EURANDOM PREPRINT SERIES
2014-003
January 18, 2014

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ISSN 1389-2355

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January 18, 2014

Abstract

In this paper we study a random walk in a one-dimensional dynamic random environment consisting of a collection of independent particles performing simple symmetric random walks in a Poisson equilibrium with density $\rho \in (0, \infty)$. At each step the random walk performs a nearest-neighbour jump, moving to the right with probability $p_\circ$ when it is on a vacant site and probability $p_\bullet$ when it is on an occupied site. Assuming that $p_\circ \in (0, 1)$ and $p_\bullet \neq \frac{1}{2}$, we show that the position of the random walk satisfies a strong law of large numbers, a functional central limit theorem and a large deviation bound, provided ρ is large enough. The proof is based on the construction of a renewal structure together with a multiscale renormalisation argument.

MSC 2010. Primary 60F15, 60K35, 60K37; Secondary 82B41, 82C22, 82C44.
Key words and phrases. Random walk, dynamic random environment, strong law of large numbers, functional central limit theorem, large deviation bound, Poisson point process, coupling, renormalisation, regeneration times.

1 Introduction and main results

Background. Random motion in a random medium is a topic of major interest in mathematics, physics and (bio-)chemistry. It has been studied at microscopic, mesoscopic and macroscopic levels through a range of different methods and techniques coming from numerical, theoretical and rigorous analysis.

Since the pioneering work of Harris [10], there has been much interest in studies of random walk in random environment within probability theory (see [13] for an overview), both for static and dynamic random environments, and a number of deep results have been proven for various types of models.

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In the case of dynamic random environments, analytic, probabilistic and ergodic techniques were invoked (see e.g. [4]–[7], [2], [21], [8], [9]), but *good mixing assumptions on the environment* remained a pivotal requirement. By good mixing we mean that the decay of space-time correlations is sufficiently fast – polynomial with a large degree – and uniform in the initial configuration. More recently, examples of dynamic random environments with non-uniform mixing have been considered (see e.g. [11], [18], [1]). However, in all of these examples either the mixing is fast enough (despite being non-uniform), or the mixing is slow but strong extra conditions on the random walk are required.

In this context, random environments consisting of a field of random walks moving independently gained significance, not only due to an abundance of models defined in this setup, but also due to the substantial mathematical challenges that arise from their study. Among various conceptual and technical difficulties, slow mixing (in other words, slow convergence of the environment to its equilibrium as seen from the walk) makes the analysis of these systems extremely difficult. In particular, in physical terms, when ballistic behaviour occurs the motion of the walk is of "pulled type" (see [24]).

In this paper we consider a dynamic random environment given by a system of independent random walks. More precisely, we consider a *walk particle* that performs a discrete-time motion on $\mathbb{Z}$ under the influence of a field of *environment particles* which themselves perform independent discrete-time simple random walks. As initial state for the environment particles we take an i.i.d. Poisson random field with mean $\rho \in (0, \infty)$. This makes the dynamic random environment invariant under translations in space and time. The jumps of the walk particle are drawn from two different random walk transition kernels on $\mathbb{Z}$, depending on whether the space-time position of the walk particle is occupied by an environment particle or not. For reasons of exposition we restrict to nearest-neighbour kernels, but our analysis easily extends to the case where the kernels have finite range.

Model. Throughout the paper we write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_- = -\mathbb{N} \cup \{0\}$. Let $\{N(x) \colon x \in \mathbb{Z}\}$ be an i.i.d. sequence of Poisson random variables with mean $\rho \in (0, \infty)$. At time $n = 0$, for each $x \in \mathbb{Z}$ place $N(x)$ environment particles at site x. Subsequently, let all the environment particles evolve independently as "lazy simple random walks" on $\mathbb{Z}$, i.e., at each unit of time the probability to step -1, 0, 1 equals $\frac{1}{2}(1-q)$, q, $\frac{1}{2}(1-q)$, respectively, for some $q \in (0, 1)$. The assumption of laziness is not crucial for our arguments, see Comment 5 below.

Let $\mathcal{T}$ be the set of space-time points covered by the trajectory of at least one environment particle. The law of $\mathcal{T}$ is denoted by P^ρ (see Section 2.1 for a detailed construction of the dynamic environment and the precise definition of $\mathcal{T}$). Note that $\mathcal{T}$ does not have good mixing properties. Indeed,

$$\mathrm{Cov}_\rho(\mathbb{1}_{(0,0)\in\mathcal{T}}, \mathbb{1}_{(0,n)\in\mathcal{T}}) \sim c(\rho) \frac{1}{n^{1/2}}, \tag{1.1}$$

where Cov_ρ denotes covariance with respect to P^ρ. See (2.9)–(2.11) in Section 2.1.

Given $\mathcal{T}$, let $X = (X_n)_{n\in\mathbb{Z}_+}$ be the nearest-neighbour random walk on $\mathbb{Z}$ starting at the origin and with transition probabilities

$$P^{\mathcal{T}}(X_{n+1} = x + 1 \mid X_n = x) = \begin{cases} p_\circ, & \text{if } (x, n) \notin \mathcal{T}, \\ p_\bullet, & \text{if } (x, n) \in \mathcal{T}, \end{cases} \tag{1.2}$$

where $p_\circ, p_\bullet \in [0, 1]$ are fixed parameters and $P^{\mathcal{T}}$ stands for the law of X conditional on $\mathcal{T}$, called the *quenched law.* The *annealed law* is given by $\mathbb{P}^\rho(\cdot) = \int P^{\mathcal{T}}(\cdot)\, P^\rho(d\mathcal{T})$.

We will denote by

$$v_\circ = 2p_\circ - 1 \quad \text{and} \quad v_\bullet = 2p_\bullet - 1 \tag{1.3}$$

the drifts at vacant and occupied sites, respectively. Following the terminology established in the literature on random walks in static random environments (see [25], [26]), we classify our model as follows.

Definition 1.1. *The model is said to be non-nestling when* $v_\circ v_\bullet > 0$. *Otherwise it is said to be nestling.*

We are now in the position to state our main results.

Theorem 1.2. *Let* $v_\bullet \neq 0$ *and* $v_\circ \neq -\text{sign}(v_\bullet)$. *Then there exist* $\rho_\star \geq 0$ *and* $\gamma > 1$ *such that for all* $\rho \geq \rho_\star$ *there exist* $v = v(v_\circ, v_\bullet, \rho) \in [v_\circ \wedge v_\bullet, v_\circ \vee v_\bullet]$ *and* $\sigma = \sigma(v_\circ, v_\bullet, \rho) \in (0, \infty)$ *such that:*

(a) $\mathbb{P}^\rho$*-almost surely,*

$$\lim_{n\to\infty} n^{-1} X_n = v. \tag{1.4}$$

(b) Under $\mathbb{P}^\rho$*, the sequence of random processes*

$$\left(\frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{n^{1/2}\sigma} \right)_{t \geq 0}, \qquad n \in \mathbb{N}, \tag{1.5}$$

converges in distribution (in the Skorohod topology) to the standard Brownian motion.

(c) For all $\varepsilon > 0$ *there exists* $c = c(v_\circ, v_\bullet, \rho, \varepsilon) \in (0, \infty)$ *such that*

$$\mathbb{P}^\rho\big(\exists\, t \geq n \colon\ |X_t - tv| > \varepsilon t\big) \leq c^{-1} e^{-c \log^\gamma n} \qquad \forall\, n \in \mathbb{N}. \tag{1.6}$$

Moreover, in the non-nestling case $\rho_\star$ *can be taken equal to* 0.

The difference between the nestling and the non-nestling case can be seen in the statement of Theorem 1.2: in the non-nestling case we can prove (a)–(c) for any $\rho \geq 0$, in the nestling case only for $\rho \geq \rho_\star$, where $\rho_\star$ will need to be large enough.

Theorem 1.2 will be obtained as a consequence of Theorems 1.4–1.5 and Remark 1.6 below. Before stating them, we give the following definition that will be central to our analysis.

Definition 1.3. *For fixed $v_\circ, v_\bullet, \rho$ and a given $v_\star \in [-1,1]$, we say that the $v_\star$-ballisticity condition holds when there exist $c = c(v_\circ, v_\bullet, v_\star, \rho) > 0$ and $\gamma = \gamma(v_\circ, v_\bullet, v_\star, \rho) > 1$ such that*

$$\mathbb{P}^\rho\big(\exists\, n \in \mathbb{N}\colon\ X_n < nv_\star - L\big) \le c^{-1} e^{-c \log^\gamma L} \qquad \forall\, L \in \mathbb{N}. \tag{1.7}$$

Condition (1.7) is reminiscent of ballisticity conditions in the literature on random walk in static random environment such as Sznitman's (T')-*condition* (see [25]).

The next theorem shows that, if the model satisfies (1.7) with $v_\star > 0$ as well as an ellipticity condition, then the asymptotic results stated in Theorem 1.2 hold.

Theorem 1.4. *Let $v_\circ, v_\bullet \in (-1,1]$ and $\rho \in (0,\infty)$. Assume that (1.7) holds for some $v_\star \in (0,1]$. Then the conclusions of Theorem 1.2 hold with $v \ge v_\star$.*

Our last theorem shows that (1.7) holds when $v_\circ \le v_\star < v_\bullet$ and ρ is large enough.

Theorem 1.5. *If $v_\circ < v_\bullet$, then for all $v_\star \in [v_\circ, v_\bullet)$ there exist $\rho_\star = \rho_\star(v_\circ, v_\bullet, v_\star) \in (0,\infty)$ and $c = c(v_\circ, v_\bullet, v_\star) \in (0,\infty)$ such that (1.7) holds with $\gamma = \frac{3}{2}$ for all $\rho \ge \rho_\star$.*

Remark 1.6. *If $v_\circ \wedge v_\bullet > 0$, then (1.7) holds for all $\rho \in (0,\infty)$ and $v_\star \in (0, v_\circ \wedge v_\bullet)$ by comparison with a homogeneous random walk with drift $v_\circ \wedge v_\bullet$. In fact, in this case the bound in the right-hand side of (1.7) can be made exponentially small in L.*

Theorem 1.2 now follows directly from Theorems 1.4–1.5 and Remark 1.6 by noting that, when $v_\bullet \neq 0$, by reflection symmetry we may without loss of generality assume that $v_\bullet > 0$. Note that Theorem 1.5 is only needed in the nestling case.

The proofs of Theorems 1.4 and 1.5 are given in Sections 4 and 3, respectively. They rely, respectively, on the construction and control of a renewal structure for the random walk trajectory, and on a multiscale renormalisation scheme. In particular, multiscale analysis plays a central role in our work. None of these techniques are new in the field, but in the context of slow mixing dynamic random environments they are novel and open up gates to future advances.

Comments.

1. It follows from Theorem 1.5 (and reflection symmetry in the case $v_\bullet < v_\circ$) that

$$\lim_{\rho\to\infty} v(v_\circ, v_\bullet, \rho) = v_\bullet, \tag{1.8}$$

where $v = v(v_\circ, v_\bullet, \rho)$ is as in (1.4). This can also be deduced from the following asymptotic weak law of large numbers derived in [12]:

$$\lim_{\rho\to\infty} \limsup_{n\to\infty} \mathbb{P}^\rho\big(|n^{-1}X_n - v_\bullet| > \varepsilon\big) = 0 \qquad \forall\, \varepsilon > 0. \tag{1.9}$$

In fact, [12] considers the version of our model in $\mathbb{Z}^d, d \ge 1$ in continuous time and with more general transition kernels.

2. It can be shown that the asymptotic speed and variance v and σ in Theorem 1.2 are continuous functions of the parameters $v_\circ$, $v_\bullet$ and ρ. See Remark 4.8 in Section 4.3.

3. We expect Theorem 1.2 to hold when $v_\circ \neq 0$, $v_\bullet \neq -\text{sign}(v_\circ)$ and ρ is small. In the non-nestling case, this already follows (for any $\rho \geq 0$) from Theorem 1.4, but in the nestling case we would have to prove the analogue of Theorem 1.5 for $v_\bullet < v_\circ$ and ρ small.

4. Our techniques can potentially be extended to higher dimensions. The restriction to the one-dimensional setting simplifies the notation and allows us to avoid some technicalities.

5. Our dynamic random environment is composed of lazy random walks evolving in discrete time. This assumption was made for convenience in order to simplify some technical steps. However, as discussed in Remark C.4, our analysis can be extended to symmetric random walks with bounded steps that are aperiodic or bipartite (in the sense of [14]), or that evolve in continuous time.

6. It is a challenge to extend Theorem 1.2 to other environments, in particular to environments where the particles are allowed to interact with each other. The renormalisation scheme is robust enough to show that the ballisticity condition (1.7) holds as long as the environment satisfies a mild decoupling inequality (see Section 3.5 for specific examples). On the other hand, the regeneration structure is more delicate, and uses model-specific features in an important way. There are, however, other situations where a regeneration strategy can be found (see Section 4.4). It remains an interesting question to determine how far these methods can be pushed.

Organization of the paper. In Section 2 we give a graphical construction of our random walk in dynamic random environment. This construction will be convenient during the proofs of our main results. In Section 3 we set up a renormalisation scheme, and use this to show that, for large densities of the particles, the random walk moves with a positive lower speed to the right. This lower speed of the random walk plays the role of a ballisticity condition and is crucial in Section 4, where we introduce a random sequence of regeneration times at which the random walk "refreshes its outlook on the random environment", and show that these regeneration times have a good tail. In Section 4.3 the regeneration times are used to prove Theorem 1.4. Appendices A–E collect a few technical facts that are needed along the way.

Acknowledgments. VS thanks O. Angel, V. Beffara and G. Kozma for fruitful discussions at the initial stages of the work, and Y. Peres and O. Zeitouni for discussions on regeneration. MH was supported by the Dutch mathematics cluster STAR during an extended visit to the Centre for Mathematics and Computer Science in Amsterdam in the Fall of 2010. FdH is supported by ERC Advanced Grant 267356 VARIS, RdS by the French ANR project MEMEMO2 10-BLAN-0125-03, VS by Brazilian CNPq grants 308787/2011-0 and 476756/2012-0 and FAPERJ grant E-26/102.878/2012-BBP, and AT by CNPq grants 306348/2012-8 and 478577/2012-5. MH thanks Microsoft Research and IMPA for hospitality and financial support. VS thank Microsoft Research, UC Berkeley, MSRI, FIM ETH-Zurich and ENS-Paris for hospitality and financial support. This work was also supported by the ESF RGLIS Excellence Network.

2 Preliminaries

In this section we give a particular construction of our model, supporting a Poisson point process on the space of two-sided trajectories of environment particles (Section 2.1) and an i.i.d. sequence of Uniform([0, 1]) random variables that are used to define our random walk (Section 2.2). This formulation is equivalent to that given in Section 1, but has the advantage of providing independence and monotonicity properties that are useful throughout the paper (see Definitions 2.1–2.2 and Remark 2.3 below).

Throughout the sequel, c denotes a positive constant that may depend on $v_\circ, v_\bullet$ and may change each time it appears. Further dependence will be made explicit: for example, $c(\eta)$ is a constant that depends on η and possibly on $v_\circ, v_\bullet$. Numbered constants $c_0, c_1, \ldots$ refer to their first appearance in the text and also depend only on $v_\circ$ and $v_\bullet$ unless otherwise indicated.

2.1 Dynamic random environment

Let

$$S = (S^{z,i})_{i\in\mathbb{N},z\in\mathbb{Z}} \quad \text{with} \quad S^{z,i} = (S^{z,i}_n)_{n\in\mathbb{Z}} \tag{2.1}$$

be a doubly-indexed collection of independent lazy simple random walks such that $S^{z,i}_0 = z$ for all $i \in \mathbb{N}$. By this we mean that the past $(S^{z,i}_n)_{n\in\mathbb{Z}_-}$ and the future $(S^{z,i}_n)_n n \in \mathbb{Z}_+$ are independent and distributed as symmetric lazy simple random walks as described in Section 1.

Let $(N(z, 0))_{z\in\mathbb{Z}}$ be a sequence of i.i.d. random variables independent of S. Then, the process $N(\cdot, n)$ defined by

$$N(x, n) = \sum_{z\in\mathbb{N}} \sum_{1\le i\le N(z,0)} \mathbb{1}_{\{S^{z,i}_n = x\}}, \qquad (x, n) \in \mathbb{Z}^2, \tag{2.2}$$

(with 0 assigned to empty sums) is a translation-invariant Markov process representing the number of environment particles at site x and time n. For any density $\rho > 0$, the process N is in equilibrium when we choose the distribution of $N(\cdot, 0)$ to be product Poisson(ρ). Denote by $\mathbb{P}^\rho$ the joint law of $N(\cdot, 0)$ and S in this case.

It will be useful to view N as a subprocess of a Poisson point process on a space of trajectories as follows. Let

$$W = \big\{w = (w(n))_{n\in\mathbb{Z}} \colon\ w(n) \in \mathbb{Z},\ |w(n + 1) - w(n)| \le 1\ \forall\, n \in \mathbb{Z}\big\}, \tag{2.3}$$

denote the set of two-sided nearest-neighbour trajectories on $\mathbb{Z}$. Endow W with the sigma-algebra $\mathcal{W}$ generated by the canonical projections $Z_n(w) = w(n)$, $n \in \mathbb{Z}$. A partition of W into disjoint measurable sets is given by $\{W_x\}_{x\in\mathbb{Z}}$, where $W_x = \{w \in W \colon\ w(0) = x\}$.

We introduce the space $\bar\Omega$ of point measures on W as (be careful to distinguish ω from w)

$$\bar\Omega = \Big\{\omega = \sum_{j\in\mathbb{Z}_+} \delta_{w_j} \colon\ w_j \in W\ \forall\, j \in \mathbb{Z}_+,\ |\omega(W_x)| < \infty\ \forall\, x \in \mathbb{Z}\Big\}, \tag{2.4}$$

and define a random point measure $\omega \in \bar{\Omega}$ by the expression

$$\omega = \sum_{z \in \mathbb{Z}} \sum_{1 \le i \le N(z,0)} \delta_{S^{z,i}}. \tag{2.5}$$

It is then straightforward to check that, under $\mathbb{P}^\rho$, ω is a Poisson point process on W with intensity measure $\rho\mu$, where

$$\mu = \sum_{x \in \mathbb{Z}} P_x \tag{2.6}$$

and P_x is the law on W (with support on W_x) under which $Z(\cdot) = (Z_n(\cdot))_{n\in\mathbb{Z}}$ is distributed as a lazy simple random walk on $\mathbb{Z}$. Moreover, we have that

$$N(x,n) = \omega(\{w \in W \colon w(n) = x\}), \qquad (x,n) \in \mathbb{Z}^2. \tag{2.7}$$

For $w \in W$, let $\mathrm{Trace}(w) = \{(w(n), n)\}_{n\in\mathbb{Z}} \subset \mathbb{Z}^2$ be the trace of w, and define the total trace of the environment trajectories as the set

$$\mathcal{T} = \mathcal{T}(\omega) = \bigcup_{z\in\mathbb{Z}} \bigcup_{1\le i\le N(z,0)} \mathrm{Trace}(S^{z,i}). \tag{2.8}$$

We can now justify (1.1). Noting that $\{(x,k) \notin \mathcal{T}\} = \{\omega(w(k) = x) = 0\}$, compute

$$\begin{aligned} \mathbb{P}^\rho\big[(0,0) \notin \mathcal{T}, (0,n) \notin \mathcal{T}\big] &= \mathbb{P}^\rho\left[\omega\big(w(0) = 0 \text{ or } w(n) = 0\big) = 0\right] \\ &= \exp\left\{-\rho\mu\big(w(0) = 0 \text{ or } w(n) = 0\big)\right\}. \end{aligned} \tag{2.9}$$

Now,

$$\begin{aligned} \mu\big(w(0) = 0 \text{ or } w(n) = 0\big) &= \mu\big(w(0) = 0\big) + \mu\big(w(n) = 0 \neq w(0)\big) \\ &= 1 + P_0(Z_n \neq 0) = 2 - P_0(Z_n = 0), \end{aligned} \tag{2.10}$$

where we used the symmetry of Z. Hence,

$$\begin{aligned} \mathrm{Cov}_\rho\left(\mathbb{1}_{\{(0,0)\in\mathcal{T}\}}, \mathbb{1}_{\{(0,n)\in\mathcal{T}\}}\right) &= \mathrm{Cov}_\rho\left(\mathbb{1}_{\{(0,0)\notin\mathcal{T}\}}, \mathbb{1}_{\{(0,n)\notin\mathcal{T}\}}\right) \\ &= e^{-2\rho}\left(e^{\rho P_0(S_n=0)} - 1\right) \sim c(\rho) n^{-\frac{1}{2}} \end{aligned} \tag{2.11}$$

since $P_0(S_n = 0) \sim c n^{-\frac{1}{2}}$.

2.2 Random walk in dynamic random environment

In order to define the random walk on our dynamic random environment, we first enlarge the probability space. To that end, let us consider a collection of i.i.d. random variables $U = (U_y)_{y\in\mathbb{Z}^2}$, independent of the previous objects, with each U_y uniformly distributed on $[0,1]$. Set $\Omega = \bar{\Omega} \times [0,1]^{\mathbb{Z}^2}$, and redefine $\mathbb{P}^\rho$ to be the probability measure giving the joint law of $N(\cdot, 0)$, S and U.

Given a realisation of ω and U and $y \in \mathbb{Z}^2$, define the random variables Y_n^y, $n \in \mathbb{Z}_+$, as follows:

$$Y_0^y = y,$$
$$Y_{n+1}^y = \begin{cases} Y_n^y + (1,1), & \text{if } Y_n^y \in \mathcal{T}(\omega) \text{ and } U_{Y_n^y} \leq p_\bullet \\ & \text{or } Y_n^y \notin \mathcal{T}(\omega) \text{ and } U_{Y_n^y} \leq p_\circ, \\ Y_n^y + (-1,1), & \text{otherwise.} \end{cases} \tag{2.12}$$

In words, $Y^y = (Y_n^y)_{n \in \mathbb{Z}_+}$ is the space-time process on $\mathbb{Z}^2$ that starts at y, always moves upwards, and is such that its horizontal projection $X^y = (X_n^y)_{n \in \mathbb{Z}_+}$ is a random walk with drift $v_\bullet = 2p_\bullet - 1$ when Y^y steps on $\mathcal{T}(\omega)$ and drift $v_\circ = 2p_\circ - 1$ otherwise. Note that Y^y depends on $\mathcal{T}(\omega)$, but this will be suppressed from the notation. Also note that, for any $y \in \mathbb{Z}^2$, the law of X^y under $\mathbb{P}^\rho$ coincides with the annealed law described in Section 1. So from now on $X = X^0$ will be the random walk in dynamic random environment that we will consider.

Definition 2.1. *For $\omega, \omega' \in \bar{\Omega}$, we say that $\omega \leq \omega'$ when $\mathcal{T}(\omega) \subset \mathcal{T}(\omega')$. We say that a random variable $f\colon \Omega \to \mathbb{R}$ is non-increasing when $f(\omega', \xi) \leq f(\omega, \xi)$ for all $\omega \leq \omega'$ and all $\xi \in [0,1]^{\mathbb{Z}^2}$. We extend this definition to events A in Ω by considering $f = \mathbf{1}_A$. Standard coupling arguments imply that $\mathbb{E}^{\rho'}(f) \leq \mathbb{E}^\rho(f)$ for all non-increasing random variables f and all $\rho \leq \rho'$.*

Definition 2.2. *We say that a random variable $f\colon \Omega \to \mathbb{R}$ has support in $B \subset \mathbb{Z}^2$ when $f(\omega, \xi) = f(\omega', \xi')$ for all $\omega', \omega \in \bar{\Omega}$ with $\mathcal{T}(\omega) \cap B = \mathcal{T}(\omega') \cap B$ and all $\xi, \xi' \in [0,1]^{\mathbb{Z}^2}$ with $\xi(v) = \xi'(v)$ for all $v \in B$.*

Remark 2.3. *The above construction provides two forms of monotonicity:*
(i) Initial position: If $x \leq x'$ have the same parity (i.e., $x' - x \in 2\mathbb{Z}$), then

$$X_i^{(x,n)} \leq X_i^{(x',n)} \qquad \forall\, n \in \mathbb{Z}\ \forall\, i \in \mathbb{Z}_+. \tag{2.13}$$

(ii) Environment: If $v_\circ \leq v_\bullet$ and $\omega \leq \omega'$, then

$$X_i^y(\omega) \leq X_i^y(\omega') \qquad \forall\, y \in \mathbb{Z}^2\ \forall\, i \in \mathbb{Z}_+. \tag{2.14}$$

3 Proof of Theorem 1.5: Renormalisation

In this section we prove Theorem 1.5, which shows the validity of the ballisticity condition in (1.7) when $v_\circ < v_\bullet$ and ρ is large enough. This will be crucial for the proof of Theorem 1.4 later.

In Section 3.2 we introduce the required notation. In Section 3.3 we devise a *renormalisation scheme* (Lemmas 3.2–3.3) to show that under a "finite-size criterion" the random walk moves ballistically, and we prove that for large enough ρ this criterion holds (Lemma 3.4). In Section 3.4 we show that the renormalisation scheme yields the large deviation bound in Theorem 1.5 (Lemma 3.5). This bound will be needed in Section 4, where we show that, as the random walk explores fresh parts of the dynamic random environment, it builds up a *regeneration structure* that serves as a "skeleton" for the proof of Theorem 1.4. In Section 3.5 we comment on possible extensions.

3.1 Space-time decoupling

In order to implement our renormalisation scheme, we need to control the dependence of events having support in two boxes that are well separated in space-time. This is the content of the following corollary of Theorem C.1, the proof of which is deferred to Appendix C.

Corollary 3.1. *Let* $B_1 = ([a,b] \times [n,m]) \cap \mathbb{Z}^2$ *and* $B_2 = ([a',b'] \times [-n',0]) \cap \mathbb{Z}^2$ *be two space-time boxes (observe that their time separation is* n*) and assume that* $n \geq c$*. Recall Definitions 2.2 and 2.1, and assume that* $f_1\colon \Omega \to [0,1]$ *and* $f_2\colon \Omega \to [0,1]$ *are non-increasing random variables with support in* B_1 *and* B_2*, respectively. Then, for any* $\rho \geq 1$*,*

$$\mathbb{E}^{\rho(1+n^{-1/16})}[f_1 f_2] \leq \mathbb{E}^{\rho(1+n^{-1/16})}[f_1]\ \mathbb{E}^{\rho}[f_2] + c\big(\mathrm{per}(B_1) + n\big)\, e^{-cn^{1/8}}, \tag{3.1}$$

where $\mathrm{per}(B_1)$ *stands for the perimeter of* B_1*.*

The decoupling in Corollary 3.1, together with the monotonicity stated in Definition 2.1, are the only assumptions on our dynamic random environment that are used in the proof of Theorem 1.5. Hence, the results in this section can in principle be extended to different dynamic random environments. (See Section 3.5 for more details.)

3.2 Scale notation

Define recursively a sequence of scales $(L_k)_{k\in\mathbb{Z}_+}$ by putting

$$L_0 = 100, \qquad L_{k+1} = \lfloor L_k^{1/2} \rfloor L_k. \tag{3.2}$$

(The choice $L_0 = 100$ has no special importance: any integer ≥ 4 will do, as long as it stays fixed.) Note that the above sequence grows super-exponentially fast: $\log L_k \sim (3/2)^k \log L_0$ as $k \to \infty$. For $L \in \mathbb{N}$, let B_L be the space-time rectangle

$$B_L = \big([-L, 2L] \times [0, L]\big) \cap \mathbb{Z}^2 \tag{3.3}$$

and I_L its middle bottom line

$$I_L = [0, L] \times \{0\} \subset B_L \tag{3.4}$$

(see Fig. 1). For $m = (r,s) \in \mathbb{Z}^2$, let

$$B_L(m) = B_L(r,s) = (r,s)L + B_L, \qquad I_L(m) = I_L(r,s) = (r,s)L + I_L. \tag{3.5}$$

For $k \in \mathbb{N}$, let

$$M_k = \big\{(r,s) \in \mathbb{Z}^2\colon\ B_{L_k}(r,s) \cap B_{L_{k+1}} \neq \emptyset\big\} \tag{3.6}$$

denote the set of all indices whose corresponding shift of the rectangle B_{L_k} still intersects the larger rectangle $B_{L_{k+1}} = B_{L_{k+1}}(0,0)$.

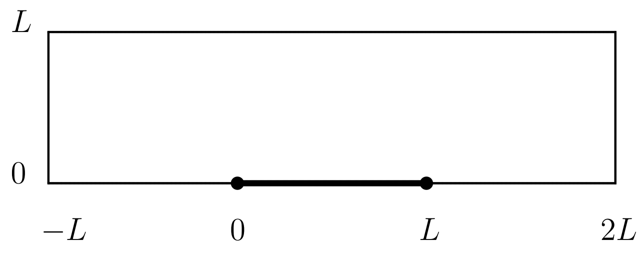

Figure 1: Picture of B_L (rectangle) and I_L (middle bottom line).

Fix $v < v_\bullet$, let $\delta = \frac{1}{2}(v_\bullet - v)$, and define recursively a sequence $(v_k)_{k\in\mathbb{N}}$ of velocities by putting

$$v_1 = v_\bullet - \delta, \qquad v_{k+1} = v_k - \delta\left(\frac{6}{\pi^2}\right)\frac{1}{k^2}. \tag{3.7}$$

Since $\sum_{k\in\mathbb{N}} 1/k^2 = \pi^2/6$, it follows that $k \mapsto v_k$ decreases strictly to v. The reason why we introduce a speed for each scale k is to allow for small errors as we change scales. (The need for this "perturbation" will become clear in (3.14) below.)

We are interested in bounding the probability of bad events A_k on which the random walk does not move to the right with speed at least v_k, namely,

$$A_k(m) = \left\{\exists\,(x,n) \in I_{L_k}(m)\colon\ X^{(x,n)}_{L_k} - x < v_k L_k\right\}, \qquad k \in \mathbb{N}, m \in \mathbb{Z}^2. \tag{3.8}$$

Note that $A_k(m)$ is defined in terms of the dynamic random environment and the random walk within $B_{L_k}(m)$, so that $\mathbb{1}_{A_k(m)}$ is a random variable with support in $B_{L_k}(m)$, in the sense of Definition 2.2. We say that $B_{L_k}(m)$ is a *slow box* when $A_k(m)$ occurs. Since we are assuming that $v_\bullet > v_\circ$ (recall (1.6)), for each k and m the random variable $\mathbb{1}_{A_k(m)}$ is non-increasing in the sense of Definition 2.1.

Define recursively a sequence $(\rho_k)_{k\in\mathbb{Z}_+}$ of densities by putting

$$\rho_0 > 0, \qquad \rho_{k+1} = (1 + L_k^{-1/16})\rho_k. \tag{3.9}$$

Again, we introduce a density for each scale k in order to allow for small errors. (The need for this "sprinkling" will become clear in (3.18) below.) Observe that ρ_k increases strictly to $\rho_\star$ defined by

$$\rho_\star = \rho_0 \prod_{l=0}^{\infty}(1 + L_l^{-1/16}) \in (\rho_0, \infty). \tag{3.10}$$

Finally, define

$$p_k = \mathbb{P}^{\rho_k}(A_k(0)) = \mathbb{P}^{\rho_k}(A_k(m)), \qquad k \in \mathbb{N}, \tag{3.11}$$

where the last equality holds for all $m \in \mathbb{Z}^2$ because of translation invariance.

3.3 Estimates on p_k

Lemmas 3.2–3.4 below show that p_k decays very rapidly with k when ρ_0 is chosen large enough.

The first step is to prove a recursion inequality that relates p_{k+1} with p_k:

Lemma 3.2. *Fix $\rho_0 \geq 1$. There is a $k_0 = k_0(\delta)$ such that, for all $k > k_0(\delta)$,*

$$p_{k+1} \leq c_1 L_k^2 \left[p_k^2 + L_k e^{-c_2 L_k^{1/8}} \right]. \tag{3.12}$$

Proof. We begin by claiming the following:

$$\begin{array}{l}\text{If } A_{k+1}(0) \text{ occurs, then there are at least three elements} \\ m_1 = (r_1, s_1),\ m_2 = (r_2, s_2),\ m_3 = (r_3, s_3) \text{ in } M_k, \text{ with} \\ s_i \neq s_j \text{ when } i \neq j, \text{ such that } A_k(m_i) \text{ occurs for } i = 1, 2, 3.\end{array} \tag{3.13}$$

The proof is by contradiction. Suppose that the claim is false. Then there are at most two elements $m = (r, s)$, $m' = (r', s')$ in M_k, with $s \neq s'$, such that $B_{L_k}(m)$ and $B_{L_k}(m')$ are slow boxes. It then follows that, for any $(x, n) \in I_{L_{k+1}}$,

$$\begin{aligned} X^{(x,n)}_{L_{k+1}} - x &= \textstyle\sum_{j=1}^{\lfloor L_k^{1/2} \rfloor} \left[X^{(x,n)}_{jL_k} - X^{(x,n)}_{(j-1)L_k} \right] \\ &\geq -2L_k + v_k L_k \left(\tfrac{L_{k+1}}{L_k} - 2 \right) \\ &\geq -4L_k + v_k L_{k+1}, \end{aligned} \tag{3.14}$$

where the terms in the sum correspond to the displacements over the $\lfloor L_k^{1/2} \rfloor$ time layers of height L_k in the box $B_{L_{k+1}}$. The term $-2L_k$ appears in the right-hand side of the first inequality because there are at most two layers (associated with the two slow boxes mentioned above) in which the total displacement of the random walk is at least $-L_k$, since the minimum speed is -1. The second inequality uses that $v_k \leq 1$. Since L_k increases faster than exponentially in k, there is a $k_0 = k_0(\delta)$ such that, for all $k \geq k_0(\delta)$,

$$\delta \left(\frac{6}{\pi^2} \right) \frac{1}{k^2} \geq \frac{4}{\lfloor L_k^{1/2} \rfloor}. \tag{3.15}$$

Hence, for all $k \geq k_0(\delta)$,

$$\begin{aligned} -4L_k + v_k L_{k+1} &= -4L_k + \left[\delta \left(\tfrac{6}{\pi^2} \right) \tfrac{1}{k^2} \right] L_{k+1} + v_{k+1} L_{k+1} \\ &= -\tfrac{4}{\lfloor L_k^{1/2} \rfloor} L_{k+1} + \left[\delta \left(\tfrac{6}{\pi^2} \right) \tfrac{1}{k^2} \right] L_{k+1} + v_{k+1} L_{k+1} \\ &\geq v_{k+1} L_{k+1}. \end{aligned} \tag{3.16}$$

Substituting this into (3.14) we get

$$X^{(x,n)}_{L_{k+1}} - x \geq v_{k+1} L_{k+1} \qquad \forall\, (x, n) \in I_{L_{k+1}}, \tag{3.17}$$

so that $A_{k+1}(0)$ cannot occur. This proves the claim (3.13).

Thus, on the event $A_{k+1}(0)$, we may assume that there exist $m_1 = (r_1, s_1)$, $m_3 = (r_3, s_3)$ in M_k such that $s_3 \geq s_1+2$, meaning that the vertical distance between $B_{L_k}(m_3)$ and $B_{L_k}(m_1)$ is at least L_k. It follows from Corollary 3.1 and the fact that the events $A_k(m)$ are non-increasing that

$$\begin{aligned}
\mathbb{P}^{\rho_{k+1}}\left(A_k(m_1) \cap A_k(m_2)\right) &\leq \mathbb{P}^{\rho_{k+1}}\left(A_k(m_1)\right)\mathbb{P}^{\rho_k}\left(A_k(m_3)\right) \\
&\qquad + c\left[\text{per}(B_{L_k}) + L_k\right] e^{-c\rho_k L_k^{1/8}} \\
&\leq \mathbb{P}^{\rho_k}\left(A_k(m_1)\right)^2 + c\left[\text{per}(B_{L_k}) + L_k\right] e^{-c\rho_k L_k^{1/8}} \\
&\leq p_k^2 + cL_k e^{-cL_k^{1/8}},
\end{aligned} \tag{3.18}$$

where $\text{per}(B_{L_k})$ denotes the perimeter of B_{L_k}, and in the last inequality we use that $\rho_k \geq \rho_0 \geq 1$. Since there are at most $c(L_{k+1}/L_k)^4 = c\lfloor L_k^{1/2} \rfloor^4$ possible choices of pairs of boxes $B_{L_k}(m_1)$ and $B_{L_k}(m_3)$ in M_k, it follows that

$$p_{k+1} \leq cL_k^2\left[p_k^2 + L_k e^{-cL_k^{1/8}}\right], \tag{3.19}$$

which completes the proof of (3.12). □

Next, we prove a recursive estimate on p_k.

Lemma 3.3. *There exists a $k_1 = k_1(\delta) \geq k_0(\delta)$ such that, for all $k \geq k_1$,*

$$p_k < e^{-\log^{3/2} L_k} \quad \Longrightarrow \quad p_{k+1} < e^{-\log^{3/2} L_{k+1}}. \tag{3.20}$$

Proof. Suppose that $p_k < e^{-\log^{3/2} L_k}$ for some $k \geq k_0(\delta)$. For such a k, Lemma 3.2 gives

$$p_{k+1} \leq c_1 L_k^2\left[p_k^2 + L_k e^{-c_2 L_k^{1/8}}\right] \leq c_1 L_k^2\left[e^{-2\log^{3/2} L_k} + L_k e^{-c_2 L_k^{1/8}}\right]. \tag{3.21}$$

Pick $k_1 = k_1(\delta) \geq k_0(\delta)$ such that

$$c_1 L_k^2\left(e^{-(1/10)\log^{3/2} L_k} + L_k e^{-c_2 L_k^{1/8} + \log^{3/2} L_{k+1}}\right) < 1 \qquad \forall\, k \geq k_1, \tag{3.22}$$

which is possible because $\lim_{k\to\infty} L_k = \infty$. Dividing (3.21) by $e^{-\log^{3/2} L_{k+1}}$, recalling from (3.2) that $L_{k+1} \leq L_k^{3/2}$ and using (3.22), we get

$$p_{k+1}\, e^{\log^{3/2} L_{k+1}} \leq c_1 L_k^2\Big[e^{-\overbrace{(2-(3/2)^{3/2})}^{>(1/10)}\log^{3/2} L_k} + L_k e^{-c_2 L_k^{1/8} + \log^{3/2} L_{k+1}}\Big] \overset{(3.22)}{<} 1, \tag{3.23}$$

which completes the proof of the (3.20). □

Finally, we show that if ρ_0 is taken large enough, then it is possible to trigger the recursive estimate in 3.3.

Lemma 3.4. *There exist ρ_0 large enough and $k_2 = k_2(\delta) \geq k_1(\delta)$ such that $p_{k_2} < e^{-\log^{3/2} L_k}$.*

Proof. Recall (3.3), (3.8) and (3.11). Recall also from (2.2) that $N(x,n)$ denotes the number of particles in our dynamic random environment that cross (x,n) (i.e., $N(x,n) = \omega(\{w \in W; w(n) = x\})$), and let

$$C_k = \Big\{ N(x,n) \geq 1 \ \forall\, (x,n) \in B_{L_k} \Big\} \tag{3.24}$$

be the event that all space-time points in B_{L_k} are occupied by a particle. Estimate

$$p_k \leq \mathbb{P}^{\rho_k}(A_k(0) \mid C_k) + \mathbb{P}^{\rho_k}(C_k^c). \tag{3.25}$$

The first term in the right-hand side of (3.25) can be estimated from above by

$$\begin{aligned} \mathbb{P}^{\rho_k}(A_k(0) \mid C_k) &\leq L_k \mathbb{P}^{\rho_k}(X^0_{L_k} < v_k L_k \mid C_k) \\ &\leq L_k \mathbb{P}^{\rho_k}(X^0_{L_k} < v_1 L_k \mid C_k) \\ &\leq L_k \mathbb{P}^{\rho_k}\Big(\tfrac{X^0_{L_k}}{L_k} < v_\bullet - \delta \mid C_k\Big), \end{aligned} \tag{3.26}$$

where the last inequality uses (3.7). On the event C_k, all the space-time points of B_{L_k} in the dynamic random environment are occupied, and so the law of $(X^0_n)_{0 \leq n \leq L_k}$ coincides with that of a nearest-neighbour random walk with drift $v_\bullet$ starting at 0. Therefore, by an elementary large deviation estimate, we have

$$\mathbb{P}^{\rho_k}(A_k(0) \mid C_k) \leq L_k e^{-c(\delta) L_k}, \qquad k \in \mathbb{N}, \tag{3.27}$$

independently of the choice of ρ_0. We can therefore choose $k_2 = k_2(\delta)$ large enough so that

$$\mathbb{P}^{\rho_k}(A_{k_2}(0) \mid C_{k_2}) \leq \tfrac{1}{2} e^{-\log^{3/2} L_{k_2}}. \tag{3.28}$$

Having fixed k_2, we next turn our attention to the second term in the right-hand side of (3.25). Recalling that, under $\mathbb{P}^{\rho_{k_2}}$, the random variables $(N(x,n))_{x \in \mathbb{Z}}$ are Poisson(ρ_{k_2}), we have $\mathbb{P}^{\rho_{k_2}}(C^c_{k_2}) \leq 3L^2_{k_2} e^{-\rho_{k_2}}$. Since this tends to zero as $\rho_0 \to \infty$ (recall (3.9)), we can take ρ_0 large enough so that

$$\mathbb{P}^{\rho_{k_2}}(C^c_{k_2}) \leq \tfrac{1}{2} e^{-\log^{3/2} L_{k_2}}. \tag{3.29}$$

Combine (3.25), (3.28) and (3.29) to get the claim. □

3.4 Large deviation bounds

Together with Lemmas 3.3–3.4, the following lemma will allow us to prove Theorem 1.5.

Define the half-plane

$$\mathcal{H}_{v,L} = \big\{ (x,n) \in \mathbb{Z}^2 \colon\ x \leq nv - L \big\}. \tag{3.30}$$

Lemma 3.5. *Fix $v < v_\bullet$ and $\rho > 0$. Suppose that $\mathbb{P}^\rho(A_k) \le e^{-\log^{3/2} L_k}$ for all $k \ge c_3$ (where A_k is defined in terms of v as in (3.8)). Then, for any $\varepsilon > 0$,*

$$\mathbb{P}^\rho\Big(\exists\, n \in \mathbb{N}\colon\ Y_n \in \mathcal{H}_{v-\varepsilon,L}\Big) \le c(\varepsilon, c_3) L^{7/2}\, e^{-c\log^{3/2} L} \qquad \forall\, L \in \mathbb{N}. \tag{3.31}$$

Proof. We first choose $k_0 = k_0(\varepsilon)$ such that $L_{k+1}/L_k > 1 + 2/\varepsilon$ for all $k \ge k_0$. A trivial observation is that we may assume that $L > 2L_{k_0 \vee c_3 + 2}$, as this would at most change the constant $c(\varepsilon, c_3)$ in (3.31). We thus choose $\check{k}$ such that

$$2L_{\check{k}+2} \le L < 2L_{\check{k}+3}. \tag{3.32}$$

Note that $\check{k} \ge k_0$ by our assumption on L.

We next define the set of indices (see Fig. 2)

$$M'_k = \{m \in M_k\colon\ B_{L_k}(m) \subseteq B_{L_{k+2}}(0)\}, \qquad k \in \mathbb{Z}_+, \tag{3.33}$$

and consider the event

$$B_{\check{k}} = \bigcap_{k \ge \check{k}} \bigcap_{m \in M'_k} A_k(m)^c. \tag{3.34}$$

This event has high probability. Indeed, according to our hypothesis on the decay of $\mathbb{P}^\rho(A_k)$, and since $\check{k} \ge c_3$, we have

$$\begin{aligned}\mathbb{P}^\rho(B^c_{\check{k}}) &\le \sum_{k \ge \check{k}} \sum_{m \in M'_k} \mathbb{P}^\rho(A_k(m)) \le \sum_{k \ge \check{k}} c\Big(\frac{L_{k+2}}{L_k}\Big)^2 e^{-\log^{3/2} L_k} \\ &\le c \sum_{l \ge L_{\check{k}}} l^{5/2} e^{-\log^{3/2} l} \le c L_{\check{k}}^{7/2} e^{-\log^{3/2} L_{\check{k}}} \le c L^{7/2} e^{-c\log^{3/2} L},\end{aligned} \tag{3.35}$$

where in the fourth inequality we use Lemma D.1, while in the last inequality we use that $L_{\check{k}} > L_{\check{k}+3}^{(3/2)^{-3}} \ge cL^{(3/2)^{-3}}$ (see (3.2) and (3.32)) and that $2L_{\check{k}} < L$. It is therefore enough to show that the event in (3.31) is contained in $B^c_{\check{k}}$.

Define the set of times

$$J_{\check{k}} = \bigcup_{k \ge \check{k}} \bigcup_{l=0}^{L_{k+2}/L_k} \{lL_k\} \subset \mathbb{Z}_+. \tag{3.36}$$

We claim that on the event $B_{\check{k}}$,

$$X_j \ge vj \qquad \forall\, j \in J_{\check{k}}. \tag{3.37}$$

To see why this is true, fix some $k \ge \check{k}$ as in the definition of $J_{\check{k}}$. It is clear that the inequality holds for $j = 0$. Suppose by induction that $X_{lL_k} \ge vlL_k$ for some $l \le L_{k+2}/L_k$. Observe that Y_{lL_k} belongs to some box $B_k(m)$ with $m \in M'_k \subset M'_{\check{k}}$. It even belongs to the corresponding interval $I_{L_k}(m)$ as defined in (3.5). Since we are on the event $A_k(m)^c$, this implies that

$$X_{(l+1)L_k} = X_{lL_k} + X_{L_k}^{Y_{lL_k}} - X_{lL_k} \ge v(l+1)L_k, \tag{3.38}$$

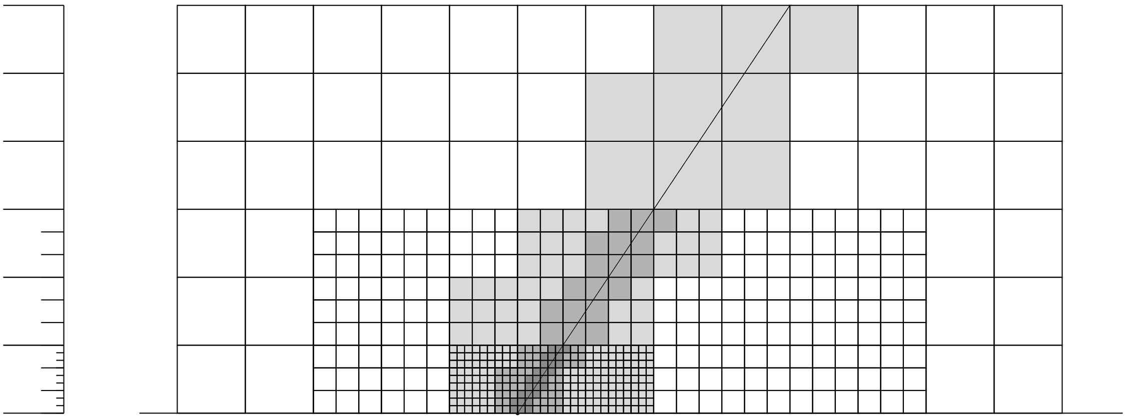

Figure 2: Illustration of space-time points $m \in M'_k$, for $k = 0, 1, 2$. The boxes $B_k(m)$ for which $I_k(m)$ intersects the line of constant speed v are shaded. The set J_0 is drawn on the left: different scales appear with different tick lengths.

which shows that the bound in (3.37) holds for $l + 1$. Since this can be done for any $k \ge \check{k}$, we have proven (3.37) by induction.

We now interpolate the statement in (3.37) to all times $n > 2L_{\check{k}+2}(> L_{\check{k}+2} + L_{\check{k}})$. More precisely, we will show that, on the event $B_{\check{k}}$,

$$X_n \ge (v - \varepsilon)n \qquad \forall\, n \ge 2L_{\check{k}+2}. \tag{3.39}$$

Indeed, given such a $n \ge L_{\check{k}+2} + L_{\check{k}}$, we fix $\bar{k}$ to be the smallest k such that

$$\exists\, l \le L_{\bar{k}+2}/L_{\bar{k}} \colon\; n \in \big[lL_{\bar{k}}, (l+1)L_{\bar{k}}\big), \tag{3.40}$$

and we write $\bar{l}$ for this unique value of l. Noting that

$$\bar{k} > \check{k} \ge k_0 \text{ and } \bar{l} \ge L_{\bar{k}+1}/L_{\bar{k}} - 1 > 2/\varepsilon, \tag{3.41}$$

we can put the above pieces together and estimate

$$\begin{aligned} X_n &= X_{\bar{l}L_{\bar{k}}} + X^{Y_{\bar{l}L_{\bar{k}}}}_{n-\bar{l}L_{\bar{k}}} - X_{\bar{l}L_{\bar{k}}} \ge v\bar{l}L_{\bar{k}} - L_{\bar{k}} = L_{\bar{k}}(v\bar{l} - 1) \\ &= L_{\bar{k}}\big((v-\varepsilon)\bar{l} + \varepsilon\bar{l} - 1\big) \ge L_{\bar{k}}\big((v-\varepsilon)\bar{l} + 1\big) \\ &\ge \max\big(L_{\bar{k}}(\bar{l}+1)(v-\varepsilon), L_{\bar{k}}\bar{l}(v-\varepsilon)\big) \ge (v-\varepsilon)n, \end{aligned} \tag{3.42}$$

where the first inequality uses (3.37), $\bar{k} \ge \check{k}$ and the definition of $\bar{l}$, the second inequality uses that $\bar{l} > 2/\varepsilon$, the third inequality uses that $v - \varepsilon \le 1$ and, for the fourth inequality, we use (3.40) considering separately the cases $v - \varepsilon \ge 0$, $v - \varepsilon < 0$. This proves (3.39).

To complete the proof, we observe that, since X is Lipschitz, having (as in (3.39)) $X_n \ge (v - \varepsilon)n$ for any $n \ge 2L_{\check{k}+2}$ we get $X_n \ge (v - \varepsilon)n - 2L_{\check{k}+2} \ge (v - \varepsilon)n - L$ for all $n \in \mathbb{Z}_+$. Thus, we have proved that the event appearing in the right-hand side of (3.31) is contained in $B^c_{\check{k}}$, so that its probability is bounded as in (3.35). □

Proof of Theorem 1.5. Put $v = v_\star + \varepsilon$, let ρ_0 be large enough to satisfy Lemma 3.4, and take $\rho_\star$ as in (3.10). Recalling that X_n is the horizontal projection of Y_n and that, by monotonicity,

$$\mathbb{P}^{\rho_\star}\left(A_k(0)\right) \leq p_k \qquad \forall\, k \in \mathbb{N}, \tag{3.43}$$

we see that Lemmas 3.3–3.5 prove the large deviation bound in Theorem 1.5. □

Remark 3.6. *Note that the speed in Lemma* 3.5 *was not chosen arbitrarily below the speed given by the law of large numbers in* (1.4)*. What we have obtained is that for any* $v < v_\bullet$ *there exists a density* $\rho_0(v)$ *such that* (3.31) *holds for* $\rho \geq \rho_0(v)$*.*

3.5 Extensions

The ballisticity statement in Theorem 1.5 holds under mild conditions on the underlying dynamic random environment. Indeed, the only assumptions we have made on the law of $\mathcal{T}$ are:

(i) The monotonicity stated in Definition 2.1 (see (3.18)).

(ii) The decoupling provided by Corollary 3.1 (used in (3.18)).

(iii) The perturbative condition $\lim_{\rho\to\infty} \mathbb{P}_\rho[0 \in \mathcal{T}] = 1$ (used to trigger (3.29)).

Let us elaborate a bit more on the space-time decoupling condition given by Corollary 3.1. This condition was designed with our particular dynamic random environment in mind, which lacks good relaxation properties. However, several dynamic random environments satisfy the simpler and stronger condition

$$\mathbb{E}^\rho[f_1 f_2] \leq \mathbb{E}^\rho[f_1]\mathbb{E}^\rho[f_2] + c\,\mathrm{per}(B_1)^c e^{-cn^\kappa}, \tag{3.44}$$

for some $\kappa > 0$ and all f_1 and f_2 with support in, respectively, $B_1 = [a, b] \times [n, m]$ and $B_2 = [a', b'] \times [-n', 0]$. It is important to observe that the constants appearing in (3.44) are not allowed to depend on ρ, since the triggering of (3.29) is done after the induction inequality of Lemma 3.3. The condition in (3.44) holds, for instance, when the dynamic random environment has a spectral gap that is bounded from below for ρ large enough. Such a property can be obtained for a variety of reversible dynamics with the help of techniques from Liggett [15].

The contact process. It can be shown that (3.44) holds for the supercritical contact process for non-increasing f_1, f_2, uniformly in infection parameters that are uniformly bounded away from the critical threshold. A proof can be developed using the graphical representation (see e.g. Remark 3.7 in [11]) and the strategy of Theorem C.1. Note, however, that the results in [11] already imply stronger results for the large deviations of the random walk in the regime of large infection parameter, namely, (1.6) with exponential decay.

Independent renewal chains. Let us mention another model for which our techniques can establish a ballistic lower bound for the random walk. Consider the probability distribution $p = (p_n)_{n\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ given by $p_n = \exp[-n^{1/4}]/Z$, where $Z = \sum_{n\in\mathbb{Z}_+} \exp[-n^{1/4}]$. Define Markov chain transition probabilities given by

$$g(l,m) = \begin{cases} \delta_{l-1}(m), & \text{if } l \in \mathbb{N}, \\ p_m, & \text{if } l = 0. \end{cases} \tag{3.45}$$

This Markov chain moves down one unit a time until it reaches zero. At zero it jumps to a random height according to distribution p. We call this the renewal chain with interarrival distribution p. It has stationary measure $q = (q_n)_{n\in\mathbb{Z}_+}$ given by

$$q_n = \frac{1}{Z'} \sum_{j\geq n} \exp[-j^{1/4}], \qquad Z' = \sum_{n\in\mathbb{Z}_+} \sum_{j\geq n} \exp[-j^{1/4}]. \tag{3.46}$$

For each site $x \in \mathbb{Z}$, we produce an independent copy $N(x,n)_{n\in\mathbb{Z}_+}$ of the above Markov chain. Denote by P_ν the law of one chain started from the probability distribution ν. We define as a dynamic random environment the field given by these chains when starting from the stationary distribution q.

We fix $\rho \geq 0$ and set $\mathcal{T} = \{(x,n)\colon\ N(x,n) < \rho\}$, so that we can define the random walk $(Y_n)_{n\in\mathbb{Z}_+}$ as in (2.12).

In order to prove Corollary 3.1 for this dynamic random environment, we would like to couple two renewal chains $N(0,n)$, $N'(0,n)$, starting, respectively, at δ_0 and q, in such a way that they coalesce at a random time T. Using Proposition 3 of [16], we obtain such a coupling with $E_{\delta_0,q}[\exp[T^{1/8}]] < \infty$ (note that p is aperiodic, i.e., $\gcd(\operatorname{supp}(p)) = 1$).

We now fix any events $A \in \sigma(N(0,m)\colon\ m \leq 0)$ and $B \in \sigma(N(0,m)\colon\ m \geq n)$, and estimate $P_q[A\cap B] - P_q[A]P_q[B]$. For this, we first check whether $N(0,m)$ reaches zero before $n/2$ and, if so, we try to couple it with an independent $N'(0,m)$ starting from the stationary distribution. This leads to

$$\begin{aligned} P_q\big[A\cap B\big] &\leq P_q[N(0,0) > n/2] + P_q\big[A\big] \sup\nolimits_{1\leq j\leq n/2} P_{\delta_j}\big[B\big] \\ &\leq P_q[N(0,0) > n/2] + P_{\delta_0,q}[T \geq n/2] + P_q\big[A\big]P_q\big[B\big] \\ &\leq P_q\big[A\big]P_q\big[B\big] + c\exp[-cn^{1/8}], \end{aligned} \tag{3.47}$$

where in the last inequality we use the definition of q and the Markov inequality for $\exp[T^{1/8}]$. Repeating this for every chain $N(x,n)$ with $x \in [a,b]$, we prove (3.44) for $\mathcal{T}$ with $\kappa = \frac{1}{8}$. It is clear that $\lim_{\rho\to\infty} P[0 \in \mathcal{T}] = 0$. Thus, the conclusion of Theorem 1.5 holds for the dynamic random environment $\mathcal{T}$.

In fact, also Theorem 1.4 holds in this case, as a simple regeneration strategy can be found; see Section 4.4. As a consequence, the statements of Theorem 1.2 are true for this example.

Remark 3.7. *Observe that $\mathcal{T}$ is not uniformly mixing. Indeed, given any $n \in \mathbb{Z}_+$, we can start our Markov chain in events with positive probability (say, $N(0,0) = 2n$) such that the information at time zero is not forgotten until time n.*

4 Proof of Theorem 1.4: Regeneration

In this section, we state and prove two results about regeneration times (Theorems 4.1–4.2) that are then used to prove Theorem 1.4 in Section 4.3. A discussion about extensions is given in Section 4.4.

In Section 4.1 we introduce some additional notation in order to define our regeneration time. This definition is made in a non-algorithmic way and does not immediately imply that the regeneration time is finite with probability 1. Nonetheless, in the latter event we are able to show in Theorem 4.1 that a renewal property holds for the law of the random walk path. The next step is to prove Theorem 4.2, which shows that the regeneration time not only is a.s. finite but also has a very good tail. This is accomplished by finding a suitable upper bound, which consists of two main steps. First, we define what we call *good record times* and show that these appear very frequently (Proposition 4.6). This is done in an algorithmic fashion, but only by exploring the system locally at each step. Second, we show that, outside a global event of small probability, if we can find a good record time then we can also find nearby an upper bound for the regeneration time.

4.1 Notation

Suppose that $\rho \in (0,\infty)$, $v_\star \in (0, v_\bullet)$ and $c \in (0,\infty)$ satisfy (1.7). Conditions for this are given in Theorem 1.5 and Remark 1.6. In the sequel we abbreviate $\mathbb{P} = \mathbb{P}^\rho$.

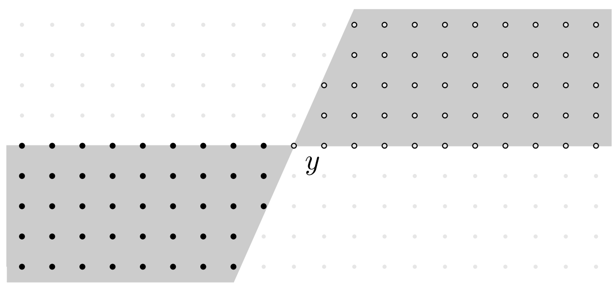

Figure 3: An illustration of the sets $\angle(y)$ (represented by white circles) and $\measuredangle(y)$ (represented by filled black circles), with $y = (x,n) \in \mathbb{Z}^2$.

Define $\bar{v} = \frac{1}{3} v_\star$. Let $\angle(x,n)$ be the cone in the first quadrant based at (x,n) with angle $\bar{v}$, i.e.,

$$\angle(x,n) = \angle(0,0) + (x,n), \text{ where } \angle(0,0) = \{(x,n) \in \mathbb{Z}_+^2; x \geq \bar{v}n\}, \tag{4.1}$$

and $\measuredangle(x,y)$ the cone in the third quadrant based at (x,n) with angle $\bar{v}$, i.e.,

$$\measuredangle(x,n) = \measuredangle(0,0) + (x,n), \text{ where } \measuredangle(0,0) = \{(x,n) \in \mathbb{Z}_-^2 : \ x < \bar{v}n\}. \tag{4.2}$$

(See Figure 3.) Note that $(0,0)$ belongs to $\angle(0,0)$ but not to $\measuredangle(0,0)$.

Define the following sets of trajectories in W:

$$
\begin{aligned}
W^{\angle}_{x,n} &= \text{ trajectories that intersect } \angle(x,n) \text{ but not } \rotatebox{180}{$\angle$}(x,n),\\
W^{\rotatebox{180}{$\angle$}}_{x,n} &= \text{ trajectories that intersect } \rotatebox{180}{$\angle$}(x,n) \text{ but not } \angle(x,n),\\
W^{+}_{x,n} &= \text{ trajectories that intersect both } \angle(x,n) \text{ and } \rotatebox{180}{$\angle$}(x,n).
\end{aligned}
\tag{4.3}
$$

Note that $W^{\angle}$, $W^{\rotatebox{180}{$\angle$}}$ and W^{+} form a partition of W. We write Y_n to denote Y^0_n. For $y \in \mathbb{Z}^2$, define the sigma-algebras

$$
\mathcal{G}^I_y = \sigma\left(\omega(A)\colon\ A \subset W^I_y, A \in \mathcal{W}\right), I = \angle, \rotatebox{180}{$\angle$}, +, \tag{4.4}
$$

and note that these are jointly independent. Also define the sigma-algebras

$$
\begin{aligned}
\mathcal{U}^{\angle}_y &= \sigma\left(U_z\colon\ z \in \angle(y)\right),\\
\mathcal{U}^{\rotatebox{180}{$\angle$}}_y &= \sigma\left(U_z\colon\ z \in \rotatebox{180}{$\angle$}(y)\right),
\end{aligned}
\tag{4.5}
$$

and set

$$
\mathcal{F}_y = \mathcal{G}^{\rotatebox{180}{$\angle$}}_y \vee \mathcal{G}^{+}_y \vee \mathcal{U}^{\rotatebox{180}{$\angle$}}_y. \tag{4.6}
$$

Next, define the *record times*

$$
R_k = \inf\{n \in \mathbb{Z}_+\colon\ Y_n \in \angle(k,0)\}, \qquad k \in \mathbb{N}, \tag{4.7}
$$

i.e., the times at which the walk enters the cone $\angle(k,0)$, and define the filtration $\mathcal{F} = (\mathcal{F}_k)_{k\in\mathbb{N}}$ by setting $\mathcal{F}_\infty = \sigma\left(\omega(A)\colon\ A \in \mathcal{W}\right) \vee \sigma(U_y\colon y \in \mathbb{Z}^2)$ and

$$
\mathcal{F}_k = \left\{B \in \mathcal{F}_\infty\colon\ \forall\, y \in \mathbb{Z}^2, \exists\, B_y \in \mathcal{F}_y \text{ with } B \cap \{Y_{R_k} = y\} = B_y \cap \{Y_{R_k} = y\}\right\}, \tag{4.8}
$$

i.e., the sigma-algebra generated by Y_{R_k}, all U_z with $z \in \rotatebox{180}{$\angle$}(Y_{R_k})$ and all $\omega(A)$ such that $A \subset W^{\rotatebox{180}{$\angle$}}_{Y_{R_k}} \cup W^{+}_{Y_{R_k}}$. In particular, $(Y_i)_{0\le i\le R_k} \in \mathcal{F}_k$.

Finally, define the event A^y, in which the walker remains inside the cone $\angle(y)$:

$$
A^y = \left\{Y^y_i \in \angle(y)\ \forall\, i \in \mathbb{Z}_+\right\}, \tag{4.9}
$$

the probability measure

$$
\mathbb{P}^{\angle}(\cdot) = \mathbb{P}\left(\ \cdot \ \middle|\ \omega\big(W^{+}_0\big) = 0,\ A^0\right), \tag{4.10}
$$

the *regeneration record index*

$$
\mathcal{I} = \inf\left\{k \in \mathbb{N}\colon\ \omega\big(W^{+}_{Y_{R_k}}\big) = 0,\ A^{Y_{R_k}} \text{ occurs }\right\} \tag{4.11}
$$

and the *regeneration time*

$$
\tau = R_{\mathcal{I}}. \tag{4.12}
$$

4.2 Regeneration theorems

The following two theorems are our key results for the regeneration times.

Theorem 4.1. *Almost surely on the event* $\{\tau < \infty\}$*, the process* $(Y_{\tau+i} - Y_\tau)_{i\in\mathbb{Z}_+}$ *under either the law* $\mathbb{P}(\,\cdot \mid \tau, (Y_i)_{0\le i\le\tau})$ *or* $\mathbb{P}^{\angle}(\,\cdot \mid \tau, (Y_i)_{0\le i\le\tau})$ *has the same distribution as that of* $(Y_i)_{i\in\mathbb{Z}_+}$ *under* $\mathbb{P}^{\angle}(\cdot)$*.*

Theorem 4.2. *There exists a constant* $c_4 > 0$ *such that*

$$\mathbb{E}\left[e^{c_4 \log^\gamma \tau}\right] < \infty \tag{4.13}$$

and the same holds under $\mathbb{P}^{\angle}$*.*

4.2.1 Proof of Theorem 4.1

Proof. First we observe that, for all $k \in \mathbb{N}$ and all bounded measurable functions f,

$$\begin{aligned}\mathbb{E}\Big[f\big((Y_{R_k+i} - Y_{R_k})_{i\in\mathbb{Z}_+}\big), A^{Y_{R_k}} \mid \mathcal{F}_k\Big]& \\ = \mathbb{E}^{\angle}\left[f((Y_i)_{i\ge 0})\right]\mathbb{P}\left(A^0 \mid \omega(W_0^+) = 0\right) \quad &\text{a.s. on } \ \omega(W^+_{Y_{R_k}}) = 0.\end{aligned} \tag{4.14}$$

Indeed, we have

$$\begin{aligned}&\mathbb{E}\left[f((Y_i^y)_{i\ge 0}), A^y, B_y, \omega(W_y^+) = 0, Y_{R_k} = y\right] \\ &\quad = \mathbb{E}^{\angle}\left[f((Y_i)_{i\ge 0})\right]\mathbb{P}\left(A^0 \mid \omega(W_0^+) = 0\right)\mathbb{P}\Big(B_y, \omega(W^+_{Y_{R_k}}) = 0, Y_{R_k} = y\Big)\end{aligned} \tag{4.15}$$

for all $B_y \in \mathcal{F}_y$ because

(1) $\omega(W_y^+), \{Y_{R_k} = y\} \in \mathcal{F}_y$.

(2) On $\omega(W_y^+) = 0$, $f((Y_i^y)_{i\ge 0})\mathbf{1}_{A^y} \in \mathcal{G}_y^{\angle} \vee \mathcal{U}_y^{\angle}$.

(3) The joint distribution of $f((Y_i^y)_{i\ge 0})$, A^y and $\omega(W_y^+)$ under $\mathbb{P}$ does not depend on y.

By summing (4.15) over $y \in \mathbb{Z}^2$, we get (4.14).

Next, let $\mathcal{F}_\tau$ be the sigma-algebra of the events before time τ, i.e., the set of all events $B \in \mathcal{F}_\infty$ such that, for each $k \in \mathbb{N}$, there exists a $B_k \in \mathcal{F}_k$ such that $B \cap \{\mathcal{I} = k\} = B_k \cap \{\mathcal{I} = k\}$. Note that τ and $(Y_i)_{0\le i\le\tau}$ are measurable with respect to $\mathcal{F}_\tau$. Let $\Gamma_k = \{\omega(W^+_{Y_{R_k}}) = 0\} \cap A^{Y_{R_k}}$, and note that for each $0 \le k \le n \in \mathbb{N}$ there exists a $D_{k,n} \in \mathcal{F}_n$ such that $\Gamma_k \cap \Gamma_n = D_{k,n} \cap \Gamma_n$. In particular, there exists a $C_n \in \mathcal{F}_n$ such that

$$\{\mathcal{I} = n\} = \bigcap_{k=1}^{n-1} \Gamma_k^c \cap \Gamma_n = C_n \cap \Gamma_n. \tag{4.16}$$

Thus, for $B \in \mathcal{F}_\tau$ and f bounded measurable, we may write

$$
\begin{aligned}
&\mathbb{E}\Big[f\big((Y_{\tau+i} - Y_\tau)_{i\in\mathbb{Z}_+}\big), B, \tau < \infty\Big] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}\Big[f\big((Y_{R_n+i} - Y_{R_n})_{i\in\mathbb{Z}_+}\big), B_n, C_n, \Gamma_n\Big] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}\Big[B_n, C_n, \omega(W^{\angle}_{Y_{R_n}}) = 0, \mathbb{E}\big[f\big((Y_{R_n+i} - Y_{R_n})_{i\in\mathbb{Z}_+}\big), A^{Y_{R_n}} \mid \mathcal{F}_n\big]\Big]. \qquad (4.17)
\end{aligned}
$$

By (4.14), the right-hand side equals

$$
\mathbb{E}^{\angle}\left[f(Y)\right] \sum_{n\in\mathbb{N}} \mathbb{P}\left(B_n, C_n, \omega(W^{\angle}_{Y_{R_n}}) = 0\right) \mathbb{P}\left(A^0 \mid \omega(W^{\angle}_0) = 0\right), \qquad (4.18)
$$

which, again by (4.14), equals

$$
\begin{aligned}
\mathbb{E}^{\angle}\left[f(Y)\right] &\sum_{n\in\mathbb{N}} \mathbb{E}\Big[B_n, C_n, \omega(W^{\angle}_{Y_{R_n}}) = 0, \mathbb{P}\left(A^{Y_{R_n}} \mid \mathcal{F}_n\right)\Big] \\
&= \mathbb{E}^{\angle}\left[f(Y)\right] \sum_{n\in\mathbb{N}} \mathbb{P}\left(B_n, \mathcal{I} = n\right) \\
&= \mathbb{E}^{\angle}\left[f(Y)\right] \mathbb{P}\left(B, \tau < \infty\right), \qquad (4.19)
\end{aligned}
$$

which proves the statement under $\mathbb{P}(\cdot)$. To extend the result to $\mathbb{P}^{\angle} = \mathbb{P}(\cdot \mid \Gamma_0)$, note that $\Gamma_0 \in \mathcal{F}_\tau$ because $\Gamma_0 \cap \Gamma_n = D_{0,n} \cap \Gamma_n$ with $D_{0,n} \in \mathcal{F}_n$, and so we may apply (4.19) to $B \cap \Gamma_0$. □

4.2.2 Proof of Theorem 4.2

In what follows the constants may depend on $v_\circ, v_\bullet$ and $v_\star, \rho$. We begin with a few preliminary lemmas.

Define the *influence field at a point* $y \in \mathbb{Z}^2$ as

$$
h(y) = \inf\Big\{l \in \mathbb{Z}_+ \colon \ \omega(W^{\angle}_y \cap W^{\angle}_{y+(l,l)}) = 0\Big\}. \qquad (4.20)
$$

Lemma 4.3. *There exist constants $c_5, c_6 > 0$ (depending on $v_\star, \rho$ only) such that, for all $y \in \mathbb{Z}^2$,*

$$
\mathbb{P}[h(y) > l] \le c_5 e^{-c_6 l}, \qquad l \in \mathbb{Z}_+. \qquad (4.21)
$$

Proof. By translation invariance, it is enough to consider the case $y = 0$. By the definition of $h(0)$, we know that

$$
\{h(0) > l\} \subseteq \big\{\exists\, y \in \urcorner(0,0), y' \in \angle(l,l) \colon \ \omega(W_{y\leftrightarrow y'}) > 0\big\}, \qquad (4.22)
$$

where $W_{(x,n)\leftrightarrow(x',n')} = \{w \in W \colon \ w(n) = x, w(n') = x'\}$ (recall (2.8)). It follows that

$$
\begin{aligned}
\mathbb{P}(h(0) > l) &\le \sum_{y\in \urcorner(0,0)} \sum_{y'\in\angle(l,l)} \mathbb{P}(\omega(W_{y\leftrightarrow y'}) > 0) \\
&= \sum_{y\in \urcorner(0,0)} \sum_{y'\in\angle(l,l)} \left(1 - e^{-\rho\mu(W_{y\leftrightarrow y'})}\right) \le \sum_{y\in \urcorner(0,0)} \sum_{y'\in\angle(l,l)} \rho\mu(W_{y\leftrightarrow y'}).
\end{aligned} \qquad (4.23)
$$

Recall from Section 2.1 that P_x stands for the law on W_x under which the family $(Z_n)_{n\in\mathbb{Z}}$ given by $Z_n(w) = w(n)$ is distributed as a two-sided simple random walk starting at x. We write $y = (x, n)$ and $y' = (x', n')$, use translation invariance of μ, and use Azuma's inequality, to get

$$\mu(W_{y\leftrightarrow y'}) = P_0(Z_{n'-n} = x' - x) \le P_0(Z_{n'-n} \ge x' - x) \le \exp\Big\{ - \frac{(x'-x)^2}{2(n'-n)}\Big\}. \quad (4.24)$$

Combining (4.23–4.24) and noting that $n' - n \le (x' - x)/\bar{v}$, we get

$$\mathbb{P}(h(0) > l) \le \rho \sum_{(x,n)\in \angle(0,0)} \sum_{(x',n')\in\angle(l,l)} \exp\left\{-\tfrac{1}{2}\bar{v}(x'-x)\right\}. \quad (4.25)$$

For fixed $x = -k$, there are at most $k/\bar{v}$ space-time points $(x, n) \in \angle(0, 0)$. Analogously, for fixed $x' = k' + l$, there are at most $(k' + 1)/\bar{v}$ space-time points $(x', t') \in \angle(l, l)$. Therefore, using (4.25) we obtain

$$\mathbb{P}(h(0) > l) \le \frac{\rho}{\bar{v}^2} \sum_{k,k'\in\mathbb{Z}_+} k(k'+1)\, e^{-\bar{v}(k+k'+l)/2} \le \frac{\rho}{\bar{v}^2}\, e^{-\bar{v}l/2}\Big(\sum_{k\in\mathbb{Z}_+}(k+1)e^{-\bar{v}k/2}\Big)^2. \quad (4.26)$$

By choosing the constants c_5 and c_6 properly, we get the claim. □

Choose

$$\delta = \frac{1}{4\log\left(\frac{1}{p_\circ \wedge p_\bullet}\right)}, \qquad \epsilon = \frac{1}{4}(c_6\delta \wedge 1), \quad (4.27)$$

and put, for $T > 1$,

$$T' = \lfloor T^\epsilon \rfloor, \qquad T'' = \lfloor \delta \log(T) \rfloor. \quad (4.28)$$

Define the *local influence field at* (x, n) as

$$h^T(x, n) = \inf\left\{ l \in \mathbb{Z}_+ \colon\ \omega(W^{\angle}_{x-T',n} \cap W^{\nearrow}_{x,n} \cap W^{\nearrow}_{x+l,n+l}) = 0\right\}. \quad (4.29)$$

Lemma 4.4. *For all $T > 1$,*

$$\mathbb{P}\left(h^T(y) > l \;\middle|\; \mathcal{F}_{y-(T',0)}\right) \le c_5 e^{-c_6 l} \ \ \mathbb{P}\text{-a.s.} \qquad \forall\, y \in \mathbb{Z}^2,\ l \in \mathbb{Z}_+, \quad (4.30)$$

where c_5, c_6 are the same constants as in Lemma 4.3.

Proof. The result follows from Lemma 4.3 by noting that $h^T(y)$ is independent of $\mathcal{F}_{y-(T',0)}$ and smaller than $h(y)$. □

We say that R_k is a *good record time (g.r.t.)* when

$$h^T(Y_{R_k}) \le T'', \quad (4.31)$$

$$U_{Y_{R_k}+(l,l)} \le p_\circ \wedge p_\bullet, \quad \forall\, l = 0, \dots, T'' - 1, \quad (4.32)$$

$$\omega(W^{\angle}_{Y_{R_k}} \cap W^{\nearrow}_{Y_{R_k}+(T'',T'')}) = 0, \quad (4.33)$$

$$\{Y_{R_{k+T''}+1}, \dots, Y_{R_{k+T'}}\} \subset \angle(Y_{R_{k+T''}}). \quad (4.34)$$

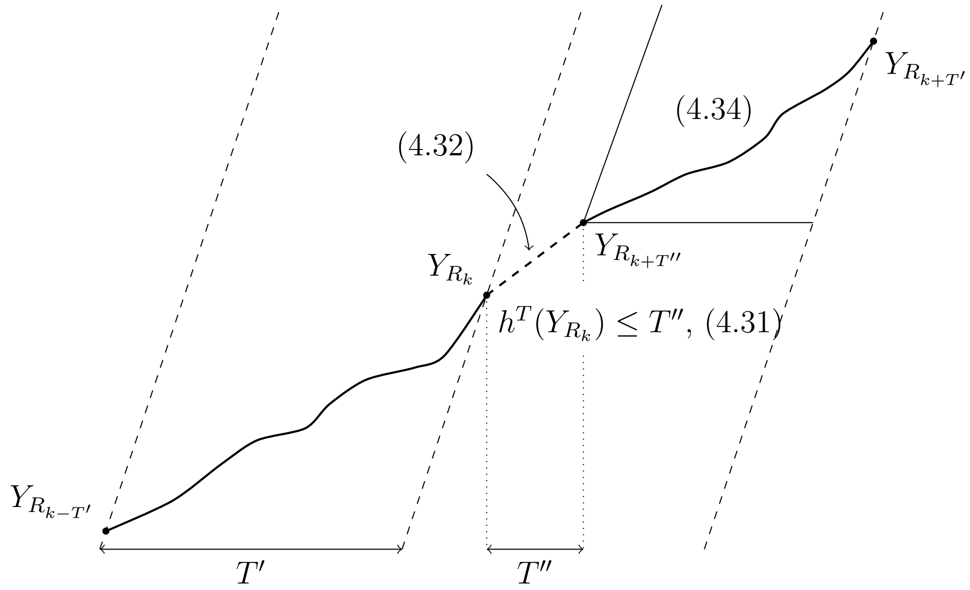

Figure 4: Illustration of a good record time R_k. Note the validity of the conditions (4.31), (4.32) and (4.34).

(See Fig. 4.)

The idea is that when R_k is a good record time, $R_{k+T''}$ is likely to be an upper bound for the regeneration time. In Proposition 4.6 below we will show that when many records are made, with high probability good record times occur. First, we need an additional lemma.

For $y \in \mathbb{Z}^2$ and $t \in \mathbb{N}$, define the space-time parallelogram

$$\mathcal{P}_t(y) = \angle(y) \cap \measuredangle(y + (2t, \lfloor t/\bar{v} \rfloor)) \tag{4.35}$$

and its right boundary

$$\partial^+ \mathcal{P}_t(y) = \{z \in \mathbb{Z}^2 \colon\ z \notin \mathcal{P}_t(y), z - (1,0) \in \mathcal{P}_t(y)\}. \tag{4.36}$$

We say that "Y^y exits $\mathcal{P}_t(y)$ through the right" when the first time i at which $Y^y_i \notin \mathcal{P}_t(y)$ satisfies $Y^y_i \in \partial^+ \mathcal{P}_t(y)$ (in particular, $X^y_i - y \geq t + \bar{v} i$).

Lemma 4.5. *There exists a constant $c_7 > 0$ such that, for all $t \in \mathbb{N}$ large enough,*

$$\mathbb{P}\left(Y^y \text{ exits } \mathcal{P}_t(y) \text{ through the right} \mid \mathcal{F}_y\right) \geq c_7 \quad \mathbb{P}\text{-a.s.} \qquad \forall\, y \in \mathbb{Z}^2. \tag{4.37}$$

Proof. If $v_\circ \geq v_\bullet$, then the claim follows from simple random walk estimates, since $0 < \bar{v} < v_\bullet$. Therefore we may assume that $v_\circ < v_\bullet$.

Reasoning as for (4.15), for fixed $y \in \mathbb{Z}^2$ we have

$$\begin{aligned}&\mathbb{P}\left(Y^y \text{ exits } \mathcal{P}_t(y) \text{ through the right} \mid \mathcal{F}_y\right) \\ &= \mathbb{P}\left(Y \text{ exits } \mathcal{P}_t(0) \text{ through the right} \,\middle|\, \omega(W^+_0) = 0\right) \ \mathbb{P}\text{-a.s. if } \omega(W^+_y) = 0.\end{aligned} \tag{4.38}$$

By monotonicity, if $\omega(W_y^+) > 0$, then Y^y can only be further to the right. Hence

$$\begin{aligned}&\mathbb{P}\left(Y^y \text{ exits } \mathcal{P}_t(y) \text{ through the right} \mid \mathcal{F}_y\right)\\ &\quad\geq \mathbb{P}\left(Y \text{ exits } \mathcal{P}_t(0) \text{ through the right} \,\middle|\, \omega(W_0^+) = 0\right) \ \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.39}$$

and we only need to show that this last probability is strictly positive. To that end, fix $L > 1$ large enough such that

$$\mathbb{P}\left(\exists\, n \in \mathbb{Z}_+ \colon \, Y_n \in \mathcal{H}_{v_\star, \lfloor L(1-v_\star)\rfloor}\right) \leq \tfrac{1}{2}\, \mathbb{P}\left(\omega(W_0^+) = 0\right), \tag{4.40}$$

which is possible by (1.7). If $t > L$, then

$$\begin{aligned}&\mathbb{P}\left(Y \text{ exits } \mathcal{P}_0(t) \text{ through the right}, \, \omega(W_0^+) = 0\right)\\ &\geq \mathbb{P}\left(U_{(i,i)} \leq p_0 \;\forall\, i = 0, \ldots, L-1, \; Y^{(L,L)} \text{ does not touch } \mathcal{H}_{v_\star,0}, \; \omega(W_0^+) = 0\right)\\ &\geq K_L \left\{\mathbb{P}\left(\omega(W_0^+) = 0\right) - \mathbb{P}\left(\exists\, n \in \mathbb{Z}_+ \colon \, Y_n \in \mathcal{H}_{v_\star, \lfloor L(1-v_\star)\rfloor}\right)\right\}\\ &\geq \tfrac{1}{2} K_L \mathbb{P}\left(\omega(W_0^+) = 0\right),\end{aligned} \tag{4.41}$$

where $K_L = \mathbb{P}\left(U_{(i,i)} \leq p_0 \;\forall\, i = 0, \ldots, L-1\right) > 0$. □

The following proposition is the main step in the proof of Theorem 4.2.

Proposition 4.6. *There exists a constant $c_8 > 0$ such that, for all $T > 1$ large enough,*

$$\mathbb{P}\left[R_k \textit{ is not a g.r.t. for all } 1 \leq k \leq T\,\right] \leq e^{-c_8 T^{1/2}}. \tag{4.42}$$

Proof. First we claim that there exists a $c > 0$ such that, for any $k \geq T'$,

$$\mathbb{P}\left[R_k \text{ is a g.r.t.} \middle| \mathcal{F}_{k-T'}\right] \geq c T^{\delta \log(p_\circ \wedge p_\bullet)} \text{ a.s.} \tag{4.43}$$

To prove (4.43), we will find $c > 0$ such that

$$\begin{aligned}&\mathbb{P}\big[(4.31) \,\big|\, \mathcal{F}_{k-T'}\big] \geq c && \text{a.s.,} && (4.44)\\ &\mathbb{P}\big[(4.32) \,\big|\, \mathcal{F}_k\big] \geq T^{\delta \log(p_\circ \wedge p_\bullet)} && \text{a.s.,} && (4.45)\\ &\mathbb{P}\big[(4.33) \,\big|\, (4.32), \mathcal{F}_k\big] \geq c && \text{a.s.,} && (4.46)\\ &\mathbb{P}\big[(4.34) \,\big|\, \mathcal{F}_{k+T''}\big] \geq c && \text{a.s.} && (4.47)\end{aligned}$$

(4.44): For $B \in \mathcal{F}_{k-T'}$, write

$$\begin{aligned}&\mathbb{P}\left(h^T(Y_{R_k}) > T'', B\right) = \sum_{y_1, y_2 \in \mathbb{Z}^2} \mathbb{P}\left(h^T(y_2) > T'', Y_{R_k} = y_2, Y_{R_{k-T'}} = y_1, B_{y_1}\right)\\ &\leq \sum_{y_1 \in \mathbb{Z}^2} \sum_{y_2 \in \partial^+ \mathcal{P}_{T'}(y_1)} \mathbb{P}\left(h^T(y_2) > T'', Y_{R_{k-T'}} = y_1, B_{y_1}\right)\\ &\quad + \sum_{y_1 \in \mathbb{Z}^2} \mathbb{P}\left(Y^{y_1} \text{ does not exit } \mathcal{P}_{T'}(y_1) \text{ through the right}, Y_{R_{k-T'}} = y_1, B_{y_1}\right)\\ &\leq \left\{c_5 T' e^{-c_6 T''} + 1 - c_7\right\} \mathbb{P}(B) \leq \left\{c_5 e^{c_6} T^{-\frac{3}{4}\delta c_6} + 1 - c_7\right\} \mathbb{P}(B),\end{aligned} \tag{4.48}$$

where the second inequality uses Lemmas 4.4–4.5 and $|\partial^+ \mathcal{P}_t(y)| \le t$, while the third inequality uses the definition of ϵ. Thus, for T large enough, (4.44) is satisfied with $c = c_7/2$.

(4.45): This follows from the fact that $(U_{Y_{R_k}+(l,l)})_{l\in\mathbb{N}_0}$ is independent of $\mathcal{F}_k$.

(4.46): We may ignore the conditioning on (4.32) since this event is independent of the others. For $B \in \mathcal{F}_k$, write

$$\begin{aligned}
&\mathbb{P}\left(\omega(W^{\angle}_{Y_{R_k}} \cap W^{+}_{Y_{R_k}+(T'',T'')}) = 0, B\right) \\
&\quad = \sum_{y\in\mathbb{Z}^2} \mathbb{P}\left(\omega(W^{\angle}_{y} \cap W^{+}_{y+(T'',T'')}) = 0, Y_{R_k} = y, B_y\right) \\
&\quad = \sum_{y\in\mathbb{Z}^2} \mathbb{P}\left(\omega(W^{\angle}_{y} \cap W^{+}_{y+(T'',T'')}) = 0\right) \mathbb{P}\left(Y_{R_k} = y, B_y\right) \\
&\quad \ge \mathbb{P}\left(\omega(W^{+}_0) = 0\right) \mathbb{P}\left(B\right), \qquad (4.49)
\end{aligned}$$

where the second equality uses the independence of $\mathcal{G}^{\angle}_y$ and $\mathcal{F}_y$, and the last step uses the monotonicity and translation invariance of ω.

(4.47): For $B \in \mathcal{F}_{k+T''}$, write

$$\begin{aligned}
&\mathbb{P}\left(\{Y_{R_{k+T''}+1}, \ldots, Y_{R_{k+T'}}\} \subset \angle(Y_{R_{k+T''}}), B\right) \\
&\quad \ge \mathbb{P}\left(Y^{Y_{R_{k+T''}}} \text{ exits } \mathcal{P}_{T'}(Y_{R_{k+T''}}) \text{ through the right}, B\right) \\
&\quad = \sum_{y\in\mathbb{Z}^2} \mathbb{P}\left(Y^{y} \text{ exits } \mathcal{P}_{T'}(y) \text{ through the right}, Y_{R_{k+T''}} = y, B_y\right) \\
&\quad \ge c_7 \mathbb{P}\left(B\right) \qquad (4.50)
\end{aligned}$$

by Lemma 4.5.

Thus, (4.43) is verified. Since $\{R_k \text{ is a g.r.t.}\} \in \mathcal{F}_{k+T'}$, we obtain, for T large enough,

$$\begin{aligned}
\mathbb{P}\left(R_k \text{ is not a g.r.t. for any } k \le T\right) &\le \mathbb{P}\left(R_{(2k+1)T'} \text{ is not a g.r.t. for any } k \le T/3T'\right) \\
&\le \exp\left\{-\frac{c}{4}\frac{T^{1+\delta\log(p_\circ\wedge p_\bullet)}}{T'}\right\} \\
&\le \exp\left\{-\frac{c}{4}T^{\frac{1}{2}}\right\} \qquad (4.51)
\end{aligned}$$

by our choice of ϵ and δ. □

We are now ready to finish the proof of Theorem 4.2.

Proof of Theorem 4.2. Since $\mathbb{P}^{\angle}(\cdot) = \mathbb{P}(\cdot|\Gamma_0)$ with $\mathbb{P}(\Gamma_0) > 0$, it is enough to prove the statement under $\mathbb{P}$. To that end, let

$$\begin{aligned}
E_1 &= \{\exists\, y \in [-T,T]\times[0,T]\cap\mathbb{Z}^2 \colon\ h(y) \ge T'\}, \\
E_2 &= \{\exists\, y \in [-T,T]\times[0,T]\cap\mathbb{Z}^2 \colon\ Y^y \text{ touches } y + \mathcal{H}_{v_\star, T'-T''}\}.
\end{aligned} \qquad (4.52)$$

Then, by Lemma 4.3, (1.7) and the union bound, there exists a $c > 0$ such that

$$\mathbb{P}\left(E_1 \cup E_2\right) \le c^{-1} e^{-c \log^\gamma T} \qquad \forall\, T > 1. \tag{4.53}$$

Next we argue that, for T large enough, if R_k is a g.r.t. with $k \le \bar{v}T$ and both E_1 and E_2 do not occur, then $\tau \le R_{k+T''} \le T$. Indeed, if $T'' \le T' \le \bar{v}T/2$, then on E_2^c we have $R_{\lfloor \bar{v}T \rfloor + T'} \le T$, since otherwise Y touches $\mathcal{H}_{v_\star, T'-T''}$. Thus, all we need to verify is that

$$\omega(W^{\not\leftarrow}_{Y_{R_{k+T''}}}) = 0, \tag{4.54}$$

and that

$$A^{Y_{R_{k+T''}}} \text{ occurs} \tag{4.55}$$

under the conditions stated.

To verify (4.55), note that $Y_{R_{k+T'}} \in [-T, T] \times [0, T] \cap \mathbb{Z}^2$ on $E_1^c \cap E_2^c$, . Therefore

$$Y_{R_{k+T'}+l} \in \angle(Y_{R_{k+T''}})\ \forall\, l \in \mathbb{Z}_+, \tag{4.56}$$

and (4.55) follows from (4.47).

To verify (4.54), first note that, by (4.44) and (4.46), we only need to check that

$$\omega(W^{\not\leftarrow}_{Y_{R_{k+T''}}} \cap W^{\not\leftarrow}_{Y_{R_k}} \cap W^{\not\rightarrow}_{Y_{R_k}-(T',0)}) \le \omega(W^{\not\leftarrow}_{Y_{R_k}} \cap W^{\not\rightarrow}_{Y_{R_k}-(T',0)}) = 0. \tag{4.57}$$

To that end, define

$$\mathcal{L} = \{(x, n) \in \mathbb{Z}^2 \colon x = \lceil \bar{v}n + k - T' \rceil, R_k - T' \le n \le R_k\}, \tag{4.58}$$

where $\lceil x \rceil$ is the smallest integer $\ge x$. Note that $\mathcal{L} \subset [-T, T] \times [0, T]$ on $E_1^c \cap E_2^c$. Furthermore, since the paths in W take nearest-neighbour steps, we have

$$\{\omega(W^{\not\leftarrow}_{Y_{R_k}} \cap W^{\not\rightarrow}_{Y_{R_k}-(T',0)}) > 0\} \subset \{\exists\, y \in \mathcal{L} \colon\ h(y) \ge T'\}, \tag{4.59}$$

which is empty on $E_1^c \cap E_2^c$. Thus, (4.57) holds.

To conclude, for T large enough we have

$$\begin{aligned}\mathbb{P}\left(\tau > T\right) &\le \mathbb{P}(E_1 \cup E_2) + \mathbb{P}\left(R_k \text{ is not a g.r.t. } \forall\, k \le \bar{v}T\right)\\ &\le c^{-1} e^{-c(\log T)^\gamma} + e^{-c_8(\bar{v}T)^{1/2}},\end{aligned} \tag{4.60}$$

which yields the claim. □

4.3 Proof of Theorem 1.4

We begin by making the following observation.

Theorem 4.7. *On an enlarged probability space there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of random times with $\tau_1 = \tau$ such that, under $\mathbb{P}$ and with $S_n = \sum_{i=1}^n \tau_i$,*

$$\left(\tau_{n+1}, (X_{S_n+s} - X_{S_n})_{0 \le s \le \tau_{n+1}}\right)_{n\in\mathbb{N}} \tag{4.61}$$

is an i.i.d. sequence, independent of $(\tau, (X_s)_{0\le s\le\tau})$, with each of its terms distributed as $(\tau, (X_s)_{0\le s\le\tau})$ under $\mathbb{P}^\angle$.

Proof. The claim follows from Theorem 4.1 and the fact that $\tau < \infty$ a.s., exactly as in Avena, dos Santos and Völlering [1, Proof of Theorem 3.8]. $\square$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We start with (c). Let

$$\iota := \mathbb{E}^{\angle}[\tau] \quad \text{and} \quad v := \frac{\mathbb{E}^{\angle}[X_\tau]}{\mathbb{E}^{\angle}[\tau]}. \tag{4.62}$$

By Theorems 4.2 and 4.7 and Lemma E.1, for any $\varepsilon > 0$ we have

$$\mathbb{P}\left(\exists\, i \geq n\colon\ |X_{S_i} - S_i v| \vee |S_i - i\iota| > \varepsilon i\right) \leq c^{-1} e^{-c \log^\gamma n} \tag{4.63}$$

for some $c > 0$.

Next define, for $t \geq 0$, k_t as the random integer such that

$$S_{k_t} \leq t < S_{k_t+1}. \tag{4.64}$$

Since $S_n > t$ if and only if $k_t < n$, for any $\varepsilon > 0$ we have

$$\mathbb{P}\left(\exists\, t \geq n\colon\ |k_t - t/\iota| > t\varepsilon\right) \leq \mathbb{P}\left(\exists\, i \geq \delta' n\colon |S_i - i\iota| > \varepsilon' i\right) \tag{4.65}$$

for some $\delta', \epsilon' > 0$ and large enough n. On the other hand, since X is Lipschitz we have

$$|X_t - tv| \leq |X_{S_{k_t}} - S_{k_t} v| + (1 + |v|)\left\{|S_{k_t} - k_t \iota| + |k_t \iota - t|\right\}, \tag{4.66}$$

and therefore for any $\varepsilon > 0$

$$\begin{aligned} \mathbb{P}\left(\exists\, t \geq n\colon |X_t - tv| > \varepsilon t\right) \leq \mathbb{P}\left(\exists\, t \geq n\colon\ |k_t - t/\iota| > t\varepsilon'\right) \\ + \mathbb{P}\left(\exists\, i \geq \delta' n\colon\ |X_{S_i} - S_i v| \vee |S_i - i\iota| > \varepsilon i\right) \end{aligned} \tag{4.67}$$

for some $\delta', \varepsilon' > 0$. Combining (4.63)–(4.67), we get (c), and (a) follows by the Borel-Cantelli lemma.

To prove (b), let $\hat{\sigma}^2$ be the variance of $X_\tau - \tau v$ under $\mathbb{P}^{\angle}$, which is finite because of (4.1) and strictly positive because $X_\tau - \tau v$ is not a.s. constant. For the process $(Y_k)_{k \in \mathbb{N}}$ defined by $Y_k = X_{S_k} - S_k v$, a functional central limit theorem with variance $\hat{\sigma}^2$ holds because, by Theorems 4.2 and 4.7, the assumptions of the Donsker-Prohorov invariance principle are satisfied.

Now consider the random time change $\varphi_n(t) = k_{nt}/n$. We claim that

$$\lim_{n \to \infty} \sup_{t \in [0,L]} \left| \varphi_n(t) - \frac{t}{\mathbb{E}^{\angle}[\tau]} \right| = 0 \quad \mathbb{P}\text{-a.s.} \quad \forall\, L > 0. \tag{4.68}$$

To prove (4.68), fix $\delta > \epsilon > 0$ and recall (4.62). Reasoning as for (4.65), we see that

$$\mathbb{P}\left(\exists\, t \in [\delta c, L]\colon\ |\varphi_n(t) - t/\iota| > \epsilon\right) \leq \mathbb{P}\left(\exists\, k \geq \delta' n\colon |S_k - k\iota| > \epsilon' k\right) \tag{4.69}$$

for some $\delta', \epsilon' > 0$ and large enough n. By (4.63), the right-hand side of (4.69) is summable, and hence

$$\limsup_{n\to\infty} \sup_{t\in[0,L]} |\varphi_n(t) - t/\iota| \le 2\delta \quad \text{a.s.} \tag{4.70}$$

Since $\delta > 0$ is arbitrary, (4.68) follows. In particular, φ_n converges a.s. in the Skorohod topology to the linear function $t \mapsto t/\mathbb{E}^{\angle}[\tau]$.

Define $Y_t^{(n)} = n^{-1/2} Y_{\lfloor nt \rfloor}$. With a time-change argument (see e.g. Billingsley [3, Eqs. (17.7)–(17.9) and Theorem 4.4]), we see that $(Y^{(n)}_{\varphi_n(t)})_{t\ge 0}$ converges weakly to a Brownian motion with variance

$$\sigma^2 = \frac{\mathbb{E}^{\angle}\left[(X_\tau - \tau v)^2\right]}{\mathbb{E}^{\angle}[\tau]}. \tag{4.71}$$

To extend this to X, note that, for any $T > 0$,

$$\begin{aligned} \sup_{0\le t\le T} \left| \frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{\sqrt{n}} - Y^{(n)}_{\varphi_n(t)} \right| &\le \frac{1}{\sqrt{n}} \sup_{0\le t\le T} \left(\left|X_{\lfloor nt \rfloor} - X_{S_{k_{nt}}}\right| + v \left|S_{k_{nt}} - \lfloor nt \rfloor\right| \right) \\ &\le \frac{|v|+1}{\sqrt{n}} \sup_{1\le k\le nT+1} |\tau_k|, \end{aligned} \tag{4.72}$$

which tends to 0 a.s. as $n \to \infty$ by Theorems 4.2 and 4.7. □

Remark 4.8. *One may check that the statement of Theorem 4.2 is uniform over compact intervals of the parameters $v_\circ$, $v_\bullet$, ρ. Using this and the formulas* (4.62) *and* (4.71)*, it is possible to show continuity of v and σ in these parameters by approximating certain relevant observables of the system up to the regeneration time by observables supported in finite space-time boxes. See e.g. Section 6.4 of [11].*

4.4 Extensions

As mentioned in the Introduction, finding a regeneration structure is usually a delicate matter as one often needs to rely on precise features of the model at hand. Approximate renewal schemes are more general, but do not usually give as much information as full regeneration.

Let us mention other examples of dynamic random environments where a renewal strategy can be found. For the simple symmetric exclusion process, such a renewal structure was developed in [1]. There, the tail of the regeneration time is controlled by imposing a non-nestling condition on the random walk drifts. Using the techniques of this section, it would be possible to extend these results (i.e., obtain Theorem 1.2) to the nestling situation if one manages to prove the analogous of Theorem 1.5 for the exclusion process.

Another example where a regeneration strategy can be found is the independent renewal chain discussed in Section 3.5. Indeed, a regeneration time can be obtained as follows. Recall that, for large enough ρ, we obtain the ballisticity condition (1.7) for some $v_\star > 0$. Retaining the notation of Section 4.1 for $\bar{v}$, $\angle(y)$, R_k and A^y, we define

$$\mathcal{I} = \inf\{k \in \mathbb{N} \colon\ A^{Y_{R_k}} \text{ occurs}\} \quad \text{and} \quad \tau := R_{\mathcal{I}}. \tag{4.73}$$

We may then verify that τ satisfies analogous properties as stated in Theorems 4.1 and 4.2. Hence, by the exact same arguments as in Section 4.3, Theorem 1.2 holds also in this case.

The remainder of this paper consists of five appendices. All we have used so far is Theorem C.1 in Appendix C (recall Section 3.1), which is a decoupling inequality, Lemma D.1 in Appendix D (recall Section 3.4), which is a tail estimate, and Lemma E.1 in Appendix E (recall Section 4.3), which is an estimate for independent random variables satisfying a certain tail assumption. Appendices A–B are preparations for Appendix C.

A Simulation with Poisson processes

In this section we recall some results from Popov and Teixeira [20] about how to simulate random processes with the help of Poisson processes. Corollary A.3 will be used in Section B to prove a mixing-type result for a collection of independent random walks (Lemma B.3 below).

Let $(\Sigma, \mathcal{B}, \mu)$ be a measure space, with Σ a locally compact Polish metric space, $\mathcal{B}$ the Borel σ-algebra on Σ, and μ a Radon measure, i.e., every compact subset of Σ has finite μ-measure. The set-up is standard for the construction of a Poisson point process on Σ. To that end, consider the space of Radon point measures on $\Sigma \times \mathbb{R}_+$,

$$M = \left\{ m = \sum_{i \in \mathbb{N}} \delta_{(z_i, v_i)} \colon\ z_i \in \Sigma, v_i \in \mathbb{R}_+, m(K) < \infty \, \forall\, K \subseteq \Sigma \times \mathbb{R}_+ \text{ compact} \right\}. \quad \text{(A.1)}$$

We can canonically construct a Poisson point process m on the measure space $(M, \mathcal{M}, \mathbb{Q})$ with intensity $\mu \otimes \mathrm{d}v$, where $\mathrm{d}v$ is the Lebesgue measure on $\mathbb{R}_+$. (For more details on this construction, see e.g. Resnick [22, Proposition 3.6].

Proposition A.1 below provides us with a way to simulate a random element of Σ by using the Poisson point process m. Although the result is intuitive, we include its proof for the sake of completeness.

Proposition A.1. *Let $g \colon \Sigma \to \mathbb{R}_+$ be a measurable function with $\int_\Sigma g(z)\mu(\mathrm{d}z) = 1$. For $m = \sum_{i \in \mathbb{N}} \delta_{(z_i, v_i)} \in M$, define*

$$\xi = \inf\{t \geq 0 \colon\ \exists\, i \in \mathbb{N} \text{ such that } tg(z_i) \geq v_i\} \quad \text{(A.2)}$$

(see Fig. 5). Then, under the law $\mathbb{Q}$ of the Poisson point process m,

(1) *A.s. there exists a single $\hat{\imath} \in \mathbb{N}$ such that $\xi g(z_{\hat{\imath}}) = v_{\hat{\imath}}$.*

(2) *$(z_{\hat{\imath}}, \xi)$ has distribution $g(z)\mu(dz) \otimes \mathrm{Exp}(1)$.*

(3) *Let $m' = \sum_{i \neq \hat{\imath}} \delta_{(z_i, v_i - \xi g(z_i))}$. Then m' has the same distribution as m and is independent of $(\xi, \hat{\imath})$.*

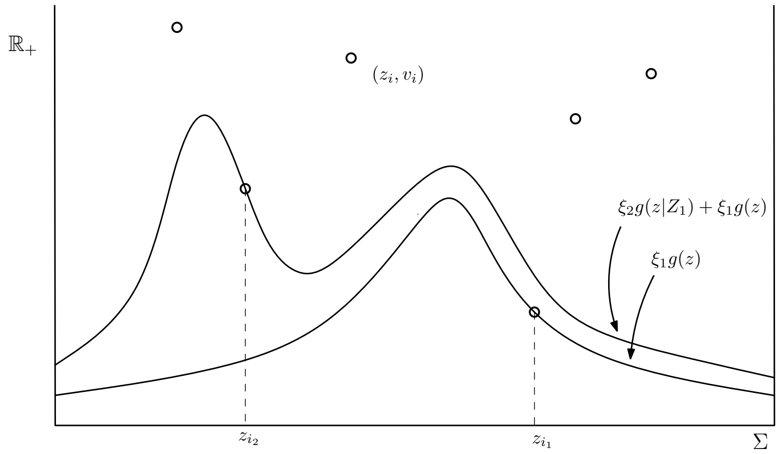

Figure 5: An example illustrating the definition of ξ and $\hat{\imath}$ in Proposition A.1, and the definition of ξ_1, Z_1 and ξ_2, Z_2 in (A.7)

.

Proof. For measurable $A \subset \Sigma$, define the random variable

$$\xi^A = \inf\{t \geq 0 \colon \exists\, i \in \mathbb{N} \text{ such that } t\mathbb{1}_A(z_i)g(z_i) \geq v_i\}. \tag{A.3}$$

Elementary properties of Poisson point processes (see e.g. Resnick [22][(a-b), pp. 130]) yield that

$$\xi^A \text{ is exponentially distributed with parameter } \textstyle\int_A g(z)\mu(dz), \text{ and if } A \text{ and } B \text{ are disjoint, then } \xi^A \text{ and } \xi^B \text{ are independent.} \tag{A.4}$$

Property (1) follows from (A.4) because Σ is separable and two independent exponential random variables are a.s. distinct. Moreover, since

$$\mathbb{Q}[\xi \geq \alpha, z_{\hat{\imath}} \in A] = \mathbb{Q}[\xi^{\Sigma\setminus A} > \xi^A \geq \alpha], \tag{A.5}$$

Property (2) also follows from (A.4) by using elementary properties of the minimum of independent exponential random variables. Thus it remains to prove Property (3).

We first claim that, given ξ, $m'' = \sum_{i \neq \hat{\imath}} \delta_{(z_i,v_i)}$ is a Poisson point process that is independent of $z_{\hat{\imath}}$ and, conditional on ξ, has intensity measure $\mathbb{1}_{\{v > \xi g(z)\}}[\mu(\mathrm{d}z) \otimes \mathrm{d}v]$. Indeed, this is a consequence of the strong Markov property for Poisson point processes together with the fact that $\{(z, v) \in \Sigma \times \mathbb{R}_+ \colon v \leq \xi g(z)\}$ is a stopping set (see Rozanov [23, Theorem 4]).

Now, given ξ, m' is a mapping of m'' (in the sense of Resnick [22, Proposition 3.7]). This mapping pulls back the measure $\mathbb{1}_{\{v > \xi g(z)\}}[\mu(\mathrm{d}z) \otimes \mathrm{d}v]$ to $\mu(\mathrm{d}z) \otimes \mathrm{d}v$. Since the latter distribution does not involve ξ, we obtain Property (A.1). $\square$

In Proposition A.2 below we use Proposition A.1 to simulate a collection $(Z_j)_{j \in \mathbb{N}}$ of independent random elements of Σ using the single Poisson point process m defined above. Formally, suppose that in some probability space $(M, \mathcal{M}, \mathcal{P})$ we are given a collection $(Z_j)_{j\in\mathbb{N}}$ of independent (not necessarily identically distibuted) random elements of Σ such that

$$\text{the distribution of } Z_j \text{ is given by } g_j(z)\mu(dz),\ j \in \mathbb{N}. \tag{A.6}$$

In the same spirit as the definition of ξ in Proposition A.1, we define what we call the *soft local time* $G = (G_j)_{j=1}^k$ associated with a sequence $(g_j)_{j=1}^k$ of measurable functions:

$$\begin{aligned} \xi_1 &= \inf\big\{t \geq 0\colon\ tg_1(z_i) \geq v_i \text{ for at least one } i \in \mathbb{N}\big\},\\ G_1(z) &= \xi_1\, g_1(z),\\ &\ \vdots \\ \xi_k &= \inf\big\{t \geq 0\colon\ tg_j(z_i) + G_{k-1}(z_i) \geq v_i \text{ for at least } k \text{ indices } i \in \mathbb{N}\big\},\\ G_k(z) &= \xi_1\, g_1(z) + \cdots + \xi_k\, g_k(z) \end{aligned} \tag{A.7}$$

(see Fig. 5 for an illustration of this recursive procedure).

Proposition A.2. *Subject to* (A.6–A.7)*,*

$$(\xi_j)_{j=1}^J \text{ are i.i.d. } \mathrm{EXP}(1). \tag{A.8}$$

$$\text{A.s. there is a unique } i_J \text{ such that } G_J(z_{i_J}) = v_{i_J}. \tag{A.9}$$

$$(z_{i_1}, \ldots, z_{i_J}) \overset{d}{=} (Z_1, \ldots, Z_J). \tag{A.10}$$

$$m' = \sum_{i\notin\{i_1,\ldots,i_J\}} \delta_{(z_i, v_i - G_J(z_i))} \text{ is distributed as } m \text{ and is independent of the above.} \tag{A.11}$$

Proof. Apply Proposition A.1 repeatedly, using induction on J. □

We close this section by exploiting the above construction to couple two collections of independent random elements of Σ using the same Poisson point process as basis. The following corollary will be needed in Appendix B.

Corollary A.3. *Let $(g_j(\cdot))_{j=1}^J$ be a a family of densities with corresponding ξ_j, G_j, i_j, $j = 1, \ldots, J$, as in* (A.7–A.8)*. Then, for any $\rho > 0$,*

$$\mathbb{Q}\left[\sum_{j\leq J} \delta_{z_{i_j}} \geq \sum_{i\colon v_i<\rho} \delta_{z_i}\right] \geq \mathbb{Q}\big[G_J \geq \rho\big]. \tag{A.12}$$

Note that the right-hand side of (A.12) only depends on the soft local time, which may e.g. be estimated through large deviation bounds.

B Simulation and domination of particles

B.1 Simple random walk

In this section we collect a few facts about the heat kernel of random walks on $\mathbb{Z}$. Let $p_n(x, x') = P_x[Z_n = x']$, $x, x' \in \mathbb{Z}$, where P_x stands for the law of a lazy simple random walk Z_n on $\mathbb{Z}$ as defined in Section 1, i.e., $p_1(0,x) > 0$ if and only if $x \in \{-1,0,1\}$ and $p_1(0,1) = p_1(0,-1)$. Then there exists constants $C, c > 0$ such that the following hold for all $n \in \mathbb{N}$:

$$\sup_{x \in Z} p_n(0,x) \leq \frac{C}{\sqrt{n}}, \tag{B.1}$$

$$|p_n(0,x) - p_n(0,x')| \leq \frac{C|x-x'|}{n} \quad \forall\, x, x' \in \mathbb{Z}, \tag{B.2}$$

$$P_0(|S_n| > \sqrt{n}\log n) \leq \, Ce^{-c\log^2 n}. \tag{B.3}$$

For (B.1), see e.g. Lawler and Limic [14, Theorem 2.4.4]. To get (B.2), use [14, Theorem 2.3.5], while (B.3) follows by an application of Bernstein's inequality.

The above observations will be used in the proof of Lemma B.2 below, which deals with the integration of the heat kernel over an evenly distributed cloud of sample points and is crucial in the proof of Theorem C.1 in Section C. In order to state this lemma, we need the following definitions.

Definition B.1. *(a) For $H \subset \mathbb{Z}$ and $L \in \mathbb{N}$, we say that a collection of intervals $\{C_i\}_{i\in I}$ is an L-paving of H when $H \subset \cup_{i \in I} C_i$ and there is an $x \in \mathbb{Z}$ such that*

$$C_i = \{[0,L) \cap \mathbb{Z} + Li + x \colon\, i \in I\}. \tag{B.4}$$

(b) We say that a collection of points $(x_j)_{j\in J} \subset \mathbb{Z}$ is ρ-dense with respect to the L-paving $\{C_i\}_{i\in I}$ when

$$\#\{j \colon\, x_j \in C_i\} \geq \rho L \quad \forall\, i \in I. \tag{B.5}$$

We know that $\sum_{x\in\mathbb{Z}} p_n(0,x) = 1$. The next lemma approximates this normalization when the sum runs over a dense collection $(x_j)_{j\in J}$.

Lemma B.2. *Let $\{C_i\}_{i\in I}$ be an L-paving of $H \subset \mathbb{Z}$ and $(x_j)_{j\in J}$ be ρ-dense collection with respect to $\{C_i\}_{i\in I}$. Then, for all $n \in \mathbb{N}$,*

$$\sum_{j\in J} p_n(0,x_j) \geq \rho \left\{ P_0\left(Z_n \in H\right) - \frac{cL\log n}{\sqrt{n}} \right\}. \tag{B.6}$$

Proof. For each $i \in I$, choose $z_i \in C_i$ such that

$$p_n(0,z_i) = \min_{x\in C_i} p_n(0,x). \tag{B.7}$$

Then we have

$$\begin{aligned} \sum_{j\in J} p_n(0,x_j) &= \sum_{i\in I} \sum_{j\colon x_j \in C_i} p_n(0,x_j) \geq \sum_{i\in I} \rho L p_n(0,z_i) \\ &\geq -\rho \sum_{i\in I} \sum_{x\in C_i} |p_n(0,x) - p_n(0,z_i)| + \rho P_0(S_n \in H). \end{aligned} \tag{B.8}$$

On the other hand, by (B.2)–(B.3) we have

$$\begin{aligned} \sum_{i\in I} \sum_{x\in C_i} |p_n(0,x) - p_n(0,z_i)| &\leq 2P_0(|S_n| > \sqrt{n}\log n) + \sum_{|x|\leq\sqrt{n}\log n} cL/n \\ &\leq cL\log n/\sqrt{n} \end{aligned} \tag{B.9}$$

and the claim follows by combining (B.8) and (B.9). □

B.2 Coupling of trajectories

Given a sequence of points $(x_j)_{j\in J}$ in $\mathbb{Z}$, let $(Z^j_n)_{n\in\mathbb{Z}_+}$, $j \in J$, be a sequence of independent simple random walks on $\mathbb{Z}$ starting at x_j, and let $\bigotimes_{j\in J} P_{x_j}$ denote their joint law. The next lemma, which will be needed in Appendix C, provides us with a way to couple the positions of these random walks at time n with a Poisson point process on $\mathbb{Z}$. This lemma is similar in flavor to [19, Proposition 4.1].

Lemma B.3. *Let $(x_j)_{j\in J} \subset \mathbb{Z}$ be ρ-dense with respect to the L-paving $\{C_i\}_{i\in I}$ of $H \subset \mathbb{Z}$. Then for any $\rho' \le \rho$ there exists a coupling $\mathbb{Q}$ of $\otimes_{j\in J} P_{x_j}$ and the law of a Poisson point process $\sum_{j'\in J'} \delta_{Y_j}$ on $\mathbb{Z}$ with intensity ρ' such that*

$$\mathbb{Q}\left[\mathbb{1}_{H'} \sum_{j\in J} \delta_{Z^j_n} \ge \mathbb{1}_{H'} \sum_{j'\in J'} \delta_{Y_{j'}}\right] \ge 1 - |H'| \exp\left\{-(\rho-\rho')L + \left(\frac{c_9 \rho L^2 \log n}{\sqrt{n}}\right)\right\} \quad \text{(B.10)}$$

for all $H' \subset \mathbb{Z}$ such that $\{z \in \mathbb{Z} \colon \operatorname{dist}(z, H') \le n\} \subset H$ and all $n \ge c_{10} L^2$.

Proof. By Corollary A.3, there exists a coupling $\mathbb{Q}$ such that

$$\mathbb{Q}\left[\mathbb{1}_{H'} \sum_{j\in J} \delta_{Z^j_n} \ge \mathbb{1}_{H'} \sum_{j'\in J'} \delta_{Y_j}\right] \ge \mathbb{Q}\big[G_J(z) \ge \rho' \ \forall\, z \in H'\big], \quad \text{(B.11)}$$

where $G_J(z) = \sum_{j\in J} \xi_j\, p_n(x_j, z)$ with $(\xi_j)_j$ i.i.d. EXP(1) random variables. We will estimate the right-hand side of (B.11) using concentration inequalities. First, noting that $P_z(Z_n \in H) = 1$ for any $z \in H'$, we use Lemma B.2 to estimate

$$E^{\mathbb{Q}}[G_J(z)] = \sum_{j\in J} p_n(z, x_j) \ge \rho\Big(1 - \frac{cL\log n}{\sqrt{n}}\Big), \qquad z \in B. \quad \text{(B.12)}$$

Next, estimate

$$\begin{aligned} \mathbb{Q}\big[\exists\, z \in H' \colon\ G_J(z) < \rho'\big] &\le |H'| \sup_{z\in H'} \mathbb{Q}[G_J(z) < \rho'] \\ &\le |H'|\, e^{\rho' L} \sup_{z\in H'} E^{\mathbb{Q}}\big[\exp\{-LG_J(z)\}\big]. \end{aligned} \quad \text{(B.13)}$$

Now note that, for any $z \in \mathbb{Z}$,

$$E^{\mathbb{Q}}\big[\exp\{-LG_J(z)\}\big] = \prod_{j\in J} E^{\mathbb{Q}}\big[\exp\{-L\xi_j p_n(x_j, z)\}\big] = \prod_{j\in J}\Big(1 + Lp_n(x_j, z)\Big)^{-1}. \quad \text{(B.14)}$$

Assuming that $n \ge (cL)^2 \vee e$, we can use (B.1) to obtain that

$$\sup_{x\in\mathbb{Z}} Lp_n(0, x) \le \frac{cL}{\sqrt{n}} \le \tfrac{1}{2}. \quad \text{(B.15)}$$

A simple Taylor expansion shows that $\log(1+u) \geq u - u^2$ for every $|u| \leq \frac{1}{2}$. Hence $1 + u \geq \exp\{u(1-u)\}$ and

$$
\begin{aligned}
\prod_{j\in J}(1+Lp_n(z,x_j))^{-1} &\leq \prod_{j\in J}\exp\big\{-Lp_n(z,x_j)\big(1-Lp_n(z,x_j)\big)\big\} \\
&\leq \exp\Big\{-\sum_{j\in J}Lp_n(z,x_j)\big(1-\sup_{x\in\mathbb{Z}}Lp_n(0,x)\big)\Big\} \\
&\leq \exp\Big\{-\rho L\big(1-\tfrac{cL\log n}{\sqrt{n}}\big)\big(1-\tfrac{cL}{\sqrt{n}}\big)\Big\} \\
&\leq \exp\Big\{-\rho L\big(1-\tfrac{2cL\log n}{\sqrt{n}}\big)\Big\}.
\end{aligned}
\tag{B.16}
$$

where the third inequality uses (B.12) and (B.15). Inserting this estimate into (B.13), we get the claim. □

C Decoupling of space-time boxes

In this section we prove a decoupling inequality on two disjoint boxes in the space-time plane $\mathbb{Z}_+ \times \mathbb{Z}$.

C.1 Correlation

Intuitively, if two events depend on what happens at far away times, then they must be close to independent due to the mixing of the dynamics. This is made precise in the following theorem.

Theorem C.1. *Let $B = ([a,b]\times[n,n'])\cap\mathbb{Z}^2$ be a space-time box as in Fig.* 6 *for some $n \geq c_{11}$, and let $D = \mathbb{Z}\times\mathbb{Z}_-$ be the space-time lower halfplane. Recall Definitions* 2.1–2.2*, and assume that $f_1\colon \Omega \to [0,1]$ and $f_2\colon \Omega\to[0,1]$ are non-decreasing random variables with support in B and D, respectively. Then, for any $\rho \geq 1$,*

$$
\mathbb{E}^{\rho(1+n^{-1/16})}[f_1f_2] \leq \mathbb{E}^{\rho(1+n^{-1/16})}[f_1]\,\mathbb{E}^{\rho}[f_2] + c\big(\mathrm{per}(B)+n\big)\,e^{-c\rho n^{1/8}}, \tag{C.1}
$$

where $\mathrm{per}(B)$ *stands for the perimeter of B.*

Note that, by the FKG-inequality, we have $\mathbb{E}^\rho[f_1f_2] \geq \mathbb{E}^\rho[f_1]\,\mathbb{E}^\rho[f_2]$. Thus, the bound in (C.1) shows that f_1, f_2 are almost uncorrelated.

To prove Theorem C.1, we need the following definition. For a box $B = ([a,b]\times[n,n'])\cap\mathbb{Z}^2$ in the space-time upper halfplane $\mathbb{Z}\times\mathbb{Z}_+$, let $\mathcal{C}(B)$ be the cone associated with B, defined as (see Fig. 6)

$$
\mathcal{C}(B) = \bigcup_{k=0}^{\infty}\big([a-k,b+k]\times\{n'-k\}\big). \tag{C.2}
$$

This cone can be interpreted as the set of points that can reach B while traveling at speed one, and encompasses every space-time point that can influence the state of B.

Given a box B and a halfplane D as in Theorem C.1, we denote by H and H' the separating segments (see Fig. 6)

$$H' = \mathcal{C}(B) \cap \big(\mathbb{Z} \times \{n\}\big), \qquad H = \mathcal{C}(B) \cap \big(\mathbb{Z} \times \{0\}\big). \tag{C.3}$$

The next lemma states a Markov-type property.

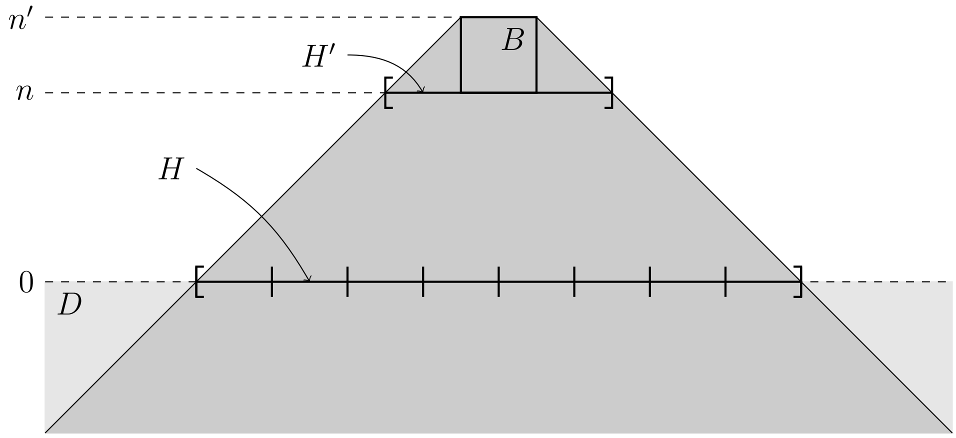

Figure 6: Box B and lower halfplane D as in Theorem C.1. The dark gray area corresponds to the cone associated with B, containing the separating segments H and H'.

Lemma C.2. *Let $B = [a, b] \times [n, n'] \cap \mathbb{Z}^2$, and let $\mathcal{C}(B)$ and H be as in* (C.2–C.3)*. Then, for any function $f\colon \Omega \to \mathbb{R}$ with support in B,*

$$\mathbb{E}\Big[f\Big|N(y), y \in D\Big] = \mathbb{E}\Big[f\Big|N(y), y \in H\Big], \tag{C.4}$$

where $N(y)$ is the number of trajectories crossing y (recall (2.2)*).*

Proof. Since $y \in \mathcal{T}$ if and only if $N(y) \geq 1$, f is a function of $(U_y, N(y))_{y \in B}$. Noting that $(N(y))_{y\in B}$ is a function of $(N(y))_{y\in H}$ and $(S_n^{y,i})_{y\in H, i\in\mathbb{N}, n\in\mathbb{Z}_+}$ only, we get the claim. ☐

C.2 Proof of Theorem C.1

In the following we will abuse notation by writing H,H' to denote also the projection on $\mathbb{Z}$ of these sets. We start by choosing an L-paving $\{I_j\}_{j\in J}$ of H, composed of segments of length $L = \lfloor n^{1/4} \rfloor$ (the choice of exponent 1/4 is arbitrary: any choice in $(0, \frac{1}{2})$ will do). Note that

$$\text{no more than } 1 + (\text{per}(B) + 2n)/L \text{ such segments are needed to cover } H, \tag{C.5}$$

i.e., we may take $|J| \leq 1 + (\text{per}(B) + 2n)/L$. Note also that $|H'| \leq \text{per}(B) + 1$.

We take c_{11} so large that

$$n > c_{11} \;\Rightarrow\; L > n^{1/4}/2, \;\; n > c_{10} L^2 \;\text{ and }\; n^{1/8} > 16 c_9 \log n. \tag{C.6}$$

Abbreviate $\rho_r = \rho(1 + rn^{-1/16})$ for $r \in [0, 1]$. Note that the two densities appearing in the statement of Theorem C.1 are $\rho = \rho_0$ and $\rho(1 + n^{-1/16}) = \rho_1$. We introduce the events

$$\mathcal{E} = \Big\{ \text{ at least } \rho_{1/2} L \text{ trajectories cross } H \cap I_j \text{ for all } j \in \mathbb{N} \Big\}. \tag{C.7}$$

Then, by Lemma C.2 and $f_1, f_2 \le 1$,

$$\begin{aligned} \mathbb{E}^{\rho_1}[f_1 f_2] &\le \mathbb{P}^{\rho_1}[\mathcal{E}^c] + \mathbb{E}^{\rho_1}\Big[f_1 1_{\mathcal{E}} \mathbb{E}^{\rho_1}\big[f_2 \big| N(y), y \in D\big]\Big] \\ &= \mathbb{P}^{\rho_1}[\mathcal{E}^c] + \mathbb{E}^{\rho_1}\Big[f_1 1_{\mathcal{E}} \mathbb{E}^{\rho_1}\big[f_2 \big| N(y), y \in H\big]\Big]. \end{aligned} \tag{C.8}$$

To estimate the first term in the right-hand side of (C.8), we do a moderate deviation estimate. For every $\theta > 0$, by (C.5),

$$\begin{aligned} \mathbb{P}^{\rho_1}[\mathcal{E}^c] &\le \big|\{j \in \mathbb{N} \colon\ I_j \cap H \neq \emptyset\}\big| \, \mathbb{P}^{\rho_1}\Big[\sum_{y \in I_j} N(y) < \rho_{1/2} L\Big] \\ &\le \Big(1 + L^{-1}(\mathrm{per}(B) + 2n)\Big) \exp\Big\{\rho_{1/2} L\theta - \rho_1 L(1 - e^{-\theta})\Big\}, \end{aligned} \tag{C.9}$$

since $N(y)$ has law Poisson(ρ_1). Since $1 - e^{-\theta} \ge \theta - \theta^2$, we have

$$\begin{aligned} \exp\Big\{\rho_{1/2} L\theta - \rho_1 L(1 - e^{-\theta})\Big\} &\le \exp\Big\{\rho_{1/2} L\theta - \rho_1 L\theta + \rho_1 L\theta^2\Big\} \\ &= \exp\Big\{L\theta\rho_0\big[(1 + \tfrac{1}{2} n^{-1/16}) - (1 + n^{-1/16}) + 2\theta\big]\Big\} \\ &= \exp\big\{L\theta\rho_0[-\tfrac{1}{2} n^{-1/16} + 2\theta]\big\}. \end{aligned} \tag{C.10}$$

Choosing $\theta = \frac{1}{8} n^{-1/16}$, we see that the above equals

$$\exp\{-\tfrac{1}{4} L\theta\rho_0 n^{-1/16}\} = \exp\{-\tfrac{1}{32}\rho_0 L n^{-1/8}\} \le \exp\{-\tfrac{1}{64}\rho_0 n^{1/8}\}. \tag{C.11}$$

where the second inequality uses that $n > c_{11}$.

Next, we employ Lemma B.3 to estimate the second term in the right-hand side of (C.8). To that end, note that on the event $\mathcal{E}$ the collection of points that hit H is $\rho_{1/2}$-dense with respect to the L-paving $\{I_j\}_{j \in J}$. Moreover, $\{z \in \mathbb{Z} \colon\ \mathrm{dist}(z, H') \le n\} \subset H$. Therefore, applying Lemma B.3 with the densities $\rho_0 \le \rho_{1/2}$ (recall (C.6)), we have, on the event $\mathcal{E}$,

$$\begin{aligned} &\sup_{\substack{\ell_y, y \in H \\ \sum_{y \in I_j} \ell_y \ge \rho_{1/2} L, \forall j}} \mathbb{E}^{\rho_1}\big[f_2 \mid N(y) = \ell_y, y \in H\big] \\ &\qquad \le \mathbb{E}^{\rho_0}\big[f_2\big] + |H'| \exp\Big\{ - (\rho_{1/2} - \rho_0)L + \frac{c_9 \rho_{1/2} L^2 \log n}{\sqrt{n}}\Big\} \\ &\qquad \le \mathbb{E}^{\rho_0}\big[f_2\big] + |H'| \exp\big\{ - (\tfrac{\rho_0}{2}) n^{-1/16} L + 2 c_9 \rho_0 \log n\big\}. \end{aligned} \tag{C.12}$$

By (C.6), we have

$$\text{r.h.s.(C.12)} \le \mathbb{E}^{\rho_0}\big[f_2\big] + |H'| \, e^{-\frac{1}{8}\rho_0 n^{1/8}}. \tag{C.13}$$

Combine (C.8–C.9), to obtain

$$\begin{aligned}\mathbb{E}^{\rho_1}[f_1 f_2] &\le c\Big(1 + L^{-1}(\mathrm{per}(B) + 2n)\Big)\, e^{-c\rho_0 n^{1/8}} \\ &\quad + \mathbb{E}^{\rho_1}[f_1]\Big(\mathbb{E}^{\rho_0}[f_2] + |H'|\, e^{-c\rho_0 n^{1/8}}\Big) \\ &\le \mathbb{E}^{\rho_1}[f_1]\,\mathbb{E}^{\rho_0}[f_2] + c(\mathrm{per}(B) + n)\, e^{-c\rho_0 n^{1/8}},\end{aligned} \tag{C.14}$$

which yields the claim in (C.1).

Remark C.3. *It is important that the constants in Theorem* C.1 *do not depend on* ρ*, in accordance with our convention. This is crucial for our proof in Section* 3 *to work. Moreover, by translation invariance we can apply the result for general boxes* B *and half-spaces* D *as long as their vertical distance is at least* c_{11}*.*

Remark C.4. *The proof of Theorem* C.1 *holds almost without modification in the more general case of aperiodic symmetric random walks with bounded steps. Indeed, since properties* (B.1)*–*(B.3) *still hold in this case, the proofs of Lemmas* B.2–B.3 *immediately extend. The only change that is needed is in the definition of* H*, which must be made larger according to the bound on the size of the random walk steps.*

In the case of bipartite random walks, however, the aforementioned lemmas are no longer true as stated. Indeed, (B.2) *does not hold for such walks. Nonetheless, Theorem* C.1 *is still true in this situation. The proof can be adapted through the following two steps:*

(1) In the statement of Lemmas B.2–B.3*, we suppose that the collection* $(x_j)_{j\in J}$ *is dense in both of the sets* $C_i \cap 2\mathbb{Z}$ *and* $C_i \cap \mathbb{Z} \setminus 2\mathbb{Z}$*. Since* (B.2) *still holds when* x, x' *have the same parity, and* (B.1) *and* (B.3) *are still valid, the proofs of the lemmas go through with this modification.*

(2) In the proof of Theorem C.1*, we modify* $\mathcal{E}$ *to be the event where enough trajectories cross both of the sets* $H \cap I_i \cap 2\mathbb{Z}$ *and* $H \cap I_i \cap \mathbb{Z} \setminus 2\mathbb{Z}$*, which allows us to use Lemmas* B.2–B.3 *with the new statements.*

In the case of continuous-time symmetric random walks with bounded jumps, (B.1)*–*(B.3) *and Lemma B.2 remain true. However, the random walk is no longer almost surely Lipschitz, and this property is used in Lemmas B.3 and C.2, Theorem C.1 and in several other places throughout the paper. Nonetheless, the random walk is still Lipschitz with high probability, and this is enough to adapt all arguments to this case.*

D Tail estimate

The following lemma consists of a simple estimate on the tail of an infinite series, used in (3.35). Its proof is inspired by similar arguments in Mörters and Peres [17, Lemma 12.9, p. 349].

Lemma D.1. *Let a be a positive integer. Then, for all $\beta > 0$ there exists a $c = c(\beta)$ such that*

$$\sum_{l>a} l^\beta e^{-\log^{3/2} l} \le c a^{\beta+1} e^{-\log^{3/2} a}. \tag{D.1}$$

Proof. It is enough to consider the case $a \ge \alpha_0 = e^{(\beta+1)^2}$. If $g(x) = x^\beta \exp\{-\log^{3/2} x\}$, then $g'(x) = x^{\beta-1} \exp\{-\log^{3/2} x\}[\beta - \frac{3}{2}\sqrt{\log x}] < 0$ for all $x \ge \alpha_0$ because $\alpha_0 > e^{(4/9)\beta^2}$. It follows that

$$\sum_{l>a} l^\beta e^{-\log^{3/2} l} \le \int_a^\infty x^\beta e^{-\log^{3/2} x} dx. \tag{D.2}$$

Let

$$\begin{aligned} D &= \max\left\{2\int_{\alpha_0}^\infty x^\beta e^{-\log^{3/2} x} dx, \frac{4}{\beta+1}\right\}, \\ f(a) &= D a^{\beta+1} e^{-\log^{3/2} a} - \int_a^\infty x^\beta e^{-\log^{3/2} x} dx. \end{aligned} \tag{D.3}$$

We claim that $\lim_{a\to\infty} f(a) = 0$, $f(\alpha_0) > 0$ and $f'(a) < 0$ for all $a \ge \alpha_0$. The first claim is immediate. The second claim follows from

$$\begin{aligned} f(\alpha_0) &= D e^{\log\alpha_0\{(\beta+1)-\sqrt{\log\alpha_0}\}} - \int_{\alpha_0}^\infty x^\beta e^{-\log^{3/2} x} dx \\ &= D - \int_{\alpha_0}^\infty x^\beta e^{-\log^{3/2} x} dx > 0, \end{aligned} \tag{D.4}$$

where the last inequality follows from $D \ge 2\int_{\alpha_0}^\infty x^\beta e^{-\log^{3/2} x} dx$. To get the third claim, note that

$$f'(a) = a^\beta e^{-\log^{3/2} a}\left[(\beta+1)D - \tfrac{3}{2}D\sqrt{\log a} + 1\right] \tag{D.5}$$

and, for $a \ge \alpha_0$, we have that $[(\beta+1)D - \frac{3}{2}D\sqrt{\log a} + 1] \le -\frac{1}{2}D(\beta+1) + 1 \le -1$, where the last inequality follows from $D \ge 4/(\beta+1)$.

In particular, $f(a) \ge 0$ for all $a \ge \alpha_0$, and hence

$$\sum_{l>a} l^\beta e^{-\log^{3/2} l} \le \int_a^\infty x^\beta e^{-\log^{6/5} x}\, dx \le D a^{\beta+1} e^{-\log^{3/2} a}, \tag{D.6}$$

which concludes the proof. □

E Rate of convergence

In this section we state and prove a basic fact about independent random variables that is used in the proof of Theorem 1.4(c) in Section 4.3.

Lemma E.1. *Let X_i, $i \in \mathbb{N}$, be independent random variables with joint law $\mathbb{P}$ such that*

$$\mathbb{E}[X_i] = 0\ \forall\, i \in \mathbb{N}, \qquad \sup_{i\in\mathbb{N}} \mathbb{E}\left[\exp\{K(\log^+|X_i|)^\gamma\}\right] < \infty \tag{E.1}$$

for some $K > 0$ and $\gamma > 1$, where $\log^+ x = \max(\log x, 0)$. Then, for all $\varepsilon > 0$, there exists a $c > 0$ such that

$$\mathbb{P}\left(\exists\, k \geq n\colon \left|\sum_{i=1}^{k} X_i\right| > \varepsilon k\right) \leq c^{-1} e^{-c \log^{\gamma} n} \quad \forall\, n \in \mathbb{N}. \tag{E.2}$$

Proof. Let

$$X_i^{(n)} = X_i \mathbb{1}_{\{|X_i| \leq \sqrt{n}\}} - \mathbb{E}[X_i \mathbb{1}_{\{|X_i| \leq \sqrt{n}\}}]. \tag{E.3}$$

We claim that

$$\mathbb{P}\left(\exists\, k \in \mathbb{N}\colon \left|\sum_{i=1}^{k} X_i - X_i^{(n)}\right| > \varepsilon k\right) \leq C e^{-c \log^{\gamma} n}. \tag{E.4}$$

Indeed, since $\mathbb{E}[X_i] = 0$, setting

$$X_i^{>(n)} = X_i \mathbb{1}_{\{|X_i| > \sqrt{n}\}} - \mathbb{E}[X_i \mathbb{1}_{\{|X_i| > \sqrt{n}\}}] \tag{E.5}$$

we have $X_i - X_i^{(n)} = X_i^{>(n)}$ for all $i \in \mathbb{N}$. Moreover, by the Marcinkiewicz-Zygmund and Minkowski inequalities,

$$\mathbb{E}\left[\left|\sum_{i=1}^{k} X_i^{>(n)}\right|^4\right]^{\frac{1}{2}} \leq C \sum_{i=1}^{k} \mathbb{E}[|X_i^{>(n)}|^4]^{\frac{1}{2}} \leq Ck\, e^{-c \log^{\gamma} n} \tag{E.6}$$

by (E.1). Therefore

$$\mathbb{P}\left(\exists\, k \in \mathbb{N}\colon \left|\sum_{i=1}^{k} X_i^{>(n)}\right| > \varepsilon k\right) \leq C e^{-c \log^{\gamma} n} \sum_{k=1}^{\infty} k^{-2} \tag{E.7}$$

and (E.4) follows. Thus, it suffices to prove (E.2) for $X_i^{(n)}$. To that end, we use Bernstein's inequality to write

$$\mathbb{P}\left(\left|\sum_{i=1}^{k} X_i^{(n)}\right| > u\right) \leq 2 \exp\left\{-c\, \frac{u^2}{k + \sqrt{n} u}\right\}. \tag{E.8}$$

Taking $u = \varepsilon k$, we obtain

$$\mathbb{P}\left(\exists\, k \geq n\colon \left|\sum_{i=1}^{k} X_i^{(n)}\right| > \varepsilon k\right) \leq 2 \sum_{k=n}^{\infty} e^{-\frac{c}{\sqrt{n}} k} = \frac{2 e^{-c\sqrt{n}}}{1 - e^{-\frac{c}{\sqrt{n}}}} \leq C e^{-c\sqrt{n}}, \tag{E.9}$$

which concludes the proof. □

References

[1] L. Avena, R.S. dos Santos and F. Völlering, Transient random walk in symmetric exclusion: limit theorems and an Einstein relation, ALEA (Lat. Am. J. Probab. Math. Stat.) 10 (2013) 693–709.

[2] A. Bandyopadhyay and O. Zeitouni, Random Walk in Dynamic Markovian Random Environment, ALEA (Lat. Am. J. Probab. Math. Stat.) 1 (2006) 205–224.

[3] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[4] C. Boldrighini, G. Cosimi, S. Frigio and A. Pellegrinotti, Computer simulations for some one-dimensional models of random walks in fluctuating random environment, J. Stat. Phys 121 (2005) 361–372.

[5] C. Boldrighini, R.A. Minlos and A. Pellegrinotti, Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment, Probab Theory Related Fields 109 (1997) 245–273.

[6] C. Boldrighini, R.A. Minlos and A. Pellegrinotti, Random walk in a fluctuating random environment with Markov evolution. in: *On Dobrushin's way. From probability theory to statistical physics*, volume 198 of Amer. Math. Soc. Transl. Ser. 2, 13–35. Amer. Math. Soc. Providence, RI, 2000.

[7] C. Boldrighini, R.A. Minlos and A. Pellegrinotti, Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive, Probab. Theory Related Fields 129 (2004) 133–156.

[8] D. Dolgopyat, G. Keller and C. Liverani, Random walk in Markovian environment, Ann. Probab. 36 (2008) 1676–1710.

[9] D. Dolgopyat and C. Liverani, Non-perturbative approach to random walk in Markovian environment, Electron. Commun. Probab. 14 (2009) 245–251.

[10] T. Harris, Diffusion with "collision" between particles, J. Appl. Probab. 2 (1965) 323–338.

[11] F. den Hollander and R. dos Santos, Scaling of a random walk on a supercritical contact process, to appear in Ann. I. Henri Poincaré. Preprint 2012. [arXiv:1209.1511]

[12] F. den Hollander, H. Kesten and V. Sidoravicius, Random walk in a high density dynamic random environment, to appear in Indagationes Mathematicae. Preprint 2013. [arXiv:1305.0923]

[13] F. den Hollander, S. A. Molchanov and O. Zeitouni, *Random media at Saint-Flour. Reprints of lectures from the Annual Saint-Flour Probability Summer School held in Saint-Flour. Probability at Saint-Flour*, Springer, Heidelberg (2012)

[14] G.F. Lawler and V. Limic, *Random Walk: A Modern Introduction*, Cambridge Studies in Advanced Mathematics 123, Cambridge University Press, Cambridge, 2010.

[15] T. Liggett, *Interacting Particle Systems*, Springer-Verlag, Berlin, 2005.

[16] T. Lindvall, On coupling of discrete renewal processes, Z. Wahrscheinlichkeitstheorie verw. Gebiete 48 (1979) 57–70.

[17] P. Mörters and Y. Peres, *Brownian Motion*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010.

[18] T. Mountford and M.E. Vares, Dynamic contact process, Preprint 2013. [arXiv:1310.7458]

[19] Y. Peres, A. Sinclair, P. Sousi and A. Stauffer, Mobile geometric graphs: detection, coverage and percolation, in: *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, 412–428, SIAM, Philadelphia, PA, 2011.

[20] S. Popov and A. Teixeira, Soft local times and decoupling of random interlacements. Preprint 2012. [arXiv:1212.1605]

[21] F. Rassoul-Agha, T. Seppäläinen, An almost sure invariance principle for random walks in a space-time random environment, Probab. Theory Related Fields 133 (2005) 299–314.

[22] S.I. Resnick, *Extreme Values, Regular Variation and Point Processes*, Springer Series in Operations Research and Financial Engineering, Springer, New York, 2008.

[23] Yu.A. Rozanov, *Markov Random Fields*, Applications of Mathematics, Springer, New York, 1982.

[24] W. van Saarloos, Front propagation into unstable states, Phys. Rep. 386 (2003) 29–222.

[25] A.S. Sznitman, Topics in random walks in random environment, School and Conference on Probability Theory, May 2002, ICTP Lecture Series, Trieste, 203–266, 2004.

[26] M. Zerner, Lyapounov exponents and quenched large deviations for multidimensional random walk in random environment, Ann. Probab. 26 (1998) 1407–1912.