-infection models.

The key problem is to show that for large  $\mu$  the green particle is unlikely to spend an appreciable amount of time in the rare space-time holes of the field of red particles. To see why this is non-trivial, consider the case  $d = 1$  with two nearest-neighbor transition kernels  $p'$  and  $p''$  of the form

$$(1.5) \quad p'(0, 1) = p = p''(0, -1), \quad p'(0, -1) = 1 - p = p''(0, 1), \quad p \in (\tfrac{1}{2}, 1),$$

for which  $v' = 2p - 1 = -v'' > 0$ . Then the green particle drifts to the right when it sees a red particle, but drifts to the left when it sees a hole. Thus, it has a tendency to linger around the boundaries of the red clusters, hopping in and out of these clusters repeatedly. To prove Theorem 1.1, we must show that the green particle does not do this in-out hopping too often. The proof in the present note uses a *multi-scale renormalization* argument, working with “good” blocks (where the green particle sees only red clusters) and “bad” blocks (where it also sees some holes). These blocks live on successive space-time scales. Estimates on how often the green particle visits the bad blocks must be uniform in the path of the green particle and must be sharp in the limit as  $\mu \rightarrow \infty$ .

**1.3. Literature.** How does Theorem 1.1 relate to the existing literature? So far, random walks in three classes of dynamic random environments have been considered: (1) *independent in time*: globally updated at each unit of time; (2) *independent in space*: locally updated according to independent single-site Markov chains; (3) *dependent in space and time*. Typically, the jumps of the walk are chosen to depend on the environment in some local manner. Most papers require additional assumptions on the environment, like a strong decay of space-time correlations (see e.g. [5], [7], [11]) or a weak influence on the walk (see e.g. [3]). In the latter case the random walk in dynamic random environment is a small perturbation of a homogeneous random walk. For more references we refer the reader to [3]. Some papers allow for a mutual interaction between the walk and the environment. For an example where the jumps of the walk depend on the environment in a non-local manner, see [9].

In [2], a strong law of large numbers was proved for finite-range random walks on a class of interacting particle systems of type (3) that satisfy a space-time mixing property called *cone-mixing*. The latter can be loosely described as the requirement that the law of the states of the interacting particle system inside a space-time cone opening upwards is close to equilibrium *conditional* on the states inside a space

plane far below the tip. The proof was based on a *regeneration-time* argument, showing that there are infinitely many space-time points at which the walk stands still for a long time, allowing the environment to lose memory. All uniquely ergodic attractive spin-flip systems for which the coupling time at the origin has finite mean are cone-mixing. However, independent random walks are *not* cone-mixing. Indeed, *particle conservation destroys the cone-mixing property*, which is why Theorem 1.1 covers new ground. Other examples of dynamic random environments that are not cone-mixing for which a strong law of large number for the random walk has been proved can be found in [4] (one-dimensional exclusion process and  $v, v' > 0$  large), [1] (one-dimensional exclusion process speeded up in time) and [8] (one-dimensional supercritical contact process).

**1.4. Open problems and outline.** It remains an open problem to extend Theorem 1.1 to a weak law of large numbers for finite  $\mu$ , i.e., to show that for every  $\mu > 0$  there exists a  $v(\mu) \in \mathbb{R}^d$  such that, for every  $\varepsilon > 0$ ,

$$(1.6) \quad \lim_{t \rightarrow \infty} P^\mu \{ \|t^{-1} \mathcal{G}(t) - v(\mu)\| > \varepsilon \} = 0.$$

The speed in (1.6) will be necessarily of the form

$$(1.7) \quad v(\mu) = \rho(\mu) v' + [1 - \rho(\mu)] v''$$

for some  $\rho(\mu) \in [0, 1]$ , the latter representing the limiting fraction of time the green particle sees a red particle. We should not expect that  $\rho(\mu) = P^\mu \{ N(0, 0) \geq 1 \} = 1 - e^{-\mu}$ . Indeed, since  $\rho(\mu)$  is a functional of the *environment process*, i.e., the environment as seen relative to the location of the walk, we should not expect that  $\rho(\mu)$  is a simple function of  $\mu$ .

To appreciate the difficulty of identifying  $\rho(\mu)$ , note that for *static* random environments  $\rho(\mu)$  can have anomalous behavior as a function of  $\mu$ . For instance, if we freeze the red particles and we let the green particle use the transition kernels in (1.5), then it is well-known (see [12]) that

$$(1.8) \quad \rho(\mu) \begin{cases} = \frac{1}{2}, & \text{if } \mu \in [\mu_c^-, \mu_c^+], \\ > \frac{1}{2}, & \text{if } \mu > \mu_c^+, \\ < \frac{1}{2}, & \text{if } \mu < \mu_c^-, \end{cases}$$

with  $0 < \mu_c^- = \log(\frac{1}{p}) < \mu_c^+ = \log(\frac{1}{1-p}) < \infty$ , resulting in  $v(\mu) = 0$  for  $\mu \in [\mu_c^-, \mu_c^+]$  and  $v(\mu) \neq 0$  elsewhere.

It would be interesting to try and extend Theorem 1.1 (and possibly also (1.6)) to the case where the dynamic random environment is the exclusion process or the zero-range process, both of which fail to be cone-mixing as well. These are natural examples that have so far defied a proper analysis.

The rest of this note is organized as follows. In Section 2 we recall several definitions from [10]. In Section 3 we state and prove two propositions showing that the green particle is unlikely to visit space-time blocks that are not well visited by red particles. In Section 4 we use these propositions to prove Theorem 1.1. In Appendix A we check the uniformity in  $\mu$  of the estimates in [10], which is needed in order to be able to take the limit  $\mu \rightarrow \infty$ .

## 2. PREPARATIONS

The proof of Theorem 1.1 will be achieved by showing that the green particle spends most of its time in space-time blocks *all* of whose points have been visited by a red particle before they are visited by the green particle. This will be done separately for “bad blocks” and “good blocks” (to be defined later) living on successive space-time scales. For the bad blocks, most of the work can be lifted from [10]. For the good blocks, a percolation-type argument will be used. In the present section we recall several definitions from [10], organized into 4 parts and leading up to a key proposition. To simplify notations, we write down the proof for  $d = 1$  and for nearest-neighbor transition kernels  $p'$  and  $p''$  only. The extension to  $d \geq 2$  and to finite-range transition kernels will be straightforward.

**1.** For  $t \geq 0$ ,  $\ell \in \mathbb{N}_0$ ,  $0 \leq s_1 < \dots < s_\ell \leq t$  and  $x_1, \dots, x_\ell \in \mathbb{Z}$ , we write  $\widehat{\pi} = \widehat{\pi}(\{s_k, x_k\}_{0 \leq k \leq \ell})$  for the space-time path that, for  $1 \leq k \leq \ell$ , jumps to  $x_k$  at time  $s_k$  and stays at  $x_k$  during the time interval  $[s_k, s_{k+1})$ , where we take  $s_0 = 0$ ,  $x_0 = 0$  and  $s_{\ell+1} = t$ , i.e., the path takes the value  $x_\ell$  on  $[s_\ell, t]$ . We only consider paths that are contained in the space interval  $\mathcal{C}(t \log t) = [-t \log t, t \log t]$ , and so the class of paths of interest is

$$(2.1) \quad \Xi(\ell, t) = \left\{ \widehat{\pi} = \widehat{\pi}(\{s_k, x_k\}_{0 \leq k \leq \ell}) : 0 = s_0 < s_1 < \dots < s_\ell \leq t, \right. \\ \left. x_k \in \mathcal{C}(t \log t), 1 \leq k \leq \ell \right\}.$$

**2.** The renormalization analysis developed in [10, Section 1 and 4] depends on the choice of a large integer  $C_0$  and a strictly increasing sequence of positive numbers  $(\gamma_r)_{r \in \mathbb{N}_0}$  bounded from above by  $\frac{1}{2}$  (for precise definitions, see (A.1–A.4) in Appendix A). These are used to define a sequence of *space-time rectangles* as follows. For  $r \in \mathbb{N}_0$ , abbreviate

$$(2.2) \quad \Delta_r = C_0^{6r},$$

and, for  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$ , define (see Fig. 1)

$$(2.3) \quad \mathcal{B}_r(i, j) = [i\Delta_r, (i+1)\Delta_r) \times [j\Delta_r, (j+1)\Delta_r),$$

$$(2.4) \quad \widetilde{\mathcal{B}}_r(i, j) = V_r(i) \times [(j-1)\Delta_r, (j+1)\Delta_r),$$

$$(2.5) \quad \mathcal{V}_r(i, j) = V_r(i) \times \{(j-1)\Delta_r\},$$

with

$$(2.6) \quad V_r(i) = [(i-3)\Delta_r, (i+4)\Delta_r).$$

The  $\mathcal{B}_r(i, j)$ 's are called  $r$ -blocks;  $\mathcal{V}_r(i, j)$  plays the role of the pedestal of  $\mathcal{B}_r(i, j)$ .

**3.** For  $r \in \mathbb{N}_0$  and  $x \in \mathbb{Z}$ , define the space interval

$$(2.7) \quad \mathcal{Q}_r(x) = [x, x + C_0^r),$$

and, for  $t \geq 0$ , let

$$(2.8) \quad U_r(x, t) = \sum_{y \in \mathcal{Q}_r(x)} N(y, t),$$

$$(2.9) \quad E^\mu \{U_r(x, t)\} = \mu |\mathcal{Q}_r(x)| = \mu C_0^r.$$

We say that  $\mathcal{B}_r(i, j)$  is *bad* if  $U_r(x, t) < \gamma_r \mu C_0^r$  for some  $(x, t)$  for which  $\mathcal{Q}_r(x) \times \{t\}$  is contained in  $\widetilde{\mathcal{B}}_r(i, j)$ , i.e., there are significantly fewer red particles than expected

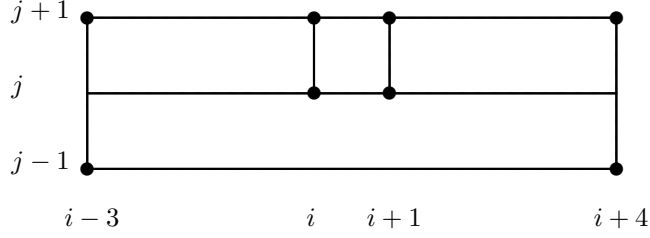


FIGURE 1. Picture of  $\mathcal{B}_r(i, j)$  (small square),  $\tilde{\mathcal{B}}_r(i, j)$  (large rectangle) and  $\mathcal{V}_r(i, j)$  (base of large rectangle), in units of  $\Delta_r$ .

in a space interval of size  $C_0^r = \Delta_r^{1/6} \ll \Delta_r$  somewhere inside the space-time block  $\tilde{\mathcal{B}}_r(i, j)$ . We say that  $\mathcal{B}_r(i, j)$  is *good* if it is not bad.

4. For  $r, \ell \in \mathbb{N}_0$ , define

$$(2.10) \quad \phi_r(\hat{\pi}) = \text{number of bad } r\text{-blocks that intersect the space-time path } \hat{\pi},$$

$$(2.11) \quad \Phi_r(\ell) = \sup_{\hat{\pi} \in \Xi(\ell, t)} \phi_r(\hat{\pi}).$$

The principal result from [10] needed in Section 3 is the following.

**Proposition 2.1.** ([10, Proposition 8, p. 2441]) *For all  $K, \varepsilon_0 > 0$  there exists an  $r_0 = r_0(K, \varepsilon_0)$  such that for all  $r \geq r_0$  there exists a  $\mu_0 = \mu_0(K, \varepsilon_0, r)$  such that for all  $r \geq r_0$  and  $\mu \geq \mu_0(K, \varepsilon_0, r)$  there exists a  $t_0 = t_0(K, \varepsilon_0, r, \mu)$  such that*

$$(2.12) \quad P^\mu \{ \Phi_r(\ell) \geq \varepsilon_0 C_0^{-6r} (t + \ell) \} \leq 2t^{-K}, \quad r \geq r_0, \mu \geq \mu_0, t \geq t_0, \ell \in \mathbb{N}_0.$$

In Appendix A we check the uniformity in  $\mu$  of the various estimates that went into the proof of Proposition 2.1.

### 3. TWO PROPOSITIONS

The proof of Theorem 1.1 in Section 4 will be built on two propositions, which are stated and proved in Sections 3.1 and 3.4, respectively. The first proposition controls the number of bad  $r$ -blocks  $\mathcal{B}_r(i, j)$  that intersect the path of the green particle up to time  $t$ , and its proof makes use of Proposition 2.1. The second proposition controls the number of good  $r$ -blocks  $\mathcal{B}_r(i, j)$  that intersect the path of the green particle up to time  $t$  and contain some point  $(x, t)$  that has no red particle coming from  $\mathcal{V}_r(i, j)$ . The proof of the second proposition requires two auxiliary lemmas, which are stated and proved in Sections 3.2 and 3.3, respectively.

**3.1. First proposition.** For  $t \geq 0$ , let  $\mathcal{E}_1(t)$  denote the event that the number of jumps by the green particle up to time  $t$  exceeds  $2t$ . Then there exist  $C_1, C_2 > 0$  such that

$$(3.1) \quad P\{\mathcal{E}_1(t)\} \leq C_1 e^{-C_2 t}.$$

Indeed, the green particle has constant jump rate 1. Therefore the number of jumps up to time  $t$  is a Poisson random variable with mean  $t$ , and the inequality is a standard large deviation bound for the Poisson distribution.

Fix  $K, \varepsilon_0 > 0$  and  $r_0 = r_0(K, \varepsilon_0)$  as in Proposition 2.1. For  $t \geq 0$ , let

$$(3.2) \quad \mathcal{H}(t) = \{\mathcal{G}(s) : 0 \leq s \leq t\}$$

and

$$(3.3) \quad \Gamma_r(t) = \{(i, j): \mathcal{B}_r(i, j) \cap \mathcal{H}(t) \neq \emptyset\}.$$

The union of the  $r$ -blocks  $\mathcal{B}_r(i, j)$  with  $(i, j) \in \Gamma_r(t)$  is a fattened-up version of the path of the green particle. We want to prove that, at large times  $t$  and high densities  $\mu$ , the green particle sees many red particles close by. In fact, we will prove a somewhat stronger statement, namely, that with a large probability in an  $r$ -block  $\mathcal{B}_r(i, j)$  that is visited by the green particle all the space-time points are visited by a red particle coming from  $\mathcal{V}_r(i, j)$ .

For  $t \geq 0$  and  $r \in \mathbb{N}_0$ , let

$$(3.4) \quad \tilde{\Gamma}_r(t) = \{(i, j): \mathcal{B}_r(i, j) \cap \mathcal{H}(t) \neq \emptyset, \mathcal{B}_r(i, j) \text{ is bad}\},$$

**Proposition 3.1.** *For  $K, \varepsilon_0 > 0$ ,  $r \geq r_0, \mu \geq \mu_0$  and  $t$  sufficiently large,*

$$(3.5) \quad P^\mu \{|\tilde{\Gamma}_r(t)| \geq 3\varepsilon_0 C_0^{-6r} t\} = P\{\phi_r(\mathcal{H}(t)) \geq 3\varepsilon_0 C_0^{-6r} t\} \leq 3t^{-K}.$$

*Proof.* The equality follows from (2.10). To obtain the inequality, we apply Proposition 2.1. This tells us that for  $r \geq r_0, \mu \geq \mu_0, t \geq t_0$  and  $\ell \in \mathbb{N}_0$ , outside an event  $\mathcal{E}_2(t)$  of probability at most  $2t^{-K}$ , we have

$$(3.6) \quad \Phi_r(\ell) = \sup_{\hat{\pi} \in \Xi(\ell, t)} \phi_r(\hat{\pi}) \leq \varepsilon_0 C_0^{-6r} (t + \ell).$$

Furthermore, since each jump has size 1, if  $\mathcal{H}(t)$  makes exactly  $\ell$  jumps with  $0 \leq \ell \leq \mathcal{C}(t \log t)$ , then  $\mathcal{H}(t) \in \Xi(\ell, t)$ . Hence, for  $\log t \geq 2$  and outside the event  $\mathcal{E}_1(t) \cup \mathcal{E}_2(t)$ , we have

$$(3.7) \quad \phi_r(\mathcal{H}(t)) \leq \Phi_r(\ell) \leq \varepsilon_0 C_0^{-6r} (t + \ell) \leq 3\varepsilon_0 C_0^{-6r} t.$$

Combine (3.1) and (3.7), and choose  $t$  so large that  $C_1 e^{-C_2 t} \leq t^{-K}$ , to get (3.5).  $\square$

**3.2. First auxiliary lemma.** The time coordinate of the green particle is just time itself. Hence, if  $T$  is some space-time set with projection  $\tilde{T}$  onto the time-axis, then the total time spent by the green particle inside  $T$  is at most the Lebesgue measure of  $\tilde{T}$ . In particular, if  $\mathcal{H}(t)$  intersects no more than  $3\varepsilon_0 C_0^{-6r} t$  bad  $r$ -blocks, then the total time that is spent by the green particle in bad  $r$ -blocks up to time  $t$  is at most

$$(3.8) \quad 3\varepsilon_0 C_0^{-6r} t \times C_0^{6r} = 3\varepsilon_0 t.$$

We want to control the set of space-time points  $(x, t)$  in a good  $r$ -block  $\mathcal{B}_r(i, j)$  that intersects  $\mathcal{H}(t)$  such that there is no red particle at  $(x, t)$  coming from  $\mathcal{V}_r(i, j)$ . We want to show that also this set is small with a large probability.

Let

$$(3.9) \quad \mathcal{F}(t) = \sigma\{N(s): 0 \leq s \leq t\},$$

be the sigma-field generated by the paths of all the red particles up to time  $t$ , and define

$$(3.10) \quad \mathcal{E}_r(i, j) = \{\exists (x, t) \in \mathcal{B}_r(i, j): \\ \text{no red particle coming from } \mathcal{V}_r(i, j) \text{ hits } (x, t)\}.$$

**Lemma 3.2.** For all  $\varepsilon_1 > 0$  and  $r \in \mathbb{N}_0$  there exists a  $\mu_1 = \mu_1(\varepsilon_1, r)$  such that for all  $\mu \geq \mu_1$ ,  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$ , and uniformly on the event

$$(3.11) \quad \mathcal{N}_r(i, j) = \left\{ \sum_{x \in V_r(i)} N(x, (j-1)\Delta_r) \geq \gamma_0 \mu \Delta_r \right\},$$

the following holds:

$$(3.12) \quad P^\mu \{ \mathcal{E}_r(i, j) \mid \mathcal{F}((j-1)\Delta_r) \} \leq \varepsilon_1.$$

*Proof.* Note that  $\mathcal{E}_r(i, j)$  depends only on the paths during the time interval  $[(j-1)\Delta_r, (j+1)\Delta_r]$  of the red particles located in the space interval  $V_r(i) = [(i-3)\Delta_r, (i+4)\Delta_r]$  at time  $(j-1)\Delta_r$ . Since the red particles are interchangeable, the conditional probability in (3.12) in fact only depends on  $N(x, (j-1)\Delta_r)$ ,  $x \in V_r(i)$ .

It is easy to see that if there are at least  $8\Delta_r$  particles in  $V_r(i)$  at time  $(j-1)\Delta_r$ , then the conditional probability  $f(r)$  that these particles after time  $(j-1)\Delta_r$  move in such a way that there is at least one of them at each point  $(x, t) \in \mathcal{B}_r(i, k)$  satisfies  $f(r) > 0$  (see Fig. 1). In particular,  $f(r)$  can be taken to be independent of the location of the red particles at time  $(j-1)\Delta_r$ . In other words, on the event

$$(3.13) \quad \sum_{x \in V_r(i)} N(x, (j-1)\Delta_r) \geq 8\Delta_r,$$

we have

$$(3.14) \quad P^\mu \{ [\mathcal{E}_r(i, j)]^c \mid \mathcal{F}((j-1)\Delta_r) \} \geq f(r).$$

Assume now that (3.11) holds. Then there are at least  $\gamma_0 \mu \Delta_r$  red particles in  $V_r(i)$  at time  $(j-1)\Delta_r$ . Order these particles in an arbitrary way and partition them into  $q = \lfloor \gamma_0 \mu / 8 \rfloor$  subsets of at least  $8\Delta_r$  particles each, ignoring what is left over. For each of these subsets the bound in (3.14) is valid. The event in the left-hand side of (3.12) occurs if and only if the event in the left-hand side of (3.14) fails for each of the  $q$  subsets. Since the paths of disjoint sets of red particles are independent, the left-hand side of (3.12) is therefore at most  $[1 - f(r)]^q$ . Now take  $\mu$  so large that  $[1 - f(r)]^{(\gamma_0 \mu / 8) - 1} \leq \varepsilon_1$ , i.e.,

$$(3.15) \quad \mu \geq \mu_1 = \frac{8}{\gamma_0} \left[ 1 + \frac{\log \varepsilon_1}{\log [1 - f(r)]} \right].$$

Then (3.12) follows.  $\square$

Note that because  $r \mapsto \gamma_r$  is non-decreasing (see Appendix A) the result in Lemma 3.2 also holds when  $\gamma_0$  is replaced by  $\gamma_r$  in (3.11). In what follows we will use the version with  $\gamma_0$ .

### 3.3. Second auxiliary lemma. Abbreviate

$$(3.16) \quad M = \gamma_0 \mu \Delta_r$$

and, for  $s \in \mathbb{N}$ ,  $i_1, \dots, i_s \in \mathbb{Z}$  and  $j_1, \dots, j_s \in \mathbb{N}$ , introduce the events

$$(3.17) \quad \mathcal{D}_r(i_1, j_1, \dots, i_s, j_s) = \bigcap_{u=1}^s \mathcal{N}_r(i_u, j_u).$$



**Lemma 3.3.** For all  $\varepsilon_1 > 0$  and  $r \in \mathbb{N}_0$  there exists a  $\mu_1 = \mu_1(\varepsilon_1, r)$  such that for all  $\mu \geq \mu_1$  the following are true.

(a) Let  $i_1, \dots, i_s$  be such that their mutual differences are all  $\geq 8$ . Then, for all  $j \in \mathbb{N}$ ,

$$(3.18) \quad \begin{aligned} & P^\mu \left\{ \mathcal{D}_r(i_1, j, \dots, i_s, j) \cap \left[ \bigcap_{u=1}^s \mathcal{E}_r(i_u, j) \right] \right\} \\ & \leq P^\mu \{ \mathcal{D}_r(i_1, j, \dots, i_s, j) \} \varepsilon_1^s \leq \varepsilon_1^s, \quad \mu \geq \mu_1(\varepsilon_1, r). \end{aligned}$$

(b) Let  $(i_1, j_1), \dots, (i_s, j_s)$  be distinct such that the sum of their componentwise mutual differences are all  $\geq 10$  and  $j_1, \dots, j_s$  all have the same parity. Then

$$(3.19) \quad P^\mu \left\{ \mathcal{D}_r(i_1, j_1, \dots, i_s, j_s) \cap \left[ \bigcap_{u=1}^s \mathcal{E}_r(i_u, j_u) \right] \right\} \leq \varepsilon_1^s, \quad \mu \geq \mu_1(\varepsilon_1, r).$$

*Proof.* (a) Write the left-hand side of (3.18) as a conditional expectation given  $\mathcal{F}((j-1)\Delta_r)$ . Since the  $j_u$ 's coincide,  $\mathcal{D}(i_1, j, \dots, i_s, j)$  only depends on the sites in  $\mathbb{Z} \times [0, (j-1)\Delta_r]$ . On the other hand,  $\mathcal{E}_r(i_u, j)$  only depends on the red particles at the points in  $\mathbb{Z} \times [(j-1)\Delta_r, (j+1)\Delta_r]$ . As in the first part of the proof of Lemma 3.2, the conditional distribution of  $\bigcap_{u=1}^s \mathcal{E}_r(i_u, j)$  given  $\mathcal{F}((j-1)\Delta_r)$  only depends on  $N(x, (j-1)\Delta_r)$ ,  $x \in \mathbb{Z}$ . In fact, the conditional distribution of  $\mathcal{E}_r(i_u, j)$  given  $\mathcal{F}((j-1)\Delta_r)$  only depends on  $N(x, (j-1)\Delta_r)$ ,  $x \in V_r(i_u)$ . The collections of red particles counted by  $N(x, (k-1)\Delta_r)$ ,  $x \in V_r(i_u)$ , for different  $i_u$  are disjoint, because the intervals  $V_r(i_u)$  for different  $i_u$  are disjoint. It follows that  $\mathcal{E}_r(i_u, j)$ ,  $1 \leq u \leq s$ , are conditionally independent given  $\mathcal{F}((j-1)\Delta_r)$ . Therefore the left-hand side of (3.18) is bounded from above by

$$(3.20) \quad \begin{aligned} & E^\mu \left\{ P^\mu \{ \mathcal{D}_r(i_1, j, \dots, i_s, j) \mid \mathcal{F}((j-1)\Delta_r) \} \right. \\ & \quad \left. \times \prod_{u=1}^s P^\mu \{ \mathcal{E}_r(i_u, j) \mid \mathcal{F}((j-1)\Delta_r) \} \right\}. \end{aligned}$$

However, on the event  $\mathcal{D}(i_1, j, \dots, i_s, j)$ , (3.11) holds for  $i \in \{i_1, \dots, i_s\}$ , and so we see from Lemma 3.2 that

$$(3.21) \quad P^\mu \{ \mathcal{E}_r(i_u, j) \mid \mathcal{F}((j-1)\Delta_r) \} \leq \varepsilon_1,$$

provided we take  $\mu \geq \mu_1 = \mu_1(\varepsilon_1, r)$ . This gives the first inequality in (3.18). The second inequality is trivial.

(b) Put

$$(3.22) \quad j = \max\{j_1, \dots, j_s\},$$

and suppose, without loss of generality, that there exists a  $1 \leq \bar{u} \leq s$  such that

$$(3.23) \quad j_u < j \text{ for } u \leq \bar{u} \text{ and } j_u = j \text{ for } j > \bar{u}.$$

Note that

$$(3.24) \quad \begin{aligned} & \mathcal{D}_r(i_1, j_1, \dots, i_s, j_s) \\ & = \mathcal{D}_r(i_1, j_1, \dots, i_{\bar{u}}, j_{\bar{u}}) \cap \mathcal{D}_r(i_{\bar{u}+1}, j_{\bar{u}+1}, \dots, i_s, j_s). \end{aligned}$$

Note further that  $j_1, \dots, j_{\bar{u}} \leq k-2$ , because all  $j_u$  have the same parity. This implies that  $\mathcal{D}_r(i_1, j_1, \dots, i_{\bar{u}}, j_{\bar{u}})$  is  $\mathcal{F}((j-1)\Delta_r)$ -measurable. Consequently, as in

the proof of part (a), on the event  $\mathcal{D}_r(i_1, j_1, \dots, i_{\bar{u}}, j_{\bar{u}})$  we have

$$\begin{aligned}
(3.25) \quad & P^\mu \left\{ \bigcap_{u=\bar{u}+1}^s \mathcal{E}_r(i_u, j_u) \mid \mathcal{F}((j-1)\Delta_r) \right\} \\
&= P^\mu \left\{ \bigcap_{u=\bar{u}+1}^s \mathcal{E}_r(i_u, j) \mid \mathcal{F}((j-1)\Delta_r) \right\} \\
&= \prod_{u=\bar{u}+1}^s P^\mu \left\{ \mathcal{E}_r(i_u, j) \mid \mathcal{F}((j-1)\Delta_r) \right\} \leq \varepsilon_1^{s-\bar{u}}.
\end{aligned}$$

By taking the conditional expectation with respect to  $\mathcal{F}((j-1)\Delta_r)$  and using part (a), we obtain

$$\begin{aligned}
(3.26) \quad & P^\mu \left\{ \mathcal{D}_r(i_1, j_1, \dots, i_s, j_s) \cap \left[ \bigcap_{u=1}^s \mathcal{E}_r(i_u, j_u) \right] \right\} \\
&\leq E^\mu \left\{ \mathcal{D}_r(i_1, j_1, \dots, i_u, j_u) \cap \left[ \bigcap_{u=1}^{\bar{u}} \mathcal{E}_r(i_u, j_u) \right] \mid \mathcal{F}((j-1)\Delta_r) \right\} \varepsilon_1^{s-\bar{u}} \\
&= P^\mu \left\{ \mathcal{D}_r(i_1, j_1, \dots, i_{\bar{u}}, j_{\bar{u}}) \cap \left[ \bigcap_{u=1}^{\bar{u}} \mathcal{E}_r(i_u, j_u) \right] \right\} \varepsilon_1^{s-\bar{u}}.
\end{aligned}$$

The proof can now be completed via a recursive argument. Indeed, the left-hand side of (3.26) deals with the probability of events indexed by  $s$  pairs  $(i_u, j_u)$  with  $j_u \leq j$  and estimates this probability in terms of probabilities of events indexed by pairs  $(i_u, j_u)$  with  $j_u \leq j-1$  (and powers of  $\varepsilon_1$ ). We can therefore iterate the estimate until it only contains powers of  $\varepsilon_1$ .  $\square$

**3.4. Second proposition.** Associated with  $\mathcal{H}(t)$  is the collection of pairs  $\Gamma_r(t)$  introduced in (3.3). Because the jumps of the green particle have size 1,  $\Gamma_r(t)$  when viewed as a subset of  $\mathbb{Z}^2$  is connected, i.e., for any two pairs  $(i', j'), (i'', j'') \in \Gamma_r(t)$  there is a path in  $\Gamma_r(t)$  that runs from  $(i', j')$  to  $(i'', j'')$ . In other words,  $\Gamma_r(t)$  is a so-called *lattice animal* containing the origin. We claim that with probability at least  $1 - C_1 e^{-C_2 t}$  this lattice animal contains at most  $\ell + t \leq 3t$  sites. Indeed, this is so because  $\mathcal{H}$  can go from an  $r$ -block to an adjacent  $r$ -block in only two ways:

- (i) It crosses one of the time lines  $j\Delta_r$ ,  $j \in \mathbb{N}$ , without making a jump. Since the time between two successive such crossings is  $\Delta_r$ , at most  $t/\Delta_r$  such crossings can occur up to time  $t$ .
- (ii) It makes a jump. By the definition of  $\Xi(\ell, t)$ , for  $\Gamma(t) \in \Xi(\ell, t)$  there are exactly  $\ell$  such jumps up to time  $t$ .

Now, it is well known that there exist constants  $C_3, C_4$  such that the number of lattice animals of size  $3t$  containing the origin is bounded from above by  $C_3 e^{C_4 t}$ . Thus, if we define

$$(3.27) \quad \mathcal{W}_r(t) = \text{collection of possible sets } \Gamma_r(t),$$

then we have proved that, with probability at least  $1 - C_1 e^{-C_2 t}$ ,

$$(3.28) \quad |\mathcal{W}_r(t)| \leq C_3 e^{C_4 t}.$$

The  $r$ -block corresponding to a point  $(i, j) \in \Gamma_r(t)$  can be either bad or good. We will call a pair  $(i, j) \in \Gamma_r(t)$  bad or good according as  $\mathcal{B}_r(i, j)$  is bad or good. It is immediate from (2.10–2.12) that, outside the event  $\mathcal{E}_1(t) \cup \mathcal{E}_2(t)$ , the number of bad  $r$ -blocks in  $\Gamma_r(t)$  is at most  $3\varepsilon_0 C_0^{-6r} t$ . Together with (3.8), this proves that  $\mathcal{H}(t)$  spends only a small fraction of its time in bad blocks. We therefore only need to deal with the subset of good pairs in  $\Gamma_r(t)$ . Of particular interest will be the following subset of  $\Gamma_r(t)$ :

$$(3.29) \quad \Lambda_r(t) = \left\{ (i, j) : \mathcal{B}_r(i, j) \cap \mathcal{H}(t) \neq \emptyset, \mathcal{B}_r(i, j) \text{ is good, } \mathcal{E}_r^*(i, j) \text{ occurs} \right\},$$

where  $\mathcal{E}_r^*(i, j)$  is the event that  $\mathcal{B}_r(i, j)$  contains a point that is not visited by any of the red particles coming from  $\mathcal{V}_r(i, j)$ .

**Proposition 3.4.** *There exist a  $C_1, \dots, C_6 > 0$  such that for all  $0 < \varepsilon_1 < 1$ ,  $\mu \geq \mu_1(\varepsilon_1, r)$  and  $t$  sufficiently large,*

$$(3.30) \quad P^\mu\{|\Lambda_r(t)| \geq \varepsilon_1 t\} \leq C_1 e^{-C_2 t} + C_3 e^{C_4 t} 2^{3t} \varepsilon_1^{\lfloor C_5 + C_6 \varepsilon_1 t \rfloor}.$$

*Proof.* The idea of the proof is to partition the event  $\{|\Lambda_r(t)| \geq \varepsilon_1 t\}$  into a union of subevents of the form analyzed in Lemma 3.3, to estimate the probability of each of these subevents by means of Lemma 3.3, and afterwards sum over all the ways to do the partition.

Consider a sample point for which  $|\Lambda_r(t)| \geq \varepsilon_1 t$ , but for which  $\mathcal{E}_1(t)$  does not occur (i.e., the green particle makes  $\leq 2t$  jumps up to time  $t$ ). Then, since  $\Lambda_r(t) \subset \Gamma_r(t) \subset \mathbb{Z}^2$ , there exist at most  $\lfloor C_5 + C_6 \varepsilon_1 t \rfloor$  points  $(i_u, j_u) \in \Lambda_r(t)$  with the sum of their componentwise differences all  $\geq 10$ , with  $C_5, C_6 > 0$  independent of  $\Lambda_r(t)$  and  $\mu$ . Since  $\mathcal{B}_r(i, j)$  is good when  $(i, j) \in \Lambda_r(t)$ , we have

$$(3.31) \quad \sum_{x \in V_r(i)} N(x, (j-1)\Delta_r) \geq \gamma_r \mu \Delta_r \geq \gamma_0 \mu \Delta_r = M,$$

i.e., the event  $\mathcal{N}_r(i, j)$  defined in (3.11) occurs. Let

$$(3.32) \quad \widehat{\Lambda}_r(t) = \{(i, j) : \mathcal{B}_r(i, j) \cap \mathcal{H}(t) \neq \emptyset, \mathcal{N}_r(i, j) \text{ and } \mathcal{E}_r^*(i, j) \text{ occur}\}.$$

Then  $\widehat{\Lambda}_r(t) \supset \Lambda_r(t)$ , and so the points  $(i_u, j_u)$ ,  $1 \leq u \leq C_5 + C_6 \varepsilon_1 t$ , all lie in  $\widehat{\Lambda}_r(t)$ . This means that  $\mathcal{D}_r(i_1, j_1, \dots, i_s, j_s)$  occurs with  $s = \lfloor C_5 + C_6 \varepsilon_1 t \rfloor$  and  $M$  as in (3.16) (recall (3.17)). In addition,

$$(3.33) \quad \cap_{u=1}^s \mathcal{E}_r(i_u, j_u)$$

occurs (recall (3.12)).

The above observations show that  $|\Lambda_r(t)| > \varepsilon_1 t$  can occur for  $\mu \geq \mu_1(\varepsilon_1, r)$  and  $t$  large enough only if, for some possible choice of the  $(i_u, j_u)$ 's,

$$(3.34) \quad \mathcal{D}(i_1, j_1, \dots, i_s, j_s) \cap \left[ \cap_{u=1}^s \mathcal{E}_r(i_u, j_u) \right]$$

occurs. Lemma 3.3 shows that, for any permissible choice of the  $(i_u, j_u)$ 's, the probability of (3.34) is at most  $\varepsilon_1^s$ . Consequently,

$$(3.35) \quad \begin{aligned} & P^\mu\{|\Lambda_r(t)| \geq \varepsilon_1 t\} \\ & \leq \sum_{\Gamma_r} \sum_{\Theta \subset \Gamma_r} P^\mu\{\mathcal{D}_r(i_1, j_1, \dots, i_s, j_s) \cap \left[ \cap_{u=1}^s \mathcal{E}_r(i_u, j_u) \right]\} \\ & \leq \sum_{\Gamma_r} \sum_{\Theta \subset \Gamma_r} \varepsilon_1^s, \end{aligned}$$

provided  $\mu \geq \mu_1(\varepsilon_1, r)$  and  $t$  is sufficiently large. Here, the sum over  $\Gamma_r$  runs over all possible realizations of the random set  $\Gamma_r(t)$ , and  $\Theta$  runs over all choices of the  $(i_u, j_u)$ 's whose sum of componentwise differences are all  $\geq 10$ , and are such that the  $j_u$ 's all have the same parity.

Now assume that our sample point lies outside  $\mathcal{E}_1(t)$ , which happens with a probability at least  $1 - C_1 e^{-C_2 t}$ . Then

$$(3.36) \quad \sum_{\Theta \subset \Gamma_r} 1 \leq 2^{3t},$$

because  $|\Gamma_r| \leq 3t$ , as we saw before. Moreover, by (3.28), we have

$$(3.37) \quad \sum_{\Gamma_r} 1 \leq C_3 e^{C_4 t},$$

since  $\Gamma_r$  is a lattice animal that contains the origin and has size at most  $3t$ . Combining these estimates, we find that for  $\mu \geq \mu_1(\varepsilon_1, r)$  and  $t$  sufficiently large,

$$(3.38) \quad P^\mu\{|\Lambda_r(t)| \geq \varepsilon_1 t\} \leq C_1 e^{-C_2 t} + C_3 e^{C_4 t} 2^{3t} \varepsilon_1^{\lfloor C_5 + C_6 \varepsilon_1 t \rfloor}.$$

□

#### 4. PROOF OF THEOREM 1.1

*Proof.* With Propositions 3.1 and 3.4 in hand, the proof of Theorem 1.1 is routine. We distinguish four possible types of  $r$ -blocks,

$$(4.1) \quad (\text{bad, occupied}), \quad (\text{bad, vacant}), \quad (\text{good, occupied}), \quad (\text{good, vacant}),$$

where an  $r$ -block  $\mathcal{B}_r(i, j)$  is called *occupied* when  $\mathcal{E}_r^*(i, j)^c$  occurs, i.e., every point in  $\mathcal{B}_r(i, j)$  is visited by a red particle coming from  $\mathcal{V}_r(i, j)$ , and is called *vacant* otherwise. The number of  $r$ -blocks of type (bad,occupied) that intersect  $\mathcal{H}(t)$  will be denoted by  $N_r(t; (\text{bad, occupied}))$ , and similarly for the other types.

We have shown in Proposition 3.1 that

$$(4.2) \quad \begin{aligned} N_r(t; (\text{bad})) &= N_r(t; (\text{bad, occupied})) + N_r(t; (\text{bad, vacant})) \\ &\leq |\tilde{\Gamma}_r(t)| \leq 3\varepsilon_0 C_0^{-6} t \end{aligned}$$

outside an event of probability at most  $3t^{-K}$ , provided  $K, \varepsilon_0 > 0$ ,  $\mu \geq \mu_0(K, \varepsilon_0, r)$  and  $t$  is sufficiently large. We have further shown in Proposition 3.4 that

$$(4.3) \quad N_r(t; (\text{good, vacant})) = |\Lambda_r(t)| \leq \varepsilon_1 t$$

outside an event of probability at most  $C_1 e^{-C_2 t} + C_3 e^{C_4 t} 2^{3t} \varepsilon_1^{\lfloor C_5 + C_6 \varepsilon_1 t \rfloor}$ , provided  $0 < \varepsilon_1 < 1$ ,  $\mu \geq \mu_1(\varepsilon_1, r)$  and  $t$  is sufficiently large. Since we can choose  $\varepsilon_0, \varepsilon_1$  arbitrarily small, it follows from (4.2–4.3) that there exists a function  $\mu \mapsto \hat{t}(\mu)$  from  $(0, \infty)$  to itself such that

$$(4.4) \quad \begin{aligned} \frac{1}{t} [N(r, t; (\text{bad, vacant})) + N(r, t; (\text{good, vacant}))] &\rightarrow 0 \\ &\text{in } P^\mu\text{-probability as } \mu, t \rightarrow \infty \text{ such that } t \geq \hat{t}(\mu). \end{aligned}$$

According to the projection argument given in the lines just before (3.8), this in turn implies that

$$(4.5) \quad \begin{aligned} \frac{1}{t} [\text{total time that } \mathcal{H}(t) \text{ is in an } r\text{-block} \\ \text{that is (bad,occupied) or (good,occupied)}] &\rightarrow 1 \\ &\text{in } P^\mu\text{-probability as } \mu, t \rightarrow \infty \text{ such that } t \geq \hat{t}(\mu). \end{aligned}$$

Finally, let  $L$  be the infinitesimal generator of the random walk performed by the green particle when the space-time trajectories of the red particles, given by  $N$  defined in (1.1), are fixed. Then

$$(4.6) \quad (LI)(x, t) = \begin{cases} v' & \text{if } N(x, t) \geq 1, \\ v'' & \text{if } N(x, t) = 0, \end{cases} \quad I(x, t) = (x, t).$$

Recall  $\mathcal{G}$  defined in (1.3). By a standard martingale property, we have (see e.g. [10](Lemma 10))

$$(4.7) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \left[ \mathcal{G}(t) - \int_0^t (LI)(\mathcal{G}(s), s) ds \right] = 0, \quad P^\mu\text{-a.s.}$$

Combining (4.5) and (4.6), we see that  $(LI)(\mathcal{G}(s), s) = v'$  on a set of  $s$ -values in  $[0, t]$  that converges in  $P^\mu$ -probability to all of  $[0, t]$ . Therefore Theorem 1.1 follows from (4.7).  $\square$

#### APPENDIX A. UNIFORMITY IN $\mu$

Our main focus in this Appendix is on [10, Section 4], and we will adopt the notation used there. Consequently, the arguments given below cannot be read independently. Moreover, in [10] at some places a choice of parameters depends on  $\mu$ . However, as we will check below, this dependence does not require  $\mu \rightarrow \infty$ .

Note that in Proposition 2.1 we choose the parameters in the order  $K, \varepsilon_0, r, \mu, t$ . Since in Theorem 1.1 we let  $t \rightarrow \infty$  first, it suffices to check that inequalities hold for large  $t$  when  $\mu$  is fixed.

1. As shown in [10, Sections 1 and 4],  $C_0$  and  $(\gamma_r)_{r \in \mathbb{N}_0}$  mentioned in Section 2, Part 2, are chosen such that

$$(A.1) \quad 0 < \gamma_0 \prod_{j=1}^{\infty} [1 - 2^{-j/4}]^{-1} \leq \frac{1}{2},$$

$$(A.2) \quad \gamma_1 = \gamma_0, \quad \gamma_{r+1} = \gamma_0 \prod_{j=1}^r [1 - C_0^{-j/4}]^{-1}, \quad r \in \mathbb{N},$$

where  $C_0$  is taken so large that, for all  $r \in \mathbb{N}$ ,

$$(A.3) \quad C_0^{-r/2} - \left(1 - \frac{C_4 r \log C_0}{C_0^r}\right) (1 - e^{-C_0^{-r/2}}) [1 - C_0^{-r/4}]^{-1} \leq -\frac{1}{2} C_0^{-3r/4},$$

$$(A.4) \quad 9C_0^{12(r+1)} \exp[-\frac{1}{2}\gamma_0 \mu C_0^{r/4}] \leq 1,$$

where  $C_4$  is the constant in [10, Lemma 5] (and  $\mu$  takes over the role of  $\mu_A$  in [10]). It is not hard to check in the proof of [10, Lemma 5] that  $C_4$  in (A.3) can be chosen independently of  $\mu$ . As was already checked in [10],  $C_4$  is also independent of  $C_0$ , so that we can choose  $(\gamma_r)_{r \in \mathbb{N}_0}$  and  $C_0$  independently of  $C_4$  as well. In other words, once a value for  $C_4$  has been determined on the basis of [10, Lemma 5], we may safely let  $\mu \rightarrow \infty$ .

2. The next place where the uniformity in  $\mu$  needs to be checked is [10, Eq. (4.16–4.17)]. For any choice of  $K > 0$  and  $K_4 > 0$ , we define  $R(t)$  by [10, Eq. (4.16)], the position at time  $t$  of the right-most red particle. The first three inequalities in [10, Eq. (4.17)] continue to be valid, uniformly in  $\mu$ . We can choose  $K_4$  to make also the fourth inequality in [10, Eq. (4.17)] valid, uniformly in  $\mu$ , by observing that  $U_r(x, t)$  in (2.8) has a Poisson distribution with mean  $\mu C_0^r$ , so that, for any  $\theta > 0$ ,

$$(A.5) \quad \begin{aligned} P^\mu \{U_r(x, t) \leq \frac{1}{2} \mu C_0^r\} &= P^\mu \left\{ e^{-\theta U_r(x, t)} \geq e^{-\frac{1}{2} \theta \mu C_0^r} \right\} \\ &\leq e^{\frac{1}{2} \theta \mu C_0^r} E^\mu \left\{ e^{-\theta U_r(x, t)} \right\} = e^{\frac{1}{2} \theta \mu C_0^r} e^{-\mu C_0^r (1 - e^{-\theta})}. \end{aligned}$$

Pick  $\theta > 0$  so small that  $(1 - e^{-\theta}) - \frac{1}{2}\theta \geq \frac{1}{4}\theta$ , to obtain

$$(A.6) \quad \sum_{r \geq R(t)} P\{U_r(x, t) \leq \frac{1}{2}\mu C_0^r\} \leq e^{-\frac{1}{4}\theta\mu C_0^{R(t)}} \sum_{r \geq R(t)} e^{-\frac{1}{4}\theta\mu C_0^{r-R(t)}} \\ \leq K_{20} e^{-\frac{1}{4}\theta\mu C_0^{R(t)}}$$

for some constant  $K_{20}$  that depends on  $C_0$  only, as long as  $\mu$  is bounded away from 0, say  $\mu \geq 2$ . It is immediate from this estimate that the sum of  $P\{U_r(x, t) \leq \frac{1}{2}\mu C_0^r\}$  over  $(x, s)$ , with  $s \in [-\Delta_r, t + \Delta_r]$  integer and  $x$  such that  $\mathcal{Q}_r(x)$  intersects  $\mathcal{C}(t \log t + 3\Delta_r)$ , when summed over  $r \geq R(t)$  is no more than  $t^{-K}$ , provided  $\mu \geq 2$  and  $K_4$  (or, equivalently,  $R(t)$ ) is sufficiently large, independently of  $\mu$ . Thus, we conclude that [10, Eq. (4.17)] is valid uniformly in  $\mu \geq 2$ .

**3.** [10, Lemmas 5–6] remain valid for  $\mu > 0$ , while also [10, Lemma 7] remains valid, even with  $C_5$  independent of  $\mu$ , as long as  $\mu$  is bounded away from 0, say  $\mu \geq 2$ . Indeed, the inequality in [10, Eq. (4.29)] is based on the estimate

$$(A.7) \quad E\{T\} = \lambda\nu^2\rho_{r+1} \leq 2^2\lambda \leq 6^2 \frac{t + \ell}{\nu\Delta_{r+1}}$$

and on Bernstein's inequality (see [10, Eq. (4.37) and subsequent lines] for the appropriate notation. In the case of a binomial random variable  $T$  corresponding to  $\lambda\nu^2$  trials with success probability  $\rho$ , Bernstein's inequality gives

$$(A.8) \quad P\{T \geq \alpha E\{T\}\} = P\{T - E\{T\} \geq (\alpha - 1)E\{T\}\} \leq \exp\left[-\frac{1}{4}(\alpha - 1)E\{T\}\right]$$

for  $\alpha > 1$  (see [6, Exercise 4.3.14]). Thus, [10, Lemma 7] holds uniformly in  $\mu \geq 2$ .

**4.** It remains to verify the uniformity of [10, Proposition 8], i.e., Proposition 2.1 above. To do so, we consider a sample point where [10, Eq. (4.39)] holds for all  $\mu \geq 2$ ,  $r \geq R(t)$  and  $\ell \in \mathbb{N}_0$ , and where [10, Eq. (4.40)] holds independently of  $\mu$ . Then there exists an  $r_0$  independent of  $\mu \geq 2$  such that [10, Eq. (4.41)] holds all  $r \in [r_0, R(t) - 1]$ ,  $\ell \in \mathbb{N}_0$ ,  $\mu \geq 2$  and  $t$  sufficiently large. Moreover, these estimates hold for all sample points outside an event of probability at most

$$(A.9) \quad t^{-K} + \sum_{r=1}^{R(t)-1} \sum_{\ell \in \mathbb{N}_0} \exp\left[-(t + \ell)C_5\kappa_0 \exp\left[\frac{1}{2}\gamma_r\mu C_0^{\frac{1}{4}}\right]\right] \leq 2t^{-K}.$$

This proves Proposition 2.1 with the required uniformity in  $\mu$ . The only requirement for  $r_0$  is that the last inequality in [10, Eq. (4.41)] holds. Thus, we only need

$$(A.10) \quad 6\kappa_0[C_0]^{12+6r} \exp\left[-\frac{\gamma_0\mu}{4}C_0^{\frac{1}{4}r}\right] \leq \varepsilon_0.$$

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