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R. Fernández, F. den Hollander, J. Martínez
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R. Fernández [1]
F. den Hollander [2]
J. Martínez [3]

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Abstract

We continue our study of Gibbs-non-Gibbs dynamical transitions. In the present paper we consider a system of Ising spins on a large discrete torus with a Kac-type interaction subject to an independent spin-flip dynamics (infinite-temperature Glauber dynamics). We show that, in accordance with the program outlined in [11], in the thermodynamic limit Gibbs-non-Gibbs dynamical transitions are *equivalent* to bifurcations in the set of global minima of the large-deviation rate function for the trajectories of the empirical density *conditional* on their endpoint. More precisely, the time-evolved measure is non-Gibbs if and only if this set is not a singleton for *some* value of the endpoint. A partial description of the possible scenarios of bifurcation is given, leading to a characterization of passages from Gibbs to non-Gibbs and vice versa, with sharp transition times.

Our analysis provides a conceptual step-up from our earlier work on Gibbs-non-Gibbs dynamical transitions for the Curie-Weiss model, where the mean-field interaction allowed us to focus on trajectories of the empirical magnetization rather than the empirical density.

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[1]Department of Mathematics, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, The Netherlands, *R.Fernandez1@uu.nl*
[2]Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA, Leiden, The Netherlands, *denholla@math.leidenuniv.nl*
[3]Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA, Leiden, The Netherlands, *martinez@math.leidenuniv.nl*

1 Introduction and main results

1.1 Background

Gibbs-non-Gibbs dynamical transitions are a surprising phenomenon. An initial Gibbsian state (e.g. a collection of interacting Ising spins) is subjected to a stochastic dynamics (e.g. a Glauber dynamics) at a temperature that is *different* from that of the initial state. For many combinations of initial and dynamical temperature, the time-evolved state is observed to become non-Gibbs after a finite time. Such a state cannot be described by any absolutely summable Hamiltonian and therefore *lacks a well-defined notion of temperature*.

The phenomenon was originally discovered by van Enter, Fernández, den Hollander and Redig [10] for *heating dynamics*, in which a low-temperature Ising model is subjected to a high-temperature Glauber dynamics. The state remains Gibbs for short times, but becomes non-Gibbs after a finite time. Remarkably, heating in this case does not lead to a succession of states with increasing temperature, but to states where the notion of temperature is *lost altogether*. Moreover, it turned out that there is a difference depending on whether the initial Ising model has zero or non-zero magnetic field. In the former case, non-Gibbsianness once lost is never recovered, while in the latter case Gibbsianness is recovered at a later time.

This initial work triggered a decade of developments. By now, results are available for a variety of interacting particle systems, both for *heating dynamics* and for *cooling dynamics*, including estimates on transition times and characterizations of the so-called *bad configurations* leading to non-Gibbsianness, i.e., the discontinuity points of the conditional probabilities. It has become clear that Gibbs-non-Gibbs transitions are the rule rather than the exception. For references we refer to the recent overview by van Enter [9].

1.2 Motivation and outline

The ubiquity of the Gibbs-non-Gibbs phenomenon calls for a better understanding of its causes and consequences. Historically, non-Gibbsianness is proved by looking at the evolving system at two times, the inital time and the final time, and applying techniques from equilibrium statistical mechanics. This is an indirect approach that does not illuminate the relation between the Gibbs-non-Gibbs phenomenon and the dynamical effects responsible for its occurrence. This unsatisfactory situation was addressed in Enter, Fernández, den Hollander and Redig [11], where possible dynamical mechanisms were proposed and a *program* was put forward to develop a theory of Gibbs-non-Gibbs transitions on *purely dynamical grounds*.

In Fernández, den Hollander and Martínez [13], building on earlier work by Külske and Le Ny [15] and Ermolaev and Külske [12], we showed that this program can be fully carried out for the Curie-Weiss model subject to an infinite-temperature dynamics. The goal of the present paper is to extend this work away from the mean-field setting by considering a model with a Kac-type interaction, i.e., Ising spins with a long-range interaction. Whereas for the Curie-Weiss model the key object was the empirical magnetization in the thermodynamic limit, for the Kac model the key object is the *empirical density* in the thermodynamic limit, which we refer to as the *profile*. Non-Gibbsianness corresponds to a discontinuous dependence of the law of the initial profile *conditional* on the final profile. The discontinuity points are called *bad profiles* (Definition 1.1 below).

Dynamically, such discontinuities are expected to arise whenever there is more than one trajectory of the profile that is *compatible* with the bad profile at the end. Indeed, this expec-

tation is confirmed and exploited in the sequel. The actual conditional trajectories are those minimizing the large-deviation rate function on the space of trajectories (Propositions 1.2–1.3 below), in the spirit of what is behind hydrodynamic scaling. The time-evolved measure is Gibbs whenever there is a single minimizing trajectory for every final profile, in which case the so-called specification kernel can be computed explicitly (Theorem 1.4 below). In contrast, if there are multiple optimal trajectories, then the choice of trajectory can be decided by an infinitesimal perturbation of the final profile, and the time-evolved measure is non-Gibbs (Theorem 1.6 below).

The rate function for the Kac model contains an action integral whose Lagrangian acts on profiles. This setting constitutes a conceptual step-up from what happens for the Curie-Weiss model, where the Lagrangian acts on magnetizations and is much easier to analyze. However, for infinite-temperature dynamics the Kac Lagrangian can be expressed as an integral of the Curie-Weiss Lagrangian with respect to the profile (Theorem 1.5 below). This link allows us to identify the possible scenarios of bifurcation (Theorem 1.7 below).

1.3 Hamiltonian

Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional unit torus. For $n \in \mathbb{N}$, let $\mathbb{T}^d_n$ be the $(1/n)$-discretization of $\mathbb{T}^d$ defined by $\mathbb{T}^d_n := \Delta^d_n/n$, with $\Delta^d_n := \mathbb{Z}^d/n\mathbb{Z}^d$ the discrete torus of size n. For $n \in \mathbb{N}$, let $\Omega_n := \{-1,+1\}^{\Delta^d_n}$ be the set of Ising-spin configurations on Δ^d_n. The energy of the configuration $\sigma := (\sigma(x))_{x\in\Delta^d_n} \in \Omega_n$ is given by the *Kac-type Hamiltonian*

$$H^n(\sigma) := -\tfrac{1}{2n^d} \sum_{x,y\in\Delta^d_n} J\left(\tfrac{x-y}{n}\right) \sigma(x)\sigma(y) - \sum_{x\in\Delta^d_n} h(\tfrac{x}{n})\,\sigma(x), \qquad \sigma \in \Omega_n, \tag{1.1}$$

where $J, h \in C(\mathbb{T}^d)$ are continuous functions on $\mathbb{T}^d$, with $J \geq 0$ symmetric and $J \not\equiv 0$. The Gibbs measure associated with H^n is

$$\mu^n(\sigma) := \frac{e^{-\beta H^n(\sigma)}}{Z^n}, \qquad \sigma \in \Omega_n, \tag{1.2}$$

with $\beta \in [0,\infty)$ the *static* inverse temperature and Z^n the normalizing partition sum.

1.4 Gibbs versus non-Gibbs

For $\Lambda \subseteq \Delta^d_n$, let $\pi^n_\Lambda\colon\ \Omega_n \to \mathcal{M}(\mathbb{T}^d_n) \subseteq \mathcal{M}(\mathbb{T}^d)$ be the *empirical density* of σ inside Λ defined by

$$\pi^n_\Lambda(\sigma) := \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sigma(x)\delta_{x/n}, \tag{1.3}$$

where $\mathcal{M}(\mathbb{T}^d_n)$ and $\mathcal{M}(\mathbb{T}^d)$ denote the set of signed measures on $\mathbb{T}^d_n$, respectively, $\mathbb{T}^d$ with total variation norm ≤ 1 endowed with the weak topology, and δ_u is the point measure at $u \in \mathbb{T}^d$. Note that $\sigma \in \Omega_n$ determines $\pi^n_\Lambda \in \mathcal{M}(\mathbb{T}^d_n)$ and vice versa.

Abbreviate (1.3) for $\Lambda = \Delta^d_n$ by π^n and for $\Lambda = \Delta^d_n\backslash\{\lfloor nu\rfloor\}$ by $\pi^{u,n}$, $u \in \mathbb{T}^d$, where $\lfloor nu\rfloor$ denotes the component-wise lower-integer part of nu. The latter is the empirical density *perforated* at $\lfloor nu\rfloor$. Abbreviate

$$\mathcal{M}^n := \pi^n(\Omega_n), \qquad \mathcal{M}^{u,n} := \pi^{u,n}(\Omega_n). \tag{1.4}$$

Note that $\mathcal{M}^n \subseteq \mathcal{M}(\mathbb{T}_n^d)$. Via π^n, the Gibbs measure μ^n on Ω_n in (1.2) induces a probability measure $\check{\mu}^n$ on $\mathcal{M}^n$ given by

$$\check{\mu}^n = \mu^n \circ (\pi^n)^{-1}. \tag{1.5}$$

Using (1.3), we can rewrite (1.1) in the form

$$H^n(\sigma) = -n^d H(\pi^n(\sigma)), \tag{1.6}$$

where in the right-hand side we introduce the notation

$$H(\nu) = \left\langle \tfrac{1}{2} J * \nu + h, \nu \right\rangle \tag{1.7}$$

$$[f * \nu](u) := \int_{\mathbb{T}^d} J(u-u')\,\nu(du'), \qquad \langle f, \nu\rangle := \int_{\mathbb{T}^d} f(u)\,\nu(du), \qquad f \in C(\mathbb{T}^d),\ \nu \in \mathcal{M}(\mathbb{T}^d). \tag{1.8}$$

Let $\lambda^n := \frac{1}{n^d}\sum_{x\in\Lambda}\delta_{x/n}$. We have $w - \lim_{n\to\infty}\lambda^n = \lambda$, where λ is the Lebesgue measure on $\mathbb{T}^d$ and $w-\lim$ stands for weak convergence. In what follows we will represent limit distributions in $\mathcal{M}(\mathbb{T}^d)$ with a Lebesgue density as measures $\alpha\lambda$ with $\alpha \in B$, where

$$B \text{ is the closed unit ball in } L^\infty(\mathbb{T}^d). \tag{1.9}$$

We will refer to α as a *profile*.

The definition of Gibbs versus non-Gibbs is the following. Given any sequence $(\rho^n)_{n\in\mathbb{N}}$ with ρ^n a probability measure on Ω_n for every $n \in \mathbb{N}$, define the single-spin conditional probabilities at site $\lfloor nu \rfloor \in \mathbb{T}^d$ as

$$\gamma^{u,n}\big(\,\cdot \mid \alpha_{n-1}^u\big) := \rho^n\big(\sigma(\lfloor nu\rfloor) = \cdot \mid \pi^{u,n}(\sigma) = \alpha_{n-1}^u\big), \qquad \alpha_{n-1}^u \in \mathcal{M}^{u,n}. \tag{1.10}$$

Definition 1.1. [Good and bad profiles, Gibbs]
(a) A profile $\alpha \in B$ is called good for $(\rho^n)_{n\in\mathbb{N}}$ if there exists a neighborhood $\mathcal{N}_\alpha$ of α in $L^\infty(\mathbb{T}^d)$ such that

$$\gamma^u\big(\,\cdot \mid \tilde{\alpha}\big) := \lim_{n\to\infty} \gamma^{u,n}(\,\cdot \mid \alpha_{n-1}^u) \tag{1.11}$$

exists for all $\tilde{\alpha} \in \mathcal{N}_\alpha$ and $u \in \mathbb{T}^d$, for all sequences $(\alpha_{n-1}^u)_{n\in\mathbb{N}}$ with $\alpha_{n-1}^u \in \mathcal{M}^{u,n}$ for every $n \in \mathbb{N}$ such that $w - \lim_{n\to\infty}\alpha_{n-1}^u = \tilde{\alpha}\lambda$, and the limit is independent of the choice of $(\alpha_{n-1}^u)_{n\in\mathbb{N}}$.
(b) A profile $\alpha \in B$ is called bad for $(\rho^n)_{n\in\mathbb{N}}$ if it is not good for $(\rho^n)_{n\in\mathbb{N}}$.
(c) $(\rho^n)_{n\in\mathbb{N}}$ is called Gibbs if it has no bad profiles in B.

Remark:
(1) Definition 1.1(a) implies continuity of $\alpha \mapsto \gamma^u(\,\cdot \mid \alpha)$ in the $L^\infty(\mathbb{T}^d)$-norm for all $u \in \mathbb{T}^d$ at good profiles. (A proof by contradiction is based on a diagonal argument.)
(2) For $(\mu^n)_{n\in\mathbb{N}}$ with μ^n defined in (1.1–1.2) all profiles $\alpha \in B$ are good with

$$\gamma^u(k \mid \alpha) = \frac{\exp[k\beta\{J*\alpha+h\}(u)]}{2\cosh[\beta\{J*\alpha+h\}(u)]}, \qquad k \in \{-1,+1\},\ \alpha \in B,\ u \in \mathbb{T}^d. \tag{1.12}$$

(The factor $\frac{1}{2}$ in (1.7) drops out because every spin is counted twice in the Hamiltonian but once in the convolution.) In particular, $(\mu^n)_{n\in\mathbb{N}}$ is Gibbs in the sense of Definition 1.1(c).

(3) Definition 1.1 assigns the notion of Gibbs to a sequence of probability measures that live on different spaces. It is different from the classical notion of Gibbs based on the Dobrushin-Lanford-Ruelle condition, which is used to define Gibbs measures on infinite lattices. Nonetheless, the quantity in (1.12) can be viewed as some sort of *specification kernel*.
(4) Definition 1.1 does not consider sequences $(\alpha^u_{n-1})_{n\in\mathbb{N}}$ whose weak limit is singular with respect to λ. In Proposition 1.2 below we will see that in the thermodynamic limit we can ignore trajectories that do not lie in the set $\{\alpha\lambda\colon\ \alpha\in B\}$ because they are too costly.

1.5 Stochastic dynamics

For fixed n, we let the spin configuration evolve according to a Glauber dynamics with generator L_n given by

$$(L_nf)(\sigma) := \sum_{x\in\Delta_n^d} c_n(x,\sigma)\,[f(\sigma^x)-f(\sigma)], \qquad f\colon\ \Omega_n\to\mathbb{R}, \tag{1.13}$$

where the spin-flip rate takes the form

$$c_n(x,\sigma) := \frac{\exp[-\frac{\beta'}{2}\{H^n(\sigma^x)-H^n(\sigma)\}]}{2\cosh[\frac{\beta'}{2}\{H^n(\sigma^x)-H^n(\sigma)\}]} \tag{1.14}$$

with σ^x the configuration obtained from σ by flipping the spin at site x, and $\beta'\in[0,\infty)$ the *dynamical* inverse temperature. We write $(\sigma_s)_{s\geq 0}$ to denote the trajectory of the spin configuration, which lives on $D_{[0,\infty)}(\Omega_n)$, the space of càdlàg paths on Ω_n endowed with the Skorohod topology.

Abbreviate $\pi^n_s := \pi^n(\sigma_s)$, and let $\bar\pi^n = (\pi^n_s)_{s\geq 0}$ denote the trajectory of the empirical density under the Glauber dynamics. For a given probability measure $\check\rho^n_0$ on $\mathcal{M}^n$ we define

$$P^n_{\check\rho^n_0} := \text{ law of } (\pi^n_s)_{s\geq 0} \text{ conditional on } \pi^n_0 \text{ being drawn according to } \check\rho^n_0, \tag{1.15}$$

which lives on $D_{[0,\infty)}(\mathcal{M}^n)$, the space of càdlàg paths on $\mathcal{M}^n$ endowed with the Skorohod topology.

1.6 Large deviation principles

For $t\geq 0$, we say that $\varphi=(\varphi_s)_{s\in[0,t]}\in C_{[0,t]}(B)$ is absolutely continuous in time when

$$\exists\,\dot\varphi=(\dot\varphi_s)_{s\in[0,t]}\in L^1_{[0,t]}(\mathbb{T}^d)\colon \quad \varphi_s(u)-\varphi_0(u)=\int_0^s \dot\varphi_r(u)\,dr \quad \forall\, s\in[0,t],\ \lambda-a.e.\ u. \tag{1.16}$$

Let us recall that a family of probability measures $(\nu^n)_{n\in\mathbb{N}}$ on a Polish space $\mathcal{X}$ satisfies a *large deviation principle* (LDP) with rate n and rate function I when $I\colon\ \mathcal{X}\to[0,\infty]$ has compact level sets, is not identically infinite, and

$$\begin{aligned} \liminf_{n\to\infty}\frac{1}{n}\log\nu^n(O) &\geq -\inf_{x\in O} I(x), \qquad O\subseteq\mathcal{X} \text{ open},\\ \limsup_{n\to\infty}\frac{1}{n}\log\nu^n(C) &\leq -\inf_{x\in C} I(x), \qquad C\subseteq\mathcal{X} \text{ closed}. \end{aligned} \tag{1.17}$$

(See Dembo and Zeitouni [5, Section 1.2].) The following LDPs can be found in Comets [3].

Proposition 1.2. *(i)* **[LDP for initial Gibbs measure]** $(\check{\mu}^n)_{n\in\mathbb{N}}$ *satisfies the LDP on* $\mathcal{M}(\mathbb{T}^d)$ *with rate* n^d *and rate function* $I_S - \inf_{\mathcal{M}(\mathbb{T}^d)} I_S$ *given by*

$$I_S(\nu) := \begin{cases} -\beta\big\langle \frac{1}{2} J * \alpha + h, \alpha\lambda\big\rangle + \langle \Phi \circ \alpha, \lambda\rangle, & \text{if } \nu = \alpha\lambda \text{ with } \alpha \in B, \\ \infty, & \text{otherwise}, \end{cases} \tag{1.18}$$

where Φ *is the relative entropy*

$$\Phi(m) := \tfrac{1+m}{2}\,\log(1+m) + \tfrac{1-m}{2}\,\log(1-m), \qquad m \in [-1,+1]. \tag{1.19}$$

(ii) **[Dynamical LDP for deterministic initial law]** *Let* $t \geq 0$ *and* $\alpha \in C(\mathbb{T}^d)$*, and let* $(\varphi_0^n)_{n\in\mathbb{N}}$ *be any sequence with* $\varphi_0^n \in \mathcal{M}^n$ *for every* $n \in \mathbb{N}$ *such that* $w - \lim_{n\to\infty} \varphi_0^n = \alpha\lambda$*. Then*

$$\Big(P^n_{\delta_{\varphi_0^n}}\Big)_{n\in\mathbb{N}} \text{ restricted to } [0,t] \tag{1.20}$$

satisfies the LDP on $D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ *with rate* n^d *and rate function* $I_D^t - \inf_{D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))} I_D^t$ *given by*

$$I_D^t(\psi) := \begin{cases} \int_0^t \mathcal{L}\big(\varphi_s, \dot{\varphi}_s\big)\,ds, & \text{if } \psi = \varphi\lambda, \text{ with } \varphi \text{ satisfying property (1.16) and } \varphi_0 \equiv \alpha, \\ \infty, & \text{otherwise}, \end{cases} \tag{1.21}$$

where

$$\mathcal{L}(p,q) := \int_{\mathbb{T}^d} L[p(u), q(u)]\,du, \qquad p \in B,\ q \in L^1(\mathbb{T}^d), \tag{1.22}$$

with

$$\begin{aligned} L[p(u),q(u)] = \tfrac{q(u)}{2}\log\left[\frac{\frac{q(u)}{2} + \sqrt{1-p(u)^2 + \big[\frac{q(u)}{2}\big]^2}}{1-p(u)}\right] - \tfrac{q(u)}{2}\,\big[\beta'(J*p+h)\big](u) \\ + \Bigg\{ -\sqrt{1-p(u)^2 + \Big[\tfrac{q(u)}{2}\Big]^2} \\ + \cosh\big[\beta'(J*p+h)\big](u) - p(u)\sinh\big[\beta'(J*p+h)\big](u)\Bigg\}. \end{aligned} \tag{1.23}$$

Note that (1.23) simplifies considerably when $\beta' = 0$ (independent spin-flip dynamics).

To ease notation, we write $I_S(\alpha)$ instead of $I_S(\nu)$ when $\nu = \alpha\lambda$ with $\alpha \in B$, and $I_D^t(\varphi)$ instead of $I_D^t(\psi)$ when $\psi = \varphi\lambda$ with $\varphi \in C_{[0,t]}(B)$, i.e., we henceforth suppress the reference measure λ from the notation.

Let $P^n = P^n_{\check{\mu}^n}$. Define

$$Q^n_{t,\alpha'}(\cdot) := P^n\big((\pi^n_s)_{s\in[0,t]} \in \cdot \mid \pi^n_t = \alpha'_n\big), \qquad t \geq 0,\ \alpha' \in B, \tag{1.24}$$

with $\alpha'_n \in \mathcal{M}^n$ the element closest to $\alpha' \in B$ in any metric that metrizes the weak topology. The following LDPs are key to our analysis. In what follows we write $f \equiv g$ when $f(u) = g(u)$ for all $u \in \mathbb{T}^d$.

Proposition 1.3. [Dynamical LDP for Gibbs initial law]
(i) For every $t \geq 0$, $(P^n)_{n\in\mathbb{N}}$ satisfies the LDP on $D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ with rate n^d and rate function $I^t - \inf_{D_{[0,T]}(\mathcal{M}(\mathbb{T}^d))} I^t$ given by

$$I^t(\varphi) := I_S(\varphi_0) + I_D^t(\varphi). \tag{1.25}$$

(ii) For every $t \geq 0$ and $\alpha' \in B$, $(Q^n_{t,\alpha'})_{n\in\mathbb{N}}$ satisfies the LDP on $D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ with rate n^d and rate function $I^{t,\alpha'} - \inf_{D_{[0,t](\mathcal{M}(\mathbb{T}^d))}} I^{t,\alpha'}$ given by

$$I^{t,\alpha'}(\varphi) := \begin{cases} I^t(\varphi), & \text{if } \varphi_t \equiv \alpha', \\ \infty, & \text{otherwise.} \end{cases} \tag{1.26}$$

The proof of Proposition 1.3 is given in Appendix A and is based on large deviation techniques coming from hydrodynamic scaling. A somewhat delicate issue is the fact that we cannot use Proposition 1.2(ii) because this has a deterministic initial condition, while in Proposition 1.3(i) the initial condition is random.

Note that, by (1.18), (1.21) and (1.25–1.26),

$$\inf_{\varphi\in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))} I^{t,\alpha'}(\varphi) = \inf_{\alpha\in B} \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\, \varphi_t\equiv\alpha'}} I^t(\varphi) = \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_t\equiv\alpha'}} I^t(\varphi). \tag{1.27}$$

1.7 Link to the specification kernel

Henceforth we only consider trajectories $\varphi \in C_{[0,t]}(B)$ satisfying (1.16), because the rate functions are infinite otherwise. The following theorem provides the fundamental link between the specification kernel in (1.11) and the minimizer of (1.27) when it is *unique*.

Theorem 1.4. [Specification kernel in absence of bifurcation] *Fix $t \geq 0$ and $\alpha' \in B$. Suppose that (1.27) has a unique minimizing path $\hat\varphi^{t,\alpha'} = (\hat\varphi^{t,\alpha'}_s)_{s\in[0,t]}$. Then the specification kernel at time t equals*

$$\gamma^u_t(k' \mid \alpha') := \frac{\sum_{k\in\{-1,+1\}} \exp\left[k\beta\{J * \hat\varphi^{t,\alpha'}_0 + h\}(u)\right] p^{u,t,\alpha'}_t(k,k')}{\sum_{j,j'\in\{-1,+1\}} \exp\left[j\beta\{J * \hat\varphi^{t,\alpha'}_0 + h\}(u)\right] p^{u,t,\alpha'}_t(j,j')}, \qquad k' \in \{-1,+1\},\ u \in \mathbb{T}^d, \tag{1.28}$$

where $p^{u,t,\alpha'}_t(j,j')$ is the probability to go from j at time 0 to j' at time t in the time-inhomogeneous Markov process on $\{-1,+1\}$ with generator $L^{u,t,\alpha'}_s$ at time $s \in [0,t]$ given by

$$\begin{aligned} (L^{u,t,\alpha'}_s f)(k) &= \frac{\exp\left[k\beta'\{J * \hat\varphi^{t,\alpha'}_s + h\}(u)\right]}{2\cosh\left[\beta'\{J * \hat\varphi^{t,\alpha'}_s + h\}(u)\right]}\,[f(-k) - f(k)], \\ & k \in \{-1,+1\},\ f\colon \{-1,+1\} \to \mathbb{R},\ u \in \mathbb{T}^d,\ s \in [0,t]. \end{aligned} \tag{1.29}$$

Remark: Note that for $\beta' = 0$ (independent spin-flip dynamics) the right-hand side of (1.29) simplifies to $\frac{1}{2}[f(-k) - f(k)]$ and that, consequently, the right-hand side of (1.28) depends on the optimal trajectory $\hat\varphi^{t,\alpha'}$ only via its initial value $\hat\varphi^{t,\alpha'}_0$, and takes the form

$$\gamma^u_t(k' \mid \alpha') = \Gamma_t\big(k', \beta\{J * \hat\varphi^{t,\alpha'}_0 + h\}(u)\big) \tag{1.30}$$

for some $\Gamma_t\colon \{-1,+1\} \times \mathbb{R} \to [0,1]$, with the property that $m \mapsto \Gamma_t(k',m)$ is continuous, strictly increasing for $k' = +1$ and strictly decreasing for $k' = -1$.

1.8 Reduction: critical trajectories

In what follows we restrict ourselves to the case of infinite-temperature dynamics, i.e., $\beta' = 0$. Let

$$\begin{aligned} \hat{\varphi}^{\alpha;t,\alpha'} &:= \operatorname{argmin}_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\,\varphi_t\equiv\alpha'}} I^t(\varphi), \\ C_{t,\alpha'}(\alpha) &:= I^t(\hat{\varphi}^{\alpha;t,\alpha'}). \end{aligned} \tag{1.31}$$

Remark: Note that

$$\inf_{\alpha\in B} C_{t,\alpha'}(\alpha) = \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_t\equiv\alpha'}} I^t(\varphi). \tag{1.32}$$

The following theorem says that $\hat{\varphi}^{\alpha;t,\alpha'}$ is unique for every $t \geq 0$ and $\alpha, \alpha' \in B$, and can be computed because the Kac model can be linked to the Curie-Weiss model treated in Fernández, den Hollander and Martínez [13]. (In the notation of that paper β is absorbed into J, h.)

Theorem 1.5. [Critical trajectories] *Let $\beta' = 0$. For every $t \geq 0$ and $\alpha, \alpha' \in B$,*

$$\hat{\varphi}^{\alpha;t,\alpha'}_s(u) = \hat{\varphi}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s), \qquad u \in \mathbb{T}^d,\ s \in [0,t], \tag{1.33}$$

where $\hat{\varphi}^{\mathrm{CW};m}_{t,m'}(s)$, $s \in [0,t]$, is the unique trajectory in $[-1,+1]$ between magnetization m at time 0 and magnetization m' at time t for the Curie-Weiss model. Accordingly (see (1.21–1.23) *and* 1.25–1.26)*),*

$$C_{t,\alpha'}(\alpha) = I_S(\alpha) + \int_{\mathbb{T}^d} du \int_0^t ds\ L^{\mathrm{CW}}\left[\hat{\varphi}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s), \dot{\hat{\varphi}}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s)\right], \tag{1.34}$$

where L^{CW} is the Lagrangian of the Curie-Weiss model. The critical points of (1.34) *(i.e., the local minima and the local maxima) satisfy the functional equation*

$$\sinh[2\beta(J*\alpha+h)](u) - \alpha(u)\cosh[2\beta(J*\alpha+h)](u) = \frac{\alpha(u)}{\tanh(2t)} - \frac{\alpha'(u)}{\sinh(2t)} \quad a.e.\ u \in \mathbb{T}^d. \tag{1.35}$$

In Theorem 1.5, the Lagrangian of the Curie-Weiss model is given by

$$L^{\mathrm{CW}}(m,\dot{m}) := -\tfrac{1}{2}\sqrt{4\,(1-m^2)+\dot{m}^2} + \tfrac{1}{2}\dot{m}\log\left(\frac{\sqrt{4\,(1-m^2)+\dot{m}^2}+\dot{m}}{2(1-m)}\right) + 1, \tag{1.36}$$

which is the same as (1.23) with $\beta' = 0$, $p(\cdot) = m$ and $q(\cdot) = \dot{m}$, and the unique trajectory is given by

$$\hat{\varphi}^{\mathrm{CW};m}_{t,m'}(s) := \frac{1}{\sinh(2t)}\Big\{m\sinh(2(t-s)) + m'\sinh(2s)\Big\}, \qquad 0 \leq s \leq t. \tag{1.37}$$

(See [13, Eqs. (1.16) and (1.28)].) The intuition behind Theorem 1.5 is that the dynamics has no spatial interaction. Consequently, we may think of $\alpha(u)$ and $\alpha'(u)$ as the local initial and final magnetization near u, and thereby reduce the minimization problem in (1.26) to that of the Curie-Weiss model.

With the help of Theorem 1.5 we are able to prove the equivalence of non-Gibbs and *bifurcation*, the latter meaning that (1.27) has more than one global minimizer. This is in accordance with the program outlined in van Enter, Fernández, den Hollander and Redig [11].

Theorem 1.6. [Equivalence of non-Gibbsianness and bifurcation] *Let $\beta' = 0$. For every $t \geq 0$, $\tilde{\alpha}' \mapsto \gamma_t^u(\,\cdot\mid \tilde{\alpha}')$ is continuous at $\alpha' \in B$ for all $u \in \mathbb{T}^d$ if and only if $\inf_{\varphi \in C_{[0,t]}(B):\ \varphi_t \equiv \alpha'} I^t(\varphi)$ has a unique minimizing path.*

Thus, non-Gibbsianness is equivalent to the occurrence of *more than one* possible history for the *same* α'.

We expect Theorem 1.6 to hold for $\beta' > 0$ as well, but the present paper deals with $\beta' = 0$ only.

1.9 Bifurcation analysis

In this section we study for which choice of J, h, β and t, α' the variational formula in the right-hand side of (1.27) has a unique global minimizer or has multiple global minimizers. According to Definition 1.1 and Theorem 1.6, this distinction classifies Gibbsianness versus non-Gibbsianness.

Theorem 1.7. *Let $\beta' = 0$ and $\langle J \rangle := \int_{\mathbb{T}^d} J(u)du$.*
(i) **[Short-time Gibbsianness]** *There exists a $t_0 = t_0(J,h) \in (0,\infty)$ such that (1.27) has a unique global minimizer $\hat{\varphi}^{t,\alpha'}$ for all $0 \leq t \leq t_0$ and all $\alpha' \in B$.*
(ii) **[Mean-field behaviour]** *If $h \equiv c \in [0,\infty)$ and $\alpha' \equiv c' \in [-1,+1]$, then the bifurcation behaviour is the same as for the Curie-Weiss model with parameters $(J^{\mathrm{CW}}, h^{\mathrm{CW}}) = (\beta\langle J \rangle, \beta c)$ and final magnetization c':*

J^{CW}	$h^{\mathrm{CW}} = 0$	$h^{\mathrm{CW}} > 0$
$(0,1]$	*No bad c' for all $t \geq 0$*	
$(1,\frac{3}{2}]$	$\varnothing$ $\{0\}$ 0 Ψ_c	$\varnothing$ $\{c'\}$ $\varnothing$ 0 Ψ_U $[-1,U_B]$ Ψ_*
$(\frac{3}{2},\infty)$	$\varnothing$ $\{\pm c'\}$ $\{0\}$ 0 Ψ_U $[-U_B,U_B]$ Ψ_c	$h < h^*$: $\varnothing$ $\{c'\}$ $\{c_1',c_2'\}$ $\{c'\}$ $\varnothing$ 0 Ψ_U $[M_B,U_B)$ Ψ_L (L_B,M_B) Ψ_T $[-1,M_T]$ Ψ_* $h \geq h^*$: $\varnothing$ $\{c'\}$ $\varnothing$ 0 Ψ_U $[-1,U_B]$ Ψ_*

The above table summarizes the results for the Curie-Weiss model studied in [13]. The center line represents the time axis. In each figure, the symbols on top indicate the set of bad magnetizations (which for the Kac-model correspond to bad constant profiles), the intervals below indicate in which range the bad magnetizations occur. For further details, in particular, a definition of the times $\Psi_U, \Psi_*, \Psi_c, \Psi_L, \Psi_T$ and the magnetizations U_B, M_B, L_B, M_T, see [13, Section 1.5.5].

Remarks:
(1) The existence of a solution of (1.27) is guaranteed by the lower semi-continuity of $\alpha \mapsto C_{t,\alpha'}(\alpha)$, which follows from the lower semi-continuity of $\varphi_0 \mapsto I_S(\varphi_0)$ and $\varphi \mapsto I_D^t(\varphi)$, together with the fact that $w - \lim_{n\to\infty} \alpha_n = \alpha$ implies $w - \lim_{n\to\infty} \hat{\varphi}^{\alpha_n;t,\alpha'} = \hat{\varphi}^{\alpha;t,\alpha'}$ in the Skorohod topology by (1.37).
(2) The claims in Theorem 1.7(ii) only concern the case where α' is constant. The problem

of deciding whether or not there exist multiple global minimizers of (1.27) when α' is not constant presents major difficulties. Similar but easier equations have been studied extensively in Comets, Eisele and Schatzman [4], De Masi, Orlandi, Presutti and Triolo [6] and Bates, Chen and Chmaj [1], with partial success. An additional complication in our case is that non-constant α' brings a non-homogeneous parameter into the problem, which makes the analysis even harder. A full analysis of the global minimizers of (1.27) as a function of J and h therefore remains a challenge.

2 Proof of Theorems 1.4–1.6

2.1 Proof of Theorem 1.4

Proof. Recall that $\pi_t^{u,n} = \pi^{u,n}(\sigma_t)$ defined below (1.3) does not depend on $\sigma_t(\lfloor nu \rfloor)$. Let $\mathbb{P}^n$ denote the law of $(\sigma_s)_{s\geq 0}$ with σ_0 distributed according to μ^n, and abbreviate $\pi_{<t}^{u,n} := (\pi_s^{u,n})_{s\in[0,t)}$ and $\xi_{<t}^{n-1} := (\xi_s^{n-1})_{s\in[0,t)}$. Write (recall (1.10))

$$\begin{aligned}
&\gamma_t^{u,n}\big(k' \mid \alpha_{n-1}'^{u}\big) := \mathbb{P}^n\Big(\sigma_t(\lfloor nu \rfloor) = k' \;\Big|\; \pi_t^{u,n} = \alpha_{n-1}'^{u}\Big) \\
&= \int_{D_{[0,t)}(\mathcal{M}^{u,n})} \mathbb{P}^n\Big(d\xi_{<t}^{n-1} \;\Big|\; \pi_t^{u,n} = \alpha_{n-1}'^{u}\Big)\mathbb{P}^n\Big(\sigma_t(\lfloor nu \rfloor) = k' \;\Big|\; \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\Big) \\
&= \int_{D_{[0,t)}(\mathcal{M}^{u,n})} \mathbb{P}^n\Big(d\xi_{<t}^{n-1} \;\Big|\; \pi_t^{u,n} = \alpha_{n-1}'^{u}\Big) \\
&\times \Bigg\{ \sum_{k=\pm 1} \mathbb{P}^n\Big(\sigma_t(\lfloor nu \rfloor) = k' \;\Big|\; \sigma_0(\lfloor nu \rfloor) = k,\, \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\Big)\mathbb{P}^n\Big(\sigma_0(\lfloor nu \rfloor) = k \;\Big|\; \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\Big)\Bigg\}.
\end{aligned} \tag{2.1}$$

We proceed by analyzing the three terms under the integral.

(1) The LDP for $(Q_{t,\alpha'}^n)_{n\in\mathbb{N}}$ in Proposition 1.3(ii), together with the assumption that (1.27) has a unique minimizing path, implies

$$w - \lim_{n\to\infty} \mathbb{P}^n\Big(\;\cdot\; \Big|\; \pi_t^{u,n} = \alpha_{n-1}'^{u}\Big) = \delta_{\hat{\varphi}_{<t}^{t,\alpha'}}(\cdot) \quad \text{on} \quad D_{[0,t)}(\mathcal{M}(\mathbb{T}^d)). \tag{2.2}$$

(2) Because $(\sigma_s(\lfloor nu \rfloor), \pi_s^{u,n})_{s\geq 0}$ is Markov, we have

$$\mathbb{P}^n\Big(\sigma_t(\lfloor nu \rfloor) = k' \;\Big|\; \sigma_0(\lfloor nu \rfloor) = k,\, \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\Big) = p_t^{\xi_{<t}^{n-1}}(k,k'), \tag{2.3}$$

where $p_t^{\xi_{<t}^{n-1}}(k,k')$ is the probability to go from k at time 0 to k' at time t in the time-inhomogeneous Markov process on $\{-1,+1\}$ with generator at time $s \in [0,t)$ given by (1.29) with $\hat{\varphi}_s^{t,\alpha'}$ replaced by ξ_s^{n-1}. Note that $\xi_{<t}^{n-1} \mapsto p_t^{\xi_{<t}^{n-1}}(k,k')$ is continuous on $D_{[0,t)}(\mathcal{M}^{u,n})$ for fixed k,k',t and u,n (recall (1.4)), and that $\lim_{n\to\infty} p_t^{\xi_{<t}^{n-1}}(k,k') = p_t^{\hat{\varphi}_{<t}^{t,\alpha'}}(k,k')$ for fixed k,k',t,α' when $\lim_{n\to\infty} \xi_{<t}^{n-1} = \hat{\varphi}_{<t}^{t,\alpha'}$ on $D_{[0,t)}(\mathcal{M}(\mathbb{T}^d))$ (recall (1.29)).

(3) Write

$$\begin{aligned}
&\mathbb{P}^n\Big(\sigma_0(\lfloor nu \rfloor) = k \;\Big|\; \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\Big) \\
&\qquad = \Big[1 + c^{u,n}(\xi_{<t}^{n-1}, k)\exp\Big(-2\beta k\{\tfrac{1}{2}J * \xi_0^{n-1} + h\}\big(\tfrac{\lfloor nu \rfloor}{n}\big)\Big)\Big]^{-1}
\end{aligned} \tag{2.4}$$

with

$$c^{u,n}(\xi^{n-1}_{<t},k) := \frac{d\mathbb{P}^{u,n}_{\xi_0^{n-1},-k}}{d\mathbb{P}^{u,n}_{\xi_0^{n-1},k}}(\xi^{n-1}_{<t}), \tag{2.5}$$

where

$$\mathbb{P}^{u,n}_{\xi_0^{n-1},k}(\cdot) = \mathbb{P}^{u,n}\big(\pi^{u,n}_{<t} \in \cdot \mid \pi^{u,n}_0 = \xi_0^{n-1}, \sigma_0(\lfloor nu \rfloor) = k\big) \tag{2.6}$$

and we use (1.1–1.2) to write

$$\frac{\mathbb{P}^n(\pi^{u,n}_0 = \xi_0^{n-1}, \sigma_0(\lfloor nu \rfloor) = -k)}{\mathbb{P}^n(\pi^{u,n}_0 = \xi_0^{n-1},\, \sigma_0(\lfloor nu \rfloor) = k)} = \exp\Big(-2\beta k\{\tfrac{1}{2}J * \xi_0^{n-1} + h\}\big(\tfrac{\lfloor nu \rfloor}{n}\big)\Big). \tag{2.7}$$

Finally, note that $\lim_{n\to\infty} c^{u,n}(\xi^{n-1}_{<t},k) = 1$ for fixed k,t and u when $\lim_{n\to\infty} \xi^{n-1}_{<t} = \hat{\varphi}^{t,\alpha'}_{<t}$ on $D_{[0,t)}(\mathcal{M}(\mathbb{T}^d))$. Indeed, (1.13–1.14) show that in the thermodynamic limit a single spin has no effect on the dynamics of the empirical density (Feller property). Combine this observation with (2.2–2.4) to get the identity in (1.28) (see Yang [16]). □

2.2 Proof of Theorem 1.5

Proof. For $\beta' = 0$ (infinite-temperature dynamics), (1.23) reduces to $\int_{\mathbb{T}^d} du\, L^{\mathrm{CW}}[p(u), q(u)]$ with L^{CW} the Curie-Weiss Lagrangian in (1.36). Hence, recalling (1.26), we have

$$\begin{aligned}
C_{t,\alpha'}(\alpha) &= \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\, \varphi_t\equiv\alpha'}} I^t(\varphi) \\
&= I_S(\alpha) + \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\, \varphi_t\equiv\alpha'}} I^t_D(\varphi) \\
&\geq I_S(\alpha) + \int_{\mathbb{T}^d} du \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\, \varphi_t\equiv\alpha'}} \int_0^t ds\; L^{\mathrm{CW}}\big[\varphi_s(u), \dot{\varphi}_s(u)\big] \\
&\geq I_S(\alpha) + \int_{\mathbb{T}^d} du \inf_{\substack{\rho\in C_{[0,t]}([-1,+1]):\\ \rho_0=\alpha(u),\, \rho_t=\alpha'(u)}} \int_0^t ds\; L^{\mathrm{CW}}\big[\rho_s, \dot{\rho}_s\big] \\
&= I_S(\alpha) + \int_{\mathbb{T}^d} du \int_0^t ds\; L^{\mathrm{CW}}\Big[\hat{\varphi}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s), \dot{\hat{\varphi}}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s)\Big],
\end{aligned} \tag{2.8}$$

which settles half of (1.34). To get equality we pick, as in (1.33),

$$\hat{\varphi}^{\alpha;t,\alpha'}_s(u) := \hat{\varphi}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s), \quad s\in[0,t],\ ,u\in\mathbb{T}^d. \tag{2.9}$$

Since $(\hat{\varphi}^{\alpha;t,\alpha'}_s)_{s\in[0,t]} \in C_{[0,t]}(B)$ verifies the restrictions $\varphi_0 \equiv \alpha$, $\varphi_t \equiv \alpha'$, it is a minimizer of the variational problem in the left-hand side of (2.8).

The derivation of (1.35) follows in the same way as for the Curie-Weiss model in [13, Section 2.1], with the Fréchet derivative replacing the standard derivative. Note that $\alpha \mapsto C_{t,\alpha'}(\alpha)$ is Fréchet differentiable on $\mathrm{int}(B)$, while the argument in Ellis [8, Section V, Theorem 5.1] shows that all its critical points lie in $\mathrm{int}(B)$. □

The following way of rewriting $C_{t,\alpha'}$ will be useful later on. Adding and subtracting $\frac{1}{4}\beta \int_{\mathbb{T}^d} du \int_{\mathbb{T}^d} dv\, J(u-v)[\alpha(u)-\alpha(v)]^2$, we may rewrite (1.18) as

$$I_S(\alpha) = \tfrac{1}{4}\beta \int_{\mathbb{T}^d} du \int_{\mathbb{T}^d} dv\, J(u-v)[\alpha(u)-\alpha(v)]^2 + \int_{\mathbb{T}^d} du\, [-\tfrac{1}{2}\beta\alpha(u)^2 - c\alpha(u)]. \tag{2.10}$$

With this formula, (1.34) reduces to

$$C_{t,\alpha'}(\alpha) = \tfrac{1}{4}\beta \int_{\mathbb{T}^d} du \int_{\mathbb{T}^d} dv\, J(u-v)[\alpha(u)-\alpha(v)]^2 + \int_{\mathbb{T}^d} du\, C^{\mathrm{CW}}_{t,\alpha'}(\alpha(u)). \tag{2.11}$$

This form clarifies the interplay between the non-local interaction and the independent spin-flip dynamics.

2.3 Proof of Theorem 1.6

As emphasized in (1.30), $\gamma^u_t(k' \mid \alpha')$ depends on α' only through $\hat{\varphi}^{t,\alpha'}_0$, the starting value of the global minimizer of $C_{t,\alpha'}$. The following lemma is the basis for the proof of Theorem 1.6. It describes the behavior of $\hat{\varphi}^{t,\alpha'}_0$ when the constraint $\alpha' \in B$ at time t is varied. Loosely speaking, it says that global minimizers are isolated, are continuous under variations of α', and can be selected by variation of α'.

Below we fix t and suppress it from the notation. In what follows we write $\hat{\alpha}(\alpha')$ to denote a global minimum of $C_{t,\alpha'}$.

Lemma 2.1. *For every $t \geq 0$ and $\alpha'_0 \in B$ there exists an open neighborhood $\mathcal{N}_{\alpha'_0}$ of α'_0 such that for all $\alpha' \in \mathcal{N}_{\alpha'_0} \setminus \{\alpha'_0\}$ the following hold:*
(a) **[Isolation of global minimizers]** *$\alpha \mapsto C_{t,\alpha'}$ has a unique global minimum at, say, $\hat{\alpha}(\alpha')$.*
(b) **[Continuity of global minimizers]** *$\alpha'' \mapsto \hat{\alpha}(\alpha'')$ is continuous at $\alpha'' = \alpha'$. If $\alpha'' \mapsto C_{t,\alpha'_0}(\alpha'')$ has a unique global minimum, then it is continuous at $\alpha'' = \alpha'_0$.*
(c) **[Selection of global minimizers]** *If C_{t,α'_0} has multiple global minima, then there are two of them, say $\hat{\alpha}_k(\alpha'_0)$ and $\hat{\alpha}_l(\alpha'_0)$, and a $\gamma' \in B$ such that*

$$\lim_{\varepsilon \downarrow 0} \hat{\alpha}(\alpha'_0 + \varepsilon\gamma') \equiv \hat{\alpha}_k(\alpha'_0), \qquad \lim_{\varepsilon \uparrow 0} \hat{\alpha}(\alpha'_0 + \varepsilon\gamma') \equiv \hat{\alpha}_l(\alpha'_0). \tag{2.12}$$

Proof. The following 3 steps describe the behavior of the minimizers under small perturbations of α' are around α'_0.

(a) Under the assumption that $\sup_{\alpha \in B} |C_{t,\alpha'} - C_{t,\alpha'_0}| \to 0$ as $\|\alpha' - \alpha'_0\|_\infty \to 0$, whenever a local minimum is emerging as α' is varied this local minimum cannot be a global minimum. Indeed, we have that

$$|C_{t,\alpha'}(\alpha) - C_{t,\alpha'_0}(\alpha)| \leq \int_{\mathbb{T}^d} du\, |C^{CW}_{t,\alpha'(u)}(\alpha(u)) - C^{CW}_{t,\alpha'_0(u)}(\alpha(u))| \leq \int_{\mathbb{T}^d} du\, \|C^{CW}_{t,\alpha'(u)} - C^{CW}_{t,\alpha'_0(u)}\|_\infty.$$

On the other hand, we know from [13] that $\|C^{CW}_{t,m'} - C^{CW}_{t,m'_0}\|_\infty \to 0$ when $m' \to m'_0$. Hence the claim follows by dominated convergence.

(b) Let $\hat{\alpha}_i(\alpha'_0)$, $i \in \mathcal{I}$, denote the global minima of C_{t,α'_0}. Each of these verifies (1.35), which may be written in the form $F(\alpha, \alpha') \equiv 0$ for some functional F. From the implicit function theorem (see e.g. Drábek and Milota [7, Theorem 4.2.1]) it follows that there exist a

neighborhood $\widetilde{\mathcal{N}}_{\alpha_0'}$ of α_0' and smooth functions $\alpha' \mapsto \overline{\alpha}_i(\alpha')$, $i \in \mathcal{I}$, on this neighborhood such that $\overline{\alpha}_i(\alpha')$, $i \in \mathcal{I}$, are minima of $C_{t,\alpha'}$, and $\lim_{\alpha' \to \alpha_0'} \overline{\alpha}_i(\alpha') \equiv \hat{\alpha}_i(\alpha_0')$.

(c) Let

$$B_i(\alpha') := C_{t,\alpha'}(\overline{\alpha}_i(\alpha')). \tag{2.13}$$

The minimal cost is

$$C_{t,\alpha'}(\hat{\alpha}(\alpha')) = \min_{i \in \mathcal{I}} B_i(\alpha'). \tag{2.14}$$

Because of the assumed multiplicity of minima at α_0', we have

$$B_i(\alpha_0') = B_j(\alpha_0'), \qquad i, j \in \mathcal{I}. \tag{2.15}$$

Expand each B_i up to first order order,

$$B_i(\alpha_0' + \varepsilon\gamma') = B(\alpha_0') + \varepsilon\big\langle [DB_i](\alpha_0'), \gamma' \big\rangle + O\big(\varepsilon \|\gamma'\|_\infty\big), \qquad \varepsilon > 0, \tag{2.16}$$

where $[DB_i](\alpha_0')$ is the Fréchet derivative. Put $G(\alpha, \alpha') := C_{t,\alpha'}(\alpha)$. Then the chain rule implies that

$$[DB_i](\alpha_0') \equiv [D_\alpha G]\big(\hat{\alpha}_i(\alpha_0'), \alpha_0'\big) \circ [D_{\alpha'} \overline{\alpha}_i](\alpha_0') + [D_{\alpha'} G]\big(\hat{\alpha}_i(\alpha_0'), \alpha_0'\big), \tag{2.17}$$

where $\circ$ denotes composition and the lower indices α, α' on the letter D refer to the variable with respect to which the derivative is taken. The first term in (2.17) vanishes due to the criticality of $\hat{\alpha}_i(\alpha_0')$. Standard calculations with Fréchet derivatives show that

$$[D_{\alpha'} G]\big(\hat{\alpha}_i(\alpha_0'), \alpha_0'\big)(u) = H^{\mathrm{CW}}\big(\hat{\alpha}_i(\alpha_0')(u), \alpha_0'(u)\big), \quad u \in \mathbb{T}^d, \tag{2.18}$$

with $H^{\mathrm{CW}}(m, m') := (\frac{\partial}{\partial m'} C^{\mathrm{CW}}_{t,m'})(m)$. The identity in (2.18) helps us to select different global minimizers by small variations of α'. Indeed, for $i \neq j$ we have $\|\hat{\alpha}_i(\alpha_0') - \hat{\alpha}_j(\alpha_0')\|_\infty > 0$, and hence there exists a $\delta > 0$ such that $\lambda(\{\hat{\alpha}_i(\alpha_0') - \hat{\alpha}_j(\alpha_0') > \delta\}) > 0$. Take $I = \{u \in \mathbb{T}^d : \hat{\alpha}_i(\alpha_0')(u) - \hat{\alpha}_j(\alpha_0')(u) > \delta\}$. Then

$$\hat{\alpha}_j(\alpha_0')(u) + \delta < \hat{\alpha}_i(\alpha_0')(u) \quad \forall\, u \in I. \tag{2.19}$$

Combining (2.17–2.19) and using the strict monotonicity of $m \mapsto H^{\mathrm{CW}}(m, m')$, we get

$$[DB_j](\alpha_0')(u) < [DB_i](\alpha_0')(u) \quad \forall\, u \in I. \tag{2.20}$$

The claim follows by picking $\gamma' \equiv 1_I$ and expressions (2.16), (2.18). □

We are now ready to prove Theorem 1.6. We continue to use the same notation as in Lemma 2.1.

Proof. Suppose that $C_{t,\alpha_0'}$ has a unique global minimizer, say $\hat{\alpha}(\alpha_0')$, and let $\mathcal{N}_{\alpha_0'}$ be the neighborhood in Lemma 2.1. Then (1.30) holds for every $\alpha' \in \mathcal{N}_{\alpha_0'}$, and the continuity of $m \mapsto \Gamma_t(k', m)$ for all t, k' gives the desired continuity of $\alpha' \mapsto \gamma_t^u(\cdot \mid \alpha')$ at $\alpha' \equiv \alpha_0'$ for all $u \in \mathbb{T}^d$. Hence α_0' is a good profile.

Conversely, suppose that $C_{t,\alpha_0'}$ has multiple global minimizers. Consider the pair $\hat{\alpha}_k(\alpha_0')$ and $\hat{\alpha}_l(\alpha_0')$ and the box I in the proof of Lemma 2.1, and put $\alpha_\epsilon'^k := \alpha_0' + \epsilon\gamma'$ for $\epsilon > 0$ and $\alpha_\epsilon'^l := \alpha_0' + \epsilon\gamma'$ for $\epsilon < 0$. Then $\gamma_t^u(\cdot \mid \alpha_\epsilon'^i) = \Gamma_t(\cdot, \beta\{J * \hat{\alpha}(\alpha_\epsilon'^i) + h\}(u))$, $i \in \{k, l\}$, and

$$\lim_{\epsilon \downarrow 0} \hat{\alpha}(\alpha_\epsilon'^k)(u) = \hat{\alpha}_k(\alpha_0')(u) \neq \hat{\alpha}_l(\alpha_0')(u) = \lim_{\epsilon \uparrow 0} \hat{\alpha}(\alpha_\epsilon'^l)(u) \quad \forall\, u \in I. \tag{2.21}$$

On the other hand, $\hat{\alpha}_k(\alpha_0')$ and $\hat{\alpha}_l(\alpha_0')$ are critical points, they satisfy (1.35) with $\alpha' \equiv \alpha_0'$, and so

$$\hat{\alpha}_k(u) \neq \hat{\alpha}_l(u) \quad \Longrightarrow \quad (J * \hat{\alpha}_k)(u) \neq (J * \hat{\alpha}_l)(u). \tag{2.22}$$

This, together with the continuity and the monotonicity of $m \mapsto \Gamma_t(k', m)$ for all t and k', forces the discontinuity

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \gamma_t^u(k' \mid \alpha_\epsilon'^k) &= \Gamma_t\big(k', \beta\{J * \hat{\alpha}_k(\alpha_0') + h\}(u)\big) \\ &\neq \Gamma_t\big(k', \beta\{J * \hat{\alpha}_l(\alpha_0') + h\}(u)\big) = \lim_{\epsilon \uparrow 0} \gamma_t^u(k' \mid \alpha_\epsilon'^l) \quad \forall\, u \in I. \end{aligned} \tag{2.23}$$

Hence α_0' is a bad profile. □

3 Proof of Theorem 1.7

Proof. Without loss of generality we may assume that $\langle J \rangle = 1$. For simplicity, we consider only $\alpha' \in C(\mathbb{T}^d)$. In that case, due to the regularization property of the convolution operator, the solutions of (1.35) may be taken to be continuous, and (1.35) must be fulfilled for all $u \in \mathbb{T}^d$. The extension to $\alpha' \notin C(\mathbb{T}^d)$ is straightforward.

(i) Let $\alpha_1, \alpha_2 \in B$ be two different solutions of (1.27). After some algebra with trigonometrical identities, we get from (1.27) that the following equation must be fulfilled:

$$\frac{2\sinh\left(\frac{A_u - B_u}{2}\right)}{a_u - b_u} \left\{ \cosh\left(\tfrac{A_u + B_u}{2}\right) - a_u \sinh\left(\tfrac{A_u + B_u}{2}\right) \right\} - \cosh\left(B_u\right) = \coth(2t) \qquad \forall\, u \in \mathbb{T}^d, \tag{3.1}$$

where $A_u = (\beta J * \alpha_1)(u) + \beta h(u)$ and $a_u = \alpha_1(u)$ (and similarly for B_u, b_u, α_2). Note that the left-hand side depends only on u and the right-hand side only on t, and that $\lim_{t \downarrow 0} \coth(2t) = \infty$. Since $|A_u|, |B_u| \leq \beta(1 + \|h\|_\infty)$ and $|a_u|, |b_u| \leq 1$, the left-hand side of (3.1) is bounded from above by

$$\frac{2\sinh\left(\frac{A_u - B_u}{2}\right)}{a_u - b_u} C_1 + C_2 \tag{3.2}$$

for some constants C_1, C_2. By taking $t > 0$ small enough, we force $a_u - b_u$ to be small for all $u \in \mathbb{T}^d$ (equivalently, $\|\alpha_1 - \alpha_2\|_\infty < \delta$). By choosing v_0 such that $|\alpha_1(v_0) - \alpha_2(v_0)| = V_0$ with $V_0 = \max_{u \in \mathbb{T}^d} |\alpha_1(u) - \alpha_2(u)|$, we get $|A_{v_0} - B_{v_0}| \leq \beta V_0$ which, together with the series expansion of sinh, leads to a contradiction.

(ii) From (2.11), whenever $\alpha' \equiv c'$ we have that

$$\inf_{\alpha \in B} C_{t,c'}(\alpha) \geq \inf_{\alpha \in B} \tfrac{1}{4}\beta \int_{\mathbb{T}^d} du \int_{\mathbb{T}^d} dv\, J(u - v)[\alpha(u) - \alpha(v)]^2 + \inf_{\alpha \in B} \int_{\mathbb{T}^d} du\, C_{t,c'}^{\mathrm{CW}}(\alpha(u)). \tag{3.3}$$

Because $J \geq 0$, the minimizers of the first term are the constant profiles. If we take the constant of the profile equal to a minimizer of $C_{t,c'}^{\mathrm{CW}}$, then the second term is also minimal. □

A Proof of Proposition 1.3

A.1 Outline

In Sections A.2–A.4 we sketch the proof of the LDP in Proposition 1.3(i) for deterministic initial conditions (as in Proposition 1.2(ii)), and explain why it remains true for random initial

conditions. We follow the line of argument in Benois, Mourragui, Orlandi, Saada and Triolo [2] rather than Comets [3], and use various results from Kipnis and Landim [14]. The strategy of the proof consists in first proving the claim for random initial conditions drawn according to $\vartheta_\kappa^n = \otimes_{x\in\mathbb{T}_n^d}\vartheta_\kappa$ with $\vartheta_\kappa = \mathrm{BER}(\kappa)$, $\kappa \in [0,1]$ (i.e., $\vartheta_\kappa(+1) = \kappa$ and $\vartheta_\kappa(-1) = 1-\kappa$), and afterwards replacing ϑ_κ^n by μ^n in (1.2) with the help of Varadhan's Lemma and Bryc's Lemma. In Section A.5 we indicate how Proposition 1.3(ii) follows.

Below we will make frequent reference to formulas in [2] and [14], so our arguments are not self-contained. We begin with the following observation.

Lemma A.1. *Suppose that μ and ν are equivalent probability measures. If P_μ and Q_ν are the laws of equivalent Markov processes with starting measures μ and ν, then*

$$\frac{dP_\mu}{dQ_\nu}(\bar\eta) = \frac{d\mu}{d\nu}(\eta_0)\,\frac{dP_\mu}{dQ_\mu}(\bar\eta) = \frac{d\mu}{d\nu}(\eta_0)\,\frac{dP_\nu}{dQ_\nu}(\bar\eta). \tag{A.1}$$

The general technique to prove an LDP relies on finding a family of mean-one positive martingales that can be written as functions of the empirical density. For Markov processes this is achieved by considering the Radon-Nikodym derivative of the original dynamics w.r.t. a small perturbation of this dynamics. It is here that Lemma A.1 comes into play: it factorizes the Radon-Nikodym derivative into a *static part* and a *dynamic part*, as in (1.25).

A.2 Upper bound

For initial condition $\gamma \in C(\mathbb{T}^d;[-1,+1])$ and potential $V \in C^{1,0}([0,t]\times\mathbb{T}^d)$, we denote by $\mathbb{P}^{n,V}_{\vartheta_\gamma^n}$ the law of the (γ,V)-perturbed inhomogeneous Markov process starting at

$$\vartheta_\gamma^n = \otimes_{x\in\mathbb{T}_n^d}\vartheta_{\chi^{-1}(\gamma(\frac{x}{n}))}, \tag{A.2}$$

where $\chi\colon [0,1]\to[-1,+1]$ is the linear map that transforms a profile taking values in $[-1,+1]$ into a profile taking values in $[0,1]$. Details about such a perturbation and its Radon-Nikodym derivative can be found in [2, Eq. (5.8)].

1. *Large deviation upper bound for compact sets.* Fix $\kappa\in[0,1]$. Let $\mathcal{K}\in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ be compact. By Lemma A.1, we have (recall the notation introduced in Section 1.5)

$$\begin{aligned}
\tfrac{1}{n^d}\log \mathbb{P}^n_{\vartheta_\kappa^n}[\bar\pi^n\in\mathcal{K}] &= \tfrac{1}{n^d}\log \mathbb{E}^{n,V}_{\vartheta_\gamma^n}\left[\left(\frac{d\mathbb{P}^n_{\vartheta_\kappa^n}}{d\mathbb{P}^{n,V}_{\vartheta_\gamma^n}}\,\mathbb{I}_{\mathcal{K}}\right)(\bar\pi^n)\right]\\
&= \tfrac{1}{n^d}\log \mathbb{E}^{n,V}_{\vartheta_\gamma^n}\left[\left(\frac{d\vartheta_\kappa^n}{d\vartheta_\gamma^n}\,\frac{d\mathbb{P}^n_{\vartheta_\kappa^n}}{d\mathbb{P}^{n,V}_{\vartheta_\kappa^n}}\,\mathbb{I}_{\mathcal{K}}\right)(\bar\pi^n)\right]\\
&= \tfrac{1}{n^d}\log \mathbb{E}^{n,V}_{\vartheta_\gamma^n}\left[e^{-n^d h_\gamma(\pi_0^n)+O_\gamma(n^{-1})}\,e^{-n^d\{\hat J_V(\bar\pi^n * l^{\varepsilon,n})+r(V,\varepsilon,n)\}}\,\mathbb{I}_{\mathcal{K}}(\bar\pi^n)\right],
\end{aligned} \tag{A.3}$$

where h_γ is the analogue of [14, Eq. (1.1), Chapter 10], $\hat J_V$ is defined in [2, Eq. (6.8)], $\varepsilon>0$ is small, $l^{\varepsilon,n}$ is an approximation of the identity for $\varepsilon\downarrow 0$, and $r(V,\varepsilon,n)$ is an error term that vanishes as $n\to\infty$ for fixed V,ε. By letting $n\to\infty$, optimizing over γ,V,ε and using the

mini-max lemma, we get

$$\begin{aligned}\limsup_{n\to\infty} \tfrac{1}{n^d} \log \mathbb{P}^n_{\vartheta^n_\kappa}[\bar\pi^n \in \mathcal{K}] &\le \inf_{\gamma,V,\varepsilon} \sup_{\bar\pi\in\mathcal{K}} \{-h_\gamma(\pi_0) - \hat J_V(\bar\pi * l^\varepsilon)\} \\ &\le \sup_{\bar\pi\in\mathcal{K}} \inf_{\gamma,V,\varepsilon} \{-h_\gamma(\pi_0) - \hat J_V(\bar\pi * l^\varepsilon)\} \\ &\le -\inf_{\bar\pi\in\mathcal{K}} \{I_S(\pi_0) + I^t_D(\bar\pi)\}.\end{aligned} \tag{A.4}$$

The last inequality uses that $\sup_\gamma h_\gamma(\pi_0) = I_S(\pi_0)$, $\sup_V \hat J_V(\tilde\pi) = I^t_D(\tilde\pi)$, and $\sup_\varepsilon I^t_D(\bar\pi * l^\varepsilon) \ge I^t_D(\bar\pi)$ by lower semi-continuity of I^t_D.

2. *Exponential tightness.* While in [14, Section 4] the initial condition is drawn from equilibrium, this is immaterial. Indeed, the proof of [2, Proposition 6.1] uses the same ideas as in [14, Section 4] even though the initial condition is deterministic. Hence the same computations apply to our case.

A.3 Lower bound

1. *Large deviation lower bound for open sets.* Fix $\kappa \in [0,1]$. Let $\mathcal{O} \in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ be open. By Lemma A.1, we have

$$\begin{aligned}\tfrac{1}{n^d} \log \mathbb{P}^n_{\vartheta^n_\kappa}[\bar\pi^n \in \mathcal{O}] &= \tfrac{1}{n^d} \log \left\{ \mathbb{E}^{n,V}_{\vartheta^n_\gamma} \left[\frac{d\mathbb{P}^n_{\vartheta^n_\kappa}}{d\mathbb{P}^{n,V}_{\vartheta^n_\gamma}}(\bar\pi^n) \,\middle|\, \bar\pi^n \in \mathcal{O} \right] \mathbb{P}^{n,V}_{\vartheta^n_\gamma}(\mathcal{O}) \right\} \\ &\ge \mathbb{E}^{n,V}_{\vartheta^n_\gamma} \left[\tfrac{1}{n^d} \log \frac{d\mathbb{P}^n_{\vartheta^n_\kappa}}{d\mathbb{P}^{n,V}_{\vartheta^n_\gamma}}(\bar\pi^n) \,\middle|\, \bar\pi^n \in \mathcal{O} \right] + \tfrac{1}{n^d} \log \mathbb{P}^{n,V}_{\vartheta^n_\gamma}(\mathcal{O}),\end{aligned} \tag{A.5}$$

where we use Jensen's inequality. By the law of large numbers for $\mathbb{P}^{n,V}_{\vartheta^n_\gamma}$, we have

$$w - \lim_{n\to\infty} \mathbb{P}^{n,V}_{\vartheta^n_\gamma} = \delta_{\bar\pi^{\gamma,V}}, \tag{A.6}$$

where $\bar\pi^{\gamma,V}$ is the solution of [2, Eq. (5.5)] with initial condition γ and potential V. (The proof of (A.6) follows in the same fashion as in [2]: all that is needed is that the laws of the random initial conditions converge to a law associated with continuous profile.) Hence, if $\bar\pi^{\gamma,V} \in \mathcal{O}$, then $\lim_{n\to\infty} \mathbb{P}^{n,V}_{\vartheta^n_\gamma}(\mathcal{O}) = 1$. After some calculations with the Radon-Nikodym derivative, we get

$$\liminf_{n\to\infty} \tfrac{1}{n^d} \log \mathbb{P}^n_{\vartheta^n_\kappa}[\bar\pi^n \in \mathcal{O}] \ge -I^t(\bar\pi^{\gamma,V}) \tag{A.7}$$

with $I^t = I_S + I^t_D$.

2. *Density arguments.* It remains to show that

$$\inf_{\substack{\gamma,V \\ \bar\pi^{\gamma,V}\in\mathcal{O}}} I^t(\bar\pi^{\gamma,V}) = \inf_{\bar\pi\in\mathcal{O}} I^t(\bar\pi). \tag{A.8}$$

In other words, $(\bar\pi^{\gamma,V})_{\gamma,V}$ is dense with respect to (ϱ^w_t, I), i.e.,

$$\begin{aligned}&\forall\, \bar\pi \in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d)) \colon\ I(\bar\pi) < \infty, \\ &\exists\, (\bar\pi^{\gamma_n,V_n})_{n\in\mathbb{N}} \colon\ \lim_{n\to\infty} \varrho^w_t(\bar\pi^{\gamma_n,V_n}, \bar\pi) = 0,\ \lim_{n\to\infty} I(\bar\pi^{\gamma_n,V_n}) = I(\bar\pi),\end{aligned} \tag{A.9}$$

where ϱ_t^w is the supremum distance in $[0,t]$ when the marginal distance is ϱ^w (any metric that metrizes the weak topology). A density argument of this type typically exploits the fact that I is lower semi-continuous and convex, but in our case $I = I^t$, which is not convex. However, in [2] density arguments are given without convexity. In order to extend these to our setting of random initial conditions, minor modifications are needed in [2, Lemma 7.5]. In particular, the space regularization of the trajectory must be done for all $s \in [0,t]$, and hence [2, Lemma 7.6] together with the arguments in [14, p. 279] prove our assertion.

A.4 Replace ϑ_κ^n by μ^n

The observations made in Sections A.2–A.3 prove the LDP in Proposition 1.3(i), but for starting measures ϑ_κ^n given by (A.2). Note that

$$\frac{d\mu^n}{d\vartheta_\kappa^n} = e^{n^d \beta H(\pi^n)} \tag{A.10}$$

with $\pi^n \mapsto H(\pi^n)$ in (1.6) continuous. Hence, by Lemma (A.1), Varadhan's Lemma and Bryc's Lemma, the LDP in Proposition 1.3(i) for starting measures μ^n follows.

A.5 Contraction principle

Proposition 1.3(ii) follows from Proposition 1.3(i) via the approximate contraction principle based on exponential approximation estimates. See Dembo and Zeitouni [5, Section 4.2].

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