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Corrected phase-type approximations of heavy-tailed queueing models in a Markovian environment

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Abstract

We develop accurate approximations of the delay distribution of the MArP/G/1 queue that capture the exact tail behavior and provide bounded relative errors. Motivated by statistical analysis, we consider the service times as a mixture of a phase-type and a heavy-tailed distribution. With the aid of perturbation analysis, we derive corrected phase-type approximations as a sum of the delay in an MArP/PH/1 queue and a heavy-tailed component depending on the perturbation parameter. We exhibit their performance with numerical examples.

1. Introduction

The evaluation of performance measures in stochastic models is a key problem that has been widely studied in the literature [1, 8, 19, 35]. In this paper, we focus on the evaluation of the delay distribution of a single server queue where customers arrive according to a Markovian Arrival Process (MArP) [9, 25] and their service times follow some general distribution. Under the presence of heavy-tailed service times, such evaluations become more challenging and sometimes even problematic [4, 11]. In such cases, it is necessary to construct approximations. In this study, we propose to modify existing approximations by adding a small refinement term, which can serve two purposes. On the one hand, the refinement term helps in constructing approximations not only with a small absolute error, but also with a small relative error. On the other hand, it gives information on the accuracy of the approximation without the modification: the smaller the refinement term, the better the pre-modified approximation.

An important generalization of the Poisson point process is the MArP. In a MArP, the arrivals are not homogenous in time, but they are determined by a Markov process $\{J_t\}_{t\geq 0}$ with a finite state space. The class of MArPs is a very rich class of point processes, containing many well-known arrival processes as special cases. A special case of a MArP is the Markov-modulated Poisson process (MMPP), which is a popular model for bursty arrivals [17]. The class of MArPs contains also the class of phase-type renewal processes, i.e. renewal processes with phase-type interarrivals [26].

It has been shown that the Laplace transform of the delay of a MArP/G/1 queue has a matrix expression analogous to the Pollazceck-Khinchine equation of an M/G/1 queue [27, 28]. However,

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these closed-form expressions are only practical in case of phase-type service times [6, 7], where the delay distribution has a phase-type representation [29] in a form which is explicit up to the solution of a matrix fixed point problem.

Since the class of phase-type distributions is dense in the class of all distributions on $(0, \infty)$ [6], a common approach to approximate the delay is by approximating the service time distribution with a phase-type one; see e.g. [15, 33]. We refer to these methods as *phase-type approximations*. There are many algorithms for phase-type approximations, which provide highly accurate approximations for the delay distribution when the service times are light-tailed. However, in many cases, a heavy-tailed distribution is most appropriate to model the service times [14, 31]. In these cases, the exponential decay of phase-type approximations gives a big relative error at the tail and the evaluation of the delay becomes more complicated. Since heavy-tailed distributions have cumbersome expressions for their Laplace transform, this prevents the usage of techniques that require transform expressions, such as [18].

In this paper, we develop approximations of the delay distribution for heavy-tailed service times that maintain the computational tractability of phase-type approximations, capture the correct tail behavior and provide small absolute and relative errors. In order to achieve these desirable characteristics, our key idea is to use a mixture model for the service times. The idea of our approach stems from fitting procedures of the service time distribution to data. Heavy-tailed statistical analysis suggests that only a small fraction of the upper-order statistics of a sample is relevant for estimating tail probabilities [30]. The remaining data set may be used to fit the bulk of the distribution, where, as we mentioned earlier, a natural choice is to fit a phase-type distribution to the remaining data set [10]. As a result, a mixture model for the service times is a natural assumption.

We now briefly explain how to derive our approximations when the service time distribution is a mixture of a phase-type distribution and a heavy-tailed one. We show that if the service time distribution is such a mixture, then the queueing delay can also be written as a mixture, in the sense that it involves the queueing delay of a model with purely phase-type service times and some additional terms related to the heavy-tailed distribution of our mixture model. Consequently, we first need to compute the delay in a MArP/PH/1 queue and afterwards use this as a base to calculate the rest of the terms involving the heavy-tailed distribution.

As a first step to derive our approximations, we write the service time distribution as perturbation of the phase-type distribution by a function that contains the heavy-tailed component. By ignoring the perturbation term and by taking the service time distribution equal to the phase-type distribution, we find the delay of a resulting simpler MArP/PH/1 queue, which is a phase-type approximation of the queueing delay. By applying perturbation analysis to all parameters that depend on the service time distribution, we can write the queueing delay as a series expansion, where the constant term is the delay of the MArP/PH/1 queue used as base and all other terms contain the heavy-tailed component.

Large deviations theory suggests that a single catastrophic event, i.e. a stationary heavy-tailed service time, is sufficient to give a non-zero tail probability for the queueing delay [14]. As we will see in Section 3.3, the second term of the series expansion of the queueing delay can be expressed in terms of such a catastrophic event. Thus, we define our approximations as the sum of the first two terms of the series expansion of the queueing delay, and we show that the addition of the second term leads to improved approximations when compared to their phase-type counterparts. In other words, the second term makes the phase-type approximation more robust so that the relative error at the tail does not explode. Therefore, we call this term correction term, and inspired by the terminology *corrected heavy traffic approximations* [7] we refer to our approximations as *corrected phase-type approximations*. In a previous study [34], we applied this approach to Poisson arrivals.

The connection between the stationary delay distribution of a MArP/G/1 queue and ruin probabilities for a risk process in a Markovian environment, where the claim sizes in the risk model correspond to the service times and the arrival process of claims is the time-reversed MArP of the queueing model, is well known [7, 8]. Thus, the corrected phase-type approximations can also be used to estimate the ruin probabilities of the above mentioned risk model. Finally, our technique can be applied to more general queueing models, i.e. queuing models with dependencies between interarrival and service times [12, 32], and also to models that allow for customers to arrive in batches (the arrival process is called then Batch Markovial Arrival Process) [22, 23, 24].

A closely related work is Adan and Kulkarni [3]. They consider a single server queue, where the interarrival times and the service times depend on a common discrete Markov Chain. In addition, they assume that a customer arrives in each phase transition, and they find a closed form expression for the delay distribution under general service time distributions. However, when there exist also phase transitions not related to arrivals of customers, their results remain valid for the evaluation of the workload. This can be seen by using the standard technique of including *dummy* customers in the model; namely customers with zero service times.

The rest of the paper is organized as follows. In Section 2, we introduce the model under consideration without assuming any special form for the service time distribution, and in Section 2.1 we find the general expressions for the Laplace transforms of the queueing delay a customer experiences upon arrival in each state. In Section 2.2, we consider service time distributions that are a mixture of a phase-type distribution and a heavy-tailed one, and we explain the idea to construct our approximations. Later in Section 3.1, we specialize the results of Section 2.1 for phase-type service times. We use as base model the phase-type model of Section 3.1, and we apply perturbation analysis to find in Section 3.2 the perturbed parameters and in Section 3.3 the desired Laplace transforms of the delay in the mixture model. Using the latter results, we construct in Section 3.4 the approximations and we discuss their properties. In Section 4, we discuss an alternative way to construct approximations for the queueing delay. Furthermore, in Section 5, we use a specific mixture service time distribution for which the exact delay distribution can be calculated and we exhibit the accuracy of our approximations through numerical experiments. Finally, in the Appendix, we give the proofs of all theorems, the necessary theory on perturbation analysis, and other related results. Due to the complexity of the formulas, we use a simple running example in order to explain the idea behind the calculations.

2. Presentation of the model

We consider a single server queue with FIFO discipline, where customers arrive according to a Markovian Arrival Process (MArP). The arrivals are regulated by a Markov process $\{J_t\}_{t\geq 0}$ with a finite state space $\mathcal{N}$, say with N states. We assume that the service time distribution of a customer is independent of the state of $\{J_t\}$ upon his arrival. For this model, we are interested in finding accurate approximations for the delay distribution.

The intensity matrix $\mathbf{D}$ governing $\{J_t\}$ is denoted by the decomposition $\mathbf{D} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)}$, where the matrix $\mathbf{D}^{(1)}$ is related to arrivals of *dummy* customers, while transitions in $\mathbf{D}^{(2)}$ are related to arrivals of *real* customers. Note that the diagonal elements of the matrix $\mathbf{D}^{(2)}$ may not be identically equal to zero. This means that if $d_{ii}^{(2)} > 0$, then a real customer arrives with rate $d_{ii}^{(2)}$ and we have a transition from state i to itself. However, phase transitions not associated with arrivals (dummy customers) from any state to itself are not allowed. Since the matrix $\mathbf{D}$ is an intensity matrix, its rows sum up to zero. Therefore, the diagonal elements of the matrix $\mathbf{D}^{(1)}$ are negative and they are defined as $d_{ii}^{(1)} = -\sum_{k\neq i} d_{ik}^{(1)} - \sum_{k=1}^{N} d_{ik}^{(2)}$.

In this paper, we are interested in modeling heavy-tailed service times. As stated earlier, motivated by statistical analysis, we assume that the service time distribution of a real customer is a mixture of a phase-type distribution, $F_p(t)$, and a heavy-tailed one, $F_h(t)$. Namely, the service time distribution of a real customer has the form

$$G_\epsilon(t) = (1-\epsilon)F_p(t) + \epsilon F_h(t), \tag{1}$$

where ϵ is typically small.

Our goal is to find the delay distribution for this mixture model. Towards this direction, we present in the next section existing results [3] for the evaluation of the delay distribution under the assumption of generally distributed service times. Ultimately, we wish to specialize these results to service times of the aforementioned form (1).

2.1 Preliminaries

Since the results of this section are valid for any service time distribution, we suppress the index ϵ and we use the notation $G(t)$ for the service time distribution of a real customer. We consider now the embedded Markov chain $\{Z_n\}_{n\geq 0}$ on the arrival epochs of customers (real and dummy) and we denote by $\mathbf{P}$ the transition probability matrix of the regulating Markov chain $\{Z_n\}$, which we assume to be irreducible. If λ_i is the exponential exit rate from state i, i.e.

$$\lambda_i = \sum_{k\neq i} d_{ik}^{(1)} + \sum_{k=1}^{N} d_{ik}^{(2)}, \tag{2}$$

the transition probabilities can be calculated by

$$p_{ij} = \frac{d_{ij}^{(1)}(1-\delta_{ij}) + d_{ij}^{(2)}}{\lambda_i}, \tag{3}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$). In addition, an arriving customer at a transition from state i to state j is tagged i. If $p_{ij} > 0$, then we define the probability

$$q_{ij}^{(1)} = \frac{d_{ij}^{(1)}(1-\delta_{ij})}{d_{ij}^{(1)}(1-\delta_{ij}) + d_{ij}^{(2)}}, \tag{4}$$

which is the probability of an arriving customer to be dummy conditioned on the event that there is a phase transition from state i to j. Similarly, conditioned on the event that there is a phase transition from i to j, the arriving customer is real with probability

$$q_{ij}^{(2)} = \frac{d_{ij}^{(2)}}{d_{ij}^{(1)}(1-\delta_{ij}) + d_{ij}^{(2)}}. \tag{5}$$

If $p_{ij} = 0$, then we define $q_{ij}^{(1)} = q_{ij}^{(2)} = 0$. Consequently, the conditional service time distribution of an arriving customer at a transition from i to j is $G_{ij}(t) = q_{ij}^{(1)} + q_{ij}^{(2)} G(t)$, and its Laplace-Stieltjes transform (LST) is $\widetilde{G}_{ij}(s) = q_{ij}^{(1)} + q_{ij}^{(2)} \widetilde{G}(s)$, $i, j = 1, \ldots, N$, where $\widetilde{G}(s)$ is the LST of the service time distribution $G(t)$ of a real customer. In matrix form, the above quantities can be written as

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N), \tag{6}$$

$$\mathbf{Q}^{(1)} = [q_{ij}^{(1)}], \tag{7}$$
$$\mathbf{Q}^{(2)} = [q_{ij}^{(2)}], \tag{8}$$
$$\widetilde{\mathbf{G}}(s) = \mathbf{Q}^{(1)} + \widetilde{G}(s)\mathbf{Q}^{(2)}. \tag{9}$$

Let now $\circ$ denote the Hadamard product between two matrices of same dimensions; i.e. if $\mathbf{B} = (b_{ij})$ and $\mathbf{C} = (c_{ij})$ are $m \times n$ matrices, then the (i, j) element of the $m \times n$ matrix $\mathbf{B} \circ \mathbf{C}$ is equal to $b_{ij}c_{ij}$. We also define the matrix

$$\mathbf{H}(s) = \widetilde{\mathbf{G}}(s) \circ \mathbf{P}\boldsymbol{\Lambda}, \tag{10}$$

which we will need later. Finally, let $\boldsymbol{\pi} = [\pi_1, \ldots, \pi_N]$ be the stationary distribution of $\{Z_n\}_{n\geq 0}$, and μ be the mean of the service time distribution $G(t)$. Then the system is stable if the mean service time of a customer is less than the mean inter-arrival times between two consecutive customers in steady state. Namely,

$$\boldsymbol{\pi}\big(\boldsymbol{\Lambda}^{-1} - \mathbf{M}\big)\mathbf{e} > 0, \tag{11}$$

where $\mathbf{M} = \mu\mathbf{Q}^{(2)} \circ \mathbf{P}$ and $\mathbf{e}$ is the column vector with appropriate dimensions and all elements equal to 1. Note that the (i, j) element of the matrix $\mathbf{Q}^{(2)} \circ \mathbf{P}$ is the unconditional probability that a real customer arrives at a transition from i to j.

From this point on, we use a simple running example so that we display the involved parameters and the derived formulas. The running example evolves progressively, which means that its parameters are introduced only once and the reader should consult a previous block of the example to recall the notation.

Running example For our running example, we consider a MArP with Erlang-2 distributed interarrival times, where the exponential phases have both rate λ ($N = 2$). Therefore, the matrices $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are given as follows:

$$\mathbf{D}^{(1)} = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix} \qquad \text{and} \qquad \mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

In this case, we have that $\lambda_1 = \lambda_2 = \lambda$, $p_{ij} = 1 - \delta_{ij}$, $q_{12}^{(1)} = q_{21}^{(2)} = 1$, and all other elements of the matrices $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ are equal to zero. Observe that we only have transitions from state 1 to state 2 and from state 2 to state 1. Therefore, in state 1 we always have arrivals of dummy customers while in state 2 we only have arrivals of real customers. Thus, only the diagonal elements of the matrix $\widetilde{\mathbf{G}}(s)$ are not equal to zero, so that $\widetilde{G}_{11}(s) = 1$ and $\widetilde{G}_{22}(s) = \widetilde{G}(s)$. Finally, the stability condition takes its known form $\lambda\mu/2 < 1$. ■

Let now V denote the steady-state workload of the system just prior to an arrival of a customer. If the arriving customer is real, then the workload just prior to its arrival equals the delay or waiting time of the customer in the queue, which we denote by W. In terms of Laplace transforms, the steady-state workload of the system just prior to an arrival of a customer in state i is found as

$$\widetilde{\phi}_i(s) = \mathbb{E}(e^{-sV}; Z = i), \qquad \Re(s) \geq 0, \quad i = 1, \ldots, N,$$

where Z is the steady-state limit of Z_n. Gathering all the above Laplace transforms $\widetilde{\phi}_i(s)$, $i = 1, \ldots, N$, we construct the transform vector

$$\widetilde{\boldsymbol{\Phi}}(s) = [\widetilde{\phi}_1(s), \ldots, \widetilde{\phi}_N(s)]. \tag{12}$$

We first provide some general theorems for the transform vector $\widetilde{\boldsymbol{\Phi}}(s)$ and we give its connection to the Laplace transform $\widetilde{w}(s)$ of the queueing delay of real customers. Later on we refine these results in order to provide more detailed information regarding the form of the elements $\widetilde{\phi}_i(s)$, $i = 1, \ldots, N$. In the following, $\mathbf{I}$ stands for the identity matrix, with appropriate dimensions.

Theorem 2.1. *Provided that the stability condition* (11) *is satisfied, the transform vector* $\widetilde{\mathbf{\Phi}}(s)$ *satisfies*

$$\widetilde{\mathbf{\Phi}}(s)\big(\mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}\big) = s\mathbf{u}, \tag{13}$$

$$\widetilde{\mathbf{\Phi}}(0)\mathbf{e} = 1, \tag{14}$$

where $\mathbf{u} = [u_1, \ldots, u_N]$ *is a vector with* N *unknown parameters that needs to be determined.*

Note that the above theorem is similar to Theorem 3.1 in [3] and so does its proof. Therefore, we omit here the proof and we refer the reader to Theorem 3.1 of [3] for more details.

A real customer arrives in state i with probability $\sum_{j=1}^{N} p_{ij} q_{ij}^{(2)} = \sum_{j=1}^{N} d_{ij}^{(2)}/\lambda_i$, and consequently a real customer arrives in the system with probability $\sum_{i=1}^{N} \pi_i \cdot \sum_{j=1}^{N} d_{ij}^{(2)}/\lambda_i$. Therefore, the following relation holds

$$\sum_{i=1}^{N} \pi_i \frac{\sum_{j=1}^{N} d_{ij}^{(2)}}{\lambda_i} \cdot \widetilde{w}(s) = \sum_{i=1}^{N} \frac{\sum_{j=1}^{N} d_{ij}^{(2)}}{\lambda_i} \widetilde{\phi}_i(s).$$

Thus, if $\boldsymbol{\omega}$ is a column vector of dimension N such that

$$\boldsymbol{\omega} = \frac{\mathbf{\Lambda}^{-1}\mathbf{D}^{(2)}\mathbf{e}}{\boldsymbol{\pi}\mathbf{\Lambda}^{-1}\mathbf{D}^{(2)}\mathbf{e}}, \tag{15}$$

the Laplace transform of the queuing delay is found as

$$\widetilde{w}(s) = \widetilde{\mathbf{\Phi}}(s)\boldsymbol{\omega}, \qquad \Re(s) \geq 0. \tag{16}$$

If $\det\big(\mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}\big)$ denotes the determinant of the square matrix $\mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}$, then for the determination of the unknown vector $\mathbf{u}$, we have the following theorem.

Theorem 2.2. *The next two statements hold:*

1. *The equation* $\det\big(\mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}\big) = 0$ *has exactly* N *solutions* $s_1, \ldots, s_N$*, with* $s_1 = 0$ *and* $\Re(s_i) > 0$ *for* $i = 2, \ldots, N$.

2. *Suppose that the stability condition* (11) *is satisfied and that the above mentioned* $N - 1$ *solutions* $s_2, \ldots, s_N$ *are distinct. Let* $\mathbf{a}_i$ *be a non-zero column vector satisfying*

$$\big(\mathbf{H}(s_i) + s_i\mathbf{I} - \mathbf{\Lambda}\big)\mathbf{a}_i = 0, \quad i = 2, \ldots, N.$$

Then $\mathbf{u}$ *is given by the unique solution to the following* N *linear equations:*

$$\mathbf{u}\mathbf{\Lambda}^{-1}\mathbf{e} = \boldsymbol{\pi}\left(\mathbf{\Lambda}^{-1} - \mathbf{M}\right)\mathbf{e}, \tag{17}$$

$$\mathbf{u}\mathbf{a}_i = 0, \qquad i = 2, \ldots, N. \tag{18}$$

Again, Theorem 2.2 is similar to Theorems 3.2 & 3.3 in [3], and therefore, its proof is omitted.

Theorem 2.2 on one hand provides us with an algorithm to calculate the vector $\mathbf{u}$ and on the other hand it guarantees that all elements of the transform vector $\widetilde{\mathbf{\Phi}}(s)$ are well-defined on the positive half-plane. To understand the latter remark observe the following. For simplicity, we set

$$\mathbf{E}(s) = \mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}. \tag{19}$$

Let $\mathcal{E}(s)$ be the adjoint matrix of $\mathbf{E}(s)$, so $\mathbf{E}(s)\cdot\mathcal{E}(s) = \det\mathbf{E}(s)\mathbf{I}$. Post-multiplying Eq. (13) with $\mathcal{E}(s)$, we have that $\widetilde{\mathbf{\Phi}}(s)\det\mathbf{E}(s) = s\mathbf{u}\mathcal{E}(s)$, and consequently

$$\widetilde{\mathbf{\Phi}}(s) = \frac{1}{\det\mathbf{E}(s)} s\mathbf{u}\mathcal{E}(s). \tag{20}$$

The first statement of Theorem 2.2 says that the determinant $\det\mathbf{E}(s)$ has the factors $s - s_i$, $i = 1,\dots,N$, in its expression. This means that the transform vector $\widetilde{\mathbf{\Phi}}(s)$ has N potential singularities on the positive half plane, as the determinant appears at the denominator. However, the second statement of Theorem 2.2 explains that the vector $\mathbf{u}$ is such that these problematic factors are canceled out.

Observe that Theorem 2.2 does not give us any information about the form of the elements of the transform vector $\widetilde{\mathbf{\Phi}}(s)$, which is the stepping stone for the construction of our approximations. For this reason, we proceed by finding an analytic expression for the aforementioned elements. It is apparent from Eq. (20) that for the evaluation of $\widetilde{\mathbf{\Phi}}(s)$ we only need $\det\mathbf{E}(s)$ and the adjoint matrix $\mathcal{E}(s)$. For the determination of these quantities, we introduce the following notation:

- As before, we denote the set of all states of the Markov process $\{J_t\}$ as $\mathcal{N} = \{1,\dots,N\}$.
- If $S \subset \Omega$, for some set $\Omega \subset \mathcal{N}$, then S^c is the complementary set of S with respect to Ω. Observe that all subset relations are used locally and that the symbol "$\subset$" does not imply strict subsets. The number of elements in a set S is denoted as $|S|$.
- For a subset S of $\mathcal{N}$ we define $\boldsymbol{\lambda}^S = \prod_{i\in S}\lambda_i$ and $\zeta^S(s) = \prod_{i\in S}(s-\lambda_i)$. We also define $\boldsymbol{\lambda}^\emptyset = \zeta^\emptyset(s) = 1$.
- Suppose that $U, W \subset \mathcal{N}$ and that $\mathbf{A}$ is a square matrix of dimension N. Then $\mathbf{A}_U^W$ is the submatrix of $\mathbf{A}$ if we keep the rows in U and the columns in W. Whenever the notation becomes very complicated, to avoid any confusion with the indices, we will denote the ith column and row of matrix $\mathbf{A}$ with $\mathbf{A}_{\bullet i}$ and $\mathbf{A}_{i\bullet}$, respectively. We also define $\det\mathbf{A}_\emptyset^\emptyset = 1$.
- Suppose that S is a subset of Ω, for some set $\Omega \subset \mathcal{N}$, and that it follows some properties, i.e. "Property 1", etc. If we want to sum with respect to S, then we write under the symbol of summation first $S \subset \Omega$, followed by the properties. Namely, we write $\sum_{\substack{S\subset\Omega \\ \text{Property 1} \\ \text{etc}}}$. In some cases, to avoid lengthy expressions we will write instead of $\sum_{\substack{S\subset\Omega \\ \text{Properties of } S}} \sum_{\substack{R\subset\Omega_1 \\ \text{Properties of } R}}$ the double sum $\sum_{\substack{S\subset\Omega \\ \text{Properties of } S; \\ R\subset\Omega_1 \\ \text{Properties of } R}}$, where R is a subset of Ω_1, for some set $\Omega_1 \subset \mathcal{N}$. We apply the same rule also for multiple sums.
- Suppose that $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of dimension N, and that U and W are two disjoint subsets of $\mathcal{N}$. For all $\Omega \subset \mathcal{N}$, we use the notation $\mathbf{A}_\Omega^U \bowtie \mathbf{B}_\Omega^W$ for the matrix that has columns the union of the columns V of matrix $\mathbf{A}$ and the columns W of matrix $\mathbf{B}$, ordered according to the index set $U \cup W$; e.g. if $\Omega = \mathcal{N} = \{1,\dots,5\}$, $U = \{1,2,4\}$, and $W = \{3,5\}$, then $\mathbf{A}_\mathcal{N}^{\{1,2,4\}} \bowtie \mathbf{B}_\mathcal{N}^{\{3,5\}} = (\mathbf{A}_{\bullet 1}, \mathbf{A}_{\bullet 2}, \mathbf{B}_{\bullet 3}, \mathbf{A}_{\bullet 4}, \mathbf{B}_{\bullet 5})$.

Using the above notation, we proceed with refining the desired quantities. More precisely, we first find $\det\mathbf{E}(s)$, then the adjoint matrix $\mathcal{E}(s)$, and finally the vector $s\mathbf{u}\mathcal{E}(s)$ that appears in the numerator of the transform vector $\widetilde{\mathbf{\Phi}}(s)$ (see Eq. (20)). Combining these results, one can easily derive $\widetilde{\mathbf{\Phi}}(s)$. We start by finding the determinant of matrix $\mathbf{E}(s)$ (see Eq. (19)).

Theorem 2.3. *The determinant of matrix* $\mathbf{E}(s)$ *can be explicitly calculated as follows:*

$$\det \mathbf{E}(s) = \sum_{S \subset \mathcal{N}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det \left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_S^S$$
$$+ \sum_{k=1}^{N} \widetilde{G}^k(s) \sum_{\substack{\Gamma \subset \mathcal{N} \\ |\Gamma|=k}} \sum_{\substack{S \subset \mathcal{N} \\ S \supset \Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det \left(\left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_S^{S \setminus \Gamma} \bowtie \left(\mathbf{Q}^{(2)} \circ \mathbf{P}\right)_S^{\Gamma} \right).$$

Proof. See Appendix B. □

Observe that the determinant $\det \mathbf{E}(s)$ is an at most N degree polynomial with respect to the LST of the service time distribution $\widetilde{G}(s)$ of a real customer. Moreover, the coefficients of this polynomial are all polynomials with respect to s. Therefore, in case $\widetilde{G}(s)$ is a rational function in s, then $\det \mathbf{E}(s)$ is also a rational function in s and its eigenvalues can be easily calculated. Furthermore, the subset Γ of $\mathcal{N}$ that appears in the second summand has at least one element, thus in the formula of $\det \mathbf{E}(s)$ it always holds that $\Gamma \neq \emptyset$.

Running example (continued) The matrix $\mathbf{E}(s)$ has elements $\mathbf{E}_{ii}(s) = s - \lambda$, $i = 1, 2$, $\mathbf{E}_{12}(s) = \lambda$, and $\mathbf{E}_{21}(s) = \lambda \widetilde{G}(s)$. We will calculate its determinant using Theorem 2.3. It holds that $\det \left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_S^S = 0$ for all subsets S of $\mathcal{N}$, except for $S = \emptyset$. Since $\Gamma \neq \emptyset$, it is evident that $\det \left(\left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_S^{S \setminus \Gamma} \bowtie \left(\mathbf{Q}^{(2)} \circ \mathbf{P}\right)_S^{\Gamma} \right) \neq 0$ only for $\Gamma = \{1\}$ and $S = \mathcal{N}$, because the 1st column of the matrix $\mathbf{Q}^{(1)}$ and the 2nd column of the matrix $\mathbf{Q}^{(2)}$ are zero. Combining all these we obtain

$$\begin{aligned} \det \mathbf{E}(s) =& \boldsymbol{\lambda}^{\emptyset} \zeta^{\mathcal{N}}(s) \det \left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_{\emptyset}^{\emptyset} + \widetilde{G}(s) \boldsymbol{\lambda}^{\mathcal{N}} \zeta^{\emptyset}(s) \det \left(\left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_{\mathcal{N}}^{\{2\}} \bowtie \left(\mathbf{Q}^{(2)} \circ \mathbf{P}\right)_{\mathcal{N}}^{\{1\}} \right) \\ =& (s - \lambda)^2 - \lambda^2 \widetilde{G}(s). \end{aligned}$$

■

In a similar manner, we find the explicit form of the adjoint matrix $\mathcal{E}(s)$ in the following theorem.

Theorem 2.4. *The adjoint matrix* $\mathcal{E}(s)$ *has elements*

$$\mathcal{E}_{ij}(s) = \begin{cases} \sum_{k=0}^{N-1} \widetilde{G}^k(s) \sum_{\substack{\Gamma \subset \mathcal{N} \setminus \{i\} \\ |\Gamma| = k; \\ S \subset \mathcal{N} \setminus \{i\} \\ S \supset \Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det \left(\left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_S^{S \setminus \Gamma} \bowtie \left(\mathbf{Q}^{(2)} \circ \mathbf{P}\right)_S^{\Gamma} \right), & i = j, \\ (-1)^{i+j} \sum_{k=1}^{N-1} \widetilde{G}^k(s) \sum_{\substack{\Gamma \subset \mathcal{N} \setminus \{i,j\} \\ |\Gamma| = k-1}} \sum_{\substack{S \subset \mathcal{N} \setminus \{i,j\} \\ S \supset \Gamma; \\ R \subset S \cap T_{ij}}} (-1)^{|R|} \boldsymbol{\lambda}^{S \cup \{j\}} \zeta^{S^c}(s) & \\ \times \det \left(\left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_{S \cup \{i\}}^{S \setminus \Gamma} \bowtie \left(\mathbf{Q}^{(2)} \circ \mathbf{P}\right)_{S \cup \{i\}}^{\Gamma \cup \{j\}} \right) & \\ +(-1)^{i+j} \sum_{k=0}^{N-2} \widetilde{G}^k(s) \sum_{\substack{\Gamma \subset \mathcal{N} \setminus \{i,j\} \\ |\Gamma| = k}} \sum_{\substack{S \subset \mathcal{N} \setminus \{i,j\} \\ S \supset \Gamma; \\ R \subset S \cap T_{ij}}} (-1)^{|R|} \boldsymbol{\lambda}^{S \cup \{j\}} \zeta^{S^c}(s) & \\ \times \det \left(\left(\mathbf{Q}^{(1)} \circ \mathbf{P}\right)_{S \cup \{i\}}^{(S \setminus \Gamma) \cup \{j\}} \bowtie \left(\mathbf{Q}^{(2)} \circ \mathbf{P}\right)_{S \cup \{i\}}^{\Gamma} \right), & i \neq j, \end{cases}$$

where $m_{ij} = \min\{i, j\}$*,* $M_{ij} = \max\{i, j\}$*, and* $T_{ij} = \{m_{ij} + 1, \ldots, M_{ij} - 1\}$*.*

Proof. See Appendix B. □

The adjoint matrix $\mathcal{E}(s)$ is equal to the transpose of the cofactor matrix of $\mathbf{E}(s)$. Therefore, similarly to $\det \mathbf{E}(s)$, each element of $\mathcal{E}(s)$ is an at most $N - 1$ degree polynomial with respect to $\widetilde{G}(s)$. This observation explains also the similarity between the formula of $\det \mathbf{E}(s)$ and the diagonal elements of $\mathcal{E}(s)$.

Running example (continued) Using the same arguments as for the evaluation of the determinant, we have for the adjoint matrix

$$\mathcal{E}_{ii}(s) = \widetilde{G}^0(s)\boldsymbol{\lambda}^{\emptyset}\zeta^{\mathcal{N}\setminus\{i\}}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{\emptyset}^{\emptyset}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{\emptyset}^{\emptyset}\right) = s-\lambda, \qquad i=1,2,$$
$$\mathcal{E}_{12}(s) = (-1)^{1+2}(-1)^{|\emptyset|}\boldsymbol{\lambda}^{\emptyset\cup\{2\}}\zeta^{\emptyset}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{\{1\}}^{\{2\}}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{\{1\}}^{\emptyset}\right) = -\lambda,$$
$$\mathcal{E}_{21}(s) = (-1)^{2+1}\widetilde{G}(s)(-1)^{|\emptyset|}\boldsymbol{\lambda}^{\emptyset\cup\{1\}}\zeta^{\emptyset}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{\{2\}}^{\emptyset}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{\{2\}}^{\{1\}}\right) = -\lambda\widetilde{G}(s).$$

■

For the evaluation of the Laplace transform $\widetilde{w}(s)$ of the queueing delay, it is only left to calculate $s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega}$ (see Eqs. (16) and (20)). Observe that the elements of the transform vector $\widetilde{\boldsymbol{\Phi}}(s)$ are defined as $\widetilde{\phi}_i(s) = s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i/\det\mathbf{E}(s)$, where $\mathbf{e}_i$ is a column vector with element equal to 1 in position i and all other elements zero. The outcome of $s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i$ is the inner product of the vector $s\mathbf{u}$ with the ith column of matrix $\mathcal{E}(s)$. Therefore, as a first step we calculate the quantities $s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i$, and we have the following theorem.

Theorem 2.5. *The numerator of the ith element of the transform vector $\widetilde{\boldsymbol{\Phi}}(s)$ takes the form*

$$\begin{aligned}
s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i = {} & su_i\sum_{k=0}^{N-1}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\|\Gamma|=k;\\S\subset\mathcal{N}\setminus\{i\}\\S\supset\Gamma}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_S^{S\setminus\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_S^{\Gamma}\right)\\
& +s\sum_{\substack{l=1\\l\neq i}}^{N}u_l(-1)^{l+i}\sum_{k=1}^{N-1}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\|\Gamma|=k-1}}\sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\S\supset\Gamma;\\R\subset S\cap T_{li}}}(-1)^{|R|}\\
& \times\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{S\setminus\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\right)\\
& +s\sum_{\substack{l=1\\l\neq i}}^{N}u_l(-1)^{l+i}\sum_{k=0}^{N-2}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\|\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\S\supset\Gamma;\\R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\\
& \times\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{\Gamma}\right).
\end{aligned}$$

Proof. See Appendix B. □

Combining now the results of the Theorems 2.3 and 2.5 by using Eq. (20), one can find the transform vector $\widetilde{\boldsymbol{\Phi}}(s)$.

Running example (continued) For each state we have

$$\mathbf{u}\mathcal{E}(s)\mathbf{e}_1 = u_1\mathcal{E}_{11}(s)+u_2\mathcal{E}_{21}(s) = u_1(s-\lambda)-u_2\lambda\widetilde{G}(s),$$
$$\mathbf{u}\mathcal{E}(s)\mathbf{e}_2 = u_1\mathcal{E}_{12}(s)+u_2\mathcal{E}_{22}(s) = -u_1\lambda+u_2(s-\lambda).$$

The transform vector $\widetilde{\boldsymbol{\Phi}}(s)$ is then

$$\widetilde{\boldsymbol{\Phi}}(s) = \left[\frac{su_1(s-\lambda)-su_2\lambda\widetilde{G}(s)}{(s-\lambda)^2-\lambda^2\widetilde{G}(s)}, \frac{-su_1\lambda+su_2(s-\lambda)}{(s-\lambda)^2-\lambda^2\widetilde{G}(s)}\right].$$

■

The following remark connects the system of equations that is required for the evaluation of $\mathbf{u}$, which was introduced in Theorem 2.2, to the adjoint matrix $\mathcal{E}(s)$.

Remark 1. The second statement of Theorem 2.2 practically says that each s_i, $i = 2, \dots, N$, is a simple eigenvalue of the matrix $\mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}$. Therefore, the column vector $\mathbf{a}_i$ belongs to the null space of the matrix $\mathbf{H}(s_i) + s_i\mathbf{I} - \mathbf{\Lambda}$. Combining the results of Theorem A.1, Remark 5 and Corollary A.2 (see Appendix A), which provide some general results with respect to the form of the null space of a singular matrix, without loss of generality we can assume that the vector $\mathbf{a}_i$ is any non-zero column of the matrix $\mathcal{E}(s_i)$. Namely, if the mth column of $\mathcal{E}(s_i)$ is such a column, then

$$\mathbf{a}_i := \mathbf{a}_i(s_i) = \big(\mathcal{E}(s_i)\big)_{\mathcal{N}}^{\{m\}}, \qquad i = 2, \dots, N. \tag{21}$$

This observation is very useful, because it allows us to calculate in a straightforward way the desired system of equations and find closed form expressions for the vector $\mathbf{u}$. In addition, since the vectors $\mathbf{a}_i$, $i = 2, \dots, N$, are matrix functions evaluated at the point $s = s_i$ we define the derivative of each $\mathbf{a}_i$ as

$$\mathbf{a}_i^{(1)} = \frac{d}{ds}\mathbf{a}_i(s)\bigg|_{s=s_i}, \qquad i = 2, \dots, N.$$

The usefulness of the latter definition will be apparent in Section 3.2, where we provide an extension of Theorem 2.2 that helps us to calculate our approximations.

Running example (continued) If s_2 is the only positive (and real) root of the equation $\det \mathbf{E}(s) = 0$, the vector $\mathbf{u}$ satisfies the system of equations (17)–(18)

$$\begin{aligned} \frac{1}{\lambda}u_1 + \frac{1}{\lambda}u_2 &= \frac{1}{\lambda} - \frac{\mu}{2}, \\ -\lambda u_1 + (s_2 - \lambda)u_2 &= 0, \end{aligned}$$

where for the derivation of the second equation we used the second column of the matrix $\mathcal{E}(s)$. Namely, we used $\mathbf{a}_2 = \big(\mathcal{E}(s_2)\big)_{\mathcal{N}}^{\{2\}}$. It is easy to verify that the solution to the above system is given by

$$\mathbf{u} = \left(\Big(1 - \frac{\lambda}{s_2}\Big)\Big(1 - \frac{\lambda\mu}{2}\Big),\ \frac{\lambda}{s_2}\Big(1 - \frac{\lambda\mu}{2}\Big)\right).$$

■

Although Theorems 2.4 and 2.5 provide explicit expressions for the transform vector, they may not be practical in cases where the LST of the service time distribution of a real customer $\widetilde{G}(s)$, which is involved in the formulas, does not have a closed form; i.e. Pareto distribution. In such cases, one would have to either consort to a numerical evaluation of $\widetilde{G}(s)$ or approximate the transform vector $\widetilde{\mathbf{\Phi}}(s)$ in some other fashion. This paper focuses on the latter approach, which we work out in detail in the following section by taking as starting point a mixture model for the service time distribution of a real customer.

2.2 Construction of the corrected phase-type approximations

We assume now that the service time distribution of a real customer is $G_\epsilon(t)$, which was defined in Eq. (1) as a mixture of a phase-type distribution and a heavy-tailed one. We will eventually show that the queueing delay can be written also as a mixture, in the sense that we can identify the queueing delay of a model with purely phase-type service times and some additional terms that involve the heavy-tailed service times. As a result, in order to derive our approximations, we first need to compute the delay in a MArP/PH/1 queue and afterwards use this as a base to further develop our approximations involving a heavy-tailed component. In the sequel, we give a more detailed description of our technique.

In terms of Laplace transforms we get for our mixture service time distribution $\widetilde{G}_\epsilon(s) = (1-\epsilon)\widetilde{F}_p(s) + \epsilon\widetilde{F}_h(s)$. As observed in Section 2.1, when the service time distribution of a real customer is of phase type, then the determinant $\det \mathbf{E}(s)$ and the elements of the adjoint matrix $\mathcal{E}(s)$ are all rational functions in s. Therefore, after the cancelation of the problematic factors $s - s_i$, $i = 1, \dots, N$, that appear in the denominator (see the analysis below Theorem 2.2), the elements of the transform vector $\widetilde{\mathbf{\Phi}}(s)$ are also rational functions in s and they can easily be inverted to find the delay distribution.

Note now that the LST of the service time distribution of a real customer $\widetilde{G}_\epsilon(s)$ can be written in the following two ways:

$$\widetilde{G}_\epsilon(s) = \widetilde{F}_p(s) + \epsilon\big(\widetilde{F}_h(s) - \widetilde{F}_p(s)\big) \qquad \text{or} \qquad \widetilde{G}_\epsilon(s) = (1-\epsilon)\widetilde{F}_p(s) + \epsilon + \epsilon\big(\widetilde{F}_h(s) - 1\big).$$

In both formulas, the LST of the service time distribution $\widetilde{G}_\epsilon(s)$ can be seen as perturbation of a phase-type distribution by a term that contains the heavy-tailed component $\widetilde{F}_h(s)$. The index ϵ is interpreted as the perturbation parameter and it used for all parameters of the system that depend on it. Next, we explain how these two different representations of the same formula can lead with the aid of perturbation analysis to two different approximations for the queuing delay.

We start our discussion with the first formula. By setting $\widetilde{F}_h(s) \equiv \widetilde{F}_p(s)$[1] in the formula, one can find with $\widetilde{G}_\epsilon(s) = \widetilde{F}_p(s)$ the workload of a simpler MAP/PH/1 queue, by specializing the formulas of Section 2.1 to phase-type service times. As a next step, we find all the parameters of the mixture model as perturbation of the simpler phase-type model, which we use as base. Then, we write the workload of the mixture model in a series expansion in ϵ, where the constant term is the workload of the MAP/PH/1 queue we used as base and all other terms contain the heavy-tailed service times.

We define our approximation by taking the first two terms of the aforementioned series, namely the up to ϵ-order terms. We call this approximation *corrected replace approximation.* The characterization "corrected" comes from the fact that the ϵ-order term corrects the tail behavior of the constant term, which as a phase-type approximation of the workload is incapable of capturing the correct tail behavior. Finally, the characterization "replace" is due to the phase-type base model we used. We give analytically all the steps to derive the corrected replace approximation in Section 3.

In a similar manner, we construct the *corrected discard approximation* by using the second formula. By setting $\widetilde{F}_h(s) \equiv 1$[2] we derive the queuing delay of the phase-type base model with service time distribution $\widetilde{G}^\bullet_\epsilon(s) = (1-\epsilon)\widetilde{F}_p(s) + \epsilon$ for a real customer, which has an atom of size ϵ at zero. Throughout the paper, we use $\widetilde{G}^\bullet_\epsilon(s)$ for the LST of the service time distribution of a real customer in the discard base model instead of $\widetilde{G}_\epsilon(s)$ to avoid confusion with the mixture model. We briefly discuss the details for the construction of the corrected discard approximation in Section 4.

In the next sections, we provide the steps to construct the corrected replace and the corrected discard approximations, which we call collectively *corrected phase-type approximations.*

3. Corrected replace approximation

In this section, we construct the corrected replace approximation. First, we calculate the queueing delay for the phase-type model that appears when we replace all the heavy-tailed customers with phase-type ones in Section 3.1; i.e. we specialize the results of Section 2.1 to phase-type service times. Later, in Section 3.2, we calculate the parameters of the mixture model with service time

[1]In other words, we assume that all of the customers come from the same phase-type distribution or equivalently that we replace all the heavy-tailed customers with phase-type ones.

[2]By setting the service time of the heavy-tailed customers equal to zero we simply discard them from the system.

distribution $\widetilde{G}_\epsilon(s)$ given by Eq. (1) as perturbation of the parameters of the corresponding phase-type model, with perturbation parameter ϵ. In Section 3.3, we find a series expansion in ϵ of the queueing delay in the mixture model with constant term the queueing delay in the phase-type base model and all higher terms involving the heavy-tailed services. Finally, in Section 3.4, we construct the corrected replace approximation by keeping only the first two terms of the aforementioned series. We start in the next section with the analysis of the replace base model; i.e. the one containing only phase-type service times.

3.1 Replace base model

When we replace the heavy-tailed customers with phase-type ones, we consider the service time distribution $\widetilde{G}_\epsilon(s) = \widetilde{F}_p(s)$ for our phase-type base model. Observe that this service time distribution is independent of the parameter ϵ, and so will be all the other parameters of this simpler model. Thus, from a mathematical point of view, the action of replacing the heavy-tailed service times with phase-type ones is equivalent to setting $\epsilon = 0$ in the mixture model.

To avoid overloading the notation, we omit the subscript "0" (which is a consequence of the fact that $\epsilon = 0$) from the parameters of the replace phase-type model and we assume that the service time distribution of a real customer is some phase-type distribution with LST $\widetilde{G}(s) := \widetilde{F}_p(s) = q(s)/p(s)$, where $q(s)$ and $p(s)$ are appropriate polynomials without common roots. The degree of $p(s)$ is M, and without loss of generality, we choose the coefficient of its highest order term to be equal to 1. Finally, the degree of the polynomial $q(s)$ is less than or equal to $M-1$. Define

$$K = \max_{k\neq 0}\left\{\max_{\Gamma\subset\mathcal{N}}\left\{\operatorname{rank}\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\setminus\Gamma}\ \bowtie\ \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big)\right\} : k = \mid\Gamma\mid,\ \text{and } \Gamma\subset S\subset\mathcal{N}\right\}. \tag{22}$$

Then, the following result holds.

Proposition 3.1. *There exist x_j and y_j, with $\Re(x_j) > 0$, $\Re(y_j) > 0$, $j = 1, \ldots, rM$, such that the Laplace transform $\widetilde{w}(s)$ of the queueing delay takes the form*

$$\widetilde{w}(s) = \frac{\mathbf{u}\boldsymbol{\omega}\prod_{j=1}^{rM}(s+y_j)}{\prod_{j=1}^{rM}(s+x_j)},$$

where the vector $\mathbf{u}$ is calculated according to Theorem 2.2 with the LST of the service times being equal to $\widetilde{F}_p(s)$, and r is some positive integer less than or equal to K defined by (22)*.*

Proof. See Appendix B. ☐

The formula of $\widetilde{w}(s)$ is a rational function that corresponds to a phase-type distribution. Applying Laplace inversion to $\widetilde{w}(s)$, we can find the exact tail probabilities of the queueing delay; namely we can find $\mathbb{P}(W > t)$.

Running example (continued) Given that we have already calculated the transform vector $\widetilde{\boldsymbol{\Phi}}(s)$, we can now calculate the Laplace transform $\widetilde{w}(s)$ of the queueing delay for phase-type customers. In our example, $K = 1$ and consequently, $r = 1$. In addition, $\boldsymbol{\omega}^T = (0, 2)$, where superscript T denotes the transpose of a vector or a matrix. Thus, $\widetilde{w}(s)$ under phase-type service times is

$$\widetilde{w}(s) = 2\widetilde{\phi}_2(s) = 2\frac{s^2u_2 - s\lambda(u_1+u_2)}{(s-\lambda)^2 - \lambda^2\widetilde{F}_p(s)} = 2\frac{s^2p(s)u_2 - sp(s)\lambda(u_1+u_2)}{(s-\lambda)^2p(s) - \lambda^2q(s)}.$$

Observe that both the numerator and the denominator of $\widetilde{w}(s)$ are polynomials of degree $M+2$. Moreover, Theorem 2.2 guarantees that 0 and s_2 are common roots of them. If $-y_j$ and $-x_j$, $j = 1, \ldots, M$, $\Re(x_j), \Re(y_j) > 0$, are the remaining roots of the numerator and the denominator, respectively, the Laplace transform of the queueing delay can be written as

$$\widetilde{w}(s) = \frac{2u_2 s(s - s_2)\prod_{j=1}^{M}(s + y_j)}{s(s - s_2)\prod_{j=1}^{M}(s + x_j)} = \frac{2u_2\prod_{j=1}^{M}(s + y_j)}{\prod_{j=1}^{M}(s + x_j)}.$$

■

As pointed out in Section 2.2, the LST of the service time distribution $\widetilde{G}_\epsilon(s)$ (see Eq. (1)) can be seen as perturbation of $\widetilde{F}_p(s)$ by the term $\epsilon\big(\widetilde{F}_h(s) - \widetilde{F}_p(s)\big)$. In the next section we write the parameters of the mixture model as perturbation of the parameters of the replace base model.

3.2 Perturbation of the parameters of the replace base model

In order to find the queueing delay in the mixture model as a series expansion in ϵ with constant term the queueing delay in the replace base model, we apply perturbation analysis to the parameters of the mixture model that depend on ϵ. Thus, we first check which of the parameters in the mixture model depend on ϵ and then we represent them as perturbation of the parameters of the replace base model.

Since the matrices $\mathbf{P}$, $\mathbf{Q}^{(1)}$, $\mathbf{Q}^{(2)}$, and $\mathbf{\Lambda}$ (see Section 2.1) depend only on the arrival process, they are invariant under any perturbation of the service time distribution. However, the matrix $\widetilde{\mathbf{G}}_\epsilon(s)$, and consequently $\mathbf{H}_\epsilon(s)$ change, and so does the stability condition (see Eqs. (9)–(11)). Let now $\widetilde{F}^e_p(s)$ and $\widetilde{F}^e_h(s)$ be the LSTs of the stationary-excess service time distributions $F^e_p(t)$ and $F^e_h(t)$, and μ_p and μ_h be the finite means of the phase-type and heavy-tailed service times, respectively. Then, we obtain

$$\widetilde{\mathbf{G}}_\epsilon(s) = \widetilde{\mathbf{G}}(s) + \epsilon s\big(\mu_p\widetilde{F}^e_p(s) - \mu_h\widetilde{F}^e_h(s)\big)\mathbf{Q}^{(2)}, \tag{23}$$

and

$$\mathbf{H}_\epsilon(s) = \widetilde{\mathbf{G}}_\epsilon(s) \circ \mathbf{P\Lambda} = \mathbf{H}(s) + \epsilon s\big(\mu_p\widetilde{F}^e_p(s) - \mu_h\widetilde{F}^e_h(s)\big)\mathbf{Q}^{(2)} \circ \mathbf{P\Lambda}. \tag{24}$$

Finally, the stability condition takes the form

$$\boldsymbol{\pi}(\mathbf{\Lambda}^{-1} - \mathbf{M}_\epsilon)\mathbf{e} > 0, \tag{25}$$

where $\mathbf{M}_\epsilon = \mathbf{M} + \epsilon s\big(\mu_h - \mu_p\big)\mathbf{Q}^{(2)} \circ \mathbf{P}$.

Under the stability condition (25), Theorem 2.1 holds for the transform vector $\widetilde{\mathbf{\Phi}}_\epsilon(s)$, for some row vector $\mathbf{u}_\epsilon$. More precisely, there exists a unique vector $\mathbf{u}_\epsilon$ such that the transform vector $\widetilde{\mathbf{\Phi}}_\epsilon(s)$ satisfies the system of equations:

$$\widetilde{\mathbf{\Phi}}_\epsilon(s)\big(\mathbf{H}_\epsilon(s) + s\mathbf{I} - \mathbf{\Lambda}\big) = s\mathbf{u}_\epsilon, \tag{26}$$

$$\widetilde{\mathbf{\Phi}}_\epsilon(0)\mathbf{e} = 1, \tag{27}$$

where the vector $\mathbf{u}_\epsilon$ is calculated according to Theorem 2.2.

Recall that the evaluation of $\mathbf{u}_\epsilon$ goes through the evaluation of the positive eigenvalues of the matrix

$$\mathbf{E}_\epsilon(s) = \mathbf{H}_\epsilon(s) + s\mathbf{I} - \mathbf{\Lambda} = \mathbf{E}(s) + \epsilon s\big(\mu_p\widetilde{F}^e_p(s) - \mu_h\widetilde{F}^e_h(s)\big)\mathbf{Q}^{(2)} \circ \mathbf{P\Lambda}. \tag{28}$$

Observe that the above representation of the matrix $\mathbf{E}_\epsilon(s)$ is a linear perturbation in ϵ of the matrix $\mathbf{E}(s)$ of the base model. Thus, according to results on perturbation of analytic matrix functions [13, 21], we have that the positive eigenvalues of the matrix $\mathbf{E}_\epsilon(s)$ and their corresponding eigenvectors are analytic functions in ϵ. Consequently, one can find a series representation in ϵ for all the involved quantities that are needed for the evaluation of the vector $\mathbf{u}_\epsilon$ (see Theorem 2.2). By using these parameters, we can find a complete series representation for the transform vector $\widetilde{\mathbf{\Phi}}_\epsilon(s)$ and by applying Laplace inversion to each term of this series we can find a formal expression for the queueing delay that is a series expansion in ϵ. As we stated earlier, we only need the first two terms of the latter series to define the corrected replace approximation. Therefore, in our analysis, we keep only the terms up to order ϵ of each involved perturbed parameter.

In the next theorem, we provide an algorithm to calculate the first order approximation in ϵ of the vector $\mathbf{u}_\epsilon$, given that we have already calculated the vector $\mathbf{u}$ of the replace base model, by specializing Theorem 2.2 to phase-type service times. We denote by $\mathbf{U}$ the square matrix of appropriate dimensions with all its elements equal to one.

Theorem 3.2. *Let $\mathbf{u}$ be the unique solution to the Eqs. (17)–(18) for the replace base model. If the roots $s_2, \ldots, s_N$ of $\det\big(\mathbf{H}(s) + s\mathbf{I} - \mathbf{\Lambda}\big) = 0$ with positive real part are simple, then*

1. *the equation $\det\big(\mathbf{H}_\epsilon(s) + s\mathbf{I} - \mathbf{\Lambda}\big) = 0$ has exactly N non-negative solutions $s_{\epsilon,1}, \ldots, s_{\epsilon,N}$, with $s_{\epsilon,1} = 0$ and $s_{\epsilon,i} = s_i - \epsilon\delta_i + O(\epsilon^2)$ for $i = 2, \ldots, N$, where*

$$\delta_i := \delta(s_i) = \frac{\sum_{j=1}^N \det\big(\mathbf{E}(s_i)_{\bullet 1}, \ldots, \mathbf{K}(s_i)_{\bullet j}, \ldots, \mathbf{E}(s_i)_{\bullet N}\big)}{\sum_{j=1}^N \det\big(\mathbf{E}(s_i)_{\bullet 1}, \ldots, \mathbf{E}^{(1)}(s_i)_{\bullet j}, \ldots, \mathbf{E}(s_i)_{\bullet N}\big)},$$

   and $\mathbf{K}(s) = s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\mathbf{Q}^{(2)} \circ \mathbf{P}\mathbf{\Lambda}$.

2. *We set $\mathbf{A} = \big(\mathbf{\Lambda}^{-1}\mathbf{e}, \mathbf{a}_2, \ldots, \mathbf{a}_N\big)$ (see Eq. (21)) and $\mathbf{c} = \big(\boldsymbol{\pi}(\mathbf{\Lambda}^{-1} - \mathbf{M})\mathbf{e}, 0, \ldots, 0\big)$, and we assume that the stability condition (25) is satisfied. Then, the vector $\mathbf{u}_\epsilon$ is the unique solution to the system of N linear equations*

$$\mathbf{u}_\epsilon\big(\mathbf{A} - \epsilon\mathbf{B} + O(\epsilon^2\mathbf{U})\big) = \mathbf{c} + \epsilon\mathbf{d}, \tag{29}$$

   where $\mathbf{B} = \big(\mathbf{0}, \delta_2\mathbf{a}_2{}^{(1)} - \mathbf{k}_2, \ldots, \delta_N\mathbf{a}_N{}^{(1)} - \mathbf{k}_N\big)$ and $\mathbf{d} = \big((\mu_p - \mu_h)\boldsymbol{\pi}\mathbf{Q}^{(2)} \circ \mathbf{P}\mathbf{e}, 0, \ldots, 0\big)$, with $\mathbf{k}_i$, $i = 2, \ldots, N$, being a column vector with coordinates

$$k_{i,j} = (-1)^{m+j}\sum_{k=1}^{N-1} \det\left(\Big(\mathbf{E}(s_i)_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{j\}}\Big)_{\bullet 1}, \ldots, \Big(\mathbf{K}(s_i)_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{j\}}\Big)_{\bullet k}, \ldots, \Big(\mathbf{E}(s_i)_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{j\}}\Big)_{\bullet N-1}\right), \quad j \in \mathcal{N},$$

   and the choice of m explained in Remark 1.

Proof. See Appendix B. □

Remark 2. When the number of states is $N = 2$, the column vector $\mathbf{k}_2$ of Theorem 3.2 is equal to

$$\mathbf{k}_2 = \big(\mathbf{K}_{22}(s_2), -\mathbf{K}_{21}(s_2)\big)^T \qquad \text{or} \qquad \mathbf{k}_2 = \big(-\mathbf{K}_{12}(s_2), \mathbf{K}_{11}(s_2)\big)^T,$$

depending on whether $m = 1$ or $m = 2$, respectively. The case $N = 1$ has been treated earlier by the authors; see [34].

Running example (continued) In order to evaluate the vector $\mathbf{u}_\epsilon$, we first need to calculate the perturbed root $s_{\epsilon,2}$, and more precisely the term δ_2. Observe that in our case only the element $\mathbf{K}_{21}(s) = s\lambda\big(\mu_p\widetilde{F}_p^e(s) - \mu_h\widetilde{F}_h^e(s)\big)$ of the matrix $\mathbf{K}(s)$ is not equal to zero. Then, the numerator of δ_2 becomes

$$\det\big(\mathbf{E}(s_2)_{\mathcal{N}}^{\{1\}}, \mathbf{K}(s_2)_{\mathcal{N}}^{\{2\}}\big) + \det\big(\mathbf{K}(s_2)_{\mathcal{N}}^{\{1\}}, \mathbf{E}(s_2)_{\mathcal{N}}^{\{2\}}\big) = -s_2\lambda^2\big(\mu_p\widetilde{F}_p^e(s_2) - \mu_h\widetilde{F}_h^e(s_2)\big),$$

and its denominator takes the form

$$\det\big(\mathbf{E}(s_2)_{\mathcal{N}}^{\{1\}}, \mathbf{E}^{(1)}(s_2)_{\mathcal{N}}^{\{2\}}\big) + \det\big(\mathbf{E}^{(1)}(s_2)_{\mathcal{N}}^{\{1\}}, \mathbf{E}(s_2)_{\mathcal{N}}^{\{2\}}\big) = 2(s_2 - \lambda) - \lambda^2\widetilde{F}_p^{(1)}(s_2),$$

because the first derivative of the matrix $\mathbf{E}(s)$ is

$$\mathbf{E}^{(1)}(s) = \begin{pmatrix} 1 & 0 \\ \lambda\widetilde{F}_p^{(1)}(s) & 1 \end{pmatrix}.$$

Combining the above, we have

$$\delta_2 = \frac{-s_2\lambda^2\big(\mu_p\widetilde{F}_p^e(s_2) - \mu_h\widetilde{F}_h^e(s_2)\big)}{2(s_2 - \lambda) - \lambda^2\widetilde{F}_p^{(1)}(s_2)}.$$

Recall that for the determination of the vector $\mathbf{a}_2$ we had used the second column of the adjoint matrix, namely we had chosen $m = 2$. Thus, according to Remark 2 the vector $\mathbf{k}_2$ is a zero column vector of dimension 2. Since ${\mathbf{a}_2}^{(1)}$ is the second column of the matrix $\mathbf{E}^{(1)}(s)$, it holds that $\mathbf{B}_{22} = \delta_2$ and all other elements of $\mathbf{B}$ are equal to zero. Finally, $\mathbf{d} = \big(\frac{1}{2}(\mu_p - \mu_h), 0\big)$. ■

By matching the coefficients of ϵ on the left and right side of Eq. (29), we can write the vector of unknown parameters $\mathbf{u}_\epsilon$ as $\mathbf{u}_\epsilon = \mathbf{u} + \epsilon\mathbf{z} + O(\epsilon^2\mathbf{e})$. The exact form of the vector $\mathbf{z}$ is given in the following lemma, which we give without proof.

Lemma 3.3. *The vector $\mathbf{u}_\epsilon$ can be written in the form*

$$\mathbf{u}_\epsilon = \mathbf{u} + \epsilon\mathbf{z} + O(\epsilon^2\mathbf{e}),$$

where

$$\mathbf{z} = \big(\mathbf{c}\mathbf{A}^{-1}\mathbf{B} + \mathbf{d}\big)\,\mathbf{A}^{-1}.$$

Running example (continued) For the evaluation of $\mathbf{z}$ we need to find the inverse of matrix $\mathbf{A}$, namely we need

$$\mathbf{A}^{-1} = \frac{\lambda}{s_2}\begin{pmatrix} s_2 - \lambda & \lambda \\ -\frac{1}{\lambda} & \frac{1}{\lambda} \end{pmatrix}. \tag{30}$$

By observing that $\mathbf{c}\mathbf{A}^{-1} = \mathbf{u}$ and by following the calculations of Lemma 3.3 we obtain

$$\mathbf{z} = \frac{\lambda}{s_2}\left[\frac{1}{2}(\mu_p - \mu_h)(s_2 - \lambda) - \frac{1}{s_2}\Big(1 - \frac{\lambda\mu_p}{2}\Big)\delta_2, \frac{\lambda}{2}(\mu_p - \mu_h) + \frac{1}{s_2}\Big(1 - \frac{\lambda\mu_p}{2}\Big)\delta_2\right].$$

■

In our analysis, we used first order perturbation with respect to the parameter ϵ. The exact same procedure can be followed if higher order terms of ϵ are desired. However, this would result to the increase of the complexity of the formulas. In the next section, we provide the formulas for the evaluation of the perturbed transform vector $\widetilde{\mathbf{\Phi}}_\epsilon(s)$ and the Laplace transform $\widetilde{w}_\epsilon(s)$ of the queueing delay.

3.3 Delay distribution of the perturbed model

If $\mathcal{E}_\epsilon(s)$ is the adjoint matrix of $\mathbf{E}_\epsilon(s)$ (see Eq. (28)), then the ith element of the transform vector $\widetilde{\mathbf{\Phi}}_\epsilon(s)$ is defined as

$$\widetilde{\phi}_{\epsilon,i}(s) = \frac{s\mathbf{u}_\epsilon \mathcal{E}_\epsilon(s)\mathbf{e}_i}{\det \mathbf{E}_\epsilon(s)}. \tag{31}$$

Therefore, to find the exact formula of $\widetilde{\phi}_{\epsilon,i}(s)$ we need to find $\det \mathbf{E}_\epsilon(s)$ and $s\mathbf{u}_\epsilon \mathcal{E}_\epsilon(s)\mathbf{e}_i$. By using the binomial identity and by omitting higher order powers of ϵ, we have that $\Big(\widetilde{F}_p(s)+\epsilon s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\Big)^k = \big(\widetilde{F}_p(s)\big)^k + \epsilon k \big(\widetilde{F}_p(s)\big)^{k-1} s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big) + O(\epsilon^2)$. We give the following lemmas without proof. The first one gives the formula for the evaluation of the denominator of the desired quantity.

Lemma 3.4. *If* $\det \mathbf{E}(s)$ *is evaluated according to Theorem 2.3 with* $\widetilde{G}(s) = \widetilde{F}_p(s)$*, then* $\det \mathbf{E}_\epsilon(s)$ *can be written as perturbation of* $\det \mathbf{E}(s)$ *as follows*

$$\det \mathbf{E}_\epsilon(s) = \det \mathbf{E}(s) + \epsilon s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big) \sum_{k=1}^{N} k\big(\widetilde{F}_p(s)\big)^{k-1} \sum_{\substack{\Gamma \subset \mathcal{N} \\ |\Gamma| = k}} \sum_{\substack{S \subset \mathcal{N} \\ S \supset \Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s)$$
$$\times \det \Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_S^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_S^{\Gamma}\Big) + O(\epsilon^2).$$

Running example (continued) Only the combination $k = 1$ with $\Gamma = \{1\}$, and $S = \mathcal{N}$ gives a non-zero coefficient for ϵ. Therefore,

$$\begin{aligned}\det \mathbf{E}_\epsilon(s) &= \det \mathbf{E}(s) + \epsilon s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\boldsymbol{\lambda}^{\mathcal{N}} \zeta^{\emptyset}(s) \det \Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\mathcal{N}}^{\{2\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\mathcal{N}}^{\{1\}}\Big) \\ &= \det \mathbf{E}(s) - \epsilon \lambda^2 s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big).\end{aligned}$$

■

The next lemma gives the numerator of each $\widetilde{\phi}_{\epsilon,i}(s)$, $i \in \mathcal{N}$.

Lemma 3.5. *If* $s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i$ *is evaluated according to Theorem 2.5 with* $\widetilde{G}(s) = \widetilde{F}_p(s)$*, then* $s\mathbf{u}_\epsilon \mathcal{E}_\epsilon(s)\mathbf{e}_i$ *can be written as perturbation of* $s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i$ *as follows*

$$s\mathbf{u}_\epsilon \mathcal{E}_\epsilon(s)\mathbf{e}_i = s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i + \epsilon s \Bigg[z_i \sum_{k=1}^{N-1} \big(\widetilde{F}_p(s)\big)^k \sum_{\substack{\Gamma \subset \mathcal{N}\setminus\{i\} \\ |\Gamma| = k; \\ S \subset \mathcal{N}\setminus\{i\} \\ S \supset \Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det \Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_S^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_S^{\Gamma}\Big)$$
$$+ z_i \sum_{S \subset \mathcal{N}\setminus\{i\}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_S^S$$
$$+ \sum_{\substack{l=1 \\ l \neq i}}^{N} z_l (-1)^{l+i} \sum_{k=1}^{N-1} \big(\widetilde{F}_p(s)\big)^k \sum_{\substack{\Gamma \subset \mathcal{N}\setminus\{l,i\} \\ |\Gamma| = k-1; \\ S \subset \mathcal{N}\setminus\{l,i\} \\ S \supset \Gamma; \\ R \subset S \cap T_{li}}} (-1)^{|R|} \boldsymbol{\lambda}^{S\cup\{i\}} \zeta^{S^c}(s) \det \Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big)$$
$$+ \sum_{\substack{l=1 \\ l \neq i}}^{N} z_l (-1)^{l+i} \sum_{k=0}^{N-2} \big(\widetilde{F}_p(s)\big)^k \sum_{\substack{\Gamma \subset \mathcal{N}\setminus\{l,i\} \\ |\Gamma| = k}} \sum_{\substack{S \subset \mathcal{N}\setminus\{l,i\} \\ S \supset \Gamma; \\ R \subset S \cap T_{li}}} (-1)^{|R|} \boldsymbol{\lambda}^{S\cup\{i\}} \zeta^{S^c}(s)$$

$$
\begin{aligned}
&\times \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big) \\
&+s\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\Bigg(u_i\sum_{k=1}^{N-1}k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\|\Gamma|=k}}\ \sum_{\substack{S\subset\mathcal{N}\setminus\{i\}\\S\supset\Gamma}}\boldsymbol{\lambda}^S\zeta^{S^c}(s) \\
&\times \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S}^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S}^{\Gamma}\Big) \\
&+\sum_{\substack{l=1\\l\neq i}}^{N}u_l(-1)^{l+i}\sum_{k=1}^{N-1}k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\|\Gamma|=k-1}}\ \sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\S\supset\Gamma;\\R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s) \\
&\times \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big) \\
&+\sum_{\substack{l=1\\l\neq i}}^{N}u_l(-1)^{l+i}\sum_{k=1}^{N-2}k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\|\Gamma|=k}}\ \sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\S\supset\Gamma;\\R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s) \\
&\times \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big)\Bigg)\Bigg]+O(\epsilon^2),
\end{aligned}
$$

where z_i, $i\in\mathcal{N}$, are the coordinates of the vector $\mathbf{z}$ given in Lemma 3.3.

Running example (continued) By doing the calculations for each state without taking into account terms that are equal to zero, we obtain:

$$
\begin{aligned}
s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\mathbf{e}_1 =& s\mathbf{u}\mathcal{E}(s)\mathbf{e}_1+\epsilon s\Bigg[z_1\boldsymbol{\lambda}^{\emptyset}\zeta^{\{2\}}(s)\det\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{\emptyset}^{\emptyset}+z_2(-1)^{2+1}\widetilde{F}_p(s)(-1)^{|\emptyset|}\boldsymbol{\lambda}^{\emptyset\cup\{1\}}\zeta^{\emptyset}(s)q_{21}^{(2)}p_{21} \\
&+s\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\Big(u_2(-1)^{2+1}(-1)^{|\emptyset|}\boldsymbol{\lambda}^{\emptyset\cup\{1\}}\zeta^{\emptyset}(s)q_{21}^{(2)}p_{21}\Big)\Bigg]+O(\epsilon^2) \\
=& s\mathbf{u}\mathcal{E}(s)\mathbf{e}_1+\epsilon s\Big(z_1(s-\lambda)-z_2\lambda\widetilde{F}_p(s)+s\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)(-\lambda u_2)\Big)+O(\epsilon^2),
\end{aligned}
$$

and

$$
\begin{aligned}
s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\mathbf{e}_2 =& s\mathbf{u}\mathcal{E}(s)\mathbf{e}_2+\epsilon s\Big[z_2\boldsymbol{\lambda}^{\emptyset}\zeta^{\{1\}}(s)\det\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{\emptyset}^{\emptyset}+z_1(-1)^{1+2}(-1)^{|\emptyset|}\boldsymbol{\lambda}^{\emptyset\cup\{2\}}\zeta^{\emptyset}(s)q_{12}^{(1)}p_{12}\Big]+O(\epsilon^2) \\
=& s\mathbf{u}\mathcal{E}(s)\mathbf{e}_2+\epsilon s\Big(-z_1\lambda+z_2(s-\lambda)\Big)+O(\epsilon^2).
\end{aligned}
$$

■

Combining the results of Lemmas 3.4–3.5, we have the following proposition for the Laplace transform $\widetilde{w}_\epsilon(s)$ of the queueing delay.

Proposition 3.6. *If $\widetilde{w}(s)$ is calculated according to Proposition 3.1 for the replace base model, then there exist unique coefficients β, γ, α_i, β_k, γ_k, $k=2,\dots,N$, and $\alpha''_{j,l}$, $\beta''_{j,l}$ and $\gamma''_{j,l}$, $j=1,\dots,\sigma$, $l=1,\dots,r_j$, such that the Laplace transform $\widetilde{w}_\epsilon(s)$ of the queueing delay of the mixture model satisfies*

$$
\widetilde{w}_\epsilon(s)=\widetilde{w}(s)+\epsilon\frac{1}{\mathbf{u}\boldsymbol{\omega}}\widetilde{w}(s)\Bigg[\Bigg(\mathbf{z}\boldsymbol{\omega}+\sum_{k=2}^{N}\frac{\alpha_k}{s-s_k}+\sum_{j=1}^{\sigma}\sum_{l=1}^{r_j}\frac{\alpha''_{j,l}\cdot(y_j)^{r_j-l+1}}{(s+y_j)^{r_j-l+1}}\Bigg)
$$

$$
+\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\left(\beta + \sum_{k=2}^{N} \frac{\beta_k}{s - s_k} + \sum_{j=1}^{\sigma}\sum_{l=1}^{r_j} \frac{\beta''_{j,l}\cdot (y_j)^{r_j-l+1}}{(s+y_j)^{r_j-l+1}}\right)
$$

$$
-\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\widetilde{w}(s)\left(\gamma + \sum_{k=2}^{N} \frac{\gamma_k}{s - s_k} + \sum_{j=1}^{\sigma}\sum_{l=1}^{r_j} \frac{\gamma''_{j,l}\cdot (y_j)^{r_j-l+1}}{(s+y_j)^{r_j-l+1}}\right)\Bigg] + O(\epsilon^2),
$$

where the vector $\mathbf{z}$ *given in Lemma 3.3.*

Proof. See Appendix B. □

Before we evaluate $\widetilde{w}_\epsilon(s)$ in our running example, we apply Laplace inversion to the coefficient of ϵ in the series expansion of $\widetilde{w}_\epsilon(s)$. We denote by $E_k(\lambda)$ the r.v. that follows an Erlang distribution with k phases and rate λ. For simplicity, we write $E(\lambda)$ for the exponential r.v. with rate λ. Finally, let B^e and C^e be the generic stationary excess phase-type and heavy-tailed service times, respectively.

Theorem 3.7. *If* $\widetilde{\theta}(s)$ *is the coefficient of* ϵ *in the series expansion of* $\widetilde{w}_\epsilon(s)$ *in Proposition 3.6, its Laplace inversion* $\Theta(t) = \mathcal{L}^{-1}\{\widetilde{\theta}(s)\}$ *is given as follows*

$$
\begin{aligned}
\Theta(t) = \frac{1}{\mathbf{u}\boldsymbol{\omega}}\Bigg[&\Big(\mathbf{z}\boldsymbol{\omega} - \sum_{k=2}^{N}\frac{\alpha_k}{s_k}\Big)\mathbb{P}(W > t) + \Big(\beta - \sum_{k=2}^{N}\frac{\beta_k}{s_k}\Big)\Big(\mu_p\mathbb{P}(W + B^e > t) - \mu_h\mathbb{P}(W + C^e > t)\Big)\\
&-\Big(\gamma - \sum_{k=2}^{N}\frac{\gamma_k}{s_k}\Big)\Big(\mu_p\mathbb{P}(W + W' + B^e > t) - \mu_h\mathbb{P}(W + W' + C^e > t)\Big)\\
&-\sum_{k=2}^{N}\frac{1}{s_k}\Bigg(\gamma_k\Big(\mu_p\mathbb{P}\big(t < W + W' + B^e < t + E(s_k)\big) - \mu_h\mathbb{P}\big(t < W + W' + C^e < t + E(s_k)\big)\Big)\\
&\quad - \beta_k\Big(\mu_p\mathbb{P}\big(t < W + B^e < t + E(s_k)\big) - \mu_h\mathbb{P}\big(t < W + C^e < t + E(s_k)\big)\Big) - \alpha_k\mathbb{P}\big(t < W < t + E(s_k)\big)\Bigg)\\
&-\sum_{j=1}^{\sigma}\sum_{l=1}^{r_j}\Bigg(\gamma''_{j,l}\Big(\mu_p\mathbb{P}\big(W + W' + B^e + E_{r_j-l+1}(y_j) > t\big) - \mu_h\mathbb{P}\big(W + W' + C^e + E_{r_j-l+1}(y_j) > t\big)\Big)\\
&\quad - \beta''_{j,l}\Big(\mu_p\mathbb{P}\big(W + B^e + E_{r_j-l+1}(y_j) > t\big) - \mu_h\mathbb{P}\big(W + C^e + E_{r_j-l+1}(y_j) > t\big)\Big)\\
&\quad - \alpha''_{j,l}\mathbb{P}\big(W + E_{r_j-l+1}(y_j) > t\big)\Bigg)\Bigg],
\end{aligned}
$$

where W' *is independent and follows the same distribution of* W.

Proof. See Appendix B. □

Remark 3. Note that an $E_k(\lambda)$ distribution ($k \geq 1$) is defined for a non-negative real valued rate λ. To state Theorem 3.7, we assumed that all the roots s_k, $k = 2, \ldots, N$, and $-y_j$, $j = 1, \ldots, rM$, are real-valued. In most systems, this assumption in not always true. Recall that the previously mentioned roots are roots of a polynomial with real coefficients (see analysis above Eq. (44)). Therefore, from the Complex Conjugate Root Theorem it holds that if e.g. s_2 is complex, then its complex conjugate $\overline{s_2}$ is also a root. Thus, we write $E_{Re(s_2)}$ instead of E_{s_2} and $E_{\overline{s_2}}$, because every parameter or function that depends on $\overline{s_2}$ appears as a complex conjugate of the corresponding quantity that depends on s_2, and their imaginary parts cancel out. The same result holds for all other roots.

Running example (continued) For the evaluation of the Laplace transform $\widetilde{w}_\epsilon(s) = \widetilde{\boldsymbol{\Phi}}_\epsilon(s)\boldsymbol{\omega}$ of the queueing delay W_ϵ, we follow the steps in the proof of Proposition 3.6. Recall that in our example, $r = 1$, and assume that only σ of the roots $-y_j$ are distinct and that the multiplicity of each of them is r_j, such that $\sum_{j=1}^{\sigma} r_j = M$.

Therefore, we first find $p(s)\det \mathbf{E}_\epsilon(s)$ and $p(s)s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\boldsymbol{\omega}$. If we set $\xi(s) = -\lambda^2 p(s)$, $\xi_1'(s) = -2\lambda p(s)$, and $\xi_2'(s) = 2(s-\lambda)p(s)$, then we obtain

$$
\begin{aligned}
p(s)\det \mathbf{E}_\epsilon(s) &= p(s)\det \mathbf{E}(s) + \epsilon s\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\xi(s) + O(\epsilon^2),\\
p(s)s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\boldsymbol{\omega} &= p(s)s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega} + \epsilon s\sum_{l=1}^{2} z_l\xi_l'(s) + O(\epsilon^2).
\end{aligned}
$$

We define the functions $d(s)$ and $n(s)$ (see Eqs. (47) and (52) respectively) as

$$
\begin{aligned}
d(s) =& \frac{\big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\xi(s)\widetilde{w}(s)}{\mathbf{u}\boldsymbol{\omega}(s-s_2)\prod_{j=1}^{\sigma}(s+y_j)^{r_j}} - \frac{\delta_2}{s-s_2},\\
n(s) =& \frac{\sum_{l=1}^{2} z_l\xi_l'(s)}{\mathbf{u}\boldsymbol{\omega}(s-s_2)\prod_{j=1}^{\sigma}(s+y_j)^{r_j}} - \frac{\delta_2}{s-s_2},
\end{aligned}
$$

where the two equivalent definitions of δ_2 (see Eqs. (46) and (51)) take the form

$$
\delta_2 = \frac{\big(\mu_p \widetilde{F}_p^e(s_2) - \mu_h \widetilde{F}_h^e(s_2)\big)\xi(s_2)\widetilde{w}(s_2)}{\mathbf{u}\boldsymbol{\omega}\prod_{j=1}^{\sigma}(s_2+y_j)^{r_j}} = \frac{\sum_{l=1}^{2} z_l\xi_l'(s_2)}{\mathbf{u}\boldsymbol{\omega}\prod_{j=1}^{\sigma}(s_2+y_j)^{r_j}}.
$$

Following the calculations after Eq. (53) we get that

$$
\begin{aligned}
\widetilde{w}_\epsilon(s) = \widetilde{w}(s) + \epsilon\frac{1}{\mathbf{u}\boldsymbol{\omega}}\widetilde{w}(s)&\left(\frac{\sum_{l=1}^{2} z_l\xi_l'(s)}{(s-s_2)\prod_{j=1}^{\sigma}(s+y_j)^{r_j}} - \big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\big)\widetilde{w}(s)\frac{\xi(s)}{(s-s_2)\prod_{j=1}^{\sigma}(s+y_j)^{r_j}}\right)\\
&+O(\epsilon^2).
\end{aligned}
\tag{32}
$$

Now, we apply simple fraction decomposition to the rational functions

$$
\frac{\sum_{l=1}^{2} z_l\xi_l'(s)}{(s-s_2)\prod_{j=1}^{\sigma}(s+y_j)^{r_j}}, \qquad \frac{\xi(s)}{(s-s_2)\prod_{j=1}^{\sigma}(s+y_j)^{r_j}}.
$$

Thus, we calculate

$$
\alpha_2 = \frac{\sum_{l=1}^{2} z_l\xi_l'(s_2)}{\prod_{j=1}^{\sigma}(s_2+y_j)^{r_j}}, \qquad \gamma_2 = \frac{\xi(s_2)}{\prod_{j=1}^{\sigma}(s_2+y_j)^{r_j}},
$$

and for $j = 1,\dots,\sigma$, $p = 1,\dots,r_j$, the coefficients $\alpha_{j,p}''$ and $\gamma_{j,p}''$, are respectively the unique solutions to the following two linear systems of r_j equations

$$
\begin{aligned}
\frac{d}{ds^n}\left[\sum_{l=1}^{2} z_l\xi_l'(s)\right]\Bigg|_{s=-y_j} &= \frac{d}{ds^n}\left[(s-s_2)\prod_{\substack{l=1\\l\neq j}}^{\sigma}(s+y_l)^{r_l}\sum_{p=1}^{r_j}\alpha_{j,p}''(y_j)^{r_j-p+1}(s+y_j)^{p-1}\right]\Bigg|_{s=-y_j},\\
\frac{d}{ds^n}\left[\xi(s)\right]\Bigg|_{s=-y_j} &= \frac{d}{ds^n}\left[(s-s_2)\prod_{\substack{l=1\\l\neq j}}^{\sigma}(s+y_l)^{r_l}\sum_{p=1}^{r_j}\gamma_{j,p}''(y_j)^{r_j-p+1}(s+y_j)^{p-1}\right]\Bigg|_{s=-y_j},
\end{aligned}
$$

$n = 0, \ldots, r_j$. In addition, the polynomial $\xi(s)$ is of degree M, and the polynomial $\sum_{l=1}^{2} z_l \xi_l'(s)$ is of degree $M+1$ with the coefficient of s^{M+1} equal to $2z_2$. Combining all these, we write Eq. (32) as

$$\begin{aligned}\widetilde{w}_\epsilon(s) =& \widetilde{w}(s) + \epsilon \frac{1}{2u_2} \widetilde{w}(s) \Bigg[\Bigg(2z_2 + \frac{\alpha_2}{s - s_2} + \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \frac{\alpha_{j,l}'' \cdot (y_j)^{r_j - l + 1}}{(s + y_j)^{r_j - l + 1}} \Bigg) \\ & - \big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s) \big) \widetilde{w}(s) \Bigg(\frac{\gamma_2}{s - s_2} + \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \frac{\gamma_{j,l}'' \cdot (y_j)^{r_j - l + 1}}{(s + y_j)^{r_j - l + 1}} \Bigg) \Bigg] + O(\epsilon^2).\end{aligned}$$

Observe that in this case $\gamma = 0$ and all β coefficients are also equal to zero. Thus, if $\widetilde{\theta}(s)$ is the coefficient of ϵ in the series expansion of $\widetilde{w}_\epsilon(s)$, we apply Theorem 3.7 to find its Laplace inversion as

$$\begin{aligned}\Theta(t) = \frac{1}{2u_2} \Bigg[& \Big(2z_2 - \frac{\alpha_2}{s_2} \Big) \mathbb{P}(W > t) + \frac{\gamma_2}{s_2} \Big(\mu_p \mathbb{P}(W + W' + B^e > t) - \mu_h \mathbb{P}(W + W' + C^e > t) \Big) \\ & - \frac{1}{s_2} \Bigg(\gamma_2 \Big(\mu_p \mathbb{P}\big(t < W + W' + B^e < t + E(s_2)\big) - \mu_h \mathbb{P}\big(t < W + W' + C^e < t + E(s_2)\big) \Big) \\ & - \alpha_2 \mathbb{P}\big(t < W < t + E(s_2)\big) \Bigg) \\ & - \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \Bigg(\gamma_{j,l}'' \Big(\mu_p \mathbb{P}\big(W + W' + B^e + E_{r_j - l + 1}(y_j) > t\big) - \mu_h \mathbb{P}\big(W + W' + C^e + E_{r_j - l + 1}(y_j) > t\big) \Big) \\ & - \alpha_{j,l}'' \mathbb{P}\big(W + E_{r_j - l + 1}(y_j) > t\big) \Bigg) \Bigg],\end{aligned}$$

where W' is independent and follows the same distribution of W. ■

By applying Laplace inversion to the first two terms of the series expansion in ϵ of the queueing delay, we obtain that the first term is a phase-type approximation of the queueing delay that results from the replace base model (see Section 3.1). In addition, the second term, which we refer to as correction term and is found explicitly in Theorem 3.7, involves linear combinations of terms that have probabilistic interpretation. More precisely, these terms with probabilistic interpretation are either tail probabilities of convoluted r.v. or probabilities for some of the aforementioned convoluted r.v. to lie between a fixed value t and the same value t shifted by an exponential time. Finally, observe that these convoluted r.v. involve the heavy-tailed stationary-excess service time r.v. C^e in a maximum appearance of one. Combining the results of Proposition 3.6 and Theorem 3.7, in the next section we define our approximations.

3.4 Corrected replace approximations

The goal of this section is to provide approximations that maintain the numerical tractability but improve the accuracy of the phase-type approximations and that are able to capture the tail behavior of the exact delay distribution. As we pointed out in the introduction, a single appearance of a stationary excess heavy-tailed service time C^e is sufficient to capture the correct tail behavior of the exact queueing delay. As we observed in Section 3.3, the correction term contains terms with single appearances of C^e. For this reason, the proposed approximation for the queueing delay is constructed by the first two terms of its respective series expansion for the queueing delay. We propose the following approximation:

Approximation 1. The corrected replace approximation of the survival function $\mathbb{P}(W_\epsilon > t)$ of the exact queueing delay is defined as

$$\begin{aligned}
\widehat{\varphi}_{r,\epsilon}(t) := \mathbb{P}(W > t) + \epsilon \frac{1}{\mathbf{u}\boldsymbol{\omega}} \Bigg[\Big(\mathbf{z}\boldsymbol{\omega} - \sum_{k=2}^{N} \frac{\alpha_k}{s_k} \Big) \mathbb{P}(W > t) \\
+ \Big(\beta - \sum_{k=2}^{N} \frac{\beta_k}{s_k} \Big) \Big(\mu_p \mathbb{P}(W + B^e > t) - \mu_h \mathbb{P}(W + C^e > t) \Big) \\
- \Big(\gamma - \sum_{k=2}^{N} \frac{\gamma_k}{s_k} \Big) \Big(\mu_p \mathbb{P}(W + W' + B^e > t) - \mu_h \mathbb{P}(W + W' + C^e > t) \Big) \\
- \sum_{k=2}^{N} \frac{1}{s_k} \Bigg(\gamma_k \Big(\mu_p \mathbb{P}\big(t < W + W' + B^e < t + E(s_k)\big) - \mu_h \mathbb{P}\big(t < W + W' + C^e < t + E(s_k)\big) \Big) \\
- \beta_k \Big(\mu_p \mathbb{P}\big(t < W + B^e < t + E(s_k)\big) - \mu_h \mathbb{P}\big(t < W + C^e < t + E(s_k)\big) \Big) \\
- \alpha_k \mathbb{P}\big(t < W < t + E(s_k)\big) \Bigg) \\
- \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \Bigg(\gamma''_{j,l} \Big(\mu_p \mathbb{P}\big(W + W' + B^e + E_{r_j - l + 1}(y_j) > t\big) - \mu_h \mathbb{P}\big(W + W' + C^e + E_{r_j - l + 1}(y_j) > t\big) \Big) \\
- \beta''_{j,l} \Big(\mu_p \mathbb{P}\big(W + B^e + E_{r_j - l + 1}(y_j) > t\big) - \mu_h \mathbb{P}\big(W + C^e + E_{r_j - l + 1}(y_j) > t\big) \Big) \\
- \alpha''_{j,l} \mathbb{P}\big(W + E_{r_j - l + 1}(y_j) > t\big) \Bigg) \Bigg],
\end{aligned}$$

where $\mathbb{P}(W > t)$ is the replace phase-type approximation of $\mathbb{P}(W_\epsilon > t)$, W' is independent and follows the same distribution of W, and the coefficients β, γ, α_k, β_k, γ_k, $k = 2, \ldots, N$, and $\alpha''_{j,l}$, $\beta''_{j,l}$ and $\gamma''_{j,l}$, $j = 1, \ldots, \sigma$, $l = 1, \ldots, r_j$, are calculated according to Proposition 3.6.

The following result shows that the corrected replace approximation makes sense rigorously.

Proposition 3.8. *If $\mathbb{P}(W > t)$ is the replace approximation of the exact queueing delay $\mathbb{P}(W_\epsilon > t)$, then as $\epsilon \to 0$, it holds that*

$$\frac{\mathbb{P}(W_\epsilon > t) - \mathbb{P}(W > t)}{\epsilon} \to \Theta(t),$$

where $\Theta(t)$ is given in Theorem 3.7.

Proof. See Appendix B. □

Although Approximation 1 gives an approximation of the queueing delay that can be calculated explicitly and is computationally tractable, it involves the evaluation of many terms. Therefore, to simplify the formula of the approximation, it makes sense to ignore terms that do not contribute significantly to the accuracy of the corrected replace approximation. Such terms seem to be the probabilities of convoluted r.v. that lie between a fixed value t and the same value t shifted by an exponential time. Therefore, we define the *simplified corrected replace approximation* as follows.

Approximation 2. The simplified corrected replace approximation of the survival function $\mathbb{P}(W_\epsilon > t)$ of the exact delay is defined as

$$\begin{aligned}
\widehat{\varphi}_{sr,\epsilon}(t) := \mathbb{P}(W > t) + \epsilon\frac{1}{\mathbf{u}\boldsymbol{\omega}}\Bigg[&\Big(\mathbf{z}\boldsymbol{\omega} - \sum_{k=2}^{N}\frac{\alpha_k}{s_k}\Big)\mathbb{P}(W > t) \\
&+ \Big(\beta - \sum_{k=2}^{N}\frac{\beta_k}{s_k}\Big)\Big(\mu_p\mathbb{P}(W + B^e > t) - \mu_h\mathbb{P}(W + C^e > t)\Big) \\
&-\Big(\gamma - \sum_{k=2}^{N}\frac{\gamma_k}{s_k}\Big)\Big(\mu_p\mathbb{P}(W + W' + B^e > t) - \mu_h\mathbb{P}(W + W' + C^e > t)\Big) \\
&-\sum_{j=1}^{\sigma}\sum_{l=1}^{r_j}\Bigg(\gamma''_{j,l}\Big(\mu_p\mathbb{P}\big(W + W' + B^e + E_{r_j-l+1}(y_j) > t\big) - \mu_h\mathbb{P}\big(W + W' + C^e + E_{r_j-l+1}(y_j) > t\big)\Big) \\
&- \beta''_{j,l}\Big(\mu_p\mathbb{P}\big(W + B^e + E_{r_j-l+1}(y_j) > t\big) - \mu_h\mathbb{P}\big(W + C^e + E_{r_j-l+1}(y_j) > t\big)\Big) \\
&- \alpha''_{j,l}\mathbb{P}\big(W + E_{r_j-l+1}(y_j) > t\big)\Bigg)\Bigg],
\end{aligned}$$

where $\mathbb{P}(W > t)$ is the replace phase-type approximation of $\mathbb{P}(W_\epsilon > t)$, W' is independent and follows the same distribution of W, and the coefficients β, γ, α_k, β_k, γ_k, $k = 2, \dots, N$, and $\alpha''_{j,l}$, $\beta''_{j,l}$ and $\gamma''_{j,l}$, $j = 1, \dots, \sigma$, $l = 1, \dots, r_j$, are calculated according to Proposition 3.6.

4. Corrected discard approximation

In this section, we construct the corrected discard approximation. There are two different approaches to obtain this approximation. In the first one, we follow the same steps as in the construction of the corrected replace approximation. Namely, we first calculate the queueing delay for the simpler phase-type model when we discard the heavy-tailed customers and then we write the queueing delay of the mixture model as perturbation of the queueing delay in the discard base model. However, here we use an alternative approach that connects the discard base model with the replace base model.

As we mentioned in Section 2.2, when we discard the heavy-tailed customers we simply consider that

$$\widetilde{G}^{\bullet}_\epsilon(s) = (1-\epsilon)\widetilde{F}_p(s) + \epsilon = \widetilde{F}_p(s) + \epsilon\big(1 - \widetilde{F}_p(s)\big) = \widetilde{F}_p(s) + \epsilon s\mu_p\widetilde{F}^e_p(s).$$

Although the service time distribution $\widetilde{G}^{\bullet}_\epsilon(s)$ has an atom at zero, the resulting delay distribution has a phase-type representation and consequently it can be directly calculated through Laplace inversion of its LST $\widetilde{w}^{\bullet}_\epsilon(s)$. However, it is difficult to apply perturbation analysis to find the connection between $\widetilde{w}^{\bullet}_\epsilon(s)$ and $\widetilde{w}_\epsilon(s)$, because both of them depend on ϵ.

Observe that $\widetilde{G}^{\bullet}_\epsilon(s)$ can be expressed as perturbation of $\widetilde{F}_p(s)$ by the term $\epsilon s\mu_p\widetilde{F}^e_p(s)$. Therefore, we can apply perturbation analysis to find a connection between $\widetilde{w}^{\bullet}_\epsilon(s)$ and $\widetilde{w}(s)$, which is the Laplace transform of the queueing delay in the replace base model, and then use the connection of $\widetilde{w}(s)$ with $\widetilde{w}_\epsilon(s)$ to establish a connection between $\widetilde{w}_\epsilon(s)$ and $\widetilde{w}^{\bullet}_\epsilon(s)$. Thus, as a first step we express the matrices in the discard base model as perturbation of the ones in the replace base model, by setting $\widetilde{F}_h(s) \equiv 1$ in the results of Section 3.2. So, we define the matrices

$$\widetilde{\mathbf{G}}^{\bullet}_\epsilon(s) = \widetilde{\mathbf{G}}(s) + \epsilon s\mu_p\widetilde{F}^e_p(s)\mathbf{Q}^{(2)},$$

$$
\begin{aligned}
\mathbf{H}^{\bullet}_{\epsilon}(s) =&\mathbf{H}(s) + \epsilon s\mu_p \widetilde{F}^e_p(s)\mathbf{Q}^{(2)} \circ \mathbf{P}\boldsymbol{\Lambda},\\
\mathbf{E}^{\bullet}_{\epsilon}(s) =&\mathbf{E}(s) + \epsilon s\mu_p \widetilde{F}^e_p(s)\mathbf{Q}^{(2)} \circ \mathbf{P}\boldsymbol{\Lambda},\\
\mathbf{M}^{\bullet}_{\epsilon} =&\mathbf{M} - \epsilon s\mu_p \mathbf{Q}^{(2)} \circ \mathbf{P}.
\end{aligned}
$$

Now, we provide a series of results for the evaluation of $\widetilde{w}^{\bullet}_{\epsilon}(s) = s\mathbf{u}^{\bullet}_{\epsilon}\mathcal{E}^{\bullet}_{\epsilon}(s)\boldsymbol{\omega}/\det \mathbf{E}^{\bullet}_{\epsilon}(s)$, which occur as corollaries of their corresponding results in Sections 3.2 and 3.3. The first two corollaries are for the evaluation of the vector $\mathbf{u}^{\bullet}_{\epsilon}$ of unknown parameters.

Corollary 4.1. *Let* $\mathbf{u}$ *be the unique solution to the Eqs.* (17)–(18) *for the replace base model. If the roots* $s_2, \dots, s_N$ *of* $\det\big(\mathbf{H}(s) + s\mathbf{I} - \boldsymbol{\Lambda}\big) = 0$ *with positive real part are simple, then*

1. *the equation* $\det\big(\mathbf{H}^{\bullet}_{\epsilon}(s) + s\mathbf{I} - \boldsymbol{\Lambda}\big) = 0$ *has exactly* N *non-negative solutions* $s^{\bullet}_{\epsilon,1}, \dots, s^{\bullet}_{\epsilon,N}$, *with* $s^{\bullet}_{\epsilon,1} = 0$ *and* $s^{\bullet}_{\epsilon,i} = s_i - \epsilon\delta^{\bullet}_i + O(\epsilon^2)$ *for* $i = 2, \dots, N$, *where*

$$
\delta^{\bullet}_i := \delta^{\bullet}(s_i) = \frac{\sum_{j=1}^{N} \det\big(\mathbf{E}(s_i)_{\bullet 1}, \dots, \mathbf{K}(s_i)_{\bullet j}, \dots, \mathbf{E}(s_i)_{\bullet N}\big)}{\sum_{j=1}^{N} \det\big(\mathbf{E}(s_i)_{\bullet 1}, \dots, \mathbf{E}^{(1)}(s_i)_{\bullet j}, \dots, \mathbf{E}(s_i)_{\bullet N}\big)},
$$

   and $\mathbf{K}(s) = s\mu_p\widetilde{F}^e_p(s)\mathbf{Q}^{(2)} \circ \mathbf{P}\boldsymbol{\Lambda}$.

2. *We set* $\mathbf{A} = \big(\boldsymbol{\Lambda}^{-1}\mathbf{e}, \mathbf{a}_2, \dots, \mathbf{a}_N\big)$ *(see Eq.* (21)*) and* $\mathbf{c} = \big(\boldsymbol{\pi}(\boldsymbol{\Lambda}^{-1} - \mathbf{M})\mathbf{e}, 0, \dots, 0\big)$, *and we assume that the stability condition* $\boldsymbol{\pi}(\boldsymbol{\Lambda}^{-1} - \mathbf{M}^{\bullet}_{\epsilon})\mathbf{e} > 0$, *is satisfied. Then, the vector* $\mathbf{u}^{\bullet}_{\epsilon}$ *is the unique solution to the system of* N *linear equations*

$$
\mathbf{u}^{\bullet}_{\epsilon}\big(\mathbf{A} - \epsilon\mathbf{B}^{\bullet} + O(\epsilon^2\mathbf{U})\big) = \mathbf{c} + \epsilon\mathbf{d}^{\bullet}, \tag{33}
$$

   where $\mathbf{B}^{\bullet} = \big(\mathbf{0}, \delta^{\bullet}_2\mathbf{a}_2{}^{(1)} - \mathbf{k}^{\bullet}_2, \dots, \delta^{\bullet}_N\mathbf{a}_N{}^{(1)} - \mathbf{k}^{\bullet}_N\big)$ *and* $\mathbf{d}^{\bullet} = \big(\mu_p\boldsymbol{\pi}\mathbf{Q}^{(2)} \circ \mathbf{P}\mathbf{e}, 0, \dots, 0\big)$, *with* $\mathbf{k}^{\bullet}_i$, $i = 2, \dots, N$, *being a column vector with coordinates*

$$
k^{\bullet}_{i,j} = (-1)^{m+j}\sum_{k=1}^{N-1} \det\left(\left(\mathbf{E}(s_i)^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet 1}, \dots, \left(\mathbf{K}(s_i)^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet k}, \dots, \left(\mathbf{E}(s_i)^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet N-1}\right), \quad j \in \mathcal{N},
$$

   and the choice of m *explained in Remark 1.*

Corollary 4.2. *The vector* $\mathbf{u}^{\bullet}_{\epsilon}$ *can be written in the form*

$$
\mathbf{u}^{\bullet}_{\epsilon} = \mathbf{u} + \epsilon\mathbf{z}^{\bullet} + O(\epsilon^2\mathbf{e}),
$$

where

$$
\mathbf{z}^{\bullet} = \big(\mathbf{c}\mathbf{A}^{-1}\mathbf{B}^{\bullet} + \mathbf{d}^{\bullet}\big)\,\mathbf{A}^{-1}.
$$

The next corollary give as the denominator of $\widetilde{w}^{\bullet}_{\epsilon}(s)$.

Corollary 4.3. *If* $\det\mathbf{E}(s)$ *is evaluated according to Theorem 2.3 with* $\widetilde{G}(s) = \widetilde{F}_p(s)$, *then* $\det\mathbf{E}^{\bullet}_{\epsilon}(s)$ *can be written as perturbation of* $\det\mathbf{E}(s)$ *as follows*

$$
\begin{aligned}
\det\mathbf{E}^{\bullet}_{\epsilon}(s) =& \det\mathbf{E}(s)\\
&+ \epsilon s\mu_p\widetilde{F}^e_p(s)\sum_{k=1}^{N} k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\\|\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\\S\supset\Gamma}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)^{S\setminus\Gamma}_{S}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)^{\Gamma}_{S}\Big)\\
&+ O(\epsilon^2).
\end{aligned}
$$

For the evaluation of the numerator of $\widetilde{w}^\bullet_\epsilon(s)$, we need the following result.

Corollary 4.4. *If $s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i$ is evaluated according to Theorem 2.5 with $\widetilde{G}(s) = \widetilde{F}_p(s)$, then $s\mathbf{u}^\bullet_\epsilon\mathcal{E}^\bullet_\epsilon(s)\mathbf{e}_i$ can be written as perturbation of $s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i$ as follows*

$$
\begin{aligned}
s\mathbf{u}^\bullet_\epsilon\mathcal{E}_\epsilon(s)\mathbf{e}_i = s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i + \epsilon s\Bigg[& z_i^\bullet \sum_{k=1}^{N-1}\big(\widetilde{F}_p(s)\big)^k \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\ |\Gamma|=k;\\ S\subset\mathcal{N}\setminus\{i\}\\ S\supset\Gamma}} \boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\setminus\Gamma}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big)\\
&+z_i^\bullet\sum_{S\subset\mathcal{N}\setminus\{i\}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^S\\
&+\sum_{\substack{l=1\\ l\neq i}}^{N} z_l^\bullet(-1)^{l+i}\sum_{k=1}^{N-1}\big(\widetilde{F}_p(s)\big)^k\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k-1;\\ S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big)\\
&+\sum_{\substack{l=1\\ l\neq i}}^{N} z_l^\bullet(-1)^{l+i}\sum_{k=0}^{N-2}\big(\widetilde{F}_p(s)\big)^k\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\\
&\quad\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big)\\
&+s\mu_p\widetilde{F}_p^e(s)\Bigg(u_i\sum_{k=1}^{N-1}k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\ |\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\setminus\{i\}\\ S\supset\Gamma}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\setminus\Gamma}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big)\\
&+\sum_{\substack{l=1\\ l\neq i}}^{N} u_l(-1)^{l+i}\sum_{k=1}^{N-1}k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k-1}}\sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\\
&\quad\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big)\\
&+\sum_{\substack{l=1\\ l\neq i}}^{N} u_l(-1)^{l+i}\sum_{k=1}^{N-2}k\big(\widetilde{F}_p(s)\big)^{k-1}\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\\
&\quad\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big)\Bigg)\Bigg]+O(\epsilon^2),
\end{aligned}
$$

where $z_i^\bullet$, $i\in\mathcal{N}$, are the coordinates of the vector $\mathbf{z}^\bullet$ given in Corollary 4.2.

Combining Corollaries 4.3 and 4.4, and Proposition 3.6, we have the following Proposition that connects the delay in the discard model $\widetilde{w}^\bullet_\epsilon(s)$ and the delay in the mixture model $\widetilde{w}_\epsilon(s)$.

Proposition 4.5. *If $\widetilde{w}^\bullet_\epsilon(s)$ is the Laplace transform of the queueing delay of the discard base model that is calculated as perturbation of $\widetilde{w}(s)$ (see Proposition 3.1), then there exist unique coefficients β, γ, α_i, β_k, γ_k, $k = 2,\ldots,N$, and $\alpha''_{j,l}$, $\beta''_{j,l}$ and $\gamma''_{j,l}$, $j = 1,\ldots,\sigma$, $l = 1,\ldots,r_j$, such that the*

Laplace transform $\widetilde{w}_\epsilon(s)$ of the queueing delay of the mixture model satisfies

$$\begin{aligned}\widetilde{w}_\epsilon(s) =&\widetilde{w}^\bullet_\epsilon(s) + \epsilon\frac{1}{\mathbf{u}^\bullet_\epsilon\boldsymbol{\omega}}\widetilde{w}^\bullet_\epsilon(s)\Bigg[\Bigg((\mathbf{z}-\mathbf{z}^\bullet)\boldsymbol{\omega} + \sum_{k=2}^N \frac{\alpha_k}{s-s_k} + \sum_{j=1}^\sigma\sum_{l=1}^{r_j}\frac{\alpha''_{j,l}\cdot(y_j)^{r_j-l+1}}{(s+y_j)^{r_j-l+1}}\Bigg)\\ &-\mu_h\widetilde{F}^e_h(s)\Bigg(\beta + \sum_{k=2}^N\frac{\beta_k}{s-s_k} + \sum_{j=1}^\sigma\sum_{l=1}^{r_j}\frac{\beta''_{j,l}\cdot(y_j)^{r_j-l+1}}{(s+y_j)^{r_j-l+1}}\Bigg)\\ &+\mu_h\widetilde{F}^e_h(s)\widetilde{w}^\bullet_\epsilon(s)\Bigg(\gamma + \sum_{k=2}^N\frac{\gamma_k}{s-s_k} + \sum_{j=1}^\sigma\sum_{l=1}^{r_j}\frac{\gamma''_{j,l}\cdot(y_j)^{r_j-l+1}}{(s+y_j)^{r_j-l+1}}\Bigg)\Bigg] + O(\epsilon^2),\end{aligned}$$

where the vector $\mathbf{z}^\bullet$ given in Corollary 4.2.

Using same arguments as in the definition of the corrected replace approximations, we define the corrected discard approximations as follows.

Approximation 3. The corrected discard approximation of the survival function $\mathbb{P}(W_\epsilon > t)$ of the exact queueing delay is defined as

$$\begin{aligned}\widehat{\varphi}^\bullet_{d,\epsilon}(t) := &\mathbb{P}(W^\bullet_\epsilon > t) + \epsilon\frac{1}{\mathbf{u}^\bullet_\epsilon\boldsymbol{\omega}}\Bigg[\Big((\mathbf{z}-\mathbf{z}^\bullet)\boldsymbol{\omega} - \sum_{k=2}^N\frac{\alpha_k}{s_k}\Big)\mathbb{P}(W^\bullet_\epsilon > t) - \Big(\beta - \sum_{k=2}^N\frac{\beta_k}{s_k}\Big)\mu_h\mathbb{P}(W^\bullet_\epsilon + C^e > t)\\ &+\Big(\gamma - \sum_{k=2}^N\frac{\gamma_k}{s_k}\Big)\mu_h\mathbb{P}(W^\bullet_\epsilon + W^{\bullet\prime}_\epsilon + C^e > t) + \sum_{k=2}^N\frac{1}{s_k}\Bigg(\gamma_k\mu_h\mathbb{P}\big(t < W^\bullet_\epsilon + W^{\bullet\prime}_\epsilon + C^e < t + E(s_k)\big)\\ &- \beta_k\mu_h\mathbb{P}\big(t < W^\bullet_\epsilon + C^e < t + E(s_k)\big) + \alpha_k\mathbb{P}\big(t < W^\bullet_\epsilon < t + E(s_k)\big)\Bigg)\\ &+\sum_{j=1}^\sigma\sum_{l=1}^{r_j}\Bigg(\gamma''_{j,l}\mu_h\mathbb{P}\big(W^\bullet_\epsilon + W^{\bullet\prime}_\epsilon + C^e + E_{r_j-l+1}(y_j) > t\big) - \beta''_{j,l}\mu_h\mathbb{P}\big(W^\bullet_\epsilon + C^e + E_{r_j-l+1}(y_j) > t\big)\\ &+ \alpha''_{j,l}\mathbb{P}\big(W^\bullet_\epsilon + E_{r_j-l+1}(y_j) > t\big)\Bigg)\Bigg],\end{aligned}$$

where $\mathbb{P}(W^\bullet_\epsilon > t)$ is the discard phase-type approximation of $\mathbb{P}(W_\epsilon > t)$, $W^{\bullet\prime}_\epsilon$ is independent and follows the same distribution of W, and the coefficients β, γ, α_k, β_k, γ_k, $k = 2, \ldots, N$, and $\alpha''_{j,l}$, $\beta''_{j,l}$ and $\gamma''_{j,l}$, $j = 1, \ldots, \sigma$, $l = 1, \ldots, r_j$, are calculated according to Proposition 3.6.

Approximation 3 can be made rigorous along the same lines as in Proposition 3.8. The simplified version of this approximation is found in the following lines.

Approximation 4. The simplified corrected discard approximation of the survival function $\mathbb{P}(W_\epsilon > t)$ of the exact queueing delay is defined as

$$\begin{aligned}\widehat{\varphi}^\bullet_{sd,\epsilon}(t) := &\mathbb{P}(W^\bullet_\epsilon > t) + \epsilon\frac{1}{\mathbf{u}^\bullet_\epsilon\boldsymbol{\omega}}\Bigg[\Big((\mathbf{z}-\mathbf{z}^\bullet)\boldsymbol{\omega} - \sum_{k=2}^N\frac{\alpha_k}{s_k}\Big)\mathbb{P}(W^\bullet_\epsilon > t) - \Big(\beta - \sum_{k=2}^N\frac{\beta_k}{s_k}\Big)\mu_h\mathbb{P}(W^\bullet_\epsilon + C^e > t)\\ &+\Big(\gamma - \sum_{k=2}^N\frac{\gamma_k}{s_k}\Big)\mu_h\mathbb{P}(W^\bullet_\epsilon + W^{\bullet\prime}_\epsilon + C^e > t) + \sum_{j=1}^\sigma\sum_{l=1}^{r_j}\Bigg(\gamma''_{j,l}\mu_h\mathbb{P}\big(W^\bullet_\epsilon + W^{\bullet\prime}_\epsilon + C^e + E_{r_j-l+1}(y_j) > t\big)\\ &- \beta''_{j,l}\mu_h\mathbb{P}\big(W^\bullet_\epsilon + C^e + E_{r_j-l+1}(y_j) > t\big) + \alpha''_{j,l}\mathbb{P}\big(W^\bullet_\epsilon + E_{r_j-l+1}(y_j) > t\big)\Bigg)\Bigg],\end{aligned}$$

where $\mathbb{P}(W^\bullet_\epsilon > t)$ is the replace phase-type approximation of $\mathbb{P}(W_\epsilon > t)$, $W^{\bullet\prime}_\epsilon$ is independent and follows the same distribution of W, and the coefficients β, γ, α_k, β_k, γ_k, $k = 2, \ldots, N$, and $\alpha''_{j,l}$, $\beta''_{j,l}$ and $\gamma''_{j,l}$, $j = 1, \ldots, \sigma$, $l = 1, \ldots, r_j$, are calculated according to Proposition 3.6.

In the next section, we perform numerical experiments to check the accuracy of the corrected phase-type and the simplified corrected phase-type approximations. In addition, we show that indeed the corrected approximations do not differ significantly from their simplified versions.

5. Numerical experiments

In Section 3.3, we pointed out that the first term of the corrected replace expansion is already a phase-type approximation of the queueing delay, a result that holds also for the discard expansion. In this section, we show that addition of the correction term leads to improved approximations that are significantly more accurate than their phase-type counterparts. Therefore, we check here the accuracy of the corrected phase-type approximations (see Definitions 1 – 4) by comparing them with the exact delay distribution and their corresponding phase-type approximations.

For the MArP arrival process of customers we choose a MMPP with two states and a MMPP with five states. Since it is more meaningful to compare approximations with exact results than with simulation outcomes, we choose the service time distribution such that we can find an exact formula for the queueing delay.

As service time distribution we use a mixture of an exponential distribution with rate ν and a heavy-tailed one that belongs to a class of long-tailed distributions introduced in [2]. The Laplace transform of the latter distribution is $\widetilde{F}_h(s) = 1 - \frac{s}{(\kappa+\sqrt{s})(1+\sqrt{s})}$, where $\mathbb{E}C = \kappa^{-1}$ and all higher moments are infinite. Furthermore, the Laplace transform of the stationary heavy-tailed claim size distribution is

$$\widetilde{F}^e_h(s) = \frac{\kappa}{(\kappa+\sqrt{s})(1+\sqrt{s})},$$

which for $\kappa \neq 1$ can take the form

$$\widetilde{F}^e_h(s) = \left(\frac{\kappa}{1-\kappa}\right)\left(\frac{1}{\kappa+\sqrt{s}} - \frac{1}{1+\sqrt{s}}\right).$$

For this combination of service time distributions, the survival queueing delay can be found explicitly, by following the same ideas as in Theorem 9 of [34].

What is left now is to fix values for the parameters of the mixture models and perform our numerical experiments. Thus, for the MMPP(2) arrival process we choose the parameters such that $\lambda_1 = 10$, $\lambda_1 = 1/2$, $p_{11} = 8/9$, and $p_{22} = 3/100$ (the rest of the parameters can be calculated using the formulas (2)–(5)). For the MMPP(5) model we choose:

$$\mathbf{P} = \begin{pmatrix} \frac{7}{27} & \frac{5}{27} & 0 & 0 & \frac{5}{9} \\ 0 & \frac{1}{29} & \frac{20}{29} & \frac{8}{29} & 0 \\ \frac{3}{25} & \frac{2}{5} & \frac{3}{10} & \frac{9}{50} & 0 \\ 0 & 0 & \frac{7}{36} & \frac{5}{18} & \frac{19}{36} \\ \frac{12}{47} & \frac{20}{47} & \frac{20}{47} & \frac{5}{47} & \frac{10}{47} \end{pmatrix},$$

and $\mathbf{\Lambda} = \mathrm{diag}\{11, 11, 13, 10, 8\}$. Although we do not have any restrictions for the parameters of the involved service time distributions, from a modeling point of view, it is counterintuitive to fit a heavy-tailed service-time distribution with a mean smaller than the mean of the phase-type service-type distribution. For this reason, we select $\kappa = 2$ and $\nu = 3$.

Finally, note that we performed extensive numerical experiments for various values of the perturbation parameter ϵ in the interval $[0.001, 0.1]$. We chose to present only the case $\epsilon = 0.01$, since the qualitative conclusions for all other values of ϵ are similar to those presented in this section.

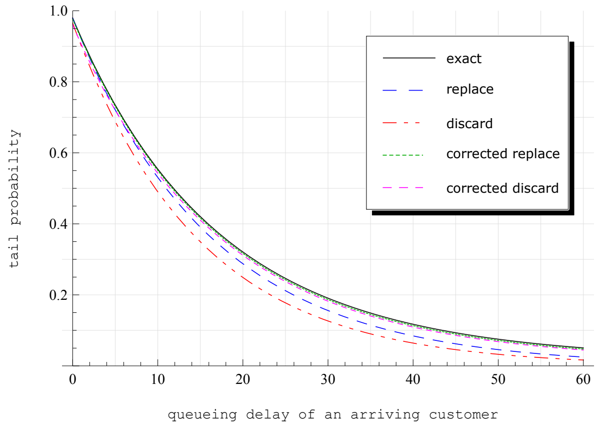

Figure 1: Exact queueing delay, phase-type, and corrected phase-type approximations for perturbation parameter 0.01, MMPP(2) arrivals, and load of the system 0.908336

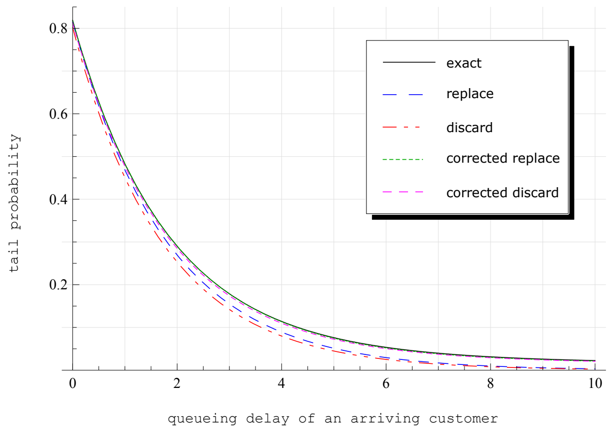

Figure 2: Exact queueing delay, phase-type, and corrected phase-type approximations for perturbation parameter 0.01, MMPP(5) arrivals, and load of the system 0.812845

For this choice of parameters, the load of the first system is equal to 0.909336 and of the second is 0.812845.

As we observe from Figures 2, the phase-type approximations (replace and discard) give accurate estimates for small values of the queueing delay, while they are incapable of capturing the correct tail behavior of the exact survival function of the queueing delay. Contrary, both corrected phase-type approximations are highly accurate and give a small relative error at the tail. More precisely, we can observe the following:

- The corrected replace approximation gives better numerical estimates than the corrected discard approximation.
- The corrected discard approximation always underestimates the exact tail probability of the queueing delay. Contrary, the corrected replace approximation may overestimate the exact survival function for small values, but always underestimates the tail of the exact queueing delay.
- The corrected phase-type approximations do not differ significantly from their simplified versions. The maximum observed absolute error between the two corrected replace approximations is smaller than 0.0011 for the MMPP(2) model and smaller than 0.00069 for the MMPP(5) model. The corresponding numbers for the corrected discard approximations are 0.0052 and 0.00167.
- Finally, we estimated the relative error at the tail for all four corrected phase-type approximations. We found that in the MMPP(2) model the relative error is smaller than 10% for all the approximations, while this number reduces to 7% in the MMPP(5) model.

6. Conclusions

To conclude, all corrected replace approximations are highly accurate and there is no significant difference between them. For this reason, the simplified versions of the approximations serve as excellent substitutes to their original corrected phase-type approximations for estimating the queueing delay. Finally, the corrected phase-type approximations give a small relative error at the tail, which can easy be verified that it is $O(\epsilon)$.

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A. Results on perturbation theory

In this section, we provide some preliminary results on linear algebra, matrix functions, and perturbation theory that are needed in our analysis. We introduce an $N \times N$ matrix function $\mathbf{E}(s)$ with a single parameter $s > 0$. We say that the matrix function $\mathbf{E}(s)$ is *regular* if $\det \mathbf{E}(s)$ is not identically zero as a function of s. In addition, if $\mathbf{E}(s)$ is regular (we denote it as $\det \mathbf{E}(s) \not\equiv 0$), then the eigenvalues of $\mathbf{E}(s)$ are the solutions of the equations $\det \mathbf{E}(s) = 0$ [13]. Throughout our analysis, we assume that the matrix $\mathbf{E}(s)$ is regular and that r is a simple eigenvalues of it. In addition, we assume that the matrix $\mathbf{E}(s)$ is analytic in the neighborhood of r. We use the notation $\mathbf{E}^{(n)}(s)$ for the nth derivative of the matrix function $\mathbf{E}(s)$. Thus, $\mathbf{E}(s)$ can be written as a Taylor series in the following form:

$$\mathbf{E}(s) = \mathbf{E}^{(0)}(r) + (s-r)\mathbf{E}^{(1)}(r) + \cdots = \sum_{n=0}^{\infty} \frac{(s-r)^n}{n!}\mathbf{E}^{(n)}(r). \tag{34}$$

To avoid redundant notation, in the forthcoming analysis we use the conventions that $\mathbf{E} = \mathbf{E}^{(0)}(r) = \mathbf{E}(r)$ and $\mathbf{E}^{(n)} = \mathbf{E}^{(n)}(r)$.

As a consequence of the fact that the multiplicity of the eigenvalue r is one, the dimension of the nullspace of $\mathbf{E}$ is equal to one. Our first goal is to find the form of the eigenvectors of the nullspace of matrix $\mathbf{E}$. The following theorem gives us exactly the form of these eigenvectors.

Theorem A.1. *If $\mathbf{C}$ is an $N \times N$ matrix with determinant equal to zero, i.e. $\det \mathbf{C} = 0$, and nullspace of dimension one, then a right $N \times 1$ eigenvector that corresponds to the simple eigenvalue zero is $\mathbf{t}$ with coordinates $t_j = (-1)^{1+j} \det \mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}}$, $j \in \mathcal{N}$.*

Proof. We need to prove that the inner product of every row of $\mathbf{C}$ with $\mathbf{t}$ is equal to zero. More precisely, if $\mathbf{c}_i$ denotes the ith row of matrix $\mathbf{C}$, we need to show that

$$\mathbf{c}_i\mathbf{t} = 0, \qquad i \in \mathcal{N}.$$

If c_{ij} is the (i,j) element of matrix $\mathbf{C}$, for the first row we have

$$\mathbf{c}_1\mathbf{t} = \sum_{j=1}^{N} c_{1j}(-1)^{1+j} \det \mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}} \stackrel{\text{def.}}{=} \det \mathbf{C} = 0.$$

For an arbitrary row $i = 2, \ldots, N$, we have

$$\mathbf{c}_i\mathbf{t} = \sum_{j=1}^{N} c_{ij}(-1)^{1+j} \det \mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}}.$$

We expand the determinant of each matrix $\mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}}$, $j \in \mathcal{N}$, in minors of the ith row of matrix $\mathbf{C}$. Observe that the ith row of the initial matrix is indexed $i-1$ in every matrix $\mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}}$, due to the removal of the first row of $\mathbf{C}$. Note also that, every column k placed to the right of the jth column of matrix $\mathbf{C}$, after the removal of the jth column is shifted one position to the left, therefore it is indexed as $k-1$. Using the notation $\mathbb{1}$ for the indicator function, after the above observations, we have

$$\mathbf{c}_i\mathbf{t} = \sum_{j=1}^{N} c_{ij}(-1)^{1+j} \det \mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}} = \sum_{j=1}^{N} c_{ij}(-1)^{1+j} \sum_{k\neq j} c_{ik}(-1)^{i-1+k-\mathbb{1}_{\{k>j\}}} \det \mathbf{C}_{\mathcal{N}\setminus\{1,i\}}^{\mathcal{N}\setminus\{j,k\}}$$

$$= (-1)^i \sum_{j=1}^{N} \sum_{k \neq j} c_{ij} c_{ik} (-1)^{j+k-\mathbb{1}_{\{k>j\}}} \det \mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}} = 0,$$

because for any two arbitrary columns m and l, with $m > l$, only the summands

$$c_{il} c_{im} (-1)^{l+m-1} \det \mathbf{C}_{\mathcal{N}\setminus\{1,i\}}^{\mathcal{N}\setminus\{l,m\}}, \quad \text{and} \quad c_{im} c_{il} (-1)^{m+l} \det \mathbf{C}_{\mathcal{N}\setminus\{1,i\}}^{\mathcal{N}\setminus\{l,m\}},$$

appear in the expression of $\mathbf{c}_i \mathbf{t}$ and they cancel out with one another. Since, all summands of the above double sum are coupled and canceled out, the double sum is equal to zero. Thus, we have proven that the inner product of any column of $\mathbf{C}$ with $\mathbf{t}$ is equal to zero. Consequently, $\mathbf{t}$ is an eigenvector of matrix $\mathbf{C}$ that corresponds to its eigenvalue zero. □

Remark 4. If the nullspace of an $N \times N$ matrix $\mathbf{C}$ has dimension one, then $\text{rank}\mathbf{C} = N - 1$. Therefore, there exists at least one submatrix of $\mathbf{C}$ such that its determinant is not equal to zero. More precisely, there exists at least one combination of row-column (m, n) with $\det \mathbf{C}_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{n\}} \neq 0$. Thus, if all determinants $\det \mathbf{C}_{\mathcal{N}\setminus\{1\}}^{\mathcal{N}\setminus\{j\}}$, $j \in \mathcal{N}$, are equal to zero, we can choose the coordinates of the right eigenvector $\mathbf{t}$, which corresponds to the eigenvalue zero, as $t_j = (-1)^{m+j} \det \mathbf{C}_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{j\}}$, $j \in \mathcal{N}$.

Remark 5. If $\mathbf{t}$ is an arbitrary eigenvector that belongs to the nullspace of $\mathbf{C}$, then any other eigenvector $\mathbf{z}$ that belongs to the same nullspace is proportional to $\mathbf{t}$. Namely, there exists $\sigma \in \mathbb{R}$ such that $\mathbf{z} = \sigma \mathbf{t}$.

From Theorem A.1 and Remark 4, we have as consequence the following corollary for the right eigenvectors of the matrix $\mathbf{E}$.

Corollary A.2. *If $m \in \mathcal{N}$ is such that $\det \mathbf{E}_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{j\}} \neq 0$ for at least one $j \in \mathcal{N}$, a right eigenvector $\mathbf{t}$ of the nullspace of $\mathbf{E}$ has coordinates*

$$t_j = (-1)^{m+j} \det \mathbf{E}_{\mathcal{N}\setminus\{m\}}^{\mathcal{N}\setminus\{j\}}, \qquad j \in \mathcal{N}.$$

We now perturb the matrix function $\mathbf{E}(s)$ by $\epsilon \mathbf{K}(s)$. Namely, we consider the matrix $\mathbf{E}(s) + \epsilon \mathbf{K}(s)$, where we assume that the matrix $\mathbf{K}(s)$ is analytic in the neighborhood of r. If $\mathbf{K}^{(n)}$ is the nth derivative of the matrix function $\mathbf{K}(s)$ at $s = r$, the Taylor series of matrix $\mathbf{K}(s)$ around r is:

$$\mathbf{K}(s) = \mathbf{K} + (s - r)\mathbf{K}^{(1)} + \cdots = \sum_{n=0}^{\infty} \frac{(s - r)^n}{n!} \mathbf{K}^{(n)}, \tag{35}$$

where $\mathbf{K}^{(n)} = \mathbf{K}^{(n)}(r)$ and $\mathbf{K} = \mathbf{K}^{(0)}$. Our goal is to find the form of the eigenvectors of the nullspace of $\mathbf{E}(s) + \epsilon \mathbf{K}(s)$. Thus, as a first step we find the roots of the equation

$$\det \big(\mathbf{E}(s) + \epsilon \mathbf{K}(s) \big) = 0. \tag{36}$$

At this point, we need the following result from perturbation theory, which gives us the root of a function $f(s)$ when it it perturbed by a small amount.

Theorem A.3. *Let r be a simple root of an analytic function $f(s)$. For some function $h(s, \epsilon)$ and for all small real values ϵ, we define the perturbed function*

$$F(s, \epsilon) = f(s) + h(s, \epsilon). \tag{37}$$

If $h(s,\epsilon)$ is analytic in s and ϵ near $(r,0)$, then $F(s,\epsilon)$ has a unique simple root $(x(\epsilon),\epsilon)$ near $(r,0)$ for all small values of ϵ. Moreover, $x(\epsilon)$ is an analytic function in ϵ, and if $\frac{\partial}{\partial s^n}h(s,0)\equiv 0$, $n=0,1,\dots,$ then it holds

$$x(\epsilon)=r-\epsilon\frac{\frac{\partial}{\partial\epsilon}h(r,0)}{f^{(1)}(r)}+O(\epsilon^2). \tag{38}$$

Proof. From the Implicit function theorem [5], we know that there exist a unique function x, with $x(0)=r$, such that for all small values of ϵ, it holds that $F\big(x(\epsilon),\epsilon\big)=0$ close to $(r,0)$. Moreover, the function x is analytic in ϵ. To find the linear Taylor polynomial approximation of $x(\epsilon)$, which is defined as

$$x(\epsilon)=x(0)+\epsilon x^{(1)}(0)+O(\epsilon^2),$$

we differentiate the function $F\big(x(\epsilon),\epsilon\big)=0$ as a function of ϵ, and by using the chain rule we obtain

$$\frac{\partial}{\partial x(\epsilon)}F\big(x(\epsilon),\epsilon\big)x^{(1)}(\epsilon)+\frac{\partial}{\partial\epsilon}F\big(x(\epsilon),\epsilon\big)=0$$
$$\Rightarrow$$
$$\big(f^{(1)}(x(\epsilon))+\frac{\partial}{\partial x(\epsilon)}h(x(\epsilon),\epsilon)\big)x^{(1)}(\epsilon)+\frac{\partial}{\partial\epsilon}h(x(\epsilon),\epsilon)=0.$$

In the latter equation, we substitute $\epsilon=0$ and we solve it with respect to $x^{(1)}(0)$. Since r is a simple root the function f, it holds that $f^{(1)}(r)\neq 0$ [20]. Thus, we have

$$f^{(1)}(r)x^{(1)}(0)+\frac{\partial}{\partial\epsilon}h(r,0)=0\ \Rightarrow\ x^{(1)}(0)=-\frac{\frac{\partial}{\partial\epsilon}h(r,0)}{f^{(1)}(r)},$$

which completes the proof. □

From Theorem A.3, we have the following lemma.

Lemma A.4. *If the functions $f(s)$ and $h(s,\epsilon)$ satisfy the assumptions of Theorem A.3, and $g(s)$ is an analytic function with $g(r)\neq 0$, then the perturbed function*

$$G(s,\epsilon)=f(s)g(s)+h(s,\epsilon)g(s),$$

has the same unique simple root $(x(\epsilon),\epsilon)$ near $(r,0)$ for all small values of ϵ with the perturbed function $F(s,\epsilon)=f(s)+h(s,\epsilon)$. Namely

$$x(\epsilon)=r-\epsilon\frac{\frac{\partial}{\partial\epsilon}h(r,0)}{f^{(1)}(r)}+O(\epsilon^2).$$

Proof. According to Theorem A.3, the unique simple root $x(\epsilon)$ of $G(s,\epsilon)$ near $(r,0)$ for all small values of ϵ satisfies

$$\begin{aligned}x(\epsilon)&=r-\epsilon\frac{\frac{\partial}{\partial\epsilon}\big(h(s,\epsilon)g(s)\big)\big|_{(r,0)}}{\frac{\partial}{\partial s}\big(f(r)g(r)\big)\big|_{s=r}}+O(\epsilon^2)=r-\epsilon\frac{\frac{\partial}{\partial\epsilon}h(r,0)g(r)}{f^{(1)}(r)g(r)+f(r)g^{(1)}(r)}+O(\epsilon^2)\\&=r-\epsilon\frac{\frac{\partial}{\partial\epsilon}h(r,0)}{f^{(1)}(r)}+O(\epsilon^2),\end{aligned}$$

because $f(r)=0$. □

We also need the following property for the determinant of a square matrix.

Proposition A.5. *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are* $N \times N$ *matrices with columns* $\mathbf{C}_{\bullet i}$ *and* $\mathbf{D}_{\bullet i}$*,* $i \in \mathcal{N}$*, respectively, then*

$$\det(\mathbf{C}_{\bullet 1} + \epsilon \mathbf{D}_{\bullet 1}, \ldots, \mathbf{C}_{\bullet N} + \epsilon \mathbf{D}_{\bullet N}) = \underbrace{\det(\mathbf{C}_{\bullet 1}, \ldots, \mathbf{C}_{\bullet N})}_{\det(\mathbf{C})} + \epsilon \sum_{i=1}^{N} \det(\mathbf{C}_{\bullet 1}, \ldots, \mathbf{D}_{\bullet i}, \ldots, \mathbf{C}_{\bullet N}) + O(\epsilon^2).$$

Proof. The result is an immediate consequence of the additive property of determinants. □

As shown in the following corollary, we can find the roots of the equation $\det\big(\mathbf{E}(s)+\epsilon\mathbf{K}(s)\big) = 0$, combining the results of Theorem A.3 and Proposition A.5.

Corollary A.6. *The number* $r_\epsilon = r - \epsilon\delta + O(\epsilon^2)$*, where*

$$\delta = \frac{\sum_{j=1}^{N} \det\big(\mathbf{E}_{\bullet 1}, \ldots, \mathbf{K}_{\bullet j}, \ldots, \mathbf{E}_{\bullet N}\big)}{\sum_{j=1}^{N} \det\big(\mathbf{E}_{\bullet 1}, \ldots, {\mathbf{E}_{\bullet j}}^{(1)}, \ldots, \mathbf{E}_{\bullet N}\big)},$$

is a simple root of the determinant $\det\big(\mathbf{E}(s) + \epsilon\mathbf{K}(s)\big) = 0$.

Proof. According to Proposition A.5,

$$\det\big(\mathbf{E}(s) + \epsilon\mathbf{K}(s)\big) = \det\mathbf{E}(s) + \epsilon\sum_{j=1}^{N} \det\big(\mathbf{E}_{\bullet 1}(s), \ldots, \mathbf{K}_{\bullet j}(s), \ldots, \mathbf{E}_{\bullet N}(s)\big) + O(\epsilon^2).$$

Note that $\det\mathbf{E}(s)$ is an analytic function in r and its derivative is defined as

$$\frac{d}{ds}\det\mathbf{E}(s) = \sum_{j=1}^{N} \det\big(\mathbf{E}_{\bullet 1}(s), \ldots, {\mathbf{E}_{\bullet j}}^{(1)}(s), \ldots, \mathbf{E}_{\bullet N}(s)\big).$$

Since r is a simple eigenvalue of $\mathbf{E}(s)$, by the definition of the multiplicity of a root of an analytic function, it holds that $\frac{d}{ds}\det\mathbf{E}(s)\,|_{s=r} \neq 0$ (see [20]). In addition, the function $\sum_{j=1}^{N} \det\big(\mathbf{E}_{\bullet 1}(s), \ldots, \mathbf{K}_{\bullet j}(s), \ldots, \mathbf{E}_{\bullet N}(s)\big)$ is also analytic in the neighborhood of r. The result is then immediate from Theorem A.3. □

According to Corollary A.6, the eigenvalue r_ϵ of the matrix $\mathbf{E}(s)+\epsilon\mathbf{K}(s)$ is simple. Consequently, the dimension of the nullspace of the matrix $\mathbf{E}(r_\epsilon)+\epsilon\mathbf{K}(r_\epsilon)$ is equal to one. We apply Theorem A.1 to find the eigenvectors of the matrix $\mathbf{E}(s) + \epsilon\mathbf{K}(s)$, that correspond to its eigenvalue r_ϵ. Before that though, we do the following simplification. From Eqs. (34)–(35) we have the Taylor expansion

$$\mathbf{E}(s) + \epsilon\mathbf{K}(s) = \sum_{n=0}^{\infty} \frac{(s-r)^n}{n!}\Big(\mathbf{E}^{(n)} + \epsilon\mathbf{K}^{(n)}\Big).$$

Evaluating this at the point $r_\epsilon = r - \epsilon\delta + O(\epsilon^2)$, we obtain

$$\mathbf{E}(r_\epsilon) + \epsilon\mathbf{K}(r_\epsilon) = \mathbf{E} - \epsilon\delta\mathbf{E}^{(1)} + \epsilon\mathbf{K} + O(\epsilon^2\mathbf{U}) = \mathbf{E} + \epsilon\big(\mathbf{K} - \delta\mathbf{E}^{(1)}\big) + O(\epsilon^2\mathbf{U}),$$

where we denote by $\mathbf{U}$ the matrix with all its elements equal to one.

Theorem A.7. *A right eigenvector of matrix* $\mathbf{E}+\epsilon\big(\mathbf{K}-\delta\mathbf{E}^{(1)}\big)$ *that corresponds to its eigenvalue* r_ϵ *is*

$$\mathbf{w}=\mathbf{t}-\epsilon\delta\mathbf{t}^{(1)}+\epsilon\mathbf{k}+O(\epsilon^2\mathbf{e}),$$

where $\mathbf{t}$ *is a right eigenvector of* $\mathbf{E}$ *defined as in Corollary A.2 and* $\mathbf{t}^{(1)}$ *is its derivative. Moreover,* $\mathbf{k}$ *is an* $N\times 1$ *vector with coordinates*

$$k_j=(-1)^{m+j}\sum_{k=1}^{N-1}\det\left(\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet 1},\dots,\left(\mathbf{K}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet k},\dots,\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet N-1}\right),\qquad j\in\mathcal{N},$$

where the choice of $m\in\mathcal{N}$ *is explained in Corollary A.2.*

Proof. According to Remark 4 and Corollary A.2, there exists an $m\in\mathcal{N}$ such that the vector $\mathbf{t}$ with coordinates

$$t_j=(-1)^{m+j}\det\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}},\qquad j\in\mathcal{N},$$

is a right eigenvector of matrix $\mathbf{E}$. We prove that a right eigenvector that corresponds to the matrix $\mathbf{E}+\epsilon\big(\mathbf{K}-\delta\mathbf{E}^{(1)}\big)$ is $\mathbf{w}$ with coordinates

$$w_j=(-1)^{m+j}\det\big(\mathbf{E}+\epsilon\big(\mathbf{K}-\delta\mathbf{E}^{(1)}\big)\big)^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}},\qquad j\in\mathcal{N}.$$

Using Proposition A.5, the above equation simplifies to

$$\begin{aligned}
w_j&=(-1)^{m+j}\det\Big(\mathbf{E}+\epsilon\big(\mathbf{K}-\delta\mathbf{E}^{(1)}\big)\Big)^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\\
&=(-1)^{m+j}\det\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}+\epsilon(-1)^{m+j}\sum_{k=1}^{N-1}\det\left(\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet 1},\dots,\left(\big(\mathbf{K}-\delta\mathbf{E}^{(1)}\big)^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet k},\dots,\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet N-1}\right)\\
&=(-1)^{m+j}\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}-\epsilon(-1)^{m+j}\delta\sum_{k=1}^{N-1}\det\left(\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet 1},\dots,\dots,\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet N-1}\right)\\
&\quad+\ \epsilon(-1)^{m+j}\sum_{k=1}^{N-1}\det\left(\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet 1},\dots,\left(\mathbf{K}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet k},\dots,\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet N-1}\right)\\
&=t_j-\epsilon\delta t_j^{(1)}+\epsilon k_j,
\end{aligned}$$

where $t_j^{(1)}=\frac{d}{ds}t_j(s)\big|_{s=r}$ and
$k_j=(-1)^{m+j}\sum_{k=1}^{N-1}\det\left(\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet 1},\dots,\left(\mathbf{K}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet k},\dots,\left(\mathbf{E}^{\mathcal{N}\setminus\{j\}}_{\mathcal{N}\setminus\{m\}}\right)_{\bullet N-1}\right)$.

Observe that $\mathbf{t}$ is not identically equal to zero, because it is an eigenvector of $\mathbf{E}$. Thus, the vector $\mathbf{w}$ is also not identically equal to zero. Therefore, according to Remark 4, $\mathbf{w}$ is an eigenvector of the matrix $\mathbf{E}+\epsilon\big(\mathbf{K}-\delta\mathbf{E}^{(1)}\big)$, which completes the proof. □

B. Proofs

of Theorem 2.3. To prove the theorem, we need formulas that result from the properties of the determinants. We define the sets $F_i=\{1,\dots,i\}$ and $L_i=\{i,\dots,N\}$, where $F_0=L_{N+1}=\emptyset$. Expansion by minors along the first row and the additive property of determinants give for $i\in\mathcal{N}$,

$$\det\mathbf{E}(s)^{L_i}_{L_i}=\widetilde{G}(s)\lambda_i\det\Big(\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)^{\{i\}}_{L_i},\mathbf{E}(s)^{L_{i+1}}_{L_i}\Big)+\lambda_i\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)^{\{i\}}_{L_i},\mathbf{E}(s)^{L_{i+1}}_{L_i}\Big)$$

$$+ (s-\lambda_i)\det \mathbf{E}(s)_{L_{i+1}}^{L_{i+1}}.$$

Suppose now that $V = \{i_1, \ldots, i_n\}$ and $W = \{j_1, \ldots, j_k\}$ are two non-overlapping ($V \cap W = \emptyset$) collections of n and k elements from $\mathcal{N}$, respectively, with $1 \leq n+k \leq N-1$. Furthermore, we choose j such that $j > \max\{l : l \in V \cup W\}$. Then, the determinant of the $(N+1-j+n+k)$-dimension square matrix $\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{W}, \mathbf{E}(s)_{V\cup W\cup L_j}^{L_j}\Big)$ satisfies,

$$\begin{aligned}
&\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{W}, \mathbf{E}(s)_{V\cup W\cup L_j}^{L_j}\Big)\\
&=\widetilde{G}(s)\lambda_j \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{W\cup\{j\}}, \mathbf{E}(s)_{V\cup W\cup L_j}^{L_{j+1}}\Big)\\
&\quad + \lambda_j \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{V\cup\{j\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{V\cup W\cup L_j}^{W}, \mathbf{E}(s)_{V\cup W\cup L_j}^{L_{j+1}}\Big)\\
&\quad + (s-\lambda_j)\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{V\cup W\cup L_{j+1}}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{V\cup W\cup L_{j+1}}^{W}, \mathbf{E}(s)_{V\cup W\cup L_{j+1}}^{L_{j+1}}\Big).
\end{aligned}$$

Note that $\det \mathbf{E}(s) = \det \mathbf{E}(s)_{L_1}^{L_1}$. The theorem is proven by applying recursively the above formulas. $\square$

of Theorem 2.4. It is known that

$$\mathcal{E}_{ij}(s) = (-1)^{i+j}\det \mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{\mathcal{N}\backslash\{i\}}.$$

The case $i=j$ is merely an application of Theorem 2.3, where instead of state space $\mathcal{N}$ we have $\mathcal{N} \setminus \{i\}$. Therefore,

$$\begin{aligned}
\mathcal{E}_{ii}(s) = &\sum_{S\subset\mathcal{N}\backslash\{i\}} \boldsymbol{\lambda}^S \zeta^{S^c}(s)\det\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_S^S\\
&+\sum_{k=1}^{N-1}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\backslash\{i\}\\|\Gamma|=k}}\ \sum_{\substack{S\subset\mathcal{N}\backslash\{i\}\\S\supset\Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_S^{S\backslash\Gamma} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_S^{\Gamma}\Big).
\end{aligned}$$

When $i \neq j$, we need to separate the two cases $i<j$ and $i>j$. We first deal with the case $i<j$. We then have,

$$\begin{aligned}
\mathcal{E}_{ij}(s) =&(-1)^{i+j}\widetilde{G}(s)\lambda_j \det\Big(\mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{F_{j-1}\backslash\{i\}}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\mathcal{N}\backslash\{j\}}^{\{j\}}, \mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{L_{j+1}}\Big)\\
&+(-1)^{i+j}\lambda_j \det\Big(\mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{F_{j-1}\backslash\{i\}}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\mathcal{N}\backslash\{j\}}^{\{j\}}, \mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{L_{j+1}}\Big).
\end{aligned}$$

We find $\mathcal{E}_{ij}(s)$ by expanding the determinants that appear above by minors along their first row. For this reason, it is important to know what is the position of the elements $\mathbf{E}_{n,n}(s) = \widetilde{G}_{nn}(s)p_{nn}\lambda_n + s - \lambda_n$, $n \in \mathcal{N} \setminus \{i,j\}$, in the above reduced matrix. Note that the elements $\mathbf{E}_{n,n}(s)$ with $n = i+1, \ldots, j-1$, are on the diagonal of matrix $\mathbf{E}(s)$. However, when $j \neq i+1$ they drop to the lower-diagonal of the matrices $\Big(\mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{F_{j-1}\backslash\{i\}}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\mathcal{N}\backslash\{j\}}^{\{j\}}, \mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{L_{j+1}}\Big)$ and $\Big(\mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{F_{j-1}\backslash\{i\}}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\mathcal{N}\backslash\{j\}}^{\{j\}}, \mathbf{E}(s)_{\mathcal{N}\backslash\{j\}}^{L_{j+1}}\Big)$

It is immediately obvious that if this displacement takes place, it will result in a change of sign for the determinants. For this reason, we split the columns of the latter matrices in the subsets F_{i-1}, T, $\{j\}$ and L_{j+1}, where $T = \{i+1, \ldots, j-1\}$. We fix some $m \in \mathcal{N} \setminus \{i,j\}$ and we separate the following cases:

1. $m \in F_{i-1}$. For every two non-overlapping collections of n and k elements from F_{m-1}, say $V = \{i_1, \dots, i_n\}$ and $W = \{j_1, \dots, j_k\}$, with $1 \le n + k \le m - 1$, it holds that

$$\begin{aligned}
&\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{(L_m \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&= \widetilde{G}(s)\lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W \cup \{m\}}, \mathbf{E}(s)_{\Omega}^{(L_{m+1} \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&+ \lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V \cup \{m\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{(L_{m+1} \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&+ (s - \lambda_m) \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{W}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{(L_{m+1} \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{\{j\}}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{j+1}}\Big),
\end{aligned}$$

and,

$$\begin{aligned}
&\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{(L_m \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&= \widetilde{G}(s)\lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W \cup \{m\}}, \mathbf{E}(s)_{\Omega}^{(L_{m+1} \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&+ \lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V \cup \{m\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{(L_{m+1} \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&+ (s - \lambda_m) \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{W}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{(L_{m+1} \cap F_{i-1}) \cup T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{\{j\}}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{j+1}}\Big),
\end{aligned}$$

where $\Omega = V \cup W \cup (L_m \cap F_{i-1}) \cup \{i\} \cup T \cup L_{j+1}$.

2. $m \in T$ with $T \neq \emptyset$ (note that $T \neq \emptyset$ when $j \neq i + 1$). For every two non-overlapping collections of n and k elements from $F_{m-1} \setminus \{i\}$, say $V = \{i_1, \dots, i_n\}$ and $W = \{j_1, \dots, j_k\}$, with $1 \le n + k \le m - 2$, it holds that

$$\begin{aligned}
&\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{L_m \cap T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&= \widetilde{G}(s)\lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W \cup \{m\}}, \mathbf{E}(s)_{\Omega}^{L_{m+1} \cap T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&+ \lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V \cup \{m\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{L_{m+1} \cap T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&- (s - \lambda_m) \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{W}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{m+1} \cap T}, \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{\{j\}}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{j+1}}\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{L_m \cap T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&= \widetilde{G}(s)\lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W \cup \{m\}}, \mathbf{E}(s)_{\Omega}^{L_{m+1} \cap T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&+ \lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V \cup \{m\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{L_{m+1} \cap T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{\{j\}}, \mathbf{E}(s)_{\Omega}^{L_{j+1}}\Big) \\
&- (s - \lambda_m) \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{W}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{m+1} \cap T}, \big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{\{j\}}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{j+1}}\Big),
\end{aligned}$$

where $\Omega = V \cup W \cup \{i\} \cup (L_m \cap T) \cup L_{j+1}$.

3. $m \in L_{j+1}$. For every two non-overlapping collections of n and k elements from $F_{m-1} \setminus \{i\}$, say $V = \{i_1, \dots, i_n\}$ and $W = \{j_1, \dots, j_k\}$, with $1 \le n + k \le m - 2$, it holds that

$$\begin{aligned}
&\det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{L_m}\Big) = \widetilde{G}(s)\lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W \cup \{m\}}, \mathbf{E}(s)_{\Omega}^{L_{m+1}}\Big) \\
&+ \lambda_m \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega}^{V \cup \{m\}} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega}^{W}, \mathbf{E}(s)_{\Omega}^{L_{m+1}}\Big) \\
&+ (s - \lambda_m) \det\Big(\big(\mathbf{Q}^{(1)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{V} \bowtie \big(\mathbf{Q}^{(2)} \circ \mathbf{P}\big)_{\Omega \setminus \{m\}}^{W}, \mathbf{E}(s)_{\Omega \setminus \{m\}}^{L_{m+1}}\Big),
\end{aligned}$$

where $\Omega = V \cup W \cup L_m$.

Using the above formulas to evaluate all the involved determinants, we find that

$$\begin{aligned}\mathcal{E}_{ij}(s) =&(-1)^{i+j}\widetilde{G}(s)\sum_{k=0}^{N-2}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i,j\}\\|\Gamma|=k;\\S\subset\mathcal{N}\setminus\{i,j\}\\S\supset\Gamma;\\R\subset S\cap T}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{j\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{S\setminus\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{\Gamma\cup\{j\}}\right)\\&+(-1)^{i+j}\sum_{k=0}^{N-2}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i,j\}\\|\Gamma|=k;\\S\subset\mathcal{N}\setminus\{i,j\}\\S\supset\Gamma;\\R\subset S\cap T}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{j\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{(S\setminus\Gamma)\cup\{j\}}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{\Gamma}\right),\end{aligned}$$

which holds even when $T=\emptyset$.

We assume now that $i>j$, and we have to calculate

$$\begin{aligned}\mathcal{E}_{ij}(s) =&(-1)^{i+j}\widetilde{G}(s)\lambda_j\det\left(\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{F_{j-1}},\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{\mathcal{N}\setminus\{j\}}^{\{j\}},\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{L_{j+1}\setminus\{i\}}\right)\\&+(-1)^{i+j}\lambda_j\det\left(\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{F_{j-1}},\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{\mathcal{N}\setminus\{j\}}^{\{j\}},\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{L_{j+1}\setminus\{i\}}\right).\end{aligned}$$

In this case, $T=\{j+1,\ldots,i-1\}$. When $T\neq\emptyset$, the elements $\mathbf{E}_{n,n}(s)=\widetilde{G}_{nn}(s)p_{nn}\lambda_n+s-\lambda_n$, with $n=j+1,\ldots,i-1$, which are on the diagonal of matrix $\mathbf{E}(s)$, move to the upper-diagonal of the matrices $\left(\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{F_{j-1}},\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{\mathcal{N}\setminus\{j\}}^{\{j\}},\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{L_{j+1}\setminus\{i\}}\right)$ and $\left(\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{F_{j-1}},\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{\mathcal{N}\setminus\{j\}}^{\{j\}},\mathbf{E}(s)_{\mathcal{N}\setminus\{j\}}^{L_{j+1}\setminus\{i\}}\right)$.

The formula is exactly the same, with $T=\{i+1,\ldots,j-1\}$. Thus, gathering all the above, for $i\neq j$

$$\begin{aligned}\mathcal{E}_{ij}(s) =&(-1)^{i+j}\sum_{k=1}^{N-1}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i,j\}\\|\Gamma|=k-1;\\S\subset\mathcal{N}\setminus\{i,j\}\\S\supset\Gamma;\\R\subset S\cap T_{ij}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{j\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{S\setminus\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{\Gamma\cup\{j\}}\right)\\&+(-1)^{i+j}\sum_{k=0}^{N-2}\widetilde{G}^k(s)\sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i,j\}\\|\Gamma|=k;\\S\subset\mathcal{N}\setminus\{i,j\}\\S\supset\Gamma;\\R\subset S\cap T_{ij}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{j\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{(S\setminus\Gamma)\cup\{j\}}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{i\}}^{\Gamma}\right),\end{aligned}$$

where $m_{ij}=\min\{i,j\}$, $M_{ij}=\max\{i,j\}$ and $T_{ij}=\{m_{ij}+1,\ldots,M_{ij}-1\}$. □

of Theorem 2.5. Observe that

$$s\mathbf{u}\mathcal{E}(s)\mathbf{e}_i =s\sum_{l=1}^{N}u_l\mathcal{E}_{li}(s)=s\sum_{\substack{l=1\\l\neq i}}^{N}u_l\mathcal{E}_{li}(s)+su_i\mathcal{E}_{ii}(s).$$

Using the definition of $\mathcal{E}_{ij}(s)$, $\forall i,j\in\mathcal{N}$, and Theorem 2.4, the result is straightforward. □

of Proposition 3.1. In this case, the determinant $\det\mathbf{E}(s)$ (see Theorem 2.3) takes the form

$$\det\mathbf{E}(s)=\sum_{S\subset\mathcal{N}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_S^S$$

$$+\sum_{k=1}^{N}\left(\frac{q(s)}{p(s)}\right)^{k}\sum_{\substack{\Gamma\subset\mathcal{N}\\|\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\\S\supset\Gamma}}\boldsymbol{\lambda}^{S}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S}^{S\backslash\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S}^{\Gamma}\right),\tag{39}$$

and the numerator of $\widetilde{w}(s)$ (see Eq. (16) and Theorem 2.5) becomes

$$\begin{aligned}
s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega} &= s\sum_{i=1}^{N}u_i\omega_i\sum_{k=0}^{N-1}\left(\frac{q(s)}{p(s)}\right)^{k}\sum_{\substack{\Gamma\subset\mathcal{N}\backslash\{i\}\\|\Gamma|=k;\\S\subset\mathcal{N}\backslash\{i\}\\S\supset\Gamma}}\boldsymbol{\lambda}^{S}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S}^{S\backslash\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S}^{\Gamma}\right)\\
&+s\sum_{i=1}^{N}\omega_i\sum_{\substack{l=1\\l\neq i}}^{N}u_l(-1)^{l+i}\sum_{k=1}^{N-1}\left(\frac{q(s)}{p(s)}\right)^{k}\sum_{\substack{\Gamma\subset\mathcal{N}\backslash\{l,i\}\\|\Gamma|=k-1;\\S\subset\mathcal{N}\backslash\{l,i\}\\S\supset\Gamma;\\R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{S\backslash\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\right)\\
&+s\sum_{i=1}^{N}\omega_i\sum_{\substack{l=1\\l\neq i}}^{N}u_l(-1)^{l+i}\sum_{k=0}^{N-2}\left(\frac{q(s)}{p(s)}\right)^{k}\sum_{\substack{\Gamma\subset\mathcal{N}\backslash\{l,i\}\\|\Gamma|=k;\\S\subset\mathcal{N}\backslash\{l,i\}\\S\supset\Gamma;\\R\subset S\cap T_{li}}}(-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{(S\backslash\Gamma)\cup\{i\}}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S\cup\{l\}}^{\Gamma}\right).
\end{aligned}\tag{40}$$

Observe that both the denominator (39) and the numerator (40) of $\widetilde{w}(s)$ are rational functions with denominators the polynomial $p(s)$ raised to some power. To simplify as much as possible the expression of $\widetilde{w}(s)$, we multiply (39) and (40) with $\big(p(s)\big)^r$, where $r\in\mathcal{N}$ is the highest possible power of $p(s)$ that is involved in the formulas. It is immediately obvious that $r\leq K$. Therefore, we multiply both (39) and (40) with $\big(p(s)\big)^r$

When multiplied with $\big(p(s)\big)^r$, the denominator of $\widetilde{w}(s)$ becomes

$$\begin{aligned}
\big(p(s)\big)^r\det\mathbf{E}(s) &= \big(p(s)\big)^r\sum_{S\subset\mathcal{N}}\boldsymbol{\lambda}^{S}\zeta^{S^c}(s)\det\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S}^{S}\\
&+\sum_{k=1}^{N}\big(q(s)\big)^k\big(p(s)\big)^{r-k}\sum_{\substack{\Gamma\subset\mathcal{N}\\|\Gamma|=k}}\sum_{\substack{S\subset\mathcal{N}\\S\supset\Gamma}}\boldsymbol{\lambda}^{S}\zeta^{S^c}(s)\det\left(\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S}^{S\backslash\Gamma}\bowtie\left(\mathbf{Q}^{(2)}\circ\mathbf{P}\right)_{S}^{\Gamma}\right).
\end{aligned}\tag{41}$$

The term $\big(p(s)\big)^r\sum_{S\subset\mathcal{N}}\boldsymbol{\lambda}^{S}\zeta^{S^c}(s)\det\left(\mathbf{Q}^{(1)}\circ\mathbf{P}\right)_{S}^{S}$ is a polynomial of degree $rM+N$. The coefficient of s^{rM+N} is found when we set $S=\emptyset$, and it is equal to 1. Let now n be the degree of the polynomial $q(s)$. Therefore, the second term of the right hand side of (41) is a polynomial of degree at most $n+(r-1)M+N-1$ (the highest order of s is found when $|S|=1$). Since $n\leq M-1$, it is immediately obvious that $\big(p(s)\big)^r\det\mathbf{E}(s)$ is a polynomial of degree $N+rM$, thus it has exactly $N+rM$ roots. From Theorem 2.2, we know that exactly $N-1$ of its roots have positive real part and that zero is also a root. We denote these roots as $s_1=0$, and $s_2,\ldots,s_N$, and we assume them to be simple. We denote the remaining rM roots with negative real part as $-x_j$, $j=1,\ldots,rM$. Consequently, the denominator of $\widetilde{w}(s)$ is written as

$$\big(p(s)\big)^r\det\mathbf{E}(s)=s\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+x_j).\tag{42}$$

Similarly, the numerator of $\widetilde{w}(s)$ becomes

$$\begin{aligned}
\big(p(s)\big)^r s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega} =& s\sum_{i=1}^{N} u_i\omega_i \big(p(s)\big)^r \sum_{S\subset\mathcal{N}\setminus\{i\}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^S \\
&+ s\sum_{i=1}^{N} u_i\omega_i \sum_{k=1}^{N-1} \big(q(s)\big)^k \big(p(s)\big)^{r-k} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\ |\Gamma|=k;\\ S\subset\mathcal{N}\setminus\{i\}\\ S\supset\Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big) \\
&+ s\sum_{i=1}^{N}\omega_i \sum_{\substack{l=1\\ l\neq i}}^{N} u_l(-1)^{l+i} \sum_{k=1}^{N-1} \big(q(s)\big)^k \big(p(s)\big)^{r-k} \\
&\times \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k-1;\\ S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}} (-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s) \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big) \\
&+ s\sum_{i=1}^{N}\omega_i \sum_{\substack{l=1\\ l\neq i}}^{N} u_l(-1)^{l+i} \sum_{k=0}^{N-2} \big(q(s)\big)^k \big(p(s)\big)^{r-k} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k;\\ S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}} (-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s) \\
&\times \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big). \qquad (43)
\end{aligned}$$

It is easy to verify that $\big(p(s)\big)^r s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega}$ is also a polynomial of degree $rM+N$. The coefficient of s^{rM+N} is equal to $\mathbf{u}\boldsymbol{\omega}$ and it is determined by the term $s\sum_{i=1}^{N} u_i\omega_i\big(p(s)\big)^r \sum_{S\subset\mathcal{N}\setminus\{i\}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)$ $\det\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^S$ for $S=\emptyset$. We know from Theorem 2.2, that the vector $\mathbf{u}$ is such that the numbers s_k, $k\in\mathcal{N}$, are also roots of the numerator of $\widetilde{w}(s)$. We denote the rest rM roots of the numerator as $-y_j$, $j=1,\dots,rM$. Therefore, the numerator of $\widetilde{w}(s)$ is written as

$$\big(p(s)\big)^r s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega} = \mathbf{u}\boldsymbol{\omega} s\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+y_j). \qquad (44)$$

Combining (42) and (44), the result is immediate. □

of Theorem 3.2. Since $\mathbf{K}(0)$ is an $N\times N$ zero matrix, it is evident that $s_{\epsilon,1}=0$ is an eigenvalue of the matrix $\mathbf{H}_\epsilon(s)+s\mathbf{I}-\boldsymbol{\Lambda}$ (see Eq. (28)). According to Corollary A.6, the numbers $s_{\epsilon,i}$, $i=2,\dots,N$, are also simple eigenvalues of this matrix. Thus, according to Theorem 2.2, there are no other roots of the equation $\det\big(\mathbf{E}(s)+\epsilon\mathbf{K}(s)\big)=0$ with non-negative real part besides the values $s_{\epsilon,i}$, $i\in\mathcal{N}$.

For the second part of proof we have the following. Using Theorem A.7, we can evaluate $N-1$ column vectors $\mathbf{w}_{\epsilon,i}$ such that

$$\big(\mathbf{H}_\epsilon(s_{\epsilon,i})+s_{\epsilon,i}\mathbf{I}-\boldsymbol{\Lambda}\big)\mathbf{w}_{\epsilon,i}=0, \qquad i=2,\dots,N.$$

Since $s_{\epsilon,i}\neq 0$, $i=2,\dots,N$, post-multiplying equation (26) with $s=s_{\epsilon,i}$ by $\mathbf{w}_{\epsilon,i}$, we obtain

$$\mathbf{u}_\epsilon\mathbf{w}_{\epsilon,i}=0, \qquad i=2,\dots,N.$$

To derive the remaining equation, we take the derivative of equation (26) with respect to s, yielding

$$\widetilde{\mathbf{\Phi}}_\epsilon(s)\big(\mathbf{H}_\epsilon{}^{(1)}(s)+\mathbf{I}\big)+\widetilde{\mathbf{\Phi}}_\epsilon^{(1)}(s)\big(\mathbf{H}_\epsilon(s)+s\mathbf{I}-\mathbf{\Lambda}\big)=\mathbf{u}_\epsilon.$$

Setting $s=0$ we get

$$\widetilde{\mathbf{\Phi}}_\epsilon(0)\big(\mathbf{H}_\epsilon{}^{(1)}(0)+\mathbf{I}\big)+\widetilde{\mathbf{\Phi}}_\epsilon^{(1)}(0)\big(\mathbf{P}-\mathbf{I}\big)\mathbf{\Lambda}=\mathbf{u}_\epsilon.$$

Post-multiplying by $\mathbf{\Lambda}^{-1}\mathbf{e}$ gives

$$\widetilde{\mathbf{\Phi}}_\epsilon(0)\big(\mathbf{H}_\epsilon{}^{(1)}(0)+\mathbf{I}\big)\mathbf{\Lambda}^{-1}\mathbf{e}+\widetilde{\mathbf{\Phi}}_\epsilon^{(1)}(0)\big(\mathbf{P}-\mathbf{I}\big)\mathbf{\Lambda}\mathbf{\Lambda}^{-1}\mathbf{e}=\mathbf{u}_\epsilon\mathbf{\Lambda}^{-1}\mathbf{e}.$$

Finally, using $(\mathbf{P}-\mathbf{I})\mathbf{e}=0$, $\mathbf{H}_\epsilon{}^{(1)}(0)=-\mathbf{M}\mathbf{\Lambda}+\epsilon\big(\mu_p-\mu_h\big)\mathbf{Q}^{(2)}\circ\mathbf{P}\mathbf{\Lambda}$ and $\widetilde{\mathbf{\Phi}}_\epsilon(0)=\boldsymbol{\pi}$ (where the latter follows from (26) with $s=0$ and the normalization equation (27)), the above can be simplified to

$$\boldsymbol{\pi}\left(\mathbf{\Lambda}^{-1}-\mathbf{M}\right)\mathbf{e}+\epsilon(\mu_p-\mu_h)\boldsymbol{\pi}\mathbf{Q}^{(2)}\circ\mathbf{P}\mathbf{e}=\mathbf{u}_\epsilon\mathbf{\Lambda}^{-1}\mathbf{e}.$$

The uniqueness of the solution follows from the general theory of Markov chains that under the condition of stability, there is a unique stationary distribution and thus also a unique solution $\widetilde{\mathbf{\Phi}}_\epsilon(s)$ to the equations (26) and (27). This completes the proof. □

of Proposition 3.6. Recall that r is the maximum power of $p(s)$ that appears in the formulas. Therefore, to use perturbation analysis, we multiply both $\det\mathbf{E}_\epsilon(s)$ and $s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\boldsymbol{\omega}$ with $\big(p(s)\big)^r$. So, if we set

$$\xi_{rM+N-1}(s)=\sum_{k=1}^{N}k\big(q(s)\big)^{k-1}\big(p(s)\big)^{r-k+1}\sum_{\substack{\Gamma\subset\mathcal{N}\\|\Gamma|=k;\\S\subset\mathcal{N}\\S\supset\Gamma}}\boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\backslash\Gamma}\bowtie\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big),\quad(45)$$

then,

$$\big(p(s)\big)^r\det\mathbf{E}_\epsilon(s)=\big(p(s)\big)^r\det\mathbf{E}(s)+\epsilon s\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\xi_{rM+N-1}(s)+O(\epsilon^2).$$

Note that the polynomial $\xi_{rM+N-1}(s)$ is of degree at most $rM+N-1$, and the coefficient of s^{rM+N-1} is equal to $\gamma=\sum_{i=1}^N\lambda_i\det\big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{\{i\}}^{\{i\}}=\sum_{i=1}^N\lambda_iq_{ii}^{(2)}p_{ii}$. Theorem 3.2 guarantees that the function $\big(p(s)\big)^r\det\mathbf{E}_\epsilon(s)$ has exactly $N-1$ roots with positive real part and it also has $s_{\epsilon,1}=0$. The roots with positive real part are of the form $s_{\epsilon,k}=s_k-\epsilon\delta_k+O(\epsilon^2)$, $k=2,\ldots,N$, where

$$\delta_k=\frac{\big(\mu_p\widetilde{F}_p^e(s_k)-\mu_h\widetilde{F}_h^e(s_k)\big)\xi_{rM+N-1}(s_k)}{\prod_{\substack{l=2\\l\neq k}}^N(s_k-s_l)\prod_{j=1}^{rM}(s_k+x_j)}=\frac{\big(\mu_p\widetilde{F}_p^e(s_k)-\mu_h\widetilde{F}_h^e(s_k)\big)\xi_{rM+N-1}(s_k)\widetilde{w}(s_k)}{\mathbf{u}\boldsymbol{\omega}\prod_{\substack{l=2\\l\neq k}}^N(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)}.\quad(46)$$

Thus, if we set

$$d(s)=\frac{\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\xi_{rM+N-1}(s)\widetilde{w}(s)}{\mathbf{u}\boldsymbol{\omega}\prod_{k=2}^N(s-s_k)\prod_{j=1}^{rM}(s+y_j)}-\sum_{k=2}^N\frac{\delta_k}{s-s_k},\quad(47)$$

the denominator of $\widetilde{w}_\epsilon(s)$ multiplied by $\big(p(s)\big)^r$ can be written as

$$\big(p(s)\big)^r\det\mathbf{E}_\epsilon(s)=s\prod_{j=1}^{rM}(s+x_j)\prod_{k=2}^N\big(s-s_k+\epsilon\delta_k+O(\epsilon^2)\big)\big(1+\epsilon d(s)+O(\epsilon^2)\big).\quad(48)$$

Note that the function $d(s)$ is well defined in the positive half plane due to the definition (46) of δ_k, $k=2,\dots,N$. Similarly, if we set

$$\xi_{i,l,rM+N-2}(s) = \mathbb{1}_{\{l=i\}} \sum_{k=1}^{N-1} k\big(q(s)\big)^{k-1}\big(p(s)\big)^{r-k+1} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\ |\Gamma|=k;\\ S\subset\mathcal{N}\setminus\{i\}\\ S\supset\Gamma}} \boldsymbol{\lambda}^S \zeta^{S^c}(s) \det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big)$$
$$+\mathbb{1}_{\{l\neq i\}}\Bigg[(-1)^{l+i}\sum_{k=1}^{N-1} k\big(q(s)\big)^{k-1}\big(p(s)\big)^{r-k+1} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k-1}} \sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}} (-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)$$
$$\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big)$$
$$+(-1)^{l+i}\sum_{k=1}^{N-2} k\big(q(s)\big)^{k-1}\big(p(s)\big)^{r-k+1} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k}} \sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}} (-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)$$
$$\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big)\Bigg], \tag{49}$$

and

$$\xi'_{i,l,rM+N-1}(s) = \mathbb{1}_{\{l=i\}}\Bigg[\big(p(s)\big)^r \sum_{S\subset\mathcal{N}\setminus\{i\}} \boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^S$$
$$+\sum_{k=1}^{N-1}\big(q(s)\big)^k\big(p(s)\big)^{r-k} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{i\}\\ |\Gamma|=k}} \sum_{\substack{S\subset\mathcal{N}\setminus\{i\}\\ S\supset\Gamma}} \boldsymbol{\lambda}^S\zeta^{S^c}(s)\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_S^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_S^{\Gamma}\Big)\Bigg]$$
$$+\mathbb{1}_{\{l\neq i\}}\Bigg[(-1)^{l+i}\sum_{k=1}^{N-1}\big(q(s)\big)^k\big(p(s)\big)^{r-k} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k-1}} \sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}} (-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)$$
$$\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{S\setminus\Gamma} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma\cup\{i\}}\Big)$$
$$+(-1)^{l+i}\sum_{k=0}^{N-2}\big(q(s)\big)^k\big(p(s)\big)^{r-k} \sum_{\substack{\Gamma\subset\mathcal{N}\setminus\{l,i\}\\ |\Gamma|=k}} \sum_{\substack{S\subset\mathcal{N}\setminus\{l,i\}\\ S\supset\Gamma;\\ R\subset S\cap T_{li}}} (-1)^{|R|}\boldsymbol{\lambda}^{S\cup\{i\}}\zeta^{S^c}(s)$$
$$\times\det\Big(\big(\mathbf{Q}^{(1)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{(S\setminus\Gamma)\cup\{i\}} \bowtie \big(\mathbf{Q}^{(2)}\circ\mathbf{P}\big)_{S\cup\{l\}}^{\Gamma}\Big)\Bigg], \tag{50}$$

then

$$\big(p(s)\big)^r s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\boldsymbol{\omega} = \big(p(s)\big)^r s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega} + \epsilon s\Bigg[\sum_{i=1}^N \omega_i \sum_{l=1}^N z_l \xi'_{i,l,rM+N-1}(s)$$
$$+ s\big(\mu_p\widetilde{F}_p^e(s) - \mu_h\widetilde{F}_h^e(s)\big)\sum_{l=1}^N \omega_i \sum_{l=1}^N u_l\xi_{i,l,rM+N-2}(s)\Bigg] + O(\epsilon^2).$$

Note that the polynomial $\sum_{l=1}^{N}\omega_i\sum_{l=1}^{N}z_l\xi'_{i,l,rM+N-1}(s)$ is of degree $rM+N-1$, and the coefficient of s^{rM+N-1} is $\mathbf{z}\boldsymbol{\omega}$. Analogously, the polynomial $s\sum_{l=1}^{N}\omega_i\sum_{l=1}^{N}u_l\times\xi_{i,l,rM+N-2}(s)$ is of degree at most $rM+N-1$, and the coefficient of s^{rM+N-1} is equal to $\beta=\sum_{l=1}^{N}\omega_i\sum_{l=1}^{N}u_l \times\Big(\mathbb{1}_{\{l=i\}}\sum_{\substack{j=1\\j\neq i}}^{N}\lambda_j q_{jj}^{(2)}p_{jj}+\mathbb{1}_{\{l\neq i\}}(-1)^{l+i}\lambda_i q_{li}^{(2)}p_{li}\Big)$. The first part is for $S=\Gamma=\{j\}$, and the second part for $S=\Gamma=\emptyset$. Theorem 3.2 guarantees that the roots $s_{\epsilon,k}$, $k\in\mathcal{N}$, are also roots of the numerator of $\widetilde{w}_\epsilon(s)$. Therefore, applying perturbation analysis to $\big(p(s)\big)^r s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\boldsymbol{\omega}$ results in an equivalent definition for each δ_k, $k=2,\ldots,N$, as

$$\delta_k=\frac{\big(\mu_p\widetilde{F}_p^e(s_k)-\mu_h\widetilde{F}_h^e(s_k)\big)s_k\sum_{i,l=1}^{N}\omega_i u_l\xi_{i,l,rM+N-2}(s_k)}{\mathbf{u}\boldsymbol{\omega}\prod_{\substack{l=2\\l\neq k}}^{N}(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)}+\frac{\sum_{i=1}^{N}\omega_i\sum_{l=1}^{N}z_l\xi'_{i,l,rM+N-1}(s_k)}{\mathbf{u}\boldsymbol{\omega}\prod_{\substack{l=2\\l\neq k}}^{N}(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)}. \tag{51}$$

Now, if we set

$$n(s)=\frac{\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)s\sum_{i,l=1}^{N}\omega_i u_l\xi_{i,l,rM+N-2}(s)}{\mathbf{u}\boldsymbol{\omega}\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+y_j)}+\frac{\sum_{i=1}^{N}\omega_i\sum_{l=1}^{N}z_l\xi'_{i,l,rM+N-1}(s)}{\mathbf{u}\boldsymbol{\omega}\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+y_j)}-\sum_{k=2}^{N}\frac{\delta_k}{s-s_k}, \tag{52}$$

the numerator of $\widetilde{w}_\epsilon(s)$ multiplied by $\big(p(s)\big)^r$ can be written as

$$\big(p(s)\big)^r s\mathbf{u}_\epsilon\mathcal{E}_\epsilon(s)\boldsymbol{\omega}=\mathbf{u}\boldsymbol{\omega}s\prod_{j=1}^{rM}(s+y_j)\prod_{k=2}^{N}\big(s-s_k+\epsilon\delta_k+O(\epsilon^2)\big)\times\big(1+\epsilon n(s)+O(\epsilon^2)\big). \tag{53}$$

Note that the function $n(s)$ is well defined in the positive half plane due to the definition (51) of δ_k, $k=2,\ldots,N$. Combining (48) and (53), we obtain

$$\begin{aligned}
\widetilde{w}_\epsilon(s)=&\frac{\mathbf{u}\boldsymbol{\omega}\prod_{j=1}^{rM}(s+y_j)}{\prod_{j=1}^{rM}(s+x_j)}\cdot\frac{1+\epsilon n(s)+O(\epsilon^2)}{1+\epsilon d(s)+O(\epsilon^2)}=\widetilde{w}(s)\big(1+\epsilon n(s)+O(\epsilon^2)\big)\big(1-\epsilon d(s)+O(\epsilon^2)\big)\\
=&\widetilde{w}(s)+\epsilon\widetilde{w}(s)\big(n(s)-d(s)\big)+O(\epsilon^2)\\
=&\widetilde{w}(s)+\epsilon\frac{1}{\mathbf{u}\boldsymbol{\omega}}\widetilde{w}(s)\Bigg(\frac{\sum_{i=1}^{N}\omega_i\sum_{l=1}^{N}z_l\xi'_{i,l,rM+N-1}(s)}{\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+y_j)}+\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\frac{s\sum_{i=1}^{N}\omega_i\sum_{l=1}^{N}u_l\xi_{i,l,rM+N-2}(s)}{\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+y_j)}\\
&-\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\widetilde{w}(s)\frac{\xi_{rM+N-1}(s)}{\prod_{k=2}^{N}(s-s_k)\prod_{j=1}^{rM}(s+y_j)}\Bigg)+O(\epsilon^2)\\
=&\widetilde{w}(s)+\epsilon\frac{1}{\mathbf{u}\boldsymbol{\omega}}\widetilde{w}(s)\Bigg[\Bigg(\mathbf{z}\boldsymbol{\omega}+\sum_{k=2}^{N}\frac{\alpha_k}{s-s_k}+\sum_{j=1}^{rM}\frac{\alpha'_j\cdot y_j}{s+y_j}\Bigg)+\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\Bigg(\beta+\sum_{k=2}^{N}\frac{\beta_k}{s-s_k}+\sum_{j=1}^{rM}\frac{\beta'_j\cdot y_j}{s+y_j}\Bigg)\\
&-\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\widetilde{w}(s)\Bigg(\gamma+\sum_{k=2}^{N}\frac{\gamma_k}{s-s_k}+\sum_{j=1}^{rM}\frac{\gamma'_j\cdot y_j}{s+y_j}\Bigg)\Bigg]+O(\epsilon^2),
\end{aligned} \tag{54}$$

where the last equality comes from simple fraction decomposition under the assumption that the roots $-y_j$, $j=1,\ldots,rM$, are simple. The coefficients $\alpha_k,\beta_k,\gamma_k$, $k=2,\ldots,N$, and $\alpha'_j,\beta'_j,\gamma'_j$, $j=1,\ldots,rM$, are as follows

$$\alpha_k=\frac{\sum_{i=1}^{N}\omega_i\sum_{l=1}^{N}z_l\xi'_{i,l,rM+N-1}(s_k)}{\prod_{\substack{l=2\\l\neq k}}^{N}(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)}, \tag{55}$$

$$\beta_k = \frac{s_k \sum_{i=1}^N \omega_i \sum_{l=1}^N u_l \xi_{i,l,rM+N-2}(s_k)}{\prod_{\substack{l=2\\l\neq k}}^N (s_k - s_l) \prod_{j=1}^{rM}(s_k + y_j)}, \tag{56}$$

$$\gamma_k = \frac{\xi_{rM+N-1}(s_k)}{\prod_{\substack{l=2\\l\neq k}}^N (s_k - s_l) \prod_{j=1}^{rM}(s_k + y_j)}, \tag{57}$$

$$\alpha'_j = \frac{\sum_{i=1}^N \omega_i \sum_{l=1}^N z_l \xi'_{i,l,rM+N-1}(-y_j)}{y_{i,j} \prod_{k=2}^N (-y_j - s_k) \prod_{\substack{l=1\\l\neq j}}^{rM}(-y_j + y_l)}, \tag{58}$$

$$\beta'_j = \frac{-\sum_{i=1}^N \omega_i \sum_{l=1}^N u_l \xi_{i,l,rM+N-2}(-y_j)}{\prod_{k=2}^N (-y_j - s_k) \prod_{\substack{l=1\\l\neq j}}^{rM}(-y_j + y_l)}, \tag{59}$$

$$\gamma'_j = \frac{\xi_{rM+N-1}(-y_j)}{y_j \prod_{k=2}^N (-y_j - s_k) \prod_{\substack{l=1\\l\neq j}}^{rM}(-y_j + y_l)}. \tag{60}$$

The above results hold when all roots $-y_j$, $j = 1, \ldots, rM$, are simple. Suppose now that only σ of the roots are distinct and that the multiplicity of root $-y_j$, $j = 1, \ldots, \sigma$, is r_j, such that $\sum_{j=1}^{\sigma} r_j = rM$. In this case,

$$\begin{aligned}\widetilde{w}_\epsilon(s) = \widetilde{w}(s) + \epsilon \frac{1}{\mathbf{u}\boldsymbol{\omega}} \widetilde{w}(s) \Bigg[&\left(\mathbf{z}\boldsymbol{\omega} + \sum_{k=2}^N \frac{\alpha_k}{s - s_k} + \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \frac{\alpha''_{j,l} \cdot (y_j)^{r_j - l + 1}}{(s + y_j)^{r_j - l + 1}} \right) \\ &+ \left(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\right) \left(\beta + \sum_{k=2}^N \frac{\beta_k}{s - s_k} + \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \frac{\beta''_{j,l} \cdot (y_j)^{r_j - l + 1}}{(s + y_{i,j})^{r_j - l + 1}} \right) \\ &- \left(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s)\right) \widetilde{w}(s) \left(\gamma + \sum_{k=2}^N \frac{\gamma_k}{s - s_k} + \sum_{j=1}^{\sigma} \sum_{l=1}^{r_j} \frac{\gamma''_{j,l} \cdot (y_j)^{r_j - l + 1}}{(s + y_j)^{r_j - l + 1}} \right) \Bigg] + O(\epsilon^2), \end{aligned} \tag{61}$$

where α_k, β_k and γ_k, $k = 2, \ldots, N$, are defined through (55)–(57). For each $j = 1, \ldots, \sigma$, the coefficients $\alpha''_{j,p}$, $p = 1, \ldots, r_j$, are the unique solution to the following linear system of r_j equations

$$\frac{d}{ds^n} \left[\sum_{i=1}^N \omega_i \sum_{l=1}^N z_l \xi'_{i,l,rM+N-1}(s) \right] \Bigg|_{s=-y_j} = \frac{d}{ds^n} \left[\prod_{k=2}^N (s - s_k) \prod_{\substack{l=1\\l\neq j}}^{\sigma} (s + y_l)^{r_l} \sum_{p=1}^{r_j} \alpha''_{j,p} \, (y_j)^{r_j - p + 1} (s + y_j)^{p-1} \right] \Bigg|_{s=-y_j}, \tag{62}$$

for $n = 0, \ldots, r_j$. Similarly, for each $j = 1, \ldots, \sigma$, the coefficients $\beta''_{j,p}$ and $\gamma''_{j,p}$, $p = 1, \ldots, r_j$, are the respective unique solutions to the following two linear system of r_j equations

$$\frac{d}{ds^n} \left[s \sum_{i=1}^N \omega_i \sum_{l=1}^N u_l \xi_{i,l,rM+N-2}(s) \right] \Bigg|_{s=-y_j} = \frac{d}{ds^n} \left[\prod_{k=2}^N (s - s_k) \prod_{\substack{l=1\\l\neq j}}^{\sigma} (s + y_l)^{r_l} \sum_{p=1}^{r_j} \beta''_{j,p} \, (y_j)^{r_j - p + 1} (s + y_j)^{p-1} \right] \Bigg|_{s=-y_j}, \tag{63}$$

$$\frac{d}{ds^n} \left[\xi_{rM+N-1}(s) \right] \Bigg|_{s=-y_j} = \frac{d}{ds^n} \left[\prod_{k=2}^N (s - s_k) \prod_{\substack{l=1\\l\neq j}}^{\sigma} (s + y_l)^{r_l} \sum_{p=1}^{r_j} \gamma''_{j,p} \, (y_j)^{r_j - p + 1} (s + y_j)^{p-1} \right] \Bigg|_{s=-y_j}, \tag{64}$$

for $n = 0, \ldots, r_j$. □

of Theorem 3.7. Here, we follow the notation we introduced in Proposition 3.6. We denote by $\widetilde{\theta}(s)$ the correction term (the coefficient of ϵ) in the expression of $\widetilde{w}_\epsilon(s)$. In order to apply Laplace inversion to $\widetilde{\theta}(s)$, we first reorder the involved terms (see Eq. (61)) as

$$\begin{aligned}\widetilde{\theta}(s) =&\frac{1}{\mathbf{u}\boldsymbol{\omega}}\widetilde{w}(s)\Bigg[\Bigg(\mathbf{z}\boldsymbol{\omega}+\beta\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)-\gamma\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\widetilde{w}(s)\Bigg)\\&+\sum_{k=2}^{N}\frac{1}{s-s_k}\Bigg(\alpha_k+\beta_k\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)-\gamma_k\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\widetilde{w}(s)\Bigg)\\&+\sum_{j=1}^{\sigma}\sum_{l=1}^{r_j}\frac{1}{(s+y_j)^{r_j-l+1}}\Bigg(\alpha''_{j,l}+\beta''_{j,l}\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)-\gamma''_{j,l}\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\widetilde{w}(s)\Bigg)\Bigg].\end{aligned}\tag{65}$$

From the above formula it is evident that only the terms in the middle bracket cannot be inverted directly as they are, because of the singularities they seem to have in the positive half plane. Thus, we treat them separately in the next lines. From the two equivalent definitions (46) and (51) of the perturbation terms δ_k, $k=2,\dots,N$, and the relations (55)–(57) we obtain that

$$\alpha_k\widetilde{w}(s_k)+\beta_k\big(\mu_p\widetilde{F}_p^e(s_k)-\mu_h\widetilde{F}_h^e(s_k)\big)\widetilde{w}(s_k)-\gamma_k\big(\mu_p\widetilde{F}_p^e(s_k)-\mu_h\widetilde{F}_h^e(s_k)\big)\big(\widetilde{w}(s_k)\big)^2=0,\quad k=2,\dots,N.$$

The above equations are equivalent to

$$\begin{aligned}0=&\alpha_k\int_{x=0}^{\infty}e^{-s_kx}d\mathbb{P}(W\le x)+\beta_{i,k}\Big(\mu_p\int_{x=0}^{\infty}e^{-s_kx}d\mathbb{P}(W+B^e\le x)-\mu_h\int_{x=0}^{\infty}e^{-s_kx}d\mathbb{P}(W+C^e\le x)\Big)\\&-\gamma_k\Big(\mu_p\int_{x=0}^{\infty}e^{-s_kx}d\mathbb{P}(W+W'+B^e\le x)-\mu_h\int_{x=0}^{\infty}e^{-s_kx}d\mathbb{P}(W+W'+C^e\le x)\Big),\end{aligned}\tag{66}$$

$k=2,\dots,N$. We first show that

$$\begin{aligned}&\mathcal{L}^{-1}\Bigg(\sum_{k=2}^{N}\frac{1}{s-s_k}\Big(\alpha_k\widetilde{w}(s)+\beta_k\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\widetilde{w}(s)-\gamma_k\big(\mu_p\widetilde{F}_p^e(s)-\mu_h\widetilde{F}_h^e(s)\big)\big(\widetilde{w}(s)\big)^2\Big)\Bigg)\\&=\sum_{k=2}^{N}\Bigg[\gamma_k\Big(\mu_p\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+W'+B^e\le y)-\mu_h\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+W'+C^e\le y)\Big)\\&\quad-\beta_k\Bigg(\mu_p\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+B^e\le y)-\mu_h\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+C^e\le y)\Bigg)-\alpha_k\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W\le y)\Bigg].\end{aligned}\tag{67}$$

Since Laplace transforms turn convolutions of functions into their product, using the property $\int_{y=0}^{\infty}f(y)dy=\int_{y=0}^{x}f(y)dy+\int_{y=x}^{\infty}f(y)dy$ and the relations (66) we obtain

$$\begin{aligned}&\mathcal{L}\Bigg\{\sum_{k=2}^{N}\Bigg[\gamma_k\Bigg(\mu_p\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+W'+B^e\le y)-\mu_h\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+W'+C^e\le y)\Bigg)\\&\quad-\beta_k\Bigg(\mu_p\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+B^e\le y)-\mu_h\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W+C^e\le y)\Bigg)-\alpha_k\int_{y=x}^{\infty}e^{s_k(x-y)}d\mathbb{P}(W\le y)\Bigg]\Bigg\}\\&=\mathcal{L}\Bigg\{\sum_{k=2}^{N}\Bigg[-\gamma_k\Bigg(\mu_p\int_{y=0}^{x}e^{s_k(x-y)}d\mathbb{P}(W+W'+B^e\le y)-\mu_h\int_{y=0}^{x}e^{s_k(x-y)}d\mathbb{P}(W+W'+C^e\le y)\Bigg)\\&\quad+\beta_k\Bigg(\mu_p\int_{y=0}^{x}e^{s_k(x-y)}d\mathbb{P}(W+B^e\le y)-\mu_h\int_{y=0}^{x}e^{s_k(x-y)}d\mathbb{P}(W+C^e\le y)\Bigg)\Bigg]+\alpha_k\int_{y=0}^{x}e^{s_k(x-y)}d\mathbb{P}(W\le y)\Bigg\}\end{aligned}$$

$$= \sum_{k=2}^{N} \frac{1}{s - s_k} \Big(\alpha_k \widetilde{w}(s) + \beta_k \big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s) \big) \widetilde{w}(s) - \gamma_k \big(\mu_p \widetilde{F}_p^e(s) - \mu_h \widetilde{F}_h^e(s) \big) \big(\widetilde{w}(s) \big)^2 \Big) \Bigg),$$

which proves (67).

To find the tail probabilities that correspond to the terms in the middle bracket of (65), we integrate the inverted Laplace transform in Eq. (67) from t to ∞, and we obtain

$$\begin{aligned}
&\sum_{k=2}^{N} \Bigg[\gamma_k \Bigg(\mu_p \int_{x=t}^{\infty} \int_{y=x}^{\infty} e^{s_k(x-y)} d\mathbb{P}(W + W' + B^e \leq y) dx - \mu_h \int_{x=t}^{\infty} \int_{y=x}^{\infty} e^{s_k(x-y)} d\mathbb{P}(W + W' + C^e \leq y) dx \Bigg) \\
&\quad - \beta_k \Bigg(\mu_p \int_{x=t}^{\infty} \int_{y=x}^{\infty} e^{s_k(x-y)} d\mathbb{P}(W + B^e \leq y) dx - \mu_h \int_{x=t}^{\infty} \int_{y=x}^{\infty} e^{s_k(x-y)} d\mathbb{P}(W + C^e \leq y) dx \Bigg) \\
&\quad - \alpha_k \int_{x=t}^{\infty} \int_{y=x}^{\infty} e^{s_k(x-y)} d\mathbb{P}(W \leq y) dx \Bigg] \\
&= \sum_{k=2}^{N} \Bigg[\gamma_k \Bigg(\mu_p \int_{y=t}^{\infty} e^{-s_k y} d\mathbb{P}(W + W' + B^e \leq y) \int_{x=t}^{y} e^{s_k x} dx - \mu_h \int_{y=t}^{\infty} e^{-s_k y} d\mathbb{P}(W + W' + C^e \leq y) \int_{x=t}^{y} e^{s_k x} dx \Bigg) \\
&\quad - \beta_k \Bigg(\mu_p \int_{y=t}^{\infty} e^{-s_k y} d\mathbb{P}(W + B^e \leq y) \int_{x=t}^{y} e^{s_k x} dx - \mu_h \int_{y=t}^{\infty} e^{-s_k y} d\mathbb{P}(W + C^e \leq y) \int_{x=t}^{y} e^{s_k x} dx \Bigg) \\
&\quad - \alpha_k \int_{y=t}^{\infty} e^{-s_k y} d\mathbb{P}(W \leq y) \int_{x=t}^{y} e^{s_k x} dx \Bigg] \\
&= \sum_{k=2}^{N} \Bigg[\frac{\gamma_k}{s_k} \Bigg(\mu_p \int_{y=t}^{\infty} d\mathbb{P}(W + W' + B^e \leq y) - \mu_h \int_{y=t}^{\infty} d\mathbb{P}(W + W' + C^e \leq y) \Bigg) \\
&\quad - \frac{\beta_k}{s_k} \Bigg(\mu_p \int_{y=t}^{\infty} d\mathbb{P}(W + B^e \leq y) - \mu_h \int_{y=t}^{\infty} d\mathbb{P}(W + C^e \leq y) \Bigg) - \frac{\alpha_k}{s_k} \int_{y=t}^{\infty} d\mathbb{P}(W \leq y) \\
&\quad - \frac{\gamma_k}{s_k} \Bigg(\mu_p \int_{y=t}^{\infty} e^{-s_k(y-t)} d\mathbb{P}(W + W' + B^e \leq y) - \mu_h \int_{y=t}^{\infty} e^{-s_k(y-t)} d\mathbb{P}(W + W' + C^e \leq y) \Bigg) \\
&\quad + \frac{\beta_k}{s_k} \Bigg(\mu_p \int_{y=t}^{\infty} e^{-s_k(y-t)} d\mathbb{P}(W + B^e \leq y) - \mu_h \int_{y=t}^{\infty} e^{-s_k(y-t)} d\mathbb{P}(W + C^e \leq y) \Bigg) \\
&\quad + \frac{\alpha_k}{s_k} \int_{y=t}^{\infty} e^{-s_k(y-t)} d\mathbb{P}(W \leq y) \Bigg] \\
&= \sum_{k=2}^{N} \frac{1}{s_k} \Bigg[-\gamma_k \Bigg(\mu_p \mathbb{P}\big(t < W + W' + B^e < t + E(s_k)\big) - \mu_h \mathbb{P}\big(t < W + W' + C^e < t + E(s_k)\big) \Bigg) \\
&\quad + \beta_k \Bigg(\mu_p \mathbb{P}\big(t < W + B^e < t + E(s_k)\big) - \mu_h \mathbb{P}\big(t < W + C^e < t + E(s_k)\big) \Bigg) + \alpha_k \mathbb{P}\big(t < W < t + E(s_k)\big) \Bigg] \\
&+ \sum_{k=2}^{N} \frac{1}{s_k} \Bigg[\gamma_k \Bigg(\mu_p \mathbb{P}(W + W' + B^e > t) - \mu_h \mathbb{P}(W + W' + C^e > t) \Bigg) - \beta_k \Bigg(\mu_p \mathbb{P}(W + B^e > t) - \mu_h \mathbb{P}(W + C^e > t) \Bigg) \\
&\quad - \alpha_k \mathbb{P}(W > t) \Bigg]. \qquad (68)
\end{aligned}$$

By using now the property $\mathcal{L}^{-1}\{\frac{a^{n+1}}{(s+a)^{n+1}}\} = \frac{1}{n!} a^{n+1} t^n \times e^{-at}$, $t \geq 0$, of the inverse Laplace transform, we see that the terms $\frac{(y_j)^{r_j - l + 1}}{(s + y_j)^{r_j - l + 1}}$ in Eq. (65) correspond to the Laplace transform of an $E_{r_j-l+1}(y_j)$ r.v. Combining all the above, the result in immediate, which completes the proof of the theorem. □

of Proposition 3.8. In Proposition 3.6, we found that

$$\widetilde{w}_\epsilon(s) = \widetilde{w}(s) + \epsilon\widetilde{\theta}(s) + O(\epsilon^2),$$

where $\widetilde{\theta}(s)$ is the Laplace-Stieltjes transform of the signed measure $\Theta(t)$ introduced in Proposition 3.7. The above equation implies that

$$\frac{\widetilde{w}_\epsilon(s) - \widetilde{w}(s)}{\epsilon} = \widetilde{\theta}(s) + o(1). \tag{69}$$

We set $n = \frac{1}{\epsilon}$ and we define the sequence of functions

$$\widetilde{v}_n(s) := \frac{1}{\epsilon}\big(\widetilde{w}_\epsilon(s) - \widetilde{w}(s)\big),$$

where $\widetilde{v}_n(s)$ is the Laplace-Stieltjes transform of the measure $V_n(t) = \frac{1}{\epsilon}\big(\mathbb{P}(W_\epsilon > t) - \mathbb{P}(W > t)\big)$. By using (69), we obtain that $\widetilde{v}_n(s) \to \widetilde{\theta}(s)$, for all $s > 0$ as $n \to \infty$ (or equivalently $\epsilon \to 0$). Thus, it follows from the Extended Continuity Theorem (see Theorem XIII.2 [16]) that $\frac{\mathbb{P}(W_\epsilon>t)-\mathbb{P}(W>t)}{\epsilon} \to \Theta(t)$, which completes the proof. □

of Theorem 4.5. The steps are exactly the same as in Proposition 3.6, but with different parameters that are in accordance to the discard base model. We first write the denominator and the numerator of $\widetilde{w}^\bullet_\epsilon(s)$ multiplied by $\big(p(s)\big)^r$ as perturbation of the respective quantities in the replace base model, and we have that

$$\big(p(s)\big)^r \det \mathbf{E}^\bullet_\epsilon(s) = \big(p(s)\big)^r \det \mathbf{E}(s) + \epsilon s\mu_p\widetilde{F}^e_p(s)\xi_{rM+N-1}(s) + O(\epsilon^2),$$

and,

$$\begin{aligned}\big(p(s)\big)^r s\mathbf{u}^\bullet_\epsilon\mathcal{E}^\bullet_\epsilon(s)\boldsymbol{\omega} =& \big(p(s)\big)^r s\mathbf{u}\mathcal{E}(s)\boldsymbol{\omega}\\ &+\epsilon s\left[\sum_{i=1}^N\sum_{l=1}^N z_l\omega_i\xi'_{i,l,rM+N-1}(s) + s\mu_p\widetilde{F}^e_p(s)\sum_{i=1}^N\sum_{l=1}^N u_l\omega_i\xi_{i,l,rM+N-2}(s)\right] + O(\epsilon^2),\end{aligned}$$

where the polynomials $\xi_{rM+N-1}(s)$, $\xi_{i,l,rM+N-2}(s)$, and $\xi'_{i,l,rM+N-1}(s)$ are defined according to the formulas (45), (49), and (50), respectively, and r is the maximum power of $p(s)$ that appears in the formulas. The $N-1$ common roots of the numerator and the denominator of $\widetilde{w}^\bullet_\epsilon(s)$ with positive real part are of the form $s^\bullet_{\epsilon,k} = s_k - \epsilon\delta^\bullet_k + O(\epsilon^2)$, $k = 2,\ldots,N$, where the two equivalent definitions of $\delta^\bullet_k$ are as follows

$$\begin{aligned}\delta^\bullet_k =& \frac{\mu_p\widetilde{F}^e_p(s_k)\xi_{rM+N-1}(s_k)\widetilde{w}(s_k)}{\mathbf{u}\boldsymbol{\omega}\prod_{\substack{l=2\\l\neq k}}^N(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)}\\ =& \frac{\mu_p\widetilde{F}^e_p(s_k)s_k\sum_{i=1}^N\sum_{l=1}^N u_l\omega_i\xi_{i,l,rM+N-2}(s_k)}{\mathbf{u}\boldsymbol{\omega}\prod_{\substack{l=2\\l\neq k}}^N(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)} + \frac{\sum_{i=1}^N\sum_{l=1}^N z_l\omega_i\xi'_{i,l,rM+N-1}(s_k)}{\mathbf{u}\boldsymbol{\omega}\prod_{\substack{l=2\\l\neq k}}^N(s_k-s_l)\prod_{j=1}^{rM}(s_k+y_j)}.\end{aligned}$$

If we set now

$$d^\bullet(s) = \frac{\mu_p\widetilde{F}^e_p(s)\xi_{rM+N-1}(s)\widetilde{w}(s)}{\mathbf{u}\boldsymbol{\omega}\prod_{k=2}^N(s-s_k)\prod_{j=1}^{rM}(s+y_j)} - \sum_{k=2}^N\frac{\delta^\bullet_k}{s-s_k},$$

and,

$$n^{\bullet}(s) = \frac{\mu_p \widetilde{F}_p^e(s) s \sum_{i=1}^N \sum_{l=1}^N u_l \omega_i \xi_{i,l,rM+N-2}(s)}{\mathbf{u}\boldsymbol{\omega} \prod_{k=2}^N (s-s_k) \prod_{j=1}^{rM}(s+y_j)} + \frac{\sum_{i=1}^N \sum_{l=1}^N z_l \omega_i \xi'_{i,l,rM+N-1}(s)}{\mathbf{u}\boldsymbol{\omega} \prod_{k=2}^N (s-s_k) \prod_{j=1}^{rM}(s+y_j)} - \sum_{k=2}^N \frac{\delta_k^{\bullet}}{s-s_k},$$

the denominator and the numerator of $\widetilde{w}_\epsilon^{\bullet}(s)$ multiplied by $\big(p(s)\big)^r$ can be written respectively as

$$\big(p(s)\big)^r \det \mathbf{E}_\epsilon^{\bullet}(s) = s \prod_{j=1}^{rM}(s+x_j) \prod_{k=2}^N \big(s - s_k + \epsilon \delta_k^{\bullet} + O(\epsilon^2)\big)\big(1 + \epsilon d^{\bullet}(s) + O(\epsilon^2)\big), \tag{70}$$

and,

$$\big(p(s)\big)^r s \mathbf{u}_\epsilon^{\bullet} \mathcal{E}_\epsilon^{\bullet}(s) \boldsymbol{\omega} = \mathbf{u}\boldsymbol{\omega} s \prod_{j=1}^{rM}(s+y_j) \prod_{k=2}^N \big(s - s_k + \epsilon \delta_k^{\bullet} + O(\epsilon^2)\big)\big(1 + \epsilon n^{\bullet}(s) + O(\epsilon^2)\big).$$

Note that both functions $d^{\bullet}(s)$ and $n^{\bullet}(s)$ are well-defined in the positive half-plane due to the definitions of $\delta_k^{\bullet}$. Combining (70) and (B) we obtain

$$\widetilde{w}_\epsilon^{\bullet}(s) = \widetilde{w}(s) \frac{1 + \epsilon n^{\bullet}(s) + O(\epsilon^2)}{1 + \epsilon d^{\bullet}(s) + O(\epsilon^2)} \quad \Rightarrow \quad \widetilde{w}(s) = \widetilde{w}_\epsilon^{\bullet}(s) \frac{1 + \epsilon d^{\bullet}(s) + O(\epsilon^2)}{1 + \epsilon n^{\bullet}(s) + O(\epsilon^2)}.$$

So,

$$\begin{aligned}
\widetilde{w}_\epsilon(s) =& \widetilde{w}(s) \frac{1 + \epsilon n(s) + O(\epsilon^2)}{1 + \epsilon d(s) + O(\epsilon^2)} = \widetilde{w}_\epsilon^{\bullet}(s) \frac{1 + \epsilon d^{\bullet}(s) + O(\epsilon^2)}{1 + \epsilon n^{\bullet}(s) + O(\epsilon^2)} \cdot \frac{1 + \epsilon n(s) + O(\epsilon^2)}{1 + \epsilon d(s) + O(\epsilon^2)} \\
=& \widetilde{w}_\epsilon^{\bullet}(s) \Big(1 + \epsilon\big((n(s) - n^{\bullet}(s)) - (d(s) - d^{\bullet}(s))\big) + O(\epsilon^2) \Big) \\
=& \widetilde{w}_\epsilon^{\bullet}(s) + \epsilon \frac{1}{\mathbf{u}\boldsymbol{\omega}} \widetilde{w}_\epsilon^{\bullet}(s) \Bigg(\frac{\sum_{i=1}^N \sum_{l=1}^N (z_l - z_l^{\bullet}) \omega_i \xi'_{i,l,rM+N-1}(s)}{\prod_{k=2}^N (s-s_k) \prod_{j=1}^{rM}(s+y_j)} - \mu_h \widetilde{F}_h^e(s) \frac{s \sum_{i=1}^N \sum_{l=1}^N u_l \omega_i \xi_{i,l,rM+N-2}(s)}{\prod_{k=2}^N (s-s_k) \prod_{j=1}^{rM}(s+y_j)} \\
&+ \mu_h \widetilde{F}_h^e(s) \widetilde{w}(s) \frac{\xi_{rM+N-1}(s)}{\prod_{k=2}^N (s-s_k) \prod_{j=1}^{rM}(s+y_j)} \Bigg) + O(\epsilon^2) \\
=& \widetilde{w}_\epsilon^{\bullet}(s) + \epsilon \frac{1}{\mathbf{u}\boldsymbol{\omega}} \widetilde{w}_\epsilon^{\bullet}(s) \Bigg[\Bigg((z_i - z_l^{\bullet}) + \sum_{k=2}^N \frac{\alpha_k^{\bullet}}{s-s_k} + \sum_{j=1}^{rM} \frac{\alpha_j^{\bullet\prime} \cdot y_j}{s+y_j} \Bigg) - \mu_h \widetilde{F}_h^e(s) \Bigg(\beta + \sum_{k=2}^N \frac{\beta_k}{s-s_k} + \sum_{j=1}^{rM} \frac{\beta'_j \cdot y_j}{s+y_j} \Bigg) \\
&+ \mu_h \widetilde{F}_h^e(s) \widetilde{w}_\epsilon^{\bullet}(s) \Bigg(\gamma + \sum_{k=2}^N \frac{\gamma_{i,k}}{s-s_k} + \sum_{j=1}^{rM} \frac{\gamma'_j \cdot y_j}{s+y_j} \Bigg) \Bigg] + O(\epsilon^2).
\end{aligned} \tag{71}$$

$\square$