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Abstract

In this paper we study a model describing a copolymer in a micro-emulsion. The copolymer consists of a random concatenation of hydrophobic and hydrophilic monomers, the micro-emulsion consists of large blocks of oil and water arranged in a percolation-type fashion. The interaction Hamiltonian assigns energy $-\alpha$ to hydrophobic monomers in oil and energy $-\beta$ to hydrophilic monomers in water, where α, β are parameters that without loss of generality are taken to lie in the cone $\{(\alpha, \beta) \in \mathbb{R}^2 \colon \alpha \geq |\beta|\}$. Depending on the values of these parameters, the copolymer either stays close to the oil-water interface (localization) or wanders off into the oil and/or the water (delocalization). We derive two variational formulas for the quenched free energy per monomer, one that is "column-based" and one that is "slope-based". Using these variational formulas we identify the phase diagram in the (α, β)-cone. There are two regimes: *supercritical* (the oil blocks percolate) and *subcritical* (the oil blocks do not percolate). The supercritical and the subcritical phase diagram each have two localized phases and two delocalized phases, separated by four critical curves meeting at a quadruple critical point. The different phases correspond to the different ways in which the copolymer can move through the micro-emulsion.

AMS 2000 *subject classifications.* 60F10, 60K37, 82B27.
Key words and phrases. Random copolymer, random micro-emulsion, free energy, percolation, variational formula, large deviations, concentration of measure.
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Remark: The part of this paper dealing with the "column-based" variational formula for the free energy has appeared as a preprint on the mathematics archive: arXiv:1204.1234.

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0 Outline

In Section 1, we introduce the model and present a variational formula for the quenched free energy per monomer, which we refer to as the *slope-based variational formula*, involving the fractions of time the copolymer moves at a given slope in the interior of the two solvents and the fraction of time it moves along the interfaces between the two solvents. This variational formula is the corner stone of our analysis. In Section 2, we identify the phase diagram. There are two regimes: *supercritical* (the oil blocks percolate) and *subcritical* (the oil blocks do not percolate). We obtain the *general structure* of the phase diagram, and state a number of properties that exhibit the *fine structure* of the phase diagram as well. The latter come in the form of theorems, hypotheses and conjectures.

In Section 3, we give a precise definition of the various ingredients that are necessary to state the slope-based variational formula, including various auxiliary quantities that are needed for its proof. Among these is the quenched free energy per monomer of the copolymer crossing a block column of a given type, whose existence and variational characterization are given in Section 4. In Section 5, we derive an auxiliary variational formula for the quenched free energy per monomer, which we refer to as the *column-based variational formula*, involving both the free energy per monomer and the fraction of time spent inside single columns of a given type, summed over the possible types. In Section 6, we use the column-based variational formula to prove the slope-based variational formula. In Section 7 we use the slope-based variational formula to prove our results for the phase diagram.

Appendices A–G collect several technical results that are needed along the way.

For more background on random polymers with disorder we refer the reader to the monographs by Giacomin [2] and den Hollander [4], and to the overview paper by Caravenna, den Hollander and Pétrélis [1].

1 Model and slope-based variational formula

In Section 1.1 we define the model, In Section 1.2 we state the slope-based variational formula.

1.1 Model

To build our model, we distinguish between three scales: (1) the *microscopic* scale associated with the size of the monomers in the copolymer (= 1, by convention); (2) the *mesoscopic* scale associated with the size of the droplets in the micro-emulsion ($L_n \gg 1$); (3) the *macroscopic* scale associated with the size of the copolymer ($n \gg L_n$).

Copolymer configurations. Pick $n \in \mathbb{N}$ and let $\mathcal{W}_n$ be the set of n-step *directed self-avoiding paths* starting at the origin and being allowed to move *upwards, downwards and to the right*, i.e.,

$$\begin{aligned}\mathcal{W}_n = \big\{\pi = (\pi_i)_{i=0}^n \in (\mathbb{N}_0 \times \mathbb{Z})^{n+1}\colon\ \pi_0 = (0,1),\\ \pi_{i+1} - \pi_i \in \{(1,0),(0,1),(0,-1)\}\ \forall\, 0 \leq i < n,\ \pi_i \neq \pi_j\ \forall\, 0 \leq i < j \leq n\big\}.\end{aligned} \tag{1.1}$$

The copolymer is associated with the path π. The i-th monomer is associated with the bond (π_{i-1}, π_i). The starting point π_0 is chosen to be $(0,1)$ for convenience.

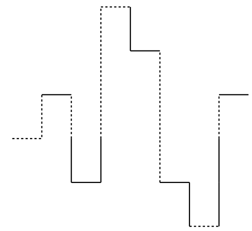

Figure 1: Microscopic disorder ω in the copolymer. Dashed bonds represent monomers of type A (hydrophobic), drawn bonds represent monomers of type B (hydrophilic).

Microscopic disorder in the copolymer. Each monomer is randomly labelled A (hydrophobic) or B (hydrophilic), with probability $\frac{1}{2}$ each, independently for different monomers. The resulting labelling is denoted by

$$\omega = \{\omega_i\colon\ i \in \mathbb{N}\} \in \{A, B\}^{\mathbb{N}} \tag{1.2}$$

and represents the *randomness of the copolymer*, i.e., $\omega_i = A$ and $\omega_i = B$ mean that the i-th monomer is of type A, respectively, of type B (see Fig. 1). We denote by $\mathbb{P}_\omega$ the law of the microscopic disorder.

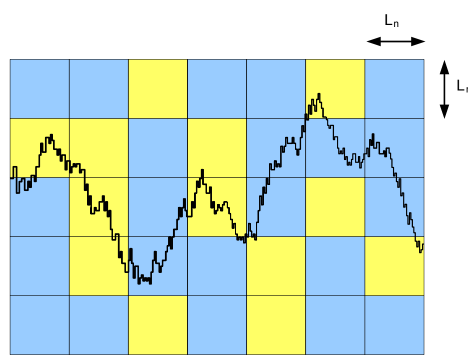

Figure 2: Mesoscopic disorder Ω in the micro-emulsion. Light shaded blocks represent droplets of type A (oil), dark shaded blocks represent droplets of type B (water). Drawn is also the copolymer, but without an indication of the microscopic disorder ω that is attached to it.

Mesoscopic disorder in the micro-emulsion. Fix $p \in (0,1)$ and $L_n \in \mathbb{N}$. Partition $(0,\infty) \times \mathbb{R}$ into square blocks of size L_n:

$$(0,\infty) \times \mathbb{R} = \bigcup_{x \in \mathbb{N}_0 \times \mathbb{Z}} \Lambda_{L_n}(x), \qquad \Lambda_{L_n}(x) = xL_n + (0, L_n]^2. \tag{1.3}$$

Each block is randomly labelled A (oil) or B (water), with probability p, respectively, $1-p$, independently for different blocks. The resulting labelling is denoted by

$$\Omega = \{\Omega(x)\colon\ x \in \mathbb{N}_0 \times \mathbb{Z}\} \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}} \tag{1.4}$$

and represents the *randomness of the micro-emulsion*, i.e., $\Omega(x) = A$ and $\Omega(x) = B$ mean that the x-th block is of type A, respectively, of type B (see Fig. 2). The law of the mesoscopic disorder is denoted by $\mathbb{P}_\Omega$ and is independent of $\mathbb{P}_\omega$. The size of the blocks L_n is assumed to be non-decreasing and to satisfy

$$\lim_{n\to\infty} L_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\log n}{n} L_n = 0, \tag{1.5}$$

i.e., the blocks are large compared to the monomer size but small compared to the copolymer size. For convenience we assume that if an A-block and a B-block are next to each other, then the interface belongs to the A-block.

Path restriction. We bound the vertical displacement on the block scale in each column of blocks by $M \in \mathbb{N}$. The value of M will be *arbitrary but fixed.* In other words, instead of considering the full set of trajectories $\mathcal{W}_n$, we consider only trajectories that exit a column through a block at most M above or M below the block where the column was entered (see Fig. 3). Formally, we partition $(0,\infty)\times\mathbb{R}$ into columns of blocks of width L_n, i.e.,

$$(0,\infty)\times\mathbb{R} = \cup_{j\in\mathbb{N}_0}\mathcal{C}_{j,L_n}, \qquad \mathcal{C}_{j,L_n} = \cup_{k\in\mathbb{Z}}\Lambda_{L_n}(j,k), \tag{1.6}$$

where C_{j,L_n} is the j-th column. For each $\pi\in\mathcal{W}_n$, we let τ_j be the time at which π leaves the $(j-1)$-th column and enters the j-th column, i.e.,

$$\tau_j = \sup\{i\in\mathbb{N}_0\colon\ \pi_i\in\mathcal{C}_{j-1,n}\} = \inf\{i\in\mathbb{N}_0\colon\ \pi_i\in\mathcal{C}_{j,n}\}-1, \qquad j=1,\dots,N_\pi-1, \tag{1.7}$$

where N_π indicates how many columns have been visited by π. Finally, we let $v_{-1}(\pi)=0$ and, for $j\in\{0,\dots,N_\pi-1\}$, we let $v_j(\pi)\in\mathbb{Z}$ be such that the block containing the last step of the copolymer in $\mathcal{C}_{j,n}$ is labelled by $(j,v_j(\pi))$, i.e., $(\pi_{\tau_{j+1}-1},\pi_{\tau_{j+1}})\in\Lambda_{L_N}(j,v_j(\pi))$. Thus, we restrict $\mathcal{W}_n$ to the subset $\mathcal{W}_{n,M}$ defined as

$$\mathcal{W}_{n,M} = \big\{\pi\in\mathcal{W}_n\colon\ |v_j(\pi)-v_{j-1}(\pi)|\le M\ \forall\, j\in\{0,\dots,N_\pi-1\}\big\}. \tag{1.8}$$

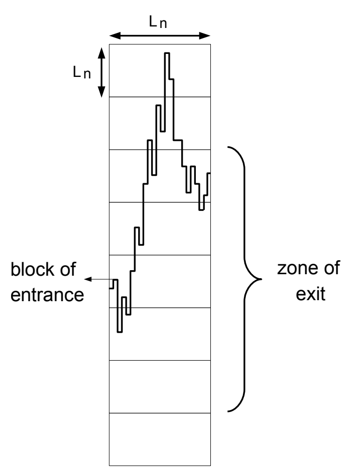

Figure 3: Example of a trajectory $\pi\in\mathcal{W}_{n,M}$ with $M=2$ crossing the column $\mathcal{C}_{0,L_n}$ with $v_0(\pi)=2$.

In Remark 1.2 below we discuss how the mesoscopic vertical restriction can be relaxed by letting $M\to\infty$.

Hamiltonian and free energy. Given ω, Ω, M and n, with each path $\pi \in \mathcal{W}_{n,M}$ we associate an *energy* given by the Hamiltonian

$$H^{\omega,\Omega}_{n,L_n}(\pi;\alpha,\beta) = \sum_{i=1}^{n} \Big(\alpha\, 1\Big\{\omega_i = \Omega^{L_n}_{(\pi_{i-1},\pi_i)} = A\Big\} + \beta\, 1\Big\{\omega_i = \Omega^{L_n}_{(\pi_{i-1},\pi_i)} = B\Big\}\Big), \tag{1.9}$$

where $\Omega^{L_n}_{(\pi_{i-1},\pi_i)}$ denotes the label of the block the step (π_{i-1},π_i) lies in. What this Hamiltonian does is count the number of AA-matches and BB-matches and assign them energy α and β, respectively, where $\alpha,\beta \in \mathbb{R}$. (Note that the interaction is assigned to bonds rather than to sites, and that we do not follow the convention of putting a minus sign in front of the Hamiltonian.) Similarly to what was done in our earlier papers [5], [6], [7], [8], without loss of generality we may restrict the interaction parameters to the cone

$$\text{CONE} = \{(\alpha,\beta) \in \mathbb{R}^2 :\ \alpha \geq |\beta|\}. \tag{1.10}$$

For $n \in \mathbb{N}$ and $M \in \mathbb{N}$, the free energy per monomer is defined as

$$f^{\omega,\Omega}_n(\alpha,\beta;M) = \tfrac{1}{n} \log Z^{\omega,\Omega}_{n,L_n}(\alpha,\beta;M) \quad \text{with} \quad Z^{\omega,\Omega}_{n,L_n}(\alpha,\beta;M) = \sum_{\pi \in \mathcal{W}_{n,M}} e^{H^{\omega,\Omega}_{n,L_n}(\pi;\alpha,\beta)}, \tag{1.11}$$

and in the limit as $n \to \infty$ the free energy per monomer is given by

$$f(\alpha,\beta;M,p) = \lim_{n\to\infty} f^{\omega,\Omega}_{n,L_n}(\alpha,\beta;M), \tag{1.12}$$

provided this limit exists ω,Ω-a.s.

Henceforth, we subtract the term $\alpha\sum_{i=1}^n 1\{\omega_i = A\}$ from the Hamiltonian, which by the law of large numbers ω-a.s. is $\frac{\alpha}{2}n(1+o(1))$ as $n\to\infty$ and corresponds to a shift of $-\frac{\alpha}{2}$ in the free energy. The latter transformation allows us to lighten the notation, starting with the Hamiltonian in (1.9), which becomes

$$H^{\omega,\Omega}_{n,L_n}(\pi;\alpha,\beta) = \sum_{i=1}^{n} \Big(\beta\, 1\{\omega_i = B\} - \alpha\, 1\{\omega_i = A\}\Big)\, 1\Big\{\Omega^{L_n}_{(\pi_{i-1},\pi_i)} = B\Big\}. \tag{1.13}$$

1.2 The slope-based variational formula for the quenched free energy per step

Theorem 1.1 below gives a variational formula for the free energy per step in (1.12). This variational formula, which is the *corner stone* of our paper, involves the fractions of time the copolymer moves at a given slope through the interior of solvents A and B and the fraction of time it moves along AB-interfaces. This variational formula will be crucial to identify the phase diagram, i.e., to identify the typical behavior of the copolymer in the micro-emulsion as a function of the parameters α,β,p. Of particular interest is the distinction between *localized phases*, where the copolymer stays close to the AB-interfaces, and *delocalized phases*, where it wanders off into the solvents A and/or B. We will see that there are several such phases.

To state Theorem 1.1 we need to introduce some further notation. With each $l \in \mathbb{R}_+ = [0,\infty)$ we associate two numbers $v_{A,l}, v_{B,l} \in [1+l,\infty)$ indicating how many steps per horizontal step the copolymer takes when traveling at slope l in solvents A and B, respectively. We further let $v_{\mathcal{I}} \in [1,\infty)$ denote the number of steps per horizontal step the copolymer takes when traveling along AB-interfaces. These numbers are gathered into the set

$$\bar{\mathcal{B}} = \{v = (v_A, v_B, v_{\mathcal{I}}) \in \mathcal{C} \times \mathcal{C} \times [1,\infty)\} \tag{1.14}$$

with

$$\mathcal{C} = \big\{ l \mapsto u_l \text{ on } \mathbb{R}_+ \colon \text{ continuous with } u_l \geq 1 + l \ \ \forall\, l \in \mathbb{R}_+ \big\}. \tag{1.15}$$

Let $\tilde{\kappa}(u,l)$ be the entropy per step carried by trajectories moving at slope l with the constraint that the total number of steps divided by the total number of horizontal steps is equal to $u \in [1+l,\infty)$ (for more details, see Section 3.1). Let $\phi_{\mathcal{I}}(u;\alpha,\beta)$ be the free energy per step when the copolymer moves along an AB-interface, with the constraint that the total number of steps divided by the total number of horizontal steps is equal to $u \in [1,\infty)$ (for more details, see Section 3.2). Let $\bar{\rho} = (\rho_A, \rho_B, \rho_{\mathcal{I}}) \in \mathcal{M}_1(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\})$, where $\bar{\rho}_A(dl)$ and $\bar{\rho}_B(dl)$ denote the fractions of horizontal steps at which the copolymer travels through solvents A and B at a slope that lies between l and $l + dl$, and $\rho_{\mathcal{I}}$ denotes the fraction of horizontal steps at which the copolymer travels along AB-interfaces. The possible $\bar{\rho}$ form a set

$$\bar{\mathcal{R}}_{p,M} \subset \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}\big) \tag{1.16}$$

that depends on p and M (for more details, see Section 3.4). With these ingredients we can now state our *slope-based variational formula.*

Theorem 1.1 [slope-based variational formula] *For every $(\alpha,\beta) \in$ CONE, $M \in \mathbb{N}$ and $p \in (0,1)$ the free energy in (1.12) exists for $\mathbb{P}$-a.e. (ω,Ω) and in $L^1(\mathbb{P})$, and is given by*

$$f(\alpha,\beta;M,p) = \sup_{\bar{\rho}\in\bar{\mathcal{R}}_{p,M}} \ \sup_{v\in\bar{\mathcal{B}}} \ \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)}, \tag{1.17}$$

where

$$\begin{aligned} \bar{N}(\bar{\rho},v) &= \int_0^\infty v_{A,l}\, \tilde{\kappa}(v_{A,l},l)\, \bar{\rho}_A(dl) + \int_0^\infty v_{B,l} \left[\tilde{\kappa}(v_{B,l},l) + \tfrac{\beta-\alpha}{2}\right] \bar{\rho}_B(dl) + v_{\mathcal{I}}\, \phi_{\mathcal{I}}(v_{\mathcal{I}};\alpha,\beta)\, \bar{\rho}_{\mathcal{I}}, \\ \bar{D}(\bar{\rho},v) &= \int_0^\infty v_{A,l}\, \bar{\rho}_A(dl) + \int_0^\infty v_{B,l}\, \bar{\rho}_B(dl) + v_{\mathcal{I}}\, \bar{\rho}_{\mathcal{I}}, \end{aligned} \tag{1.18}$$

with the convention that $\bar{N}(\bar{\rho},v)/\bar{D}(\bar{\rho},v) = -\infty$ when $\bar{D}(\bar{\rho},v) = \infty$.

Remark 1.2 We are unable to prove the existence of the quenched free energy per step $f(\alpha,\beta;p)$ of the *free model*, i.e., the model with no restriction on the mesoscopic vertical displacement. By monotonicity,

$$f(\alpha,\beta;\infty,p) = \lim_{M\to\infty} f(\alpha,\beta;M,p) = \sup_{M\in\mathbb{N}} f(\alpha,\beta;M,p) \tag{1.19}$$

exists for all α,β and p. Taking the supremum over $M \in \mathbb{N}$ on both sides of (1.17), we obtain a variational formula for $f(\alpha,\beta;\infty,p)$, namely,

$$f(\alpha,\beta;\infty,p) = \sup_{\bar{\rho}\in\bar{\mathcal{R}}_{p,\infty}} \ \sup_{v\in\bar{\mathcal{B}}} \ \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)} \tag{1.20}$$

with $\bar{\mathcal{R}}_{p,\infty} = \cup_{M\in\mathbb{N}}\bar{\mathcal{R}}_{p,M}$. Clearly, we have $f(\alpha,\beta;p) \geq f(\alpha,\beta;\infty,p)$, and we expect that equality holds. Indeed, if the inequality would be strict, then the free energy per step of the free model would be controlled by trajectories whose mesoscopic vertical displacements are unbounded. The energetic gain the copolymer may obtain from a large vertical displacement in a given column comes from the fact that it may reach a height where the mesoscopic disorder is more favorable. However, the energetic penalty associated with such a displacement is large as well (see Lemma C.6 in Appendix C). Therefore we do not expect such trajectories to be optimal, in which case $f(\alpha,\beta;p)$ is indeed given by the same variational formula as in (1.20).

1.3 Discussion

The variational formula in (1.17-1.18) is tractable, to the extent that the $\tilde{\kappa}$-function is known explicitly, the $\phi_{\mathcal{I}}$-function has been studied in depth in the literature (and much is known about it), while the set $\bar{\mathcal{B}}$ is simple. *The key difficulty of* (1.17–1.18) *resides in the set* $\bar{\mathcal{R}}_{p,M}$*, whose structure is not easy to control.* However, it turns out that we need to know *relatively little* about this set in order to identify the phase diagram.

In Appendix F we will show that the supremum in (1.17) is attained at some (not necessarily unique) $\bar{\rho} \in \bar{\mathcal{R}}_{p,M}$ and some unique $v \in \bar{\mathcal{B}}$. Each maximizer corresponds to the copolymer having a specific way to configure itself optimally within the micro-emulsion.

Column-based variational formula. The *slope-based variational formula* in Theorem 1.1 will be obtained by combining two auxiliary variational formulas. Both formulas involve the free energy per step $\psi(\Theta,u_\Theta;\alpha,\beta)$ when the copolymer crosses a block column of a given type Θ, taking values in a type space $\overline{\mathcal{V}}_M$, for a given $u_\Theta \in \mathbb{R}^+$ that indicates how many steps on scale L_n the copolymer makes in this column type. A precise definition of this free energy per block column will be given in Section 3.3.2.

The first auxiliary variational formula is stated in Section 3 (Proposition 3.5) and gives an expression for $\psi(\Theta,u_\Theta;\alpha,\beta)$ that involves the entropy $\tilde{\kappa}(\cdot,l)$ of the copolymer moving at a given slope l and the quenched free energy per monomer $\phi_{\mathcal{I}}$ of the copolymer near a *single linear interface*. Consequently, the free energy of our model with a *random geometry* is directly linked to the free energy of a model with a *non-random geometry*. This will be crucial for our analysis of the phase diagram in Section 2. The microscopic disorder manifests itself only through the free energy of the linear interface model.

The second auxiliary variational formula is stated in Section 5 (Proposition 5.1). It is referred to as the *column-based variational formula*, and provides an expression for $f(\alpha,\beta;M,p)$ by using the block-column free energies $\psi(\Theta,u_\Theta;\alpha,\beta)$ for $\Theta \in \overline{\mathcal{V}}_M$ and by weighting each column type with the frequency $\rho(d\Theta)$ at which it is visited by the copolymer. The numerator is the total free energy, the denominator is the total number of monomers (both on the mesoscopic scale). The variational formula optimizes over $(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M} \in \mathcal{B}_{\overline{\mathcal{V}}_M}$ and $\rho \in \mathcal{R}_{p,M}$. The reason why these two suprema appear in (1.17) is that, as a consequence of assumption (1.5), the *mesoscopic scale carries no entropy*: all the entropy comes from the microscopic scale, through the free energy per monomer in single columns.

Removal of the corner restriction. In our earlier papers [5], [6], [7], [8], we allowed the configurations of the copolymer to be given by the subset of $\mathcal{W}_n$ consisting of those paths that enter pairs of blocks through a common corner, exit them at one of the two corners diagonally opposite and in between stay confined to the two blocks that are seen upon entering. The latter is an *unphysical restriction* that was adopted to simplify the model. In these papers we derived a variational formula for the free energy per step that had a simpler structure. We analyzed this variational formula as a function of α, β, p and found that there are two regimes, *supercritical* and *subcritical*, depending on whether the oil blocks percolate or not along the coarse-grained self-avoiding path. In the supercritical regime the phase diagram turned out to have two phases, in the subcritical regime it turned out to have four phases, meeting at two tricritical points.

In Section 2 we show how the variational formula in Theorem 1.1 can be used to identify the phase diagram. It turns out that there are two types of phases: *localized phases* (where the copolymer spends a positive fraction of its time near the AB-interfaces) and *delocalized phases* (where it spends a zero fraction near the AB-interfaces). Which of these phases occurs depends on the parameters α, β, p. It is energetically favorable for the copolymer to stay close to the AB-interfaces, where it has the possibility of placing more than half of its monomers in their preferred solvent (by switching sides when necessary), but this comes with a loss of entropy. The competition between energy and entropy is controlled by the energy parameters α, β (determining the reward of switching sides) and by the density parameter p (determining the density of the AB-interfaces). It turns out that the phase diagram is different in the supercritical and the subcritical regimes, where the A-blocks percolate, respectively, do not percolate. The phase diagram is richer than for the model with the corner restriction.

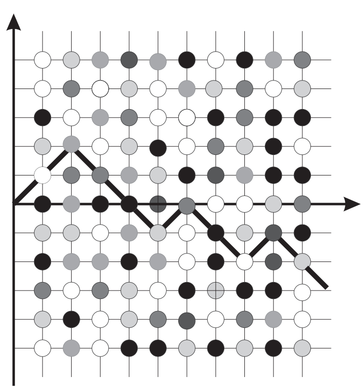

Figure 4: Picture of a directed polymer with bulk disorder. The different shades of black, grey and white represent different values of the disorder.

Comparison with the directed polymer with bulk disorder. A model of a polymer with disorder that has been studied intensively in the literature is the *directed polymer with bulk disorder*. Here, the set of paths is

$$\mathcal{W}_n = \big\{\pi = (i,\pi_i)_{i=0}^n \in (\mathbb{N}_0 \times \mathbb{Z}^d)^{n+1} \colon \pi_0 = 0,\, \|\pi_{i+1}-\pi_i\| = 1\ \forall\, 0 \le i < n\big\}, \tag{1.21}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{Z}^d$, and the Hamiltonian is

$$H_n^\omega(\pi) = \lambda \sum_{i=1}^n \omega(i,\pi_i), \tag{1.22}$$

where $\lambda > 0$ is a parameter and $\omega = \{\omega(i,x) \colon i \in \mathbb{N},\, x \in \mathbb{Z}^d\}$ is a field of i.i.d. $\mathbb{R}$-valued random variables with zero mean, unit variance and finite moment generating function, where $\mathbb{N}$ is time and $\mathbb{Z}^d$ is space (see Fig. 4). This model can be viewed as a version of a copolymer in a micro-emulsion where the droplets are of the *same* size as the monomers. For this model *no variational formula is known for the free energy*, and the analysis relies on the application of martingale techniques (for details, see e.g. den Hollander [4], Chapter 12).

In our model (which is restricted to $d = 1$ and has self-avoiding paths that may move north, south and east instead of north-east and south-east), the droplets are much larger than the monomers. This causes a *self-averaging of the microscopic disorder*, both when the copolymer moves inside one of the solvents and when it moves near an interface. Moreover, since the copolymer is much larger than the droplets, also *self-averaging of the mesoscopic disorder* occurs. This is why the free energy can be expressed in terms of a variational formula, as in Theorem 1.1. This variational formula acts as a *jumpboard* for a detailed analysis of the phase diagram. Such a detailed analysis is lacking for the directed polymer with bulk disorder.

The directed polymer in random environment has two phases: a *weak disorder phase* (where the quenched and the annealed free energy are asymptotically comparable) and a *strong disorder phase* (where the quenched free energy is asymptotically smaller than the annealed free energy). The strong disorder phase occurs in dimension $d = 1, 2$ for all $\lambda > 0$ and in dimension $d \geq 3$ for $\lambda > \lambda_c$, with $\lambda_c \in [0, \infty]$ a critical value that depends on d and on the law of the disorder. It is predicted that in the strong disorder phase the copolymer moves within a narrow corridor that carries sites with high energy (recall our convention of not putting a minus sign in front of the Hamiltonian), resulting in *superdiffusive* behavior in the spatial direction. We expect a similar behavior to occur in the localized phases of our model, where the polymer targets the AB-interfaces. It would be interesting to find out how far the coarsed-grained path in our model travels vertically as a function of n.

2 Phase diagram

In Section 2.1 we identify the *general structure* of the phase diagram. The results in this section are valid for the free energy $f(\alpha, \beta; M, p)$ with $M \in \mathbb{N} \cup \{\infty\}$, i.e., for the model where the mesoscopic vertical displacement is $\leq M$ and for the limiting model obtained by letting $M \to \infty$ (recall (1.20)), which we believe to coincide with the free model (recall Remark 1.2). In particular, we show that there is a localized phase $\mathcal{L}$ in which AB-localization occurs, and a delocalized phase $\mathcal{D}$ in which no AB-localization occurs. In Section 2.2, we focus on the free energy $f(\alpha, \beta; M, p)$ with $M \in \mathbb{N}$ of the restricted model and obtain various results for the *fine structure* of the phase diagram, both for the supercritical regime $p > p_c$ and for the subcritical regime $p < p_c$, where p_c denotes the critical threshold for directed bond percolation in the positive quandrant of $\mathbb{Z}^2$. This fine structure comes in the form of theorems, hypotheses and conjectures, which we discuss in Section 2.3. The reason why in Section 2.2 we do not consider the limiting case $M = \infty$ is that, contrary to what we find in Appendix F for the variational formula in (1.17), the supremum of the variational formula in (1.20) is not a priori attained at some $\bar{\rho} \in \bar{\mathcal{R}}_{p,\infty}$. This makes the content of the hypotheses harder to understand and harder to exploit.

2.1 General structure

Throughout this section, $M \in \mathbb{N} \cup \{\infty\}$, but we suppress the M-dependence from the notation. To state the general structure of the phase diagram, we need to define a reduced version of the free energy, called the *delocalized free energy* $f_{\mathcal{D}}$, obtained by taking into account those trajectories that, when moving along an AB-interface, are delocalized in the A-solvent. The latter amounts to replacing the linear interface free energy $\phi_{\mathcal{I}}(v_{\mathcal{I}};\alpha,\beta)$ in (1.17) by the entropic constant lower bound $\tilde{\kappa}(v_{\mathcal{I}},0)$. Thus, we define

$$f_{\mathcal{D}}(\alpha,\beta;p) = \sup_{\bar\rho\in\bar{\mathcal{R}}_p}\ \sup_{v\in\bar{\mathcal{B}}} \frac{\bar N_{\mathcal{D}}(\bar\rho,v)}{\bar D_{\mathcal{D}}(\bar\rho,v)} \tag{2.1}$$

with

$$\bar N_{\mathcal{D}}(\bar\rho,v) = \int_0^\infty v_{A,l}\,\tilde\kappa(v_{A,l},l)\,[\bar\rho_A+\bar\rho_{\mathcal{I}}\,\delta_0](dl) + \int_0^\infty v_{B,l}\,\left[\tilde\kappa(v_{B,l},l)+\tfrac{\beta-\alpha}{2}\right]\bar\rho_B(dl), \tag{2.2}$$

$$\bar D_{\mathcal{D}}(\bar\rho,v) = \int_0^\infty v_{A,l}\,[\bar\rho_A+\bar\rho_{\mathcal{I}}\,\delta_0](dl) + \int_0^\infty v_{B,l}\,\bar\rho_B(dl), \tag{2.3}$$

provided $\bar D_{\mathcal{D}}(\bar\rho,v)<\infty$. Note that $f_{\mathcal{D}}(\alpha,\beta;p)$ depends on (α,β) through $\alpha-\beta$ only.

We partition the CONE into the two phases $\mathcal{D}$ and $\mathcal{L}$ defined by

$$\begin{aligned}\mathcal{L} &= \{(\alpha,\beta)\in \text{CONE}\colon\ f(\alpha,\beta;\,p) > f_{\mathcal{D}}(\alpha,\beta;p)\},\\ \mathcal{D} &= \{(\alpha,\beta)\in \text{CONE}\colon\ f(\alpha,\beta;\,p) = f_{\mathcal{D}}(\alpha,\beta;p)\}.\end{aligned} \tag{2.4}$$

The localized phase $\mathcal{L}$ corresponds to large values of β, for which the energetic reward to spend some time travelling along AB-interfaces exceeds the entropic penalty to do so. The delocalized phase $\mathcal{D}$, on the other hand, corresponds to small values of β, for which the energetic reward does not exceed the entropic penalty.

For $\alpha\geq 0$, let J_α be the halfline in CONE defined by

$$J_\alpha = \{(\alpha+\beta,\beta)\colon\ \beta\in[-\tfrac{\alpha}{2},\infty)\}. \tag{2.5}$$

Theorem 2.1 *(a) For every $\alpha\in(0,\infty)$ there exists a $\beta_c(\alpha)\in(0,\infty)$ such that*

$$\begin{aligned}\mathcal{L}\cap J_\alpha &= \{(\alpha+\beta,\beta)\colon\ \beta\in(\beta_c(\alpha),\infty)\},\\ \mathcal{D}\cap J_\alpha &= \big\{(\alpha+\beta,\beta)\colon\ \beta\in[-\tfrac{\alpha}{2},\beta_c(\alpha)]\big\}.\end{aligned} \tag{2.6}$$

(b) Inside phase $\mathcal{D}$ the free energy f is a function of $\alpha-\beta$ only, i.e., f is constant on $J_\alpha\cap\mathcal{D}$ for all $\alpha\in(0,\infty)$.

2.2 Fine structure

Throughout this section $M\in\mathbb{N}$, but once again we suppress the M-dependence from the notation. This section is organized as follows. In Section 2.2.1, we consider the supercritical regime $p>p_c$, and state a theorem. Subject to two hypotheses, we show that the delocalized phase $\mathcal{D}$ (recall (2.4)) splits into two subphases $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2$. We give a characterization of the critical curve $\alpha\mapsto\beta_c(\alpha)$ (recall (2.6)) in terms of the single linear free energy and state some properties of this curve. Subsequently, we formulate a conjecture stating that the localized phase $\mathcal{L}$ also splits into two subphases $\mathcal{L}=\mathcal{L}_1\cup\mathcal{L}_2$, which are saturated, respectively, non-saturated. In Section 2.2.2, we consider the subcritical regime $p<p_c$, and state several conjectures concerning the splitting of the localized phase $\mathcal{L}$ and of the delocalized phase $\mathcal{D}$.

For $p\in(0,1)$ and $(\alpha,\beta)\in$ CONE, let $\mathcal{O}_{p,\alpha,\beta}$ denote the subset of $\bar{\mathcal{R}}_p$ containing those $\bar\rho$ that maximize the variational formula in (1.17), i.e.,

$$\mathcal{O}_{p,\alpha,\beta} = \left\{\bar\rho\in\bar{\mathcal{R}}_p\colon\ f(\alpha,\beta;p) = \sup_{v\in\bar{\mathcal{B}}}\frac{\bar N(\bar\rho,v)}{\bar D(\bar\rho,v)}\right\}. \tag{2.7}$$

For $c \in (0,\infty)$, define $v(c) = (v_A(c), v_B(c), v_{\mathcal{I}}(c)) \in \bar{\mathcal{B}}$ as

$$v_{A,l}(c) = \chi_l^{-1}(c), \qquad l \in [0,\infty), \tag{2.8}$$
$$v_{B,l}(c) = \chi_l^{-1}\big(c + \tfrac{\alpha-\beta}{2}\big), \quad l \in [0,\infty), \tag{2.9}$$
$$v_{\mathcal{I}}(c) = z, \qquad \partial_u^-(u\,\phi_{\mathcal{I}}(u))(z) \geq c \geq \partial_u^+(u\,\phi_{\mathcal{I}}(u))(z), \tag{2.10}$$

where

$$\chi_l(v) = \big(\partial_u(u\,\tilde{\kappa}(u,l)\big)(v) \tag{2.11}$$

and χ_l^{-1} denotes the inverse function. Lemma B.1(v-vi) ensures that $v \mapsto \chi_l(v)$ is one-to-one between $(1+l,\infty)$ and $(0,\infty)$. The existence and uniqueness of z in (2.10) follow from the strict concavity of $u \mapsto u\phi_{\mathcal{I}}(u)$ (see Lemma3.3) and Lemma C.1 (see (C.1-C.2)). We will prove in Proposition 7.1 that the maximizer $v \in \bar{\mathcal{B}}$ of (1.17) necessarily belongs to the familly $\{v(c)\colon\ c \in (0,\infty)\}$.

2.2.1 Supercritical regime

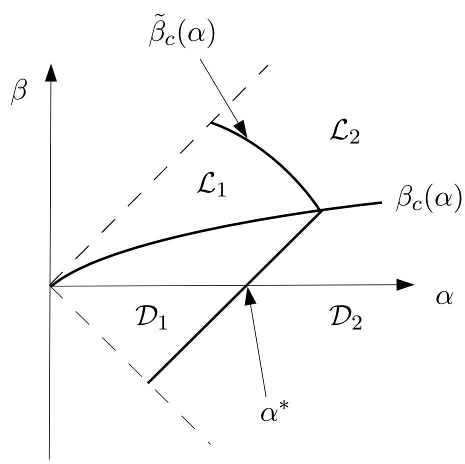

Figure 5: Qualitative picture of the phase diagram in the supercritical regime $p > p_c$.

Let $\mathcal{T}_p$ be the subset of $\bar{\mathcal{R}}_p$ containing those $\bar{\rho}$ that have a strictly positive B-component and are relevant for the variational formula in (1.17), i.e.,

$$\mathcal{T}_p = \left\{ \bar{\rho} \in \bar{\mathcal{R}}_p\colon\ \bar{\rho}_B([0,\infty)) > 0,\ \int_0^\infty (1+l)\,[\bar{\rho}_A + \bar{\rho}_B](dl) < \infty \right\}. \tag{2.12}$$

Note that $\mathcal{T}_p$ does not depend on (α,β).

Splitting of the $\mathcal{D}$-phase. We partition $\mathcal{D}$ into two phases: $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. To that end we introduce the *delocalized A-saturated free energy*, denoted by $f_{\mathcal{D}_2}(p)$, which is obtained by restricting the supremum in (2.1) to those $\bar{\rho} \in \bar{\mathcal{R}}_p$ that do not charge B. Such $\bar{\rho}$, which we call A-saturated, exist because $p > p_c$, allowing for trajectories that do not visit B-blocks. Thus, $f_{\mathcal{D}_2}(p)$ is defined as

$$f_{\mathcal{D}_2}(p) = \sup_{\substack{\bar\rho \in \bar{\mathcal{R}}_p \\ \bar\rho_B([0,\infty))=0}} \sup_{v \in \bar{\mathcal{B}}} \frac{\bar N_{\mathcal{D}_2}(\bar\rho, v)}{\bar D_{\mathcal{D}}(\bar\rho, v)} \tag{2.13}$$

with

$$\bar N_{\mathcal{D}_2}(\bar\rho, v) = \int_0^\infty v_{A,l}\, \tilde\kappa(v_{A,l}, l)\, [\bar\rho_A + \bar\rho_{\mathcal{I}}\, \delta_0](dl), \tag{2.14}$$

provided $D_{\mathcal{D}}(\bar\rho, v) < \infty$. Note that $f_{\mathcal{D}_2}(p)$ is a constant that does not depend on (α, β).

With the help of this definition, we can split the $\mathcal{D}$-phase defined in (2.4) into two parts:

- The $\mathcal{D}_1$-phase corresponds to small values of β and small to moderate values of α. In this phase there is no AB-localization and no A-saturation. For the variational formula in (1.17) this corresponds to the restriction where the AB-localization term disappears while the A-block term and the B-block term contribute, i.e.,

$$\mathcal{D}_1 = \big\{(\alpha, \beta) \in \mathrm{CONE}\colon\ f(\alpha, \beta;\, p) = f_{\mathcal{D}}(\alpha, \beta;\, p) > f_{\mathcal{D}_2}(p)\big\}. \tag{2.15}$$

- The $\mathcal{D}_2$-phase corresponds to small values of β and large values of α. In this phase there is no AB-localization but A-saturation occurs. For the variational formula in (1.17) this corresponds to the restriction where the AB-localization term disappears and the B-block term as well, i.e.,

$$\mathcal{D}_2 = \big\{(\alpha, \beta) \in \mathrm{CONE}\colon\ f(\alpha, \beta;\, p) = f_{\mathcal{D}_2}(p)\big\}. \tag{2.16}$$

To state our main result for the delocalized part of the phase diagram we need two hypotheses:

Hypothesis 1 *For all $p > p_c$ and all $\alpha \in (0, \infty)$ there exists a $\bar\rho \in \mathcal{O}_{p,\alpha,0}$ such that $\bar\rho_{\mathcal{I}} > 0$.*

Hypothesis 2 *For all $p > p_c$,*

$$\sup_{\bar\rho \in \mathcal{T}_p} \frac{\int_0^\infty g(l)\, [\bar\rho_A + \bar\rho_{\mathcal{I}}\, \delta_0](dl)}{\int_0^\infty (1+l)\, \bar\rho_B(dl)} < \infty, \tag{2.17}$$

where

$$g(l) = \bar v_{A,l} \left[\tilde\kappa(\bar v_{A,l}, l) - f_{\mathcal{D}_2}\right] \tag{2.18}$$

and $\bar v = v(f_{\mathcal{D}_2})$ as defined in (2.8–2.10).

Hypothesis 1 will allow us to derive an expression for $\beta_c(\alpha)$ in (2.6). Hypothesis 2 will allow us to show that $\mathcal{D}_1$ and $\mathcal{D}_2$ are non-empty.

Remark 2.4 The function g has the following properties: (1) $g(0) > 0$; (2) g is strictly decreasing on $[0, \infty)$; (3) $\lim_{l \to \infty} g(l) = -\infty$. Property (2) follows from Lemma B.1(ii) and the fact that $u \mapsto u\tilde\kappa(u, l)$ is concave (see Lemma B.1(i)). Property (3) follows from $f_{\mathcal{D}_2} > 0$, Lemma B.1(iv) and the fact that $\bar v_{A,l} \geq 1 + l$ for $l \in [0, \infty)$. Property (1) follows from property (2) because $\int_0^\infty g(l)[\hat\rho_A + \hat\rho_I \delta_0](dl) = 0$ for all $\hat\rho$ maximizing (2.13).

Let

$$\alpha^* = \sup\{\alpha \geq 0\colon\ f_{\mathcal{D}}(\alpha, 0;\, p) > f_{\mathcal{D}_2}(p)\}. \tag{2.19}$$

Theorem 2.5 *(a) If Hypothesis* 2 *holds, then* $\alpha^* \in (0,\infty)$.
(b) For every $\alpha \in [0,\alpha^*)$,

$$J_\alpha \cap \mathcal{D}_1 = J_\alpha \cap \mathcal{D} = \{(\alpha+\beta,\beta)\colon\ \beta \in [-\tfrac{\alpha}{2},\, \beta_c(\alpha)]. \tag{2.20}$$

(c) For every $\alpha \in [\alpha^*,\infty)$,

$$J_\alpha \cap \mathcal{D}_2 = J_\alpha \cap \mathcal{D} = \{(\alpha+\beta,\beta)\colon\ \beta \in [-\tfrac{\alpha}{2}, \beta_c(\alpha)]\}. \tag{2.21}$$

(d) If Hypothesis 1 *holds, then for every* $\alpha \in [0,\infty)$

$$\beta_c(\alpha) = \inf\big\{\beta > 0\colon\ \phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta) > \tilde{\kappa}(\bar{v}_{A,0},0)\big\} \quad \textit{with} \quad \bar{v} = v(f_{\mathcal{D}}(\alpha,0;\,p)). \tag{2.22}$$

(e) $\alpha \mapsto \beta_c(\alpha)$ *is concave, continuous, non-decreasing and bounded from above on* $[\alpha^*,\infty)$.
(f) Inside phase $\mathcal{D}_1$ *the free energy* f *is a function of* $\alpha-\beta$ *only, i.e.,* f *is constant on* $J_\alpha \cap \mathcal{D}_1$ *for all* $\alpha \in [0,\alpha^*]$.
(g) Inside phase $\mathcal{D}_2$ *the free energy* f *is constant.*

Splitting of the $\mathcal{L}$-phase. We partition $\mathcal{L}$ into two phases: $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. To that end we introduce the *localized A-saturated free energy*, denoted by $f_{\mathcal{L}_2}$, which is obtained by restricting the supremum in (1.17) to those $\bar{\rho} \in \bar{\mathcal{R}}_p$ that do not charge B, i.e.,

$$f_{\mathcal{L}_2}(\alpha,\beta;\,p) = \sup_{\substack{\bar{\rho}\in\bar{\mathcal{R}}_p \\ \bar{\rho}_B([0,\infty))=0}} \ \sup_{v\in\bar{\mathcal{B}}} \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)}, \tag{2.23}$$

provided $D(\bar{\rho},v) < \infty$.

With the help of this definition, we can split the $\mathcal{L}$-phase defined in (2.4) into two parts:

- The $\mathcal{L}_1$-phase corresponds to small to moderate values of α and large values of β. In this phase AB-localization occurs, but A-saturation does not, so that the free energy is given by the variational formula in (1.17) without restrictions, i.e.,

$$\mathcal{L}_1 = \big\{(\alpha,\beta) \in \text{CONE}\colon\ f(\alpha,\beta;p) > \max\{f_{\mathcal{D}}(\alpha,\beta;p),\ f_{\mathcal{L}_2}(\alpha,\beta;p)\}\big\}. \tag{2.24}$$

- The $\mathcal{L}_2$-phase corresponds to large values of α and β. In this phase both AB-localization and A-saturation occur. For the variational formula in (1.17) this corresponds to the restriction where the contribution of B-blocks disappears, i.e.,

$$\mathcal{L}_2 = \big\{(\alpha,\beta) \in \text{CONE}\colon\ f(\alpha,\beta;p) = f_{\mathcal{L}_2}(\alpha,\beta;p) > f_{\mathcal{D}}(\alpha,\beta;\,p)\big\}. \tag{2.25}$$

Conjecture 2.6 *(a) For every* $\alpha \in (0,\alpha^*]$ *there exists a* $\tilde{\beta}_c(\alpha) \in (\beta_c(\alpha),\infty)$ *such that*

$$\begin{aligned} \mathcal{L}_1 \cap J_\alpha &= \{(\alpha+\beta,\beta)\colon\ \beta \in (\beta_c(\alpha),\tilde{\beta}_c(\alpha)]\}, \\ \mathcal{L}_2 \cap J_\alpha &= \big\{(\alpha+\beta,\beta)\colon\ \beta \in [\tilde{\beta}_c(\alpha),\infty)\big\}. \end{aligned} \tag{2.26}$$

(b) For every $\alpha \in (\alpha^*,\infty)$, *the set* $\mathcal{L}_1 \cap J_\alpha = \emptyset$.

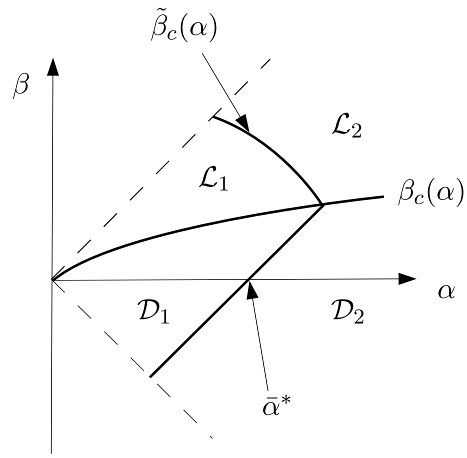

Figure 6: Qualitative picture of the phase diagram in the subcritical regime $p < p_c$.

2.2.2 Subcritical regime

Splitting of the $\mathcal{D}$-phase. Let

$$K = \inf_{\bar{\rho} \in \bar{\mathcal{R}}_p} \rho_B([0,\infty)). \tag{2.27}$$

Note that $K > 0$ because $p_c < p_c$. We again partition $\mathcal{D}$ into two phases: $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. To that end we introduce the *delocalized maximally A-saturated free energy*, denoted by $f_{\mathcal{D}_2}(p)$, which is obtained by restricting the supremum in (2.1) to those $\bar{\rho} \in \bar{\mathcal{R}}_p$ achieving K. Thus, $f_{\mathcal{D}_2}(p)$ is defined as

$$f_{\mathcal{D}_2}(p) = \sup_{\substack{\bar{\rho} \in \bar{\mathcal{R}}_p \\ \bar{\rho}_B([0,\infty)) = K}} \sup_{v \in \bar{\mathcal{B}}} \frac{\bar{N}_{\mathcal{D}}(\bar{\rho}, v)}{\bar{D}_{\mathcal{D}}(\bar{\rho}, v)}, \tag{2.28}$$

provided $D_{\mathcal{D}}(\bar{\rho}, v) < \infty$. Note that, contrary to what we had in the supercritical regime, $f_{\mathcal{D}_2}(p)$ depends on (α, β).

With the help of this definition, we can split the $\mathcal{D}$-phase defined in (2.4) into two parts:

- The $\mathcal{D}_1$-phase corresponds to small values of β and small to moderate values of α. In this phase there is no AB-localization and no maximal A-saturation. For the variational formula in (1.17) this corresponds to the restriction where the AB-localization term disappears while the A-block term and the B-block term contribute, i.e.,

$$\mathcal{D}_1 = \big\{(\alpha, \beta) \in \text{CONE}\colon\ f(\alpha, \beta; p) = f_{\mathcal{D}}(\alpha, \beta; p) > f_{\mathcal{D}_2}(p)\big\}. \tag{2.29}$$

- The $\mathcal{D}_2$-phase corresponds to small values of β and large values of α. In this phase there is no AB-localization and maximal A-saturation. For the variational formula in (1.17) this corresponds to the restriction where the AB-localization term disappears and the B-block term is minimal, i.e.,

$$\mathcal{D}_2 = \big\{(\alpha, \beta) \in \text{CONE}\colon\ f(\alpha, \beta; p) = f_{\mathcal{D}_2}(p)\big\}. \tag{2.30}$$

Let

$$\bar{\alpha}^* = \inf\big\{\alpha \geq 0\colon\ \exists\, \rho_B([0,\infty)) = K\ \forall\, \rho \in \mathcal{O}_{p,\alpha',0}\ \forall\, \alpha' \geq \alpha\big\}. \tag{2.31}$$

Conjecture 2.7 *(a)* $\bar{\alpha}^* \in (0,\infty)$.
(b) Theorems 2.5*(b,c,d,f) hold with* α^* *replaced by* $\bar{\alpha}^*$.
(c) Theorem 2.5*(g) does not hold.*

Splitting of the $\mathcal{L}$-phase. We again partition $\mathcal{L}$ into two phases: $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. To that end we introduce the *localized maximally A-saturated free energy*, denoted by $f_{\mathcal{L}_2}$, which is obtained by restricting the supremum in (1.17) to those $\bar{\rho} \in \bar{\mathcal{R}}_p$ achieving K. Thus, $f_{\mathcal{L}_2}(\alpha,\beta;\,p)$ is defined as

$$f_{\mathcal{L}_2}(\alpha,\beta;\,p) = \sup_{\substack{\bar{\rho}\in\bar{\mathcal{R}}_p \\ \bar{\rho}_B([0,\infty))=K}}\ \sup_{v\in\bar{\mathcal{B}}} \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)}, \tag{2.32}$$

provided $D(\bar{\rho},v) < \infty$.

With the help of this definition, we can split the $\mathcal{L}$-phase defined in (2.4) into two parts:

- The $\mathcal{L}_1$-phase corresponds to small to moderate values of α and large values of β. In this phase AB-localization occurs, but maximal A-saturation does not, so that the free energy is given by the variational formula in (1.17) without restrictions, i.e.,

$$\mathcal{L}_1 = \big\{(\alpha,\beta) \in \text{CONE}\colon\ f(\alpha,\beta;p) > \max\{f_{\mathcal{D}}(\alpha,\beta;p),\ f_{\mathcal{L}_2}(\alpha,\beta;p)\}\big\}. \tag{2.33}$$

- The $\mathcal{L}_2$-phase corresponds to large values of α and β. In this phase both AB-localization and maximal A-saturation occur. For the variational formula in (1.17) this corresponds to the restriction where the contribution of B-blocks is minimal, i.e.,

$$\mathcal{L}_2 = \big\{(\alpha,\beta) \in \text{CONE}\colon\ f(\alpha,\beta;p) = f_{\mathcal{L}_2}(\alpha,\beta;p) > f_{\mathcal{D}}(\alpha,\beta;\,p)\big\}. \tag{2.34}$$

Conjecture 2.8 *Conjecture* 2.6 *holds with* $\bar{\alpha}^*$ *instead of* α^*.

2.3 Proof of the hypotheses

Hypothesis 1 can be understood as follows. At $(\alpha,0) \in$ CONE, the BB-interaction is vanishes while the AA-interaction does not, and we have seen earlier that there is no localization of the copolymer along AB-interfaces when $\beta = 0$. Consequently, when the copolymer moves at a non-zero slope $l \in \mathbb{R}\setminus\{0\}$ it necessarily reduces the time it spends in the B-solvent. To be more specific, let $\bar{\rho} \in \mathcal{R}_{p,M}$ be a maximizer of the variational formula in (1.17), and assume that the copolymer moves in the emulsion by following the strategy of displacement associated with $\bar{\rho}$. Consider the situation in which the copolymer moves upwards for awhile at slope $l > 0$ and over a horizontal distance $h > 0$, and subsequently changes direction to move downward at slope $l' < 0$ and over a horizontal distance $h' > 0$. This change of vertical direction is necessary to pass over a B-block, otherwise it would be entropically more advantageous to move at slope $(hl + h'l')/(h+h')$ over an horizontal distance $h + h'$ (by the concavity of $\tilde{\kappa}$ in Lemma B.1(i)). Next, we observe (see Fig. 7) that when the copolymer passes over a B-block, the best strategy in terms of entropy is to follow the AB-interface (consisting of this

B-block and the A-solvent above it) without being localized, i.e., the copolymer performs a long excursion into the A-solvent but the two ends of this excursion are located on the AB-interface. This long excursion is counted in $\bar{\rho}_{\mathcal{I}}$. Consequently, Hypothesis 1 ($\bar{\rho}_{\mathcal{I}} > 0$) will be satisfied if we can show that the copolymer necessarily spends a strictly positive fraction of its time performing such changes of vertical direction. But, by the ergodicity of ω and Ω, this has to be the case.

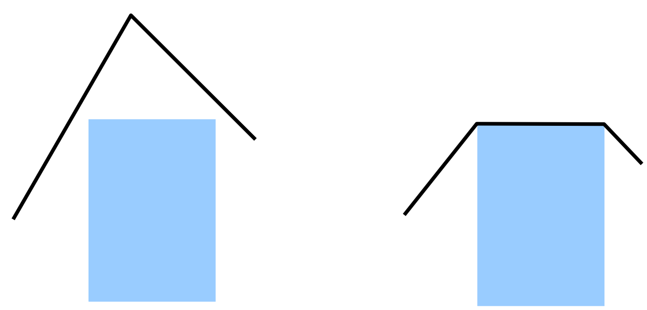

Figure 7: Entropic optimization when the copolymer passes over a B-block.

The statement of Hypothesis 2 is technical, but can be rephrased in a simpler way. Recall Remark 2.4 and note that there is an $l_0 \in (0, \infty)$ such that $g > 0$ on $[0, l_0)$ and $g < 0$ on (l_0, ∞). Assume by contradiction that Hypothesis 2 fails and that the ratio in (2.17) is unbounded. Then, by spending an arbitrarily small amount of time in the B-solvent, the copolymer can improve the best saturated strategies by moving some of the mass of $\bar{\rho}_A(l_0, \infty)$ to $\bar{\rho}_A(0, l_0)$, and that the entropic gain of this transformation is arbitrarily larger than the time spent in the B-solvent. In other words, failure of Hypothesis 2 means that spending an arbitrarily small fraction of time in the B-solvent allows the copolymer to travel flatter when it is in the A-solvent during a fraction of the time that is arbitrarily larger than the fraction of the time it spends in the B-solvent. This means that, instead of going around some large cluster of the B-solvent, the copolymer simply crosses it straight to travel flatter. However, the fact that large subcritical clusters scale as round balls contradicts this scenario, because it means that the time needed to go around the cluster is of the same order as the time required to cross the cluster, which makes the unboundedness of the ratio in (2.17) impossible.

3 Key ingredients

In Section 3.1, we define the entropy per step $\tilde{\kappa}(u, l)$ carried by trajectories moving at slope $l \in \mathbb{R}_+$ with the constraint that the total number of steps divided by the total number of horizontal steps is equal to $u \in [1 + l, \infty)$ (Proposition 3.1 below). In Section 3.2, we define the free energy per step $\phi_{\mathcal{I}}(\mu)$ of a copolymer in the vicinity of an AB-interface with the constraint that the total number of steps divided by the total number of horizontal steps is equal to $\mu \in [1, \infty)$ (Proposition 3.2 below). In Section 3.3, we combine the definitions in Sections 3.1–3.2 to obtain a variational formula for the free energy per step in single columns of different types (Proposition 3.5 below). In Section 3.4 we define the set of probability laws introduced in (1.16), which is a key ingredient of the slope-based variational formula in Theorem 1.1. Finally, in Section 3.5, we prove that the quenched free energy per step $f(\alpha, \beta; p)$ is strictly positive on CONE.

3.1 Path entropies at given slope

Path entropies. We define the entropy of a path crossing a single column. To that aim, we set

$$\begin{aligned} \mathcal{H} &= \{(u,l) \in [0,\infty) \times \mathbb{R} \colon\ u \geq 1 + |l|\}, \\ \mathcal{H}_L &= \big\{(u,l) \in \mathcal{H} \colon\ l \in \tfrac{\mathbb{Z}}{L},\, u \in 1 + |l| + \tfrac{2\mathbb{N}}{L}\big\}, \qquad L \in \mathbb{N}, \end{aligned} \tag{3.1}$$

and note that $\mathcal{H} \cap \mathbb{Q}^2 = \cup_{L\in\mathbb{N}} \mathcal{H}_L$. For $(u,l) \in \mathcal{H}$, we denote by $\mathcal{W}_L(u,l)$ the set containing those paths $\pi = (0,-1) + \widetilde{\pi}$ with $\widetilde{\pi} \in \mathcal{W}_{uL}$ (recall (1.1)) for which $\pi_{uL} = (L, lL)$ (see Fig. 8). The entropy per step associated with the paths in $\mathcal{W}_L(u,l)$ is given by

$$\tilde{\kappa}_L(u,l) = \tfrac{1}{uL} \log |\mathcal{W}_L(u,l)|. \tag{3.2}$$

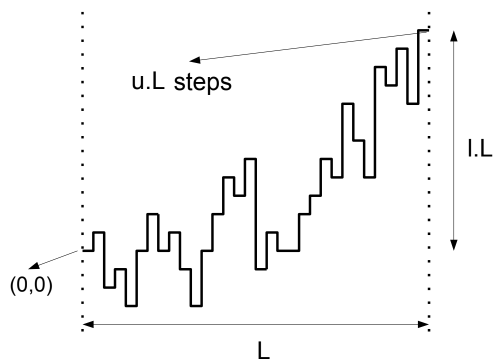

Figure 8: A trajectory in $\mathcal{W}_L(u,l)$.

The following propositions will be proven in Appendix A.

Proposition 3.1 *For all $(u,l) \in \mathcal{H} \cap \mathbb{Q}^2$ there exists a $\tilde{\kappa}(u,l) \in [0, \log 3]$ such that*

$$\lim_{\substack{L\to\infty \\ (u,l)\in\mathcal{H}_L}} \tilde{\kappa}_L(u,l) = \sup_{\substack{L\in\mathbb{N} \\ (u,l)\in\mathcal{H}_L}} \tilde{\kappa}_L(u,l) = \tilde{\kappa}(u,l). \tag{3.3}$$

An explicit formula is available for $\tilde{\kappa}(u,l)$, namely,

$$\tilde{\kappa}(u,l) = \begin{cases} \kappa(u/|l|, 1/|l|), & l \neq 0, \\ \hat{\kappa}(u), & l = 0, \end{cases} \tag{3.4}$$

where $\kappa(a,b)$, $a \geq 1 + b$, $b \geq 0$, and $\hat{\kappa}(\mu)$, $\mu \geq 1$, are given in [5], Section 2.1, in terms of elementary variational formulas involving entropies (see [5], proof of Lemmas 2.1.1–2.1.2). The two formulas in (3.4) allow us to extend $(u,l) \mapsto \tilde{\kappa}(u,l)$ to a continuous and strictly concave function on $\mathcal{H}$ (see Lemma B.1).

3.2 Free energy for a linear interface

Free energy along a single linear interface. To analyze the free energy per monomer in a single column we need to first analyze the free energy per monomer when the path moves in the vicinity of an AB-interface. To that end we consider a *single linear interface* $\mathcal{I}$ separating a solvent B in the lower halfplane from a solvent A in the upper halfplane (the latter is assumed to include the interface itself).

For $L \in \mathbb{N}$ and $\mu \in 1 + \frac{2\mathbb{N}}{L}$, let $\mathcal{W}_L^{\mathcal{I}}(\mu) = \mathcal{W}_L(\mu, 0)$ denote the set of μL-step directed self-avoiding paths starting at $(0,0)$ and ending at $(L,0)$. Recall (1.2) and define

$$\phi_L^{\omega,\mathcal{I}}(\mu) = \frac{1}{\mu L} \log Z_{L,\mu}^{\omega,\mathcal{I}} \quad \text{and} \quad \phi_L^{\mathcal{I}}(\mu) = \mathbb{E}[\phi_L^{\omega,\mathcal{I}}(\mu)], \tag{3.5}$$

with

$$\begin{aligned} Z_{L,\mu}^{\omega,\mathcal{I}} &= \sum_{\pi \in \mathcal{W}_L^{\mathcal{I}}(\mu)} \exp\left[H_L^{\omega,\mathcal{I}}(\pi)\right], \\ H_L^{\omega,\mathcal{I}}(\pi) &= \sum_{i=1}^{\mu L} \big(\beta\, 1\{\omega_i = B\} - \alpha\, 1\{\omega_i = A\}\big)\; 1\{(\pi_{i-1},\pi_i) < 0\}, \end{aligned} \tag{3.6}$$

where $(\pi_{i-1},\pi_i) < 0$ means that the i-th step lies in the lower halfplane, strictly below the interface (see Fig. 9).

Proposition 3.2 ([5], Section 2.2.2)
For all $(\alpha,\beta) \in \text{CONE}$ and $\mu \in \mathbb{Q} \cap [1,\infty)$ there exists a $\phi_{\mathcal{I}}(\mu) = \phi_{\mathcal{I}}(\mu;\alpha,\beta) \in \mathbb{R}$ such that

$$\lim_{\substack{L\to\infty \\ \mu \in 1+\frac{2\mathbb{N}}{L}}} \phi_L^{\omega,\mathcal{I}}(\mu) = \phi_{\mathcal{I}}(\mu) \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega \textit{ and in } L^1(\mathbb{P}). \tag{3.7}$$

It is easy to check (via concatenation of trajectories) that $\mu \mapsto \mu\phi_{\mathcal{I}}(\mu;\alpha,\beta)$ is concave. For technical reasons we need to assume that it is *strictly concave*, a property we believe to be true but are unable to verify:

Lemma 3.3 *For all $(\alpha,\beta) \in \text{CONE}$ the function $\mu \mapsto \mu\phi_{\mathcal{I}}(\mu;\alpha,\beta)$ is strictly concave on $[1,\infty)$.*

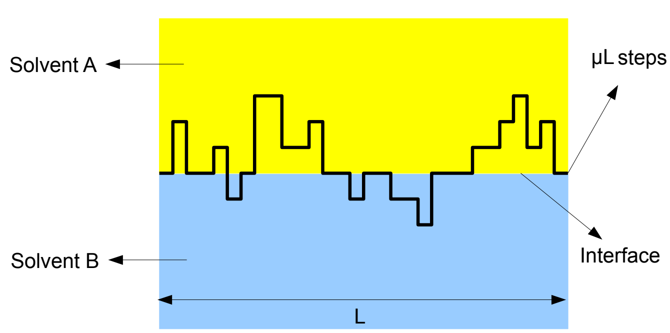

Figure 9: Copolymer near a single linear interface.

Proof. To show that strict concavity holds we argue by contradiction. Suppose that there is an interval $[\mu_1,\mu_2]$ on which $\mu \mapsto \mu\phi_{\mathcal{I}}(\mu;\alpha,\beta)$ is linear. Then $\phi_{\mathcal{I}}(\mu) > \tilde{\kappa}(\mu,0)$ for all

$\mu \in [\mu_1, \mu_2]$ except in at most two points (because $\mu \mapsto \mu\tilde{\kappa}(\mu, 0)$ is strictly concave by Lemma B.1(i)). Therefore we may assume that $\phi_{\mathcal{I}}((\mu_1 + \mu_2)/2) > \tilde{\kappa}((\mu_1 + \mu_2)/2, 0)$, and with the assumed linearity we get

$$\lim_{L\to\infty} \frac{1}{2L} \log \left(Z^{\omega,\mathcal{I}}_{L,\mu_1} Z^{\theta_1^{\mu} L(\omega),\mathcal{I}}_{L,\mu_2} \right) - \frac{1}{2L} \log Z^{\omega,\mathcal{I}}_{2L,(\mu_1+\mu_2)/2} = 0, \tag{3.8}$$

where θ is the left-shift acting on sequences of letters. Write $P^{\omega,\mathcal{I}}_{2L,(\mu_1+\mu_2)/2}$ to denote the Gibbs measure on $\mathcal{W}^{\mathcal{I}}_{2L}((\mu_1 + \mu_2)/2)$ associated with the Hamiltonian $H^{\omega,\mathcal{I}}_{2L}(\pi)$ defined as in (3.6). A consequence of (3.8) is that

$$P^{\omega,\mathcal{I}}_{2L,(\mu_1+\mu_2)/2}(\pi_{\mu_1 L} = (L, 0)) \tag{3.9}$$

does not decay exponentially as $L \to \infty$. However, the fact that $\phi_{\mathcal{I}}((\mu_1 + \mu_2)/2) > \tilde{\kappa}((\mu_1 + \mu_2)/2, 0)$ implies that the copolymer is localized under $P^{\omega,\mathcal{I}}_{2L,(\mu_1+\mu_2)/2}$, and therefore the excursions away from the origin are exponentially tight. Under the event $\{\pi_{\mu_1 L} = (L, 0)\}$ we necessarily have that the excursions constituting the first L horizontal steps of the path have a total length of $\mu_1 L$. But $\mu_1 < (\mu_1 + \mu_2)/2$ means that the ratio of the total number of steps and the number of horizontal steps is small for the excursions constituting the first $\mu_1 L$ steps of the path. But ω is ergodic, and therefore the average of the ratio over the trajectory is necessarily $(\mu_1 + \mu_2)/2$. □

3.3 Free energy in a single column and variational formulas

In this section, we prove the convergence of the free energy per step in a single column (Proposition 3.4) and derive a variational formula for this free energy with the help of Propositions 3.1–3.2. The variational formula takes different forms (Propositions 3.5), depending on *whether there is or is not an AB-interface between the heights where the copolymer enters and exits the column, and in the latter case whether an AB-interface is reached or not.*

In what follows we need to consider the randomness in a single column. To that aim, we recall (1.6), we pick $L \in \mathbb{N}$ and once Ω is chosen, we can record the randomness of $C_{j,L}$ as

$$\Omega_{(j,\,\cdot\,)} = \{\Omega_{(j,l)} \colon\ l \in \mathbb{Z}\}. \tag{3.10}$$

We will also need to consider the randomness of the j-th column seen by a trajectory that enters $\mathcal{C}_{j,L}$ through the block $\Lambda_{j,k}$ with $k \neq 0$ instead of $k = 0$. In this case, the randomness of $\mathcal{C}_{j,L}$ is recorded as

$$\Omega_{(j,k+\,\cdot\,)} = \{\Omega_{(j,k+l)} \colon\ l \in \mathbb{Z}\}. \tag{3.11}$$

Pick $L \in \mathbb{N}$, $\chi \in \{A, B\}^{\mathbb{Z}}$ and consider $\mathcal{C}_{0,L}$ endowed with the disorder χ, i.e., $\Omega(0, \cdot) = \chi$. Let $(n_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}$ be the successive heights of the AB-interfaces in $\mathcal{C}_{0,L}$ divided by L, i.e.,

$$\cdots < n_{-1} < n_0 \leq 0 < n_1 < n_2 < \dots. \tag{3.12}$$

and the j-th interface of $\mathcal{C}_{0,L}$ is $\mathcal{I}_j = \{0, \dots, L\} \times \{n_j L\}$ (see Fig. 10). Next, for $r \in \mathbb{N}_0$ we set

$$k_{r,\chi} = 0 \text{ if } n_1 > r \text{ and } k_{r,\chi} = \max\{i \geq 1 \colon n_i \leq r\} \text{ otherwise}, \tag{3.13}$$

while for $r \in -\mathbb{N}$ we set

$$k_{r,\chi} = 0 \text{ if } n_0 \leq r \text{ and } k_{r,\chi} = \min\{i \leq 0 \colon n_i \geq r + 1\} - 1 \text{ otherwise}. \tag{3.14}$$

Thus, $|k_{r,\chi}|$ is the number of AB-interfaces between heigths 1 and rL in $\mathcal{C}_{0,L}$.

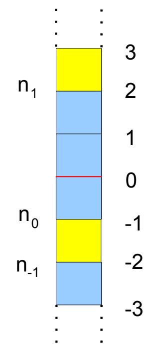

Figure 10: Example of a column with disorder $\chi = (\dots, \chi(-3), \chi(-2), \chi(-1), \chi(0), \chi(1), \chi(2), \dots) = (\dots, B, A, B, B, B, A, , \dots)$. In this example, for instance, $k_{-2,\chi} = -1$ and $k_{1,\chi} = 0$.

3.3.1 Free energy in a single column

Column crossing characteristics. Pick $L, M \in \mathbb{N}$, and consider the first column $\mathcal{C}_{0,L}$. The type of $\mathcal{C}_{0,L}$ is determined by $\Theta = (\chi, \Xi, x)$, where $\chi = (\chi_j)_{j\in\mathbb{Z}}$ encodes the type of each block in $\mathcal{C}_{0,L}$, i.e., $\chi_j = \Omega_{(0,j)}$ for $j \in \mathbb{Z}$, and (Ξ, x) indicates which trajectories π are taken into account. In the latter, Ξ is given by $(\Delta\Pi, b_0, b_1)$ such that the vertical increment in $\mathcal{C}_{0,L}$ on the block scale is $\Delta\Pi$ and satisfies $|\Delta\Pi| \leq M$, i.e., π enters $\mathcal{C}_{0,L}$ at $(0, b_0 L)$ and exits $\mathcal{C}_{0,L}$ at $(L, (\Delta\Pi + b_1)L)$. As in (3.13) and (3.14), we set $k_\Theta = k_{\Delta\Pi,\chi}$ and we let $\mathcal{V}_{\text{int}}$ be the set containing those Θ satisfying $k_\Theta \neq 0$. Thus, $\Theta \in \mathcal{V}_{\text{int}}$ means that the trajectories crossing $\mathcal{C}_{0,L}$ from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ necessarily hit an AB-interface, and in this case we set $x = 1$. If, on the other hand, $\Theta \in \mathcal{V}_{\text{nint}} = \mathcal{V} \setminus \mathcal{V}_{\text{int}}$, then we have $k_\Theta = 0$ and we set $x = 1$ when the set of trajectories crossing $\mathcal{C}_{0,L}$ from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ is restricted to those that do not reach an AB-interface before exiting $\mathcal{C}_{0,L}$, while we set $x = 2$ when it is restricted to those trajectories that reach at least one AB-interface before exiting $\mathcal{C}_{0,L}$. To fix the possible values taken by $\Theta = (\chi, \Xi, x)$ in a column of width L, we put $\mathcal{V}_{L,M} = \mathcal{V}_{\text{int},L,M} \cup \mathcal{V}_{\text{nint},L,M}$ with

$$\begin{aligned}
\mathcal{V}_{\text{int},L,M} &= \Big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A, B\}^{\mathbb{Z}} \times \mathbb{Z} \times \big\{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\big\}^2 \times \{1\}\colon \\
&\qquad\qquad |\Delta\Pi| \leq M,\ k_{\Delta\Pi,\chi} \neq 0\Big\}, \\
\mathcal{V}_{\text{nint},L,M} &= \Big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A, B\}^{\mathbb{Z}} \times \mathbb{Z} \times \big\{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\big\}^2 \times \{1, 2\}\colon \\
&\qquad\qquad |\Delta\Pi| \leq M,\ k_{\Delta\Pi,\chi} = 0\Big\}.
\end{aligned} \tag{3.15}$$

Thus, the set of all possible values of Θ is $\mathcal{V}_M = \cup_{L\geq 1} \mathcal{V}_{L,M}$, which we partition into $\mathcal{V}_M = \mathcal{V}_{\text{int},M} \cup \mathcal{V}_{\text{nint},M}$ (see Fig. 11) with

$$\begin{aligned}
\mathcal{V}_{\text{int},M} &= \cup_{L\in\mathbb{N}}\ \mathcal{V}_{\text{int},L,M} \\
&= \Big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A, B\}^{\mathbb{Z}} \times \mathbb{Z} \times (\mathbb{Q}_{(0,1]})^2 \times \{1\}\colon\ |\Delta\Pi| \leq M,\ k_{\Delta\Pi,\chi} \neq 0\Big\}, \\
\mathcal{V}_{\text{nint},M} &= \cup_{L\in\mathbb{N}}\ \mathcal{V}_{\text{nint},L,M} \\
&= \Big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A, B\}^{\mathbb{Z}} \times \mathbb{Z} \times (\mathbb{Q}_{(0,1]})^2 \times \{1, 2\}\colon\ |\Delta\Pi| \leq M,\ k_{\Delta\Pi,\chi} = 0\Big\},
\end{aligned} \tag{3.16}$$

where, for all $I \subset \mathbb{R}$, we set $\mathbb{Q}_I = I \cap \mathbb{Q}$. We define the closure of $\mathcal{V}_M$ as $\overline{\mathcal{V}}_M = \overline{\mathcal{V}}_{\text{int},M} \cup \overline{\mathcal{V}}_{\text{nint},M}$ with

$$\begin{aligned}\overline{\mathcal{V}}_{\text{int},M} &= \big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A,B\}^{\mathbb{Z}} \times \mathbb{Z} \times [0,1]^2 \times \{1\} \colon \ |\Delta\Pi| \leq M,\ k_{\Delta\Pi,\chi} \neq 0\big\},\\ \overline{\mathcal{V}}_{\text{nint},M} &= \big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A,B\}^{\mathbb{Z}} \times \mathbb{Z} \times [0,1]^2 \times \{1,2\} \colon \ |\Delta\Pi| \leq M,\ k_{\Delta\Pi,\chi} = 0\big\}.\end{aligned} \tag{3.17}$$

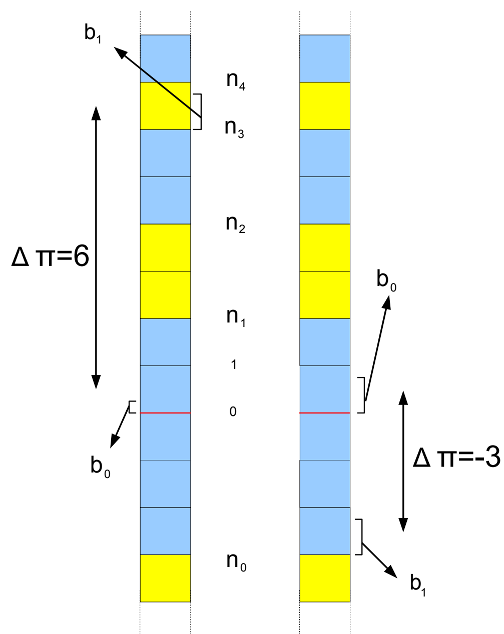

Figure 11: Labelling of coarse-grained paths and columns. On the left the type of the column is in $\mathcal{V}_{\text{int},M}$, on the right it is in $\mathcal{V}_{\text{nint},M}$ (with $M \geq 6$).

Time spent in columns. We pick $L, M \in \mathbb{N}$, $\Theta = (\chi, \Delta\Pi, b_0, b_1, x) \in \mathcal{V}_{L,M}$ and we specify the total number of steps that a trajectory crossing the column $\mathcal{C}_{0,L}$ of type Θ is allowed to make. For $\Theta = (\chi, \Delta\Pi, b_0, b_1, 1)$, set

$$t_\Theta = 1 + \text{sign}(\Delta\Pi)\,(\Delta\Pi + b_1 - b_0)\, 1_{\{\Delta\Pi \neq 0\}} + |b_1 - b_0|\, 1_{\{\Delta\Pi = 0\}}, \tag{3.18}$$

so that a trajectory π crossing a column of width L from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ makes a total of uL steps with $u \in t_\Theta + \frac{2\mathbb{N}}{L}$. For $\Theta = (\chi, \Delta\Pi, b_0, b_1, 2)$ in turn, recall (3.12) and let

$$t_\Theta = 1 + \min\{2n_1 - b_0 - b_1 - \Delta\Pi, 2|n_0| + b_0 + b_1 + \Delta\Pi\}, \tag{3.19}$$

so that a trajectory π crossing a column of width L and type $\Theta \in \mathcal{V}_{\text{nint},L,M}$ from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ and reaching an AB-interface makes a total of uL steps with $u \in t_\Theta + \frac{2\mathbb{N}}{L}$.

At this stage, we can fully determine the set $\mathcal{W}_{\Theta,u,L}$ consisting of the uL-step trajectories π that are considered in a column of width L and type Θ. To that end, for $\Theta \in \mathcal{V}_{\text{int},L,M}$ we map the trajectories $\pi \in \mathcal{W}_L(u, \Delta\Pi + b_1 - b_0)$ onto $\mathcal{C}_{0,L}$ such that π enters $\mathcal{C}_{0,L}$ at $(0, b_0 L)$ and exits $\mathcal{C}_{0,L}$ at $(L, (\Delta\Pi + b_1)L)$ (see Fig. 12), and for $\Theta \in \mathcal{V}_{\text{nint},L,M}$ we remove, dependencing on

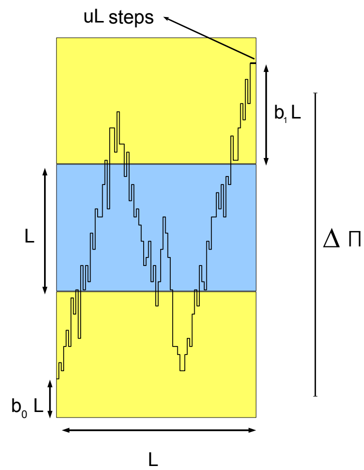

Figure 12: Example of a uL-step path inside a column of type $(\chi, \Delta\Pi, b_0, b_1, 1) \in \mathcal{V}_{\mathrm{int},L}$ with disorder $\chi = (\dots, \chi(0), \chi(1), \chi(2), \dots) = (\dots, A, B, A, \dots)$, vertical displacement $\Delta\Pi = 2$, entrance height b_0 and exit height b_1.

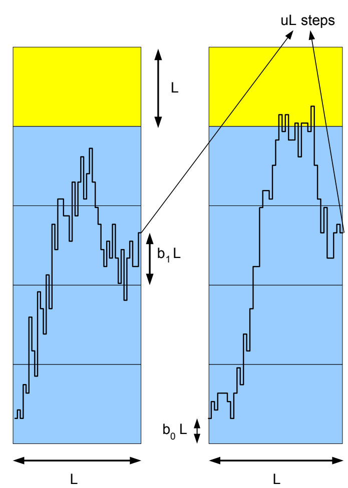

Figure 13: Two examples of a uL-step path inside a column of type $(\chi, \Delta\Pi, b_0, b_1, 1) \in \mathcal{V}_{\mathrm{nint},L}$ (left picture) and $(\chi, \Delta\Pi, b_0, b_1, 2) \in \mathcal{V}_{\mathrm{nint},L}$ (right picture) with disorder $\chi = (\dots, \chi(0), \chi(1), \chi(2), \chi(3), \chi(4), \dots) = (\dots, B, B, B, B, A, \dots)$, vertical displacement $\Delta\Pi = 2$, entrance height b_0 and exit height b_1.

$x \in \{1,2\}$, those trajectories that reach or do not reach an AB-interface in the column (see Fig. 13). Thus, for $\Theta \in \mathcal{V}_{\mathrm{int},L,M}$ and $u \in t_\Theta + \frac{2\mathbb{N}}{L}$, we let

$$\mathcal{W}_{\Theta,u,L} = \big\{\pi = (0, b_0 L) + \widetilde{\pi}\colon\ \widetilde{\pi} \in \mathcal{W}_L(u, \Delta\Pi + b_1 - b_0)\big\}, \tag{3.20}$$

and, for $\Theta \in \mathcal{V}_{\mathrm{nint},L,M}$ and $u \in t_\Theta + \frac{2\mathbb{N}}{L}$,

$$\begin{aligned} \mathcal{W}_{\Theta,u,L} &= \big\{\pi \in (0, b_0 L) + \mathcal{W}_L(u, \Delta\Pi + b_1 - b_0)\colon\ \pi \text{ reaches no } AB\text{-interface}\big\} \text{ if } x_\Theta = 1,\\ \mathcal{W}_{\Theta,u,L} &= \big\{\pi \in (0, b_0 L) + \mathcal{W}_L(u, \Delta\Pi + b_1 - b_0)\colon\ \pi \text{ reaches an } AB\text{-interface}\big\} \text{ if } x_\Theta = 2, \end{aligned} \tag{3.21}$$

with x_Θ the last coordinate of $\Theta \in \mathcal{V}_M$. Next, we set

$$\begin{aligned} \mathcal{V}^*_{L,M} &= \Big\{(\Theta, u) \in \mathcal{V}_{L,M} \times [0,\infty)\colon\ u \in t_\Theta + \tfrac{2\mathbb{N}}{L}\Big\},\\ \mathcal{V}^*_M &= \big\{(\Theta, u) \in \mathcal{V}_M \times \mathbb{Q}_{[1,\infty)}\colon\ u \geq t_\Theta\big\},\\ \overline{\mathcal{V}}^*_M &= \big\{(\Theta, u) \in \overline{\mathcal{V}}_M \times [1,\infty)\colon\ u \geq t_\Theta\big\}, \end{aligned} \tag{3.22}$$

which we partition into $\mathcal{V}^*_{\mathrm{int},L,M} \cup \mathcal{V}^*_{\mathrm{nint},L,M}$, $\mathcal{V}^*_{\mathrm{int},M} \cup \mathcal{V}^*_{\mathrm{nint},M}$ and $\overline{\mathcal{V}}^*_{\mathrm{int},M} \cup \overline{\mathcal{V}}^*_{\mathrm{nint},M}$. Note that for every $(\Theta,u) \in \mathcal{V}^*_M$ there are infinitely many $L \in \mathbb{N}$ such that $(\Theta,u) \in \mathcal{V}^*_{L,M}$, because $(\Theta,u) \in \mathcal{V}^*_{qL,M}$ for all $q \in \mathbb{N}$ as soon as $(\Theta,u) \in \mathcal{V}^*_{L,M}$.

Restriction on the number of steps per column. In what follows, we set

$$\text{EIGH} = \{(M,m) \in \mathbb{N} \times \mathbb{N}\colon\ m \geq M+2\}, \tag{3.23}$$

and, for $(M,m) \in \text{EIGH}$, we consider the situation where the number of steps uL made by a trajectory π in a column of width $L \in \mathbb{N}$ is bounded by mL. Thus, we restrict the set $\mathcal{V}_{L,M}$ to the subset $\mathcal{V}^m_{L,M}$ containing only those types of columns Θ that can be crossed in less than mL steps, i.e.,

$$\mathcal{V}^m_{L,M} = \{\Theta \in \mathcal{V}_{L,M}\colon\ t_\Theta \leq m\}. \tag{3.24}$$

Note that the latter restriction only conconcerns those Θ satisfying $x_\Theta = 2$. When $x_\Theta = 1$ a quick look at (3.18) suffices to state that $t_\Theta \leq M+2 \leq m$. Thus, we set $\mathcal{V}^m_{L,M} = \mathcal{V}^m_{\mathrm{int},L,M} \cup \mathcal{V}^m_{\mathrm{nint},L,M}$ with $\mathcal{V}^m_{\mathrm{int},L,M} = \mathcal{V}_{\mathrm{int},L,M}$ and with

$$\mathcal{V}^m_{\mathrm{nint},L,M} = \Big\{\Theta \in \{A,B\}^{\mathbb{Z}} \times \mathbb{Z} \times \big\{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\big\}^2 \times \{1,2\}\colon \\ |\Delta\Pi| \leq M,\ k_\Theta = 0 \ \text{ and } t_\Theta \leq m\Big\}. \tag{3.25}$$

The sets $\mathcal{V}^m_M = \mathcal{V}^m_{\mathrm{int},M} \cup \mathcal{V}^m_{\mathrm{nint},M}$ and $\overline{\mathcal{V}}^m_M = \overline{\mathcal{V}}^m_{\mathrm{int},M} \cup \overline{\mathcal{V}}^m_{\mathrm{nint},M}$ are obtained by mimicking (3.16–3.17). In the same spirit, we restrict $\mathcal{V}^*_{L,M}$ to

$$\mathcal{V}^{*,m}_{L,M} = \{(\Theta,u) \in \mathcal{V}^*_{L,M}\colon\ \Theta \in \mathcal{V}^m_{L,M}, u \leq m\} \tag{3.26}$$

and $\mathcal{V}^*_{L,M} = \mathcal{V}^*_{\mathrm{int},L,M} \cup \mathcal{V}^*_{\mathrm{nint},L,M}$ with

$$\begin{aligned} \mathcal{V}^{*,m}_{\mathrm{int},L,M} &= \Big\{(\Theta,u) \quad \in \mathcal{V}^m_{\mathrm{int},L,M} \times [1,m]\colon\ u \in t_\Theta + \tfrac{2\mathbb{N}}{L}\Big\},\\ \mathcal{V}^{*\,m}_{\mathrm{nint},L,M} &= \Big\{(\Theta,u) \quad \in \mathcal{V}^m_{\mathrm{nint},L,M} \times [1,m]\colon\ u \in t_\Theta + \tfrac{2\mathbb{N}}{L}\Big\}. \end{aligned} \tag{3.27}$$

We set also $\mathcal{V}^{*,m}_M = \mathcal{V}^{*,m}_{\mathrm{int},M} \cup \mathcal{V}^{*,m}_{\mathrm{nint},M}$ with $\mathcal{V}^{*,m}_{\mathrm{int},M} = \cup_{L\in\mathbb{N}} \mathcal{V}^{*,m}_{\mathrm{int},L,M}$ and $\mathcal{V}^{*,m}_{\mathrm{nint},M} = \cup_{L\in\mathbb{N}} \mathcal{V}^{*,m}_{\mathrm{nint},L,M}$, and rewrite these as

$$\begin{aligned}\mathcal{V}^{*,m}_{\mathrm{int},M} &= \big\{(\Theta,u) \in \mathcal{V}^{m}_{\mathrm{int},M} \times \mathbb{Q}_{[1,m]}\colon\ u \geq t_\Theta\big\},\\ \mathcal{V}^{*,m}_{\mathrm{nint},M} &= \big\{(\Theta,u) \in \mathcal{V}^{m}_{\mathrm{nint},M} \times \mathbb{Q}_{[1,m]}\colon\ u \geq t_\Theta\big\}.\end{aligned} \tag{3.28}$$

We further set $\overline{\mathcal{V}}^{*}_M = \overline{\mathcal{V}}^{*,m}_{\mathrm{int},M} \cup \overline{\mathcal{V}}^{*,m}_{\mathrm{nint},M}$ with

$$\begin{aligned}\overline{\mathcal{V}}^{*,m}_{\mathrm{int},M} &= \big\{(\Theta,u) \in \overline{\mathcal{V}}^{m}_{\mathrm{int},M} \times [1,m]\colon\ u \geq t_\Theta\big\},\\ \overline{\mathcal{V}}^{*,m}_{\mathrm{nint},M} &= \Big\{(\Theta,u) \in \overline{\mathcal{V}}^{m}_{\mathrm{nint},M} \times [1,m]\colon\ u \geq t_\Theta\Big\}.\end{aligned} \tag{3.29}$$

Existence and uniform convergence of free energy per column. Recall (3.20), (3.21) and, for $L \in \mathbb{N}$, $\omega \in \{A,B\}^{\mathbb{N}}$ and $(\Theta,u) \in \mathcal{V}^*_{L,M}$, we associate with each $\pi \in \mathcal{W}_{\Theta,u,L}$ the energy

$$H^{\omega,\chi}_{uL,L}(\pi) = \sum_{i=1}^{uL} \big(\beta\, 1\{\omega_i = B\} - \alpha\, 1\{\omega_i = A\}\big)\, 1\Big\{\chi^L_{(\pi_{i-1},\pi_i)} = B\Big\}, \tag{3.30}$$

where $\chi^L_{(\pi_{i-1},\pi_i)}$ indicates the label of the block containing (π_{i-1},π_i) in a column with disorder χ of width L. (Recall that the disorder in the block is part of the type of the block.) The latter allows us to define the quenched free energy per monomer in a column of type Θ and size L as

$$\psi^\omega_L(\Theta,u) = \frac{1}{uL} \log Z^\omega_L(\Theta,u) \quad \text{with} \quad Z^\omega_L(\Theta,u) = \sum_{\pi\in\mathcal{W}_{\Theta,u,L}} e^{H^{\omega,\chi}_{uL,L}(\pi)}. \tag{3.31}$$

Abbreviate $\psi_L(\Theta,u) = \mathbb{E}[\psi^\omega_L(\Theta,u)]$, and note that for $M \in \mathbb{N}$, $m \geq M+2$ and $(\Theta,u) \in \mathcal{V}^{*,m}_{L,M}$ all $\pi \in \mathcal{W}_{\Theta,u,L}$ necessarily remain in the blocks $\Lambda_L(0,i)$ with $i \in \{-m+1,\dots,m-1\}$. Consequently, the dependence on χ of $\psi^\omega_L(\Theta,u)$ is restricted to those coordinates of χ indexed by $\{-m+1,\dots,m-1\}$. The following proposition will be proven in Section 4.

Proposition 3.4 *For every $M \in \mathbb{N}$ and $(\Theta,u) \in \mathcal{V}^*_M$ there exists a $\psi(\Theta,u) \in \mathbb{R}$ such that*

$$\lim_{\substack{L\to\infty\\ (\Theta,u)\in\mathcal{V}^*_{L,M}}} \psi^\omega_L(\Theta,u) = \psi(\Theta,u) = \psi(\Theta,u;\alpha,\beta) \quad \omega - a.s. \tag{3.32}$$

Moreover, for every $(M,m) \in$ EIGH the convergence is uniform in $(\Theta,u) \in \mathcal{V}^{*,m}_M$.

Uniform bound on the free energies. Pick $(\alpha,\beta) \in$ CONE, $n \in \mathbb{N}$, $\omega \in \{A,B\}^{\mathbb{N}}$, $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, and let $\bar{\mathcal{W}}_n$ be any non-empty subset of $\mathcal{W}_n$ (recall (1.1)). Note that the quenched free energies per monomer introduced until now are all of the form

$$\psi_n = \tfrac{1}{n} \log \sum_{\pi\in\bar{\mathcal{W}}_n} e^{H_n(\pi)}, \tag{3.33}$$

where $H_n(\pi)$ may depend on ω and Ω and satisfies $-\alpha n \leq H_n(\pi) \leq \alpha n$ for all $\pi \in \bar{\mathcal{W}}_n$ (recall that $|\beta| \leq \alpha$ in CONE). Since $1 \leq |\bar{\mathcal{W}}_n| \leq |\mathcal{W}_n| \leq 3^n$, we have

$$|\psi_n| \leq \log 3 + \alpha =^{\mathrm{def}} C_{\mathrm{uf}}(\alpha). \tag{3.34}$$

The uniformity of this bound in n, ω and Ω allows us to average over ω and/or Ω or to let $n \to \infty$.

3.3.2 Variational formulas for the free energy in a single column

We next show how the free energies per column can be expressed in terms of a variational formula involving the path entropy and the single interface free energy defined in Sections 3.1 and 3.2. Throughout this section $M \in \mathbb{N}$ is fixed.

For $\Theta \in \overline{\mathcal{V}}_M$ we need to specify $l_{A,\Theta}$ and $l_{B,\Theta}$, the minimal vertical distances the copolymer must cross in blocks of type A and B, respectively, when crossing a column of type Θ.

Vertical distance to be crossed in columns of class int**.** Pick $\Theta \in \overline{\mathcal{V}}_{\mathrm{int},M}$ and put

$$
\begin{aligned}
l_1 &= 1_{\{\Delta\Pi>0\}}(n_1 - b_0) + 1_{\{\Delta\Pi<0\}}(b_0 - n_0),\\
l_j &= 1_{\{\Delta\Pi>0\}}(n_j - n_{j-1}) + 1_{\{\Delta\Pi<0\}}(n_{-j+2} - n_{-j+1}) \quad \text{for} \quad j \in \{2,\dots,|k_\Theta|\},\\
l_{|k_\Theta|+1} &= 1_{\{\Delta\Pi>0\}}(\Delta\Pi + b_1 - n_{k_\Theta}) + 1_{\{\Delta\Pi<0\}}(n_{k_\Theta+1} - \Delta\Pi - b_1),
\end{aligned} \tag{3.35}
$$

i.e., l_1 is the vertical distance between the entrance point and the first interface, l_i is the vertical distance between the i-th interface and the $(i+1)$-th interface, and $l_{|k_\Theta|+1}$ is the vertical distance between the last interface and the exit point.

Recall that $\Theta = (\chi, \Delta\Pi, b_0, b_1, x)$, and let $l_{A,\Theta}$ and $l_{B,\Theta}$ correspond to the minimal vertical distance the copolymer must cross in blocks of type A and B, respectively, in a column with disorder χ when going from $(0, b_0)$ to $(1, \Delta\Pi + b_1)$, i.e.,

$$
\begin{aligned}
l_{A,\Theta} &= 1_{\{\Delta\Pi>0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{j-1})=A\}} + 1_{\{\Delta\Pi<0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{-j+1})=A\}},\\
l_{B,\Theta} &= 1_{\{\Delta\Pi>0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{j-1})=B\}} + 1_{\{\Delta\Pi<0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{-j+1})=B\}}.
\end{aligned} \tag{3.36}
$$

Vertical distance to be crossed in columns of class nint**.** Depending on χ and $\Delta\Pi$, we further partition $\overline{\mathcal{V}}_{\mathrm{nint},M}$ into four parts

$$
\overline{\mathcal{V}}_{\mathrm{nint},A,1,M} \cup \overline{\mathcal{V}}_{\mathrm{nint},A,2,M} \cup \overline{\mathcal{V}}_{\mathrm{nint},B,1,M} \cup \overline{\mathcal{V}}_{\mathrm{nint},B,2,M}, \tag{3.37}
$$

where $\overline{\mathcal{V}}_{\mathrm{nint},A,x,M}$ and $\overline{\mathcal{V}}_{\mathrm{nint},B,x,M}$ contain those columns with label x for which all the blocks between the entrance and the exit block are of type A and B, respectively. Pick $\Theta \in \mathcal{V}_{\mathrm{nint},M}$. In this case, there is no AB-interface between b_0 and $\Delta\Pi + b_1$, which means that $\Delta\Pi < n_1$ if $\Delta\Pi \geq 0$ and $\Delta\Pi \geq n_0$ if $\Delta\Pi < 0$ (n_0 and n_1 being defined in (3.12)).

For $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},A,1,M}$ we have $l_{B,\Theta} = 0$, whereas $l_{A,\Theta}$ is the vertical distance between the entrance point $(0, b_0)$ and the exit point $(1, \Delta\Pi + b_1)$, i.e.,

$$
l_{A,\Theta} = 1_{\{\Delta\Pi\geq 0\}}(\Delta\Pi - b_0 + b_1) + 1_{\{\Delta\Pi<0\}}(|\Delta\Pi| + b_0 - b_1) + 1_{\{\Delta\Pi=0\}}|b_1 - b_0|, \tag{3.38}
$$

and similarly for $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$ we have obviously $l_{A,\Theta} = 0$ and

$$
l_{B,\Theta} = 1_{\{\Delta\Pi\geq 0\}}(\Delta\Pi - b_0 + b_1) + 1_{\{\Delta\Pi<0\}}(|\Delta\Pi| + b_0 - b_1) + 1_{\{\Delta\Pi=0\}}|b_1 - b_0|. \tag{3.39}
$$

For $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},A,2,M}$, in turn, we have $l_{B,\Theta} = 0$ and $l_{A,\Theta}$ is the minimal vertical distance a trajectory has to cross in a column with disorder χ, starting from $(0, b_0)$, to reach the closest AB-interface before exiting at $(1, \Delta\Pi + b_1)$, i.e.,

$$
l_{A,\Theta} = 1_{\{\Delta\Pi\geq 0\}}(\Delta\Pi - b_0 + b_1) + 1_{\{\Delta\Pi<0\}}(|\Delta\Pi| + b_0 - b_1) + 1_{\{\Delta\Pi=0\}}|b_1 - b_0|, \tag{3.40}
$$

and similarly for $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},B,2,M}$ we have $l_{A,\Theta} = 0$ and

$$l_{B,\Theta} = 1_{\{\Delta\Pi \geq 0\}}(\Delta\Pi - b_0 + b_1) + 1_{\{\Delta\Pi < 0\}}(|\Delta\Pi| + b_0 - b_1) + 1_{\{\Delta\Pi = 0\}}|b_1 - b_0|. \tag{3.41}$$

Variational formula for the free energy in a column. We abbreviate $(h) = (h_A, h_B, h_{\mathcal{I}})$ and $(a) = (a_A, a_B, a_{\mathcal{I}})$. Note that the quantity h_x indicates the fraction of horizontal steps made by the copolymer in solvent x for $x \in \{A, B\}$ and along AB-interfaces for $x = \mathcal{I}$. Similarly, a_x indicates the total number of steps made by the copolymer in solvent x for $x \in \{A, B\}$ and along AB-interfaces for $x = \mathcal{I}$. For $(l_A, l_B) \in [0, \infty)^2$ and $u \geq l_A + l_B + 1$, we put

$$\begin{aligned}\mathcal{L}(l_A, l_B; u) = \big\{(h), (a) \in [0,1]^3 \times [0,\infty)^3\colon\ & h_A + h_B + h_{\mathcal{I}} = 1,\ a_A + a_B + a_{\mathcal{I}} = u \\ & a_A \geq h_A + l_A,\ a_B \geq h_B + l_B,\ a_{\mathcal{I}} \geq h_{\mathcal{I}}\big\}.\end{aligned} \tag{3.42}$$

For $l_A \in [0, \infty)$ and $u \geq 1 + l_A$, we set

$$\begin{aligned}\mathcal{L}_{\mathrm{nint},A,2}(l_A; u) &= \big\{(h), (a) \in \mathcal{L}(l_A, 0; u)\colon\ h_B = a_B = 0\big\},\\ \mathcal{L}_{\mathrm{nint},A,1}(l_A; u) &= \big\{(h), (a) \in \mathcal{L}(l_A, 0; u)\colon\ h_B = a_B = h_{\mathcal{I}} = a_{\mathcal{I}} = 0\big\},\end{aligned} \tag{3.43}$$

and, for $l_B \in [0, \infty)$ and $u \geq 1 + l_B$, we set

$$\begin{aligned}\mathcal{L}_{\mathrm{nint},B,2}(l_B; u) &= \big\{(h), (a) \in \mathcal{L}(0, l_B; u)\colon\ h_A = a_A = 0\big\},\\ \mathcal{L}_{\mathrm{nint},B,1}(l_B; u) &= \big\{(h), (a) \in \mathcal{L}(0, l_B; u)\colon\ h_A = a_A = h_{\mathcal{I}} = a_{\mathcal{I}} = 0\big\}.\end{aligned} \tag{3.44}$$

The following proposition will be proved in Section 4. The free energy per step in a single column is given by the following variational formula.

Proposition 3.5 *For all $\Theta \in \overline{\mathcal{V}}_M$ and $u \geq t_\Theta$,*

$$\psi(\Theta, u; \alpha, \beta) = \sup_{(h),(a) \in \mathcal{L}(\Theta;\, u)} \frac{a_A\, \tilde{\kappa}\big(\frac{a_A}{h_A}, \frac{l_A}{h_A}\big) + a_B\, \big[\tilde{\kappa}\big(\frac{a_B}{h_B}, \frac{l_B}{h_B}\big) + \frac{\beta-\alpha}{2}\big] + a_{\mathcal{I}}\, \phi_{\mathcal{I}}\big(\frac{a_{\mathcal{I}}}{h_{\mathcal{I}}}\big)}{u}, \tag{3.45}$$

with

$$\begin{aligned}\mathcal{L}_{\Theta,u} &= \mathcal{L}(l_A, l_B; u) && \text{if } \Theta \in \overline{\mathcal{V}}_{\mathrm{int},M},\\ \mathcal{L}_{\Theta,u} &= \mathcal{L}_{\mathrm{nint},k,x}(l_k; u) && \text{if } \Theta \in \overline{\mathcal{V}}_{\mathrm{nint},k,x,M},\ k \in \{A, B\} \text{ and } x \in \{1, 2\}.\end{aligned} \tag{3.46}$$

The importance of Proposition 3.5 lies in the fact that it *expresses the free energy in a single column in terms of the path entropy in a single column $\tilde{\kappa}$ and the free energy along a single linear interface $\phi_{\mathcal{I}}$*, which were defined in Sections 3.1–3.2 and are well understood.

3.4 Mesoscopic percolation frequencies

In Section 3.4.1, we associate with each path $\pi \in \mathcal{W}_L$ a coarse-grained path that records the mesoscopic displacement of π in each column. In Section 3.4.2, we define a set of probability laws providing the frequencies with which each type of column can be crossed by the copolymer. This set will be used in Section 5 to state and prove the column-based variational formula. Finally, in Section 3.4.3, we introduce a set of probability laws providing the fractions of horizontal steps that the copolymer can make when travelling inside each solvent with a given slope or along an AB interface. This latter subset appears in the slope-based variational formula.

3.4.1 Coarse-grained paths

For $x \in \mathbb{N}_0 \times \mathbb{Z}$ and $n \in \mathbb{N}$, let $c_{x,n}$ denote the center of the block $\Lambda_{L_n}(x)$ defined in (1.3), i.e.,

$$c_{x,n} = xL_n + (\tfrac{1}{2}, \tfrac{1}{2})L_n, \tag{3.47}$$

and abbreviate

$$(\mathbb{N}_0 \times \mathbb{Z})_n = \{c_{x,n} \colon\ x \in \mathbb{N}_0 \times \mathbb{Z}\}. \tag{3.48}$$

Let $\widehat{\mathcal{W}}$ be the set of *coarse-grained paths* on $(\mathbb{N}_0 \times \mathbb{Z})_n$ that start at $c_{0,n}$, are self-avoiding and are allowed to jump up, down and to the right between neighboring sites of $(\mathbb{N}_0 \times \mathbb{Z})_n$, i.e., the increments of $\widehat{\Pi} = (\widehat{\Pi}_j)_{j\in\mathbb{N}_0} \in \widehat{\mathcal{W}}$ are $(0, L_n), (0, -L_n)$ and $(L_n, 0)$. (These paths are the coarse-grained counterparts of the paths π introduced in (1.1).) For $l \in \mathbb{N} \cup \{\infty\}$, let $\widehat{\mathcal{W}}_l$ be the set of l-step coarse-grained paths.

Recall, for $\pi \in \mathcal{W}_n$, the definitions of N_π and $(v_j(\pi))_{j\leq N_\pi-1}$ given below (1.7). With π we associate a coarse-grained path $\widehat{\Pi} \in \widehat{\mathcal{W}}_{N_\pi}$ that describes how π moves with respect to the blocks. The construction of $\widehat{\Pi}$ is done as follows: $\widehat{\Pi}_0 = c_{(0,0)}$, $\widehat{\Pi}$ moves vertically until it reaches $c_{(0,v_0)}$, moves one step to the right to $c_{(1,v_0)}$, moves vertically until it reaches $c_{(1,v_1)}$, moves one step to the right to $c_{(2,v_1)}$, and so on. The vertical increment of $\widehat{\Pi}$ in the j-th column is $\Delta\widehat{\Pi}_j = (v_j - v_{j-1})L_n$ (see Figs. 11–13).

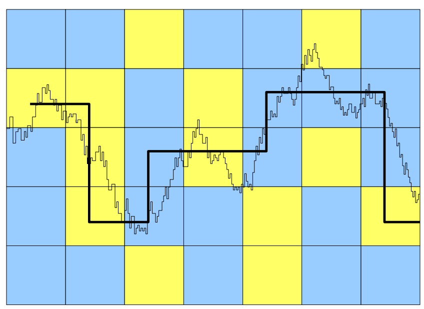

Figure 14: Example of a coarse-grained path.

To characterize a path π, we will often use the sequence of vertical increments of its associated coarse-grained path $\widehat{\Pi}$, modified in such a way that it does not depend on L_n anymore. To that end, with every $\pi \in \mathcal{W}_n$ we associate $\Pi = (\Pi_k)_{k=0}^{N_\pi-1}$ such that $\Pi_0 = 0$ and,

$$\Pi_k = \sum_{j=0}^{k-1} \Delta\Pi_j \quad \text{with} \quad \Delta\Pi_j = \frac{1}{L_n}\Delta\widehat{\Pi}_j, \qquad j = 0, \dots, N_\pi - 1. \tag{3.49}$$

Pick $M \in \mathbb{N}$ and note that $\pi \in \mathcal{W}_{n,M}$ if and only if $|\Delta\Pi_j| \leq M$ for all $j \in \{0, \dots, N_\pi - 1\}$.

3.4.2 Percolation frequencies along coarse-grained paths.

Given $M \in \mathbb{N}$, we denote by $\mathcal{M}_1(\overline{\mathcal{V}}_M)$ the set of probability measures on $\overline{\mathcal{V}}_M$. Pick $\Omega \in \{A, B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $\Pi \in \mathbb{Z}^{\mathbb{N}_0}$ such that $\Pi_0 = 0$ and $|\Delta\Pi_i| \leq M$ for all $i \geq 0$ and $b = (b_j)_{j\in\mathbb{N}_0} \in (\mathbb{Q}_{(0,1]})^{\mathbb{N}_0}$. Set $\Theta_{\text{traj}} = (\Xi_j)_{j\in\mathbb{N}_0}$ with

$$\Xi_j = \big(\Delta\Pi_j, b_j, b_{j+1}\big), \qquad j \in \mathbb{N}_0, \tag{3.50}$$

let

$$\mathcal{X}_{\Pi,\Omega} = \big\{x \in \{1,2\}^{\mathbb{N}_0} \colon\ (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i) \in \mathcal{V}_M \ \ \forall\, i \in \mathbb{N}_0\big\}, \tag{3.51}$$

and for $x \in \mathcal{X}_{\Pi,\Omega}$ set

$$\Theta_j = \big(\Omega(j,\Pi_j+\cdot),\Delta\Pi_j,b_j,b_{j+1},x_j\big), \qquad j \in \mathbb{N}_0. \tag{3.52}$$

With the help of (3.52), we can define the empirical distribution

$$\rho_N(\Omega,\Pi,b,x)(\Theta) = \frac{1}{N}\sum_{j=0}^{N-1} 1_{\{\Theta_j=\Theta\}}, \quad N \in \mathbb{N},\ \Theta \in \overline{\mathcal{V}}_M, \tag{3.53}$$

Definition 3.6 *For $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $M \in \mathbb{N}$, let*

$$\begin{aligned}\mathcal{R}^{\Omega}_{M,N} = \big\{&\rho_N(\Omega,\Pi,b,x) \text{ with } b=(b_j)_{j\in\mathbb{N}_0} \in (\mathbb{Q}_{(0,1]})^{\mathbb{N}_0},\\ &\Pi=(\Pi_j)_{j\in\mathbb{N}_0} \in \{0\}\times\mathbb{Z}^{\mathbb{N}} \colon\ |\Delta\Pi_j| \le M \quad \forall\, j\in\mathbb{N}_0,\\ &x=(x_j)_{j\in\mathbb{N}_0} \in \{1,2\}^{N_0} \colon\ \big(\Omega(j,\Pi_j+\cdot),\Delta\Pi_j,b_j,b_{j+1},x_j\big) \in \mathcal{V}_M\big\}\end{aligned} \tag{3.54}$$

and

$$\mathcal{R}^{\Omega}_M = \text{closure}\Big(\cap_{N'\in\mathbb{N}}\cup_{N\ge N'}\mathcal{R}^{\Omega}_{M,N}\Big), \tag{3.55}$$

both of which are subsets of $\mathcal{M}_1(\overline{\mathcal{V}}_M)$.

Proposition 3.7 *For every $p \in (0,1)$ and $M \in \mathbb{N}$ there exists a closed set $\mathcal{R}_{p,M} \subsetneq \mathcal{M}_1(\overline{\mathcal{V}}_M)$ such that*

$$\mathcal{R}^{\Omega}_M = \mathcal{R}_{p,M} \text{ for } \mathbb{P}\text{-a.e. } \Omega. \tag{3.56}$$

Proof. Note that, for every $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, the set $\mathcal{R}^{\Omega}_M$ does not change when finitely many variables in Ω are changed. Therefore $\mathcal{R}^{\Omega}_M$ is measurable with respect to the tail σ-algebra of Ω. Since Ω is an i.i.d. random field, the claim follows from Kolmogorov's zero-one law. Because of the constraint on the vertical displacement, $\mathcal{R}_{p,M}$ does not coincide with $\mathcal{M}_1(\overline{\mathcal{V}}_M)$. □

Each probability measure $\rho \in \mathcal{R}_{p,M}$ is associated with a strategy of displacement of the copolymer on the mesoscopic scale. As mentioned above, the growth rate of the square blocks in (1.5) ensures that no entropy is carried by the mesoscopic displacement, and this justifies the optimization over $\mathcal{R}_{p,M}$ in the column-based variational formula.

3.4.3 Fractions of horizontal steps per slope

In this section, we introduce $\bar{\mathcal{R}}_{p,M}$ as the counterpart of $\mathcal{R}_{p,M}$ for the slope-based variational formula. To that aim, we define

$$\begin{aligned}\mathcal{E} = \big\{(h_{A,\Theta},h_{B,\Theta},h_{\mathcal{I},\Theta})_{\Theta\in\overline{\mathcal{V}}_M} \in ([0,1]^3)^{\overline{\mathcal{V}}_M} \colon\ &h_{A,\Theta}+h_{B,\Theta}+h_{\mathcal{I},\Theta} = 1\ \forall\,\Theta,\\ &\Theta \mapsto h_{k,\Theta} \text{ Borel } \forall\, k \in \{A,B,\mathcal{I}\},\\ &h_{k,\Theta} > 0 \text{ if } l_{k,\Theta} > 0\ \forall\, k \in \{A,B\},\\ &h_{k,\Theta} = 1 \ \text{ if } \ \Theta \in \mathcal{V}_{\text{nint},k,1,M},\\ &h_{\mathcal{I},\Theta}+h_{k,\Theta} = 1 \ \text{ if } \ \Theta \in \mathcal{V}_{\text{nint},k,2,M}\big\}.\end{aligned} \tag{3.57}$$

With each $\rho \in \mathcal{R}_{p,M}$ and $h \in \mathcal{E}$ associate $G_{\rho,h} \in \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}\big)$, defined by

$$\begin{aligned} G_{\rho,h,A}(dl) &= \int_{\bar{\mathcal{V}}_M} h_{A,\Theta}\, 1\Big\{\tfrac{l_{A,\Theta}}{h_{A,\Theta}} \in dl\Big\}\, \rho(d\Theta), \\ G_{\rho,h,B}(dl) &= \int_{\bar{\mathcal{V}}_M} h_{B,\Theta}\, 1\Big\{\tfrac{l_{B,\Theta}}{h_{B,\Theta}} \in dl\Big\}\, \rho(d\Theta), \\ G_{\rho,h,\mathcal{I}} &= \int_{\bar{\mathcal{V}}_M} h_{\mathcal{I},\Theta}\, \rho(d\Theta), \end{aligned} \tag{3.58}$$

where $l_{k,\Theta}/h_{k,\Theta} = 0$ by convention if $h_{k,\Theta} = 0$ for $\Theta \in \overline{\mathcal{V}}_M$ and $k \in \{A, B\}$. The set $\bar{\mathcal{R}}_{p,M}$ in (1.17) is defined as

$$\bar{\mathcal{R}}_{p,M} = \text{Closure}\,\Big\{\bar{\rho} \in \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}\big)\colon\ \exists\, \rho \in \mathcal{R}_{p,M}, h \in \mathcal{E}\colon\ \bar{\rho} = G_{\rho,h}\Big\}, \tag{3.59}$$

For $\bar{\rho} \in \bar{\mathcal{R}}_{p,M}$, let $\bar{\rho}_A$, $\bar{\rho}_B$ and $\bar{\rho}_{\mathcal{I}}$ denote the restriction of $\bar{\rho}$ to $\mathbb{R}_+$, $\mathbb{R}_+$ and $\{\mathcal{I}\}$, respectively, as in (1.18). The measures $\bar{\rho}_A(dl)$, $\bar{\rho}_B(dl)$ represent the fraction of horizontal steps made by the copolymer when it moves at slope l in solvent A, respectively, B. The number $\bar{\rho}_{\mathcal{I}}$ represents the fraction of horizontal steps made by the copolymer when it moves along the AB-interface.

3.5 Positivity of the free energy

It is easy to prove that for all $p \in (0, 1)$, $M \in \mathbb{N}$ and $(\alpha, \beta) \in$ CONE the two variational formulas (the slope-based variational formula stated in (1.17) and the column-based variational formula stated in (5.2) below and proved in Section 5) are strictly positive, i.e.,

$$f(\alpha, \beta; M, p) > 0. \tag{3.60}$$

To prove that the variational formula in (1.17) is strictly positive, we define $\bar{\rho}_{\text{hor}} \in \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}\big)$ as

$$\bar{\rho}_{\text{hor}} = p^2 \delta_{A,0}(dl) + (1-p)^2 \delta_{B,0}(dl) + 2p(1-p)\delta_{\mathcal{I}}. \tag{3.61}$$

When moving along the x-axis, the pairs of blocks appearing above and below the x-axis have density p^2 for type AA, density $(1-p)^2$ for type BB, and density $2p(1-p)$ for types AB and BA. Consequently, $\bar{\rho}_{\text{hor}}$ belongs to $\bar{\mathcal{R}}_p$ and (1.17) implies that, for any choice of $v_A, v_B \geq 1$, the variational formula in (1.17) is at least

$$\frac{[p^2 + 2p(1-p)]\, v_A\, \tilde{\kappa}(v_A, 0) + (1-p)^2\, v_B\, [\tilde{\kappa}(v_B, 0) + \frac{\beta-\alpha}{2}]}{[p^2 + 2p(1-p)]\, v_A + (1-p)^2\, v_B}. \tag{3.62}$$

Thus, it suffices to pick $v_B = 1$, to recall that $\lim_{u\to\infty} u\tilde{\kappa}(u, 0) = \infty$ (Lemma B.1(iv)), and to choose v_A large enough so that (3.62) becomes strictly positive.

To prove that the variational formula in (5.2) is strictly positive, we can argue similarly, taking both sequences $(\Pi_i)_{i\in\mathbb{N}_0}$ and $(b_i)_{i\in\mathbb{N}_0}$ constant and equal to 0.

4 Proof of Propositions 3.4–3.5

In this section we prove Propositions 3.4 and 3.5, which were stated in Sections 3.3.1 and 3.3.2 and contain the precise definition of the key ingredients of the variational formula in Theorem 5.1. In Section 5 we will use these propositions to prove Theorem 5.1.

In Section 4.1 we associate with each trajectory π in a column a sequence recording the indices of the AB-interfaces successively visited by π. The latter allows us to state a key proposition, Proposition 4.1 below, from which Propositions 3.4 and 3.5 are straightforward consequences. In Section 4.2 we give an outline of the proof of Proposition 4.1, in Sections 4.3–4.5 we provide the details.

4.1 Column crossing characteristic

4.1.1 The order of the visits to the interfaces

Pick $(M, m) \in$ EIGH. To prove Propositions 3.4 and 3.5, instead of considering $(\Theta, u) \in \mathcal{V}_M^{*,m}$, we will restrict to $(\Theta, u) \in \mathcal{V}_{\text{int},M}^{*,m}$. Our proof can be easily extended to $(\Theta, u) \in \mathcal{V}_{\text{nint},M}^{*,m}$.

Pick $(\Theta, u) \in \mathcal{V}_{\text{int},M}^{*,m}$, recall (3.12) and set $\mathcal{J}_{\Theta,u} = \{\mathcal{N}_{\Theta,u}^{\downarrow}, \dots, \mathcal{N}_{\Theta,u}^{\uparrow}\}$, with

$$\begin{aligned} \mathcal{N}_{\Theta,u}^{\uparrow} &= \max\{i \geq 1 \colon n_i \leq u\} \quad \text{and} \quad \mathcal{N}_{\Theta,u}^{\uparrow} = 0 \quad \text{if} \quad n_1 > u. \\ \mathcal{N}_{\Theta,u}^{\downarrow} &= \min\{i \leq 0 \colon |n_i| \leq u\} \quad \text{and} \quad \mathcal{N}_{\Theta,u}^{\downarrow} = 1 \quad \text{if} \quad |n_0| > u. \end{aligned} \tag{4.1}$$

Next pick $L \in \mathbb{N}$ so that $(\Theta, u) \in \mathcal{V}_{\text{int},L,M}^{*}$ and recall that for $j \in \mathcal{J}_{\Theta,u}$ the j-th interface of the Θ-column is $\mathcal{I}_j = \{0, \dots, L\} \times \{n_j L\}$. Note also that $\pi \in \mathcal{W}_{\Theta,u,L}$ makes uL steps inside the column and therefore can not reach the AB-interfaces labelled outside $\{\mathcal{N}_{\Theta,u}^{\downarrow}, \dots, \mathcal{N}_{\Theta,u}^{\uparrow}\}$.

First, we associate with each trajectory $\pi \in \mathcal{W}_{\Theta,u,L}$ the sequence $J(\pi)$ that records the indices of the interfaces that are successively visited by π. Next, we pick $\pi \in \mathcal{W}_{\Theta,u,L}$, and define τ_1, J_1 as

$$\tau_1 = \inf\{i \in \mathbb{N} \colon \exists j \in \mathcal{J}_{\Theta,u} \colon \pi_i \in \mathcal{I}_j\}, \qquad \pi_{\tau_1} \in \mathcal{I}_{J_1}, \tag{4.2}$$

so that $J_1 = 0$ (respectively, $J_1 = 1$) if the first interface reached by π is $\mathcal{I}_0$ (respectively, $\mathcal{I}_1$). For $i \in \mathbb{N} \setminus \{1\}$, we define τ_i, J_i as

$$\tau_i = \inf\big\{t > \tau_{i-1} \colon \exists j \in \mathcal{J}_{\Theta,u} \setminus \{J_{i-1}\}, \pi_i \in \mathcal{I}_j\big\}, \qquad \pi_{\tau_i} \in \mathcal{I}_{J_i}, \tag{4.3}$$

so that the increments of $J(\pi)$ are restricted to -1 or 1. The length of $J(\pi)$ is denoted by $m(\pi)$ and corresponds to the number of jumps made by π between neighboring interfaces before time uL, i.e., $J(\pi) = (J_i)_{i=1}^{m(\pi)}$ with

$$m(\pi) = \max\{i \in \mathbb{N} \colon \tau_i \leq uL\}. \tag{4.4}$$

Note that $(\Theta, u) \in \mathcal{V}_{\text{int},M}^{*,m}$ necessarily implies $k_\Theta \leq m(\pi) \leq u \leq m$. Set

$$\mathcal{S}_r = \{j = (j_i)_{i=1}^r \in \mathbb{Z}^{\mathbb{N}} \colon j_1 \in \{0, 1\}, j_{i+1} - j_i \in \{-1, 1\} \ \forall\, 1 \leq i \leq r-1\}, \qquad r \in \mathbb{N}, \tag{4.5}$$

and, for $\Theta \in \mathcal{V}$, $r \in \{1, \dots, m\}$ and $j \in \mathcal{S}_r$, define

$$\begin{aligned} l_1 &= 1_{\{j_1=1\}}(n_1 - b_0) + 1_{\{j_1=0\}}(b_0 - n_0), \\ l_i &= |n_{j_i} - n_{j_{i-1}}| \text{ for } i \in \{2, \dots, r\}, \\ l_{r+1} &= 1_{\{j_r = k_\Theta + 1\}}(n_{k_\Theta+1} - \Delta\Pi - b_1) + 1_{\{j_r = k_\Theta\}}(\Delta\Pi + b_1 - n_{k_\Theta}), \end{aligned} \tag{4.6}$$

so that $(l_i)_{i\in\{1,\dots,r+1\}}$ depends on Θ and j. Set

$$\begin{aligned} \mathcal{A}_{\Theta,j} &= \{i \in \{1, \dots, r+1\} \colon A \text{ between}\, \mathcal{I}_{j_{i-1}} \text{ and}\, \mathcal{I}_{j_i}\}, \\ \mathcal{B}_{\Theta,j} &= \{i \in \{1, \dots, r+1\} \colon B \text{ between}\, \mathcal{I}_{j_{i-1}} \text{ and}\, \mathcal{I}_{j_i}\}, \end{aligned} \tag{4.7}$$

and set $l_{\Theta,j} = (l_{A,\Theta,j}, l_{B,\Theta,j})$ with

$$l_{A,\Theta,j} = \textstyle\sum_{i\in\mathcal{A}_{\Theta,j}} l_i,\ \ l_{B,\Theta,j} = \sum_{i\in\mathcal{B}_{\Theta,j}} l_i. \tag{4.8}$$

For $L \in \mathbb{N}$ and $(\Theta,u) \in \mathcal{V}^{*,m}_{\mathrm{int},L,M}$, we denote by $\mathcal{S}_{\Theta,u,L}$ the set $\{J(\pi), \pi \in \mathcal{W}_{\Theta,u,L}\}$. It is not difficult to see that a sequence $j \in \mathcal{S}_r$ belongs to $\mathcal{S}_{\Theta,u,L}$ if and only if it satisfies the two following conditions. First, $j_r \in \{k_\Theta, k_\Theta + 1\}$, since j_r is the index of the interface last visited before the Θ-column is exited. Second, $u \geq 1 + l_{A,\Theta,j} + l_{B,\Theta,j}$ because the number of steps taken by a trajectory $\pi \in \mathcal{W}_{\Theta,u,L}$ satisfying $J(\pi) = j$ must be large enough to ensures that all interfaces $\mathcal{I}_{j_s}$, $s \in \{1,\dots,r\}$, can be visited by π before time uL. Consequently, $\mathcal{S}_{\Theta,u,L}$ does not depend on L and can be written as $\mathcal{S}_{\Theta,u} = \cup_{r=1}^m \mathcal{S}_{\Theta,u,r}$, where

$$\mathcal{S}_{\Theta,u,r} = \{j \in \mathcal{S}_r\colon j_r \in \{k_\Theta, k_\Theta+1\}, u \geq 1 + l_{A,\Theta,j} + l_{B,\Theta,j}\}. \tag{4.9}$$

Thus, we partition $\mathcal{W}_{\Theta,u,L}$ according to the value taken by $J(\pi)$, i.e.,

$$\mathcal{W}_{\Theta,u,L} = \bigcup_{r=1}^{m} \bigcup_{j\in\mathcal{S}_{\Theta,u,r}} \mathcal{W}_{\Theta,u,L,j}, \tag{4.10}$$

where $\mathcal{W}_{\Theta,u,L,j}$ contains those trajectories $\pi \in \mathcal{W}_{\Theta,u,L}$ for which $J(\pi) = j$.

Next, for $j \in \mathcal{S}_{\Theta,u}$, we define (recall (3.30))

$$\psi_L^\omega(\Theta,u,j) = \frac{1}{uL}\log Z_L^\omega(\Theta,u,j), \qquad \psi_L(\Theta,u,j) = \mathbb{E}\big[\psi_L^\omega(\Theta,u,j)\big], \tag{4.11}$$

with

$$Z_L^\omega(\Theta,u,j) = \sum_{\pi\in\mathcal{W}_{\Theta,u,L,j}} e^{H^{\omega,\chi}_{uL,L}(\pi)}. \tag{4.12}$$

For each $L \in \mathbb{N}$ satisfying $(\Theta,u) \in \mathcal{V}^{*,m}_{\mathrm{int},L,M}$ and each $j \in \mathcal{S}_{\Theta,u}$, the quantity $l_{A,\Theta,j}L$ (respectively, $l_{B,\Theta,j}L$) corresponds to the minimal vertical distance a trajectory $\pi \in \mathcal{W}_{\Theta,u,L,j}$ has to cross in solvent A (respectively, B).

4.1.2 Key proposition

For simplicity we give the proof for the case $(\Theta,u) \in \mathcal{V}^{*,m}_{\mathrm{int},M}$. The extension to $(\Theta,u) \in \mathcal{V}^{*,m}_{\mathrm{nint},M}$ is straightforward.

Recalling (3.45) and (4.8), we define the free energy associated with Θ, u, j as

$$\begin{aligned}\psi(\Theta,u,j) &= \psi_{\mathrm{int}}(u, l_{\Theta,j}) \\ &= \sup_{(h),(u)\in\mathcal{L}(l_{\Theta,j};\,u)} \frac{u_A\,\tilde\kappa\big(\frac{u_A}{h_A},\frac{l_{A,\Theta,j}}{h_A}\big) + u_B\,\big[\tilde\kappa\big(\frac{u_B}{h_B},\frac{l_{B,\Theta,j}}{h_B}\big) + \frac{\beta-\alpha}{2}\big] + u_I\,\phi(\frac{u^I}{h^I})}{u}.\end{aligned} \tag{4.13}$$

Proposition 4.1 below states that $\lim_{L\to\infty}\psi_L(\Theta,u,j) = \psi(\Theta,u,j)$ uniformly in $(\Theta,u) \in \mathcal{V}^{*,m}_{\mathrm{int},M}$ and $j \in \mathcal{S}_{\Theta,u}$.

Proposition 4.1 *For every $M, m \in \mathbb{N}$ such that $m \geq M+2$ and every $\varepsilon > 0$ there exists an $L_\varepsilon \in \mathbb{N}$ such that*

$$\big|\psi_L(\Theta,u,j) - \psi(\Theta,u,j)\big| \leq \varepsilon \quad \forall\,(\Theta,u) \in \mathcal{V}^{*,m}_{\mathrm{int},L,M},\ \ j \in \mathcal{S}_{\Theta,u},\ \ L \geq L_\varepsilon. \tag{4.14}$$

Proof of Propositions 3.4 and 3.5 subject to Proposition 4.1. Pick $\varepsilon > 0$, $L \in \mathbb{N}$ and $(\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}$. Recall (3.36) and note that $l_A(\Theta)L$ and $l_B(\Theta)L$ are the minimal vertical distances the trajectories of $\mathcal{W}_{\Theta,u,L}$ have to cross in blocks of type A, respectively, B. For simplicity, in what follows the Θ-dependence of l_A and l_B will be suppressed. In other words, l_A and l_B are the two coordinates of $l_{\Theta,f}$ (recall (4.8)) with $f = (1, 2, \dots, |k_\Theta|)$ when $\Delta\Pi \geq 0$ and $f = (0, -1, \dots, -|k_\Theta| + 1)$ when $\Delta\Pi < 0$, so (3.45) and (4.13) imply

$$\psi_{\text{int}}(u, l_A, l_B) = \psi(\Theta, u, f). \tag{4.15}$$

Hence Propositions 3.4 and 3.5 will be proven once we show that $\lim_{L\to\infty} \psi_L(\Theta, u) = \psi(\Theta, u, f)$ uniformly in $(\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}$. Moreover, a look at (4.13), (4.15) and (3.45) allows us to assert that for every $j \in \mathcal{S}_{\Theta,u}$ we have $\psi(\Theta, u, j) \leq \psi(\Theta, u, f)$. The latter is a consequence of the fact that $l \mapsto \tilde{\kappa}(u, l)$ decreases on $[0, u-1]$ (see Lemma B.1(ii) in Appendix A) and that

$$\begin{aligned} l_A &= l_{A,\Theta,f} = \min\{l_{A,\Theta,j} \colon\ j \in \mathcal{S}_{\Theta,u}\}, \\ l_B &= l_{B,\Theta,f} = \min\{l_{B,\Theta,j} \colon\ j \in \mathcal{S}_{\Theta,u}\}. \end{aligned} \tag{4.16}$$

By applying Proposition 4.1 we have, for $L \geq L_\varepsilon$,

$$\begin{aligned} &\psi_L(\Theta, u, j) \leq \psi(\Theta, u, f) + \varepsilon \qquad \forall\, (\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M},\ \forall\, j \in \mathcal{S}_{\Theta,u}, \\ &\psi_L(\Theta, u, f) \geq \psi(\Theta, u, f) - \varepsilon \qquad \forall\, (\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}. \end{aligned} \tag{4.17}$$

The second inequality in (4.17) allows us to write, for $L \geq L_\varepsilon$,

$$\psi(\Theta, u, f) - \varepsilon \leq \psi_L(\Theta, u, f) \leq \psi_L(\Theta, u) \qquad \forall\, (\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}. \tag{4.18}$$

To obtain the upper bound we introduce

$$\mathcal{A}_{L,\varepsilon} = \Big\{\omega\colon\ |\psi^\omega_L(\Theta, u, j) - \psi_L(\Theta, u, j)| \leq \varepsilon \quad \forall\, (\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M},\ \forall\, j \in \mathcal{S}_{\Theta,u}\Big\}, \tag{4.19}$$

so that

$$\begin{aligned} \psi_L(\Theta, u) &\leq \mathbb{E}\big[1_{\mathcal{A}^c_{L,\varepsilon}}\, \psi^\omega_L(\Theta, u)\big] + \mathbb{E}\big[1_{\mathcal{A}_{L,\varepsilon}}\, \psi^\omega_L(\Theta, u)\big] \\ &\leq C_{\text{uf}}(\alpha)\, \mathbb{P}(\mathcal{A}^c_{L,\varepsilon}) + \tfrac{1}{uL}\mathbb{E}\Big[1_{\mathcal{A}_{L,\varepsilon}}\ \log \textstyle\sum_{j\in\mathcal{S}_{\Theta,u}} e^{uL(\psi_L(\Theta,u,j)+\varepsilon)}\Big], \end{aligned} \tag{4.20}$$

where we use (3.34) to bound the first term in the right-hand side, and the definition of $\mathcal{A}_{L,\varepsilon}$ to bound the second term. Next, with the help of the first inequality in (4.17) we can rewrite (4.20) for $L \geq L_\varepsilon$ and $(\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}$ in the form

$$\psi_L(\Theta, u) \leq C_{\text{uf}}(\alpha)\, \mathbb{P}(\mathcal{A}^c_{L,\varepsilon}) + \tfrac{1}{uL} \log|\cup_{r=1}^m \mathcal{S}_r| + \psi(\Theta, u, f) + 2\varepsilon. \tag{4.21}$$

At this stage we want to prove that $\lim_{L\to\infty} \mathbb{P}(\mathcal{A}^c_{L,\varepsilon}) = 0$. To that end, we use the concentration of measure property in (D.3) in Appendix D with $l = uL$, $\Gamma = \mathcal{W}_{\Theta,u,L,j}$, $\eta = \varepsilon uL$, $\xi_i = -\alpha 1\{\omega_i = A\} + \beta 1\{\omega_i = B\}$ for all $i \in \mathbb{N}$ and $T(x, y) = 1\{\chi^{L_n}_{(x,y)} = B\}$. We then obtain that there exist $C_1, C_2 > 0$ such that, for all $L \in \mathbb{N}$, $(\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}$ and $j \in \mathcal{S}_{\Theta,u}$,

$$\mathbb{P}\big(|\psi^\omega_L(\Theta, u, j) - \psi_L(\Theta, u, j)| > \varepsilon\big) \leq C_1\, e^{-C_2\, \varepsilon^2\, uL}. \tag{4.22}$$

The latter inequality, combined with the fact that $|\mathcal{V}^{*,m}_{\mathrm{int},L,M}|$ grows polynomialy in L, allows us to assert that $\lim_{L\to\infty}\mathbb{P}(\mathcal{A}^c_{L,\varepsilon})=0$. Next, we note that $|\cup_{r=1}^m \mathcal{S}_r|<\infty$, so that for L_ε large enough we obtain from (4.21) that, for $L\geq L_\varepsilon$,

$$\psi_L(\Theta,u)\leq \psi(\Theta,u,f)+3\varepsilon \qquad \forall\,(\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{int},L,M}. \tag{4.23}$$

Now (4.18) and (4.23) are sufficient to complete the proof of Propositions 3.4–3.5 in the case $(\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{int},M}$. As mentioned earlier, the proof for the case $(\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{nint},M}$ is entirely similar. □

4.2 Structure of the proof of Proposition 4.1

Intermediate column free energies. Let

$$G^m_M=\big\{(L,\Theta,u,j)\colon\ (\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{int},L,M},\ j\in\mathcal{S}_{\Theta,u}\big\}, \tag{4.24}$$

and define the following order relation.

Definition 4.2 *For $g,\widetilde{g}\colon\, G^m_M\mapsto\mathbb{R}$, write $g\prec\widetilde{g}$ when for every $\varepsilon>0$ there exists an $L_\varepsilon\in\mathbb{N}$ such that*

$$g(L,\Theta,u,j)\leq\widetilde{g}(L,\Theta,u,j)+\varepsilon \qquad \forall\,(L,\Theta,u,j)\in G^m_M\colon\ L\geq L_\varepsilon. \tag{4.25}$$

Recall (4.11) and (4.13), set

$$\psi_1(L,\Theta,u,j)=\psi_L(\Theta,u,j), \qquad \psi_4(L,\Theta,u,j)=\psi(\Theta,u,j), \tag{4.26}$$

and note that the proof of Proposition 4.1 will be complete once we show that $\psi_1\prec\psi_4$ and $\psi_4\prec\psi_1$. In what follows, we will focus on $\psi_1\prec\psi_4$. Each step of the proof can be adapted to obtain $\psi_4\prec\psi_1$ without additional difficulty.

In the proof we need to define two intermediate free energies ψ_2 and ψ_3, in addition to ψ_1 and ψ_4 above. Our proof is divided into 3 steps, organized in Sections 4.3–4.5, and consists of showing that $\psi_1\prec\psi_2\prec\psi_3\prec\psi_4$.

Additional notation. Before stating Step 1, we need some further notation. First, we partition $\mathcal{W}_{\Theta,u,L,j}$ according to the total number of steps and the number of horizontal steps made by a trajectory along and in between AB-interfaces. To that end, we assume that $j\in\mathcal{S}_{\Theta,u,r}$ with $r\in\{1,\dots,m\}$, we recall (4.6) and we let

$$\begin{aligned}\mathcal{D}_{\Theta,L,j}&=\big\{(d_i,t_i)_{i=1}^{r+1}\colon\ d_i\in\mathbb{N}\text{ and }t_i\in d_i+l_iL+2\mathbb{N}_0\ \forall\,1\leq i\leq r+1\big\},\\ \mathcal{D}^{\mathcal{I}}_r&=\big\{(d^{\mathcal{I}}_i,t^{\mathcal{I}}_i)_{i=1}^{r}\colon\ d^{\mathcal{I}}_i\in\mathbb{N}\text{ and }t^{\mathcal{I}}_i\in d^{\mathcal{I}}_i+2\mathbb{N}_0\ \forall\,1\leq i\leq r\big\},\end{aligned} \tag{4.27}$$

where d_i,t_i denote the number of horizontal steps and the total number of steps made by the trajectory between the $(i-1)$-th and i-th interfaces, and $d^{\mathcal{I}}_i,t^{\mathcal{I}}_i$ denote the number of horizontal steps and the total number of steps made by the trajectory along the i-th interface. For $(d,t)\in\mathcal{D}_{\Theta,L,j}$, $(d^{\mathcal{I}},t^{\mathcal{I}})\in\mathcal{D}^{\mathcal{I}}_r$ and $1\leq i\leq r$, we set $T_0=0$ and

$$\begin{aligned}V_i&=\sum_{j=1}^{i}t_j+\sum_{j=1}^{i-1}t^{\mathcal{I}}_j, \qquad i=1,\dots,r,\\ T_i&=\sum_{j=1}^{i}t_j+\sum_{j=1}^{i}t^{\mathcal{I}}_j, \qquad i=1,\dots,r,\end{aligned} \tag{4.28}$$

so that V_i, respectively, T_i indicates the number of steps made by the trajectory when reaching, respectively, leaving the i-th interface.

Next, we let $\theta\colon \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}^{\mathbb{N}}$ be the left-shift acting on infinite sequences of real numbers and, for $u \in \mathbb{N}$ and $\omega \in \{A,B\}^{\mathbb{N}}$, we put

$$H_u^{\omega}(B) = \sum_{i=1}^{u} \big[\beta\, 1_{\{\omega_i = B\}} - \alpha\, 1_{\{\omega_i = A\}}\big]. \tag{4.29}$$

Finally, we recall that

$$\psi_1(L,\Theta,u,j) = \tfrac{1}{uL}\, \mathbb{E}[\log Z_1^{\omega}(L,\Theta,u,j)], \tag{4.30}$$

where the partition function defined in (3.31) has been renamed Z_1 and can be written in the form

$$Z_1^{\omega}(L,\Theta,u,j) = \sum_{(d,t)\in\mathcal{D}_{\Theta,L,j}} \;\sum_{(d^{\mathcal{I}},t^{\mathcal{I}})\in\mathcal{D}_r^{\mathcal{I}}} A_1\, B_1\, C_1, \tag{4.31}$$

where (recall (4.7) and (3.5))

$$\begin{aligned} A_1 &= \prod_{i\in\mathcal{A}_{\Theta,j}} e^{t_i\,\tilde\kappa_{d_i}\left(\frac{t_i}{d_i},\frac{l_iL}{d_i}\right)} \prod_{i\in\mathcal{B}_{\Theta,j}} e^{t_i\,\tilde\kappa_{d_i}\left(\frac{t_i}{d_i},\frac{l_iL}{d_i}\right)}\, e^{H_{t_i}^{\theta^{T_{i-1}}(w)}(B)}, \\ B_1 &= \prod_{i=1}^{r} e^{t_i^{\mathcal{I}}\,\phi_{d_i^{\mathcal{I}}}^{\theta^{V_i}(w)}\left(\frac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\right)}, \\ C_1 &= 1_{\left\{\sum_{i=1}^{r+1} d_i + \sum_{i=1}^{r} d_i^{\mathcal{I}} = L\right\}}\, 1_{\left\{\sum_{i=1}^{r+1} t_i + \sum_{i=1}^{r} t_i^{\mathcal{I}} = uL\right\}}. \end{aligned} \tag{4.32}$$

It is important to note that a simplification has been made in the term A_1 in (4.32). Indeed, this term is not $\tilde\kappa_{d_i}(\cdot,\cdot)$ defined in (3.2), since the latter does not take into account the vertical restrictions on the path when it moves from one interface to the next. However, the fact that two neighboring AB-interfaces are necessarily separated by a distance at least L allows us to apply Lemma A.5 in Appendix A.2, which ensures that these vertical restrictions can be removed at the cost of a negligible error.

To show that $\psi_1 \prec \psi_2 \prec \psi_3 \prec \psi_4$, we fix $(M,m) \in \text{EIGH}$ and $\varepsilon > 0$, and we show that there exists an $L_\varepsilon \in \mathbb{N}$s such that $\psi_k(L,\Theta,u,j) \le \psi_{k+1}(L,\Theta,u,j) + \varepsilon$ for all $(L,\Theta,u,j) \in G_M^m$ and $L \ge L_\varepsilon$. The latter will complete the proof of Proposition 4.1.

4.3 Step 1

In this step, we remove the ω-dependence from $Z_1^{\omega}(L,\Theta,u,j)$. To that aim, we put

$$\psi_2(L,\Theta,u,j) = \frac{1}{uL} \log Z_2(L,\Theta,u,j) \tag{4.33}$$

with

$$Z_2(L,\Theta,u,j) = \sum_{(d,t)\in\mathcal{D}_{\Theta,L,j}} \;\sum_{(d^{\mathcal{I}},t^{\mathcal{I}})\in\mathcal{D}_r^{\mathcal{I}}} A_2\; B_2\; C_2, \tag{4.34}$$

where

$$A_2 = \prod_{i\in\mathcal{A}_{\Theta,j}} e^{t_i\,\tilde\kappa_{d_i}\left(\frac{t_i}{d_i},\,\frac{l_iL}{d_i}\right)} \prod_{i\in\mathcal{B}_{\Theta,j}} e^{t_i\,\tilde\kappa_{d_i}\left(\frac{t_i}{d_i},\,\frac{l_iL}{d_i}\right)}\, e^{\frac{\beta-\alpha}{2}\,t_i}, \tag{4.35}$$

$$B_2 = \prod_{i=1}^{r} e^{t_i^{\mathcal{I}}\,\phi_{d_i^{\mathcal{I}}}\left(\frac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\right)},$$

$$C_2 = C_1.$$

Next, for $n\in\mathbb{N}$ we define

$$\mathcal{A}_{\varepsilon,n} = \Big\{\exists\, 0\le t,s\le n\colon\ t\ge \varepsilon n,\ \big|H_t^{\theta^s(\omega)}(B) - \tfrac{\beta-\alpha}{2}t\big| > \varepsilon t\Big\},$$

$$\mathcal{B}_{\varepsilon,n} = \Big\{\exists\, 0\le t,d,s\le n\colon\ t\in d+2\mathbb{N}_0,\ t\ge\varepsilon n,\ \big|\phi_d^{\theta^s(w)}(\tfrac{t}{d}) - \phi_d(\tfrac{t}{d})\big| > \varepsilon\Big\}. \tag{4.36}$$

By applying Cramér's theorem for i.i.d. random variables (see e.g. den Hollander [3], Chapter 1), we obtain that there exist $C_1(\varepsilon), C_2(\varepsilon) > 0$ such that

$$\mathbb{P}\big(\big|H_t^{\theta^s(w)}(B) - \tfrac{\beta-\alpha}{2}t\big| > \varepsilon t\big) \le C_1(\varepsilon)\, e^{-C_2(\varepsilon)t}, \qquad t,s\in\mathbb{N}. \tag{4.37}$$

By using the concentration of measure property in (D.3) in Appendix D with $l=t$, $\Gamma = \mathcal{W}_d^{\mathcal{I}}(\frac{t}{d})$, $T(x,y) = 1\{(x,y)<0\}$, $\eta=\varepsilon t$ and $\xi_i = -\alpha 1\{\omega_i = A\} + \beta 1\{\omega_i = B\}$ for all $i\in\mathbb{N}$, we find that there exist $C_1, C_2 > 0$ such that

$$\mathbb{P}\big(\big|\phi_d^{\theta^s(w)}(\tfrac{t}{d}) - \phi_d(\tfrac{t}{d})\big| > \varepsilon\big|\big) \le C_1\, e^{-C_2\,\varepsilon^2 t}, \qquad t,d,s\in\mathbb{N},\ t\in d+2\mathbb{N}_0. \tag{4.38}$$

With the help of (3.34) and (4.30) we may write, for $(L,\Theta,u,j)\in G_M^m$,

$$\psi_1(L,\Theta,u,j) \le C_{\mathrm{uf}}(\alpha)\,\mathbb{P}\big(\mathcal{A}_{\varepsilon,mL}\cup\mathcal{B}_{\varepsilon,mL}\big) + \tfrac{1}{uL}\,\mathbb{E}\big[1_{\{\mathcal{A}^c_{\varepsilon,mL}\cap\mathcal{B}^c_{\varepsilon,mL}\}}\,\log Z_1^{\omega}(L,\Theta,u,j)\big]. \tag{4.39}$$

With the help of (4.37) and (4.38), we get that $\mathbb{P}(\mathcal{A}_{\varepsilon,mL})\to 0$ and $\mathbb{P}(\mathcal{B}_{\varepsilon,mL})\to 0$ as $L\to\infty$. Moreover, from ((4.31)-(4.36)) it follows that, for $(L,\Theta,u,j)\in G_M^m$ and $\omega\in\mathcal{A}^c_{\varepsilon,mL}\cap\mathcal{B}^c_{\varepsilon,ML}$,

$$Z_1^{\omega}(L,\Theta,u,j) \le Z_2(L,\Theta,u,j)\, e^{\varepsilon uL}. \tag{4.40}$$

The latter completes the proof of $\psi_1 \prec \psi_2$.

4.4 Step 2

In this step, we concatenate the pieces of trajectories that travel in A-blocks, respectively, B-blocks, respectively, along the AB-interfaces and replace the finite-size entropies and free energies by their infinite-size counterparts. Recall the definition of $l_{A,\Theta,j}$ and $l_{B,\Theta,j}$ in (4.8) and define, for $(L,\Theta,u,j)\in G_M^m$, the sets

$$\mathcal{J}_{\Theta,L,j} = \Big\{\big(a_A,h_A,a_B,h_B\big)\in\mathbb{N}^4\colon\ a_A\in l_{A,\Theta,j}L+h_A+2\mathbb{N}_0,\ a_B\in l_{B,\Theta,j}L+h_B+2\mathbb{N}_0\Big\}, \tag{4.41}$$

$$\mathcal{J}^{\mathcal{I}} = \Big\{\big(a^{\mathcal{I}},h^{\mathcal{I}}\big)\in\mathbb{N}^2\colon\ a^{\mathcal{I}}\in h^{\mathcal{I}}+2\mathbb{N}_0\Big\},$$

and put $\psi_3(L,\Theta,u,j) = \frac{1}{uL}\log Z_3(L,\Theta,u,j)$ with

$$Z_3(L,\Theta,u,j) = \sum_{(a,h)\in\mathcal{J}_{\Theta,L,j}} \sum_{(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}} A_3\, B_3\, C_3, \tag{4.42}$$

where

$$\begin{aligned} A_3 &= e^{a_A\,\tilde\kappa\left(\frac{a_A}{h_A},\,\frac{l_{A,\Theta,j}L}{h_A}\right)}\, e^{a_B\,\tilde\kappa\left(\frac{a_B}{h_B},\,\frac{l_{B,\Theta,j}L}{h_B}\right)}\, e^{\frac{\beta-\alpha}{2}\,a_B},\\ B_3 &= e^{a^{\mathcal{I}}\,\phi\left(\frac{a^{\mathcal{I}}}{h^{\mathcal{I}}}\right)},\\ C_3 &= 1_{\{a_A+a_B+a^{\mathcal{I}}=uL\}}\, 1_{\{h_A+h_B+h^{\mathcal{I}}=L\}}. \end{aligned} \tag{4.43}$$

In order to establish a link between ψ_2 and ψ_3 we define, for $(a,h)\in\mathcal{J}_{\Theta,L,j}$ and $(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}$,

$$\begin{aligned} \mathcal{P}_{(a,h)} &= \left\{(t,d)\in\mathcal{D}_{\Theta,L,j}\colon\ \textstyle\sum_{i\in\mathcal{A}_{\Theta,j}}(t_i,d_i) = (a_A,h_A),\ \sum_{i\in\mathcal{B}_{\Theta,j}}(t_i,d_i) = (a_B,h_B)\right\},\\ \mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})} &= \left\{(t^{\mathcal{I}},d^{\mathcal{I}})\in\mathcal{D}^{\mathcal{I}}_r\colon\ \textstyle\sum_{i=1}^r (t^{\mathcal{I}}_i,d^{\mathcal{I}}_i) = (a^{\mathcal{I}},h^{\mathcal{I}})\right\}. \end{aligned} \tag{4.44}$$

Then we can rewrite Z_2 as

$$Z_2(L,\Theta,u,j) = \sum_{(a,h)\in\mathcal{J}_{\Theta,L,j}} \sum_{(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}} C_3 \sum_{(t,d)\in\mathcal{P}_{(a,h)}} \sum_{(t^{\mathcal{I}},d^{\mathcal{I}})\in\mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})}} A_2\, B_2. \tag{4.45}$$

To prove that $\psi_2 \prec \psi_3$, we need the following lemma.

Lemma 4.3 *For every $\eta>0$ there exists an $L_\eta\in\mathbb{N}$ such that, for every $(L,\Theta,u,j)\in G^m_M$ with $L\geq L_\eta$ and every $(d,t)\in\mathcal{D}_{\Theta,L,j}$ and $(d^{\mathcal{I}},t^{\mathcal{I}})\in\mathcal{D}^{\mathcal{I}}_r$ satisfying $\sum_{i=1}^{r+1} d_i + \sum_{i=1}^r d^{\mathcal{I}}_i = L$ and $\sum_{i=1}^{r+1} t_i + \sum_{i=1}^r t^{\mathcal{I}}_i = uL$,*

$$\begin{aligned} t_i\,\tilde\kappa\big(\tfrac{t_i}{d_i},\,\tfrac{l_iL}{d_i}\big) - \eta uL \leq t_i\,\tilde\kappa_{d_i}\big(\tfrac{t_i}{d_i},\,\tfrac{l_iL}{d_i}\big) &\leq t_i\,\tilde\kappa\big(\tfrac{t_i}{d_i},\,\tfrac{l_iL}{d_i}\big) + \eta uL \quad i=1,\dots,r+1,\\ t^{\mathcal{I}}_i\phi(\tfrac{t^{\mathcal{I}}_i}{d^{\mathcal{I}}_i}) - \eta uL \leq t^{\mathcal{I}}_i\phi_{d^{\mathcal{I}}_i}(\tfrac{t^{\mathcal{I}}_i}{d^{\mathcal{I}}_i}) &\leq t^{\mathcal{I}}_i\phi(\tfrac{t^{\mathcal{I}}_i}{d^{\mathcal{I}}_i}) + \eta uL \quad i=1,\dots,r. \end{aligned} \tag{4.46}$$

Proof. By using Lemmas A.1 and C.2 in Appendix A, we have that there exists a $\tilde L_\eta\in\mathbb{N}$ such that, for $L\geq\tilde L_\eta$, $(u,l)\in\mathcal{H}_L$ and $\mu\in 1+\frac{2\mathbb{N}}{L}$,

$$|\tilde\kappa_L(u,l) - \tilde\kappa(u,l)| \leq \eta, \qquad |\phi^{\mathcal{I}}_L(\mu) - \phi^{\mathcal{I}}(\mu)| \leq \eta. \tag{4.47}$$

Moreover, Lemmas 3.1, B.1(ii–iii), C.1(ii) and C.2 ensure that there exists a $v_\eta>1$ such that, for $L\geq 1$, $(u,l)\in\mathcal{H}_L$ with $u\geq v_\eta$ and $\mu\in 1+\frac{2\mathbb{N}}{L}$ with $\mu\geq v_\eta$,

$$0\leq\tilde\kappa_L(u,l)\leq\eta, \qquad 0\leq\phi_L(\mu)\leq\eta. \tag{4.48}$$

Note that the two inequalities in (4.48) remain valid when $L=\infty$. Next, we set $r_\eta = \eta/(2v_\eta C_{\mathrm{uf}})$ and $L_\eta = \tilde L_\eta/r_\eta$, and we consider $L\geq L_\eta$. Because of the left-hand side of (4.47), the two inequalities in the first line of (4.46) hold when $d_i\geq r_\eta L\geq\tilde L_\eta$. We deal with the case $d_i\leq r_\eta L$ by considering first the case $t_i\leq\eta uL/2C_{\mathrm{uf}}$, which is easy because $\tilde\kappa_{d_i}$ and $\tilde\kappa$ are uniformly bounded by C_{uf} (see (3.34)). The case $t_i\geq\eta uL/2C_{\mathrm{uf}}$ gives $t_i/d_i\geq uv_\eta\geq v_\eta$, which by the left-hand side of (4.48) completes the proof of the first line in (4.46). The same observations applied to $t_i^{\mathcal{I}}, d_i^{\mathcal{I}}$ combined with the right-hand side of (4.47) and (4.48) provide the two inequalities in the second line in (4.46). □

To prove that $\psi_2 \prec \psi_3$, we apply Lemma 4.3 with $\eta = \varepsilon/(2m+1)$ and we use (4.35) to obtain, for $L \geq L_{\varepsilon/(2m+1)}$, $(d,t) \in \mathcal{D}_{\Theta,L,j}$ and $(d^{\mathcal{I}}, t^{\mathcal{I}}) \in \mathcal{D}_r^{\mathcal{I}}$,

$$\begin{aligned} A_2 &\leq \prod_{i \in \mathcal{A}_{\Theta,j}} e^{t_i\, \tilde\kappa\left(\frac{t_i}{d_i}, \frac{l_i L}{d_i}\right) + \frac{\varepsilon u L}{2m+1}} \prod_{i \in \mathcal{B}_{\Theta,j}} e^{t_i\, \tilde\kappa\left(\frac{t_i}{d_i}, \frac{l_i L}{d_i}\right) + t_i\, \frac{\beta-\alpha}{2} + \frac{\varepsilon u L}{2m+1}}, \\ B_2 &\leq \prod_{i=1}^r e^{t_i^{\mathcal{I}}\, \phi\left(\frac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\right) + \frac{\varepsilon u L}{2m+1}}. \end{aligned} \tag{4.49}$$

Next, we pick $(a,h) \in \mathcal{J}_{\Theta,L,j}$, $(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}$, $(t,d) \in \mathcal{P}_{(a,h)}$ and $(t^{\mathcal{I}}, d^{\mathcal{I}}) \in \mathcal{Q}_{(a^{\mathcal{I}}, h^{\mathcal{I}})}$, and we use the concavity of $(a,b) \mapsto a\tilde\kappa(a,b)$ and $\mu \mapsto \phi^{\mathcal{I}}(\mu)$ (see Lemma B.1 in Appendix A and Lemma C.1 in Appendix C) to rewrite (4.49) as

$$\begin{aligned} A_2 &\leq e^{a_A\, \tilde\kappa\left(\frac{a_A}{h_A}, \frac{l_{A,\Theta,j} L}{h_A}\right) + a_B\, \tilde\kappa\left(\frac{a_B}{h_B}, \frac{l_{B,\Theta,j} L}{h_B}\right) + \frac{\beta-\alpha}{2} a_B + \frac{\varepsilon (r+1) u L}{2m+1}} = A_3\, e^{\frac{\varepsilon (r+1) u L}{2m+1}}, \\ B_2 &\leq e^{a^{\mathcal{I}}\, \phi^{\mathcal{I}}\left(\frac{a^{\mathcal{I}}}{h^{\mathcal{I}}}\right) + \frac{\varepsilon r u L}{2m+1}} = B_3\, e^{\frac{\varepsilon r u L}{2m+1}}. \end{aligned} \tag{4.50}$$

Moreover, r, which is the number of AB interfaces crossed by the trajectories in $\mathcal{W}_{\Theta,u,j,L}$, is at most m (see (4.10)), so that (4.50) allows us to rewrite (4.45) as

$$Z_2(L,\Theta,u,j) \leq e^{\varepsilon u L} \sum_{(a,h) \in \mathcal{J}_{\Theta,L,j}} \sum_{(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}} C_3\, |\mathcal{P}_{(a,h)}|\, |\mathcal{Q}_{(a^{\mathcal{I}}, h^{\mathcal{I}})}|\, A_3\, B_3. \tag{4.51}$$

Finally, it turns out that $|\mathcal{P}_{(a,h)}| \leq (uL)^{8r}$ and $|\mathcal{Q}_{(a^{\mathcal{I}}, h^{\mathcal{I}})}| \leq (uL)^{8r}$. Therefore, since $r \leq m$, (4.42) and (4.51) allow us to write, for $(L,\Theta,u,j) \in G_M^m$ and $L \geq L_{\varepsilon/2m+1}$,

$$Z_2(L,\Theta,u,j) \leq (mL)^{16m} Z_3(L,\Theta,u,j). \tag{4.52}$$

The latter is sufficient to conclude that $\psi_2 \prec \psi_3$.

4.5 Step 3

For every $(L,\Theta,u,j) \in G_M^m$ we have, by the definition of $\mathcal{L}(l_{A,\Theta,j}, l_{B,\Theta,j}; u)$ in (3.42), that $(a,h) \in \mathcal{J}_{\Theta,L,j}$ and $(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}$ satisfying $a_A + a_B + a^{\mathcal{I}} = uL$ and $h_A + h_B + h^{\mathcal{I}} = L$ also satisfy

$$\Big(\big(\tfrac{a_A}{L}, \tfrac{a_B}{L}, \tfrac{a^{\mathcal{I}}}{L}\big), \big(\tfrac{h_A}{L}, \tfrac{h_B}{L}, \tfrac{h^{\mathcal{I}}}{L}\big) \Big) \in \mathcal{L}(l_{A,\Theta,j}, l_{B,\Theta,j}; u). \tag{4.53}$$

Hence, (4.53) and the definition of $\psi_{\mathcal{I}}$ in (3.45) ensure that, for this choice of (a,h) and $(a^{\mathcal{I}}, h^{\mathcal{I}})$,

$$A_3 B_3 \leq e^{uL\psi_{\mathcal{I}}(u, l_{A,\Theta,j}, l_{B,\Theta,j})}. \tag{4.54}$$

Because of C_3, the summation in (4.42) is restricted to those $(a,h) \in \mathcal{J}_{\Theta,L,j}$ and $(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}$ for which $a_A, a_B, a^{\mathcal{I}} \leq uL$ and $h_A, h_B, h^{\mathcal{I}} \leq L$. Hence, the summation is restricted to a set of cardinality at most $(uL)^3 L^3$. Consequently, for all $(L,\Theta,u,j) \in G_M^m$ we have

$$Z_3(L,\Theta,u,j) = \sum_{(a,h) \in \mathcal{J}_{\Theta,L,j}} \sum_{(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}} A_4\, B_4\, C_4 \leq (mL)^3 L^3\, e^{u\, L\, \psi_{\mathcal{I}}(u, l_{A,\Theta,j}, l_{B,\Theta,j})}. \tag{4.55}$$

The latter implies that $\psi_3 \prec \psi_4$ since $\psi_4 = \psi_{\mathcal{I}}(u,\, l_{A,\Theta,j},\, l_{B,\Theta,j})$ by definition (recall (4.13) and (4.26)).

5 Column-based variational formula

To derive the *slope-based variational formula* that is the cornerstone of our analysis, we state and prove in this section an auxiliary variational formula for the quenched free energy per step that involves the fraction of the time spent by the copolymer in each type of block columns and the free energy per step of the copolymer in a given block column. This auxiliary variational formula will be used in Section 6 in combination with Proposition 3.5 to complete the proof of the *slope-based variational formula.*

With each $\Theta \in \overline{\mathcal{V}}_M$ we associate a quantity $u_\Theta \in [t_\Theta, \infty)$ indicating how many *steps on scale* L_n the copolymer makes in columns of type Θ, where t_Θ is the minimal number of steps required to cross a column of type Θ. These numbers are gathered into the set

$$\mathcal{B}_{\overline{\mathcal{V}}_M} = \big\{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M} \in \mathbb{R}^{\overline{\mathcal{V}}_M} \colon\ u_\Theta \geq t_\Theta \ \forall\, \Theta \in \overline{\mathcal{V}}_M,\ \Theta \mapsto u_\Theta \text{ continuous}\big\}, \tag{5.1}$$

where the continuity in Θ is with respect to the distance d_M defined in (C.7) in Appendix C.2. We recall Proposition 3.5, which identifies the free energy per step $\psi(\Theta, u_\Theta; \alpha, \beta)$ associated with the copolymer when crossing a column of type Θ in u_Θ steps, and we recall that the set $\mathcal{R}_{p,M}$ introduced in Section 3.4.2 gathers the *frequencies with which different types of columns can be visited by the copolymer.*

Proposition 5.1 *(column-based variational formula) For every* $(\alpha, \beta) \in$ CONE*,* $M \in \mathbb{N}$ *and* $p \in (0,1)$ *the free energy in* (1.12) *exists for* $\mathbb{P}$*-a.e.* (ω, Ω) *and in* $L^1(\mathbb{P})$*, and is given by*

$$f(\alpha, \beta;\, M, p) = \sup_{\rho\in\mathcal{R}_{p,M}}\ \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M}\in\mathcal{B}_{\overline{\mathcal{V}}_M}} \frac{N(\rho,u)}{D(\rho,u)}, \tag{5.2}$$

where

$$\begin{aligned} N(\rho,u) &= \int_{\overline{\mathcal{V}}_M} u_\Theta\, \psi(\Theta, u_\Theta; \alpha, \beta)\, \rho(d\Theta),\\ D(\rho,u) &= \int_{\overline{\mathcal{V}}_M} u_\Theta\, \rho(d\Theta), \end{aligned} \tag{5.3}$$

with the convention that $N(\rho,u)/D(\rho,u) = -\infty$ *when* $D(\rho,u) = \infty$.

The present section is technically involved because it goes through a sequence of approximation steps in which the self-averaging of the free energy with respect to ω and Ω in the limit as $n \to \infty$ is proven, and the various ingredients of the variational formula in Theorem 5.1 that were constructed in Section 3 are put together.

In Section 5.1 we introduce additional notation and state Propositions 5.2, 5.3 and 5.14 from which Theorem 5.1 is a straightforward consequence. Proposition 5.2, which deals with $(M, m) \in$ EIGH, is proven in Section 5.2 and the details of the proof are worked out in Sections 5.2.1–5.2.5, organized into 5 Steps that link intermediate free energies. We pass to the limit $m \to \infty$ with Propositions 5.3 and 5.4 which are proven in Section 5.3 and 5.4, respectively.

5.1 Proof of Theorem 5.1

5.1.1 Additional notation

Pick $(M,m) \in$ EIGH and recall that Ω and ω are independent, i.e., $\mathbb{P} = \mathbb{P}_\omega \times \mathbb{P}_\Omega$. For $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $\omega \in \{A,B\}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $(\alpha,\beta) \in$ CONE, define

$$f_{1,n}^{\omega,\Omega}(M,m;\alpha,\beta) = \tfrac{1}{n}\log Z_{1,n,L_n}^{\omega,\Omega}(M,m) \quad \text{with} \quad Z_{1,n,L_n}^{\omega,\Omega}(M,m) = \sum_{\pi\in\mathcal{W}_{n,M,}^m} e^{H_{n,L_n}^{\omega,\Omega}(\pi)}, \tag{5.4}$$

where $\mathcal{W}_{n,M}^m$ contains those paths in $\mathcal{W}_{n,M}$ that, in each column, make at most mL_n steps. We also restrict the set $\mathcal{R}_{p,M}$ in (3.6) to those limiting empirical measures whose support is included in $\overline{\mathcal{V}}_M^m$, i.e., those measures charging the types of column that can be crossed in less than mL_n steps only. To that aim we recall (3.54) and define, for $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $N \in \mathbb{N}$,

$$\begin{aligned} \mathcal{R}_{M,N}^{\Omega,m} = \big\{\rho_N(\Omega,\Pi,b,x) \text{ with } & b = (b_j)_{j\in\mathbb{N}_0} \in (\mathbb{Q}_{(0,1]})^{\mathbb{N}_0},\\ & \Pi = (\Pi_j)_{j\in\mathbb{N}_0} \in \{0\}\times\mathbb{Z}^{\mathbb{N}}\colon\ |\Delta\Pi_j| \leq M \quad \forall\, j\in\mathbb{N}_0,\\ & x = (x_j)_{j\in\mathbb{N}_0} \in \{1,2\}^{N_0}\colon\ \big(\Omega(j,\Pi_j+\cdot),\Delta\Pi_j,b_j,b_{j+1},x_j\big) \in \mathcal{V}_M^m\big\} \end{aligned} \tag{5.5}$$

which is a subset of $\mathcal{R}_{M,N}^{\Omega}$ and allows us to define

$$\mathcal{R}_M^{\Omega,m} = \text{closure}\Big(\cap_{N'\in\mathbb{N}}\cup_{N\geq N'}\mathcal{R}_{M,N}^{\Omega,m}\Big), \tag{5.6}$$

which, for $\mathbb{P}$-a.e. Ω is equal to $\mathcal{R}_{p,M}^m \subsetneq \mathcal{R}_{p,M}$.

At this stage, we further define,

$$f(M,m;\alpha,\beta) = \sup_{\rho\in\mathcal{R}_{p,M}^m}\ \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M^m}\in\mathcal{B}_{\overline{\mathcal{V}}_M^m}} V(\rho,u), \tag{5.7}$$

where

$$V(\rho,u) = \frac{\int_{\overline{\mathcal{V}}_M^m} u_\Theta\,\psi(\Theta,u_\Theta;\alpha,\beta)\,\rho(d\Theta)}{\int_{\overline{\mathcal{V}}_M^m} u_\Theta\,\rho(d\Theta)}, \tag{5.8}$$

where (recall (3.25))

$$\mathcal{B}_{\overline{\mathcal{V}}_M^m} = \Big\{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M^m} \in \mathbb{R}^{\overline{\mathcal{V}}_M^m}\colon \Theta\mapsto u_\Theta \in \mathcal{C}^0\big(\overline{\mathcal{V}}_M^m,\mathbb{R}\big),\, t_\Theta \leq u_\Theta \leq m\ \forall\,\Theta\in\overline{\mathcal{V}}_M^m\Big\}, \tag{5.9}$$

and where $\overline{\mathcal{V}}_M^m$ is endowed with the distance d_M defined in (C.7) in Appendix C.2.

Let $\mathcal{W}_{n,M}^{*,m} \subset \mathcal{W}_{n,M}^m$ be the subset consisting of those paths whose endpoint lies at the boundary between two columns of blocks, i.e., satisfies $\pi_{n,1} \in \mathbb{N}L_n$. Recall (5.4), and define $Z_{n,L_n}^{*,\omega,\Omega}(M)$ and $f_{1,n}^{*,\omega,\Omega}(M,m;\alpha,\beta)$ as the counterparts of $Z_{n,L_n}^{\omega,\Omega}(M,m)$ and $f_{1,n}^{\omega,\Omega}(M,m;\alpha,\beta)$ when $\mathcal{W}_{n,M}^m$ is replaced by $\mathcal{W}_{n,M}^{*,m}$. Then there exists a constant $c>0$, depending on α and β only, such that

$$\begin{aligned} & Z_{1,n,L_n}^{\omega,\Omega}(M,m)e^{-cL_n} \leq Z_{1,n,L_n}^{*,\omega,\Omega}(M,m) \leq Z_{1,n,L_n}^{\omega,\Omega}(M,m),\\ & n\in\mathbb{N},\ \omega\in\{A,B\}^{\mathbb{N}},\ \Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}. \end{aligned} \tag{5.10}$$

The left-hand side of the latter inequality is obtained by changing the last L_n steps of each trajectory in $\mathcal{W}_{n,M}^m$ to make sure that the endpoint falls in $L_n\mathbb{N}$. The energetic and entropic cost of this change are obviously $O(L_n)$. By assumption, $\lim_{n\to\infty} L_n/n = 0$, which together with (5.10) implies that the limits of $f_{1,n}^{\omega,\Omega}(M,m;\alpha,\beta)$ and $f_{1,n}^{*,\omega,\Omega}(M,m;\alpha,\beta)$ as $n\to\infty$ are the same. In the sequel we will therefore restrict the summation in the partition function to $\mathcal{W}_{n,M}^{*,m}$ and drop the $*$ from the notations.

Finally, let

$$\begin{aligned} f_{1,n}^{\Omega}(M,m;\alpha,\beta) &= \mathbb{E}_\omega\big[f_{1,n}^{\omega,\Omega}(M,m;\alpha,\beta)\big],\\ f_{1,n}(M,m;\alpha,\beta) &= \mathbb{E}_{\omega,\Omega}\big[f_{1,n}^{\omega,\Omega}(M,m;\alpha,\beta)\big], \end{aligned} \tag{5.11}$$

and recall (1.11) to set $f_n^{\Omega}(M;\alpha,\beta) = \mathbb{E}_\omega[f_n^{\omega,\Omega}(M;\alpha,\beta)]$.

5.1.2 Key Propositions

Theorem 5.1 is a consequence of Propositions 5.2, 5.3 and 5.4 stated below and proven in Sections 5.2.1–5.2.5, Sections 5.3.1–5.3.3 and Section 5.4, respectively.

Proposition 5.2 *For all $(M,m)\in \text{EIGH}$,*

$$\lim_{n\to\infty} f_{1,n}^{\Omega}(M,m;\alpha,\beta) = f(M,m;\alpha,\beta) \quad \textit{for } \mathbb{P}-a.e.\,\Omega. \tag{5.12}$$

Proposition 5.3 *For all $M\in\mathbb{N}$,*

$$\lim_{n\to\infty} f_{1,n}^{\Omega}(M;\alpha,\beta) = \sup_{m\geq M+2} f(M,m;\alpha,\beta) \quad \textit{for } \mathbb{P}-a.e.\,\Omega. \tag{5.13}$$

Proposition 5.4 *For all $M\in\mathbb{N}$,*

$$\sup_{m\geq M+2} f(M,m;\alpha,\beta) = \sup_{\rho\in\mathcal{R}_{p,M}} \ \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M}\in\mathcal{B}_{\overline{\mathcal{V}}_M}} V(\rho,u), \tag{5.14}$$

where, in the righthand side of (5.14), we recognize the variational formula of Theorem 5.1 and with $\mathcal{B}_{\overline{\mathcal{V}}_M}$ defined in (3.15).

Proof of Theorem 5.1 subject to Propositions 5.2, 5.3 and 5.4. The proof of Theorem 5.1 will be complete once we show that for all $(M,m)\in\text{EIGH}$

$$\lim_{n\to\infty} |f_n^{\omega,\Omega}(M,m;\alpha,\beta) - f_n^{\Omega}(M,m;\alpha,\beta)| = 0 \quad \text{for } \mathbb{P}-a.e.\ (\omega,\Omega). \tag{5.15}$$

To that aim, we note that for all $n\in\mathbb{N}$ the Ω-dependence of $f_n^{\omega,\Omega}(M,m;\alpha,\beta)$ is restricted to $\{\Omega_x\colon\ x\in G_n\}$ with $G_n=\{0,\dots,\frac{n}{L_n}\}\times\{-\frac{n}{L_n},\dots,\frac{n}{L_n}\}$. Thus, for $n\in\mathbb{N}$ and $\varepsilon>0$ we set

$$A_{\varepsilon,n} = \{|f_n^{\omega,\Omega}(M;\alpha,\beta) - f_n^{\Omega}(M;\alpha,\beta)| > \varepsilon)\}, \tag{5.16}$$

and by independence of ω and Ω we can write

$$\begin{aligned} \mathbb{P}_{\omega,\Omega}(A_{\varepsilon,n}) &= \textstyle\sum_{\Upsilon\in\{A,B\}^{G_n}} \mathbb{P}_{\omega,\Omega}(A_{\varepsilon,n}\cap\{\Omega_{G_n}=\Upsilon\})\\ &= \textstyle\sum_{\Upsilon\in\{A,B\}^{G_n}} \mathbb{P}_\omega(|f_n^{\omega,\Upsilon}(M;\alpha,\beta) - f_n^{\Upsilon}(M;\alpha,\beta)|>\varepsilon)\ \mathbb{P}_\Omega(\{\Omega_{G_n}=\Upsilon\}). \end{aligned} \tag{5.17}$$

At this stage, for each $n\in\mathbb{N}$ we can apply the concentration inequality (D.3) in Appendix D with $\Gamma=\mathcal{W}_{n,M}^{m}$, $l=n$, $\eta=\varepsilon n$,

$$\xi_i = -\alpha\,1\{\omega_i=A\} + \beta\,1\{\omega_i=B\}, \qquad i\in\mathbb{N}, \tag{5.18}$$

and with $T(x,y)$ indicating in which block step (x,y) lies in. Therefore, there exist $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ and all $\Upsilon \in \{A,B\}^{G_n}$ we have

$$\mathbb{P}_\omega(|f_n^{\omega,\Upsilon}(M;\alpha,\beta) - f_n^{\Upsilon}(M;\alpha,\beta)| > \varepsilon) \leq C_1 e^{-C_2\varepsilon^2 n}, \tag{5.19}$$

which, together with (5.17) yields $\mathbb{P}_{\omega,\Omega}(A_{\varepsilon,n}) \leq C_1 e^{-C_2\varepsilon^2 n}$ for all $n \in \mathbb{N}$. By using the Borel-Cantelli Lemma, we obtain (5.15). □

5.2 Proof of Proposition 5.2

Pick $(M,m) \in \text{EIGH}$ and $(\alpha,\beta) \in \text{CONE}$. In Steps 1–2 in Sections 5.2.1–5.2.2 we introduce an intermediate free energy $f_{3,n}^{\Omega}(M,m;\alpha,\beta)$ and show that

$$\lim_{n\to\infty} |f_{1,n}^{\Omega}(M,m;\alpha,\beta) - f_{3,n}^{\Omega}(M,m;\alpha,\beta)| = 0 \qquad \forall\, \Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}. \tag{5.20}$$

Next, in Steps 3–4 in Sections 5.2.3–5.2.4 we show that

$$\limsup_{n\to\infty} f_{3,n}^{\Omega}(M,m;\alpha,\beta) = f(M,m;\alpha,\beta) \qquad \text{for } \mathbb{P}-a.e.\ \Omega, \tag{5.21}$$

while in Step 5 in Section 5.2.5 we prove that

$$\liminf_{n\to\infty} f_{3,n}^{\Omega}(M,m;\alpha,\beta) = \limsup_{n\to\infty} f_{3,n}^{\Omega}(M,m;\alpha,\beta) \qquad \text{for } \mathbb{P}-a.e.\ \Omega. \tag{5.22}$$

Combing (5.20–5.22) we get

$$\liminf_{n\to\infty} f_{1,n}^{\Omega}(M,m;\alpha,\beta) = \limsup_{n\to\infty} f_{1,n}^{\Omega}(M,m;\alpha,\beta) = f(M,m;\alpha,\beta) \qquad \text{for } \mathbb{P}-a.e.\ \Omega, \tag{5.23}$$

which completes the proof of Proposition 5.2.

In the proof we need the following order relation.

Definition 5.5 *For $g, \widetilde{g}\colon \mathbb{N}^3 \times \text{CONE} \mapsto \mathbb{R}$, write $g \prec \widetilde{g}$ if for all $(M,m) \in \text{EIGH}$, $(\alpha,\beta) \in \text{CONE}$ and $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that*

$$g(n,M,m;\alpha,\beta) \leq \widetilde{g}(n,M,m;\alpha,\beta) + \varepsilon \qquad \forall\, n \geq n_\varepsilon. \tag{5.24}$$

The proof of (5.20) will be complete once we show that $f_1^{\Omega} \prec f_3^{\Omega}$ and $f_3^{\Omega} \prec f_1^{\Omega}$ for all $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$. We will focus on $f_1^{\Omega} \prec f_3^{\Omega}$, since the proof of the latter can be easily adapted to obtain $f_3^{\Omega} \prec f_1^{\Omega}$. To prove $f_1^{\Omega} \prec f_3^{\Omega}$ we introduce another intermediate free energy f_2^{Ω}, and we show that $f_1^{\Omega} \prec f_2^{\Omega}$ and $f_2^{\Omega} \prec f_3^{\Omega}$.

For $L \in \mathbb{N}$, let

$$\mathcal{D}_L^M = \left\{\Xi = (\Delta\Pi, b_0, b_1) \in \{-M,\dots,M\} \times \{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\}^2\right\}. \tag{5.25}$$

For $L, N \in \mathbb{N}$, let

$$\widetilde{\mathcal{D}}_{L,N}^M = \Big\{\Theta_{\text{traj}} = (\Xi_i)_{i\in\{0,\dots,N-1\}} \in (\mathcal{D}_L^M)^N \colon\ b_{0,0} = \tfrac{1}{L},\ b_{0,i} = b_{1,i-1}\ \forall\, 1 \leq i \leq N-1\Big\}, \tag{5.26}$$

and with each $\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L,N}$ associate the sequence $(\Pi_i)_{i=0}^N$ defined by $\Pi_0 = 0$ and $\Pi_i = \sum_{j=0}^{i-1} \Delta\Pi_j$ for $1 \le i \le N$. Next, for $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L,N}$, set

$$\mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega} = \big\{x \in \{1,2\}^{\{0,\dots,N-1\}} \colon (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i) \in \mathcal{V}^m_M \ \forall\, 0 \le i \le N-1\big\}, \tag{5.27}$$

and, for $x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}$, set

$$\Theta_i = (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i) \quad \text{for} \quad i \in \{0,\dots,N-1\} \tag{5.28}$$

and

$$\mathcal{U}^{M,m,L}_{\Theta_{\rm traj},x,n} = \Big\{u = (u_i)_{i\in\{0,\dots,N-1\}} \in [1,m]^N \colon u_i \in t_{\Theta_i} + \tfrac{2\mathbb{N}}{L} \ \ \forall\, 0 \le i \le N-1,\ \sum_{i=0}^{N-1} u_i = \tfrac{n}{L}\Big\}. \tag{5.29}$$

Note that $\mathcal{U}^{M,m,L}_{\Theta_{\rm traj},x,n}$ is empty when $N \notin \big[\frac{n}{mL},\frac{n}{L}\big]$.

For $\pi \in \mathcal{W}^m_{n,M}$, we let N_π be the number of columns crossed by π after n steps. We denote by $(u_0(\pi),\dots,u_{N_\pi-1}(\pi))$ the time spent by π in each column divided by L_n, and we set $\widetilde{u}_0(\pi) = 0$ and $\widetilde{u}_j(\pi) = \sum_{k=0}^{j-1} u_k(\pi)$ for $1 \le j \le N_\pi$. With these notations, the partition function in (5.4) can be rewritten as

$$Z^{\omega,\Omega}_{1,n,L_n}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \ \sum_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}^M_{L_n,N}} \ \sum_{x\in\mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}} \ \sum_{u\in\mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}} A_1, \tag{5.30}$$

with (recall (3.31))

$$A_1 = \prod_{i=0}^{N-1} Z^{\theta^{\widetilde{u}_i L_n}(\omega)}_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i). \tag{5.31}$$

5.2.1 Step 1

In this step we average over the disorder ω in each column. To that end, we set

$$f^{\Omega}_{2,n}(M,m;\alpha,\beta) = \tfrac{1}{n}\log Z^{\Omega}_{2,n,L_n}(M,m) \tag{5.32}$$

with

$$Z^{\Omega}_{2,n,L_n}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \ \sum_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}^M_{L_n,N}} \ \sum_{x\in\mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}} \ \sum_{u\in\mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}} A_2, \tag{5.33}$$

where

$$A_2 = \prod_{i=0}^{N-1} e^{\mathbb{E}_\omega\big[\log Z^{\theta^{\widetilde{u}_i}(\omega)}_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\big]} = \prod_{i=0}^{N-1} e^{u_i L_n \psi_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)}. \tag{5.34}$$

Note that the ω-dependence has been removed from $Z^{\Omega}_{2,n,L_n}(M,m)$.

To prove that $f^\Omega_1 \prec f^\Omega_2$, we need to show that for all $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that, for $n \ge n_\varepsilon$ and all Ω,

$$\mathbb{E}_\omega\big[\log Z^{\omega,\Omega}_{1,n,L_n}(M,m)\big] \le \log Z^{\Omega}_{2,n,L_n}(M,m) + \varepsilon n. \tag{5.35}$$

To this end, we rewrite $Z^{\omega,\Omega}_{1,n,L_n}(M,m)$ as

$$Z^{\omega,\Omega}_{1,n,L_n}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \sum_{x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega}} \sum_{u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}} A_2 \, \frac{A_1}{A_2}, \tag{5.36}$$

where we note that

$$\frac{A_1}{A_2} = \prod_{i=0}^{N-1} e^{u_i L_n \left[\psi^{\theta^{\tilde{u}_i L_n}(\omega)}_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i) - \psi_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\right]}. \tag{5.37}$$

In order to average over ω, we apply a concentration of measure inequality. Set

$$\mathcal{K}_n = \bigcup_{N=n/mL_n}^{n/L_n} \bigcup_{\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \bigcup_{x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega}} \bigcup_{u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}} \Big\{ |\log A_1 - \log A_2| \geq \varepsilon n \Big\}, \tag{5.38}$$

and note that $\omega \in \mathcal{K}_n^c$ implies that $Z^{\omega,\Omega}_{1,n,L_n}(M,m) \leq e^{\varepsilon n} Z^{\Omega}_{2,n,L_n}(M,m)$. Consequently, we can write

$$\begin{aligned} \mathbb{E}_\omega\big[\log Z^{\omega,\Omega}_{1,n,L_n}(M,m)\big] &= \mathbb{E}_\omega\big[\log Z^{\omega,\Omega}_{1,n,L_n}(M,m)\, 1_{\{\mathcal{K}_n\}}\big] + \mathbb{E}_\omega\big[\log Z^{\omega,\Omega}_{1,n,L_n}(M,m)\, 1_{\{\mathcal{K}_n^c\}}\big] \\ &\leq \mathbb{E}_\omega\big[\log Z^{\omega,\Omega}_{1,n,L_n}(M,m)\, 1_{\{\mathcal{K}_n\}}\big] + \log Z^{\Omega}_{2,n,L_n}(M,m) + \varepsilon n. \end{aligned} \tag{5.39}$$

We can now use the uniform bound in (3.34) to control the first term in the right-hand side of (5.39), to obtain

$$\mathbb{E}_\omega\big[\log Z^{\omega,\Omega}_{1,n,L_n}(M,m)\big] \leq \log Z^{\Omega}_{2,n,L_n}(M,m) + \varepsilon n + C_{\text{uf}}(\alpha)\, n\, \mathbb{P}_\omega(\mathcal{K}_n). \tag{5.40}$$

Therefore the proof of this step will be complete once we show that $\mathbb{P}_\omega(\mathcal{K}_n)$ vanishes as $n \to \infty$.

Lemma 5.6 *There exist $C_1, C_2 > 0$ such that, for all $\varepsilon > 0$, $n \in \mathbb{N}$, $N \in \{\frac{n}{mL_n}, \dots, \frac{n}{L_n}\}$, $\Omega \in \{A,B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}$, $x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega}$ and $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}$,*

$$\mathbb{P}_\omega(|\log A_1 - \log A_2| \geq \varepsilon n) \leq C_1 e^{-C_2 \varepsilon^2 n}. \tag{5.41}$$

Proof. Pick $\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}$, $x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega}$ and $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}$, and consider the subset Γ of $\mathcal{W}^m_{n,M}$ consisting of those paths of length n that first cross the $(\Omega(0,\cdot),\Xi_0,x_0)$ column such that $\pi_0 = (0,1)$ and $\pi_{\tilde{u}_1 L_n} = (1, \Pi_1 + b_{1,0})L_n$, then cross the $(\Omega(1,\cdot),\Xi_1,x_1)$ column such that $\pi_{\tilde{u}_1 L_n + 1} = (1 + 1/L_n, \Pi_1 + b_{1,0})L_n$ and $\pi_{\tilde{u}_2 L_n} = (2, \Pi_2 + b_{1,1})L_n$, and so on. We can apply the concentration of measure inequality stated in (D.3) to the set Γ defined above, with $l = n$, $\eta = \varepsilon n$,

$$\xi_i = -\alpha\, 1\{\omega_i = A\} + \beta\, 1\{\omega_i = B\}, \qquad i \in \mathbb{N}, \tag{5.42}$$

and with $T(x,y)$ indicating in which block step (x,y) lies in. After noting that $\mathbb{E}_\omega(\log A_1) = \log A_2$, we obtain that there exist $C_1, C_2 > 0$ such that, for all $n \in \mathbb{N}$, $N \in \{\frac{n}{mL_n}, \dots, \frac{n}{L_n}\}$, $\Omega \in \{A,B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}$, $x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega}$ and $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}$,

$$\mathbb{P}\big(|\log A_1 - \log A_2| \geq \varepsilon\, n\big) \leq C_1\, e^{-C_2\, \varepsilon^3\, n}. \tag{5.43}$$

□

It now suffices to remark that

$$\big|\{(N, \Theta_{\text{traj}}, x, u)\colon\ N \in \{\tfrac{n}{mL_n}, \dots, \tfrac{n}{L_n}\}, \Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N},\ x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega}, u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}\}\big| \tag{5.44}$$

grows subexponentially in n to obtain that $f_1^\Omega \prec f_2^\Omega$ for all Ω.

5.2.2 Step 2

In this step we replace the finite-size free energy ψ_{L_n} by its limit ψ. To do so we introduce a third intermediate free energy,

$$f_{3,n}^{\Omega}(M,m;\alpha,\beta) = \mathbb{E}\big[\tfrac{1}{n}\log Z_{3,n,L_n}^{\Omega}(M,m)\big], \tag{5.45}$$

where

$$Z_{3,n,L_n}^{\Omega}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}_{L_n,N}^{M}} \sum_{x\in\mathcal{X}_{\Theta_{\rm traj},\Omega}^{M,m}} \sum_{u\in\mathcal{U}_{\Theta_{\rm traj},x,n}^{M,m,L_n}} A_3 \tag{5.46}$$

with

$$A_3 = \prod_{i=0}^{N-1} e^{u_i L_n \psi(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)}. \tag{5.47}$$

For all Ω,

$$\frac{A_2}{A_3} = \prod_{i=0}^{N-1} e^{u_i L_n \left[\psi_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)-\psi(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\right]}, \tag{5.48}$$

and, for all $i\in\{0,\dots,N-1\}$, we have $(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\in\mathcal{V}_M^{*,m}$, so that Proposition 3.4 can be applied.

5.2.3 Step 3

In this step we want the variational formula (5.7) to appear. Recall (3.53) and define, for $n\in\mathbb{N}$, $(M,m)\in$ EIGH, $N\in\{\frac{n}{mL_n},\dots,\frac{n}{L_n}\}$, $\Theta_{\rm traj}\in\widetilde{\mathcal{D}}_{L_n,N}^{M}$ and $x\in\mathcal{X}_{\Theta_{\rm traj},\Omega}^{M,m}$,

$$\Theta_j = (\Omega(j,\Pi_j+\cdot),\Xi_j,x_j), \qquad j=0,\dots,N-1, \tag{5.49}$$

and

$$\rho_{\Theta_{\rm traj},x}^{\Omega}(\Theta,\Theta') = \frac{1}{N}\sum_{j=1}^{N} 1_{\left\{(\Theta_{j-1},\Theta_j)=(\Theta,\Theta')\right\}}, \tag{5.50}$$

and, for $u\in\mathcal{U}_{\Theta_{\rm traj},x,n}^{M,m,L_n}$,

$$H^{\Omega}(\Theta_{\rm traj},x,u) = \sum_{j=0}^{N-1} u_j\,\psi(\Theta_j,u_j). \tag{5.51}$$

In terms of these quantities we can rewrite $Z_{3,n,L_n}^{\Omega}(M,m)$ in (5.46) as

$$Z_{3,n,L_n}^{\Omega}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}_{L_n,N}^{M}} \sum_{x\in\mathcal{X}_{\Theta_{\rm traj},\Omega}^{M,m}} \sum_{u\in\mathcal{U}_{\Theta_{\rm traj},x,n}^{M,m,L_n}} e^{L_n\,H^{\Omega}(\Theta_{\rm traj},x,u)}. \tag{5.52}$$

For $n\in\mathbb{N}$, denote by

$$N_n^{\Omega}, \qquad \Theta_{{\rm traj},n}^{\Omega}\in\widetilde{\mathcal{D}}_{L_n,N_n^{\Omega}}^{M}, \qquad x_n^{\Omega}\in\mathcal{X}_{\Theta_{{\rm traj},n}^{\Omega},\Omega}^{M,m}, \qquad u_n^{\Omega}\in\mathcal{U}_{\Theta_{{\rm traj},n}^{\Omega},x_n^{\Omega},n}^{M,m,L_n}, \tag{5.53}$$

the indices in the summation set of (5.52) that maximize $H^\Omega(\Theta_{\text{traj}}, x, u)$. For ease of notation we put

$$\Theta^\Omega_{\text{traj},n} = (\Xi^n_j)_{j=0}^{N^\Omega_n - 1}, \quad x^\Omega_n = (x^n_j)_{j=0}^{N^\Omega_n - 1}, \quad u^\Omega_n = (u^n_j)_{j=0}^{N^\Omega_n - 1}, \tag{5.54}$$

and

$$c_n = \Big|\{(N, \Theta_{\text{traj}}, x, u)\colon \tfrac{n}{mL_n} \le N \le \tfrac{n}{L_n},\, \Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N},\, x \in \mathcal{X}^{M,m}_{\Theta_{\text{traj}},\Omega},\, u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\text{traj}},x,n}\}\Big|. \tag{5.55}$$

Then we can estimate

$$\frac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m) \le \frac{1}{n} \log c_n + \tfrac{L_n}{n} \sum_{j=0}^{N^\Omega_n - 1} u^n_j\, \psi(\Theta^n_j, u^n_j). \tag{5.56}$$

We next note that $u \mapsto u\psi(\Theta, u)$ is concave for all $\Theta \in \overline{\mathcal{V}}_M$ (see Lemma C.4). Hence, after setting

$$v^n_\Theta = \sum_{j=0}^{N^\Omega_n - 1} 1_{\{\Theta^n_j = \Theta\}}\, u^n_j, \qquad d^n_\Theta = \sum_{j=0}^{N^\Omega_n - 1} 1_{\{\Theta^n_j = \Theta\}}, \qquad \Theta \in \overline{\mathcal{V}}^m_M, \tag{5.57}$$

we can estimate

$$\sum_{j=0}^{N^\Omega_n - 1} 1_{\{\Theta^n_j = \Theta\}}\, u^n_j\, \psi(\Theta^n_j, u^n_j) \le v^n_\Theta\, \psi\big(\Theta, \tfrac{v^n_\Theta}{d^n_\Theta}\big) \quad \text{for} \quad \Theta \in \overline{\mathcal{V}}^m_M\colon d^n_\Theta \ge 1. \tag{5.58}$$

Next, we recall (5.50) and we set $\rho_n = \rho^\Omega_{\Theta^\Omega_{\text{traj},n}, x^\Omega_n}$, so that $\rho_{n,1}(\Theta) = d^n_\Theta / N^\Omega_n$ for all $\Theta \in \overline{\mathcal{V}}^m_M$. Since $\{\Theta \in \overline{\mathcal{V}}^m_M\colon d^n_\Theta \ge 1\}$ is a finite subset of $\overline{\mathcal{V}}^m_M$, we can easily extend $\Theta \mapsto v^n_\Theta / d^n_\Theta$ from $\{\Theta \in \overline{\mathcal{V}}_M\colon d^n_\Theta \ge 1\}$ to $\overline{\mathcal{V}}^m_M$ as a continuous function. Moreover, $\sum_{j=0}^{N^\Omega_n - 1} u^n_j = n/L_n$ implies that $N^\Omega_n \int_{\overline{\mathcal{V}}^m_M} v^n_\Theta / d^n_\Theta\, \rho_{n,1}(d\Theta) = n/L_n$, which, together with (5.56) and (5.58) gives

$$\tfrac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m) \le \sup_{u \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}} \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta\, \psi(\Theta, u_\Theta)\, \rho_n(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta\, \rho_n(d\Theta)} + o(1), \qquad n \to \infty, \tag{5.59}$$

where we use that $\lim_{n\to\infty} \frac{1}{n} \log c_n = 0$. In what follows, we abbreviate the first term in the right-hand side of the last display by l_n. We want to show that $\limsup_{n\to\infty} \frac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m) \le f(M,m;\alpha,\beta)$. To that end, we assume that $\frac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m)$ converges to some $t \in \mathbb{R}$ and we prove that $t \le f(M,m;\alpha,\beta)$. Since $(l_n)_{n\in\mathbb{N}}$ is bounded and $\overline{\mathcal{V}}^m_M$ is compact, it follows from the definition of l_n that along an appropriate subsequence both $l_n \to l_\infty \ge t$ and $\rho_n \to \rho_\infty \in \mathcal{R}^m_{p,M}$ as $n \to \infty$. Hence, the proof will be complete once we show that

$$l_\infty \le \sup_{u \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}} V(\rho_\infty, u), \tag{5.60}$$

because the right-hand side in (5.60) is bounded from above by $f(M,m;\alpha,\beta)$.

Recall (3.18) and, for $\Theta \in \overline{\mathcal{V}}^m_M$ and $y \in \mathbb{R}$, define

$$u^{M,m}_\Theta(y) = \begin{cases} t_\Theta & \text{if } \partial^+_u(u\,\psi(\Theta,u))(t_\Theta) \le y, \\ m & \text{if } \partial^-_u(u\,\psi(\Theta,u))(m) \ge y, \\ z & \text{otherwise, with } z \text{ such that } \partial^-_u(u\,\psi(\Theta,u))(z) \ge y \ge \partial^+_u(u\,\psi(\Theta,u))(z), \end{cases} \tag{5.61}$$

where z is unique by strict concavity of $u \to u\psi(\Theta, u)$ (see Lemma C.2).

Lemma 5.7 *(i) For all $y \in \mathbb{R}$ and $(M, m) \in$ EIGH, $\Theta \mapsto u_\Theta^{M,m}(y)$ is continuous on $(\overline{\mathcal{V}}_M^m, d_M)$, where d_M is defined in* (C.7) *in Appendix* C.
(ii) For all $(M, m) \in$ EIGH and $\Theta \in \overline{\mathcal{V}}_M^m$, $y \mapsto u_\Theta^{M,m}(y)$ is continuous on $\mathbb{R}$.

Proof. The proof uses the strict concavity of $u \to u\psi(\Theta, u)$ (see Lemma C.2).

(i) The proof is by contradiction. Pick $y \in \mathbb{R}$, and pick a sequence $(\Theta_n)_{n\in\mathbb{N}}$ in $\overline{\mathcal{V}}_M^m$ such that $\lim_{n\to\infty} \Theta_n = \Theta_\infty \in \overline{\mathcal{V}}_M^m$. Suppose that $u_{\Theta_n}^{M,m}(y)$ does not tend to $u_{\Theta_\infty}^{M,m}(y)$ as $n \to \infty$. Then, by choosing an appropriate subsequence, we may assume that $\lim_{n\to\infty} u_{\Theta_n}^{M,m}(y) = u_1 \in [t_{\Theta_\infty}, m]$ with $u_1 < u_{\Theta_\infty}^{M,m}(y)$. The case $u_1 > u_{\Theta_\infty}^{M,m}(y)$ can be handled similarly.

Pick $u_2 \in (u_1, u_{\Theta_\infty}^{M,m}(y))$. For n large enough, we have $u_{\Theta_n}^{M,m}(y) < u_2 < u_{\Theta_\infty}^{M,m}(y)$. By the definition of $u_{\Theta_n}^{M,m}(y)$ in (5.61) and the strict concavity of $u \mapsto u\psi(\Theta_n, u)$ we have, for n large enough,

$$\partial_u^+ (u\,\psi(\Theta_n, u))(u_{\Theta_n}^{M,m}(y)) > \frac{u_{\Theta_\infty}^{M,m}(y)\psi(\Theta_n, u_{\Theta_\infty}^{M,m}(y)) - u_2\psi(\Theta_n, u_2)}{u_{\Theta_\infty}^{M,m}(y) - u_2}. \tag{5.62}$$

Let $n \to \infty$ in (5.62) and use the strict concavity once again, to get

$$\liminf_{n\to\infty} \partial_u^+ (u\,\psi(\Theta_n, u))(u_{\Theta_n}^{M,m}(y)) > \partial_u^- (u\,\psi(\Theta_\infty, u))(u_{\Theta_\infty}^{M,m}(y)). \tag{5.63}$$

If $u_{\Theta_\infty}^{M,m}(y) \in (t_{\Theta_\infty}, m]$, then (5.61) implies that the right-hand side of (5.63) is not smaller than y. Hence (5.63) yields that $\partial_u^+ (u\,\psi(\Theta_n, u))(u_{\Theta_n}^{M,m}(y)) > y$ for n large enough, which implies that $u_{\Theta_n}^{M,m}(y) = m$ by (5.61). However, the latter inequality contradicts the fact that $u_{\Theta_n}^{M,m}(y) < u_2 < u_{\Theta_\infty}^{M,m}(y)$ for n large enough. If $u_{\Theta_\infty}^{M,m}(y) = t_{\Theta_\infty}$, then we note that $\lim_{n\to\infty} t_{\Theta_n} = t_{\Theta_\infty}$, which again contradicts that $t_{\Theta_n} \le u_{\Theta_n}^{M,m}(y) < u_2 < u_{\Theta_\infty}^{M,m}(y)$ for n large enough.

(ii) The proof is again by contradiction. Pick $\Theta \in \overline{\mathcal{V}}_M^m$, and pick an infinite sequence $(y_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n = y_\infty \in \mathbb{R}$ and such that $u_\Theta^{M,m}(y_n)$ does not converge to $u_\Theta^{M,m}(y_\infty)$. Then, by choosing an appropriate subsequence, we may assume that there exists a $u_1 < u_\Theta^{M,m}(y_\infty)$ such that $\lim_{n\to\infty} u_\Theta^{M,m}(y_n) = u_1$. The case $u_1 > u_\Theta^{M,m}(y_\infty)$ can be treated similarly.

Pick $u_2, u_3 \in (u_1, u_\Theta^{M,m}(y_\infty))$ such that $u_2 < u_3$. Then, for n large enough, we have

$$t_\Theta \le u_\Theta^{M,m}(y_n) < u_2 < u_3 < u_\Theta^{M,m}(y_\infty) \le m. \tag{5.64}$$

Combining (5.61) and (5.64) with the strict concavity of $u \mapsto u\psi(\Theta, u)$ we get, for n large enough,

$$y_n > \partial_u^+ (u\,\psi(\Theta, u))(u_2) > \partial_u^- (u\,\psi(\Theta, u))(u_3) > y_\infty, \tag{5.65}$$

which contradicts $\lim_{n\to\infty} y_n = y_\infty$. □

We resume the line of proof. Recall that $\rho_{n,1}$, $n \in \mathbb{N}$, charges finitely many $\Theta \in \overline{\mathcal{V}}_M^m$. Therefore the continuity and the strict concavity of $u \mapsto u\psi(\Theta, u)$ on $[t_\Theta, m]$ for all $\Theta \in \overline{\mathcal{V}}_M^m$ (see Lemma C.4) imply that the supremum in (5.59) is attained at some $u_n^{M,m} \in \mathcal{B}_{\overline{\mathcal{V}}_M^m}$ that satisfies $u_n^{M,m}(\Theta) = u_\Theta^{M,m}(l_n)$ for $\Theta \in \overline{\mathcal{V}}_M^m$. Set $u_\infty^{M,m}(\Theta) = u_\Theta^{M,m}(l_\infty)$ for $\Theta \in \overline{\mathcal{V}}_M^m$ and note that $(l_n)_{n\in\mathbb{N}}$ may be assumed to be monotone, say, non-decreasing. Then the concavity of $u \mapsto u\psi(\Theta, u)$ for $\Theta \in \overline{\mathcal{V}}_M^m$ implies that $(u_n^{M,m})_{n\in\mathbb{N}}$ is a non-increasing sequence of functions on $\overline{\mathcal{V}}_M^m$. Moreover, $\overline{\mathcal{V}}_M^m$ is a compact set and, by Lemma 5.7(ii), $\lim_{n\to\infty} u_n^{M,m}(\Theta) = u_\infty^{M,m}(\Theta)$ for $\Theta \in \overline{\mathcal{V}}_M^m$. Therefore Dini's theorem implies that $\lim_{n\to\infty} u_n^{M,m} = u_\infty^{M,m}$ uniformly on $\overline{\mathcal{V}}_M^m$. We estimate

$$\begin{aligned}
&\left| l_n - \int_{\overline{\mathcal{V}}_M^m} u_\infty^{M,m}(\Theta)\, \psi(\Theta, u_\infty^{M,m}(\Theta))\rho_\infty(d\Theta) \right| \\
&\qquad \le \int_{\overline{\mathcal{V}}_M^m} \left| u_n^{M,m}(\Theta)\, \psi(\Theta, u_n^{M,m}(\Theta)) - u_\infty^{M,m}(\Theta)\, \psi(\Theta, u_\infty^{M,m}(\Theta)) \right| \rho_n(d\Theta) \\
&\qquad + \Big| \int_{\overline{\mathcal{V}}_M^m} u_\infty^{M,m}(\Theta)\, \psi(\Theta, u_\infty^{M,m}(\Theta))\, \rho_n(d\Theta) - \int_{\overline{\mathcal{V}}_M^m} u_\infty^{M,m}(\Theta)\, \psi(\Theta, u_\infty^{M,m}(\Theta))\, \rho_\infty(d\Theta) \Big|.
\end{aligned} \tag{5.66}$$

The second term in the right-hand side of (5.66) tends to zero as $n \to \infty$ because, by Lemma 5.7(i), $\Theta \mapsto u_\infty^{M,m}(\Theta)$ is continuous on $\overline{\mathcal{V}}_M^m$ and because ρ_n converges in law to ρ_∞ as $n \to \infty$. The first term in the right-hand side of (5.66) tends to zero as well, because $(\Theta, u) \mapsto u\psi(\Theta, u)$ is uniformly continuous on $\overline{\mathcal{V}}_M^{*,m}$ (see Lemma C.3) and because we have proved above that $u_n^{M,m}$ converges to $u_\infty^{M,m}$ uniformly on $\overline{\mathcal{V}}_M^m$. This proves (5.60), and so Step 3 is complete.

5.2.4 Step 4

In this step we prove that

$$\limsup_{n\to\infty} f_{3,n}^{\Omega}(M, m; \alpha, \beta) \ge f(M, m; \alpha, \beta) \;\text{ for } \mathbb{P} - a.e.\ \Omega. \tag{5.67}$$

Note that the proof will be complete once we show that

$$\limsup_{n\to\infty} f_{3,n}^{\Omega}(M, m, \alpha, \beta) \ge V(\rho, u) \text{ for } \rho \in \mathcal{R}_{p,M}^m, u \in \mathcal{B}_{\overline{\mathcal{V}}_M^m}. \tag{5.68}$$

Pick $\Omega \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $\rho \in \mathcal{R}_{p,M}^{\Omega,m}$ and $u \in \mathcal{B}_{\overline{\mathcal{V}}_M^m}$. By the definition of $\mathcal{R}_{p,M}^{\Omega,m}$, there exists a strictly increasing subsequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that, for all $k \in \mathbb{N}$, there exists an

$$N_k \in \left\{ \frac{n_k}{mL_{n_k}}, \dots, \frac{n_k}{L_{n_k}} \right\}, \tag{5.69}$$

a $\Theta_{\text{traj}}^k \in \widetilde{\mathcal{D}}_{L_{n_k}, N_k}^M$ and a $x^k \in \mathcal{X}_{\Theta_{\text{traj}}^k, \Omega}^{M,m}$ such that $\rho_k \stackrel{\text{def}}{=} \rho_{\Theta_{\text{traj}}^k, x^k}^{\Omega}$ (see (5.50)) converges in law to ρ as $k \to \infty$. Recall (5.26), and note that

$$\Xi_j^k = \big(\Delta\Pi_j^k,\, b_j^k,\, b_{j+1}^k\big), \quad j = 0, \dots, N_k - 1, \tag{5.70}$$

with $\Delta\Pi_j^k \in \{-M, \dots, M\}$ and $b_j^k \in (0, 1] \cap \frac{\mathbb{N}}{L_{n_k}}$ for $j = 0, \dots, N_k$. For ease of notation we define

$$\Theta_j^k = \big(\Omega(j, \Pi_j^k + \cdot), \Xi_j^k, x_j^k\big) \quad \text{with} \quad \Pi_j^k = \sum_{i=0}^{j-1} \Delta\Pi_i^k, \qquad j = 0, \dots, N_k - 1, \tag{5.71}$$

and

$$v_k = N_k \int_{\Theta \in \mathcal{V}_M^m} u_\Theta\, \rho_{k,1}(d\Theta) = \sum_{j=0}^{N_k - 1} u_{\Theta_j^k}, \tag{5.72}$$

where we recall that $u = (u_\Theta)_{\Theta \in \overline{\mathcal{V}}_M^m}$ was fixed at the beginning of the section.

Next, we recall that $\lim_{n\to\infty} L_n/n = 0$ and that L_n is non-decreasing (see (1.5)). Thus, L_n is constant on intervals. On those intervals, n/L_n takes constant increments. The latter implies that there exists an $\widetilde{n}_k \in \mathbb{N}$ satisfying

$$0 \le v_k - \tfrac{\widetilde{n}_k}{L_{\widetilde{n}_k}} \le \tfrac{1}{L_{\widetilde{n}_k}} \quad \text{and therefore} \quad 0 \le v_k L_{\widetilde{n}_k} - \widetilde{n}_k \le 1. \tag{5.73}$$

Next, for $j = 0, \dots, N_k - 1$ we pick $\overline{b_j^k} \in (0,1] \cap \frac{N}{L_{\widetilde{n}_k}}$ such that $|\overline{b_j^k} - b_j^k| \le \frac{1}{L_{\widetilde{n}_k}}$, define

$$\overline{\Xi_j^k} = \big(\Delta\Pi_j^k, \overline{b_j^k}, \overline{b_{j+1}^k}\big), \quad \overline{\Theta_j^k} = \big(\Omega(j, \Pi_j^k + \cdot), \overline{\Xi_j^k}, x_j^k\big), \tag{5.74}$$

and pick

$$s_j^k \in t_{\overline{\Theta_j^k}} + \frac{2\mathbb{N}}{L_{\widetilde{n}_k}} \quad \text{such that} \quad |s_j^k - u_{\Theta_j^k}| \le 2/L_{\widetilde{n}_k}. \tag{5.75}$$

We use (5.72) to write

$$L_{\widetilde{n}_k} \sum_{j=0}^{N_k-1} s_j^k = L_{\widetilde{n}_k}\left(v_k + \sum_{j=0}^{N_k-1} (s_j^k - u_{\Theta_j^k})\right) = L_{\widetilde{n}_k}(I + II). \tag{5.76}$$

Next, we note that (5.73) and (5.75) imply that $|L_{\widetilde{n}_k} I - \widetilde{n}_k| \le 1$ and $|L_{\widetilde{n}_k} II| \le 2N_k$. The latter in turn implies that, by adding or subtracting at most 3 steps per colum, the quantities s_j^k for $j = 0, \dots, N_k - 1$ can be chosen in such a way that $\sum_{j=0}^{N_k-1} s_j^k = \widetilde{n}_k / L_{\widetilde{n}_k}$.

Next, set

$$\overline{\Theta_{\text{traj}}^k} = (\overline{\Xi_j^k})_{j=0}^{N_k-1} \in \widetilde{\mathcal{D}}_{L_{\widetilde{n}_k}, N_k}^M, \quad s^k = (s_j^k)_{j=0}^{N_k-1} \in \mathcal{U}_{\overline{\Theta_{\text{traj}}^k}, x^k, \widetilde{n}_k}^{M,m,L_{\widetilde{n}_k}}, \tag{5.77}$$

and recall (5.46) to get $f_3^\Omega(\widetilde{n}_k, M) \ge R_k$ with

$$R_k = \frac{L_{\widetilde{n}_k} H^\Omega\big(\overline{\Theta_{\text{traj}}^k}, x^k, s^k\big)}{\widetilde{n}_k} = \frac{\sum_{j=0}^{N_k-1} s_j^k \,\psi\Big(\overline{\Theta_j^k},\, s_j^k\Big)}{\sum_{j=0}^{N_k-1} s_j^k} = \frac{R_{\text{nu}}^k}{R_{\text{de}}^k}. \tag{5.78}$$

Further set

$$R_k' = \frac{R_{\text{nu}}'^k}{R_{\text{de}}'^k} = \frac{\int_{\overline{\mathcal{V}}_M^m} u_\Theta \,\psi(\Theta, u_\Theta)\rho_k(d\Theta)}{\int_{\overline{\mathcal{V}}_M^m} u_\Theta \,\rho_k(d\Theta)}, \tag{5.79}$$

and note that $\lim_{k\to\infty} R_k' = V(\rho, u)$, since $\lim_{k\to\infty} \rho_k = \rho$ by assumption and $\Theta \mapsto u_\Theta$ is continuous on $\mathcal{V}_M^m$. We note that R_k' can be rewritten in the form

$$R_k' = \frac{R_{\text{nu}}'^k}{R_{\text{de}}'^k} = \frac{\sum_{j=0}^{N_k-1} u_{\Theta_j^k}\,\psi(\Theta_j^k,\, u_{\Theta_j^k})}{\sum_{j=0}^{N_k-1} u_{\Theta_j^k}}. \tag{5.80}$$

Now recall that $\lim_{k\to\infty} n_k = \infty$. Since $N_k \ge n_k / M L_{n_k}$, it follows that $\lim_{k\to\infty} N_k = \infty$ as well. Moreover, $N_k \le \widetilde{n}_k / L_{\widetilde{n}_k}$ with $\lim_{k\to\infty} \widetilde{n}_k = \infty$. Therefore (5.72–5.73) allow us to conclude that $R_{\text{de}}^k = \widetilde{n}_k / L_{\widetilde{n}_k} = R_{\text{de}}'^k[1 + o(1)]$.

Next, note that $\mathcal{H}_M$ is compact, and that $(\Theta, u) \mapsto u\psi(\Theta, u)$ is continuous on $\mathcal{H}_M$ and therefore is uniformly continuous. Consequently, for all $\varepsilon > 0$ there exists an $\eta > 0$ such that, for all $(\Theta, u), (\Theta', u') \in \mathcal{H}_M$ satisfying $|\Theta - \Theta'| \le \eta$ and $|u - u'| \le \eta$,

$$|u\psi(\Theta, u) - u'\psi(\Theta', u')| \le \varepsilon. \tag{5.81}$$

We recall (5.74), which implies that $d_M(\overline{\Theta^k_j}, \Theta_j) \leq 2/L_{\widetilde{n}_k}$ for all $j \in \{0, \dots, N_k-1\}$, we choose k large enough to ensure that $2/L_{\widetilde{n}_k} \leq \eta$, and we use (5.81), to obtain

$$R^k_{\mathrm{nu}} = \sum_{j=0}^{N_k-1} s^k_j \, \psi\Big(\overline{\Theta^k_j}, \, s^k_j\Big) = \sum_{j=0}^{N_k-1} u_{\Theta^k_j} \, \psi\big(\Theta^k_j, \, u_{\Theta^k_j}\big) + T = R'^k_{\mathrm{nu}} + T, \tag{5.82}$$

with $|T| \leq \varepsilon N_k$. Since $\lim_{k\to\infty} R'_k = V(\rho, u)$ and $\sum_{j=0}^{N_k-1} u_{\Theta^k_j} = v_k \geq \widetilde{n}_k / L_{\widetilde{n}_k}$ (see (5.73)), if $V(\rho, u) \neq 0$, then $\big|R'^k_{\mathrm{nu}}\big| \geq \mathrm{Cst.}\ \widetilde{n}_k / L_{\widetilde{n}_k}$, whereas $|T| \leq \varepsilon N_k \leq \varepsilon \widetilde{n}_k / L_{\widetilde{n}_k}$ for k large enough. Hence $T = o(R'^k_{\mathrm{nu}})$ and

$$\frac{R^k_{\mathrm{nu}}}{R^k_{\mathrm{de}}} = \frac{R'^k_{\mathrm{nu}}\,[1+o(1)]}{R'^k_{\mathrm{de}}\,[1+o(1)]} \to V(\rho, u), \qquad k \to \infty. \tag{5.83}$$

Finally, if $V(\rho, u) = 0$, then $R'^k_{\mathrm{nu}} = o(R'^k_{\mathrm{de}})$ and $T = o(R'^k_{\mathrm{de}})$, so that R_k tends to 0. This completes the proof of Step 4.

5.2.5 Step 5

In this step we prove (5.22), suppressing the (α, β)-dependence from the notation. For $\Omega \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}^2}$, $n \in \mathbb{N}$, $N \in \{n/mL_n, \dots, n/L_n\}$ and $r \in \{-NM, \dots, NM\}$, we recall (5.26) and define

$$\widetilde{\mathcal{D}}^{M,m,r}_{L,N} = \Big\{\Theta_{\mathrm{traj}} \in \widetilde{\mathcal{D}}^{M,m}_{L,N} \colon \Pi_N = r\Big\}, \tag{5.84}$$

where we recall that $\Pi_N = \sum_{j=0}^{N-1} \Delta\Pi_j$. We set

$$f^{\Omega}_{3,n}(M, m, N, r) = \tfrac{1}{n} \log Z^{\Omega}_{3,n,L_n}(N, M, m, r) \tag{5.85}$$

with

$$Z^{\Omega}_{3,n,L_n}(N, M, m, r) = \sum_{\Theta_{\mathrm{traj}} \in \widetilde{\mathcal{D}}^{M,m,r}_{L_n,N}} \; \sum_{x \in \mathcal{X}^{M,m}_{\Theta_{\mathrm{traj}},\Omega}} \; \sum_{u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\mathrm{traj}},n}} A_3, \tag{5.86}$$

where A_3 is defined in (5.47). We further set $f_3(\cdot) = \mathbb{E}_\Omega\big(f^{\Omega}_3(\cdot)\big)$.

5.2.6 Concentration of measure

In the first part of this step we prove that for all $(M, m, \alpha, \beta) \in \textsc{eigh} \times \textsc{cone}$ there exist $c_1, c_2 > 0$ (depending on (M, m, α, β) only) such that, for all $n \in \mathbb{N}$, $N \in \{n/(mL_n), \dots n/L_n\}$ and $r \in \{-NM, \dots, NM\}$,

$$\begin{aligned} &\mathbb{P}_\Omega\Big(\big|f^{\Omega}_{3,n}(M,m) - f_{3,n}(M,m)\big| > \varepsilon\Big) \leq c_1 \, e^{-\frac{c_2 \varepsilon^2 n}{L_n}}, \\ &\mathbb{P}_\Omega\Big(\big|f^{\Omega}_{3,n}(M,m,N,r) - f_{3,n}(M,m,N,r)\big| > \varepsilon\Big) \leq c_1 \, e^{-\frac{c_2 \varepsilon^2 n}{L_n}}. \end{aligned} \tag{5.87}$$

We only give the proof of the first inequality. The second inequality is proved in a similar manner. The proof uses Theorem D.1. Before we start we note that, for all $n \in \mathbb{N}$, $(M, m) \in \textsc{eigh}$ and $\Omega \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $f^{\Omega}_{3,n}(M, m)$ only depends on

$$\mathcal{C}^{\Omega}_{0,L_n}, \dots, \mathcal{C}^{\Omega}_{n/L_n,L_n} \quad \text{with} \quad \mathcal{C}^{\Omega}_{j,L_n} = (\Omega(j,i))_{i=-n/L_n}^{n/L_n}. \tag{5.88}$$

We apply Theorem D.1 with $\mathcal{S} = \{0, \ldots, n/L_n\}$, with $X_i = \{A, B\}^{\{-\frac{n}{L_n}, \ldots, \frac{n}{L_n}\}}$ and with μ_i the uniform measure on X_i for all $i \in \mathcal{S}$. Note that $|f^{\Omega_1}_{3,n}(M,m) - f^{\Omega_2}_{3,n}(M,m)| \leq 2C_{\rm uf}(\alpha)m\frac{L_n}{n}$ for all $i \in \mathcal{S}$ and all Ω_1, Ω_2 satisfying $\mathcal{C}^{\Omega_1}_{j,n} = \mathcal{C}^{\Omega_2}_{j,n}$ for all $j \neq i$. After we set $c = 2C_{\rm uf}(\alpha)m$ we can apply Theorem D.1 with $D = c^2 L_n/n$ to get (5.87).

Next, we note that the first inequality in (5.87), the Borel-Cantelli lemma and the fact that $\lim_{n\to\infty} n/L_n \log n = \infty$ (recall (1.5)) imply that, for all $(M,m) \in$ EIGH,

$$\lim_{n\to\infty} \Big[f^{\Omega}_{3,n}(M,m) - f_{3,n}(M,m) \Big] = 0 \quad \text{for } \mathbb{P}-a.e.\ \Omega. \tag{5.89}$$

Therefore (5.22) will be proved once we show that

$$\liminf_{n\to\infty} f_{3,n}(M,m) = \limsup_{n\to\infty} f_{3,n}(M,m). \tag{5.90}$$

To that end, we first prove that, for all $n \in \mathbb{N}$ and all $(M,m) \in$ EIGH, there exist an $N_n \in \{n/mL_n, \ldots, n/L_n\}$ and an $r_n \in \{-MN_n, \ldots, MN_n\}$ such that

$$\lim_{n\to\infty} \Big[f_{3,n}(M,m) - f_{3,n}(M,m,N_n,r_n) \Big] = 0. \tag{5.91}$$

The proof of (5.91) is done as follows. Pick $\varepsilon > 0$, and for $\Omega \in \{A,B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $n \in \mathbb{N}$ and $(M,m) \in$ EIGH, denote by N^{Ω}_n and r^{Ω}_n the maximizers of $f^{\Omega}_{3,n}(M,m,N,r)$. Then

$$f^{\Omega}_{3,n}\big(M,m,N^{\Omega}_n, r^{\Omega}_n\big) \leq f^{\Omega}_{3,n}(M,m) \leq \tfrac{1}{n}\log(\tfrac{n^2}{L_n^2}) + f^{\Omega}_{3,n}\big(M,m,N^{\Omega}_n, r^{\Omega}_n\big), \tag{5.92}$$

so that, for n large enough and every Ω,

$$0 \leq f^{\Omega}_{3,n}(M,m) - f^{\Omega}_{3,n}\big(M,m,N^{\Omega}_n, r^{\Omega}_n\big) \leq \varepsilon. \tag{5.93}$$

For $n \in \mathbb{N}$, $N \in \{n/mL_n, \ldots, n/L_n\}$ and $r \in \{-NM, \ldots, NM\}$, we set

$$A_{n,N,r} = \{\Omega \colon (N^{\Omega}_n, r^{\Omega}_n) = (N,r)\}. \tag{5.94}$$

Next, denote by N_n, r_n the maximizers of $\mathbb{P}(A_{n,N,r})$. Note that (5.91) will be proved once we show that, for all $\varepsilon > 0$, $|f_{3,n}(M,m) - f_{3,n}(M,m,N_n,r_n)| \leq \varepsilon$ for n large enough. Further note that $\mathbb{P}(A_{n,N_n,r_n}) \geq L_n^2/n^2$ for all $n \in \mathbb{N}$. For every Ω we can therefore estimate

$$|f_{3,n}(M,m) - f_{3,n}(M,m,N_n,r_n)| \leq I + II + III \tag{5.95}$$

with

$$\begin{aligned} I &= |f_{3,n}(M,m) - f^{\Omega}_{3,n}(M,m)|, \\ II &= |f^{\Omega}_{3,n}(M,m) - f^{\Omega}_{3,n}(M,m,N_n,r_n)|, \\ III &= |f^{\Omega}_{3,n}(M,m,N_n,r_n) - f_{3,n}(M,m,N_n,r_n)|. \end{aligned} \tag{5.96}$$

Hence, the proof of (5.91) will be complete once we show that, for n large enough, there exists an $\Omega_{\varepsilon,n}$ for which I, II and III in (5.96) are bounded from above by $\varepsilon/3$.

To that end, note that, because of (5.87), the probabilities $\mathbb{P}(\{I > \varepsilon/3\})$ and $\mathbb{P}(\{III > \varepsilon/3\})$ are bounded from above by $c_1 e^{-c_2\varepsilon^2 n/9L_n}$, while

$$\mathbb{P}(\{II > \varepsilon\}) \leq \mathbb{P}(A^c_{n,N_n,r_n}) \leq 1 - (L_n^2/n^2), \qquad n \in \mathbb{N}. \tag{5.97}$$

Since $\lim_{n\to\infty} n/L_n \log n = \infty$, we have $\mathbb{P}(\{I, II, III \leq \varepsilon/3\}) > 0$ for n large enough. Consequently, the set $\{I, II, III \leq \varepsilon/3\}$ is non-empty and (5.91) is proven.

5.2.7 Convergence

It remains to prove (5.90). Assume that there exist two strictly increasing subsequences $(n_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ and two limits $l_2 > l_1$ such that $\lim_{k\to\infty} f_{3,n_k}(M,m) = l_2$ and $\lim_{k\to\infty} f_{3,t_k}(M,m) = l_1$. By using (5.91), we have that for every $k \in \mathbb{N}$ there exist $N_k \in \{n_k/mL_{n_k}, \dots, n_k/L_{n_k}\}$ and $r_k \in \{-MN_k, \dots, MN_k\}$ such that $\lim_{k\to\infty} f_{3,n_k}(M,m,N_k,r_k) = l_2$. Denote by

$$(\Theta^{k,\Omega}_{\mathrm{traj,max}}, x^{k,\Omega}_{\max}, u^{k,\Omega}_{\max}) \in \widetilde{\mathcal{D}}^{M,r_k}_{L_{n_k},N_k} \times \mathcal{X}^{M,m}_{\Theta^{k,\Omega}_{\mathrm{traj,max}},\Omega} \times \mathcal{U}^{M,m,L_n}_{\Theta^{k,\Omega}_{\mathrm{traj,max}},x^{k,\Omega}_{\max},n_k} \tag{5.98}$$

the maximizer of $H^\Omega(\Theta_{\mathrm{traj}}, x, u)$. We recall that $\Theta_{\mathrm{traj}}, x$ and u take their values in sets that grow subexponentially fast in n_k, and therefore

$$\lim_{k\to\infty} \tfrac{L_{n_k}}{n_k}\, \mathbb{E}_\Omega\big[H^\Omega(\Theta^{k,\Omega}_{\mathrm{traj,max}}, x^{k,\Omega}_{\max}, u^{k,\Omega}_{\max})\big] = l_2. \tag{5.99}$$

Since $l_2 > l_1$, we can use (5.99) and the fact that $\lim_{k\to\infty} n_k/L_{n_k} = \infty$ to obtain, for k large enough,

$$\mathbb{E}_\Omega\big[H^\Omega(\Theta^{k,\Omega}_{\mathrm{traj,max}}, x^{k,\Omega}_{\max}, u^{k,\Omega}_{\max})\big] + (\beta-\alpha) \geq \tfrac{n_k}{L_{n_k}}\big(l_1 + \tfrac{l_2-l_1}{2}\big). \tag{5.100}$$

(The term $\beta-\alpha$ in the left-hand side of (5.100) is introduced for later convenience only.) Next, pick $k_0 \in \mathbb{N}$ satisfying (5.100), whose value will be specified later. Similarly to what we did in (5.75) and (5.76), for $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $k \in \mathbb{N}$ we associate with

$$\Theta^{k_0,\Omega}_{\mathrm{traj,max}} = \big(\Delta\Pi^{k_0,\Omega}_j, b^{k_0,\Omega}_{0,j}, b^{k_0,\Omega}_{1,j}\big)_{j=0}^{N_{k_0}-1} \in \widetilde{\mathcal{D}}^{M,r_{k_0}}_{L_{n_{k_0}},N_{k_0}} \tag{5.101}$$

and

$$x^{k_0,\Omega}_{\max} = \big(x^{k_0,\Omega}_j\big)_{j=0}^{N_{k_0}-1} \in \mathcal{X}^{M,m}_{\Theta^{k_0,\Omega}_{\mathrm{traj,max}},\Omega} \tag{5.102}$$

and

$$u^{k_0,\Omega}_{\max} = \big(u^{k_0,\Omega}_j\big)_{j=0}^{N_{k_0}-1} \in \mathcal{U}^{M,m,L_{n_{k_0}}}_{\Theta^{k_0,\Omega}_{\mathrm{traj,max}},x^{k_0,\Omega}_{\max},n_{k_0}} \tag{5.103}$$

the quantities

$$\overline{\Theta}^{k,\Omega}_{\mathrm{traj}} = \big(\Delta\Pi^{k_0,\Omega}_j, \overline{b}^{\,k,\Omega}_{0,j}, \overline{b}^{\,k,\Omega}_{1,j}\big)_{j=0}^{N_{k_0}-1} \in \widetilde{\mathcal{D}}^{M,r_{k_0}}_{L_{t_k},N_{k_0}} \tag{5.104}$$

and

$$\overline{u}^{\,k,\Omega} = \big(\overline{u}^{\,k,\Omega}_j\big)_{j=0}^{N_{k_0}-1} \in \mathcal{U}^{M,m,L_{t_k}}_{\overline{\Theta}^{k,\Omega}_{\mathrm{traj}},x^{k_0,\Omega}_{\max},*} \tag{5.105}$$

(where $*$ will be specified later), so that

$$\big|\overline{b}^{\,k,\Omega}_{0,j} - b^{k_0,\Omega}_{0,j}\big| \leq \tfrac{1}{L_{t_k}}, \quad \big|\overline{b}^{\,k,\Omega}_{1,j} - b^{k_0,\Omega}_{1,j}\big| \leq \tfrac{1}{L_{t_k}}, \quad \big|\overline{u}^{\,k,\Omega}_j - u^{k_0,\Omega}_j\big| \leq \tfrac{2}{L_{t_k}}, \quad j=0,\dots,N_{k_0}-1. \tag{5.106}$$

Next, put $\overline{s}^{\,\Omega}_k = L_{t_k}\sum_{j=0}^{N_{k_0}-1} \overline{u}^{\,k,\Omega}_j$, which we substitute for $*$ above. The uniform continuity in Lemma C.3 allows us to claim that, for k large enough and for all Ω,

$$\Big|\overline{u}^{\,k,\Omega}_j\ \psi\Big(\overline{\Theta}^{k,\Omega}_j, \overline{u}^{\,k,\Omega}_j\Big) - u^{k_0,\Omega}_j\ \psi\Big(\Theta^{k_0,\Omega}_j, u^{k_0,\Omega}_j\Big)\Big| \leq \tfrac{l_2-l_1}{4}, \tag{5.107}$$

where we recall that, as in (5.71), for all $j = 0,\dots,N_{k_0}-1$,

$$\begin{aligned}\overline{\Theta}^{k,\Omega}_j &= \Big(\Omega\big(j, \Pi^{k_0,\Omega}_j + \cdot\big),\, \Delta\Pi^{k_0,\Omega}_j,\, \overline{b}^{\,k,\Omega}_{0,j},\, \overline{b}^{\,k,\Omega}_{1,j},\, x^{k_0,\Omega}_j\Big),\\ \Theta^{k_0,\Omega}_j &= \Big(\Omega\big(j, \Pi^{k_0,\Omega}_j + \cdot\big),\, \Delta\Pi^{k_0,\Omega}_j,\, b^{k_0,\Omega}_{0,j},\, b^{k_0,\Omega}_{1,j},\, x^{k_0,\Omega}_j\Big).\end{aligned} \tag{5.108}$$

Recall (5.51). An immediate consequence of (5.107) is that

$$\big|H^\Omega(\overline{\Theta}^{k,\Omega}_{\rm traj}, x^{k_0,\Omega}_{\max}, \overline{u}^{\,k,\Omega}) - H^\Omega(\Theta^{k_0,\Omega}_{\rm traj,max}, x^{k_0,\Omega}_{\max}, u^{k_0,\Omega}_{\max})\big| \leq N_{k_0}\,\tfrac{l_2-l_1}{4}. \tag{5.109}$$

Hence we can use (5.100), (5.109) and the fact that $N_{k_0} \leq n_{k_0}/L_{n_{k_0}}$, to conclude that, for k large enough,

$$\mathbb{E}_\Omega\big[H^\Omega(\overline{\Theta}^{k,\Omega}_{\rm traj}, x^{k_0,\Omega}_{\max}, \overline{u}^{\,k,\Omega})\big] + (\beta-\alpha) \geq \tfrac{n_{k_0}}{L_{n_{k_0}}}\big(l_1 + \tfrac{l_2-l_1}{4}\big). \tag{5.110}$$

At this stage we add a column at the end of the group of N_{k_0} columns in such a way that the conditions $\widehat{b}^{k,\Omega}_{1,N_{k_0}-1} = \widehat{b}^{k,\Omega}_{0,N_{k_0}}$ and $\widehat{b}^{k,\Omega}_{1,N_{k_0}} = 1/L_{t_k}$ are satisfied. We put

$$\widehat{\Xi}^{k,\Omega}_{N_{k_0}} = \big(\Delta\Pi^{k_0,\Omega}_{N_{k_0}}, \widehat{b}^{k,\Omega}_{0,N_{k_0}}, \widehat{b}^{k,\Omega}_{1,N_{k_0}}\big) = \big(0, \widehat{b}^{k,\Omega}_{1,N_{k_0}-1}, \tfrac{1}{L_{t_k}}\big), \tag{5.111}$$

and we let $\widehat{\Theta}^{k,\Omega}_{\rm traj} \in \widetilde{\mathcal{D}}^{M,r_{k_0}}_{L_{t_k},N_{k_0}+1}$ be the concatenation of $\overline{\Theta}^{k,\Omega}_{\rm traj}$ (see (5.104)) and $\widehat{\Xi}^{k,\Omega}_{N_{k_0}}$. We let $\widehat{x}^{k_0,\Omega} \in \mathcal{X}^{M,m}_{\widehat{\Theta}^{k,\Omega}_{\rm traj},\Omega}$ be the concatenation of $x^{k_0,\Omega}_{\max}$ and 0. We further let

$$\widehat{s}^{\,\Omega}_k = \overline{s}^{\,\Omega}_k + \Big[1 + b^{k,\Omega}_{1,N_{k_0}-1} - \tfrac{1}{L_{t_k}}\Big] L_{t_k}, \tag{5.112}$$

and we let $\widehat{u}^{k,\Omega} \in \mathcal{U}^{M,m,L_{t_k}}_{\widehat{\Theta}^{k,\Omega}_{\rm traj},\widehat{x}^{k_0,\Omega},\widehat{s}^{\Omega}_k}$ be the concatenation of $\overline{u}^{\,k,\Omega}$ (see (5.105)) and

$$\widehat{u}^{k,\Omega}_{N_{k_0}} = 1 + (b^{k,\Omega}_{1,N_{k_0}-1} - \tfrac{1}{L_{t_k}}). \tag{5.113}$$

Next, we note that the right-most inequality in (5.106), together with the fact that

$$\sum_{j=0}^{N_{k_0}-1} u^{k_0,\Omega}_j = n_{k_0}/L_{n_{k_0}}, \tag{5.114}$$

allow us to asset that $|\overline{s}^{\,\Omega}_k - L_{t_k} n_{k_0}/L_{n_{k_0}}| \leq 2N_{k_0}$. Therefore the definition of $\widehat{s}^{\,\Omega}_k$ in (5.112) implies that

$$\widehat{s}^{\,\Omega}_k = L_{t_k}\frac{n_{k_0}}{L_{n_{k_0}}} + \widehat{m}^{\Omega}_k \quad \text{with} \quad |\widehat{m}^{\Omega}_k| \leq 2N_{k_0} + 2L_{t_k}. \tag{5.115}$$

Moreover,

$$H^\Omega\big(\widehat{\Theta}^{k,\Omega}_{\rm traj}, \widehat{x}^{k_0,\Omega}, \widehat{u}^{k,\Omega}\big) \geq H^\Omega\big(\overline{\Theta}^{k,\Omega}_{\rm traj}, x^{k_0,\Omega}_{\max}, \overline{u}^{\,k,\Omega}\big) + (\beta-\alpha), \tag{5.116}$$

because $\widehat{u}^{k,\Omega}_{N_{k_0}} \leq 2$ by definition (see (5.113)) and the free energies per columns are all bounded from below by $(\beta-\alpha)/2$. Hence, (5.110) and (5.116) give that for all Ω there exist a

$$\widehat{\Theta}^{k,\Omega}_{\rm traj} \in \widetilde{\mathcal{D}}^{M,r_{k_0}}_{L_{t_k},N_{k_0}+1} \colon\ b_{1,N_{k_0}} = \tfrac{1}{L_{t_k}}, \tag{5.117}$$

an $\widehat{x}^{k_0,\Omega} \in \mathcal{X}^{M,m}_{\widehat{\Theta}^{k,\Omega}_{\rm traj},\Omega}$ and a $\widehat{u}^{k,\Omega} \in \mathcal{U}^{M,m,L_{t_k}}_{\widehat{\Theta}^{k,\Omega}_{\rm traj},\widehat{x}^{k_0,\Omega},\widehat{s}^{\Omega}_k}$ such that, for k large enough,

$$\mathbb{E}_\Omega\big[H(\widehat{\Theta}^{k,\Omega}_{\rm traj}, \widehat{x}^{k_0,\Omega}, \widehat{u}^{k,\Omega})\big] \geq \tfrac{n_{k_0}}{L_{n_{k_0}}}\big(l_1 + \tfrac{l_2-l_1}{4}\big). \tag{5.118}$$

Next, we subdivide the disorder Ω into groups of $N_{k_0}+1$ consecutive columns that are successively translated by r_{k_0} in the vertical direction, i.e., $\Omega = (\Omega_1, \Omega_2, \dots)$ with (recall (3.10))

$$\Omega_j = \big(\Omega(i,\,(j-1)\,r_{k_0} + \cdot)\big)_{i=(j-1)(N_{k_0}+1)}^{j(N_{k_0}+1)-1}, \tag{5.119}$$

and we let q_k^Ω be the unique integer satisfying

$$\widehat{s}_k^{\Omega_1} + \widehat{s}_k^{\Omega_2} + \cdots + \widehat{s}_k^{\Omega_{q_k}} \leq t_k < \widehat{s}_k^{\Omega_1} + \cdots + \widehat{s}_k^{\Omega_{q_k+1}}, \tag{5.120}$$

where we suppress the Ω-dependence of q_k. We recall that

$$f_{3,t_k}^\Omega(M,m) = \mathbb{E}\Bigg[\frac{1}{t_k} \log \sum_{N=t_k/mL_{t_k}}^{t_k/L_{t_k}} \sum_{\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}_{L_{t_k},N}^M} \sum_{x \in \mathcal{X}_{\Theta_{\text{traj}},\Omega}^{M,m}} \sum_{u \in \mathcal{U}_{\Theta_{\text{traj}},\,x,\,t_k}^{M,m,\,L_{t_k}}} e^{L_{t_k}\, H^\Omega(\Theta_{\text{traj}},x,u)}\Bigg], \tag{5.121}$$

set $\widetilde{t}_k^\Omega = \widehat{s}_k^{\Omega_1} + \widehat{s}_k^{\Omega_2} + \cdots + \widehat{s}_k^{\Omega_{q_k}}$, and concatenate

$$\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega} = \Big(\widehat{\Theta}_{\text{traj}}^{k,\Omega_1}, \widehat{\Theta}_{\text{traj}}^{k,\Omega_2}, \dots, \widehat{\Theta}_{\text{traj}}^{k,\Omega_{q_k}}\Big) \in \widetilde{\mathcal{D}}_{L_{t_k},\,q_k(N_{k_0}+1),}^{M,} \tag{5.122}$$

and

$$\widehat{x}_{\text{tot}}^{k,\Omega} = \big(\widehat{x}^{k_0,\Omega_1}, \widehat{x}^{k_0,\Omega_2}, \dots, \widehat{x}^{k_0,\Omega_{q_k}}\big) \in \mathcal{X}_{\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega}\Omega}^{M,m}. \tag{5.123}$$

and

$$\widehat{u}_{\text{tot}}^{k,\Omega} = \big(\widehat{u}^{k,\Omega_1}, \widehat{u}^{k,\Omega_2}, \dots, \widehat{u}^{k,\Omega_{q_k}}\big) \in \mathcal{U}_{\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega},\widehat{x}_{\text{tot}}^{k,\Omega},\widetilde{t}_k^\Omega}^{M,m,L_{t_k}}. \tag{5.124}$$

It still remains to complete $\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega}$, $\widehat{x}_{\text{tot}}^{k,\Omega}$ and $\widehat{u}_{\text{tot}}^{k,\Omega}$ such that the latter becomes an element of $\mathcal{U}_{\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega},\widehat{x}_{\text{tot}}^{k,\Omega},t_k}^{M,m,L_{t_k}}$. To that end, we recall (5.120), which gives $t_k - \widetilde{t}_k^\Omega \leq \widehat{s}_k^{\Omega_{q_k+1}}$. Then, using (5.115), we have that there exists a $c > 0$ such that

$$t_k - \widetilde{t}_k^\Omega \leq c L_{t_k} \tfrac{n_{k_0}}{L_{n_{k_0}}}. \tag{5.125}$$

Therefore we can complete $\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega}$, $\widehat{x}_{\text{tot}}^{k,\Omega}$ and $\widehat{u}_{\text{tot}}^{k,\Omega}$ with

$$\Theta_{\text{rest}} \in \mathcal{D}_{L_{t_k},\,g_k^\Omega}^M, \qquad x_{\text{rest}} \in \mathcal{X}_{\Theta_{\text{rest}},\Omega}^{M,m}, \qquad u_{\text{rest}} \in \mathcal{U}_{\Theta_{\text{rest}},x_{\text{rest}},t_k-\widetilde{t}_k^\Omega}^{M,m,L_{t_k}}, \tag{5.126}$$

such that, by (5.125), the number of columns g_k^Ω involved in Θ_{rest} satisfies $g_k^\Omega \leq c n_{k_0}/L_{n_{k_0}}$. Henceforth $\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega}$, $\widehat{x}_{\text{tot}}^{k,\Omega}$ and $\widehat{u}_{\text{tot}}^{k,\Omega}$ stand for the quantities defined in (5.122) and (5.124), and concatenated with $\Theta_{\text{rest}}, x_{\text{rest}}$ and u_{rest} so that they become elements of

$$\mathcal{D}_{L_{t_k},\,q_k(N_{k_0}+1)+g_k^\Omega}^M, \qquad \mathcal{X}_{\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega},\Omega}^{M,m}, \qquad \mathcal{U}_{\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega},\widehat{x}_{\text{tot}}^{k,\Omega},t_k}^{M,m,L_{t_k}}, \tag{5.127}$$

respectively. By restricting the summation in (5.45) to $\widehat{\Theta}_{\text{traj,tot}}^{k,\Omega}$, $\widehat{x}_{\text{tot}}^{k,\Omega}$ and $\widehat{u}_{\text{tot}}^{k,\Omega}$, we get

$$f_{3,t_k}(M,m) \geq \frac{L_{t_k}}{t_k} \mathbb{E}_\Omega\Bigg[\sum_{j=1}^{q_k} H^{\Omega_j}(\widehat{\Theta}_{\text{traj}}^{k,\Omega_j}, \widehat{x}^{k_0,\Omega_j}, \widehat{u}^{k,\Omega_j}) + H(\Theta_{\text{rest}}, x_{\text{rest}}, u_{\text{rest}})\Bigg], \tag{5.128}$$

where the term $H(\Theta_{\text{rest}}, x_{\text{rest}}, u_{\text{rest}})$ is negligible because, by (5.125), $(t_k - \widetilde{t}_k^\Omega)/t_k$ vanishes as $k \to \infty$, while all free energies per column are bounded from below by $(\beta - \alpha)/2$. Pick $\varepsilon > 0$ and recall (5.115). Choose k_0 such that $2L_{n_{k_0}}/n_{k_0} \leq \varepsilon/2$ and note that, for k large enough,

$$\widehat{s}_k^\Omega \in \Big[L_{t_k} \tfrac{n_{k_0}}{L_{n_{k_0}}}(1-\varepsilon), L_{t_k} \tfrac{n_{k_0}}{L_{n_{k_0}}}(1+\varepsilon)\Big]. \tag{5.129}$$

By (5.120), we therefore have

$$q_k \in \Big[\tfrac{t_k L_{n_{k_0}}}{L_{t_k} n_{k_0}} \tfrac{1}{1+\varepsilon}, \tfrac{t_k L_{n_{k_0}}}{L_{t_k} n_{k_0}} \tfrac{1}{1-\varepsilon}\Big] = [a,b]. \tag{5.130}$$

Recalling (5.128), we obtain

$$f_{3,t_k}(M,m) \geq \frac{L_{t_k}}{t_k} \mathbb{E}_\Omega \Big[\sum_{j=1}^{a} H^{\Omega_j}(\widehat{\Theta}^{k,\Omega_j}_{\text{traj}}, \widehat{x}^{k_0,\Omega_j}, \widehat{u}^{k,\Omega_j}) - \sum_{j=a}^{b} \Big| H^{\Omega_j}(\widehat{\Theta}^{k,\Omega_j}_{\text{traj}}, \widehat{x}^{k_0,\Omega_j}, \widehat{u}^{k,\Omega_j}) \Big| \Big], \tag{5.131}$$

and, consequently,

$$f_{3,t_k}(M,m) \geq \tfrac{L_{n_{k_0}}}{n_{k_0}(1+\varepsilon)} \, \mathbb{E}\,\Omega \Big[H^{\Omega}(\widehat{\Theta}^{k,\Omega}_{\text{traj}}, \widehat{x}^{k_0,\Omega}, \widehat{u}^{k,\Omega}) \Big] - \frac{L_{t_k}}{t_k}(b-a)(N_{k_0}+1) m^{\frac{\beta-\alpha}{2}}, \tag{5.132}$$

and, by (5.118),

$$f_{3,t_k}(M,m) \geq \frac{l_1 + \frac{l_2 - l_1}{4}}{1+\varepsilon} - (\tfrac{1}{1-\varepsilon} - \tfrac{1}{1+\varepsilon})(b-a) m^{\frac{\beta-\alpha}{2}}. \tag{5.133}$$

After taking ε small enough, we may conclude that $\liminf_{k\to\infty} f_{3,t_k}(M,m) > l_1$, which completes the proof.

5.3 Proof of Proposition 5.3

Pick $(M,m) \in \text{EIGH}$ and note that, for every $n \in \mathbb{N}$, the set $\mathcal{W}^m_{n,M}$ is contained in $\mathcal{W}_{n,M}$. Thus, by using Proposition 5.2 we obtain

$$\begin{aligned} \liminf_{n\to\infty} f^{\Omega}_{1,n}(M;\alpha,\beta) &\geq \sup_{m \geq M+2} \liminf_{n\to\infty} f^{\Omega}_{1,n}(M,m;\alpha,\beta) \\ &= \sup_{m\geq M+2} f(M,m;\alpha,\beta) \quad \text{ for } \mathbb{P} - a.e.\,\Omega. \end{aligned} \tag{5.134}$$

Therefore, the proof of Proposition 5.3 will be complete once we show that

$$\limsup_{n\to\infty} f^{\Omega}_{1,n}(M;\alpha,\beta) \leq \sup_{m\geq M+2} \limsup_{n\to\infty} f^{\Omega}_{1,n}(M,m;\alpha,\beta) \quad \text{ for } \mathbb{P} - a.e.\,\Omega. \tag{5.135}$$

We will not prove (5.135) in full detail, but only give the main steps in the proof. The proof consists in showing that, for m large enough, the pieces of the trajectory in a column that exeed mL_n steps do not contribute substantially to the free energy.

Recall (5.25–5.30) and use (5.30) with $m = \infty$, i.e.,

$$Z^{\omega,\Omega}_{n,L_n}(M) = \sum_{N=1}^{n/L_n} \sum_{\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \sum_{x \in \mathcal{X}^{M,\infty}_{\Theta_{\text{traj}},\Omega}} \sum_{u \in \mathcal{U}^{M,\infty,L_n}_{\Theta_{\text{traj}},x,n}} A_1. \tag{5.136}$$

With each $(N, \Theta_{\text{traj}}, x, u)$ in (5.136), we associate the trajectories obtained by concatenating N shorter trajectories $(\pi_i)_{i\in\{0,\dots,N-1\}}$ chosen in $(\mathcal{W}_{\Theta_i,u_i,L_n})_{i\in\{0,\dots,N-1\}}$, respectively. Thus, the quantity A_1 in (5.136) corresponds to the restriction of the partition function to the trajectories associated with $(N, \Theta_{\text{traj}}, x, u)$. In order to discriminate between the columns in which more than mL_n steps are taken and those in which less are taken, we rewrite A_1 as $A_2\widetilde{A}_2$ with

$$A_2 = \prod_{i\in V_{u,m}} Z_{L_n}^{\omega_{I_i}}(\Theta_i, u_i), \qquad \widetilde{A}_2 = \prod_{i\in \widetilde{V}_{u,m}} Z_{L_n}^{\omega_{I_i}}(\Theta_i, u_i), \tag{5.137}$$

with $\widetilde{u}_i = \sum_{k=0}^{i-1} u_k$, $\Theta_i = (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i)$ and $I_i = \{\widetilde{u}_i L_n,\dots,\widetilde{u}_{i+1}L_n - 1\}$ for $i \in \{0,\dots,N-1\}$, with $\omega_I = (\omega_i)_{i\in I}$ for $I\subset\mathbb{N}$, where $\{0,\dots,N-1\}$ is partitioned into

$$\widetilde{V}_{u,m}\cup V_{u,m} \quad \text{with} \quad \widetilde{V}_{u,m} = \{i\in\{0,\dots,N-1\}\colon\ u_i > m\}. \tag{5.138}$$

For all $(N,\Theta_{\text{traj}},x,u)$, we rewrite $\widetilde{V}_{u,m}$ in the form of an increasing sequence $\{i_1,\dots,i_{\widetilde{k}}\}$ and we drop the (u,m)-dependence of $\widetilde{k}$ for simplicity. We also set $\widetilde{u} = u_{i_1}+\cdots+u_{i_{\widetilde{k}}}$, which is the total number of steps taken by a trajectory associated with $(N,\Theta_{\text{traj}},x,u)$ in those columns where more than mL_n steps are taken. Finally, for $s\in\{1,\dots,\widetilde{k}\}$ we partition I_{i_s} into

$$\begin{aligned} J_{i_s}\cup\widetilde{J}_{i_s} \quad \text{with} \quad & J_{i_s} = \{\widetilde{u}_{i_s}L_n,\dots,(\widetilde{u}_{i_s}+M+2)L_n\}, \\ & \widetilde{J}_{i_s} = \{(\widetilde{u}_{i_s}+M+2)L_n+1,\dots,\widetilde{u}_{i_s+1}L_n-1\}, \end{aligned} \tag{5.139}$$

and we partition $\{1,\dots,n\}$ into

$$J\cup\widetilde{J} \quad \text{with} \quad \widetilde{J} = \cup_{s=1}^{\widetilde{k}}\widetilde{J}_{i_s}, \qquad J = \{1,\dots,n\}\setminus\widetilde{J}, \tag{5.140}$$

so that $\widetilde{J}$ contains the label of the steps constituting the pieces of trajectory exeeding $(M+2)L_n$ steps in those columns where more than mL_n steps are taken.

5.3.1 Step 1

In this step we replace the pieces of trajectories in the columns indexed in $\widetilde{V}_{u,m}$ by shorter trajectories of length $(M+2)L_n$. To that aim, for every $(N,\Theta_{\text{traj}},x,u)$ we set

$$\widehat{A}_2 = \prod_{i\in\widetilde{V}_{u,m}} Z_{L_n}^{\omega_{J_i}}(\Theta_i', M+2) \tag{5.141}$$

with $\Theta_i' = (\Omega(i,\Pi_i+\cdot),\Xi_i,1)$. We will show that for all $\varepsilon>0$ and for m large enough, the event

$$B_n = \{\omega\colon\ \widetilde{A}_2 \le \widehat{A}_2\, e^{3\varepsilon n} \text{ for all } (N,\Theta_{\text{traj}},x,u)\} \tag{5.142}$$

satisfies $\mathbb{P}_\omega(B_n)\to 1$ as $n\to\infty$.

Pick, for each $s\in\{1,\dots,\widetilde{k}\}$, a trajectory π_s in the set $\mathcal{W}_{\Theta_{i_s},u_{i_s},L_n}$. By concatenating them we obtain a trajectory in $\mathcal{W}_{\widetilde{u}L_n}$ satisfying $\pi_{\widetilde{u}L_n,1} = \widetilde{k}L_n$. Thus, the total entropy carried by those pieces of trajectories crossing the columns indexed in $\{i_1,\dots,i_{\widetilde{k}}\}$ is bounded above by

$$\textstyle\prod_{s=1}^{\widetilde{k}} |\mathcal{W}_{\Theta_{i_s},u_{i_s},L_n}| \le \big|\{\pi\in\mathcal{W}_{\widetilde{u}L_n}\colon\ \pi_{\widetilde{u}L_n,1} = \widetilde{k}L_n\}\big|. \tag{5.143}$$

Since $\widetilde{u}/\widetilde{k}\ge m$, we can use Lemma A.2 in Appendix A to assert that, for m large enough, the right-hand side of (5.143) is bounded above by $e^{\varepsilon n}$.

Moreover, we note that an $\widetilde{u}L_n$-step trajectory satisfying $\pi_{\widetilde{u}L_n,1} = \widetilde{k}L_n$ makes at most $\widetilde{k}L_n+\widetilde{u}$ excursions in the B solvent because such an excursion requires at least one horizontal step or at least L_n vertical steps. Therefore, by using the inequalities $\widetilde{k}L_n \le n/m$ and $\widetilde{u} \le n/L_n$ we obtain that, for n large enough, the sum of the Hamiltonians associated with $(\pi_1, \dots, \pi_{\widetilde{k}})$ is bounded from above, uniformly in $(N, \Theta_{\text{traj}}, x, u)$ and $(\pi_1, \dots, \pi_{\widetilde{k}})$, by

$$\textstyle\sum_{s=1}^{\widetilde{k}} H_{u_{i_s}L_n, L_n}^{\omega_{I_{i_s}}, \Omega(i_s, \Pi_{i_s}+\cdot)}(\pi_s) \le \max\{\sum_{i\in I} \xi_i \colon\ I \in \cup_{r=1}^{2n/m} \mathcal{E}_{n,r}\}, \tag{5.144}$$

with $\mathcal{E}_{n,r}$ defined in (E.1) in Appendix E and $\xi_i = \beta 1_{\{\omega_i = A\}} - \alpha 1_{\{\omega_i = B\}}$ for $i \in \mathbb{N}$. At this stage we use the definition in (E.3) and note that, for all $\omega \in \mathcal{Q}_{n,m}^{\varepsilon/\beta, (\alpha-\beta)/2+\varepsilon}$, the right-hand side in (5.144) is smaller than εn. Consequently, for m and n large enough we have that, for all $\omega \in \mathcal{Q}_{n,m}^{\varepsilon/\beta, (\alpha-\beta)/2+\varepsilon}$,

$$\widetilde{A}_2 \le e^{2\varepsilon n} \quad \text{for all} \quad (N, \Theta_{\text{traj}}, x, u). \tag{5.145}$$

Recalling (3.34) and noting that $\widetilde{k}L_n \le n/m$, we can write

$$\widehat{A}_2 \ge e^{-\widetilde{k}(M+2)L_n C_{\text{uf}}(\alpha)} \ge e^{-n\frac{M+2}{m}C_{\text{uf}}(\alpha)}, \tag{5.146}$$

and therefore, for m large enough, for all n and all $(N, \Theta_{\text{traj}}, x, u)$ we have $\widehat{A}_2 \ge e^{-\varepsilon n}$.

Finally, use (5.145) and (5.146) to conclude that, for m and n large enough, $\mathcal{Q}_{n,m}^{\varepsilon/\beta, (\alpha-\beta)/2+\varepsilon}$ is a subset of B_n. Thus, Lemma E.1 ensures that, for m large enough, $\lim_{n\to\infty} P_\omega(B_n) = 1$.

5.3.2 Step 2

Let $(\widetilde{w}_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernouilli trials, independent of ω, Ω. For $(N, \Theta_{\text{traj}}, x, u)$ we set $\widehat{u} = \widetilde{u} - \widetilde{k}(M+2)$. In Step 1 we have removed $\widehat{u}L_n$ steps from the trajectories associated with $(N, \Theta_{\text{traj}}, x, u)$ so that they have become trajectories associated with $(N, \Theta_{\text{traj}}, x', u)$. In this step, we will concatenate the trajectories associated with $(N, \Theta_{\text{traj}}, x', u)$ with an $\widehat{u}L_n$-step trajectory to recover a trajectory that belongs to $\mathcal{W}_{n,M}^m$.

For $\Omega \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $t, N \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$P_A^\Omega(N,k)(t) = \frac{1}{t}\sum_{j=0}^{t-1} 1_{\{\Omega(N+j,k)=A\}} \tag{5.147}$$

be the proportion of A-blocks on the k^{th} line and between the N^{th} and the $(N+t-1)^{\text{th}}$ column of Ω. Pick $\eta > 0$ and $j \in \mathbb{N}$, and set

$$S_{\eta,j} = \bigcup_{N=0}^{j} \bigcup_{k=-m_1 N}^{m_1 N} \bigcup_{t \ge \eta j} \Big\{P_A^\Omega(N,k)(t) \le \frac{p}{2}\Big\}. \tag{5.148}$$

By a straightforward application of Cramer's Theorem for i.i.d. random variables, we have that $\sum_{j\in\mathbb{N}} P_\Omega(S_{\eta,j}) < \infty$. Therefore, using the Borel-Cantelli Lemma, it follows that for $\mathbb{P}_\Omega$-a.e. Ω, there exists a $j_\eta(\Omega) \in \mathbb{N}$ such that $\Omega \notin S_{\eta,j}$ as soon as $j \ge j_\eta(\Omega)$. In what follows, we consider $\eta = \varepsilon/\alpha m$ and we take n large enough so that $n/L_n \ge j_{\varepsilon/\alpha m}(\Omega)$, and therefore $\Omega \notin S_{\frac{n}{L_n}, \frac{\varepsilon}{\alpha m}}$.

Pick (N, Θ, x, u) and consider one trajectory $\widehat{\pi}$, of length $\widehat{u}L_n$, starting from $(N, \Pi_N + b_N)L_n$, staying in the coarsed-grained line at height Π_N, crossing the B-blocks in a straight line and the A-blocks in mL_n steps. The number of columns crossed by $\widehat{\pi}$ is denoted by $\widehat{N}$

and satisfies $\widehat{N} \geq \widehat{u}/m$. If $\widehat{u}L_n \leq \varepsilon n/\alpha$, then the Hamiltonian associated with $\widehat{\pi}$ is clearly larger than $-\varepsilon n$. If $\widehat{u}L_n \geq \varepsilon n/\alpha$ in turn, then

$$H_{\widehat{u}L_n,L_n}^{\widetilde{w},\Omega(N+\cdot,\Pi_N)}(\widehat{\pi}) \geq -\alpha L_n \widehat{N}\big[1 - P_A^{\Omega}(N,\Pi_N)(\widehat{N})\big]. \tag{5.149}$$

Since $N \leq n/L_n$, $|\Pi_N| \leq m_1 N$ and $\widehat{N} \geq \varepsilon n/(\alpha m L_n)$, we can use the fact that $\Omega \notin S_{\frac{n}{L_n},\frac{\varepsilon}{\alpha m}}$ to obtain

$$P_A^{\Omega}(N,\Pi_N)(\widehat{N}) \geq \frac{p}{2}. \tag{5.150}$$

At this point it remains to bound $\widehat{N}$ from above, which is done by noting that

$$\widehat{N}\big[m P_A^{\Omega}(N,\Pi_N)(\widehat{N}) + 1 - P_A^{\Omega}(N,\Pi_N)(\widehat{N})\big] = \widehat{u} \leq \tfrac{n}{L_n}. \tag{5.151}$$

Hence, using (5.150) and (5.151), we obtain $\widehat{N} \leq 2n/pmL_n$ and therefore the right-hand side of (5.149) is bounded from below by $-\alpha(2-p)n/pm$, which for m large enough is larger than $-\varepsilon n$.

Thus, for n and m large enough and for all (N,Θ,x,u), we have a trajectory $\widehat{\pi}$ at which the Hamiltonian is bounded from below by $-\varepsilon n$ that can be concatenated with all trajectories associated with (N,Θ,x',u) to obtain a trajectory in $\mathcal{W}_{n,M}^{m}$. Consequently, recalling (5.139), for n and m large enough we have

$$A_2\widehat{A_2} \leq e^{\varepsilon n} Z_{n,L_n}^{(\omega_J,\widetilde{\omega}),\Omega}(M,m) \qquad \forall\,(N,\Theta,x,u). \tag{5.152}$$

5.3.3 Step 3

In this step, we average over the microscopic disorders $\omega,\widetilde{\omega}$. Use (5.152) to note that, for n and m large enough and all $\omega \in B_n$, we have

$$Z_{n,L_n}^{\omega,\Omega}(M) \leq e^{4\varepsilon n} \sum_{N=1}^{n/L_n} \sum_{\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}_{L_n,N}^{M}} \sum_{x \in \mathcal{X}_{\Theta_{\text{traj}},\Omega}^{M,\infty}} \sum_{u \in \mathcal{U}_{\Theta_{\text{traj}},x,n}^{M,\infty,L_n}} Z_{n,L_n}^{(\omega_J,\widetilde{\omega}),\Omega}(M,m). \tag{5.153}$$

We use (D.3) to claim that there exists $C_1, C_2 > 0$ so that for all $n \in \mathbb{N}$, all $m \in \mathbb{N}$ and all J,

$$\mathbb{P}_{\omega,\widetilde{\omega}}\Big(\Big|\tfrac{1}{n}\log Z_{n,L_n}^{(\omega_J,\widetilde{\omega}),\Omega}(M,m) - f_{1,n}^{\Omega}(M,m)\Big| \geq \varepsilon\Big) \leq C_1 e^{-C_2\varepsilon^2 n}. \tag{5.154}$$

We set also

$$D_n = \bigcap_{(N,\Theta_{\text{traj}},x,u)} \Big\{\Big|\tfrac{1}{n}\log Z_{n,L_n}^{(\omega_J,\widetilde{\omega}),\Omega}(M,m) - f_{1,n}^{\Omega}(M,m)\Big| \leq \varepsilon\Big\}, \tag{5.155}$$

recall the definition of c_n in (5.55) (used with (M,∞)), and use (5.154) and the fact that c_n grows subexponentially, to obtain $\lim_{n\to\infty} \mathbb{P}_{\omega,\widetilde{\omega}}(D_n^c) = 0$. For all $(\omega,\widetilde{\omega})$ satisfying $\omega \in B_n$ and $(\omega,\widetilde{\omega}) \in D_n$, we can rewrite (5.153) as

$$Z_{n,L_n}^{\omega,\Omega}(M) \leq c_n\; e^{n f_{1,n}^{\Omega}(M,m) + 5\varepsilon n}. \tag{5.156}$$

As a consequence, recalling (3.34), for m large enough we have

$$f_n^{\Omega}(M;\alpha,\beta) \leq \mathbb{P}(B_n^c \cup D_n^c)\, C_{\text{uf}}(\alpha) + \frac{\log c_n}{n} + \frac{1}{n}\mathbb{E}\Big(1_{\{B_n \cup D_n\}}\big(n f_{1,n}^{\Omega}(M,m) + 5\varepsilon n\big)\Big). \tag{5.157}$$

Since $\mathbb{P}(B_n^c \cup D_n^c)$ and $(\log c_n)/n$ vanish when $n \to \infty$, it suffices to apply Proposition 5.2 and to let $\varepsilon \to 0$ to obtain (5.135). This completes the proof of Proposition 5.3.

5.4 Proof of Proposition 5.4

Note that, for all $m \geq M+2$, we have $\mathcal{R}^m_{p,M} \subset \mathcal{R}_{p,M}$. Moreover, any $(u_\Theta)_{\Theta \in \overline{\mathcal{V}}^m_M} \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}$ can be extended to $\overline{\mathcal{V}}_M$ so that it belongs to $\mathcal{B}_{\overline{\mathcal{V}}_M}$. Thus,

$$\sup_{m \geq M+2} f(M, m; \alpha, \beta) \leq \sup_{\rho \in \mathcal{R}_{p,M}} \sup_{(u) \in \mathcal{B}_{\overline{\mathcal{V}}_M}} V(\rho, u). \tag{5.158}$$

As a consequence, it suffices to show that for all $\rho \in \mathcal{R}_{p,M}$ and $(u_\Theta)_{\Theta \in \overline{\mathcal{V}}_M} \in \mathcal{B}_{\overline{\mathcal{V}}_M}$,

$$V(\rho, u) \leq \sup_{m \geq M+2} \sup_{\rho \in \mathcal{R}^m_{p,M}} \sup_{(u) \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}} V(\rho, u). \tag{5.159}$$

If $\int_{\overline{\mathcal{V}}_M} u_\Theta \, \rho(d\Theta) = \infty$, then (5.159) is trivially satisfied since $V(\rho, u) = -\infty$. Thus, we can assume that $\rho(\overline{\mathcal{V}}_M \setminus D_M) = 1$, where $D_M = \{\Theta \in \overline{\mathcal{V}}_M \colon \chi_\Theta \in \{A^{\mathbb{Z}}, B^{\mathbb{Z}}\}, x_\Theta = 2\}$. Since $\int_{\overline{\mathcal{V}}_M} u_\Theta \, \rho(d\Theta) < \infty$ and since (recall (3.34)) $\psi(\Theta, u)$ is uniformly bounded by $C_{\mathrm{uf}}(\alpha)$ on $(\Theta, u) \in \overline{\mathcal{V}}^*_M$, we have by dominated convergence that for all $\varepsilon > 0$ there exists an $m_0 \geq M+2$ such that, for all $m \geq m_0$,

$$V(\rho, u) \leq \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \psi(\Theta, u_\Theta) \rho(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta)} + \tfrac{\varepsilon}{2}. \tag{5.160}$$

Since $\rho(\overline{\mathcal{V}}_M \setminus D_M) = 1$ and since $\cup_{m \geq M+2} \overline{\mathcal{V}}^m_M = \overline{\mathcal{V}}_M \setminus D_M$, we have $\lim_{m\to\infty} \rho(\overline{\mathcal{V}}^m_M) = 1$. Moreover, for all $m \geq m_0$ there exists a $\widehat{\rho}_m \in \mathcal{R}^m_{p,M}$ such that $\widehat{\rho}_m = \rho_m + \overline{\rho}_m$, with ρ_m the restriction of ρ to $\overline{\mathcal{V}}^m_M$ and $\overline{\rho}_m$ charging only those Θ satisfying $x_\Theta = 1$. Since all $\Theta \in \overline{\mathcal{V}}_M$ with $x_\Theta = 1$ also belong to $\overline{\mathcal{V}}^{M+2}_M$, we can state that $\overline{\rho}_m$ only charges $\overline{\mathcal{V}}^{M+2}_M$. Therefore

$$V(\widehat{\rho}_m, u) = \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \psi(\Theta, u_\Theta) \rho(d\Theta) + \int_{\overline{\mathcal{V}}^{M+2}_M} u_\Theta \psi(\Theta, u_\Theta) \overline{\rho}_m(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta) + \int_{\overline{\mathcal{V}}^{M+2}_M} u_\Theta \overline{\rho}_m(d\Theta)}. \tag{5.161}$$

Since $\Theta \mapsto u_\Theta$ is continuous on $\overline{\mathcal{V}}_M$, there exists an $R > 0$ such that $u_\Theta \leq R$ for all $\Theta \in \overline{\mathcal{V}}^{M+2}_M$. Therefore we can use (5.160) and (5.161) to obtain, for $m \geq m_0$,

$$V(\widehat{\rho}_m, u) \geq (V(\rho, u) - \tfrac{\varepsilon}{2}) \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta) + \int_{\overline{\mathcal{V}}^{M+2}_M} u_\Theta \overline{\rho}_m(d\Theta)} - R\, C_{\mathrm{uf}}(\alpha)\, (1 - \rho(\overline{\mathcal{V}}^m_M)). \tag{5.162}$$

The fact that $\overline{\rho}_m(\mathcal{V}^{M+2}_M) = \rho(\overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}^m_M)$ for all $m \geq m_0$ impliess that $\lim_{m\to\infty} \overline{\rho}_m(\mathcal{V}^{M+2}_M) = 0$. Consequently, the right-hand side in (5.162) tends to $V(\rho, u) - \varepsilon/2$ as $m \to \infty$. Thus, there exists a $m_1 \geq m_0$ such that $V(\widehat{\rho}_{m_1}, u) \geq V(\rho, u) - \varepsilon$. Finally, we note that there exists a $m_2 \geq m_1 + 1$ such that $u_\Theta \leq m_2$ for all $\Theta \in \overline{\mathcal{V}}^{m_1}_M$, which allows us to extend $(u_\Theta)_{\Theta \in \overline{\mathcal{V}}^{m_1}_M}$ to $\overline{\mathcal{V}}^{m_2}_M$ such that $(u_\Theta)_{\Theta \in \overline{\mathcal{V}}^{m_2}_M} \in \mathcal{B}_{\overline{\mathcal{V}}^{m_2}_M}$. It suffices to note that $\widehat{\rho}_{m_1} \in \mathcal{R}^{m_1}_{p,M} \subset \mathcal{R}^{m_2}_{p,M}$ to conclude that

$$V(\rho, u) \leq f(M, m_2; \alpha, \beta) + \varepsilon. \tag{5.163}$$

6 Proof of Theorem 1.1: slope-based variational formula

We are now ready to show how (5.2) can be transformed into (1.17).

Let $\mathcal{F}_{\overline{\mathcal{V}}_M}$ and $\bar{\mathcal{F}}$ be the counterparts of $\mathcal{B}_{\overline{\mathcal{V}}_M}$ and $\bar{\mathcal{B}}$ for Borel functions instead of continuous functions, i.e.,

$$\mathcal{F}_{\overline{\mathcal{V}}_M} = \big\{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M} \in \mathbb{R}^{\overline{\mathcal{V}}_M}\colon\ u_\Theta \geq t_\Theta\ \forall\,\Theta \in \overline{\mathcal{V}}_M,\ \Theta \mapsto u_\Theta \text{ Borel}\big\} \tag{6.1}$$

and

$$\bar{\mathcal{F}} = \{v = (v_A, v_B, v_\mathcal{I}) \in \bar{\mathcal{D}} \times \bar{\mathcal{D}} \times [1,\infty)\}, \tag{6.2}$$

where

$$\bar{\mathcal{D}} = \{l \mapsto v_l \text{ on } [0,\infty) \text{ Lebesgue measurable with } v_l \geq 1 + l \quad \forall\, l \geq 0\}\,. \tag{6.3}$$

The proof of Theorem 1.1 is divided into 4 steps, organized as Sections 6.1–6.4. In Step 1 we show that the supremum over $\mathcal{B}_{\overline{\mathcal{V}}_M}$ in (5.2) may be extended to $\mathcal{F}_{\overline{\mathcal{V}}_M}$, i.e.,

$$\sup_{\rho\in\mathcal{R}_{p,M}}\ \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M}\in\mathcal{B}_{\overline{\mathcal{V}}_M}} \frac{N(\rho,u)}{D(\rho,u)} = \sup_{\rho\in\mathcal{R}_{p,M}}\ \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M}\in\mathcal{F}_{\overline{\mathcal{V}}_M}} \frac{N(\rho,u)}{D(\rho,u)}. \tag{6.4}$$

In Step 2 we show that the supremum over $\bar{\mathcal{B}}$ in (1.17) may be extended to $\bar{\mathcal{F}}$, i.e.,

$$\sup_{\bar\rho\in\bar{\mathcal{R}}_{p,M}}\ \sup_{v\in\bar{\mathcal{B}}} \frac{\bar N(\bar\rho,v)}{\bar D(\bar\rho,v)} = \sup_{\bar\rho\in\bar{\mathcal{R}}_{p,M}}\ \sup_{v\in\bar{\mathcal{F}}} \frac{\bar N(\bar\rho,v)}{\bar D(\bar\rho,v)}. \tag{6.5}$$

In Steps 3 and 4 we use Proposition 5.1 to show that

$$f(\alpha,\beta;M,p) \geq \sup_{\bar\rho\in\bar{\mathcal{R}}_{p,M}}\ \sup_{v\in\bar{\mathcal{F}}} \frac{\bar N(\bar\rho,v)}{\bar D(\bar\rho,v)}, \tag{6.6}$$

$$f(\alpha,\beta;M,p) \leq \sup_{\bar\rho\in\bar{\mathcal{R}}_{p,M}}\ \sup_{v\in\bar{\mathcal{F}}} \frac{\bar N(\bar\rho,v)}{\bar D(\bar\rho,v)}. \tag{6.7}$$

Along the way we will need a few technical facts, which are collected in Appendices C–G.

6.1 Step 1: extension of the variational formula

For $c \in (0,\infty)$, let $u(c) = (u_\Theta(c))_{\Theta\in\overline{\mathcal{V}}_M}$ be the counterpart of the function $v(c)$ introduced in (2.8-2.10). For $\Theta \in \overline{\mathcal{V}}_M$ and $c \in (0,\infty)$, set

$$u_\Theta(c) = \begin{cases} t_\Theta & \text{if } \partial_u^+(u\,\psi(\Theta,u))(t_\Theta) \leq c,\\ z & \text{otherwise, with } z \text{ such that } \partial_u^-(u\,\psi(\Theta,u))(z) \geq c \geq \partial_u^+(u\,\psi(\Theta,u))(z), \end{cases} \tag{6.8}$$

where z exists and is finite by Lemma C.7 in Appendix C, and is unique by the strict concavity of $u \to u\psi(\Theta,u)$ for $\Theta \in \overline{\mathcal{V}}_M$ (see Lemma C.4 in Appendix C). The fine properties of $\Theta \mapsto u_\Theta(c)$ are given in Lemma B.4 in Appendix B.

For $(\alpha,\beta) \in$ CONE and $\rho \in \mathcal{M}_1(\overline{\mathcal{V}}_M)$ such that $\int_{\overline{\mathcal{V}}_M} t_\Theta\,\rho(d\Theta) < \infty$, set

$$g(\rho;\alpha,\beta) = \sup_{u\in\mathcal{F}_{\overline{\mathcal{V}}_M}} \frac{N(\rho,u)}{D(\rho,u)}, \tag{6.9}$$

with the convention that $N(\rho,u)/D(\rho,u) = -\infty$ when $D(\rho,u) = \infty$. The equality in (6.4) is a straightforward consequence of the following lemma.

Lemma 6.1 *For* $(\alpha,\beta) \in \text{CONE}$ *and* $\rho \in \mathcal{M}_1(\overline{\mathcal{V}}_M)$ *such that* $g(\rho;\alpha,\beta) > 0$,

$$g(\rho;\alpha,\beta) = \frac{N(\rho,\bar{u})}{D(\rho,\bar{u})} \quad \textit{with } \bar{u} = u(g(\rho;\alpha,\beta)). \tag{6.10}$$

Moreover, $u = \bar{u}$ *for* ρ*-a.e.* $\Theta \in \overline{\mathcal{V}}_M$ *for all* $u \in \mathcal{F}_{\overline{\mathcal{V}}_M}$ *satisfying* $g(\rho;\alpha,\beta) = \frac{N(\rho,u)}{D(\rho,u)}$.

Proof. The following lemma will be needed in the proof.

Lemma 6.2 *For* $(\alpha,\beta) \in \text{CONE}$ *and* $\varepsilon > 0$ *there exists a* $t_\varepsilon > 0$ *such that, for all* $\rho \in \mathcal{M}_1(\overline{\mathcal{V}}_M)$ *and all* $u \in \mathcal{F}_{\overline{\mathcal{V}}_M}$ *satisfying* $D(\rho,u) \in (t_\varepsilon,\infty)$,

$$\frac{N(\rho,u)}{D(\rho,u)} \leq \varepsilon. \tag{6.11}$$

Proof. Pick $\varepsilon > 0$. By Lemma C.6, there exists a $C_\varepsilon > 0$ such that $\psi(\Theta,u) \leq \varepsilon/2$ for $\Theta \in \overline{\mathcal{V}}_M$ and $u \geq \max\{C_\varepsilon, t_\Theta\}$. For $R \in (0,\infty)$, set $B^-(R) = \{\Theta \in \overline{\mathcal{V}}_M \colon u_\Theta \leq R\}$ and $B^+(R) = \{\Theta \in \overline{\mathcal{V}}_M \colon u_\Theta > R\}$, and write

$$\frac{N(\rho,u)}{D(\rho,u)} = \frac{\int_{B^-(C_\varepsilon)} u_\Theta \psi(\Theta,u_\Theta)\rho(d\Theta)}{D(\rho,u)} + \frac{\int_{B^+(C_\varepsilon)} u_\Theta \psi(\Theta,u_\Theta)\rho(d\Theta)}{D(\rho,u)}. \tag{6.12}$$

By the definition of C_ε and since $u_\Theta \geq t_\Theta$ for all $\Theta \in \overline{\mathcal{V}}_M$, we can bound the second term in the right-hand side of (6.12) by $\varepsilon/2 > 0$. The first term in the right-hand side of (6.12) in turn can be bounded from above by $C_\varepsilon C_{\text{uf}}(\alpha)/D(\rho,u)$ (recall (3.34)). Consequently, it suffices to choose $t_\varepsilon = 2C_\varepsilon C_{\text{uf}}(\alpha)/\varepsilon$ to complete the proof. □

We resume the proof of Lemma 6.1. By assumption, we know that $g(\rho) > 0$, which entails that $\int_{\overline{\mathcal{V}}_M} t_\Theta \rho(d\Theta) < \infty$. Thus, Lemma B.4(iv) tells us that $D(\rho,u(c)) < \infty$ for all $c > 0$. We argue by contradiction. Suppose that $\frac{N(\rho,\bar{u})}{D(\rho,\bar{u})} < g(\rho)$, and pick $u \in \mathcal{F}_{\overline{\mathcal{V}}_M}$ such that $D(\rho,u) < \infty$. Write

$$\frac{N(\rho,u)}{D(\rho,u)} = \frac{N(\rho,\bar{u}) + [N(\rho,u) - N(\rho,\bar{u})]}{D(\rho,\bar{u}) + [D(\rho,u) - D(\rho,\bar{u})]}, \tag{6.13}$$

where

$$N(\rho,u) - N(\rho,\bar{u}) = \int_{\overline{\mathcal{V}}_M} u_\Theta \psi(\Theta,u_\Theta) - \bar{u}_\Theta \psi(\Theta,\bar{u}_\Theta)\, \rho(d\Theta). \tag{6.14}$$

The strict concavity of $u \mapsto u\,\psi(\Theta,u)$ on $[t_\Theta,\infty)$ for every $\Theta \in \overline{\mathcal{V}}_M$, together with the definition of $\bar{u}$ in (6.10), allows us to estimate

$$N(\rho,u) - N(\rho,\bar{u}) \leq g(\rho) \int_{\overline{\mathcal{V}}_M} (u_\Theta - \bar{u}_\Theta)\ \rho(d\Theta). \tag{6.15}$$

Consequently, (6.13) becomes

$$\frac{N(\rho,u)}{D(\rho,u)} \leq \frac{N(\rho,\bar{u}) + g(\rho)[\bar{D}(\rho,u) - \bar{D}(\rho,\bar{u})]}{D(\rho,\bar{u}) + [D(\rho,u) - D(\rho,\bar{u})]}. \tag{6.16}$$

Define $G = x \mapsto [N(\rho,\bar{u}) + g(\bar{\rho})x]/[D(\rho,\bar{u}) + x]$ on $(-D(\rho,\bar{u}),\infty)$. Note that $N(\rho,\bar{u})/D(\rho,\bar{u}) < g(\rho)$ implies that G is strictly increasing with $\lim_{x\to\infty} G(x) = g(\rho)$. Use Lemma 6.2 to assert that $\mathbb{N}(\rho,u)/D(\rho,u) \leq \frac{1}{2}g(\rho)$ when $D(\rho,u) \geq t_{\frac{1}{2}g(\rho)}$. But then, for all u satisfying $D(\rho,u) \leq t_{\frac{g(\rho)}{2}}$, (6.16) gives

$$\frac{N(\rho,u)}{D(\rho,u)} \leq G\Big(t_{\frac{g(\rho)}{2}} - D(\rho,\bar{u})\Big) < g(\rho). \tag{6.17}$$

Consequently,

$$\sup_{u\in\mathcal{F}_{\overline{\mathcal{V}}_M}} \frac{N(\rho,u)}{D(\rho,u)} \leq \max\left\{\frac{g(\rho)}{2}, G\Big(t_{\frac{g(\rho)}{2}} - D(\rho,\bar{u})\Big)\right\} < g(\rho), \tag{6.18}$$

which is a contradiction, and so $g(\rho) = N(\rho,\bar{u})/D(\rho,\bar{u})$.

It remains to prove that if $u \in \mathcal{F}_{\overline{\mathcal{V}}_M}$ satisfies $g(\rho) = N(\rho,u)/D(\rho,u)$, then $u = \bar{u}$ for ρ-a.e. $\Theta \in \overline{\mathcal{V}}_M$. We proceed again by contradiction, i.e., we suppose that a such u is not equal to $\bar{u}$ for ρ-a.e. $\Theta \in \overline{\mathcal{V}}_M$. In this case, both inequalities in (6.15) and (6.16) are strict, which immediately yields that $\frac{N(\rho,u)}{D(\rho,u)} < g(\rho)$. □

6.2 Step 2: extension of the reduced variational formula

Recall (2.8–2.10) and, for $(\alpha,\beta) \in$ CONE and $\bar{\rho} \in \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}\big)$ such that $\int_0^\infty (1+l)\,[\bar{\rho}_A + \bar{\rho}_B](dl) < \infty$, set

$$h(\bar{\rho};\alpha,\beta) = \sup_{v\in\bar{\mathcal{F}}} \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)}. \tag{6.19}$$

Recall (2.8-2.10). The equality in (6.5) is a straightforward consequence of the following lemma.

Lemma 6.3 *For $(\alpha,\beta) \in$ CONE and $\bar{\rho} \in \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}\big)$ such that $h(\bar{\rho};\,\alpha,\beta) > 0$,*

$$h(\bar{\rho};\alpha,\beta) = \frac{\bar{N}(\bar{\rho},\bar{v})}{\bar{D}(\bar{\rho},\bar{v})}, \quad \textit{with } \bar{v} = v(h(\bar{\rho};\alpha,\beta)). \tag{6.20}$$

For $v \in \bar{\mathcal{F}}$ satisfying $h(\bar{\rho};\alpha,\beta) = \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)}$, $v = \bar{v}$ for $\bar{\rho}$-a.e. $(k,l) \in \{A,B\} \times [0,\infty)$ or $k = \mathcal{I}$.

Proof. The proof is similar to that of Lemma 6.1. The counterpart of Lemma 6.2 is obtained by showing that for $(\alpha,\beta) \in$ CONE and $\varepsilon > 0$ there exists a $t_\varepsilon > 0$ such that, for all $\bar{\rho} \in \mathcal{M}_1\big(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\}$ and all $v \in \bar{\mathcal{F}}$ satisfying $\bar{D}(\bar{\rho},v) \in (t_\varepsilon,\infty)$,

$$\frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)} \leq \varepsilon. \tag{6.21}$$

The proof of (6.20) is similar to the proof of Lemma 6.2 and relies mainly on Lemmas B.1(ii–iii) and on the limit given in Lemma C.1(ii).

It remains to show that $h(\bar{\rho};\alpha,\beta) = \frac{\bar{N}(\bar{\rho},\bar{v})}{\bar{D}(\bar{\rho},\bar{v})}$ and that $v \in \bar{\mathcal{F}}$ satisfying $h(\bar{\rho};\alpha,\beta) = \frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)}$ necessarily satisfies $v = \bar{v}$ for $\bar{\rho}$-a.e. $(k,l) \in \{A,B\} \times [0,\infty)$ or $k = \mathcal{I}$. The proofs are similar to their counterparts in Lemma 6.1 and require the strict concavity of $u \mapsto u\tilde{\kappa}(u,l)$ for $l \in \mathbb{R}$ and of $u \mapsto u\phi_{\mathcal{I}}(u)$, as well as the definition of $\bar{v}$ in (2.8–2.10). □

6.3 Step 3: lower bound

The inequality in (6.6) is a straightforward consequence of the following lemma.

Lemma 6.4 *For all* $(\alpha, \beta) \in \mathrm{CONE}$, $\bar{\rho} \in \bar{\mathcal{R}}_{p,M}$ *and* $v = (v_A, v_B, v_{\mathcal{I}}) \in \bar{\mathcal{F}}$ *there exists* $\rho \in \mathcal{R}_{p,M}$ *and* $u = (u_\Theta)_{\Theta \in \overline{\mathcal{V}}_M} \in \mathcal{F}_{\overline{\mathcal{V}}_M}$ *satisfying*

$$\frac{N(\bar{\rho}, v)}{D(\bar{\rho}, v)} \le \frac{N(\rho, u)}{D(\rho, u)}. \tag{6.22}$$

Proof. Since $\bar{\rho} \in \bar{\mathcal{R}}_{p,M}$, there exist $\rho \in \mathcal{R}_{p,M}$ and $h \in \mathcal{E}$ such that $\bar{\rho} = G_{\rho,h}$. For $\Theta \in \overline{\mathcal{V}}_M$ and $k \in \{A, B\}$, set $d_{k,\Theta} = l_{k,\Theta}/h_{k,\Theta}$ if $h_{k,\Theta} > 0$ and $d_{k,\Theta} = 0$ otherwise. Put

$$u_\Theta = h_{A,\Theta}\, v_{A,d_{A,\Theta}} + h_{B,\Theta}\, v_{B,\, d_{A,\Theta}} + h_{\mathcal{I},\Theta}\, v_{\mathcal{I}}, \quad \Theta \in \overline{\mathcal{V}}_M. \tag{6.23}$$

To prove (6.22), we recall (3.58) and integrate (6.23) against ρ. Since $\bar{\rho} = G_{\rho,h}$, it follows that

$$D(\bar{\rho}, v) = \int_{\overline{\mathcal{V}}_M} u_\Theta\, \rho(d\Theta) = D(\rho, u). \tag{6.24}$$

Since $h \in \mathcal{E}$ we can assert that

$$(h_{A,\Theta}, h_{B,\Theta}, h_{\mathcal{I},\Theta}), (h_{A,\Theta}\, v_{A,\, d_{A,\Theta}}, h_{B,\Theta}\, v_{B,\, d_{B,\Theta}}, h_{\mathcal{I},\Theta}\, v_{\mathcal{I}}) \in \mathcal{L}(\Theta;\, u_\Theta), \quad \Theta \in \overline{\mathcal{V}}_M, \tag{6.25}$$

which, with the help of (3.45), allows us to write

$$\begin{aligned} u_\Theta\, \Psi(\Theta, u_\Theta) \ge{}& h_{A,\Theta}\ v_{A,\, d_{A,\Theta}}\ \tilde{\kappa}(v_{A,\, d_{A,\Theta}}, d_{A,\Theta}) \\ &+ h_{B,\Theta}\, v_{B,\, d_{B,\Theta}}\ \Big[\tilde{\kappa}(v_{B,\, d_{B,\Theta}}, d_{B,\Theta}) + \tfrac{\beta-\alpha}{2}\Big] + h_{\mathcal{I},\Theta}\, v_{\mathcal{I}}\, \phi_{\mathcal{I}}(v_{\mathcal{I}}; \alpha, \beta). \end{aligned} \tag{6.26}$$

After integrating (6.26) against ρ and using that $\bar{\rho} = G_{\rho,u}$, we obtain

$$\begin{aligned} \int_{\overline{\mathcal{V}}_M} u_\Theta \psi(\Theta, u_\Theta) \rho(d\Theta) \ge \Bigg[& \int_0^\infty v_{A,l}\, \kappa(v_{A,l}, l)\, \rho_A(dl) \\ &+ \int_0^\infty v_{B,l}\, \big[\kappa(v_{B,l}, l) + \tfrac{\beta-\alpha}{2}\big]\, \rho_B(dl) + \rho_{\mathcal{I}}\, v_{\mathcal{I}}\, \phi_{\mathcal{I}}(v_{\mathcal{I}}; \alpha, \beta)\Bigg]. \end{aligned} \tag{6.27}$$

Thus, (6.22) is immediate from (6.24) and (6.27). □

6.4 Step 4: upper bound

The proof of (6.7) is a straightforward consequence of the following lemma.

Lemma 6.5 *For all* $(\alpha, \beta) \in \mathrm{CONE}$, $\rho \in \mathcal{R}_{p,M}$ *and* $u \in \mathcal{B}_{\overline{\mathcal{V}}_M}$, *there exist* $\bar{\rho} \in \bar{\mathcal{R}}_{p,M}$ *and* $v \in \bar{\mathcal{F}}$ *such that*

$$\frac{N(\rho, u)}{D(\rho, u)} \le \frac{N(\bar{\rho}, v)}{D(\bar{\rho}, v)}. \tag{6.28}$$

Proof. Since $u \in \mathcal{B}_{\overline{\mathcal{V}}_M}$, Proposition G.1 in Appendix G allows us to state that there exist $h \in \mathcal{E}$ and $r \in \mathcal{U}(h)$ such that

$$\begin{aligned} u_\Theta\, \psi(\Theta, u_\Theta) = h_{A,\Theta}\, r_{A,\Theta}\, \tilde\kappa\big(r_{A,\Theta}, \tfrac{l_{A,\Theta}}{h_{A,\Theta}}\big) + h_{B,\Theta}\, r_{B,\Theta}\, \big[\tilde\kappa\big(r_{B,\Theta}, \tfrac{l_{B,\Theta}}{h_{B,\Theta}}\big) + \tfrac{\beta-\alpha}{2}\big] \\ + h_{\mathcal{I},\Theta}\, r_{\mathcal{I},\Theta}\, \phi_{\mathcal{I}}(r_{\mathcal{I},\Theta}), \qquad \forall \Theta \in \overline{\mathcal{V}}_M, \end{aligned} \tag{6.29}$$

and

$$h_{A,\Theta}\, r_{A,\Theta} + h_{B,\Theta}\, r_{B,\Theta} + h_{\mathcal{I},\Theta}\, r_{\mathcal{I},\Theta} = u_\Theta, \qquad \forall \Theta \in \overline{\mathcal{V}}_M. \tag{6.30}$$

Define $\rho_{A,h}, \rho_{B,h}, \rho_{\mathcal{I},h}$ to be the probability measures on $\overline{\mathcal{V}}_M$ given by

$$\frac{d\rho_{k,h}}{d\rho}(\Theta) = \frac{h_{k,\Theta}}{\int_{\overline{\mathcal{V}}_M} h_{k,\Theta}\, \rho(d\Theta)}, \quad k \in \{A, B, \mathcal{I}\}. \tag{6.31}$$

For $l \in \mathbb{R}_+$, set

$$v_{A,l} = E_{\rho_{A,h}}\big[r_{A,\Theta} \,\big|\, \tfrac{l_{A,\Theta}}{h_{A,\Theta}} = l\,\big], \quad v_{B,l} = E_{\rho_{B,h}}\big[r_{B,\Theta} \,\big|\, \tfrac{l_{B,\Theta}}{h_{B,\Theta}} = l\,\big], \tag{6.32}$$

and

$$v_{\mathcal{I}} = E_{\rho_{\mathcal{I},h}}\big[r_{\mathcal{I},\Theta}\,\big]. \tag{6.33}$$

The fact that $r \in \mathcal{U}(h)$ implies that $v_{\mathcal{I}} \geq 1$ and $v_{k,l} \geq 1 + l$ for $l \in \mathbb{R}_+$ and $k \in \{A, B\}$. Moreover, the Borel measurability of $\Theta \mapsto h_{k,\Theta}$ for $k \in \{A, B\}$ implies the Lebesgue measurability of $l \mapsto v_{k,l}$ for $k \in \{A, B\}$. Therefore, $(v_A, v_B, v_{\mathcal{I}}) \in \bar{\mathcal{F}}$.

By the concavity of $a \mapsto a\tilde\kappa(a, b)$ and $\mu \mapsto \mu\phi_{\mathcal{I}}(\mu)$, we obtain that

$$\begin{aligned} &E_{\rho_{A,h}}\Big[r_{A,\Theta}\, \tilde\kappa(r_{A,\Theta}, l) \,\Big|\, \tfrac{l_{A,\Theta}}{h_{A,\Theta}} = l\Big] \leq v_{A,l}\, \tilde\kappa(v_{A,l}, l), \\ &E_{\rho_{B,h}}\Big[r_{B,\Theta}\, \big(\tilde\kappa(r_{B,\Theta}, l) + \tfrac{\beta-\alpha}{2}\big) \,\Big|\, \tfrac{l_{B,\Theta}}{h_{B,\Theta}} = l\Big] \leq v_{B,l}\, \big[\tilde\kappa(v_{B,l}, l) + \tfrac{\beta-\alpha}{2}\big], \\ &E_{\rho_{\mathcal{I},h}}\Big[r_{\mathcal{I},\Theta}\, \phi_{\mathcal{I}}(r_{\mathcal{I},\Theta})\Big] \leq v_{\mathcal{I}}\, \phi_{\mathcal{I}}(v_{\mathcal{I}}). \end{aligned} \tag{6.34}$$

Integrate (6.29) against ρ, to obtain

$$\begin{aligned} \int_{\overline{\mathcal{V}}_M} u_\Theta\, \psi(\Theta, u_\Theta)\, \rho(d\Theta) = {} & \textstyle\int_{\overline{\mathcal{V}}_M} h_{A,\Theta}\, \rho(d\Theta)\ E_{\rho_{A,h}}\big[r_{A,\Theta}\, \tilde\kappa\big(r_{A,\Theta}, \tfrac{l_{A,\Theta}}{h_{A,\Theta}}\big)\big] \\ & + \textstyle\int_{\overline{\mathcal{V}}_M} h_{\mathcal{I},\Theta}\, \rho(d\Theta)\ E_{\rho_{\mathcal{I},h}}\big[r_{\mathcal{I},\Theta}\, \phi_{\mathcal{I}}(r_{\mathcal{I},\Theta})\big] \\ & + \textstyle\int_{\overline{\mathcal{V}}_M} h_{B,\Theta}\, \rho(d\Theta)\ E_{\rho_{B,h}}\big[r_{B,\Theta}\, \big(\tilde\kappa\big(r_{B,\Theta}, \tfrac{l_{B,\Theta}}{h_{B,\Theta}}\big) + \tfrac{\beta-\alpha}{2}\big)\big]. \end{aligned} \tag{6.35}$$

Set $\bar\rho = G_{\rho,h}$. In the right-hand side of (6.35) take the conditional expectation with respect to $\frac{l_{A,\Theta}}{h_{A,\Theta}}$ and $\frac{l_{B,\Theta}}{h_{B,\Theta}}$ in the first term and the second term, respectively. Then use the inequalities in (6.34), to obtain

$$\begin{aligned} \int_{\overline{\mathcal{V}}_M} u_\Theta\, \psi(\Theta, u_\Theta)\, \rho(d\Theta) \leq {} & \int_0^\infty v_{A,l}\, \tilde\kappa(v_{A,l}, l)\, \bar\rho_A(dl) \\ & + \int_0^\infty v_{B,l}\, \big[\tilde\kappa(v_{B,l}, l) + \tfrac{\beta-\alpha}{2}\big]\, \bar\rho_B(dl) + \bar\rho_{\mathcal{I}}\, v_{\mathcal{I}}\, \phi_{\mathcal{I}}(v_{\mathcal{I}}, \alpha, \beta). \end{aligned} \tag{6.36}$$

Similarly, integrate (6.30) against ρ and take the conditional expectation with respect to $\frac{l_{A,\Theta}}{h_{A,\Theta}}$ and $\frac{l_{B,\Theta}}{h_{B,\Theta}}$, to obtain

$$\int_{\bar{\mathcal{V}}_M} u_\Theta\, \rho(d\Theta) = \int_0^\infty v_{A,l}\, \bar{\rho}_A(dl) + \int_0^\infty v_{B,l}\, \bar{\rho}_B(dl) + \bar{\rho}_{\mathcal{I}}\, v_{\mathcal{I}}. \tag{6.37}$$

At this point, (6.35) and (6.37) allow us to conclude that $N(\rho,u)/D(\rho,u) \leq N(\bar{\rho},v)/D(\bar{\rho},v)$. Since $v \in \bar{\mathcal{F}}$, this completes the proof. □

7 Phase diagrams: proof of Theorems 2.1 and 2.5

7.1 Proof of Theorem 2.1

We first state and prove a proposition that compares f, $f_{\mathcal{D}}$ and $f_{\mathcal{D}_2}$, and deals with the regularity and the monotonicity of $f_{\mathcal{D}}$. Recall the definition of α^* in (2.19).

Proposition 7.1 *(i)* $f(\alpha,\beta) = f_{\mathcal{D}}(\alpha,\beta)$ *for* $(\alpha,\beta) \in \text{CONE}\colon\ \beta \leq 0$.
(ii) $x \mapsto f_{\mathcal{D}}(x,0)$ *is continuous, convex and non-increasing on* $[0,\infty)$.
(iii) $f_{\mathcal{D}}(x,0) > f_{\mathcal{D}_2}$ *for* $x \in [0,\alpha^*)$ *and* $f_{\mathcal{D}}(x,0) = f_{\mathcal{D}_2}$ *for* $x \in [\alpha^*,\infty)$.

Proof. (i) Note that for $(\alpha,\beta) \in \text{CONE}\colon \beta \leq 0$ and $v \geq 1$ we have $\phi^{\mathcal{I}}(v,\alpha,\beta) = \tilde{\kappa}(v,0)$, because the Hamiltonian in (3.6) is always non-positive. Thus, (1.17) and (2.1) imply (i).

(ii) Since $(\alpha,\beta) \mapsto f(\alpha,\beta)$ is convex on $\mathbb{R}^2$ (being the pointwise limit of a sequence of convex functions; see (1.12)) and is everywhere finite, it is also continuous. Therefore (i) implies that $x \in [0,\infty) \mapsto f_{\mathcal{D}}(x,0)$ is continous and convex. The monotonicity of $x \mapsto f_{\mathcal{D}}(x,0)$ can be read off directly from (2.1).

(iii) It is obvious from (2.1) and (2.13) that $f_{\mathcal{D}}(x,0) \geq f_{\mathcal{D}_2}$ for every $x \in [0,\infty)$. Recall (2.19). Since $x \mapsto f_{\mathcal{D}}(x,0)$ is continuous and non-increasing, it follows that $f_{\mathcal{D}}(x,0) > f_{\mathcal{D}_2}$ for $x \in [0,\alpha^*)$ and $f_{\mathcal{D}}(x,0) = f_{\mathcal{D}_2}$ for $x \in [\alpha^*,\infty)$. □

We are now ready to give the proof of Theorem 2.1.

Proof. (a) Pick $\alpha \geq 0$ and note that every element of J_α can be written in the form $(\alpha+\beta,\beta)$ (with $\beta \geq -\alpha/2$), so that $f_{\mathcal{D}}$ is constant and equal to $f_{\mathcal{D}}(\alpha,0)$ on J_α. By the convexity of $(\alpha,\beta) \mapsto f(\alpha,\beta)$ and by Proposition 7.1(i), we know that $g_\alpha\colon\ \beta \mapsto f(\alpha+\beta,\beta) - f_{\mathcal{D}}(\alpha,0)$ is convex and equal to 0 when $\beta \leq 0$. Therefore g_α is non-decreasing, and we can define

$$\beta_c(\alpha) = \inf\{\beta \geq 0\colon\ f(\alpha+\beta,\beta) > f_{\mathcal{D}}(\alpha,0)\}, \tag{7.1}$$

so that $(\alpha+\beta,\beta) \in \mathcal{D}$ if and only if $\beta \leq \beta_c(\alpha)$. It remains to check that $\beta_c(\alpha) < \infty$.

To that aim, pick any $\bar{\rho} \in \bar{\mathcal{R}}_p$ such that $\bar{\rho}_{\mathcal{I}} > 0$ and any $v \in \bar{\mathcal{B}}$ such that $v_{\mathcal{I}} > 1$ and $\bar{D}(\bar{\rho},v) < \infty$, recall (1.18), and note that $\lim_{\beta\to\infty} \bar{N}(\alpha+\beta,\beta;\bar{\rho},v) = \infty$ because $\lim_{\beta\to\infty} \phi_{\mathcal{I}}(v_{\mathcal{I}};\alpha+\beta,\beta) = \infty$. The last observation is obtained by considering a trajectory in $\mathcal{W}_{v_{\mathcal{I}}L}$ that starts at $(0,0)$ ends at $(L,0)$, and in between stays in the A-solvent except when the microscopic disorder ω has 3 consecutive B-monomers, in which case the trajectory makes an excursion of size 3: one step south, one step east and one step north, inside the B-solvent. Such a trajectory has energy $\beta c L$ for some $c > 0$.

(b) This is a straightforward consequence of the fact that $f_{\mathcal{D}}(\alpha,\beta) = f_{\mathcal{D}}(\alpha-\beta,0)$ for $(\alpha,\beta) \in$ CONE. □

7.2 Proof of Theorem 2.5

Proof. (a) We want to show that $\alpha^* \in (0,\infty)$. To that aim, we first prove that $f_{\mathcal{D}}(0,0) > f_{\mathcal{D}_2}$, which by the continuity of $x \mapsto f_{\mathcal{D}}(x,0)$ implies that $\alpha^* > 0$. It is easy to see that $p\delta_{A,0}(dl) + (1-p)\delta_{B,0}(dl) \in \bar{\mathcal{R}}_p$, since this corresponds to trajectories travelling along the x-axis while staying on one side. Thus, (2.1) implies that $f_{\mathcal{D}}(0,0) \geq \tilde{\kappa}(u^*,0)$, where u^* is the unique maximizer of $u \mapsto \tilde{\kappa}(u,0)$ on $[1,\infty)$. Moreover, by Lemma B.1(ii), we have $\tilde{\kappa}(u,l) \leq \tilde{\kappa}(u^*,0)$ for every $l \in [0,\infty)$ and $u \geq 1+l$. Since $\delta_{A,0}(dl)$ does not belong to $\mathcal{R}_p$, it follows that $f_{\mathcal{D}_2} < f_{\mathcal{D}}(0,0)$, and therefore the continuity of $x \mapsto f_{\mathcal{D}}(x,0)$ implies that $\alpha^* > 0$.

It remains to show that $\alpha^* < \infty$. Recall Hypothesis 2. We argue by contradiction. Assume that $f_{\mathcal{D}}(n,0) > f_{\mathcal{D}_2}$ for all $n \in \mathbb{N}$. Then Proposition F.1 tells us that there exists a sequence $(\bar{\rho}_n)_{n\in\mathbb{N}}$ in $\mathcal{T}_p$ such that

$$f(n,0) = f_{\mathcal{D}}(n,0) = \frac{\bar{N}_{\mathcal{D}}(\bar{\rho}_n, v_n;\, n, 0)}{\bar{D}_{\mathcal{D}}(\bar{\rho}_n, v_n)} > f_{\mathcal{D}_2} > 0, \quad n \in \mathbb{N}, \tag{7.2}$$

with $v_n = v(f_{\mathcal{D}}(n,0))$, where we recall (2.8–2.10). For simplicity, we write $f_2 = f_{\mathcal{D}_2}$ until the end of the proof. Since $f_{\mathcal{D}}(n,0) > f_2$ for $n \in \mathbb{N}$, Lemma B.3(ii) yields $v_{n,A,l} \leq \bar{v}_{A,l}$ for $l \in [0,\infty), n \in \mathbb{N}$. Note that Lemma B.3 is stated for fixed $(\alpha,\beta) \in$ CONE, which is not the case here because $(\alpha,\beta) = (n,0)$. However, in the present case (ii) remains true for v_A since, by definition, the value taken by $v_{A,l}(c)$ for $l \in [0,\infty)$ and $c \in (0,\infty)$ does not depend on (α,β).

We can write

$$\begin{aligned} f_{\mathcal{D}}(n,0) - f_2 = & \frac{\int_0^\infty v_{n,A,l}[\tilde{\kappa}(v_{n,A,l},l) - f_2](\bar{\rho}_{n,A} + \bar{\rho}_{n,\mathcal{I}}\,\delta_0)(dl)}{D_{\mathcal{D}}(\bar{\rho}_n, v_n)} \\ & + \frac{\int_0^\infty v_{n,B,l}[\tilde{\kappa}(v_{n,B,l},l) - \frac{n}{2} - f_2]\bar{\rho}_{n,B}(dl)}{D_{\mathcal{D}}(\bar{\rho}_n, v_n)}, \end{aligned} \tag{7.3}$$

and the concavity of $v \mapsto v\tilde{\kappa}(v,l)$, together with the fact that $v_{n,A,l} \leq \bar{v}_{A,l}$ for all $l \in [0,\infty)$ and $\partial_v(v\tilde{\kappa}(v,l))(\bar{v}_{A,l}) = f_2$, implies that

$$\bar{v}_{A,l}\tilde{\kappa}(\bar{v}_{A,l},l) - v_{n,A,l}\tilde{\kappa}(v_{n,A,l},l) \geq f_2(\bar{v}_{A,l} - v_{n,A,l}). \tag{7.4}$$

Since $\tilde{\kappa}$ is uniformly bounded from above and $v_{n,B,l} \geq 1+l$ for every $l \in [0,\infty)$, we can claim that, for n large enough,

$$v_{n,B,l}[\tilde{\kappa}(v_{n,B,l},l) - \tfrac{n}{2} - f_2] \leq -\tfrac{n}{4}(1+l), \quad l \in [0,\infty). \tag{7.5}$$

Consequently, (7.2) and (7.3–7.5) allow us to write

$$\int_0^\infty \bar{v}_{A,l}[\tilde{\kappa}(\bar{v}_{A,l},l) - f_2](\bar{\rho}_{n,A} + \bar{\rho}_{n,\mathcal{I}}\delta_0)(dl) - \frac{n}{4}\int_0^\infty 1 + l\,\bar{\rho}_{n,B}(dl) > 0 \text{ for } n \text{ large enough}, \tag{7.6}$$

which clearly contradicts Hypothesis 2 because $\bar{\rho}_n \in \mathcal{T}_p$ for $n \in \mathbb{N}$. The proof is therefore complete.

(b-c) By the definition of $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ in (2.4), (2.15) and (2.16), we know that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$. Thus, Theorem 2.1(a) implies that (b) and (c) will be proven once we show that $J_\alpha \cap \mathcal{D}_2 = \emptyset$ for $\alpha \in [0,\alpha^*)$ and that $J_\alpha \cap \mathcal{D}_1 = \emptyset$ for $\alpha \in [\alpha^*,\infty)$. Moreover, Theorem 2.1(b) tells us that $f_{\mathcal{D}}$ is constant and equal to $f_{\mathcal{D}}(\alpha,0)$ on each J_α with $\alpha \in [0,\infty)$.

Consequently it suffices to show that $f_{\mathcal{D}}(\alpha,0) > f_{\mathcal{D}_2}$ for $\alpha \in [0,\alpha^*)$ and $f_{\mathcal{D}}(\alpha,0) = f_{\mathcal{D}_2}$ for $\alpha \in [\alpha^*,\infty)$. But this is precisely what Proposition 7.1(iii) states.

(d) Pick $\alpha \in [0,\infty)$ and assume that Hypothesis 1 holds. Then there exists a $\bar{\rho}_\alpha \in \mathcal{O}_{p,\alpha,0}$ such that $\bar{\rho}_{\alpha,\mathcal{I}} > 0$. Set $\bar{v} = v(f_{\mathcal{D}}(\alpha,0))$ and

$$\tilde{\beta}_c(\alpha) = \inf\big\{\beta > 0\colon\ \phi_{\mathcal{I}}(\bar{v}_{A,0};\beta+\alpha,\beta) > \tilde{\kappa}(\bar{v}_{A,0},0)\big\}, \tag{7.7}$$

and the proof will be complete as soon as we show that $\tilde{\beta}_c(\alpha) = \beta_c(\alpha)$ (recall (2.6)). Note that, by the convexity of $\beta \to \phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta)$, and since $\phi_{\mathcal{I}}(\bar{v}_{A,0};\beta+\alpha,\beta) = \tilde{\kappa}(\bar{v}_{A,0},0)$ for $\beta \le 0$, we necessarily have that $\phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta) > \tilde{\kappa}(\bar{v}_{A,0},0)$ for all $\beta > \tilde{\beta}_c(\alpha)$. From Propositions 7.1(i) and F.1(2), we have that

$$f(\alpha,0) = f_{\mathcal{D}}(\alpha,0) = \tfrac{\bar{N}_{\mathcal{D}}(\bar{\rho}_\alpha,\bar{v})}{\bar{D}_{\mathcal{D}}(\bar{\rho}_\alpha,\bar{v})}, \tag{7.8}$$

and

$$\bar{N}_{\mathcal{D}}(\bar{\rho}_\alpha,\bar{v}) = \int_0^\infty \bar{v}_{A,l}\,\tilde{\kappa}(\bar{v}_{A,l},l)\,[\bar{\rho}_{\alpha,A} + \bar{\rho}_{\alpha,\mathcal{I}}\,\delta_0](dl) + \int_0^\infty \bar{v}_{B,l}\,\big[\tilde{\kappa}(\bar{v}_{B,l},l) - \tfrac{\alpha}{2}\big]\,\bar{\rho}_{\alpha,B}(dl). \tag{7.9}$$

By the definition of $\bar{v} = v(f_{\mathcal{D}}(\alpha,0))$ in (2.8-2.10), we have that $\partial_v(v\,\tilde{\kappa}(v,0))(\bar{v}_{A,0}) = f_{\mathcal{D}}(\alpha,0)$. For notational reasons we suppress the dependence on α of $f_{\mathcal{D}}$.

First, assume that $\phi_{\mathcal{I}}(\bar{v}_{A,0};\beta+\alpha,\beta) = \tilde{\kappa}(\bar{v}_{A,0},0)$ (we also suppress the dependence on $(\beta+\alpha,\beta)$). Then, since $v \to v\phi_{\mathcal{I}}(v)$ and $v \to v\tilde{\kappa}(v,0)$ are both concave and $\phi_{\mathcal{I}}(v) \ge \tilde{\kappa}(v,0)$ for all $v \ge 1$, we have that $v \to v\phi_{\mathcal{I}}(v)$ is differentiable at $\bar{v}_{A,0}$ and

$$\partial_v[v\,\tilde{\kappa}(v,0)](\bar{v}_{A,0}) = \partial_v[v\,\phi_{\mathcal{I}}(v)](\bar{v}_{A,0}) = f_{\mathcal{D}}. \tag{7.10}$$

Thus, for any $\bar{\rho} \in \bar{\mathcal{R}}_p$ and $v \in \bar{\mathcal{B}}$, we set $\tilde{v} \in \bar{\mathcal{B}}$ such that $\tilde{v} \equiv v$, except for $\tilde{v}_{\mathcal{I}}$, which takes the value $\bar{v}_{A,0}$. In other words,

$$\begin{aligned}\frac{\bar{N}(\bar{\rho},v)}{\bar{D}(\bar{\rho},v)} &= \frac{\bar{N}_{\mathcal{D}}(\bar{\rho},\tilde{v}) + \bar{\rho}_{\mathcal{I}}[v_{\mathcal{I}}\phi_{\mathcal{I}}(v_{\mathcal{I}}) - \bar{v}_{A,0}\tilde{\kappa}(\bar{v}_{A,0},0)]}{\bar{D}_{\mathcal{D}}(\bar{\rho},\tilde{v}) + \bar{\rho}_{\mathcal{I}}[v_{\mathcal{I}} - \bar{v}_{A,0}]}\\ &\le \frac{\bar{N}_{\mathcal{D}}(\bar{\rho},\tilde{v}) + \bar{\rho}_{\mathcal{I}}f_{\mathcal{D}}(v_{\mathcal{I}} - \bar{v}_{A,0})}{\bar{D}_{\mathcal{D}}(\bar{\rho},\tilde{v}) + \bar{\rho}_{\mathcal{I}}(v_{\mathcal{I}} - \bar{v}_{A,0})},\end{aligned} \tag{7.11}$$

where we use (7.10), the concavity of $v \to v\phi_{\mathcal{I}}(v)$ and the fact that $\phi_{\mathcal{I}}(\bar{v}_{A,0}) = \tilde{\kappa}(\bar{v}_{A,0},0)$ by assumption. At this stage we recall that, by definition, $\frac{\bar{N}_{\mathcal{D}}(\bar{\rho},\tilde{v})}{\bar{D}_{\mathcal{D}}(\bar{\rho},\tilde{v})} \le f_{\mathcal{D}}$. Hence (7.11) entails that $\frac{N(\bar{\rho},v)}{D(\bar{\rho},v)} \le f_{\mathcal{D}}$. Thus, $\beta_c(\alpha) \ge \tilde{\beta}_c(\alpha)$.

The other inequality is much easier, because if we consider β such that $\phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta) > \tilde{\kappa}(\bar{v}_{A,0},0)$, then $\bar{N}(\bar{\rho}_\alpha,\bar{v}) > \bar{N}_{\mathcal{D}}(\bar{\rho}_\alpha,\bar{v})$, because $\bar{\rho}_{\mathcal{I},\alpha} > 0$. As a consequence, $f(\alpha+\beta,\beta) > f_{\mathcal{D}}(\alpha,0)$, so that $\beta > \beta_c(\alpha)$, and therefore $\beta_c(\alpha) \le \tilde{\beta}_c(\alpha)$.

(e) We recall that, for $\alpha \in [\alpha^*,\infty)$ we have $\bar{v} = v(f_{D_2})$ and therefore $\bar{v}_{A,0}$ is constant. In (c) we proved that $\beta_c(\alpha) = \tilde{\beta}_c(\alpha)$ on $[\alpha^*,\infty)$ with

$$\tilde{\beta}_c(\alpha) = \inf\big\{\beta > 0\colon\ \phi_{\mathcal{I}}(\bar{v}_{A,0};\beta+\alpha,\beta) > \tilde{\kappa}(\bar{v}_{A,0},0)\big\}. \tag{7.12}$$

The definition of $\tilde{\beta}_c(\alpha)$ in (7.12) can be extended to $\alpha \in [0,\infty)$. Since $\alpha^* > 0$, the proof of (d) will be complete once we show that $\alpha \mapsto \tilde{\beta}_c(\alpha)$ is concave, continuous and non-decreasing on $(0,\infty)$ and that $\lim_{\alpha\to\infty}\tilde{\beta}_c(\alpha) < \infty$.

By using the same argument as the one we used in the proof of Theorem 2.1(a), we can claim that $\lim_{\beta\to\infty}\phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta)=\infty$ for every $\alpha\in[0,\infty)$. Consequently, $\tilde{\beta}_c(\alpha)\in[0,\infty)$ for every $\alpha\in[0,\infty)$. Moreover, the convexity of $(\alpha,\beta)\mapsto\phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha,\beta)$ implies the convexity of $(\alpha,\beta)\mapsto\phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta)-\tilde{\kappa}(\bar{v}_{A,0},0)$, which is also non-negative. Therefore, the set $\{(\alpha,\beta)\colon\ \alpha\in[0,\infty),\beta\in[-\frac{\alpha}{2},\tilde{\beta}_c(\alpha)]\}$ is convex, and consequently $\alpha\mapsto\tilde{\beta}_c(\alpha)$ is concave on $[0,\infty)$. This concavity yields that $\alpha\mapsto\tilde{\beta}_c(\alpha)$ is continuous on $(0,\infty)$, and since it is bounded from below by 0, also that it is non-decreasing.

It remains to show that $\lim_{\alpha\to\infty}\tilde{\beta}_c(\alpha)<\infty$. To that aim, we define $\tilde{\beta}_c(\infty)$ by choosing $\alpha=\infty$ in (7.12). Since $\phi_{\mathcal{I}}(\bar{v}_{A,0};\infty,\beta)\leq\phi_{\mathcal{I}}(\bar{v}_{A,0};\alpha+\beta,\beta)$ for every $\alpha\geq0$ and $\beta\in[-\frac{\alpha}{2},\infty)$, it follows that $\tilde{\beta}_c(\alpha)\leq\tilde{\beta}_c(\infty)$ for every $\alpha\in(0,\infty)$. Therefore it suffices to prove that $\tilde{\beta}_c(\infty)<\infty$. But this is a consequence of the fact that $\lim_{\alpha\to\infty}\phi_{\mathcal{I}}(\bar{v}_{A,0};\infty,\beta)=\infty$. This limit is obtained by using again the same argument as the one we used in the proof of Theorem 2.1(a).

(f) This is a straightforward consequence of the fact that $f=f_{\mathcal{D}}$ on $\mathcal{D}_1$ and $f_{\mathcal{D}}$ is a function of $\alpha-\beta$.

(g) This is a direct consequence of the definition of the $\mathcal{D}_2$-phase in (2.16) and the fact that $f_{\mathcal{D}_2}$ does not depend on α and β (see (2.13)). □

A Uniform convergence of path entropies

In Appendix A.1 we state a basic lemma (Lemma A.1) about uniform convergence of path entropies in a single column. This lemma is proved with the help of three additional lemmas (Lemmas A.2–A.4), which are proved in Appendix A.3. The latter ends with an elementary lemma (Lemma B.1) that allows us to extend path entropies from rational to irrational parameter values. In Appendix A.2, we extend Lemma A.1 to entropies associated with sets of paths fullfilling certain restrictions on their vertical displacement.

A.1 Basic lemma

We recall the definition of $\widetilde{\kappa}_L$, $L\in\mathbb{N}$, in (3.2) and $\widetilde{\kappa}$ in (3.3).

Lemma A.1 *For every $\varepsilon>0$ there exists an $L_\varepsilon\in\mathbb{N}$ such that*

$$|\tilde{\kappa}_L(u,l)-\tilde{\kappa}(u,l)|\leq\varepsilon \text{ for } L\geq L_\varepsilon \text{ and } (u,l)\in\mathcal{H}_L. \tag{A.1}$$

Proof. With the help of Lemma A.2 below we get rid of those $(u,l)\in\mathcal{H}\cap\mathbb{Q}^2$ with u large, i.e., we prove that $\lim_{u\to\infty}\kappa_L(u,l)=0$ uniformly in $L\in\mathbb{N}$ and $(u,l)\in\mathcal{H}_L$. Lemma A.3 in turn deals with the moderate values of u, i.e., u bounded away from infinity and $1+|l|$. Finally, with Lemma A.4 we take into account the small values of u, i.e., u close to $1+|l|$. To ease the notation we set, for $\eta\geq0$ and $M>1$,

$$\mathcal{H}_{L,\eta,M}=\{(u,l)\in\mathcal{H}_L\colon 1+|l|+\eta\leq u\leq M\},\qquad \mathcal{H}_{\eta,M}=\{(u,l)\in\mathcal{H}\colon 1+|l|+\eta\leq u\leq M\}. \tag{A.2}$$

Lemma A.2 *For every $\varepsilon>0$ there exists an $M_\varepsilon>1$ such that*

$$\tfrac{1}{uL}\log\big|\{\pi\in\mathcal{W}_{uL}\colon \pi_{uL,1}=L\}\big|\leq\varepsilon \qquad \forall\, L\in\mathbb{N},\ u\in1+\tfrac{\mathbb{N}}{L}\colon u\geq M_\varepsilon. \tag{A.3}$$

Lemma A.3 *For every $\varepsilon > 0$, $\eta > 0$ and $M > 1$ there exists an $L_{\varepsilon,\eta,M} \in \mathbb{N}$ such that*

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}(u,l)| \le \varepsilon \qquad \forall\, L \ge L_{\varepsilon,\eta,M},\, (u,l) \in \mathcal{H}_{L,\eta,M}. \tag{A.4}$$

Lemma A.4 *For every $\varepsilon > 0$ there exist $\eta_\varepsilon \in (0, \frac{1}{2})$ and $L_\varepsilon \in \mathbb{N}$ such that*

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}_L(u+\eta,l)| \le \varepsilon \qquad \forall\, L \ge L_\varepsilon,\, (u,l) \in \mathcal{H}_L,\, \eta \in (0,\eta_\varepsilon) \cap \tfrac{2\mathbb{N}}{L}. \tag{A.5}$$

Note that, after letting $L \to \infty$ in Lemma A.4, we get

$$|\tilde{\kappa}(u,l) - \tilde{\kappa}(u+\eta,l)| \le \varepsilon \qquad \forall\, (u,l) \in \mathcal{H} \cap \mathbb{Q}^2,\, \eta \in (0,\eta_\varepsilon) \cap \mathbb{Q}. \tag{A.6}$$

Pick $\varepsilon > 0$ and $\eta_\varepsilon \in (0, \frac{1}{2})$ as in Lemma A.4. Note that Lemmas A.2–A.3 yield that, for L large enough, (A.1) holds on $\{(u,l) \in \mathcal{H}_L \colon u \ge 1 + |l| + \frac{\eta_\varepsilon}{2}\}$. Next, pick $L \in \mathbb{N}$, $(u,l) \in \mathcal{H}_L \colon u \le 1 + |l| + \frac{\eta_\varepsilon}{2}$ and $\eta_L \in (\frac{\eta_\varepsilon}{2}, \eta_\varepsilon) \cap \frac{2\mathbb{N}}{L}$, and write

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}(u,l)| \le A + B + C, \tag{A.7}$$

where

$$A = |\tilde{\kappa}_L(u,l) - \tilde{\kappa}_L(u+\eta_L,l)|, \quad B = |\tilde{\kappa}_L(u+\eta_L,l) - \tilde{\kappa}(u+\eta_L,l)|, \quad C = |\tilde{\kappa}(u+\eta_L,l) - \tilde{\kappa}(u,l)|. \tag{A.8}$$

By (A.6), it follows that $C \le \varepsilon$. As mentioned above, the fact that $(u+\eta_L, l) \in \mathcal{H}_L$ and $u + \eta_L \ge |l| + \frac{\eta_\varepsilon}{2}$ implies that, for L large enough, $B \le \varepsilon$ uniformly in $(u,l) \in \mathcal{H}_L \colon u \le 1 + |l| + \frac{\eta_\varepsilon}{2}$. Finally, from Lemma A.4 we obtain that $A \le \varepsilon$ for L large enough, uniformly in $(u,l) \in \mathcal{H}_L \colon u \le 1 + |l| + \frac{\eta_\varepsilon}{2}$. This completes the proof of Lemma A.1. □

A.2 A generalization of Lemma A.1

In Section 5 we sometimes needed to deal with subsets of trajectories of the following form. Recall (3.1), pick $L \in \mathbb{N}$, $(u,l) \in \mathcal{H}_L$ and $B_0, B_1 \in \frac{Z}{L}$ such that

$$B_1 \ge 0 \vee l \ge 0 \wedge l \ge B_0 \quad \text{and} \quad B_1 - B_0 \ge 1. \tag{A.9}$$

Denote by $\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)$ the subset of $\mathcal{W}_L(u,l)$ containing those trajectories that are constrained to remain above $B_0 L$ and below $B_1 L$ (see Fig. 15), i.e.,

$$\widetilde{\mathcal{W}}_L(u,l,B_0,B_1) = \big\{\pi \in \mathcal{W}_L(u,l) \colon\ B_0 L < \pi_{i,2} < B_1 L \text{ for } i \in \{1,\dots,uL-1\}\big\}, \tag{A.10}$$

and let

$$\widetilde{\kappa}_L(u,l,B_0,B_1) = \frac{1}{uL} \log |\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)| \tag{A.11}$$

be the entropy per step carried by the trajectories in $\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)$. With Lemma A.5 below we prove that the effect on the entropy of the restriction induced by B_0 and B_1 in the set $\widetilde{\mathcal{W}}_L(u,l)$ vanishes uniformly as $L \to \infty$.

Lemma A.5 *For every $\varepsilon > 0$ there exists an $L_\varepsilon \in \mathbb{N}$ such that, for $L \ge L_\varepsilon$, $(u,l) \in \mathcal{H}_L$ and $B_0, B_1 \in \mathbb{Z}/L$ satisfying $B_1 - B_0 \ge 1$, $B_1 \ge \max\{0,l\}$ and $B_0 \le \min\{0,l\}$,*

$$|\tilde{\kappa}_L(u,l,B_0,B_1) - \tilde{\kappa}_L(u,l)| \le \varepsilon. \tag{A.12}$$

Proof. The key fact is that $B_1 - B_0 \ge 1$. The vertical restrictions $B_1 \ge \max\{0,l\}$ and $B_0 \le \min\{0,l\}$ gives polynomial corrections in the computation of the entropy, but these corrections are harmless because $(B_1 - B_0)L$ is large. □

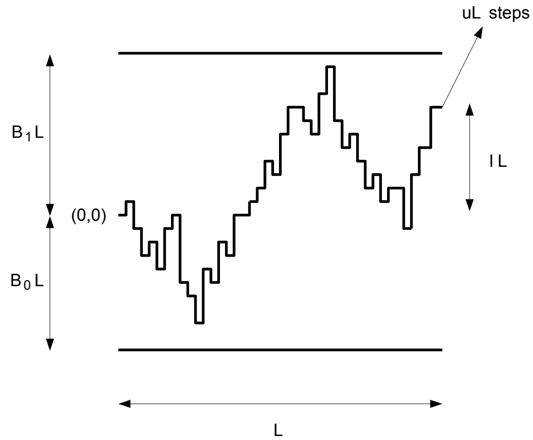

Figure 15: A trajectory in $\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)$.

A.3 Proofs of Lemmas A.2–A.4

A.3.1 Proof of Lemma A.2

The proof relies on the following expression:

$$v_{u,L} = \big|\{\pi \in \mathcal{W}_{uL} \colon \pi_{uL,1} = L\}\big| = \sum_{r=1}^{L+1} \binom{L+1}{r}\binom{(u-1)L}{r} 2^r, \tag{A.13}$$

where r stands for the number of vertical stretches made by the trajectory (a vertical stretch being a maximal sequence of consecutive vertical steps). Stirling's formula allows us to assert that there exists a $g\colon [1,\infty) \to (0,\infty)$ satisfying $\lim_{u\to\infty} g(u) = 0$ such that

$$\binom{uL}{L} \leq e^{g(u)uL}, \qquad u \geq 1,\ L \in \mathbb{N}. \tag{A.14}$$

Equations (A.13–A.14) complete the proof.

A.3.2 Proof of Lemma A.3

We first note that, since u is bounded from above, it is equivalent to prove (A.4) with $\tilde\kappa_L$ and $\tilde\kappa$, or with G_L and G given by

$$G(u,l) = u\tilde\kappa(u,l), \qquad G_L(u,l) = u\tilde\kappa_L(u,l), \quad (u,l) \in \mathcal{H}_L. \tag{A.15}$$

Via concatenation of trajectories, it is straightforward to prove that G is $\mathbb{Q}$-concave on $\mathcal{H}\cap\mathbb{Q}^2$, i.e.,

$$G(\lambda(u_1,l_1)+(1-\lambda)(u_2,l_2)) \geq \lambda G(u_1,l_1)+(1-\lambda)G(u_2,l_2),\ \ \lambda \in \mathbb{Q}_{[0,1]},\ (u_1,l_1),(u_2,l_2) \in \mathcal{H}\cap\mathbb{Q}^2. \tag{A.16}$$

Therefore G is Lipschitz on every $K\cap\mathcal{H}\cap\mathbb{Q}^2$ with $K \subset \mathcal{H}^0$ (the interior of $\mathcal{H}$) compact. Thus, G can be extended on $\mathcal{H}^0$ to a function that is Lipschitz on every compact subset in $\mathcal{H}^0$.

Pick $\eta > 0$, $M > 1$, $\varepsilon > 0$, and choose $L_\varepsilon \in \mathbb{N}$ such that $1/L_\varepsilon \leq \varepsilon$. Since $\mathcal{H}_{\eta,M} \subset \mathcal{H}^0$ is compact, there exists a $c > 0$ (depending on η, M) such that G is c-Lipschitz on $\mathcal{H}_{\eta,M}$. Moreover, any point in $\mathcal{H}_{\eta,M}$ is at distance at most ε from the finite lattice $\mathcal{H}_{L_\varepsilon,\eta,M}$. Lemma 3.1 therefore implies that there exists a $q_\varepsilon \in \mathbb{N}$ satisfying

$$|G_{qL_\varepsilon}(u,l) - G(u,l)| \leq \varepsilon \qquad \forall\,(u,l) \in \mathcal{H}_{L_\varepsilon,\eta,M},\ q \geq q_\varepsilon. \tag{A.17}$$

Let $L' = q_\varepsilon L_\varepsilon$, and pick $q \in \mathbb{N}$ to be specified later. Then, for $L \geq qL'$ and $(u,l) \in \mathcal{H}_{L,\eta,M}$, there exists an $(u',l') \in \mathcal{H}_{L_\varepsilon,\eta,M}$ such that $|(u,l)-(u',l')|_\infty \leq \varepsilon$, $u > u'$, $|l| \geq |l'|$ and $u - u' \geq |l| - |l'|$. We recall (3.3) and write

$$0 \leq G(u,l) - G_L(u,l) \leq A + B + C, \tag{A.18}$$

with

$$A = |G(u,l) - G(u',l')|, \quad B = |G(u',l') - G_{L'}(u',l')|, \quad C = G_{L'}(u',l') - G_L(u,l). \tag{A.19}$$

Since G is c-Lipschitz on $\mathcal{H}_{\eta,M}$, and since $|(u,l)-(u',l')|_\infty \leq \varepsilon$, we have $A \leq c\varepsilon$. By (A.17) we have that $B \leq \varepsilon$. Therefore only C remains to be considered. By Euclidean division, we get that $L = sL' + r$, where $s \geq q$ and $r \in \{0,\dots,L'-1\}$. Pick $\pi_1,\pi_2,\dots,\pi_s \in \mathcal{W}_{L'}(u',|l'|)$, and concatenate them to obtain a trajectory in $\mathcal{W}_{sL'}(u',|l'|)$. Moreover, note that

$$\begin{aligned} uL - u'sL' &= (u-u')sL' + ur \\ &\geq (|l|-|l'|)sL' + (1+|l|)r = (L - sL') + (|l|L - s|l'|L'), \end{aligned} \tag{A.20}$$

where we use that $L - sL' = r$, $u - u' \geq |l| - |l'|$ and $u \geq 1 + |l|$. Thus, (A.20) implies that any trajectory in $\mathcal{W}_{L'}(u',|l'|)$ can be concatenated with an $(uL - u'sL')$-step trajectory, starting at $(sL', s|l'|L')$ and ending at $(L, |l|L)$, to obtain a trajectory in $\mathcal{W}_L(u,|l|)$. Consequently,

$$G_L(u,l) \geq \tfrac{s}{L}\log\kappa_{L'}(u',l') \geq \tfrac{s}{s+1}G_{L'}(u',l'). \tag{A.21}$$

But $s \geq q$ and therefore $G_{L'}(u',l') - G_L(u,l) \leq \frac{1}{q}G_{L'}(u',l') \leq \frac{1}{q}M\log 3$ (recall that $\log 3$ is an upper bound for all entropies per step). Thus, by taking q large enough, we complete the proof.

A.3.3 Proof of Lemma A.4

Pick $L \in \mathbb{N}$, $(u,l) \in \mathcal{H}_L$, $\eta \in \frac{2\mathbb{N}}{L}$, and define the map $T\colon \mathcal{W}_L(u,l) \mapsto \mathcal{W}_L(u+\eta,l)$ as follows. Pick $\pi \in \mathcal{W}_L(u,l)$, find its first vertical stretch, and extend this stretch by $\frac{\eta L}{2}$ steps. Then, find the first vertical stretch in the opposite direction of the stretch just extended, and extend this stretch by $\frac{\eta L}{2}$ steps. The result of this map is $T(\pi) \in \mathcal{W}_L(u+\eta,l)$, and it is easy to verify that T is an injection, so that $|\mathcal{W}_L(u,l)| \leq |\mathcal{W}_L(u+\eta,l)|$.

Next, define a map $\widetilde{T}\colon \mathcal{W}_L(u+\eta,l) \mapsto \mathcal{W}_L(u,l)$ as follows. Pick $\pi \in \mathcal{W}_L(u+\eta,l)$ and remove its first $\frac{\eta L}{2}$ steps north and its first $\frac{\eta L}{2}$ steps south. The result is $\widetilde{T}(\pi) \in \mathcal{W}_L(u,l)$, but $\widetilde{T}$ is not injective. However, we can easily prove that for every $\varepsilon > 0$ there exist $\eta_\varepsilon > 0$ and $L_\varepsilon \in \mathbb{N}$ such that, for all $\eta < \eta_\varepsilon$ and all $L \geq l_\varepsilon$, the number of trajectories in $\mathcal{W}_L(u+\eta,l)$ that are mapped by $\widetilde{T}$ to a particular trajectory in $\pi \in \mathcal{W}_L(u,l)$ is bounded from above by $e^{\varepsilon L}$, uniformly in $(u,l) \in \mathcal{H}_L$ and $\pi \in \mathcal{W}_L,(u,l)$.

This completes the proof of Lemmas A.2–A.4.

B Entropic properties

Recall Lemma 3.1, where $(u,l) \mapsto \tilde{\kappa}(u,l)$ is defined on $\mathcal{H} \cap \mathbb{Q}^2$.

Lemma B.1 *(i) $(u,l) \mapsto u\tilde{\kappa}(u,l)$ extends to a continuous and strictly concave function on $\mathcal{H}$.*
(ii) For all $u \in [1,\infty)$, $l \mapsto \tilde{\kappa}(u,l)$ is strictly increasing on $[-u+1,0]$ and strictly decreasing on $[0,u-1]$.
(iii) For all $l \in \mathbb{R}$, $\lim_{u\to\infty} \tilde{\kappa}(u,l) = 0$.
(iv) $\lim_{|l|\to\infty} \tilde{\kappa}(u,l) = 0$ uniformly in $u \geq 1+|l|$.
(v) For all $l \in \mathbb{R}$, $u \mapsto u\tilde{\kappa}(u,l)$ is continuous, strictly concave, strictly increasing on $[1+|l|,\infty)$ and $\lim_{u\to\infty} u\tilde{\kappa}(u,l) = \infty$.
(vi) For all $l \in \mathbb{R}$, $u \mapsto u\tilde{\kappa}(u,l)$ is analytic on $(1+|l|,\infty)$ and

$$\lim_{v\to\infty} \partial_u(u\tilde{\kappa}(u,l))(v) = 0, \tag{B.1}$$

$$\lim_{v\to 1+l} \partial_u(u\tilde{\kappa}(u,l))(v) = \partial_u^+(u\tilde{\kappa}(u,l))(1+|l|) = \infty. \tag{B.2}$$

Lemma B.2 *For all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\partial_u(u\tilde{\kappa}(u,l))(v) \leq \varepsilon, \quad \text{for } l \in [0,\infty),\ v \geq R_\varepsilon \vee 2+l. \tag{B.3}$$

Recall the definition of $\{v(c), c \in (0,\infty)\}$ in (2.8-2.10).

Lemma B.3 *(i) For all $c \in (0,\infty)$, $v(c) \in \bar{\mathcal{B}}$.*
(ii) For $(k,l) \in \{A,B\} \times (0,\infty)$, $c \mapsto v_{k,l}(c)$ is strictly decreasing and $c \mapsto v_{\mathcal{I}}(c)$ is non-increasing.
(iii) If $(c_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ satisfies $\lim_{n\to\infty} c_n = c_\infty \in (0,\infty)$, then $v(c_n)$ converges pointwise to $v(c_\infty)$.
(iv) $D(\bar{\rho}, v(c)) < \infty$ for all $\bar{\rho} \in \mathcal{M}_1(\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{\mathcal{I}\})$ satisfying $\int_0^\infty (1+l)(\bar{\rho}_A + \bar{\rho}_B)(dl) < \infty$ and all $c \in (0,\infty)$.

Recall the definition of $\{u(c), c \in (0,\infty)\}$ in (6.8).

Lemma B.4 *(i) For all $c \in (0,\infty)$, $u(c) \in \mathcal{B}_{\overline{\mathcal{V}}_M}$.*
(ii) For all $\Theta \in \overline{\mathcal{V}}_M$, $c \mapsto u_\Theta(c)$ is non-increasing on $(0,\infty)$.
(iii) If $(c_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ satisfies $\lim_{n\to\infty} c_n = c_\infty \in (0,\infty)$, then $u(c_n)$ converges pointwise to $u(c_\infty)$.
(iv) $D(\rho, u(c)) < \infty$ for all $\rho \in \mathcal{M}_1(\overline{\mathcal{V}}_M)$ satisfying $\int_{\overline{\mathcal{V}}_M} t_\Theta\, \rho(d\Theta) < \infty$ and all $c \in (0,\infty)$.

B.1 Proofs of Lemmas B.1–B.4

B.1.1 Proof of Lemma B.1

(i) In the proof of Lemma A.1 we have shown that $\tilde{\kappa}$ can be extended to $\mathcal{H}^0$ in such a way that $(u,l) \mapsto u\tilde{\kappa}(u,l)$ is continuous and concave on $\mathcal{H}^0$. Lemma A.4 allows us to extend $\tilde{\kappa}$ to the boundary of $\mathcal{H}$, in such a way that continuity and concavity of $(u,l) \mapsto u\tilde{\kappa}(u,l)$ hold on all of $\mathcal{H}$. To obtain the strict concavity, we recall the formula in (3.4), i.e.,

$$u\tilde{\kappa}(u,l) = \begin{cases} u\kappa(u/|l|, 1/|l|), & l \neq 0, \\ u\hat{\kappa}(u), & l = 0, \end{cases} \tag{B.4}$$

where $(a,b) \mapsto a\kappa(a,b)$, $a \geq 1+b$, $b \geq 0$, and $\mu \mapsto \mu\hat{\kappa}(\mu)$, $\mu \geq 1$, are given in [5], Section 2.1, and are strictly concave. In the case $l \neq 0$, (B.4) provides strict concavity of $(u,l) \mapsto u\tilde{\kappa}(u,l)$ on $\mathcal{H}^+ = \{(u,l) \in \mathcal{H} \colon l > 0\}$ and on $\mathcal{H}^- = \{(u,l) \in \mathcal{H} \colon l < 0\}$, while in the case $l = 0$ it provides strict concavity on $\overline{\mathcal{H}} = \{(u,0), u \geq 1\}$. We already know that $(u,l) \mapsto u\tilde{\kappa}(u,l)$ is concave on $\mathcal{H}$, which, by the strict concavity on $\mathcal{H}^+$, $\mathcal{H}^-$ and $\overline{\mathcal{H}}$, implies strict concavity of $(u,l) \mapsto u\tilde{\kappa}(u,l)$ on $\mathcal{H}$.

(ii) This follows from the strict concavity of $l \mapsto \tilde{\kappa}(u,l)$ and from the fact that $\tilde{\kappa}(u,l) = \tilde{\kappa}(u,-l)$ for $(u,l) \in \mathcal{H}$.

(iii-iv) These are direct consequences of Lemma A.2.

(v) By (i) we have that $u \mapsto u\tilde{\kappa}(u,l)$ is continuous and strictly concave on $[1+|l|,\infty)$. Therefore, proving that $\lim_{u\to\infty} u\tilde{\kappa}(u,l) = \infty$ is sufficient to obtain that $u \mapsto u\tilde{\kappa}(u,l)$ is strictly increasing. It is proven in [5], Lemma 2.1.2 (iii), that $\lim_{\mu\to\infty} u\hat{\kappa}(u) = \infty$, so that (B.4) completes the proof for $l = 0$. If $l \neq 0$, then we use (B.4) again and the variational formula in the proof of [5], Lemma 2.1.1, to check that $\lim_{a\to\infty} a\kappa(a,b) = \infty$ for all $b > 0$.

(vi) To get the analyticity on $(1+|l|,\infty)$, we use (B.4) and the analyticity of $(a,b) \mapsto a\kappa(a,b)$ and $\mu \mapsto \mu\hat{\kappa}(\mu)$ inside their domain of definition (see [5], Section 2.1).

We note that for every $l \in \mathbb{R}$,

$$u\phi_{\mathcal{I}}(u) \geq u\tilde{\kappa}(u,0) \geq u\tilde{\kappa}(u,l), \quad u \in [1+|l|,\infty), \tag{B.5}$$

where the first inequality is well known and the second inequality comes from Lemma B.1(ii). Since, by Lemma B.1(v), $u \mapsto u\tilde{\kappa}(u,l)$ is concave and increasing on $[1+|l|,\infty)$, (C.1) and (5.154) imply (B.1).

It remains to prove (B.2). To that aim, we recall that an explicit formula is available for $\tilde{\kappa}(u,l)$, namely,

$$\tilde{\kappa}(u,l) = \kappa(u/|l|, 1/|l|), \quad \text{for } l \neq 0, \tag{B.6}$$

where $\kappa(a,b)$, $a \geq 1+b$, $b \geq 0$ is given in [5], Section 2.1 (in the proof of Lemmas 2.1.1–2.1.2). The latter formula allows us to compute $\partial_u(u\tilde{\kappa}(u,l))(1+l+\varepsilon,l) = G\big(1+\frac{1}{l}+\frac{\varepsilon}{l},\frac{1}{l}\big)$ with

$$G(a,b) = \tfrac{1}{2}\log\left[\frac{(a+1-b)(a-1-b)}{(a+1-b-2\delta_{a,b})(a-1-b-2\varepsilon_{a,b})}\right] \tag{B.7}$$

and with

$$\begin{aligned} \delta_{a,b} &= \tfrac{b}{2(1+b)}\Big[(a+1) - \big((a-b)^2 + (b^2-1)\big)^{1/2}\Big] \\ \varepsilon_{a,b} &= \tfrac{b}{2(1-b)}\Big[-(a-1) + \big((a-b)^2 + b^2 - 1\big)^{1/2}\Big], \end{aligned} \tag{B.8}$$

so that the proof of (B.2) will be complete once we show that for all $b > 0$ it holds that $\lim_{\varepsilon\to 0^+} G(1+b+\varepsilon,b) = \infty$. The latter is achieved by using first (B.8) to check that $\delta_{1+b+\varepsilon,b} = \frac{b}{1+b} + \big(\frac{1}{2} - \frac{1}{1+b}\big)\varepsilon + o(\varepsilon)$ and $\varepsilon_{1+b+\varepsilon,b} = \frac{\varepsilon}{2} + o(\varepsilon)$ as $\varepsilon \to 0^+$, and then by substituting these two expansions into (B.7) at $(a,b) = (1+b+\varepsilon,b)$, which implies the result after a straightforward computation.

B.1.2 Proof of Lemma B.2

The proof is based on the following lemma.

Lemma B.5

$$\lim_{l\to\infty} \partial_u\big[u\tilde{\kappa}(u,l)\big](2+l,l)=0. \tag{B.9}$$

Proof. We recall (B.6–B.8), and we note that $\partial_u(u\tilde{\kappa}(u,l))(2+l,l)=G\big(1+\frac{2}{l},\frac{1}{l}\big)$. Thus, the proof of Lemma B.5 will be complete once we show that $\lim_{b\to 0^+} G(1+2b,b)=0$. The latter is achieved by using (B.7) and (B.8) to compute

$$G(1+2b,b)=\tfrac{1}{2}\log\left[\frac{(2+b)b}{\Big[2+b\Big(1-\frac{2}{1+b}+o(b)\Big)\Big](b+o(b))}\right] \tag{B.10}$$

which immediately implies the result. □

We resume the proof of Lemma B.2. Once Lemma B.5 is proven, we use the concavity of $u\mapsto u\tilde{\kappa}(u,l)$ for $l\in\mathbb{R}$ to obtain that for $\varepsilon>0$ there exists a $l_\varepsilon>0$ such that $\partial_u[u\tilde{\kappa}(u,l)](u,l)\leq \varepsilon$ for all $l\colon\ |l|\geq l_\varepsilon$ and $u\geq 2+l$. Thus, it remains to show that there exists a $R_\varepsilon>0$ such that $\partial_u[u\tilde{\kappa}(u,l)](u,l)\leq\varepsilon$ for $l\in[0,l_\varepsilon]$ and $u\geq R_\varepsilon$. By contradiction, if we assume that the latter does not hold, then there exists $\varepsilon>0$ and two sequences $(l_n)_{n\in\mathbb{N}}\in[0,l_\varepsilon]^{\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ such that $u_n\geq 1+l_n$ for $n\in\mathbb{N}$ and $\lim_{n\to\infty}u_n=\infty$ and such that $\partial_u[u\tilde{\kappa}(u,l)](u_n,l_n)\geq\varepsilon$ for $n\in\mathbb{N}$. As a consequence, we can write

$$u_n\tilde{\kappa}(u_n,l_n)-(1+l_n)\tilde{\kappa}(1+l_n,l_n)\geq\varepsilon(u_n-1-l_n), \tag{B.11}$$

and, with the help of Lemma B.1(ii), we obtain

$$u_n\tilde{\kappa}(u_n,0)\geq u_n\tilde{\kappa}(u_n,l_n)\geq\varepsilon(u_n-1-l_\varepsilon),\quad \text{for } n\in\mathbb{N}, \tag{B.12}$$

which clearly contradicts Lemma B.1(iii) because $\lim_{n\to\infty}u_n=\infty$.

B.1.3 Proof of Lemma B.3

(i) We must prove that $l\mapsto v_{A,l}(c)$ and $l\mapsto v_{B,l}(c)$ are continuous on $[0,\infty)$. We give the proof for v_A, the proof for v_B being similar. Let $(l_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\infty)$ such that $\lim_{n\to\infty}l_n=l_\infty\in[0,\infty)$. We want to prove that $\lim_{n\to\infty}v_{A,l_n}(c)=v_{A,l_\infty}(c)$. For simplicity, we set $v_n=v_{A,l_n}(c)$ for $n\in\mathbb{N}$ and $v_\infty=v_{A,l_\infty}(c)$. We also set $g_n(u)=u\tilde{\kappa}(u,l_n)$ for $n\in\mathbb{N}$ and $u\geq 1+l_n$ and $g_\infty(u)=u\tilde{\kappa}(u,l_\infty)$ for $u\geq 1+l_\infty$. By Lemmas B.1(i) and (v), we know that g_n converges pointwise to g_∞ as $n\to\infty$, and that g_n and g_∞ are strictly concave. Consequently, $\partial_u(g_n)$ converges pointwise to $\partial_u(g_\infty)$. We argue by contradiction. Suppose that v_n does not converge to v_∞. Then there exists an $\eta>0$ such that $v_n\geq v_\infty+\eta$ along a subsequence or $v_n\leq v_\infty-\eta$ along a subsequence. Suppose for simplicity that $v_n\leq v_\infty-\eta$ for $n\in\mathbb{N}$. Then the strict concavity of g_n implies that $\partial_u(g_n)(v_\infty-\eta)\leq\partial_u(g_n)(v_n)=c$, and therefore, letting $n\to\infty$ and using the strict concavity of g_∞, we obtain $\partial_u(g_\infty)(v_\infty)<\partial_u(g_\infty)(v_\infty-\eta)\leq c$. This provides the contradiction, because $\partial_u(g_\infty)(v_\infty)=c$ by definition. The proof is similar when we assume that $v_n\geq v_\infty+\eta$ for $n\in\mathbb{N}$.

(ii) For $(k,l)\in\{A,B\}\times[0,\infty)$, this is a straightforward consequence of the definition of $v(c)$ in (2.8-2.9), of the strict concavity of $u\mapsto u\tilde{\kappa}(u,l)$ and of the continuity of $u\mapsto\partial_u(u\tilde{\kappa}(u,l))$

for every $l \in [0,\infty)$ (see Lemma B.1(v-vi)). For $c \mapsto v_{\mathcal{I}}(c)$ we do not have strict monotonicity because $u \mapsto \partial_u(u\phi_{\mathcal{I}}(u))$ is not proven to be continuous.

(iii) Similarly to what we did in (i), we consider $(c_n)_{n\in\mathbb{N}}$ a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} c_n = c_\infty \in (0,\infty)$, and we want to show that $\lim_{n\to\infty} v_{k,l}(c_n) = v_{k,l}(c_\infty)$ for $k \in \{A,B\}$ and $l \in [0,\infty)$ and $\lim_{n\to\infty} v_{\mathcal{I}}(c_n) = v_{\mathcal{I}}(c_\infty)$. Again we argue by contradiction. Suppose, for instance, that $v_{\mathcal{I}}(c_n)$ does not converge to $v_{\mathcal{I}}(c_\infty)$. Then there exists an $\eta > 0$ such that $v_{\mathcal{I}}(c_n) \le v_{\mathcal{I}}(c_\infty) - \eta$ or $v_{\mathcal{I}}(c_n) \ge v_{\mathcal{I}}(c_\infty) + \eta$ along a subsequence. Suppose for simplicity that $v_{\mathcal{I}}(c_n) \ge v_{\mathcal{I}}(c_\infty)+\eta$. Then $\partial_u^-(u\phi_{\mathcal{I}}(u))(v_{\mathcal{I}}(c_\infty)+\eta) \ge \partial_u^-(u\phi_{\mathcal{I}}(u))(v_{\mathcal{I}}(c_n)) \ge c_n$ for $n \in \mathbb{N}$. Let $n \to \infty$ to obtain $\partial_u^+(u\phi_{\mathcal{I}}(u))(v_{\mathcal{I}}(c_\infty)) > \partial_u^-(u\phi_{\mathcal{I}}(u))(v_{\mathcal{I}}(c_\infty)+\eta) \ge c_\infty$, which contradicts the definition of $v_{\mathcal{I}}(c_\infty)$ in (2.8-2.10). The proof is similar when we assume that $v_{\mathcal{I}}(c_n) \le v_{\mathcal{I}}(c_\infty) - \eta$ for $n \in \mathbb{N}$.

(iv) This is a consequence of Lemma B.2, which implies that for all $c \in (0,\infty)$ there exists a $l_c \in [0,\infty)$ such that $v_{A,l}(c) \le 2+l$ for all $l \ge l_c$. Moreover, (2.8-2.9) and the fact that $(\alpha,\beta) \in$ CONE entail that $v_{B,l}(c) \le v_{A,l}(c)$ for $l \in [0,\infty)$, and therefore $\int_0^\infty (1+l)(\bar\rho_A + \bar\rho_B)(dl) < \infty$ combined with the finitness of $v_{\mathcal{I}}(c)$ imply $\bar D(\bar\rho, v(c)) < \infty$.

B.1.4 Proof of Lemma B.4

(i) The proof is similar to that of Lemma B.3(i), except for the fact that when we consider $\Theta_n \to \Theta_\infty$ as $n \to \infty$ in $\overline{\mathcal{V}}_M$, we have (by Lemma C.3) the pointwise convergence of $g_n(u) = u\psi(\Theta_n, u)$ to $g_\infty(u) = u\psi(\Theta_\infty, u)$, but we do not have the pointwise convergence of $\partial g_n(u)$ to $\partial g_\infty(u)$ since g_∞ is not a priori differentiable. However, the strict concavity and the pointwise convergence of g_n towards g_∞ gives us

$$\partial^- g_\infty(u) \ge \limsup_{n\to\infty} \partial^- g_n(u) \ge \liminf_{n\to\infty} \partial^+ g_n(u) \ge \partial^+ g_\infty(u), \tag{B.13}$$

with which we can easily mimick the proof in Lemma B.3(i)

(ii) The proof is similar to that of Lemma B.3(ii), except for the fact that the monotonicity of $c \mapsto u_\Theta(c)$ is not proven to be strict because $u \mapsto \partial(u\psi(\Theta,u))$ is not proven to be continuous.

(iii) We mimick the proof of Lemma B.3(iii). Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} c_n = c_\infty \in (0,\infty)$, and assume that there exists an $\eta > 0$ such that $u_\Theta(c_n) \ge u_\Theta(c_\infty)+\eta$ along a subsequence. Then $\partial_u^-(u\psi(\Theta,u))(u_\Theta(c_\infty)+\eta) \ge \partial_u^-(u\psi(\Theta,u))(u_\Theta(c_n)) \ge c_n$ for $n \in \mathbb{N}$. Let $n \to \infty$ to obtain $\partial_u^+(u\psi(\Theta,u))(u_\Theta(c_\infty)) > \partial_u^-(u\psi(\Theta,u))(u_\Theta(c_\infty)+\eta) \ge c_\infty$, which contradicts the definition of $u_\Theta(c_\infty)$ in (6.8).

(iv) The proof is similar to that of Lemma B.3(iv). The role of Lemma B.2 is taken over by Lemma C.8

C Properties of free energies

C.1 Free energy along a single linear interface

Also the free energy $\mu \mapsto \phi^{\mathcal{I}}(\mu;\alpha,\beta)$ defined in Proposition 3.2 can be extended from $\mathbb{Q} \cap [1,\infty)$ to $[1,\infty)$, in such a way that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu;\alpha,\beta)$ is concave and continous on $[1,\infty)$. By concatenating trajectories, we can indeed check that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu;\alpha,\beta)$ is concave on

$\mathbb{Q} \cap [1, \infty)$. Therefore it is Lipschitz on every compact subset of $(1, \infty)$ and can be extended to a concave and continuous function on $(1, \infty)$. The continuity at $\mu = 1$ comes from the fact that $\phi^{\mathcal{I}}(1; \alpha, \beta) = 0$ and $\lim_{\mu \downarrow 1} \phi^{\mathcal{I}}(\mu) = 0$, which is obtained by using Lemma E.1 below.

Lemma C.1 *For all* $(\alpha, \beta) \in$ CONE*:*
(i) $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ *is strictly increasing on* $[1, \infty)$ *and* $\lim_{\mu \to \infty} \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta) = \infty$*.*
(ii) $\lim_{\mu \to \infty} \phi^{\mathcal{I}}(\mu; \alpha, \beta) = 0$*.*
(iii)

$$\lim_{v \to \infty} \partial_u^-(u\phi_{\mathcal{I}}(u;\, \alpha, \beta))(v) = 0, \tag{C.1}$$

$$\lim_{v \to 1} \partial_u^+(u\phi_{\mathcal{I}}(u;\, \alpha, \beta))(v) = \partial_u^+(u\phi_{\mathcal{I}}(u;\, \alpha, \beta))(1) = \infty. \tag{C.2}$$

Proof. (i) Clearly, $\phi^{\mathcal{I}}(\mu; \alpha, \beta) \geq \tilde{\kappa}(\mu, 0)$ for $\mu \geq 1$. Therefore Lemma B.1(iv) implies that $\lim_{\mu \to \infty} \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta) = \infty$. Thus, the concavity of $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is sufficient to obtain that it is strictly increasing on $[1, \infty)$.
(ii) See [6], Lemma 2.4.1(i).
(iii) To prove (C.1), we pick $\chi \in \{A, B\}^{\mathbb{Z}}$ such that $\chi(0) = A$ and $\chi(-1) = B$. We recall (3.37) and consider $\Theta = (\chi, 0, 0, 0, 2) \in \bar{\mathcal{V}}_{\text{nint},A,2,M}$ such that $l_A(\Theta) = l_B(\Theta) = 0$. By Proposition 3.5, we have

$$u\psi(\Theta_2, u) \geq u\phi_{\mathcal{I}}(u), \quad u \in [1, \infty), \tag{C.3}$$

and (C.3), together with Lemma C.7 and the concavity and monotonicity of $u \mapsto u\phi_{\mathcal{I}}(u)$, imply (C.1).

It remains to prove (C.2). For all $(\alpha, \beta) \in$ CONE we know that $u \mapsto u\phi_{\mathcal{I}}(u; \alpha, \beta)$ is continuous and strictly concave on $[1, \infty)$. Therefore we necessarily have

$$\lim_{v \to 1^+} \partial_u^+(u\phi_{\mathcal{I}}(u))(v) = \partial_u^+(u\phi_{\mathcal{I}}(u))(1). \tag{C.4}$$

Moreover, since $(u\phi_{\mathcal{I}}(u))(1) = (u\tilde{\kappa}(u, 0))(1) = 0$ and since $\phi_{\mathcal{I}}(u) \geq \tilde{\kappa}(u, 0)$ for $u \geq 1$, we have $\partial_u^+(u\phi_{\mathcal{I}}(u))(1) \geq \partial_u^+(u\tilde{\kappa}(u, 0))(1)$ and (B.2) gives $\partial_u^+(u\tilde{\kappa}(u, 0))(1) = \infty$, which completes the proof of (C.2). □

Recall Claim 3.3, in which we assumed that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is strictly concave on $[1, \infty)$. The next lemma states that the convergence of the average quenched free energy $\phi_L^{\mathcal{I}}$ to $\phi^{\mathcal{I}}$ as $L \to \infty$ is uniform on $\mathbb{Q} \cap [1, \infty)$.

Lemma C.2 *For every* $(\alpha, \beta) \in$ CONE *and* $\varepsilon > 0$ *there exists an* $L_\varepsilon \in \mathbb{N}$ *such that*

$$|\phi_L(\mu) - \phi(\mu)| \leq \varepsilon \qquad \forall\, \mu \in 1 + \tfrac{2\mathbb{N}}{L},\, L \geq L_\varepsilon. \tag{C.5}$$

Proof. Similarly to what we did for Lemma A.1, the proof can be done by treating separately the cases μ large, moderate and small. We leave the details to the reader. □

C.2 Free energy in a single column

We can extend $(\Theta, u) \mapsto \psi(\Theta, u)$ from $\mathcal{V}_M^*$ to $\overline{\mathcal{V}}_M^*$ by using the variational formula in (3.45) and by recalling that $\tilde{\kappa}$ and $\phi^{\mathcal{I}}$ have been extended to $\mathcal{H}$ and $[1, \infty)$ in Appendices A.3 and C.1.

Pick $M \in \mathbb{N}$ and recall (3.15). Define a distance d_M on $\overline{\mathcal{V}}_M$ as follows. Pick $\Theta_1, \Theta_2 \in \overline{\mathcal{V}}_M$, abbreviate

$$\Theta_1 = (\chi_1, \Delta\Pi_1, b_{0,1}, b_{1,1}, x_1), \qquad \Theta_2 = (\chi_2, \Delta\Pi_2, b_{0,2}, b_{1,2}, x_2), \tag{C.6}$$

and define

$$d_M(\Theta_1, \Theta_2) = \sum_{j\in\mathbb{Z}} \frac{1_{\{\chi_1(j)\neq\chi_2(j)\}}}{2^{|j|}} + |\Delta\Pi_1 - \Delta\Pi_2| + |b_{0,1} - b_{0,2}| + |b_{1,1} - b_{1,2}| + |x_1 - x_2| \tag{C.7}$$

so that $\widetilde{d}_M((\Theta_1, u_1), (\Theta_2, u_2)) = \max\{|u_1 - u_2|, d_M(\Theta_1, \Theta_2)\}$ is a distance on $\overline{\mathcal{V}}_M^{*,m}$ for which $\overline{\mathcal{V}}_M^{*,m}$ is compact.

Lemmas C.3 and C.4 below are proven in Section C.3.

Lemma C.3 *For every* $(M, m) \in \text{EIGH}$ *and* $(\alpha, \beta) \in \text{CONE}$,

$$(u, \Theta) \mapsto u\,\psi(\Theta, u; \alpha, \beta) \tag{C.8}$$

is uniformly continuous on $\overline{\mathcal{V}}_M^{*,m}$ *endowed with* $\widetilde{d}_M$.

Lemma C.4 *For every* $\Theta \in \overline{\mathcal{V}}_M$, *the function* $u \mapsto u\psi(\Theta, u)$ *is continuous and strictly concave on* $[t_\Theta, \infty)$.

Below we list several results that were used in Section 6. The proofs of these result are given in Section C.3. Proposition C.5 below says that the free energy per column associated with the Hamiltonian given by $(\beta - \alpha)/2$ times the time spent by the copolymer in the B-solvent is a good aproximation of $\psi(\Theta, u)$ when $u \to \infty$ uniformly in $\Theta \in \overline{\mathcal{V}}_M$. This proof of this proposition will be given in Section C.3.3.

Proposition C.5 *For all* $(\alpha, \beta) \in \text{CONE}$ *and all* $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *and* $L_\varepsilon \in \mathbb{N}$ *such that*

$$\Big|\psi(\Theta, u) - \tfrac{1}{uL} \log \sum_{\pi\in\mathcal{W}_{\Theta,u,L}} e^{T(\pi)\frac{\beta-\alpha}{2}}\Big| \leq \varepsilon, \quad \Theta \in \overline{\mathcal{V}}_M, \quad u \geq t_\Theta \vee R_\varepsilon, \quad L \geq L_\varepsilon, \tag{C.9}$$

where $T(\pi) = \sum_{i=1}^{uL} 1\{\chi^L_{(\pi_{i-1},\pi_i)} = B\}$ *is the time spent by* π *in solvent* B.

Lemmas C.6–C.8 below are consequences of Lemma C.4 and Proposition C.5. The proofs of Lemmas C.6 and C.8 will be given in Sections C.3.4 and C.3.6. Lemma C.6 shows that $\psi(\Theta, u)$ is bounded from above uniformly in $\Theta \in \overline{\mathcal{V}}_M$ as $u \to \infty$. Lemma C.7 identifies the limit of $\partial_u^-(u\,\psi(\Theta, u))$ as $u \to \infty$ for $\Theta \in \overline{\mathcal{V}}_M$. Lemma C.8 is the counterpart of Lemma C.6 for $\partial_u^-(u\psi(\Theta, u))$ instead of $\psi(\Theta, u)$.

Lemma C.6 *For all* $(\alpha, \beta) \in \text{CONE}$ *and* $\varepsilon > 0$ *there exists a* $C_\varepsilon > 0$ *such that*

$$\psi(\Theta, u) \leq \begin{cases} \varepsilon & \text{if } \ \Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_{\text{nint},B,1,M}, \quad u \geq t_\Theta \vee C_\varepsilon, \\ \frac{\beta-\alpha}{2} + \varepsilon & \text{if } \ \Theta \in \overline{\mathcal{V}}_{\text{nint},B,1,M}, \quad u \geq t_\Theta \vee C_\varepsilon, \end{cases} \tag{C.10}$$

Lemma C.7 *For all* $(\alpha,\beta)\in$ CONE,

$$\lim_{v\to\infty}\partial_u^+(u\psi(\Theta,u))(v)=\begin{cases}0 & \text{if}\quad \Theta\in\overline{\mathcal{V}}_M\setminus\overline{\mathcal{V}}_{\mathrm{nint},B,1,M},\\ \frac{\beta-\alpha}{2} & \text{if}\quad \Theta\in\overline{\mathcal{V}}_{\mathrm{nint},B,1,M}.\end{cases} \tag{C.11}$$

Lemma C.8 *For all* $(\alpha,\beta)\in$ CONE *and* $\varepsilon>0$ *there exists a* $V_\varepsilon>0$ *such that*

$$\partial_u^-(u\psi(\Theta,u))(v)\leq\begin{cases}\varepsilon & \text{if}\quad \Theta\in\overline{\mathcal{V}}_M\setminus\overline{\mathcal{V}}_{\mathrm{nint},B,1,M},\quad v\geq 2t_\Theta\vee V_\varepsilon,\\ \frac{\beta-\alpha}{2}+\varepsilon & \text{if}\quad \Theta\in\overline{\mathcal{V}}_{\mathrm{nint},B,1,M},\quad v\geq 2t_\Theta\vee V_\varepsilon.\end{cases} \tag{C.12}$$

C.3 Proof of Lemmas C.3–C.8

C.3.1 Proof of Lemma C.3

Pick $(M,m)\in$ EIGH. By the compactness of $\overline{\mathcal{V}}_M^{*,m}$, it suffices to show that $(u,\Theta)\mapsto u\,\psi(\Theta,u)$ is continuous on $\overline{\mathcal{V}}_M^{*,m}$. Let $(\Theta_n,u_n)=(\chi_n,\Delta\Pi_n,b_{0,n},b_{1,n},u_n)$ be the general term of an infinite sequence that tends to $(\Theta,u)=(\chi,\Delta\Pi,b_0,b_1,u)$ in $(\overline{\mathcal{V}}_M^{*,m},\widetilde{d}_M)$. We want to show that $\lim_{n\to\infty}u_n\psi(\Theta_n,u_n)=u\psi(\Theta,u)$. By the definition of $\widetilde{d}_M$, we have $\chi_n=\chi$ and $\Delta\Pi_n=\Delta\Pi$ for n large enough. We assume that $\Theta\in\mathcal{V}_{\mathrm{int}}$, so that $\Theta_n\in\mathcal{V}_{\mathrm{int}}$ for n large enough as well. The case $\Theta\in\mathcal{V}_{\mathrm{nint}}$ can be treated similarly.

Set

$$\mathcal{R}_m=\{(a,h,l)\in[0,m]\times[0,1]\times\mathbb{R}\colon h+|l|\leq a\} \tag{C.13}$$

and note that $\mathcal{R}_m$ is a compact set. Let $g\colon\mathcal{R}_m\mapsto[0,\infty)$ be defined as $g(a,h,l)=a\,\widetilde{\kappa}(\frac{a}{h},\frac{l}{h})$ if $h>0$ and $g(a,h,l)=0$ if $h=0$. The continuity of $\widetilde{\kappa}$, stated in Lemma B.1(i), ensures that g is continuous on $\{(a,h,l)\in\mathcal{R}_m\colon h>0\}$. The continuity at all $(a,0,l)\in\mathcal{R}_m$ is obtained by recalling that $\lim_{u\to\infty}\tilde{\kappa}(u,l)=0$ uniformly in $l\in[-u+1,u-1]$ (see Lemma B.1(ii-iii)) and that $\widetilde{\kappa}$ is bounded on $\mathcal{H}$.

In the same spirit, we may set $\mathcal{R}'_m=\{(u,h)\in[0,m]\times[0,1]\colon\ h\leq u\}$ and define $g'\colon\mathcal{R}'_m\mapsto[0,\infty)$ as $g'(u,h)=u\,\phi^{\mathcal{I}}(\frac{u}{h})$ for $h>0$ and $g'(u,h)=0$ for $h=0$. With the help of Lemma C.1 we obtain the continuity of g' on $\mathcal{R}'_m$ by mimicking the proof of the continuity of g on $\mathcal{R}_m$.

Note that the variational formula in (3.45) can be rewriten as

$$u\,\psi(\Theta,u)=\sup_{(h),(a)\in\mathcal{L}(l_A,l_B;\,u)}Q((h),(a),l_A,l_B), \tag{C.14}$$

with

$$Q((h),(a),l_A,l_B)=g(a_A,h_A,l_A)+g(a_B,h_B,l_B)+a_B\,\tfrac{\beta-\alpha}{2}+g^{'}(a^{\mathcal{I}},h^{\mathcal{I}}), \tag{C.15}$$

and with l_A and l_B defined in (3.36). Note that $\mathcal{L}(l_A,\,l_B;\,u)$ is compact, and that $(h),(a)\mapsto Q((h),(a),l_A,l_B)$ is continuous on $\mathcal{L}(l_A,\,l_B;\,u)$ because g and g' are continuous on $\mathcal{R}_m$ and $\mathcal{R}^{'}_m$, respectively. Hence, the supremum in (C.14) is attained.

Pick $\varepsilon>0$, and note that g and g' are uniformly continuous on $\mathcal{R}_m$ and $\mathcal{R}'_m$, which are compact sets. Hence there exists an $\eta_\varepsilon>0$ such that $|g(a,h,l)-g(a',h',l')|\leq\varepsilon$ and $|g'(u,b)-g'(u',b')|\leq\varepsilon$ when $(a,h,l),(a',h',l')\in\mathcal{R}_m$ and $(u,b),(u',b')\in\mathcal{R}'_m$ are such that $|a-a'|,|h-h'|,|l-l'|,|u-u'|$ and $|b-b'|$ are bounded from above by η_ε.

Since $\lim_{n\to\infty}(\Theta_n, u_n) = (\Theta, u)$ we also have that $\lim_{n\to\infty} b_{0,n} = b_0$, $\lim_{n\to\infty} b_{1,n} = b_1$ and $\lim_{n\to\infty} u_n = u$. Thus, $\lim_{n\to\infty} l_{A,n} = l_A$ and $\lim_{n\to\infty} l_{B,n} = l_B$, and therefore $|l_{A,n} - l_A| \leq \eta_\varepsilon$, $|l_{B,n} - l_B| \leq \eta_\varepsilon$ and $|u_n - u| \leq \eta_\varepsilon$ for $n \geq n_\varepsilon$ large enough.

For $n \in \mathbb{N}$, let $(h_n), (a_n) \in \mathcal{L}(l_{A,n},\, l_{B,n};\, u_n)$ be a maximizer of (C.14) at (Θ_n, u_n), and note that, for $n \geq n_\varepsilon$, we can choose $(\widetilde{h}_n), (\widetilde{a}_n) \in \mathcal{L}(l_A,\, l_B;\, u)$ such that $|\widetilde{a}_{A,n} - a_{A,n}|$, $|\widetilde{a}_{B,n} - a_{B,n}|$, $|\widetilde{a}^{\mathcal{I}}_n - a^{\mathcal{I}}_n|$, $|\widetilde{h}_{A,n} - h_{A,n}|$, $|\widetilde{h}_{B,n} - h_{B,n}|$ and $|\widetilde{h}^{\mathcal{I}}_n - h^{\mathcal{I}}_n|$ are bounded above by η_ε. Consequently,

$$u_n\psi(\Theta_n, u_n) - u\psi(\Theta, u) \leq Q((h_n), (a_n), l_{A,n}, l_{B,n}) - Q((\widetilde{h}_n), (\widetilde{a}_n), l_A, l_B) \leq 3\varepsilon. \tag{C.16}$$

We bound $u\psi(\Theta, u) - u_n\psi(\Theta_n, u_n)$ from above in a similar manner, and this suffices to obtain the claim.

C.3.2 Proof of lemma C.4

The continuity is a straightforward consequence of Lemma C.3: simply fix Θ and let $m \to \infty$. To prove the strict concavity, we note that the cases $\Theta \in \mathcal{V}_{\mathrm{int},M}$ and $\Theta \in \mathcal{V}_{\mathrm{nint},M}$ can be treated similarly. We will therefore focus on $\Theta \in \mathcal{V}_{\mathrm{int},M}$.

For $l \in \mathbb{R}$, let

$$\mathcal{N}_l = \{(a, h) \in [0, \infty) \times [0, 1] \colon a \geq h + |l|\}, \quad \mathcal{N}_l^+ = \{(a, h) \in \mathcal{N}_l \colon h > 0\}, \tag{C.17}$$

and let $g_l \colon \mathcal{N}_l \mapsto [0, \infty)$ be defined as $g_l(a, h) = a\,\tilde{\kappa}(\frac{a}{h}, \frac{l}{h})$ for $h > 0$ and $g_l(a, h) = 0$ for $h = 0$. For $l \neq 0$, the strict concavity of $(u, l) \mapsto u\tilde{\kappa}(u, l)$ on $\mathcal{H}$, stated in Lemma B.1(i), immediately yields that g_l is strictly concave on $\mathcal{N}_l^+$ and concave on $\mathcal{N}_l$. Consequently, for all $(a_1, h_1) \in \mathcal{N}_l^+$ and $(a_2, h_2) \in \mathcal{N}_l \setminus \mathcal{N}_l^+$, g_l is strictly concave on the segment $[(u_1, h_1), (u_2, h_2)]$.

Define also $\widetilde{g} \colon \mathcal{N}_0 \mapsto [0, \infty)$ as $\widetilde{g}(a, h) = a\,\phi^{\mathcal{I}}(\frac{a}{h})$ for $h > 0$ and $\widetilde{g}(a, h) = 0$ for $h = 0$. The strict concavity of $u \mapsto u\phi^{\mathcal{I}}(u)$ and of $u \mapsto u\tilde{\kappa}(u, 0)$ on $[1, \infty)$, stated in Claim 3.3 and in Lemma B.1, immediately yield that $\widetilde{g}$ and g_0 are concave on $\mathcal{N}_0$ and that, for $h > 0$, $a \mapsto \widetilde{g}(a, h)$ and $a \mapsto g_0(a, h)$ are strictly concave on $[h, \infty)$

Similarly to what we did in (C.14), we can rewrite the variational formula in (3.45) as

$$u\,\psi(\Theta, u) = \sup_{(h),(a)\in\mathcal{L}(l_A,\, l_B;\, u)} \widetilde{Q}((h), (a)) \tag{C.18}$$

with

$$\widetilde{Q}((h), (a)) = g_{l_A}(a_A, h_A) + g_{l_B}(a_B, h_B) + a_B\,\tfrac{\beta-\alpha}{2} + \widetilde{g}(u - a_A - a_B, 1 - h_A - h_B), \tag{C.19}$$

and the supremum in (C.18) is attained. In what follows we will restrict the proof to the case $l_A, l_B > 0$ for the following reason. If $l_k = 0$ for $k \in \{A, B\}$, then the inequality $g_0 \leq \widetilde{g}$ and the concavity of $\widetilde{g}$ ensure that there exists a $(h), (a) \in \mathcal{L}(l_A,\, l_B;\, u)$ maximizing (C.18) and satisfying $h_k = a_k = 0$, which allows to copy the proof below after removing the k-th coordinate in $(h), (a)$.

Next, we show that if $(h), (a) \in \mathcal{L}(l_A,\, l_B;\, u)$ realizes the maximum in (C.18), then $(h), (a) \notin \widetilde{\mathcal{L}}(l_A,\, l_B;\, u)$ with

$$\widetilde{\mathcal{L}}(l_A,\, l_B;\, u) = \widetilde{\mathcal{L}}_A(l_A,\, l_B;\, u) \cup \widetilde{\mathcal{L}}_B(l_A,\, l_B;\, u) \cup \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\, l_B;\, u) \tag{C.20}$$

and

$$\begin{aligned}\widetilde{\mathcal{L}}_A(l_A,\, l_B;\, u) &= \{(h),(a) \in \mathcal{L}(l_A,\, l_B;\, u)\colon\; h_A = 0 \;\text{ and }\; a_A > l_A\},\\ \widetilde{\mathcal{L}}_B(l_A,\, l_B;\, u) &= \{(h),(a) \in \mathcal{L}(l_A,\, l_B;\, u)\colon\; h_B = 0 \;\text{ and }\; a_B > l_B\},\\ \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\, l_B;\, u) &= \{(h),(a) \in \mathcal{L}(l_A,\, l_B;\, u)\colon\; h_I = 0 \;\text{ and }\; a_I > 0\}. \end{aligned} \tag{C.21}$$

Assume that $(h),(a) \in \widetilde{\mathcal{L}}(l_A,\, l_B;\, u)$, and that $h_A > 0$ or $h^{\mathcal{I}} > 0$. For instance, $(h),(a) \in \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\, l_B;\, u)$ and $h_A > 0$. Then, by Lemma B.1(iv), $\widetilde{Q}$ strictly increases when a_A is replaced by $a_A+a^{\mathcal{I}}$ and $a^{\mathcal{I}}$ by 0. This contradicts the fact that $(h),(a)$ is a maximizer. Next, if $(h),(a) \in \widetilde{\mathcal{L}}(l_A,\, l_B;\, u)$ and $h_A = h^{\mathcal{I}} = 0$, then $h_B = 1$, and the first case is $(h),(a) \in \widetilde{\mathcal{L}}_A(l_A,\, l_B;\, u)$, while the second case is $(h),(a) \in \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\, l_B;\, u)$. In the second case, as before, we replace a_A by $a_A+a^{\mathcal{I}}$ and $a^{\mathcal{I}}$ by 0, which does not change $\widetilde{Q}$ but yields that $a_A > l_A$ and therefore brings us back to the first case. In this first case, we are left with an expression of the form

$$Q((h),(a)) = g_{l_B}(a_B, 1) + a_B\, \tfrac{\beta-\alpha}{2} \tag{C.22}$$

with $h_A = h^{\mathcal{I}} = 0$ and $a_A > l_A$. Thus, if we can show that there exists an $x \in (0,1)$ such that

$$g_{l_A}(a_A, x) + g_{l_B}(a_B, 1-x) > g_{l_B}(a_B, 1), \tag{C.23}$$

then we can claim that $(h),(a)$ is not a maximizer of (C.18) and the proof for $(h),(a) \notin \widetilde{\mathcal{L}}(l_A,\, l_B;\, u)$ will be complete.

To that end, we recall (3.4), which allows us to rewrite the left-hand side in (C.23) as

$$g_{l_A}(a_A, x) + g_{l_B}(a_B, 1-x) = a_A\, \kappa\big(\tfrac{a_A}{l_A}, \tfrac{x}{l_A}\big) + a_B\, \kappa\big(\tfrac{a_B}{l_B}, \tfrac{1-x}{l_B}\big) + a_B\, \tfrac{\beta-\alpha}{2}. \tag{C.24}$$

We recall [5], Lemma 2.1.1, which claims that κ is defined on $\textsc{dom} = \{(a,b)\colon a \geq 1+b, b \geq 0\}$, is analytic on the interior of $\textsc{dom}$ and is continuous on $\textsc{dom}$. Moreover, in the proof of this lemma, an expression for $\partial_b\, \kappa(a,b)$ is provided, which is valid on the interior of $\textsc{dom}$. From this expression we can easily check that if $a > 1$, then $\lim_{b\to 0} \partial_b\, \kappa(a,b) = \infty$. Therefore, by the continuity of κ on $(a_A/l_A, 0)$ with $a_A/l_A > 1$ we can assert that the derivative with respect to x of the left-hand side in (C.24) at $x = 0$ is infinite, and therefore there exists an $x > 0$ such that (C.23) is satisfied.

It remains to prove the strict concavity of $u \mapsto u\psi(\Theta, u)$ with $\Theta \in \mathcal{V}_{\text{int},M}$. Pick $u_1 > u_2 \geq t_\Theta$, and let $(h_1),(a_1) \in \mathcal{L}(l_A,\, l_B;\, u_1)$ and $(h_2),(a_2) \in \mathcal{L}(l_A,\, l_B;\, u_2)$ be maximizers of (C.18) for u_1 and u_2, respectively. We can write

$$\begin{aligned}(a_1),(h_1) &= \big(a_{A,1}, a_{B,1}, a_1^{\mathcal{I}}\big), \big(h_{A,1}, h_{B,1}, h_1^{\mathcal{I}}\big),\\ (a_2),(h_2) &= \big(a_{A,2}, a_{B,2}, a_2^{\mathcal{I}}\big), \big(h_{A,2}, h_{B,2}, h_2^{\mathcal{I}}\big).\end{aligned} \tag{C.25}$$

Thus, $(\frac{a_1+a_2}{2}), (\frac{h_1+h_2}{2}) \in \mathcal{L}(l_A,\, l_B;\, \frac{u_1+u_2}{2})$ and, with the help of the concavity of $g_{l_A}, g_{l_B}, \widetilde{g}$ proven above, we can write

$$\tfrac{u_1+u_2}{2}\, \psi(\Theta, \tfrac{u_1+u_2}{2}) \geq \widetilde{Q}((\tfrac{a_1+a_2}{2}), (\tfrac{h_1+h_2}{2})) \geq \tfrac{1}{2}\big(u_1\, \psi(\Theta, u_1) + u_2\, \psi(\Theta, u_2)\big). \tag{C.26}$$

At this stage, we assume that the right-most inequality in (C.26) is an equality and show that this leads to a contradiction, after which Lemma C.4 will be proven.

We have proven above that $(a_1),(h_1) \notin \widetilde{\mathcal{L}}(l_A,\, l_B;\, u_1)$ and $(a_2),(h_2) \notin \widetilde{\mathcal{L}}(l_A,\, l_B;\, u_2)$. Thus, we can use (C.19) and the strict concavity of g_{l_A}, g_{l_B} on $\mathcal{N}_{l_A}^+, \mathcal{N}_{l_B}^+$ and the concavity of $\widetilde{g}$ on $\mathcal{N}_0$ to conclude that necessarily

$$(a_{A,1}, h_{A,1}) = (a_{A,2}, h_{A,2}) \quad \text{and} \quad (a_{B,1}, h_{B,1}) = (a_{B,2}, h_{B,2}). \tag{C.27}$$

As a consequence, we recall that $u_1 > u_2$ and we can write

$$u_1^{\mathcal{I}} = u_1 - a_{A,1} - a_{B,2} > u_2 - a_{A,2} - a_{B,2} = u_2^{\mathcal{I}} \geq 0, \tag{C.28}$$

and therefore, since $(a_1),(h_1) \notin \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\, l_B;\, u_1)$, it follows that $h_1^{\mathcal{I}} > 0$ such that (recall (C.27))

$$h_1^{\mathcal{I}} = 1 - h_{A,1} - h_{B,1} = 1 - h_{A,2} - h_{B,2} = h_2^{\mathcal{I}} > 0. \tag{C.29}$$

Hence we can use the strict concavity of $a \mapsto \widetilde{g}(a, h_1^{\mathcal{I}})$ to conclude that $u_1^{\mathcal{I}} = u_2^{\mathcal{I}}$, which clearly contradicts (C.28).

C.3.3 Proof of Proposition C.5

The proof is performed with the help of Lemma E.1 stated in section E. For this reason we use some notations introduced in Lemma E.1.

We pick $\gamma, \eta > 0$ (which will be specified later), and we let $\widehat{K} \in \mathbb{N}$ be the integer in Lemma E.1 associated with $\alpha, \beta, \eta, \gamma$. For $\Theta \in \overline{\mathcal{V}}_M$, $u \geq t_\Theta$ and $\pi \in \mathcal{W}_{\Theta,u,L}$, we let N_π be the number of excursions of π in solvent B in columns of type Θ. We further let also $(I_\pi) = (I_\pi(1), \ldots, I_\pi(N_\pi))$ be the sequence of consecutive intervals in $\{1, \ldots, uL\}$ on which π makes these N_π excursions in B, so that $(I_\pi) \in \mathcal{E}_{uL,N_\pi}$ and $T(\pi) = \sum_{i=1}^{N_\pi} |I_\pi(i)|$.

Pick $\Theta \in \overline{\mathcal{V}}_M$, $u \geq t_\Theta$ and partition $\mathcal{W}_{\Theta,u,L}$ into two parts:

$$V_{u,L,\gamma}^{\Theta,+} = \{\pi \in \mathcal{W}_{\Theta,u,L} \colon\, T(\pi) \geq \gamma u L\} \quad \text{and} \quad V_{u,L,\gamma}^{\Theta,-} = \{\pi \in \mathcal{W}_{\Theta,u,L} \colon\, T(\pi) \leq \gamma u L\}. \tag{C.30}$$

There exists a $c > 0$, depending on α, β only, such that

$$\big|H_L^{\Theta,\omega}(\pi) - T(\pi)\tfrac{\beta-\alpha}{2}\big| \leq cT(\pi) \leq c\gamma uL, \quad \pi \in V_{u,L,\gamma}^{\Theta,-}. \tag{C.31}$$

Since any excursion in solvent B requires at least 1 horizontal steps or L vertical steps, we have that $N_\pi \leq u + L$ for $\pi \in \mathcal{W}_{\Theta,u,L}$. Since $u + L \leq uL/\widehat{K}$ as soon as $u, L \geq 2\widehat{K}$, it follows that

$$I(\pi) \in \cup_{N=1}^{uL/\widehat{K}} \{I \in \mathcal{E}_{uL,N} \colon\, T(I) \geq \gamma uL\}, \quad L \geq 2\widehat{K}, \quad u \geq t_\Theta \vee 2\widehat{K}, \quad \pi \in V_{u,L,\gamma}^{\Theta,+}, \tag{C.32}$$

and therefore $\omega \in Q_{uL,\widehat{K}}^{\gamma,\eta}$ implies that $|H_L^{\Theta,\omega}(\pi) - T(\pi)\frac{\beta-\alpha}{2}| \leq \eta uL$ for $\pi \in V_{u,L,\gamma}^{\Theta,+}$. Consequently, for $\omega \in Q_{uL,\widehat{K}}^{\gamma,\eta}$, we have

$$\big|H_L^{\Theta,\omega}(\pi) - T(\pi)\tfrac{\beta-\alpha}{2}\big| \leq uL(\eta + c\gamma), \quad \Theta \in \overline{\mathcal{V}}_M, \quad u \geq 2\widehat{K} \vee t_\Theta, \quad L \geq 2\widehat{K}, \quad \pi \in \mathcal{W}_{\Theta,u,L}. \tag{C.33}$$

Rewrite

$$\psi_L(\Theta, u) = \mathbb{E}\Big[\tfrac{1}{uL} \log \sum_{\pi \in \mathcal{W}_{\Theta,u,L}} e^{H_L^{\Theta,\omega}(\pi)} \big| \mathcal{Q}_{uL,\widehat{K}}^{\gamma,\eta}\Big] + \mathbb{P}\Big(\big(\mathcal{Q}_{uL,\widehat{K}}^{\gamma,\eta}\big)^c\Big)\ \Delta, \tag{C.34}$$

where Δ is an error term given by

$$\Delta = \mathbb{E}\Big[\tfrac{1}{uL}\log \sum_{\pi\in\mathcal{W}_{\Theta,u,L}} e^{H_L^{\Theta,\omega}(\pi)}\big|\big(\mathcal{Q}^{\gamma,\eta}_{uL,\widehat{K}}\big)^c\Big] - \mathbb{E}\Big[\tfrac{1}{uL}\log \sum_{\pi\in\mathcal{W}_{\Theta,u,L}} e^{H_L^{\Theta,\omega}(\pi)}\big|\mathcal{Q}^{\gamma,\eta}_{uL,\widehat{K}}\Big]. \tag{C.35}$$

By (3.34), we obtain that $|\Delta| \leq 2C_{\mathrm{uf}}$.

To conclude, we set $\eta = \varepsilon/3$, $\gamma = \varepsilon/3c$. By Lemma E.1, there exists an $L_\varepsilon \in \mathbb{N}$ such that, for $u \geq 2\widehat{K} \vee t_\Theta$ and $L \geq L_\varepsilon$, we have $\mathbb{P}\big(\big(\mathcal{Q}^{\gamma,\eta}_{uL,\widehat{K}}\big)^c\big) \leq \varepsilon/6\, C_{\mathrm{uf}}$. Thus, we can use (C.33) and (C.34) to complete the proof of Proposition C.5.

C.3.4 Proof of Lemma C.6

Pick $\varepsilon > 0$. By applying Proposition C.5 with $\varepsilon/2$, we see that there exists an $R_{\varepsilon/2} > 0$ such that

$$\psi(\Theta,u) \leq \limsup_{L\to\infty} \tfrac{1}{uL}\log \sum_{\pi\in\mathcal{W}_{\Theta,u,L}} e^{T(\pi)\frac{\beta-\alpha}{2}} + \tfrac{\varepsilon}{2}, \quad \Theta\in\overline{\mathcal{V}}_M, \quad u \geq t_\Theta \vee R_{\varepsilon/2}. \tag{C.36}$$

We first consider the case $\Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$. Since $(\alpha,\beta) \in \textsc{cone}$, we can use (C.36) to obtain

$$\psi(\Theta,u) \leq \limsup_{L\to\infty} \tfrac{1}{uL}\log |\mathcal{W}_{\Theta,u,L}| + \tfrac{\varepsilon}{2}, \quad u \geq t_\Theta \vee R_{\varepsilon/2}. \tag{C.37}$$

Thus, (C.37) and Lemma A.2 imply that there exists a $C_\varepsilon \geq R_{\varepsilon/2}$ such that $\psi(\Theta,u) \leq \varepsilon$ when $u \geq t_\Theta \vee C_\varepsilon$ and $\Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$. The case $\Theta \in \overline{\mathcal{V}}\mathrm{nint}, B, 1, M$ can be treated similarly after noticing that $T(\pi) = uL$ for $\pi \in \mathcal{W}_{\Theta,u,L}$ and $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$.

C.3.5 Proof of Lemma C.7

The proof is a straightforward consequence of the strict concavity of $u \mapsto u\psi(\Theta,u)$ for $\Theta \in \overline{\mathcal{V}}_M$, Proposition C.5 and Lemma A.2.

C.3.6 Proof of Lemma C.8

Pick $\varepsilon > 0$. The proof will be complete once we show the following two properties:

(1) There exists a $T_\varepsilon > 0$ such that

$$\partial_u^-(u\psi(\Theta,u))(2t_\Theta) \leq \begin{cases} \varepsilon & \text{if} \quad \Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M} \colon t_\Theta \geq T_\varepsilon, \\ \frac{\beta-\alpha}{2} + \varepsilon & \text{if} \quad \Theta \in \overline{\mathcal{V}}_{\mathrm{nint},B,1,M} \colon t_\Theta \geq T_\varepsilon. \end{cases} \tag{C.38}$$

(2) For all $T > 0$ there exists a $V_{\varepsilon,T} > 0$ such that

$$\partial_u^-(u\psi(\Theta,u))(v) \leq \begin{cases} \varepsilon & \text{if} \quad \Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M} \colon t_\Theta \leq T, \quad v \geq t_\Theta \vee V_{\varepsilon,T}, \\ \frac{\beta-\alpha}{2} + \varepsilon & \text{if} \quad \Theta \in \overline{\mathcal{V}}_{\mathrm{nint},B,1,M} \colon t_\Theta \leq T, \quad v \geq t_\Theta \vee V_{\varepsilon,T}. \end{cases} \tag{C.39}$$

We prove (C.39) for the case $\Theta \in \mathcal{V}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$ (the case $\Theta \in \mathcal{V}_{\mathrm{nint},B,1,M}$ can be treated similarly). To that aim, we assume that there exists a sequence $(\Theta_n)_{n\in\mathbb{N}}$ in $\mathcal{V}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$ such that $t_{\Theta_n} \leq T$ for $n \in \mathbb{N}$ and a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \geq t_{\Theta_n}$ for $n \in \mathbb{N}$, $\lim_{n\to\infty} u_n = \infty$ and

$$\partial_u^-(u\psi(\Theta_n, u))(u_n) \geq \varepsilon, \quad n \in \mathbb{N}. \tag{C.40}$$

By concavity of $u \mapsto u\psi(\Theta_n, u)$ for $n \in \mathbb{N}$ (see Lemma C.4), we have

$$u_n\psi(\Theta_n, u_n) - t_{\Theta_n}\psi(\Theta_n, t_{\Theta_n}) \geq \varepsilon\,(u_n - t_{\Theta_n}), \quad n \in \mathbb{N}. \tag{C.41}$$

Therefore, the uniform bound on free energies in (3.34) and the inequality $t_{\Theta_n} \leq T$ allow us to rewrite (C.41) as

$$\psi(\Theta_n, u_n) \geq \varepsilon - \frac{T(C_{\mathrm{uf}} + \varepsilon)}{u_n}, \quad n \in \mathbb{N}, \tag{C.42}$$

which contradicts Lemma C.6 because $\lim_{n\to\infty} u_n = \infty$.

It remains to prove (C.38). This is done in a similar manner for the case $\Theta \in \mathcal{V}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$ (the case $\Theta \in \mathcal{V}_{\mathrm{nint},B,1,M}$ can again be treated similarly), by assuming that there exists a sequence $(\Theta_n)_{n\in\mathbb{N}}$ in $\mathcal{V}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$ such that $\lim_{n\to\infty} t_{\Theta_n} = \infty$ and

$$\partial_u^-(u\psi(\Theta_n, u))(2t_{\Theta_n}) \geq \varepsilon, \quad n \in \mathbb{N}. \tag{C.43}$$

Thus, similarly as in (C.41–C.42), the concavity of $u \mapsto u\psi(\Theta_n, u)$ and (C.43) give

$$\psi(\Theta_n, 2t_{\Theta_n}) \geq \frac{\varepsilon}{2} + \frac{\psi(\Theta_n, t_{\Theta_n})}{2}, \quad n \in \mathbb{N}. \tag{C.44}$$

At this point we use Proposition C.5 to assert that there exist $R_\varepsilon > 0$ and $L_\varepsilon \in \mathbb{N}$ such that, for n satisfying $t_{\Theta_n} \geq R_\varepsilon$ and $L \geq L_\varepsilon$, we have

$$\begin{aligned} \psi(\Theta_n, t_{\Theta_n}) &\geq \tfrac{1}{t_{\Theta_n}L} \log \sum_{\pi\in\mathcal{W}_{\Theta,t_{\Theta_n},L}} e^{T(\pi)\frac{\beta-\alpha}{2}} - \tfrac{\varepsilon}{4}, \\ \psi(\Theta_n, 2t_{\Theta_n}) &\leq \tfrac{1}{2t_{\Theta_n}L} \log \sum_{\pi\in\mathcal{W}_{\Theta_n,2t_{\Theta_n},L}} e^{T(\pi)\frac{\beta-\alpha}{2}} + \tfrac{\varepsilon}{4}. \end{aligned} \tag{C.45}$$

By using (C.44–C.45), we obtain that, for $t_{\Theta_n} \geq R_\varepsilon$ and $L \geq L_\varepsilon$,

$$\tfrac{1}{2t_{\Theta_n}L} \log \sum_{\pi\in\mathcal{W}_{\Theta_n,2t_{\Theta_n},L}} e^{T(\pi)\frac{\beta-\alpha}{2}} \geq \tfrac{1}{2t_{\Theta_n}L} \log \sum_{\pi\in\mathcal{W}_{\Theta,t_{\Theta_n},L}} e^{T(\pi)\frac{\beta-\alpha}{2}} + \tfrac{\varepsilon}{8}, \tag{C.46}$$

uses some key ingredients that are provided which we can rewrite as

$$\begin{aligned} \tfrac{1}{2t_{\Theta_n}L} \log |\mathcal{W}_{\Theta_n,2t_{\Theta_n},L}| &+ \tfrac{\beta-\alpha}{4t_{\Theta_n}L} \min\{T(\pi), \pi \in \mathcal{W}_{\Theta_n,2t_{\Theta_n},L}\} \\ &\geq \tfrac{\beta-\alpha}{4t_{\Theta_n}L} \min\{T(\pi), \pi \in \mathcal{W}_{\Theta_n,t_{\Theta_n},L}\} + \tfrac{\varepsilon}{8}. \end{aligned} \tag{C.47}$$

Since $\Theta_n \in \mathcal{V}_M \setminus \overline{\mathcal{V}}_{\mathrm{nint},B,1,M}$, there exist $\pi_1 \in \mathcal{W}_{\Theta_n,t_{\Theta_n},L}$ and $\pi_2 \in \mathcal{W}_{\Theta_n,2t_{\Theta_n},L}$ such that

$$\begin{aligned} T(\pi_1) &= l_B(\Theta_n) = \min\{T(\pi), \pi \in \mathcal{W}_{\Theta_n,t_{\Theta_n},L}\}, \\ T(\pi_2) &= l_B(\Theta_n) = \min\{T(\pi), \pi \in \mathcal{W}_{\Theta_n,2t_{\Theta_n},L}\}. \end{aligned} \tag{C.48}$$

Thus, for $t_{\Theta_n} \geq R_\varepsilon$ and $L \geq L_\varepsilon$, the inequality in (C.47) becomes

$$\tfrac{1}{2t_{\Theta_n}L} \log |\mathcal{W}_{\Theta_n,2t_{\Theta_n},L}| \geq \tfrac{\varepsilon}{8}, \tag{C.49}$$

which obviously contradicts Lemma A.2.

D Concentration of measure

Let $\mathcal{S}$ be a finite set and let $(X_i, \mathcal{A}_i, \mu_i)_{i\in\mathcal{S}}$ be a family of probability spaces. Consider the product space $X = \prod_{i\in\mathcal{S}} X_i$ endowed with the product σ-field $\mathcal{A} = \otimes_{i\in\mathcal{S}}\mathcal{A}_i$ and with the product probability measure $\mu = \otimes_{i\in\mathcal{S}}\mu_i$.

Theorem D.1 (Talagrand [9]) *Let $f\colon X \mapsto \mathbb{R}$ be integrable with respect to $(\mathcal{A}, \mu)$ and, for $i \in \mathcal{S}$, let $d_i > 0$ be such that $|f(x) - f(y)| \le d_i$ when $x, y \in X$ differ in the i-th coordinate only. Let $D = \sum_{i\in\mathcal{S}} d_i^2$. Then, for all $\varepsilon > 0$,*

$$\mu\left\{x \in X\colon \left|f(x) - \int f d\mu\right| > \varepsilon\right\} \le 2e^{-\frac{\varepsilon^2}{2D}}. \tag{D.1}$$

The following corollary of Theorem D.1 was used several times in the paper. Let $(\alpha, \beta) \in$ CONE and let $(\xi_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernouilli trials taking the values $-\alpha$ and β with probability $\frac{1}{2}$ each. Let $l \in \mathbb{N}$, $T\colon\ \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2\colon |x - y| = 1\} \to \{0, 1\}$ and $\Gamma \subset \mathcal{W}_l$ (recall (1.1)). Let $F_l\colon\ [-\alpha, \alpha]^l \to \mathbb{R}$ be such that

$$F_l(x_1, \dots, x_l) = \log \sum_{\pi\in\Gamma} e^{\sum_{i=1}^l x_i\, T((\pi_{i-1},\pi_i))}. \tag{D.2}$$

For all $x, y \in [-\alpha, \alpha]^l$ that differ in one coordinate only we have $|F_l(x)-F_l(y)| \le 2\alpha$. Therefore we can use Theorem D.1 with $\mathcal{S} = \{1, \dots, l\}$, $X_i = [-\alpha, \alpha]$ and $\mu_i = \frac{1}{2}(\delta_{-\alpha} + \delta_\beta)$ for all $i \in \mathcal{S}$, and $D = 4\alpha^2 l$, to obtain that there exist $C_1, C_2 > 0$ such that, for every $l \in \mathbb{N}$, $\Gamma \subset \mathcal{W}_n$ and $T\colon\ \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2\colon |x - y| = 1\} \to \{0, 1\}$,

$$\mathbb{P}\big(|F_l(\xi_1, \dots, \xi_m) - \mathbb{E}(F_l(\xi_1, \dots, \xi_m))| > \eta\big) \le C_1 e^{-\frac{C_2\eta^2}{l}}. \tag{D.3}$$

E Large deviation estimate

Let $(\xi_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernouilli trials taking values β and $-\alpha$ with probability $\frac{1}{2}$ each. For $N \le n \in \mathbb{N}$, denote by $\mathcal{E}_{n,N}$ the set of all ordered sequences of N disjoint and non-empty intervals included in $\{1, \dots, n\}$, i.e.,

$$\begin{aligned} \mathcal{E}_{n,N} = \big\{(I_j)_{1\le j\le N} \subset \{1, \dots, n\}\colon I_j = \{\min I_j, \dots, \max I_j\}\ \forall\, 1 \le j \le N,\\ \max I_j < \min I_{j+1}\ \forall\, 1 \le j \le N - 1 \text{ and } I_j \ne \emptyset\ \forall\, 1 \le j \le N\big\}. \end{aligned} \tag{E.1}$$

For $(I) \in \mathcal{E}_{n,N}$, let $T(I) = \sum_{j=1}^N |I_j|$ be the cumulative length of the intervals making up (I). Pick $\gamma > 0$ and $K \in \mathbb{N}$, and denote by $\widehat{\mathcal{E}}^\gamma_{n,K}$ the set of those (I) in $\cup_{1\le N\le(n/K)}\, \mathcal{E}_{n,N}$ that have a cumulative length larger than γn, i.e.,

$$\widehat{\mathcal{E}}^\gamma_{n,K} = \cup_{N=1}^{n/K}\big\{(I) \in \mathcal{E}_{n,N}\colon\ T(I) \ge \gamma n\big\}. \tag{E.2}$$

Next, for $\eta > 0$ set

$$\mathcal{Q}^{\gamma,\eta}_{n,K} = \bigcap_{(I)\in\widehat{\mathcal{E}}^\gamma_{n,K}} \left\{\sum_{j=1}^N \sum_{i\in I_j} \xi_i \le \left(\tfrac{\beta-\alpha}{2} + \eta\right) T(I)\right\}. \tag{E.3}$$

Lemma E.1 *For all* $(\alpha,\beta) \in \text{CONE}$, $\gamma > 0$ *and* $\eta > 0$ *there exists an* $\widehat{K} \in \mathbb{N}$ *such that, for all* $K \geq \widehat{K}$,

$$\lim_{n\to\infty} P((\mathcal{Q}_{n,K}^{\gamma,\eta})^c) = 0. \tag{E.4}$$

Proof. An application of Cramér's theorem for i.i.d. random variables gives that there exists a $c_\eta > 0$ such that, for every $(I) \in \widehat{\mathcal{E}}_{n,K}^{\gamma}$,

$$\mathbb{P}_\xi\bigg(\sum_{j=1}^{N}\sum_{i\in I_j} \xi_i \geq (\tfrac{\beta-\alpha}{2} + \eta)\, T(I)\bigg) \leq e^{-c_\eta T(I)} \leq e^{-c_\eta \gamma n}, \tag{E.5}$$

where we use that $T(I) \geq \gamma n$ for every $(I) \in \widehat{\mathcal{E}}_{n,K}^{\gamma}$. Therefore

$$\mathbb{P}_\xi((\mathcal{Q}_{n,K}^{\gamma,\eta})^c) \leq |\widehat{\mathcal{E}}_{n,K}^{\gamma}| e^{-c(\eta)\gamma n}, \tag{E.6}$$

and it remains to bound $|\widehat{\mathcal{E}}_{n,K}^{\gamma}|$ as

$$\widehat{\mathcal{E}}_{n,K}^{\gamma} = \sum_{N=1}^{n/K} \big|\big\{(I) \in \mathcal{E}_{n,N} \colon T(I) \geq \gamma n\big\}\big| \leq \sum_{N=1}^{n/K} \binom{n}{2N}, \tag{E.7}$$

where we use that choosing $(I) \in \mathcal{E}_{n,N}$ amounts to choosing in $\{1,\dots,n\}$ the end points of the N disjoint intervals. Thus, the right-hand side of (E.7) is at most $(n/K)\binom{n}{2n/K}$, which for K large enough is $o(e^{c(\eta)\gamma n})$ as $n \to \infty$. □

F On the maximizers of the slope-based variational formula

In this appendix we prove that the supremum of the variational formula in (1.17) is attained at some $\bar\rho \in \bar{\mathcal{R}}_{p,M}$ and for a unique $\bar v \in \bar{\mathcal{B}}$. For ease of notation we suppress the M,p-dependence of $f(\alpha,\beta;M,p)$.

Recall (6.19) and for $p \in (0,1)$ and $(\alpha,\beta) \in \text{CONE}$, let $\mathcal{O}_{p,M,\alpha,\beta}$ be the subset of $\bar{\mathcal{R}}_{p,M}$ containing those $\bar\rho$ that maximize the variational formula in (1.17), i.e.,

$$f(\alpha,\beta) = h(\bar\rho;\,\alpha,\beta) = \sup_{v\in\bar{\mathcal{B}}} \frac{\bar N(\bar\rho,v)}{\bar D(\bar\rho,v)} \quad \text{for } \bar\rho \in \mathcal{O}_{p,M,\alpha,\beta}. \tag{F.1}$$

Recall (2.8–2.10) and set

$$\bar v = v(f(\alpha,\beta)). \tag{F.2}$$

Theorem F.1 *For all* $p \in (0,1)$ *and* $(\alpha,\beta) \in \text{CONE}$ *the following hold:*
(1) The set $\mathcal{O}_{p,M,\alpha,\beta}$ *is non-empty.*
(2) For all $\bar\rho \in \mathcal{O}_{p,M,\alpha,\beta}$ *and all* $v \in \bar{\mathcal{B}}$ *satisfying* $f(\alpha,\beta) = \bar N(\bar\rho,v)/\bar D(\bar\rho,v)$, $v = \bar v$ *for* $\bar\rho$*-a.e.* $(k,l) \in \{A,B\} \times [0,\infty)$ *or* $k = \mathcal{I}$.

Proof. The following proposition will be proven in Section F.1 below and tells us that the maximum of the old variational formula in (2.15) is attained for some $\rho \in \mathcal{R}_{p,M}$. Recall the definition of $g(\rho;\alpha,\beta)$ for $\rho \in \mathcal{R}_{p,M}$ in (6.9).

Theorem F.2 *For $(\alpha, \beta) \in$ CONE, there exists a $\rho \in \mathcal{R}_{p,M}$ such that $f(\alpha, \beta) = g(\rho; \alpha, \beta)$.*

We give the proof of Theorem F.1 subject to Theorem F.2. To that aim, we pick $(\alpha, \beta) \in$ CONE and note that, by Theorem F.2, there exists a $\hat{\rho} \in \mathcal{R}_{p,M}$ such that $f(\alpha, \beta) = g(\hat{\rho}; \alpha, \beta)$. In what follows, we suppress the (α, β)-dependence of $g(\hat{\rho}; \alpha, \beta)$.

Since $f(\alpha, \beta) = g(\hat{\rho})$, (3.60) ensures that $g(\hat{\rho}) > 0$, and by applying Lemma 6.1 we obtain that

$$f(\alpha, \beta) = \frac{N(\hat{\rho}, u(f(\alpha, \beta)))}{D(\hat{\rho}, u(f(\alpha, \beta)))}. \tag{F.3}$$

Apply Lemma 6.5, which ensures that there exist a $\bar{\rho} \in \bar{\mathcal{R}}_p$ and a $v \in \bar{\mathcal{F}}$ such that

$$\frac{N(\hat{\rho}, u(f(\alpha, \beta)))}{D(\hat{\rho}, u(f(\alpha, \beta)))} \le \frac{\bar{N}(\bar{\rho}, v)}{\bar{D}(\bar{\rho}, v)}. \tag{F.4}$$

Then $h(\bar{\rho}) > 0$, and we use Lemma 6.3, which tells us that

$$\frac{\bar{N}(\bar{\rho}, v)}{\bar{D}(\bar{\rho}, v)} \le \frac{\bar{N}(\bar{\rho}, v(h(\bar{\rho})))}{\bar{D}(\bar{\rho}, v(h(\bar{\rho})))}. \tag{F.5}$$

Now (F.3-F.5) and the variational formula in (1.17) are sufficient to complete the proof of (1). The proof of (2) is a straightforward consequence of Lemma 6.1. □

F.1 Proof of Theorem F.2

We give the proof of Theorem F.2 subject to the following lemma, which will be proven in Section F.1.1 below.

Lemma F.3 *For all $t > 0$ and $u \in \mathcal{B}_{\overline{\mathcal{V}}_M}$ there exists an $m_0 \in \mathbb{N}$ such that, for all $\rho \in \mathcal{R}_{p,M}$ and $v \in \mathcal{B}_{\overline{\mathcal{V}}_M}$ satisfying $v \le u$ and $N(\rho, v)/D(\rho, v) \ge t$, there exists a $\tilde{\rho} \in \mathcal{R}_{p,M}^{m_0}$ such that $N(\tilde{\rho}, v)/D(\tilde{\rho}, v) \ge N(\rho, v)/D(\rho, v)$.*

Let $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{R}_{p,M}$ be such that $n \mapsto g(\rho_n; \alpha, \beta)$ is increasing with $\lim_{n\to\infty} g(\rho_n; \alpha, \beta) = f(\alpha, \beta)$. Obviously we can choose $(\rho_n)_{n\in\mathbb{N}}$ such that $g(\rho_n; \alpha, \beta) \ge f(\alpha, \beta)/2$ for all $n \in \mathbb{N}$. Thus, with the help of Lemma 6.1, we obtain

$$g(\rho_n; \alpha, \beta) = \frac{N(\rho_n, u(g(\rho_n)))}{D(\rho_n, u(g(\rho_n)))}, \quad n \in \mathbb{N}. \tag{F.6}$$

Apply Lemma F.3 to see that there exists an $m_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists an $\hat{\rho}_n \in \mathcal{R}_{p,M}^{m_0}$ such that

$$\frac{N(\hat{\rho}_n, u(g(\rho_n)))}{D(\hat{\rho}_n, u(g(\rho_n)))} \ge \frac{N(\rho_n, u(g(\rho_n)))}{D(\rho_n, u(g(\rho_n)))}. \tag{F.7}$$

A straightforward consequence of (F.7) is that

$$\lim_{n\to\infty} \frac{N(\hat{\rho}_n, u(g(\rho_n)))}{D(\hat{\rho}_n, u(g(\rho_n)))} = f(\alpha, \beta). \tag{F.8}$$

Moreover, $\hat{\rho}_n \in \mathcal{M}_1(\overline{\mathcal{V}}_M^{m_0})$ for all $n \ge n_0$, and since $\overline{\mathcal{V}}_M^{m_0}$ is compact we have that $\hat{\rho}_n$ converges weakly to $\rho_\infty \in \mathcal{R}_{p,M}^{m_0}$ along a subsequence. Lemma B.4 implies that $n \mapsto u(g(\rho_n))$ is non-increasing and converges pointwise to $u(f(\alpha, \beta))$ as $n \to \infty$. Since $\overline{\mathcal{V}}_M^{m_0}$ is compact, Dini's

Theorem tells us that the convergence of $u(g(\rho_n))$ to $u(f(\alpha,\beta))$ is uniform on $\overline{\mathcal{V}}_M^{m_0}$. Therefore, using the uniform continuity of $(u,\Theta)\mapsto u\psi(\Theta,u)$ (see Lemma C.3), we obtain

$$f(\alpha,\beta)=\frac{N(\rho_\infty,u(f(\alpha,\beta)))}{D(\rho_\infty,u(f(\alpha,\beta)))}, \tag{F.9}$$

which completes the proof of Theorem F.2.

F.1.1 Proof of Lemma F.3

First, we state and prove Claim F.4 below, which will be needed to prove Lemma F.3. Pick $m\geq M+2$, and note that for $\Theta=(\chi,\Delta\Pi,b_0,b_1,x)\in\overline{\mathcal{V}}_M\setminus\overline{\mathcal{V}}_M^m$ we necessarily have $x_\Theta=2$. Define $T_m\colon\overline{\mathcal{V}}_M\mapsto\overline{\mathcal{V}}_M^m$ as

$$T_m(\Theta)=\begin{cases}\Theta & \text{if } \Theta\in\overline{\mathcal{V}}_M^m,\\ \widetilde{\Theta}=(\chi,\Delta\Pi,b_0,b_1,1) & \text{if } \Theta=(\chi,\Delta\Pi,b_0,b_1,2)\in\overline{\mathcal{V}}_M\setminus\overline{\mathcal{V}}_M^m,\end{cases} \tag{F.10}$$

Claim F.4 *For all $\rho\in\mathcal{R}_{p,M}$ and $m\in\mathbb{N}\colon m\geq M+2$, $\rho\circ T_m^{-1}\in\mathcal{R}_{p,M}^m$.*

Proof. First note that $T_m\colon\overline{\mathcal{V}}_M\mapsto\overline{\mathcal{V}}_M^m$ is continuous with respect to the d_M-distance. Next, pick $\rho\in\mathcal{R}_{p,M}$. By the definition of $\mathcal{R}_{p,M}$, there exists a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ and $(\Pi_j^k)_{j\in\mathbb{N}_0}$, $(b_j^k)_{j\in\mathbb{N}_0}$, $(x_j^k)_{j\in\mathbb{N}_0}$ such that $\rho=\lim_{k\to\infty}\rho_{N_k}(\Omega,\Pi^k,b^k,x^k)$. The continuity of T_m implies that

$$\rho\circ T_m^{-1}=\lim_{k\to\infty}\rho_{N_k}(\Omega,\Pi^k,b^k,x^k)\circ T_m^{-1}, \tag{F.11}$$

and we can easily check that

$$\rho_{N_k}(\Omega,\Pi^k,b^k,x^k)\circ T_m^{-1}=\rho_{N_k}(\Omega,\Pi^k,b^k,\tilde{x}^k), \tag{F.12}$$

where for $j,k\in\mathbb{N}_0$ we define

$$\tilde{x}_j^k=\begin{cases}x_j^k & \text{if}\quad (\Omega(j,\cdot),\Delta\Pi_j^k,b_j^k,b_{j+1}^k\tilde{x}_j^k)\in\overline{\mathcal{V}}_M^m,\\ 1 & \text{otherwise.}\end{cases} \tag{F.13}$$

Consequently, $\rho\circ T_m^{-1}\in\mathcal{R}_{p,M}$. □

We resume the proof of Lemma F.3. Pick $t>0$, $\rho\in\mathcal{R}_{p,M}$, $u\in\mathcal{B}_{\overline{\mathcal{V}}_M}$ and $v\in\mathcal{B}_{\overline{\mathcal{V}}_M}$ satisfying $v\leq u$ and $N(\rho,v)/D(\rho,v)\geq t$. Pick $m\in\mathbb{N}\colon m\geq M+2$, whose value will be specified later, and set $\rho_m=\rho\circ T_m^{-1}$, which belongs to $\mathcal{R}_{p,M}$ by Claim F.4. Write

$$\frac{N(\rho_m,v)}{D(\rho_m,v)}-\frac{N(\rho,v)}{D(\rho,v)}=\int_0^1 G'(t)dt \quad\text{with}\quad G(t)=\frac{A+tB}{c+tD} \tag{F.14}$$

with

$$A=\int_{\overline{\mathcal{V}}_M} v_\Theta\psi(\Theta,v_\Theta)\,\rho(d\Theta) \qquad B=\int_{\overline{\mathcal{V}}_M\setminus\overline{\mathcal{V}}_M^m} v_{\widetilde{\Theta}}\,\psi(\widetilde{\Theta},v_{\widetilde{\Theta}})-v_\Theta\psi(\Theta,v_\Theta)\,\rho(d\Theta) \tag{F.15}$$

$$C=\int_{\overline{\mathcal{V}}_M} v_\Theta\,\rho(d\Theta) \qquad D=\int_{\overline{\mathcal{V}}_M\setminus\overline{\mathcal{V}}_M^m} v_{\widetilde{\Theta}}-v_\Theta\,\rho(d\Theta). \tag{F.16}$$

Note that the sign of the derivative $G'(t)$ is constant and equal to the sign of

$$B - \frac{A}{C}D = \int_{\overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_M^m} v_\Theta \left[\frac{A}{C}\left(1 - \frac{v_{\widetilde{\Theta}}}{v_\Theta}\right) - \psi(\Theta, v_\Theta) + \frac{v_{\widetilde{\Theta}}}{v_\Theta}\, \psi(\widetilde{\Theta}, v_{\widetilde{\Theta}}) \right] \rho(d\Theta). \tag{F.17}$$

Therefore Lemma F.3 will be proven once we check that for m large enough the right-hand side of (F.17) is strictly positive, uniformly in $v \leq u$. To that aim, we recall Lemma C.6, which tells us that $\psi(\Theta, v_\Theta) \leq t/2$ for every $\Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_M^m$, provided m is chosen large enough (because $v_\Theta \geq t_\Theta \geq m$), and we recall (3.34), which tells us that $\psi(\widetilde{\Theta}, v_{\widetilde{\Theta}}) \leq C_{\mathrm{uf}}(\alpha)$ for $\Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_M^m$. We further note that

$$v_{\widetilde{\Theta}} \leq \max\left\{ u_\Theta \colon\ \Theta \in \overline{\mathcal{V}}_M^{M+2} \right\} < \infty \quad \text{for every } \Theta \in \overline{\mathcal{V}}_M, \tag{F.18}$$

which, together with the fact that $\frac{A}{C} = N(\rho, v)/D(\rho, v) \geq t > 0$ and $v_\Theta \geq t_\Theta \geq m$ for $\Theta \in \overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}_M^m$, ensures that for m large enough the right-hand side of (F.17) is strictly positive, uniformly in $v \leq u$. This completes the proof of Lemma F.3.

G Uniqueness of the maximizers of the variational formula

In this appendix we first prove, with the help of Lemma G.2, that for $\Theta \in \overline{\mathcal{V}}_M$ and $u \geq t_\Theta$ the variational formula in Proposition 3.5 has unique maximizers. This uniqueness implies that, for a given column type and a given time spent in the column, the copolymer has a unique way to move through the column. We next use this uniqueness to show, with the help of Proposition G.2, that for $u \in \mathcal{B}_{\overline{\mathcal{V}}_M}$ the maximizers of (3.45) are Borel functions of $\Theta \in \overline{\mathcal{V}}_M$.

Recall (3.57) and pick $h \in \mathcal{E}$. Set

$$\begin{aligned} \mathcal{U}(h) = \big\{ (r_{A,\Theta}, r_{B,\Theta}, r_{\mathcal{I},\Theta})_{\Theta \in \overline{\mathcal{V}}_M} \in ([0,\infty)^3)^{\overline{\mathcal{V}}_M} \colon\ & r_{k,\Theta} \geq 1 + \tfrac{l_{k,\Theta}}{h_{k,\Theta}}\ \forall\, k \in \{A,B\}\ \forall \Theta \in \overline{\mathcal{V}}_M, \\ & r_{\mathcal{I},\Theta} \geq 1\ \forall\, k \in \{A,B\}\ \forall \Theta \in \overline{\mathcal{V}}_M, \\ & \Theta \mapsto r_{k,\Theta} \text{ Borel } \forall\, k \in \{A,B,\mathcal{I}\} \big\}, \end{aligned} \tag{G.1}$$

where we recall that $\frac{l_{k,\Theta}}{h_{k,\Theta}} = 0$ by convention when $l_{k,\Theta} = h_{k,\Theta} = 0$.

Proposition G.1 *For all $u \in \mathcal{B}_{\overline{\mathcal{V}}_M}$ there exist $h \in \mathcal{E}$ and $r \in \mathcal{U}(h)$ such that, for all $\Theta \in \overline{\mathcal{V}}_M$,*

$$\begin{aligned} u_\Theta\, \psi(\Theta, u_\Theta) = {} & h_{A,\Theta}\, r_{A,\Theta}\, \tilde{\kappa}\big(r_{A,\Theta}, \tfrac{l_{A,\Theta}}{h_{A,\Theta}}\big) \\ & + h_{B,\Theta}\, r_{B,\Theta} \left[\tilde{\kappa}\big(r_{B,\Theta}, \tfrac{l_{B,\Theta}}{h_{B,\Theta}}\big) + \tfrac{\beta - \alpha}{2} \right] + h_{\mathcal{I},\Theta}\, r_{\mathcal{I},\Theta}\, \phi_{\mathcal{I}}(r_{\mathcal{I},\Theta}), \end{aligned} \tag{G.2}$$

and

$$h_{A,\Theta}\, r_{A,\Theta} + h_{B,\Theta}\, r_{B,\Theta} + h_{\mathcal{I},\Theta}\, r_{\mathcal{I},\Theta} = u_\Theta. \tag{G.3}$$

Proof. For $l \in \mathbb{R}$, let

$$\mathcal{N}_l = \{(a,h) \in [0,\infty) \times [0,1] \colon a \geq h + |l|\}, \quad \mathcal{N}_l^+ = \{(a,h) \in \mathcal{N}_l \colon h > 0\}, \tag{G.4}$$

let $g_l \colon \mathcal{N}_l \mapsto [0,\infty)$ be defined as $g_l(a,h) = a\, \tilde{\kappa}(\frac{a}{h}, \frac{l}{h})$ for $h > 0$ and $g_l(a,h) = 0$ for $h = 0$, and let $\tilde{g} \colon \mathcal{N}_0 \mapsto [0,\infty)$ be defined as $\tilde{g}(a,h) = a\, \phi_{\mathcal{I}}(\frac{a}{h})$ for $h > 0$ and $\tilde{g}(a,h) = 0$ for $h = 0$. We can rewrite (3.45) as

$$u\psi(\Theta, u; \alpha, \beta) = \sup_{(h),(a) \in \mathcal{L}(\Theta; u)} f_{l_A, l_B}\big[(h), (a)\big] \tag{G.5}$$

with

$$f_{l_A,l_B}\big[(h),(a)\big] = g_{l_A}(a_A,h_A) + g_{l_B}(a_B,h_B) + a_B\,\tfrac{\beta-\alpha}{2} + \tilde{g}(a_{\mathcal{I}},h_{\mathcal{I}}). \tag{G.6}$$

Lemma G.2 shows that, subject to some additional conditions, the maximizer in the right-hand side of (G.5) is unique. This allows us to prove the continuity of this maximizer as a function of Θ on each subset of a finite partition of $\overline{\mathcal{V}}_M$, which implies the Borel measurability of this maximizer and completes the proof of Proposition G.1.

Lemma G.2 *For all $\Theta \in \overline{\mathcal{V}}_M$ and $u \geq t_\Theta$ there exists a unique $(\bar{h}),(\bar{a}) \in \mathcal{L}(\Theta;u)$ satisfying:*
(i) $u\,\psi(\Theta,u;\alpha,\beta) = f_{l_A,l_B}[(\bar{h}),(\bar{a})]$.
(ii) $\bar{h}_k > 0$ if $\bar{a}_k > 0$ for $k \in \{A,B,\mathcal{I}\}$.
(iii) $\bar{a}_k = \bar{h}_k = 0$ if $\bar{l}_k = 0$ for $\Theta \in \overline{\mathcal{V}}_{\mathrm{int},M}$ and $k \in \{A,B\}$.
(iv) $\bar{a}_k = \bar{h}_k = 0$ if $\bar{l}_k = 0$ for $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},k,2,M}$ and $k \in \{A,B\}$.

Proof. We prove existence and uniqueness.

Existence. The existence of a $(h_1),(a_1) \in \mathcal{L}(\Theta;u)$ satisfying (i) is ensured by the continuity of f_{l_A,l_B} and the compactness of $\mathcal{L}(\Theta;u)$. Assume that $\Theta \in \overline{\mathcal{V}}_{\mathrm{int},M}$, $l_A = 0$ and $(h_{1,A},a_{1,A}) \neq (0,0)$. Then

$$\begin{aligned} g_0(a_{1,A},h_{1,A}) + \tilde{g}(a_{1,\mathcal{I}},h_{1,\mathcal{I}}) &\leq \tilde{g}(a_{1,A},h_{1,A}) + \tilde{g}(a_{1,\mathcal{I}},h_{1,\mathcal{I}}) \\ &\leq 2\,\tilde{g}\big(\tfrac{a_{1,A}+a_{1,\mathcal{I}}}{2},\tfrac{h_{1,A}+h_{1,\mathcal{I}}}{2}\big) = \tilde{g}\big(a_{1,A}+a_{1,\mathcal{I}},h_{1,A}+h_{1,\mathcal{I}}\big), \end{aligned} \tag{G.7}$$

where we use the inequality $g_0 \leq \tilde{g}$ and the concavity of $\tilde{g}$. Thus, by setting $(h_2),(a_2) = (0,h_{1,B},h_{1,A}+h_{1,\mathcal{I}}),(0,a_{1,B},a_{1,A}+a_{1,\mathcal{I}})$, we obtain that $(h_2),(a_2) \in \mathcal{L}(\Theta;u)$, satisfies (iii) and

$$f_{l_A,l_B}((h_2),(a_2)) \geq f_{l_A,l_B}((h_1),(a_1)), \tag{G.8}$$

which implies that $(h_2),(a_2)$ also satisfies (i). The case $\Theta \in \overline{\mathcal{V}}_{\mathrm{int},M}$, $l_B = 0$ and the case $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},k,2,M}$, $l_k = 0$, $k \in \{A,B\}$, can be treated similarly, to conclude that there exist $(h),(a) \in \mathcal{L}(\Theta;u)$ satisfying (i), (iii–iv). We will show that (ii) follows from these as well. The proof will be given for the case $\Theta \in \overline{\mathcal{V}}_{\mathrm{int},M}$ and $l_A,l_B > 0$, since (iii) already indicates that $h_k = a_k = 0$ if $l_k = 0$ for $k \in \{A,B\}$ and $\Theta \in \overline{\mathcal{V}}_{\mathrm{int},M}$. The case $\Theta \in \overline{\mathcal{V}}_{\mathrm{nint},M}$ can be treated similarly.

In the proof of Lemma C.4 we showed that $(h),(a) \in \mathcal{L}(\Theta,u)$ maximizing (G.5) necessarily satisfies $h_k > 0$ if $a_k > l_k$ for $k \in \{A,B\}$ and $h_{\mathcal{I}} > 0$ if $a_{\mathcal{I}} > 0$. Thus, we only need to exclude the cases $h_k = 0$ and $a_k = l_k > 0$ for $k \in \{A,B\}$. We will therefore assume that $h_B = 0$ and $a_B = l_B$, and prove that this leads to a contradiction. The case $h_A = 0$ and $a_A = l_A$ is easier to deal with. We finally assume that $a_{\mathcal{I}} > h_{\mathcal{I}} > 0$ (the case $a_{\mathcal{I}} = h_{\mathcal{I}}$ being easier). We pick $c > 1$ and $x > 0$ small enough to ensure that $a_{\mathcal{I}} - cx > h_{\mathcal{I}} - x > 0$, and we set $(h)_x,(a)_x = (h_A,x,h_{\mathcal{I}}-x),(a_A,l_B+cx,a_I-cx)$. The proof will be complete once we show that for x small enough the quantity

$$f_{l_A,l_B}((h)_x,(a)_x) - f_{l_A,l_B}((h),(a)) = g_{l_B}(l_B+cx,x) - V_x + cx\left(\tfrac{\beta-\alpha}{2}\right) \tag{G.9}$$

is strictly positive with $V_x = \tilde{g}(a_{\mathcal{I}},h_{\mathcal{I}}) - \tilde{g}(a_{\mathcal{I}}-cx,h_{\mathcal{I}}-x)$.

At this stage, we note that $\mu \mapsto \mu\phi_{\mathcal{I}}(\mu)$ is concave on $[1,\infty)$, and therefore is Lipshitz on any interval $[r,t]$ with $r > 1$. Since $a_{\mathcal{I}}/h_{\mathcal{I}} > 0$, there exists a $C > 0$, depending on $(a_{\mathcal{I}},h_{\mathcal{I}})$ only, such that $V_x \leq Cx$ for x small enough. Therefore (G.9) becomes

$$f_{l_A,l_B}((h)_x,(a)_x) - f_{l_A,l_B}((h),(a)) \geq g_{l_B}(l_B+cx,x) - \big(C + c\,\tfrac{\beta-\alpha}{2}\big)\,x \tag{G.10}$$

for x small enough. By the concavity of g_{l_B}, and since $g_{l_B}(l_B+cx,0)=0$, we can write $g_{l_B}(l_B+cx,x)\geq x\,\partial_2 g_{l_B}(l_B+cx,x)$ for $x>0$. By the definition of g_{l_B}, and with (3.4), we obtain that

$$\partial_2 g_{l_B}(l_B+cx,x)=\big(1+\tfrac{cx}{l_B}\big)\partial_2\kappa\big(1+\tfrac{cx}{l_B},\tfrac{cx}{l_B}\big). \tag{G.11}$$

We now recall [5], Lemma 2.1.1, which claims that κ is defined on $\text{DOM}=\{(a,b)\colon a\geq 1+b, b\geq 0\}$ and is analytic on the interior of DOM. Moreover, in the proof of this lemma, an expression for $\partial_b\,\kappa(a,b)$ is provided that is valid on the interior of DOM. From this expression, and since $c>1$, we can check that $\lim_{s\downarrow 0}\partial_2\kappa(1+cs,s)=\infty$, which suffices to conclude that the right-hand side of (G.9) is strictly positive for x small enough. This completes the proof of the existence in Lemma G.2.

Uniqueness. The uniqueness of $(\bar{h}),(\bar{a})$ is a straightforward consequence of the strict concavity of g_{l_A} and g_{l_B} when $l_A\neq 0$ and $l_B\neq 0$ and of the concavity of g_0 and $\widetilde{g}$. We will not write out the proof in detail, because it requires us to distinguish between the cases $\Theta\in\overline{\mathcal{V}}_{\text{int},M}$ and $\Theta\in\overline{\mathcal{V}}_{\text{nint},M}$, between $l_k=0$ and $l_k\neq 0$, $k\in\{A,B\}$, and also between $x_\Theta=1$ and $x_\Theta=2$. The latter distinctions are tedious, but no technical diffulties arise. □

We resume the proof of Proposition G.1. We pick $u\in\mathcal{B}_{\overline{\mathcal{V}}_M}$, and for each $\Theta\in\overline{\mathcal{V}}_M$ we apply Lemma G.2 at Θ,u_Θ, to obtain a $(\bar{h})_\Theta,(\bar{a})_\Theta\in\mathcal{L}(\Theta;u_\Theta)$ satisfying (i–iv). We set $(\bar{h})\colon\,\Theta\in\overline{\mathcal{V}}_M\mapsto\bar{h}_\Theta$ and $(\bar{a})\colon\,\Theta\in\overline{\mathcal{V}}_M\mapsto\bar{a}_\Theta$, and we recall (3.57). If we can show that $\Theta\mapsto(\bar{h})_\Theta$ is Borel, then it follows that $(\bar{h})\in\mathcal{E}$, because (ii) and the fact that $(\bar{h})_\Theta,(\bar{a})_\Theta\in\mathcal{L}(\Theta;u_\Theta)$ for $\Theta\in\overline{\mathcal{V}}_M$ ensure that the other conditions required to belong to $\mathcal{E}$ are fulfilled by $(\bar{h})$. Moreover, if we can we show that $\Theta\mapsto(\bar{a})_\Theta$ is Borel, then the proof of Proposition G.1 will be complete, because we can set

$$(\bar{r}_A(\Theta),\bar{r}_B(\Theta),\bar{r}_{\mathcal{I}}(\Theta))=\Big(\tfrac{\bar{a}_A(\Theta)}{\bar{h}_A(\Theta)},\tfrac{\bar{a}_B(\Theta)}{\bar{h}_B(\Theta)},\tfrac{\bar{a}_{\mathcal{I}}(\Theta)}{\bar{h}_{\mathcal{I}}(\Theta)}\Big),\quad \Theta\in\overline{\mathcal{V}}_M, \tag{G.12}$$

with the convention $\bar{r}_k(\Theta)=1$ when $\bar{a}_k(\Theta)=\bar{h}_k(\Theta)=0$ for $k\in\{A,B,\mathcal{I}\}$, after which $(\bar{r})\in\mathcal{U}(h)$ and $(\bar{h}),(\bar{r})$ satisfy (G.2) and (G.3).

To complete the proof it remains to show that $\Theta\mapsto(\bar{h})_\Theta,(\bar{a})_\Theta$ is Borel. Recall the partition

$$\overline{\mathcal{V}}_M=\overline{\mathcal{V}}_{\text{int},M}\cup\big(\cup_{(x,k)\in\{1,2\}\times\{A,B\}}\overline{\mathcal{V}}_{\text{int},k,x,M}\big), \tag{G.13}$$

and partition these five subsets in the right-hand side of (G.13) into smaller subsets depending on the values taken by l_A and l_B. For $\overline{\mathcal{V}}_{\text{int},M}$, this gives

$$\begin{aligned}\overline{\mathcal{V}}_{\text{int},M}=&\{\Theta\in\overline{\mathcal{V}}_{\text{int},M}\colon l_A,l_B>0\}\cup\{\Theta\in\overline{\mathcal{V}}_{\text{int},M}\colon l_A>0,l_B=0\}\\&\cup\{\Theta\in\overline{\mathcal{V}}_{\text{int},M}\colon l_A=0,l_B>0\}\cup\{\Theta\in\overline{\mathcal{V}}_{\text{int},M}\colon l_A=l_B=0\},\end{aligned} \tag{G.14}$$

and on each of these subsets the fact that $(\bar{h})_\Theta,(\bar{a})_\Theta$ are the unique elements in $\mathcal{L}(\Theta;u_\Theta)$ satisfying (i–iv) implies that $\Theta\mapsto(\bar{h})_\Theta,(\bar{a})_\Theta$ are continuous and therefore Borel. Since each subsets in the right-hand side of (G.14) belongs to the Borel σ-field generated by d_M (recall (C.7)), we can conclude that $\Theta\mapsto(\bar{h})_\Theta,(\bar{a})_\Theta$ are Borel on $\overline{\mathcal{V}}_M$. This completes the proof of Proposition G.1. □

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