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BERMAN-KONSOWA PRINCIPLE FOR REVERSIBLE MARKOV JUMP PROCESSES

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Abstract. In this paper we prove a version of the Berman-Konsowa principle for reversible Markov jump processes on Polish spaces. The Berman-Konsowa principle provides a variational formula for the capacity of a pair of disjoint measurable sets. There are two versions, one involving a class of probability measures for random finite paths from one set to the other, the other involving a class of finite unit flows from one set to the other. The Berman-Konsowa principle complements the Dirichlet principle and the Thomson principle, and turns out to be especially useful for obtaining sharp estimates on crossover times in metastable interacting particle systems.

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1. Introduction

Section 1.1 provides the motivation, Section 1.2 formulates the setting, Section 1.3 states the main theorems, while Section 1.4 discusses these theorems and places them in their proper context.

1.1. **Motivation.** The motivation for the present paper comes from the theory of *metastability* for interacting particle systems, i.e., systems consisting of a large number of interacting random components evolving according to a Markovian random dynamics on a space of configurations. As time evolves, the system moves through different subregions of its configuration space, corresponding to different "thermodynamic phases". Typically, in a metastable setting, on short time scales the system reaches a quasi-equilibrium inside a single subregion, while on long time scales it makes rapid transitions between different subregions, with crossover times that are exponentially distributed on the scale of their mean. The task of mathematics is to analyze such systems in detail, and to explain the experimentally observed universality in their metastable behavior. This is a conceptual program of great challenge.

There are two main approaches to metastability: (1) the *pathwise approach*, initiated by Freidlin and Wentzell [14], in which a detailed description is given of the trajectories of the system, and the focus is on identifying the most likely trajectories and to estimate their probabilities; (2) the *potential-theoretic approach*, initiated by

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Bovier, Eckhoff, Gayrard and Klein [8], in which metastability is viewed as a sequence of visits of the trajectory to different metastable sets, and the focus is on a precise analysis of the respective hitting probabilities and hitting times of these sets with the help of potential theory. Phrased differently, the problem of understanding the metastable behavior of Markov processes is translated to the study of equilibrium potentials and capacities of *electric networks* (Doyle and Snell [12]).

More precisely, the configurations of the system are viewed as the vertices of the network and the transitions between pairs of configurations as the edges of the network. The transition probabilities are represented by the conductances associated with the edges. In this language, the hitting probability of a target set of configurations as a function of the starting configuration can be expressed in terms of the equilibrium potential on the network when the potential is put to one on the vertices of the target set and to zero on the starting vertex. The average hitting time of the target set can then be expressed in terms of the equilibrium potential and the capacity associated with the target set and the starting vertex. For metastable sets it turns out that *the average hitting time is essentially the inverse of the capacity.*

A key observation in the potential-theoretic approach is the fact that capacities can be estimated by exploiting powerful *variational principles.* In fact, dual variational principles are available that express the capacity both as an infimum over potentials (*Dirichlet principle*) and as a supremum over flows (*Thomson principle*). This opens up the possibility to derive sharp upper bounds and lower bounds on the capacity via a judicious choice of test functions. In fact, with the proper physical insight, test functions can be found for which the upper bound and the lower bound are asymptotically equivalent (in an appropriate limit corresponding to a metastable regime). Consequently, with the help of the potential-theoretic approach asymptotic estimates of the average crossover time can be derived that are much sharper than those typically obtainable with the help of the pathwise approach.

Both the Dirichlet principle and the Thomson principle have been key tools in electric network theory for many years (Doyle and Snell [12]). More recently, Berman and Konsowa [2] proved two variational formulas for *finite* electric networks, one in terms of probability measures on paths from one set to another (or their associated flows) the other in terms of random cuts (or their associated coboundaries). The Berman-Konsowa principle has been instrumental in obtaining sharp bounds for interacting particle systems with complex interactions (Bovier, den Hollander and Nardi [6], Bianchi, Bovier and Ioffe [3], Bovier, den Hollander and Spitoni [7]). *In the present paper we generalize the Berman-Konsowa principle from finite spaces to Polish spaces for reversible Markov jump processes.* Our principal motivation is an application to metastability for continuum interacting particle systems (den Hollander and Jansen [11]).

For an overview on the potential-theoretic approach to metastability we refer the reader to the monograph by Bovier and den Hollander [5].

1.2. **Setting.** Let $(\Omega, \mathcal{F})$ be a Polish space. Let $X = (X_t)_{t \geq 0}$ be a continuous-time irreducible *recurrent* Markov jump process on Ω with transition rates $k(x, \mathrm{d}y)$ and reversible invariant measure $\mu(\mathrm{d}x)$ (Stroock [26]), i.e., $k(x, \mathrm{d}y)$ is the rate to jump from x to a neighborhood $\mathrm{d}y$ of y, and

$$\mu(\mathrm{d}x)k(x,\mathrm{d}y) = \mu(\mathrm{d}y)k(y,\mathrm{d}x). \tag{1.1}$$

Define

$$K(\mathrm{d}x,\mathrm{d}y) = \mu(\mathrm{d}x)k(x,\mathrm{d}y), \tag{1.2}$$

which is a non-negative symmetric measure on $\Omega\times\Omega$. We assume that $K(\Omega\times\Omega) < \infty$ and that $k(x,\Omega) > 0$ for μ-a.a. x. For measurable $C, D \subset \Omega$, we think of $K(C\times D) = K(D\times C)$ as the *conductance* of X between C and D. As long as X_0 is drawn from a probability measure that is absolutely continuous w.r.t. μ, X makes finitely jumps in every finite time-interval (see Appendix A).

For $f\colon\ \Omega\to\mathbb{R}$ measurable and bounded, define

$$(Lf)(x) = \int_\Omega \big[f(y) - f(x)\big]k(x,\mathrm{d}y) \tag{1.3}$$

and

$$\mathcal{E}(f) = \int_\Omega \big[-(Lf)(x)\big]f(x)\mu(\mathrm{d}x) = \frac{1}{2}\int_{\Omega\times\Omega}\big[f(y)-f(x)\big]^2 K(\mathrm{d}x,\mathrm{d}y), \tag{1.4}$$

which are the infinitesimal generator (with domain $\mathcal{D}(L)$) and the Dirichlet form (with domain $\mathcal{D}(\mathcal{E})$) associated with X (see Appendix A for technical details).

Later we will exploit the fact that X can be constructed as a random time-change of a discrete-time Markov chain $Z = (Z_n)_{n\in\mathbb{N}}$ with transition kernel

$$\frac{k(x,\mathrm{d}y)}{k(x,\Omega)}. \tag{1.5}$$

The finiteness of K ($K(\Omega\times\Omega)<\infty$) is equivalent to Z being *positive recurrent*. It is possible that Z is positive recurrent while X is null-recurrent ($\mu(\Omega)=\infty$). See, in particular, the comments after Proposition A.4. At the end of Section 1.3 the case of *recurrent* X with *null-recurrent* Z and the case of *transient* X will be included, i.e., the extension to $K(\Omega\times\Omega)=\infty$ will be made.

Throughout the sequel, $A, B\subset\Omega$ are *disjoint* and measurable. The harmonic function and the capacity of the pair (A,B) associated with X are defined as

$$h_{AB}(x) = \begin{cases} \mathbb{P}_x(\tau_A < \tau_B), & x\in(A\cup B)^c,\\ 1, & x\in A,\\ 0, & x\in B,\end{cases} \tag{1.6}$$

and

$$\mathrm{cap}\,(A,B) = \int_A (-Lh_{AB})(x)\,\mu(\mathrm{d}x), \tag{1.7}$$

where $\tau_C = \inf\{t>0\colon\ X_t\in C\}$ is the first hitting time of C, $\mathbb{P}_x$ is the law of X given $X_0 = x$, and $(-Lh_{AB})(x)$ is the *equilibrium charge* at $x\in A$. The *Dirichlet principle* says that the capacity satisfies the variational formula (Fukushima, Oshima and Masayoshi [15])

$$\mathrm{cap}\,(A,B) = \inf_{h\in\mathcal{V}_{AB}} \mathcal{E}(h) \tag{1.8}$$

with

$$\mathcal{V}_{AB} = \big\{h\colon\ \Omega\to\mathbb{R}\mid h|_A = 1,\ h|_B = 0,\ 0\le h\le 1\ \mu\text{-a.e.}, \mathcal{E}(h)<\infty\big\}, \tag{1.9}$$

and has $h = h_{AB}$ as its unique minimizer. For completeness the proof is given in Appendix A.

1.3. **Theorems.** Let Γ_{AB} be the set of finite paths from A to B, i.e.,

$$\Gamma_{AB} = \bigcup_{n\in\mathbb{N}} \big\{\gamma = (\gamma_0,\dots,\gamma_n) \mid \gamma_0 \in A,\ \gamma_n \in B,\ \gamma_i \notin B \text{ for } i = 1,\dots,n-1\big\}. \tag{1.10}$$

Write $(x,y) \in \gamma$ when there is a $j \in \mathbb{N}$ such that $(x,y) = (\gamma_{j-1},\gamma_j)$. Let $\mathbb{P}$ be a probability measure on Γ_{AB}. Then there is a unique measure $\Phi = \Phi_{\mathbb{P}}$ on $\Omega\times\Omega$ such that for every bounded and measurable $f\colon\ \Omega\times\Omega \to [0,\infty)$,

$$\mathbb{E}\Bigg[\sum_{(x,y)\in\gamma} f(x,y)\Bigg] = \int_{\Omega\times\Omega} f(x,y)\Phi(\mathrm{d}x,\mathrm{d}y), \tag{1.11}$$

where γ is the random element of Γ_{AB} whose law is $\mathbb{P}$. Picking $f(x,y) = \mathbf{1}_{C\times D}(x,y)$, $C,D \subset \Omega$, we see that $\Phi(C\times D)$ is the expected number of edges (x,y) in the random path γ with $x \in C$ and $y \in D$. In particular, Φ is finite if and only if the expected length of γ is finite.

Our first main result is the following theorem.

Theorem 1.1 (Berman-Konsowa principle: path version)**.** *Let $\mathcal{P}_{AB}^K$ be the set of probability measures $\mathbb{P}$ on Γ_{AB} such that $\Phi_{\mathbb{P}}$ is absolutely continuous with respect to K. Then*

$$\operatorname{cap}(A,B) = \sup_{\mathbb{P}\in\mathcal{P}_{AB}^K} \mathbb{E}\left[\left(\sum_{(x,y)\in\gamma} \frac{\mathrm{d}\Phi_{\mathbb{P}}}{\mathrm{d}K}(x,y)\right)^{-1}\right]. \tag{1.12}$$

This identity remains true when the supremum is restricted to the smaller set of probability measures on finite self-avoiding paths from A to B.

In order to state our second main result we need the following definition of a flow along edges.

Definition 1.2. *A* unit flow *from A to B is a sigma-finite measure Φ on $\Omega\times\Omega$ such that*

1. $\Phi(A\times\Omega) = 1$ *and* $\Phi(\Omega\times A) = 0$.
2. $\Phi(\Omega\times B) = 1$ *and* $\Phi(B\times\Omega) = 0$.
3. $\Phi(\Omega\times C) = \Phi(C\times\Omega)$ *for all measurable* $C \subset \Omega\backslash(A\cup B)$.
4. *There is a measurable $\chi \subset \Omega\times\Omega$ such that $\Phi(\chi^{\mathrm{c}}) = 0$ and $(y,x) \notin \chi$ for all $(x,y) \in \chi$.*

Conditions (1–2) say that the total flow out of A and into B is 1 while the total flow into A and out of B is 0. Condition (3) says that the flow is *divergence-free* in $\Omega\backslash(A\cup B)$, which we refer to as *Kirchhoff's law* (the terminology used for discrete spaces). Condition (4) says that an edge and its reverse cannot not lie in the support of Φ simultaneously, i.e., the flow is *oriented*.

The flow is called *loop-free* when the set χ in condition (4) can be chosen such that it contains no loops, i.e., if $(\gamma_0,\dots,\gamma_n)$ is a finite sequence with $\gamma_n = \gamma_0$, then $(\gamma_{j-1},\gamma_j) \notin \chi$ for some $j = 1,\dots,n$.

Let Φ be a unit flow from A to B, let $\nu(C) = \Phi(C\times\Omega)$ be its left marginal, and let $\ell(x,\mathrm{d}y)$ be any probability transition kernel such that $\Phi(\mathrm{d}x,\mathrm{d}y) = \nu(\mathrm{d}x)\ell(x,\mathrm{d}y)$. Let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be the Markov chain with initial law $\mathbb{P}^{\Phi}(Y_0 \in C) = \nu(A\cap C)$, $C \subset \Omega$ measurable, and probability transition kernel ℓ, and put $\tau_B^Y = \min\{n \in \mathbb{N} \mid Y_n \in B\}$. In Section 2 we will show that the law of Y_n conditioned on $\tau_B^Y > n$ is absolutely continuous w.r.t. ν. Therefore changes of ℓ on ν-null sets do not affect the law of Y stopped in B, i.e., the law of the stopped process is uniquely determined by the flow Φ. We also show that if Φ is finite, i.e., $\Phi(\Omega\times\Omega)<\infty$, then $\mathbb{E}^{\Phi}[\tau_B^Y]<\infty$, i.e., Y is positive recurrent. The latter implies that $\tau_B^Y<\infty$ $\mathbb{P}^{\Phi}$-a.s.

Theorem 1.3 (Berman-Konsowa principle: flow version)**.** *Let $\mathcal{U}_{AB}^K$ be the set of unit flows from A to B that are absolutely continuous w.r.t. K and finite. Then*

$$\operatorname{cap}(A,B)=\sup_{\Phi\in\mathcal{U}_{AB}^K}\mathbb{E}^{\Phi}\left[\left(\sum_{n=1}^{\tau_B^Y}\frac{\mathrm{d}\Phi}{\mathrm{d}K}(Y_{n-1},Y_n)\right)^{-1}\right]. \tag{1.13}$$

The identity remains true when the supremum is restricted to the smaller set of loop-free unit flows.

An important example of a loop-free unit flow is the *harmonic flow* defined by

$$\Phi_{AB}(\mathrm{d}x,\mathrm{d}y)=\frac{1}{\operatorname{cap}(A,B)}\big[h_{AB}(x)-h_{AB}(y)\big]_+K(\mathrm{d}x,\mathrm{d}y). \tag{1.14}$$

We will show that this flow has finite self-avoiding paths and is a maximizer of (1.13). It is not necessarily the unique maximizer. Any $\mathbb{P}$ such that $\Phi_{\mathbb{P}}=\Phi_{AB}$ is a maximizer of (1.12).

We close with the statement that the assumption of positive recurrence of Z, which was made below (1.5), can be dropped. We only require that $k(x,\mathrm{d}y)$ admits a sigma-finite measure $\mu(\mathrm{d}x)$ satisfying (1.1), and do allow for $K(\Omega\times\Omega)=\infty$.

Theorem 1.4. *Suppose that $K((A\cup B)\times\Omega)<\infty$. Then Theorems* 1.1 *and* 1.3 *extend to $K(\Omega\times\Omega)=\infty$, i.e., to recurrent X with null-recurrent Z and to transient X.*

For transient X, capacity can be *defined* by (1.8), h_{AB} can be *defined* as the unique minimizer of (1.8), and (1.7) can be *shown* to hold (see Appendix A) . However, in general no explicit expression is available for h_{AB} in terms of hitting times, as in (1.6) for recurrent X. In fact, the natural analogue of (1.6), namely, the function g_{AB} given by $g_{AB}(x)=\mathbb{P}_x(\tau_A<\tau_B,\tau_A<\infty)$, is *not* the minimizer of (1.8) (see Lemma A.7 below).

1.4. **Discussion.** We place the results from Section 1.3 in their proper context.

1. We have shown that the *Berman-Konsowa principle* holds for reversible Markov jump process on general Polish spaces. It provides dual variational formulas for the capacity, the first running over probability measures $\mathbb{P}$ on the set of finite paths connecting A and B, the second running over unit flows Φ from A to B. Each probability measure $\mathbb{P}$ gives rise to a unit flow $\Phi=\Phi_{\mathbb{P}}$. Conversely, each unit flow Φ gives rise to a path measure $\mathbb{P}=\mathbb{P}_{\Phi}$, though not uniquely.

2. The Berman-Konsowa principle complements the *Dirichlet principle* in (1.8), and also the *Thomson principle*, which says that

$$\frac{1}{\operatorname{cap}(A,B)}=\inf_{\Phi\in\mathcal{U}_{AB}^K}\int_{\Omega\times\Omega}\Big(\frac{\mathrm{d}\Phi}{\mathrm{d}K}\Big)^2\mathrm{d}K \tag{1.15}$$

with the harmonic flow from (1.14) as the unique minimizer. For completeness, the proof of the Thomson principle is given in Appendix A.

3. Just as for finite state spaces, the Berman-Konsowa bound improves the Thomson bound. Indeed, for any $\mathbb{P} \in \mathcal{P}_{AB}^K$ we have, by (1.11–1.12) and Jensen's inequality,

$$\begin{aligned}\operatorname{cap}(A,B) &\geq \mathbb{E}\left[\left(\sum_{(x,y)\in\gamma} \frac{\mathrm{d}\Phi_{\mathbb{P}}}{\mathrm{d}K}(x,y)\right)^{-1}\right] \geq \left(\mathbb{E}\left[\sum_{(x,y)\in\gamma} \frac{\mathrm{d}\Phi_{\mathbb{P}}}{\mathrm{d}K}(x,y)\right]\right)^{-1} \\ &= \left(\int_{\Omega\times\Omega} \frac{\mathrm{d}\Phi_{\mathbb{P}}}{\mathrm{d}K}\,\mathrm{d}\Phi_{\mathbb{P}}\right)^{-1} = \left(\int_{\Omega\times\Omega} \left(\frac{\mathrm{d}\Phi_{\mathbb{P}}}{\mathrm{d}K}\right)^2 \mathrm{d}K\right)^{-1}.\end{aligned} \tag{1.16}$$

4. The Dirichlet principle and the Thomson principle complement each other: upper bounds on capacities can be obtained by choosing test potentials, lower bounds by choosing test unit flows. The Berman-Konsowa principle is stronger than the Thomson principle in that it leads to better bounds, even though the suprema are the same. This is particularly helpful for obtaining *approximations* of capacities.

5. In order to derive approximations of capacities it is possible to work with "leaky flows" instead of unit flows, i.e., flows for which the condition "flow out of A = flow into $B = 1$" is fulfilled with a small error. Indeed, it is possible to quantify the discrepancy between suprema for leaky flows and suprema for unit flows in terms of this error, and this allows for greater *flexibility* in the approximation procedure. We refer the reader to the monograph by Bovier and den Hollander [5] for further details.

The remainder of this paper is organized as follows. In Section 2 we give the proof of Theorems 1.1 and 1.3, and their extension in Theorem 1.4. In Appendix A we list some technical facts about Dirichlet forms, give the proof of the Dirichlet principle and the Thomson principle in the general setting considered in this paper, and show that capacities can be approximated via truncation. In Appendix B we give the interpretation of the three variational principles for finite electric networks. The two appendices take up about half of the paper and rely on basic results from the literature.

2. Proofs

Sections 2.1 and 2.2 state and prove a proposition and a lemma that are needed in the proofs of Theorems 1.1 and 1.3 in Section 2.4. Section 2.3 looks at the harmonic flow. Throughout this section, A and B are fixed disjoint measurable subsets of Ω.

2.1. **A preparatory proposition.**

The following proposition paves the way for the proofs of Theorems 1.1 and 1.3.

Proposition 2.1. *Let Φ be a unit flow from A to B, $\nu(C) = \Phi(C\times\Omega)$, $C \subset \Omega$ measurable, its left marginal, and $\ell(x,\mathrm{d}y)$ a probability transition kernel such that $\Phi(\mathrm{d}x,\mathrm{d}y) = \nu(\mathrm{d}x)\ell(x,\mathrm{d}y)$. Let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be the Markov chain with initial law $\mathbb{P}^\Phi(Y_0 \in C) = \nu(A\cap C)$, $C \subset \Omega$ measurable, and probability transition kernel ℓ, and let $\tau_B^Y = \min\{n \in \mathbb{N} \mid Y_n \in B\}$. Then:*

(1) *For all bounded non-negative measurable functions f,*

$$\int_{\Omega\times\Omega} f(x,y)\Phi(\mathrm{d}x,\mathrm{d}y) \geq \mathbb{E}^{\Phi}\left[\sum_{n=1}^{\tau_B^Y} f(Y_{n-1},Y_n)\right]. \tag{2.1}$$

(2) *If Φ is absolutely continuous w.r.t. K, then the Berman-Konsowa bound holds:*

$$\mathrm{cap}\,(A,B) \geq \mathbb{E}^{\Phi}\left[\left(\sum_{n=1}^{\tau_B^Y} \frac{\mathrm{d}\Phi}{\mathrm{d}K}(Y_{n-1},Y_n)\right)^{-1}\right]. \tag{2.2}$$

(3) *If the flow is loop-free, then the paths $(Y_n)_{0\leq n\leq \tau_B^Y}$ are self-avoiding $\mathbb{P}^{\Phi}$-a.s.*
(4) *If $\Phi(\Omega\times\Omega)<\infty$, then $\mathbb{E}^{\Phi}[\tau_B^Y]<\infty$ and so $\tau_B^Y<\infty$ $\mathbb{P}^{\Phi}$-a.s.*

Proof. First we check that the measure $\mathbb{P}^{\Phi}$ does not depend on the precise choice of ℓ. To this aim, we prove that for all $n\in\mathbb{N}$ the measure $\nu_n(C)=\mathbb{P}(Y_n\in C,\,\tau_B^Y\geq n+1)$ is absolutely continuous w.r.t. ν. The proof is by induction on n.

For $n=0$, the statement is true by the definition of Y_0. Suppose it is true for some $n\in\mathbb{N}_0$. Because $\Phi(\Omega\times A)=0$, Y never returns to A and we have $\nu_{n+1}(A)=0$. Hence we need only look at ν-null sets $C\subset\Omega\backslash(A\cup B)$. Thus, let $C\subset\Omega\backslash(A\cup B)$ with $\nu(C)=0$. Kirchhoff's law yields $\Phi(\Omega\times C)=\Phi(C\times\Omega)=\nu(C)=0$, and

$$\begin{aligned}\nu_{n+1}(C) &= \mathbb{P}(Y_{n+1}\in C,\,\tau_B^Y\geq n+2)=\mathbb{P}(Y_{n+1}\in C,\,\tau_B^Y\geq n+1)\\ &= \int_{(\Omega\backslash B)\times C}\nu_n(\mathrm{d}x)\ell(x,\mathrm{d}y)\\ &= \int_{(\Omega\backslash B)\times C}\frac{\mathrm{d}\nu_n}{\mathrm{d}\nu}(x)\Phi(\mathrm{d}x,\mathrm{d}y)=\int_{\Omega\backslash B}\frac{\mathrm{d}\nu_n}{\mathrm{d}\nu}(x)\Phi(\mathrm{d}x,C)=0.\end{aligned} \tag{2.3}$$

It follows that ν_{n+1} is absolutely continuous w.r.t. ν. To conclude, we note that the equation $\Phi(\mathrm{d}x,\mathrm{d}y)=\nu(\mathrm{d}x)\ell(x,\mathrm{d}y)$ determines $\ell(x,C)$ up to changes for x in ν-null sets. Since Y does not see ν-null sets except possibly in B, the law of Y stopped upon reaching B is unaffected by this ambiguity.

(1) Let $f\colon\,\Omega\times\Omega\to[0,\infty)$ be bounded and measurable. We prove by induction on n that

$$\int_{\Omega\times\Omega} f(x,y)\Phi(\mathrm{d}x,\mathrm{d}y)=\mathbb{E}\left[\sum_{k=1}^{n\wedge\tau_B} f(Y_{k-1},Y_k)\right]+R_n \tag{2.4}$$

with remainder term

$$R_n=\int_{\Omega\times(A\cup B)^{\mathrm{c}}}\Phi(\mathrm{d}x_0,\mathrm{d}x_1)F_n(x_1), \tag{2.5}$$

where

$$F_n(x_1)=\int_{[(A\cup B)^{\mathrm{c}}]^n\times\Omega}\ell(x_1,\mathrm{d}x_2)\times\cdots\times\ell(x_n,\mathrm{d}x_{n+1})f(x_n,x_{n+1}). \tag{2.6}$$

For $n=0$ the claim is obvious. Suppose that (2.4–2.5) hold for some $n\in\mathbb{N}_0$. Then

$$\begin{aligned}
R_n &= \int_{\Omega\times(A\cup B)^c} \nu(\mathrm{d}x_0)\ell(x_0,\mathrm{d}x_1)F_n(x_1)\\
&= \int_{(A\cup B)\times(A\cup B)^c} \nu(\mathrm{d}x_0)\ell(x_0,\mathrm{d}x_1)F_n(x_1)\\
&\qquad + \int_{\Omega\times[(A\cup B)^c]^2} \Phi(\mathrm{d}x_{-1},\mathrm{d}x_0)\ell(x_0,\mathrm{d}x_1)F_n(x_1)\\
&= \int_{(A\cup B)\times(A\cup B)^c} \nu(\mathrm{d}x_0)\ell(x_0,\mathrm{d}x_1)F_n(x_1) + R_{n+1},
\end{aligned} \tag{2.7}$$

where in the second equality we use Kirchhoff's law to rewrite $\nu(\mathrm{d}x_0)$ as the right marginal of Φ. Hence we obtain

$$R_n = \mathbb{E}\Big[f(Y_n,Y_{n+1})\mathbf{1}_{\{\tau_B^Y\geq n+1\}}\Big] + R_{n+1}, \tag{2.8}$$

where we use that $\Phi(B\times\Omega)=0$. The inequality in (2.1) follows by estimating $R_n\geq 0$ and letting $n\to\infty$ in (2.4).

(2) Recall (1.4). Let $\phi=\mathrm{d}\Phi/\mathrm{d}K$ and estimate, for any $h\in\mathcal{V}_{AB}$,

$$\begin{aligned}
\mathcal{E}(h) &\geq \tfrac12 \int_{\Omega\times\Omega} \big[h(y)-h(x)\big]^2 \mathbf{1}_{\{\phi(x,y)>0\}} K(\mathrm{d}x,\mathrm{d}y)\\
&= \tfrac12 \int_{\Omega\times\Omega} \frac{\big[h(y)-h(x)\big]^2}{\phi(x,y)} \mathbf{1}_{\{\phi(x,y)>0\}} \Phi(\mathrm{d}x,\mathrm{d}y)\\
&\geq \mathbb{E}^\Phi\left[\tfrac12\sum_{n=1}^{\tau_B^Y} \frac{\big[h(Y_n)-h(Y_{n-1})\big]^2}{\phi(Y_{n-1},Y_n)} \mathbf{1}_{\{\phi(Y_{n-1},Y_n)>0\}}\right],
\end{aligned} \tag{2.9}$$

where the second inequality uses (2.1). Take the infimum over $h\in\mathcal{V}_{AB}$ and use (1.8), to obtain

$$\mathrm{cap}\,(A,B) \geq \mathbb{E}^\Phi\left[\inf_{h\in\mathcal{V}_{AB}} \tfrac12 \sum_{n=1}^{\tau_B^Y} \frac{\big[h(Y_n)-h(Y_{n-1})\big]^2}{\phi(Y_{n-1},Y_n)} \mathbf{1}_{\{\phi(Y_{n-1},Y_n)>0\}}\right]. \tag{2.10}$$

The infimum under the expectation can be easily computed, and equals

$$\left(\sum_{n=1}^{\tau_B^Y} \phi(Y_{n-1},Y_n)\right)^{-1} \tag{2.11}$$

because $h(Y_0)=1$ and $h(Y_{\tau_B^Y})=0$ (see (2.27) below). Hence (2.2) holds.

(3) Let $\chi\subset\Omega\times\Omega$ be a loop-free measurable set with $\Phi(\chi^c)=0$. By construction, the path $(Y_0,\dots,Y_n)$ with $n\leq\tau_B^Y$ has only transitions $(Y_{j-1},Y_j)\in\chi$ a.s. Since χ is loop-free, Y stopped upon reaching B is self-avoiding.

(4) Apply the inequality in (2.1) to the constant function $f\equiv 1$. This gives

$$\mathbb{E}^\Phi\left[\tau_B^Y\right] = \mathbb{E}^\Phi\left[\sum_{n\in\mathbb{N}} \mathbf{1}_{\{\tau_B^Y\geq n\}}\right] \leq \Phi(\Omega\times\Omega). \tag{2.12}$$

If Φ is finite, then τ_B^Y has finite expectation and hence is finite $\mathbb{P}^\Phi$-a.s. $\square$

2.2. **A discrepancy lemma.** To get equalities in (2.1–2.2) we need an extra argument, which is based on the following lemma.

Lemma 2.2. *Let Φ be a unit flow from A to B that is finite, i.e., $\Phi(\Omega\times\Omega)<\infty$, and let $Y=(Y_n)_{n\in\mathbb{N}_0}$ be the associated Markov chain as in Proposition* 2.1. *Let $\tilde\Phi$ and Ψ be the measures on $\Omega\times\Omega$ defined by*

$$\begin{aligned}\int_{\Omega\times\Omega} f(x,y)\tilde\Phi(\mathrm{d}x,\mathrm{d}y) &= \mathbb{E}^{\Phi}\left[\sum_{n\in\mathbb{N}} f(Y_{n-1},Y_n)\mathbf{1}_{\{\tau_B^Y\geq n\}}\right],\\ \Psi &= \Phi-\tilde\Phi.\end{aligned}\tag{2.13}$$

Then $\tilde\Phi$ is a unit flow from A to B, and Ψ satisfies Kirchhoff's law, i.e., $\Psi(C\times\Omega)=\Psi(\Omega\times C)$ for all $C\subset\Omega$.

Proof. Proposition 2.1 tells us that $\tilde\Phi\leq\Phi$, and so Ψ is a non-negative measure on $\Omega\times\Omega$, satisfying $\Psi\leq\Phi$. Both $\tilde\Phi$ and Ψ inherit Kirchhoff's law on $(A\cup B)^{\mathrm{c}}$ from Φ. They also inherit the properties $\Phi(\Omega\times A)=0$ and $\Phi(B\times\Omega)=0$. Furthermore,

$$\tilde\Phi(A\times\Omega)=\mathbb{P}^{\Phi}\big((Y_0,Y_1)\in A\times\Omega\big)=1,\tag{2.14}$$

where we use that Y does not return to A after time 0. Thus, $n=1$ is the only summand contributing to (2.13). Similarly,

$$\tilde\Phi(\Omega\times B)=\mathbb{P}^{\Phi}(\tau_B^Y<\infty)=1,\tag{2.15}$$

where we use that the unit flow is finite and that by Proposition 2.1 the hitting time τ_B^Y is $\mathbb{P}^{\Phi}$-a.s. finite. Therefore $\tilde\Phi$ is a unit flow from A to B. Moreover,

$$\Psi(A\times\Omega)=\Phi(A\times\Omega)-\tilde\Phi(A\times\Omega)=1-1=0\tag{2.16}$$

and $\Psi(\Omega\times B)=0$. It follows that $\Psi(C\times\Omega)=\Psi(\Omega\times C)$ for all measurable $C\subset\Omega$. □

2.3. **The harmonic flow.**

Lemma 2.3. *Φ_{AB} is a finite loop-free unit flow from A to B.*

Proof. We check that properties (1)–(4) in Definition 1.2 hold for $\Phi=\Phi_{AB}$.

(1) By (1.14), we have

$$\Phi_{AB}(\Omega\times A)=\frac{1}{\mathrm{cap}\,(A,B)}\int_{\Omega\times A}\big[h_{AB}(x)-h_{AB}(y)\big]_+K(\mathrm{d}x,\mathrm{d}y)=0\tag{2.17}$$

because $h_{AB}(x)\leq 1$ for $x\in\Omega$ and $h_{AB}(y)=1$ for $y\in A$. A similar argument shows that $\Phi_{AB}(B\times\Omega)=0$.

(2) By (1.2–1.3), (1.7) and (1.14), we have

$$\begin{aligned}\Phi_{AB}(A\times\Omega)&=\frac{1}{\mathrm{cap}\,(A,B)}\int_A\mu(\mathrm{d}x)\int_\Omega\big[h_{AB}(x)-h_{AB}(y)\big]k(x,\mathrm{d}y)\\&=\frac{1}{\mathrm{cap}\,(A,B)}\int_A\mu(\mathrm{d}x)\big(-Lh_{AB}\big)(x)=1\end{aligned}\tag{2.18}$$

because $h_{AB}(x)=1$ for $x\in A$ and $h_{AB}(y)\leq 1$ for $y\in\Omega$. The fact that $\Phi_{AB}(\Omega\times B)=1$ follows from the symmetry relations $h_{BA}=1-h_{AB}$, $\Phi_{AB}(C\times D)=\Phi_{BA}(D\times C)$ and $\mathrm{cap}\,(A,B)=\mathrm{cap}\,(B,A)$.

(3) Let $C \subset \Omega\backslash(A \cup B)$. Because of the symmetry of K, we have

$$\begin{aligned}\Phi(\Omega \times C) &= \int_{\Omega\times C} \big[h_{AB}(x) - h_{AB}(y)\big]_+ K(\mathrm{d}y, \mathrm{d}x)\\ &= \int_{C\times\Omega} \big[h_{AB}(x) - h_{AB}(y)\big]_- K(\mathrm{d}x, \mathrm{d}y).\end{aligned} \tag{2.19}$$

Therefore, by (1.2–1.3) and the fact that $[\cdot]_+ - [\cdot]_- = [\cdot]$,

$$\begin{aligned}\Phi(C \times \Omega) - \Phi(\Omega \times C) &= \int_{C\times\Omega} \big[h_{AB}(x) - h_{AB}(y)\big] K(\mathrm{d}x, \mathrm{d}y)\\ &= \int_C \mu(\mathrm{d}x)(-Lh_{AB})(x) = 0,\end{aligned} \tag{2.20}$$

because μ is invariant. Thus, Kirchhoff's law holds.

(4) Define

$$\chi = \big\{(x, y) \in \Omega \times \Omega \mid h_{AB}(x) > h_{AB}(y)\big\}. \tag{2.21}$$

Clearly, $\Phi_{AB}(\chi^{\mathrm{c}}) = 0$ and χ contains no loops. Hence Φ_{AB} is a loop-free unit flow from A to B. The flow is finite because

$$\Phi_{AB}(\Omega \times \Omega) = \frac{1}{\mathrm{cap}\,(A, B)} \int_{\Omega\times\Omega} \big[h_{AB}(x) - h_{AB}(y)\big]_+ K(\mathrm{d}x, \mathrm{d}y) \leq \frac{K(\Omega \times \Omega)}{\mathrm{cap}\,(A, B)} < \infty, \tag{2.22}$$

where we use that $h_{AB} \leq 1$ and $K(\Omega \times \Omega) < \infty$. □

Lemma 2.4. *Y with law $\mathbb{P}^{\Phi_{AB}}$ satisfies $\tau_B^Y < \infty$ a.s., its paths are self-avoiding, and equality holds in* (2.2).

Proof. We proceed as in Lemma 2.2 and exploit the fact that $(h_{AB}(Y_n))_{n\in\mathbb{N}_0}$ is $\mathbb{P}^{\Phi_{AB}}$-a.s. strictly decreasing. Define Ψ_{AB} as in Lemma 2.2 for $\Phi = \Phi_{AB}$, i.e.,

$$\Psi_{AB}(f) = \int_{\Omega\times\Omega} f \mathrm{d}\Psi_{AB} = \int_{\Omega\times\Omega} f \mathrm{d}\Phi_{AB} - \mathbb{E}^{\Phi_{AB}} \left[\sum_{n=1}^{\tau_B^Y} f(Y_{n-1}, Y_n)\right]. \tag{2.23}$$

We have to show that $\Psi_{AB} = 0$. We already know that Ψ_{AB} is a finite measure satisfying Kirchhoff's law in all of Ω. Suppose that $\Psi_{AB}(\Omega \times \Omega) > 0$. Let $\hat{\nu}(\mathrm{d}x)$ be its marginal and $\hat{\ell}(x, \mathrm{d}y)$ any probability transition kernel such that $\hat{\nu}(\mathrm{d}x)\hat{\ell}(x, \mathrm{d}y) = \Psi_{AB}(\mathrm{d}x, \mathrm{d}y)$. Without loss of generality we may assume that $\hat{\nu}(\Omega) = 1$. Let $\hat{Y} = (\hat{Y}_n)_{n\in\mathbb{N}}$ be a stationary Markov process with probability transition kernel $\hat{\ell}$ and initial distribution $\mathbb{P}(\hat{Y}_0 \in C) = \hat{\nu}(C)$, $C \subset \Omega$ measurable. A Poincaré-recurrence-type argument shows that $\hat{Y}$ returns to C infinitely often for every C with $\hat{\nu}(C) > 0$. On the other hand, we know that the set $\{(x, y) \in \Omega \times \Omega \mid h_{AB}(x) \leq h_{AB}(y)\}$ has Φ_{AB}-measure 0. Since $\Psi_{AB}(f) \leq \Phi_{AB}(f)$, this set also has Ψ_{AB}-measure 0. It therefore follows that there is an $m > 0$ such that

$$\Psi_{AB}\Big(\big\{(x, y) \in \Omega \times \Omega \mid h_{AB}(x) > m > h_{AB}(y)\big\}\Big) > 0. \tag{2.24}$$

Put $C = \{x \in \Omega \mid h_{AB}(x) > m\}$. Then $\hat{\nu}(C) = 1$ and $\Psi_{AB}(C \times C) = 0$. Therefore once $\hat{Y}$ has left C it cannot come back to C, contradicting the fact that it returns to C infinitely often. Thus, the assumption $\Psi_{AB}(\Omega \times \Omega) > 0$ leads to a contradiction. We conclude that $\Psi_{AB} = 0$, and so there is equality in (2.1).

To show that equality holds in (2.2), i.e., Φ_{AB} is a maximizer of (1.13), we compute the right-hand side of (2.2). Note that $(\mathrm{d}\Phi_{AB}/\mathrm{d}K)(x, y) = [h_{AB}(x) -$

$h_{AB}(y)]_+$, and recall that $\mathbb{P}^{\Phi_{AB}}$-a.s. $(h_{AB}(Y_n))_{n\in\mathbb{N}_0}$ is strictly increasing (until it reaches B). We have

$$\begin{aligned}
&\frac{1}{\operatorname{cap}(A,B)}\,\mathbb{E}^{\Phi_{AB}}\left[\left(\sum_{n=1}^{\tau_B^Y}\frac{\mathrm{d}\Phi_{AB}}{\mathrm{d}K}(Y_{n-1},Y_n)\right)^{-1}\right]\\
&\quad=\mathbb{E}^{\Phi_{AB}}\left[\left(\sum_{n=1}^{\tau_B^Y}\big[h_{AB}(Y_{n-1})-h_{AB}(Y_n)\big]\right)^{-1}\right]\\
&\quad=\mathbb{E}^{\Phi_{AB}}\left[\Big(h_{AB}(Y_0)-h_{AB}\big(Y_{\tau_B^Y}\big)\Big)^{-1}\right]=\mathbb{E}^{\Phi_{AB}}\left[(1-0)^{-1}\right]=1,
\end{aligned}\tag{2.25}$$

which is the desired result. □

2.4. **Proof of Theorems 1.1 and 1.3.**

Proof. Theorem 1.3 follows from Proposition 2.1(2–4) and Lemma 2.4. To prove Theorem 1.1 we argue as follows.

Let $\mathbb{P}\in\mathcal{P}^K_{AB}$ and put $\phi=\mathrm{d}\Phi_{\mathbb{P}}/\mathrm{d}K$. Pick $h\in\mathcal{D}(\mathcal{E})$ such that $h|_A=1$ and $h|_B=0$. Then, by (1.4) and (1.11),

$$\begin{aligned}
\mathcal{E}(h)&\geq\tfrac12\int_{\Omega\times\Omega}\big[h(y)-h(x)\big]^2\mathbf{1}_{\{\phi(x,y)>0\}}K(\mathrm{d}x,\mathrm{d}y)\\
&=\tfrac12\int_{\Omega\times\Omega}\frac{\big[h(y)-h(x)\big]^2}{\phi(x,y)}\mathbf{1}_{\{\phi(x,y)>0\}}\Phi_{\mathbb{P}}(\mathrm{d}x,\mathrm{d}y)\\
&=\mathbb{E}\left[\tfrac12\sum_{(x,y)\in\gamma}\frac{\big[h(y)-h(x)\big]^2}{\phi(x,y)}\mathbf{1}_{\{\phi(x,y)>0\}}\right]\\
&=\mathbb{E}\left[\tfrac12\sum_{(x,y)\in\gamma}\frac{\big[h(y)-h(x)\big]^2}{\phi(x,y)}\right],
\end{aligned}\tag{2.26}$$

where in the last line we use that $\phi(x,y)>0$ for all $(x,y)\in\gamma$ for $\mathbb{P}$-a.a. paths γ. The solution to the one-dimensional harmonic problem is trivial, namely, for every $\gamma=(\gamma_0,\dots,\gamma_\tau)\in\Gamma_{AB}$ we have

$$\begin{aligned}
&\inf\left\{\tfrac12\sum_{(x,y)\in\gamma}\frac{\big[h(y)-h(x)\big]^2}{\phi(x,y)}\;\Big|\;h\colon\gamma\to[0,1],\;h(\gamma_0)=1,\;h(\gamma_\tau)=0\right\}\\
&\quad=\left(\sum_{(x,y)\in\gamma}\phi(x,y)\right)^{-1}.
\end{aligned}\tag{2.27}$$

Combining (2.26–2.27) and recalling (1.8), we get

$$\operatorname{cap}(A,B)\geq\mathbb{E}\left[\left(\sum_{(x,y)\in\gamma}\frac{\mathrm{d}\Phi_{\mathbb{P}}}{\mathrm{d}K}(x,y)\right)^{-1}\right].\tag{2.28}$$

Next, recall that $\mathbb{P}^{\Phi_{AB}}$ is the probability measure on Γ_{AB} associated with the harmonic flow Φ_{AB}. We know that $\mathbb{P}^{\Phi_{AB}}$ is a probability measure on finite self-avoiding paths from A to B, and so we have

$$\begin{aligned}\mathbb{E}^{\Phi_{AB}}\left[\left(\sum_{(x,y)\in\gamma}\frac{\mathrm{d}\Phi_{AB}}{\mathrm{d}K}(x,y)\right)^{-1}\right] &= \mathbb{E}^{\Phi_{AB}}\left[\left(\sum_{n=1}^{\tau_B^Y}\frac{\mathrm{d}\Phi_{AB}}{\mathrm{d}K}(Y_{n-1},Y_n)\right)^{-1}\right] \\ &= \mathrm{cap}\,(A,B),\end{aligned} \tag{2.29}$$

where the first equality uses the definition of Y in Proposition 2.1 and the second equality uses (2.25). Thus, equality is achieved in (2.28) for $\mathbb{P} = \mathbb{P}^{\Phi_{AB}}$. This completes the proof of Theorem 1.1. □

2.5. **Proof of Theorem 1.4.** The extension of Theorems 1.1 and 1.3 from positive recurrent X to null-recurrent and transient X proceeds via a *truncation argument.* In Appendix A.4 we let $(\Omega_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable subsets of Ω with $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$ such that $A\cup B\subset\Omega_n$ and $K(\Omega_n\times\Omega_n)<\infty$ for all $n\in\mathbb{N}$. We show that if $\mathrm{cap}_n(A,B)$ denotes the capacity for the reversible Markov jump process X^n obtained from X by *suppressing* jumps outside Ω_n, then

$$\lim_{n\to\infty}\mathrm{cap}_n(A,B) = \mathrm{cap}\,(A,B) \tag{2.30}$$

when $K((A\cup B)\times\Omega)<\infty$.

In order to prove the extension of Theorem 1.3, we argue as follows. Let h^n_{AB} be the harmonic function on Ω_n, and let Φ^n_{AB} be the harmonic flow on Ω_n given by (1.14):

$$\Phi^n_{AB}(\mathrm{d}x,\mathrm{d}y) = \frac{1}{\mathrm{cap}_n(A,B)}\left[h^n_{AB}(x)-h^n_{AB}\right]_+\mathbf{1}_{\Omega_n}(x)1_{\Omega_n}(y)\,K(\mathrm{d}x,\mathrm{d}y). \tag{2.31}$$

Note that, for all $n\in\mathbb{N}$, $\Phi^n_{AB}\ll K$ and $\Phi^n_{AB}\in\mathcal{U}^K_{AB}$, where the latter is the set of finite unit flows on Ω. Therefore, by Theorem 1.3, we have

$$\mathrm{cap}_n(A,B) = \mathbb{E}^{\Phi^n_{AB}}\left[\left(\sum_{n=1}^{\tau_B^Y}\frac{\mathrm{d}\Phi^n_{AB}}{\mathrm{d}K}(Y_{n-1},Y_n)\right)^{-1}\right], \tag{2.32}$$

where we use that, for all $n\in\mathbb{N}$, X^n has finite total conductance (i.e., is positive recurrent). Combining (2.30) and (2.32), we obtain

$$\mathrm{cap}\,(A,B) \le \sup_{\Phi\in\mathcal{U}^K_{AB}}\mathbb{E}^{\Phi}\left[\left(\sum_{n=1}^{\tau_B^Y}\frac{\mathrm{d}\Phi}{\mathrm{d}K}(Y_{n-1},Y_n)\right)^{-1}\right]. \tag{2.33}$$

But the reverse inequality was already proved in Proposition 2.1(2).

In order to prove the extension of Theorem 1.1, we argue as follows. Equation (1.12) with $\geq$ instead of $=$ is a consequence of Proposition 2.1(2). For the reverse inequality, note that in (2.32) we have $\Phi^n_{AB} = \Phi_{\mathbb{P}^n}$ with $\mathbb{P}^n = \mathbb{P}^{\Phi^n_{AB}}\in\mathcal{P}^K_{AB}$. Passing to the limit $n\to\infty$, we obtain (2.33) with Φ replaced by $\Phi_{\mathbb{P}}$ and the supremum over $\Phi\in\mathcal{U}^K_{AB}$ replaced by the supremum over $\mathbb{P}\in\mathcal{P}^K_{AB}$.

Appendix A. Potential-theoretic ingredients

In Section A.1 we provide the details of the construction of the Markov jump process $X = (X_t)_{t\geq 0}$ introduced in Section 1.2. In Section A.2 we check that the Dirichlet principle in (1.8) has a unique solution. In Sections A.3 and A.5 we give the proof of the Dirichlet principle and the Thomson principle. In Section A.4 we show that the capacity can be obtained as the limit of certain truncated capacities. This is crucial for the extension from finite total conductances ($K(\Omega\times\Omega) < \infty$, corresponding to X with positive recurrent Z) to infinite total conductances ($K(\Omega\times\Omega) = \infty$).

A.1. **Jump process.** Recall that a *kernel* on $(\Omega, \mathcal{F})$ (with Ω the state space and $\mathcal{F}$ the Borel σ-algebra) is a map $k\colon \Omega\times\mathcal{F} \to [0,\infty)$ such that $x \mapsto k(x, B)$ is measurable for every $B \in \mathcal{F}$, and $k(x,\cdot)$ is a measure for every $x \in \Omega$. We assume that $k(x,\Omega) < \infty$ for all $x \in \Omega$, and that $k(x, \mathrm{d}y)$ admits a σ-finite reversible measure μ, i.e., $\mu(\mathrm{d}x)k(x, \mathrm{d}y) = \mu(\mathrm{d}y)k(x, \mathrm{d}y) = K(\mathrm{d}x, \mathrm{d}y)$. Thus, K is the unique measure on the product space $\Omega \times \Omega$ such that

$$K(C \times D) = \int_C \mu(\mathrm{d}x)k(x, D), \qquad C, D \subset \Omega \text{ measurable.} \tag{A.1}$$

Set

$$\lambda(x) = k(x, \Omega) \tag{A.2}$$

and assume that $\lambda(x) > 0$ for μ-a.a. $x \in \Omega$. The *minimal jump process* $X = (X_t)_{t\geq 0}$ associated with $k(x, \mathrm{d}y)$ is defined as follows. Let $(Q_t(x, \mathrm{d}y))_{t\geq 0}$ be the minimal solution of the backward Kolmogorov equation (existence and uniqueness are proven in Feller [13, Chapter 3]). Thus, $Q_0(x, \mathrm{d}y) = \delta_x(\mathrm{d}y)$ is the identity kernel and, for all $t \geq 0$, $x \in \Omega$ and $C \subset \Omega$ measurable,

$$\frac{\partial Q_t}{\partial t}(x, C) = -\lambda(x)Q_t(x, C) + \int_\Omega k(x, \mathrm{d}y)Q_t(y, C). \tag{A.3}$$

The right-hand side is written as $(LQ_t(\cdot, C))(x)$ with L the generator of X. The minimal solution satisfies $Q_t(x, \Omega) \leq 1$ for all $x \in \Omega$, $t \geq 0$ and therefore defines a Markov process $(X_t)_{t\geq 0}$ with a possibly finite lifetime $\zeta > 0$.

Next, we specify the domains of the Dirichlet form and the generator, and check that μ is a reversible measure. Let L_0 be the operator in $L^2(\Omega, \mu)$ with domain

$$\mathcal{D}(L_0) = \Big\{ f \in L^2(\Omega,\mu) \mid \exists\, n \in \mathbb{N}\colon\ \mu\big(\{x \in \Omega \mid f(x) \neq 0,\ \lambda(x) > n\}\big) = 0 \Big\} \tag{A.4}$$

and $L_0 f = Lf$ as in (1.3). Let $\mathcal{E}^*$ be the quadratic form with domain

$$\mathcal{D}(\mathcal{E}^*) = \Big\{ f \in L^2(\Omega,\mu) \mid \int_{\Omega\times\Omega} \big[f(y) - f(x)\big]^2 K(\mathrm{d}x, \mathrm{d}y) < \infty \Big\} \tag{A.5}$$

and $\mathcal{E}^*(f) = \mathcal{E}(f)$ as in (1.4). Set

$$\|f\|_1 = \Big(\int_\Omega f^2 \mathrm{d}\mu + \mathcal{E}(f)\Big)^{1/2}. \tag{A.6}$$

Note that $\mathcal{D}(L_0) \subset \mathcal{D}(\mathcal{E})$. Indeed, if $f \in L^2(\Omega, \mu)$ is supported in $\{x \in \Omega \mid \lambda(x) \leq n\}$, then

$$\int_{\Omega\times\Omega} \big[f(y) - f(x)\big]^2 K(\mathrm{d}x, \mathrm{d}y) \leq 4\int_\Omega f(x)^2\lambda(x)\mu(\mathrm{d}x) \leq 4n \int_\Omega f(x)^2 \mu(\mathrm{d}x) < \infty \tag{A.7}$$

by the inequality $(a-b)^2 \le 2(a^2+b^2)$ and the symmetry of K. Let $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E}^*)$ be the closure of $\mathcal{D}(L_0)$ with respect to $\|\cdot\|_1$.

For explosive processes the operator L_0 can have several self-adjoint extensions. The next proposition says that the generator L of the minimal jump process X is the *Friedrichs extension* of L_0. For discrete state spaces, this result was shown by Silverstein [24], [23].

Proposition A.1. (Chen [10, Theorem 3.6]) *The following hold:*
(1) $\mathcal{E}$ *with domain* $\mathcal{D}(\mathcal{E})$ *is a closed quadratic form in* $L^2(\Omega,\mu)$*.*
(2) *There is a unique self-adjoint operator* L *with domain* $\mathcal{D}(L) \subset \mathcal{D}(\mathcal{E})$ *such that* $\mathcal{E}(f) = \int_\Omega f(-Lf)\mathrm{d}\mu$ *for all* $f \in \mathcal{D}(L)$*. The operator* L *is an extension of* L_0*.*
(3) *Let* $(Q_t)_{t\ge 0}$ *be the minimal solution of the backward Kolmogorov equation in* (A.3)*. For all* $t \ge 0$ *and all* $f \in L^2(\Omega,\mu)$*,*

$$(Q_t f)(x) = \big(e^{tL} f\big)(x) \quad \mu\text{-a.e.} \tag{A.8}$$

Part (3) shows that $(Q_t)_{t\ge 0}$ is self-adjoint in $L^2(\Omega,\mu)$:

Corollary A.2. *The measure* μ *is a reversible measure for the minimal jump process* X*.*

Finally, we cite the two conditions for non-explosion that we will need later.

Proposition A.3. (Chen [10, Corollary 3.7]) *The following are equivalent:*
(1) $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\mathcal{E}^*)$*, i.e.,* $\mathcal{D}(L_0)$ *is dense in* $\mathcal{D}(\mathcal{E}^*)$ *with respect to* $\|\cdot\|_1$*.*
(2) L_0 *has no self-adjoint extensions other than the operator* L *defined in Proposition* A.1*.*
(3) X *is non-explosive:* $\mathbb{P}_x(\zeta < \infty) = 0$ *for* μ*-a.a.* $x \in \Omega$*, with* ζ *denoting the lifetime.*

Proposition A.4. (Chen [10, Corollary 3.8]) *If* $K(\Omega\times\Omega) = \int_\Omega \lambda(x)\mu(\mathrm{d}x) < \infty$*, then* X *is non-explosive in the sense of Proposition* A.3*.*

The non-explosion criterion of Proposition A.4 amounts to *positive recurrence* of the underlying jump chain, which will appear in the analysis of the Dirichlet problem given below. Indeed, let $Z = (Z_n)_{n\in\mathbb{N}_0}$ be the *jump chain* associated with X, i.e., $Z_0 = X_0$ and $Z_n = X_{t_n+}$ with t_n+ the time right after the (random) time t_n of the n-th jump of X. If $\lambda(y) = 0$ for some y then it is possible that X makes only finitely a finite number N of jumps, in which case we set $Z_m = Z_N$, $m \ge N$. With this convention Z is a Markov chain with probability transition kernel

$$p(x,\mathrm{d}y) = \begin{cases} \lambda(x)^{-1} k(x,\mathrm{d}y), & \lambda(x) > 0,\\ \delta_x(\mathrm{d}y), & \lambda(x) = 0, \end{cases} \tag{A.9}$$

We have $k(x,\mathrm{d}y) = \lambda(x)p(x,\mathrm{d}y)$, and so $\nu(\mathrm{d}x) = \lambda(x)\mu(\mathrm{d}x)$ is a reversible measure for the jump chain Z. The condition $K(\Omega\times\Omega) < \infty$ holds if and only ν has finite mass, which for irreducible Markov chains on discrete state spaces is equivalent to positive recurrence (Stroock [26]). More generally, if $K(\Omega\times\Omega) < \infty$, then a Poincaré-recurrence-type argument shows Z, and hence also X, visits every set of positive measure infinitely often.

A.2. **Dirichlet problem.** We say that $f\colon \Omega \to \mathbb{R}$ solves the Dirichlet problem when

$$f = 1 \text{ on } A, \quad f = 0 \text{ on } B, \quad -Lf = 0 \text{ on } \Omega\backslash(A\cup B). \tag{A.10}$$

We recall the definition of the hitting time $\tau_A = \inf\{t > 0 \mid X_t \in A\}$. Let $Z = (Z_n)_{n\in\mathbb{N}_0}$ (with the convention $\inf \emptyset = \infty$) be the jump chain associated with $X = (X_t)_{t\geq 0}$, and set $\sigma_C = \inf\{n \in \mathbb{N}_0 \mid Z_n \in C\}$ for $C \subset \Omega$.

Proposition A.5. *Let $A, B \subset \Omega$ be disjoint measurable sets. Then*

$$g_{AB}(x) = \mathbb{P}_x(\tau_A < \zeta,\, \tau_A < \tau_B) \tag{A.11}$$

is the minimal non-negative solution of the Dirichlet problem. If $\mathbb{P}_x(\tau_{A\cup B} < \infty) = 1$ for μ-a.a. $x \in \Omega$, then g_{AB} is the unique bounded solution of the Dirichlet problem (up to μ-null sets).

Proof. First, note that $g_{AB}(x) = 1$ for all $x \in A$ and $g_{AB}(x) = 0$ for $x \in B$. This is because the jump rates $\lambda(x)$ are finite, so that $\mathbb{P}_x$-a.s. there is an $\varepsilon > 0$ such that $X_t = x$ for all $t \in [0, \varepsilon)$. Next, note that g_{AB} can be written in terms of the jump chain as

$$g_{AB}(x) = \mathbb{P}_x(\sigma_A < \infty,\, \sigma_A < \sigma_B). \tag{A.12}$$

The reader may check that $\{\tau_A < \zeta\} = \{\sigma_A < \infty\}$: if X explodes before hitting A or has infinite life-time and never hits A, then Z makes infinitely many jumps without hitting A, and vice-versa. For $x \in \Omega\backslash(A \cup B)$,

$$g_{AB}(x) = \mathbb{P}_x(Z_1 \in A) + \int_{\Omega\backslash A} p(x, \mathrm{d}y)\, \mathbb{P}_y(\sigma_A < \infty,\, \sigma_A < \sigma_B) = \big(pg_{AB}\big)(x). \tag{A.13}$$

Multiplying with $\lambda(x)$, we get

$$\begin{aligned} 0 = \lambda(x)\big[g_{AB}(x) - (pg_{AB})(x)\big] &= \int_\Omega \lambda(x)\big[g_{AB}(x) - g_{AB}(y)\big]\, p(x, \mathrm{d}y) \\ &= \int_\Omega \big[g_{AB}(x) - g_{AB}(y)\big]\, k(x, \mathrm{d}y) = (-Lg_{AB})(x) = 0, \qquad x \in \Omega\backslash(A \cup B). \end{aligned} \tag{A.14}$$

The proof that g_{AB} is the minimal solution is analogous to the proof for discrete state spaces (see e.g. Norris [21, Chapter 4.2]). Indeed, let $h \geq 0$ be another solution of the Dirichlet problem and $x \in \Omega\backslash(A \cup B)$. Induction on n shows that for all $n \in \mathbb{N}_0$,

$$h(x) = \sum_{k=0}^{n} \mathbb{P}_x(\sigma_A = k, \sigma_B \geq k+1) + r_n(x) \tag{A.15}$$

with

$$r_n(x) = \int_{[(A\cup B)^c]^n \times \Omega} p(x, \mathrm{d}y_1) p(y_1, \mathrm{d}y_2) \cdots p(y_n, \mathrm{d}y_{n+1}) h(y_{n+1}). \tag{A.16}$$

We estimate $r_n(x) \geq 0$, let $n \to \infty$, and obtain $h(x) \geq g_{AB}(x)$. Hence g_{AB} is the minimal non-negative solution.

Suppose that $\mathbb{P}_x(\sigma_{A\cup B} < \infty) = \infty$, and let h be a bounded solution of the Dirichlet problem. Then (A.15) holds and

$$|r_n(x)| \leq \|h(x)\|_\infty \mathbb{P}_x\big(\sigma_{A\cup B} \geq n+1\big), \tag{A.17}$$

which tends to zero as $n \to \infty$. It follows that $h(x) = g_{AB}(x)$. □

Proposition A.5 shows that if $K(\Omega \times \Omega) < \infty$ and X is irreducible, then g_{AB} is the unique bounded solution of the Dirichlet problem. For transient X, the solution is *not* unique: for example, new solutions are obtained by adding multiples of the function $x \mapsto 1 - \mathbb{P}_x(\tau_{A\cup B} < \zeta)$ (see also (A.24) below).

For non-explosive recurrent X, g_{AB} is exactly the function h_{AB} defined in (1.6), and is equal to the unique minimizer in the Dirichlet principle. For transient X, g_{AB} is no longer the minimizer (see Lemma A.7 below).

A.3. **Dirichlet principle.** In this section we prove the Dirichlet principle. For $K(\Omega \times \Omega) < \infty$, i.e., positive recurrent X, the proof is simple and analogous to proofs for finite state spaces. A key role is played by the Green identity

$$\begin{aligned}\tfrac{1}{2}\int_{\Omega\times\Omega} &\big[f(x)-f(y)\big]\big[g(x)-g(y)\big]K(\mathrm{d}x,\mathrm{d}y)\\ &= \int_\Omega f(x)\big[g(x)-g(y)\big]\mu(\mathrm{d}x)k(x,\mathrm{d}y) \qquad \text{(A.18)}\\ &= \int_\Omega f(x)\big(-Lg\big)(x)\mu(\mathrm{d}x),\end{aligned}$$

which holds for all bounded f and g when K is finite. When K is infinite, i.e., when Z associated with X is null-recurrent or transient, the integrals in the (A.18) need not be absolutely convergent and the identity may fail. (A formal analogue is the identity $\int_{-\infty}^{\infty} f'(x)g'(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)[-g''(x)]\mathrm{d}x$: the integration by parts works only if there are no boundary terms from infinity.) Using Cauchy-Schwarz, we see that the condition $\int_\Omega f(x)^2\lambda(x)\mu(\mathrm{d}x) < \infty$ is sufficient to ensure that the Green identity stays true. But in general the minimizer of the Dirichlet principle need not satisfy this condition and therefore we need to proceed with caution.

Our strategy is as follows. First we show that the variational formula in (1.8) has a unique minimizer h_{AB} (Lemma A.8 below). Next we check that the minimizer solves the Dirichlet problem (Lemma A.7 below), and that the minimum is the limit of "truncated" minima (Lemma A.8 below). Finally we show that $\mathcal{E}(h_{AB}) = \int_A \big(-Lh_{AB}\big)\mathrm{d}\mu$ (Lemma A.9 below).

Recall the Dirichlet principle in (1.8), with the definition of $\mathcal{V}_{AB}$ in (1.9). We assume that $\mathcal{V}_{AB}$ is non-empty or, equivalently, $\mathrm{cap}\,(A,B) < \infty$.

Lemma A.6. *The Dirichlet form restricted to $\mathcal{V}_{AB}$ has a unique minimizer h_{AB} i.e., there is a unique (up to μ-null sets) $h_{AB} \in \mathcal{V}_{AB}$ such that* $\mathrm{cap}\,(A,B) = \mathcal{E}(h_{AB})$.

Proof. The lemma is proved via standard convexity, lower semi-continuity and compactness arguments. Note that $\mathcal{V}_{AB}$ is a convex set. Since $\phi \mapsto \phi^2$ is strictly convex, we have that $\mathcal{E}(tf+(1-t)g) \le t\mathcal{E}(f)+(1-t)\mathcal{E}(g)$ for all $f,g \in \mathcal{V}_{AB}$ and $t \in (0,1)$, and there is equality if and only if $f(x)-f(y) = g(x)-g(y)$ for μ-a.a. $x,y \in \Omega$. The restriction $f|_A = 0 = g|_A$ and the irreducibility of X therefore imply that $f = g$ almost everywhere. Hence $\mathcal{E}$ is strictly convex on $\mathcal{V}_{AB}$.

Now let $(f_n)_{n\in\mathbb{N}}$ be a minimizing sequence of functions in $\mathcal{V}_{AB}$ for $\mathcal{E}$. Then $(x,y) \mapsto (f_n(x)-f_n(y))_{n\in\mathbb{N}}$ defines a sequence of functions in $L^2(\Omega\times\Omega,K)$ that is bounded in L^2-norm. The Banach-Alaoglu theorem ensures the existence of a subsequence $(x,y) \mapsto (f_{n_j}(x)-f_{n_j}(y))_{j\in\mathbb{N}}$ that converges weakly in $L^2(\Omega\times\Omega,K)$, i.e., there is a function $H \in L^2(\Omega\times\Omega,K)$ such that

$$\lim_{j\to\infty}\int_{\Omega\times\Omega}\big[f_{n_j}(x)-f_{n_j}(y)\big]G(x,y)K(\mathrm{d}x,\mathrm{d}y) \qquad \text{(A.19)}$$

$$= \int_{\Omega\times\Omega} H(x,y)G(x,y)K(\mathrm{d}x,\mathrm{d}y) \qquad \forall\, G \in L^2(\Omega\times\Omega,K). \qquad \text{(A.20)}$$

The limit function H inherits the following properties (all statements are up to K-null sets):

- $H(x,y)=1$ on $A\times B$.
- $H(x,y)=0$ on $A\times A$ and $B\times B$.
- $H(x,y)\geq 0$ on $A\times\Omega$ and $\Omega\times B$.
- $H(x,y)+H(y,z)=H(x,z)$ almost everywhere.

Let $b\in B$ be an arbitrary reference point, and set $h(x)=H(x,b)$. Because of the above properties and $\mu(B)>0$, we can choose b such that the following hold: $h=1$ on A, $h=0$ on B, $0\leq h\leq 1$, and $H(x,y)=h(x)-h(y)$ almost everywhere, i.e., $h\in\mathcal{V}_{AB}$. Since the L^2-norm is lower semi-continuous with respect to weak convergence (see Lieb and Loss [18][Theorem 2.11]), we have

$$\operatorname{cap}(A,B)\leq\mathcal{E}(h)\leq\liminf_{j\to\infty}\mathcal{E}(f_{n_j})=\inf_{f\in\mathcal{V}_{AB}}\mathcal{E}(f)=\operatorname{cap}(A,B),\tag{A.21}$$

so $\mathcal{E}(h)=\operatorname{cap}(A,B)$ and h is a minimizer. Because of the strict convexity of $\mathcal{E}$, it follows that h_{AB} is the unique minimizer. ☐

Lemma A.7. (1) *The minimizer h_{AB} solves the Dirichlet problem.*
(2) *If X is recurrent, then $h_{AB}(x)=\mathbb{P}_x(\tau_A<\tau_B)$.*
(3) *If X is transient, then h_{AB} is different from the minimal solution $x\mapsto\mathbb{P}_x(\tau_A<\zeta,\,\tau_A<\tau_B)$ of the Dirichlet problem.*

Proof. (1) Suppose by contradiction that h_{AB} does not solve the Dirichlet problem. Then we can find a function $f\in L^2(\Omega,\mu)$ such that $f=0$ on $A\cup B$ and $\int_\Omega(-Lh_{AB})f\mathrm{d}\mu<0$. Set $F(x,y)=f(x)$, $x,y\in\Omega$. Then $F\in L^2(\Omega\times\Omega,K)$ Cauchy-Schwarz ensures that the integral $\int_{\Omega\times\Omega}f(x)\big[h_{AB}(x)-h_{AB}(y)\big]K(\mathrm{d}x,\mathrm{d}y)$ is absolutely convergent, and

$$\begin{aligned}\mathcal{E}(h_{AB}+\varepsilon f)&=\mathcal{E}(h_{AB})+\varepsilon\int_{\Omega\times\Omega}f(x)\big[h_{AB}(x)-h_{AB}(y)\big]K(\mathrm{d}x,\mathrm{d}y)+\varepsilon^2\mathcal{E}(f)\\&=\mathcal{E}(h_{AB})+\varepsilon\int_\Omega f(x)\big[-Lh_{AB}\big](x)\mu(\mathrm{d}x)+\varepsilon^2\mathcal{E}(f).\end{aligned}\tag{A.22}$$

Choosing ε small enough we obtain that $\mathcal{E}(h_{AB}+\varepsilon f)<\mathcal{E}(h_{AB})$, which is a contradiction.

(2) If $\mathbb{P}_x(\sigma_{A\cup B}<\infty)=1$ for μ-a.a. x, then Proposition A.5 implies that the minimizer is equal to the unique solution of the Dirichlet problem $h_{AB}(x)=\mathbb{P}_x(\tau_A<\tau_B)$.

(3) The bijection $\mathcal{V}_{AB}\mapsto\mathcal{V}_{BA}$, $h\mapsto 1-h$ leaves the Dirichlet form unchanged, because $\mathcal{E}(h)=\mathcal{E}(1-h)$. It follows that $\operatorname{cap}(A,B)=\operatorname{cap}(B,A)$ and, because the minimizer is unique,

$$h_{AB}(x)=1-h_{BA}(x)\quad\text{ for }\mu\text{-a.a. }x.\tag{A.23}$$

On the other hand, the minimal solution $g_{AB}(x)=\mathbb{P}_x(\tau_A<\zeta,\ \tau_A<\tau_B)$ satisfies

$$\begin{aligned}1-g_{BA}(x)&=1-\mathbb{P}_x\big(\tau_B<\zeta,\,\tau_B<\tau_A\big)\\&=\mathbb{P}_x\big(\tau_A<\zeta,\,\tau_A<\tau_B\big)+\mathbb{P}_x\big(\tau_{A\cup B}\geq\zeta\big)\\&=g_{AB}(x)+1-\mathbb{P}_x\big(\tau_{A\cup B}<\zeta\big).\end{aligned}\tag{A.24}$$

It follows that $1 - g_{BA} \neq g_{AB}$ for transient X, and in view of (A.23) we have $g_{AB} \neq h_{AB}$. □

A.4. **Truncation approximation.** Next we show that the capacity can be obtained as the limit of certain truncated capacities. This will help us prove the Berman-Konsowa principle for transient and null recurrent X, and provide the missing step to show that $\mathrm{cap}\,(A,B) = \int_A \big(-Lh_{AB}\big)\mathrm{d}\mu$.

Throughout the sequel we assume that $K((A\cup B)\times\Omega) < \infty$. Let $(\Omega_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable subsets of Ω with $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$ such that $A\cup B \subset \Omega_n$ and $K(\Omega_n\times\Omega_n) < \infty$ for all $n\in\mathbb{N}$ [1]. Define the truncated kernel $k_n(x,\mathrm{d}y) = \mathbf{1}_{\Omega_n\times\Omega_n}(x,y)k(x,\mathrm{d}y)$, $x,y\in\Omega$, and the truncated measure K_n as

$$K_n(C\times D) = K\big((C\cap\Omega_n)\times(D\cap\Omega_n)\big) = \int_{C\times D}\mu(\mathrm{d}x)k_n(x,\mathrm{d}y), \qquad C,D\subset\Omega. \tag{A.25}$$

Set

$$\mathcal{E}_n(f) = \tfrac12\int_{\Omega\times\Omega}\big[f(x)-f(y)\big]^2K_n(\mathrm{d}x,\mathrm{d}y) = \tfrac12\int_{\Omega_n\times\Omega_n}\big[f(x)-f(y)\big]^2K(\mathrm{d}x,\mathrm{d}y). \tag{A.26}$$

The truncated kernel k_n is associated with a reversible jump process for which jumps outside Ω_n are suppressed.

Remark. The theory of Mosco convergence (see Mosco [19]) can be used to show that for non-explosive processes the truncated semi-groups and resolvents converge to those of the full process (see Barlow, Bass, Chen and Kassmann [1] for references). For our purpose, however, it will be enough to check that the truncated capacities converge.

Let $\mathrm{cap}_n(A,B)$ be the capacity of the truncated process and h^n_{AB} the corresponding minimizer.

Lemma A.8 (Convergence of truncated capacities)**.** *Suppose that $K((A\cup B)\times\Omega) < \infty$. Then the following hold:*
(1) *$n\mapsto \mathrm{cap}_n(A,B)$ is non-decreasing and converges to* $\mathrm{cap}\,(A,B)$.
(2) *$(x,y)\mapsto h^n_{AB}(x)-h^n_{AB}(y)$ converges weakly in $L^2(\Omega\times\Omega,K)$ as $n\to\infty$, i.e.,*

$$\begin{aligned}\lim_{n\to\infty}&\int_{\Omega_n\times\Omega_n}\big[h^n_{AB}(x)-h^n_{AB}(y)\big]G(x,y)K(\mathrm{d}x,\mathrm{d}y) \\ &= \int_{\Omega\times\Omega}\big[h_{AB}(x)-h_{AB}(y)\big]G(x,y)K(\mathrm{d}x,\mathrm{d}y) \qquad \forall\, G\in L^2(\Omega\times\Omega,K).\end{aligned} \tag{A.27}$$

Proof. (1) We have $\mathcal{E}_n(f)\le\mathcal{E}_{n+1}(f)\le\mathcal{E}(f)$ for all $f\in\mathcal{V}_{AB}$. Hence

$$\mathrm{cap}_n(A,B)\le\mathrm{cap}_{n+1}(A,B)\le\mathrm{cap}\,(A,B). \tag{A.28}$$

Note that $\mathcal{E}_n(h^n_{AB}) = \mathrm{cap}_n(A,B)\le\mathrm{cap}\,(A,B)<\infty$. It follows that the sequence of functions $(x,y)\mapsto(H_n(x,y))_{n\in\mathbb{N}}$ given by

$$H_n(x,y) = \big[h^n_{AB}(x)-h^n_{AB}(y)\big]\mathbf{1}_{\Omega_n}(x)\mathbf{1}_{\Omega_n}(y), \qquad x,y\in\Omega, \tag{A.29}$$

---

[1] When $A\cup B\subset\Omega_n$ fails, extend Ω_n to $\Omega_n' = \Omega_n\cup(A\cup B)$ and note that $K(\Omega_n'\times\Omega_n')<\infty$ because $K(\Omega_n\times\Omega_n)<\infty$ and $K((A\cup B)\times\Omega)<\infty$.

is bounded in $L^2(\Omega \times \Omega, K)$. As in the proof of Lemma A.6, the Banach-Alaoglu theorem and the lower semi-continuity of the L^2-norm with respect to weak convergence show that, upon passing to a subsequence, we may assume that $H_n(x,y)$ converges weakly in $L^2(\Omega \times \Omega, K)$ to $h(x) - h(y)$ for some $h \in \mathcal{V}_{AB}$, and

$$\begin{aligned}\mathrm{cap}\,(A,B) &\leq \tfrac{1}{2} \int_{\Omega\times\Omega} \big[h(x)-h(y)\big]^2 K(\mathrm{d}x,\mathrm{d}y)\\ &\leq \liminf_{n\to\infty} \tfrac{1}{2} \int_{\Omega\times\Omega} H_n(x,y)^2 K(\mathrm{d}x,\mathrm{d}y) = \lim_{n\to\infty} \mathrm{cap}\,_n(A,B). \qquad \text{(A.30)}\end{aligned}$$

Together with the inequality in (A.28), this implies that $\lim_{n\to\infty} \mathrm{cap}\,_n(A,B) = \mathrm{cap}\,(A,B)$ and $\mathcal{E}(h) = \mathrm{cap}\,(A,B)$, and hence $h = h_{AB}$.
(2) We see from (A.30) that any accumulation point of $H_n(x,y)$ equals $h_{AB}(x) - h_{AB}(y)$, which implies (A.27). □

Lemma A.9. *The equilibrium charges defined by*

$$Q_A = \int_A \big(-Lh_{AB}\big)(x)\mu(\mathrm{d}x), \quad Q_B = \int_B \big(-Lh_{AB}\big)(x)\mu(\mathrm{d}x), \qquad \text{(A.31)}$$

satisfy $Q_A = -Q_B = \mathrm{cap}\,(A,B)$.

Proof. Write

$$\begin{aligned}\mathcal{E}(h_{AB}) &= \lim_{n\to\infty} \tfrac{1}{2} \int_{\Omega_n\times\Omega_n} \big[h^n_{AB}(x) - h^n_{AB}(y)\big]\big[h_{AB}(x) - h_{AB}(y)\big] K(\mathrm{d}x,\mathrm{d}y)\\ &= \lim_{n\to\infty} \int_{\Omega_n\times\Omega_n} h_{AB}(x)\big[h^n_{AB}(x) - h^n_{AB}(y)\big] K(\mathrm{d}x,\mathrm{d}y)\\ &= \lim_{n\to\infty} \int_{\Omega_n} h_{AB}(x)\big(-L^n h^n_{AB}\big)(x)\mu(\mathrm{d}x)\\ &= \lim_{n\to\infty} \int_A \big(-L^n h^n_{AB}\big)(x)\mu(\mathrm{d}x) = \int_A (-Lh_{AB})(x)\mu(\mathrm{d}x),\end{aligned} \qquad \text{(A.32)}$$

where L^n is the generator of the truncated process on Ω_n. The first equality uses (A.27) with $G(x,y) = h_{AB}(x) - h_{AB}(y)$, the fourth equality use that h^n_{AB} and h_{AB} solve the Dirichlet problem in (A.10), the fifth equality uses (A.27) with $G(x,y) = 1_A(x)$. Thus we have shown that $\mathrm{cap}\,(A,B) = \mathcal{E}(h_{AB}) = Q_A$. For Q_B we apply (A.23) and note that

$$Q_B = \int_B \big(-Lh_{AB}\big)\mathrm{d}\mu = \int_B \big(Lh_{BA}\big)\mathrm{d}\mu = -\mathrm{cap}\,(B,A) = -\mathrm{cap}\,(A,B) \qquad \text{(A.33)}$$

(see the proof of Lemma A.7). □

A.5. **Thomson principle.** Let h_{AB} be the unique minimizer in the Dirichlet principle and Φ_{AB} the associated flow.

Proposition A.10 (Thomson principle)**.** *For $A, B \subset \Omega$ disjoint,*

$$\frac{1}{\mathrm{cap}\,(A,B)} = \min_{\Phi\in\mathcal{U}^K_{AB}} \left[\int_{\Omega\times\Omega} \frac{\mathrm{d}\Phi}{\mathrm{d}K}(x,y)\right]^2 K(\mathrm{d}x,\mathrm{d}y) \qquad \text{(A.34)}$$

and the harmonic flow Φ_{AB} in (1.14) *is the unique minimizer.*

Proof. For finite K, we have already checked in Lemma 2.3 that Φ_{AB} is a unit flow. For infinite K the proof is similar. The conditions $\Phi_{AB}(A\times\Omega)=\Phi_{AB}(\Omega\times B)=1$ follow from Lemma A.9.

Let $h_n=h^n_{AB}$ be the truncated harmonic function introduced in Section A.3. By Lemma A.8, upon passing to a subsequence, we may assume that $h_n(x)-h_n(y)\to h_{AB}(x)-h_{AB}(y)$ weakly in $L^2(\Omega\times\Omega,K)$ as $n\to\infty$. Let Φ be a unit flow, and let

$$\phi(x,y)=\frac{\mathrm{d}\Phi}{\mathrm{d}K}(x,y)-\frac{\mathrm{d}\Phi}{\mathrm{d}K}(y,x) \tag{A.35}$$

be the antisymmetrized Radon-Nikodým derivative of Φ with respect to K. Since the flow is directed, we have

$$\phi(x,y)=\begin{cases}\frac{\mathrm{d}\Phi}{\mathrm{d}K}(x,y), & (x,y)\in\chi,\\ -\frac{\mathrm{d}\Phi}{\mathrm{d}K}(y,x), & (y,x)\in\chi,\\ 0, & \text{otherwise},\end{cases} \tag{A.36}$$

for some measurable $\chi\subset\Omega\times\Omega$ chosen such that the cases are mutually exclusive (see item (4) in Def. 1.2). Note that

$$\mathcal{E}(\Phi)=\int_{\Omega\times\Omega}\left[\frac{\mathrm{d}\Phi}{\mathrm{d}K}(x,y)\right]^2K(\mathrm{d}x,\mathrm{d}y)=\frac{1}{2}\int_{\Omega\times\Omega}\phi(x,y)^2K(\mathrm{d}x,\mathrm{d}y). \tag{A.37}$$

By a slight abuse of notation we use the same letter $\mathcal{E}$ for the quadratic forms on flows and on functions. Assume that Φ has finite energy, i.e., $\phi\in L^2(\Omega\times\Omega,K)$. Let $h^n_{AB}(x)$ be as in the proof of Lemma A.8 and $h_n(x):=h^n_{AB}(x)\mathbf{1}_{\Omega_n}(x)$. We have $h_n(x)-h_n(y)\in L^2(\Omega\times\Omega,K)$ and $h_n\in L^2(\Omega,\nu)$ with $\nu(\mathrm{d}x)=\lambda(x)\mu(\mathrm{d}x)$ the marginal of K. By exploiting the symmetry of K and the anti-symmetry of ϕ, we get

$$\begin{aligned}\tfrac{1}{2}\int_{\Omega\times\Omega}\big[h_n(x)-h_n(y)\big]\phi(x,y)K(\mathrm{d}x,\mathrm{d}y)&=\int_{\Omega\times\Omega}h_n(x)\phi(x,y)K(\mathrm{d}x,\mathrm{d}y)\\&=\int_{\Omega\times\Omega}h_n(x)M_\Phi(\mathrm{d}x),\end{aligned} \tag{A.38}$$

where $M_\Phi(\mathrm{d}x)$ is the anti-symmetrized marginal, i.e., $M_\Phi(C)=\Phi(C\times\Omega)-\Phi(\Omega\times C)$, $C\subset\Omega$ measurable. Since Φ is a unit flow, we have $M_\Phi(C)=0$ for all $C\subset\Omega\backslash(A\cup B)$ and $M_\Phi(A)=1$. It follows that

$$\int_{\Omega\times\Omega}h_n(x)M_\Phi(\mathrm{d}x)=M_\Phi(A)=1 \tag{A.39}$$

and, taking the limit $n\to\infty$ in (A.38), we obtain

$$\tfrac{1}{2}\int_{\Omega\times\Omega}\big[h_{AB}(x)-h_{AB}(y)\big]\phi(x,y)K(\mathrm{d}x,\mathrm{d}y)=1. \tag{A.40}$$

The Cauchy-Schwarz inequality yields

$$\Big[\mathrm{cap}\,(A,B)\Big]^{1/2}\Big[\mathcal{E}(\Phi)\Big]^{1/2}\geq 1 \tag{A.41}$$

with equality if and only if $\phi(x,y)=c(h_{AB}(x)-h_{AB}(y))$ for some $c>0$ and K-a.a. (x,y). The unit flow condition fixes the constant as $c=1/\mathrm{cap}\,(A,B)$, and so

$$\frac{\mathrm{d}\Phi}{\mathrm{d}K}(x,y)=\phi(x,y)_+-\phi(x,y)_-=\frac{1}{\mathrm{cap}\,(A,B)}\big[h_{AB}(x)-h_{AB}(y)\big]_+. \tag{A.42}$$

Thus, we have found that $\mathcal{E}(\Phi) \geq 1/\mathrm{cap}\,(A,B)$ for all unit flows, with equality if and only if $\Phi = \Phi_{AB}$. $\square$

Appendix B. Physical interpretation of variational principles

The Dirichlet principle and the Thomson principle have well-known physical interpretations in terms of electric networks (see e.g. Doyle and Snell [12], Peres [22], Gaudillière [16]). In Section B.1 we recall these interpretations, while in Section B.2 we give an interpretation of the Berman-Konsowa principle that is based on three ingredients: (1) resistances in series add up; (2) conductances in parallel add up; (3) each network has an equivalent network that is richer but simpler, consisting of chains of resistors in parallel. The latter interpretation, which was suggested in Bovier [4], is worked out in detail.

B.1. **Dirichlet and Thomson.** For the sake of exposition we assume that Ω is *finite*, and write K_{xy} for $K(\{(x,y)\})$. Let G be the undirected graph with vertex set Ω and edge set $E' = \{\{x,y\} \in \Omega \times \Omega \mid K_{xy} > 0\}$. For later purpose, we write E for the corresponding set of directed edges, i.e., $E = \{(x,y) \in \Omega \mid K_{xy} > 0\}$. The network consists of a set of resistors: each edge $\{x,y\}$ of G has resistance $R_{xy} = 1/K_{xy}$. Recall *Ohm's law*: voltages $(V_x)_{x\in\Omega}$ induce currents $(i_{xy})_{(x,y)\in E}$. The current flows from high to low voltage, and the intensity of the current is $K_{xy}|V_x - V_y|$. We adopt the convention that $i_{xy} = 0$ when $V_x < V_y$, so that $i_{xy} = K_{xy}(V_x - V_y)_+$.[2] Furthermore, by the *Joule effect*, the current through the resistor network dissipates energy at a rate

$$\mathcal{P} = \sum_{(x,y)\in E} i_{xy}\big(V_x - V_y\big)_+ = \tfrac{1}{2} \sum_{x,y\in\Omega} K_{xy}\big(V_x - V_y\big)^2 = \sum_{(x,y)\in E} R_{xy} i_{xy}^2, \tag{B.1}$$

which is the *electric power* (measured in Watt). The factor $\frac{1}{2}$ can be removed when we replace $(V_x - V_y)$ by $(V_x - V_y)_+$. In (B.1) we recognize the Dirichlet form $\mathcal{E}(h)$ in (1.4) applied to the function $h(x) = V_x$.

Now, let A and B be two disjoint subsets of Ω. We extend the network by adding two points to Ω: the *source* s and the *sink* s'. The edge set is enriched by connecting all points from A to s and all points from B to s'. The new edges are assigned resistance zero (infinite conductivity). As a consequence, the vertices in A and B have the same voltage as the source and sink ("wiring"). Moreover, the flow through the new edges does not dissipate energy, so that the total energy dissipation is still given by (B.1).

It is a standard problem from electrical engineering to determine the *effective resistance* R_{eff} (or effective conductance $C_{\mathrm{eff}} = 1/R_{\mathrm{eff}}$) between the source and the sink, defined as follows. Fix V_s and $V_{s'}$, the voltages of the source and the sink, assume $V_s > V_{s'}$, and set $U = V_s - V_{s'}$. Let

$$\big((V_x)_{x\in\Omega}, (i_{xy})_{\{x,y\}\in E}\big) \tag{B.2}$$

solve the following set of equations:

- Wiring to source and sink: $V_a = V_s$ for all $a \in A$, $V_b = V_{s'}$ for all $b \in B$.
- Ohm's law: $i_{xy} = K_{xy}(V_x - V_y)_+$ for all $\{x,y\} \in E$.

[2] Another convention is to take $i_{xy} < 0$ when $V_x < V_y$, so that the sign of i carries the information of the direction of the current.

- Kirchhoff's law:

$$i_{sz} + \sum_{\substack{x\in\Omega:\\ \{x,z\}\in E}} i_{xz} = i_{zs'} + \sum_{\substack{y\in\Omega:\\ \{z,y\}\in E}} i_{zy} \qquad \forall\, z \in \Omega. \tag{B.3}$$

For most networks there will be a unique solution, which satisfies $V_{s'} \leq V_x \leq V_s$ for all $x \in \Omega$. Ohm's law implies that nothing flows into the source (or into A) and nothing flows out of the sink (or out of B). The total flow out of the source is

$$I = \sum_{a\in A} i_{sa} = \sum_{a\in A}\sum_{y\in\Omega} i_{ay}. \tag{B.4}$$

On finite networks, Kirchhoff's law implies that the current out of the source equals the current into the sink, i.e.,

$$I = \sum_{b\in B} i_{bs'} = \sum_{b\in B}\sum_{x\in\Omega} i_{xb}. \tag{B.5}$$

The effective resistance is

$$R_{\mathrm{eff}} = \frac{U}{I} = \frac{V_s - V_{s'}}{I}, \tag{B.6}$$

which depends on the sets A and B, and on the conductances K_{xy}, but not on the voltages V_s, $V_{s'}$. Therefore we can evaluate the effective resistance for $V_s = 1$, $V_{s'} = 0$, in which case the voltage distribution $(V_x)_{x\in\Omega}$ is equal to the harmonic function $h_{AB}(x)$ in (1.6), and the "capacity" in (1.7) is equal to

$$\mathrm{cap}\,(A,B) = -\sum_{a\in A}\sum_{y\in\Omega} K_{ay}(V_y - V_a) = \sum_{a\in A}\sum_{y\in\Omega} i_{ay} = I. \tag{B.7}$$

Comparing this expression with (B.6) and remembering our choice $U = V_s - V_{s'} = 1 - 0 = 1$, we see that

$$\mathrm{cap}\,(A,B) = \frac{I}{U} = \frac{1}{R_{\mathrm{eff}}} = C_{\mathrm{eff}}, \tag{B.8}$$

i.e., our mathematical capacity is equal to the effective conductance of the network.

Remark. The use of the word "capacity", though legitimate from a mathematical point of view, is not quite appropriate in the physical context of electric networks. Indeed, the current-voltage relation of a capacitor is $I(t) = C\frac{\mathrm{d}U}{\mathrm{d}t}(t)$ with $C > 0$ the capacity: this is clearly inconsistent with our relation $I = C_{\mathrm{eff}}U$. The word "capacity" becomes legitimate when the Dirichlet form is interpreted as an electrostatic energy rather than a dissipation rate – but in this context the "conductances" must be replaced by "dielectric permittivities". The probabilist vocabulary is a hybrid of two distinct physical pictures, and the physicist reader must be aware of this potential source of confusion.

The electric power in (B.1), evaluated for the voltage and current distribution solving the set of equations described above, equals

$$\mathcal{P} = UI = C_{\mathrm{eff}}U^2 = R_{\mathrm{eff}}I^2. \tag{B.9}$$

The Dirichlet principle says that minimization of the electric power over all voltage distributions with net voltage $U = V_s - V_{s'}$ yields the effective conductance, while the Thompson principle says that minimization of the power over all current distributions with net current $I = 1$ yields the effective resistance.

B.2. **Berman-Konsowa.** The interpretation of the Berman-Konsowa principle is more involved. We start with two simple examples.

The first example consists of a finite chain of resistors *in series*, e.g., $\Omega = \{0, 1, \dots, N\}$, $A = \{0\}$, $B = \{N\}$, $K_{xy} > 0$ for $|x - y| = 1$ and $K_{xy} = 0$ otherwise. Resistances in series add up, hence $R_{\text{eff}} = \sum_{j=1}^{N} R_{j-1,j}$ or, equivalently,

$$C_{\text{eff}} = \left(\sum_{j=1}^{N} \frac{1}{K_{j-1,j}} \right)^{-1}. \tag{B.10}$$

We recognize the right-hand side of (1.12) from Theorem 1.1 evaluated for the deterministic path $\gamma = (0, 1, \dots, N)$.

The second example consists of resistors *in parallel.* Suppose that $\Omega = A \cup B$, so that we may think of the network as a set of resistors R_{ab} in parallel between the source and the sink. Conductances in parallel add up, hence

$$C_{\text{eff}} = \sum_{a \in A} \sum_{b \in B} K_{ab}. \tag{B.11}$$

On the other hand, let $(p_{ab})_{a \in A, b \in B}$ be such that $\sum_{a \in A, b \in B} p_{ab} = 1$ and $p_{ab} > 0$ for all $a \in A$, $b \in B$. The weights p_{ab} determine a probability measure on "paths" of length 1 from A to B. The right-hand side of (1.12) equals

$$\sum_{a \in A,\, b \in B} p_{ab} \Big(\frac{p_{ab}}{K_{ab}} \Big)^{-1} = \sum_{a \in A,\, b \in B} K_{ab}. \tag{B.12}$$

Hence the Berman-Konsowa principle reproduces the additivity of resistances (in series) or conductances (in parallel). Now consider an arbitrary finite network and fix a probability measure $\mathbb{P}$ on Γ_{AB}, the set of finite paths from A to B. Let $\Phi_{\mathbb{P}}$ be the associated flow. We construct a new network as follows:

(1) With each path $\gamma \in \Gamma$, associate a chain of resistors in series with reduced resistances $R^{\gamma}_{xy} = R_{xy} \Phi_{\mathbb{P}}(x, y)/\mathbb{P}(\gamma)$, $(x, y) \in \gamma$. The effective conductance of this chain is

$$C^{\mathbb{P}}_{\text{eff}}(\gamma) = \left(\sum_{(x,y) \in \gamma} R^{\gamma}_{xy} \right)^{-1} = \mathbb{P}(\gamma) \left(\sum_{(x,y) \in \gamma} \frac{\Phi_{\mathbb{P}}(x, y)}{K_{xy}} \right)^{-1}. \tag{B.13}$$

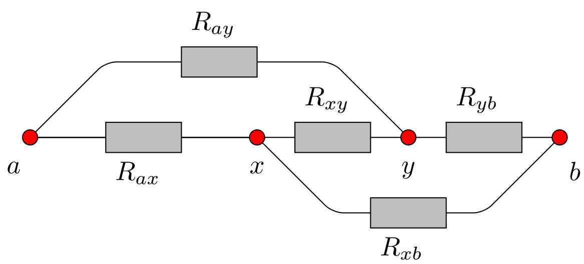
(2) Put the chains in parallel, so that they do not overlap. As a consequence, the equivalent network in general has more vertices than the original network (Figure 1). The total conductance is

$$C^{\mathbb{P}}_{\text{eff}} = \sum_{\gamma \in \Gamma} C^{\mathbb{P}}_{\text{eff}}(\gamma). \tag{B.14}$$

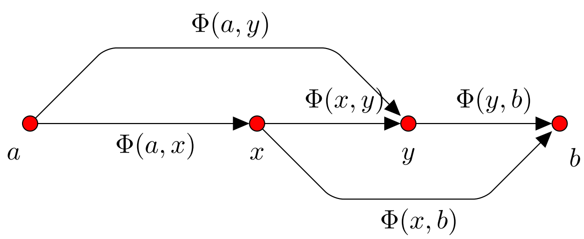
The Berman-Konsowa principle says that

$$C_{\text{eff}} = \max_{\mathbb{P}} C^{\mathbb{P}}_{\text{eff}}. \tag{B.15}$$

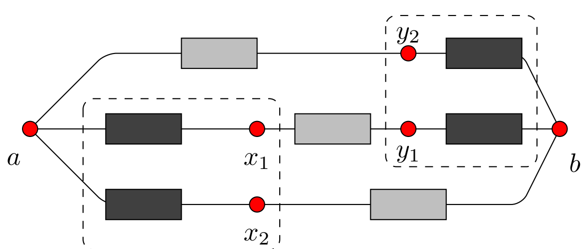
Hence the effective conductance of the original network equals that of a simpler equivalent network obtained by putting chains in parallel. The factors $\Phi_{\mathbb{P}}(x, y)/\mathbb{P}(\gamma)$ in the reduced resistances R^{γ}_{xy} compensate for the splitting of one edge (resistor) in the original network into multiple edges (resistors) of the equivalent network.



(a) The original network has four vertices a, x, y, b. Each edge $e = (ij)$ is associated with a resistance R_{ij}.



(b) An ab-unit flow $\Phi(i, j)$ determines a stochastic matrix $\ell(i, j)$ and a probability measure $\mathbb{P}$ on paths from a to b. E.g., $\ell(a, x) = \Phi(a, x)$, $\ell(x, b) = \Phi(x, b)/M(x)$ with $M(x) = \Phi(a, x)$, and $\mathbb{P}(a \to x \to b) = \ell(a, x)\ell(x, b)$.



(c) The new network consists of parallel chains of resistances; each chain comes from a path γ allowed under $\mathbb{P}$. An edge or vertex visited by more than one path is split accordingly (dashed boxes).

FIGURE 1. Electric network constructed out of a unit flow: (a) the original network, (b) a unit flow, (c) the associated new network.

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