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Abstract

Recently, the scaling limit of cluster sizes for critical inhomogeneous random graphs of rank-1 type having finite variance but infinite third moment degrees was obtained [7]. It was proved that when the degrees obey a power law with exponent $\tau \in (3,4)$, the sequence of clusters ordered in decreasing size and multiplied through by $n^{-(\tau-2)/(\tau-1)}$ converges as $n \to \infty$ to a sequence of decreasing non-degenerate random variables.

Here, we study the tails of the limit of the rescaled largest cluster, i.e., the probability that the scaling limit of the largest cluster takes a large value u, as a function of u. This extends a related result of Pittel [29] for the Erdős-Rényi random graph to the setting of rank-1 inhomogeneous random graphs with infinite third moment degrees. We make use of delicate large deviations and weak convergence arguments.

Key words: critical random graphs, power-law degrees, inhomogeneous networks, thinned Lévy processes, exponential tilting, large deviations
MSC2000 subject classification. 60C05, 05C80, 90B15.

1 Introduction

The Erdős-Rényi random graph $G(n,p)$ on the vertex set $[n] := \{1,\dots,n\}$ is constructed by including each of the $\binom{n}{2}$ possible edges with probability p, independently of all other edges. Erdős and Rényi discovered the double-jump phenomenon: The size of the largest component was shown to be, in probability, of order $\log n$, $n^{2/3}$, or n, depending on whether the average vertex degree was less than, close to, or more than one. In 1984 Bollobás [9] and subsequently Łuczak [27] showed for the scaling window $p = (1+\lambda n^{-1/3})/n$, that the largest component is of the order $n^{2/3}$. Since then, the critical, or near-critical behavior of random graphs has received tremendous attention (see [2, 4, 10, 18, 26]). Let $(\mathcal{C}_{(i)})_{i\geq 1}$ denote the connected components of $G(n,p)$, ordered in size, i.e., $|\mathcal{C}_{\max}| = |\mathcal{C}_{(1)}| \geq |\mathcal{C}_{(2)}| \geq \dots$ Aldous [2] proved the following result:

Theorem 1.1 (Aldous [2]). *For $p = (1+\lambda n^{-1/3})/n$, $\lambda \in \mathbb{R}$ fixed, and $n \to \infty$,*

$$\Big(|\mathcal{C}_{(1)}|n^{-2/3}, |\mathcal{C}_{(2)}|n^{-2/3}, \dots\Big) \xrightarrow{d} (\gamma_1(\lambda), \gamma_2(\lambda), \dots), \tag{1.1}$$

where $\gamma_1(\lambda) > \gamma_2(\lambda) > \dots$ are the ordered excursions of the reflected version of the process $(W_t^\lambda)_{t\geq 0} \equiv (W_t + \lambda t - t^2/2)_{t\geq 0}$ with $(W_t)_{t\geq 0}$ a standard Wiener process.

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Theorem 1.1 says that the ordered connected components in the critical Erdős-Rényi random graph are described by the ordered excursions of the reflected version of $(W_t^\lambda)_{t\geq 0}$. The strict inequalities between the scaling limits of the ordered cluster follows from the local limit theorem proved in [24]. In [29] an exact formula was derived for the distribution function of the limiting variable $\gamma_1(\lambda)$ (of the largest component) and various asymptotic results were obtained, including

$$\mathbb{P}(\gamma_1(\lambda) > u) = \frac{1}{\sqrt{2\pi} u^{3/2}} \mathrm{e}^{-\frac{1}{8}u(u-2\lambda)^2}(1+o(1)), \quad u \to \infty. \tag{1.2}$$

The result in (1.2) gives sharp asymptotics for the largest component in the critical Erdős-Rényi graph. It was rederived in [23] by studying the excursions of the scaling limit of the exploration process that is used to describe the limits in Theorem 1.1. In this paper, we follow a similar path, but then for a class of inhomogeneous random graphs and its scaling limit, and extend (1.2) to this setting.

Several recent works have studied inhomogeneity in random graphs and how it changes the critical nature. In our model, the vertices have a weight associated to them, and the weight of a vertex moderates its degree. Therefore, by choosing these weights appropriately, we can generate random graphs with highly variable degrees. For our class of random graphs, it is shown in [22, Theorem 1.1] that when the weights do not vary too much, the critical behavior is similar to the one in the Erdős-Rényi random graph. See in particular the recent works [6, 30], where it was shown that if the degrees have finite *third* moment, then the scaling limit for the largest critical components in the critical window are essentially the same (up to a trivial rescaling) as for the Erdős-Rényi random graph in Theorem 1.1.

When the degrees have *infinite* third moment, instead, it was shown in [22, Theorem 1.2] that the sizes of the largest critical clusters are quite different. In [7] scaling limits were obtained for the sizes of the largest components at criticality for rank-1 inhomogeneous random graphs with power-law degrees with power-law exponent $\tau \in (3,4)$. For $\tau \in (3,4)$, the degrees have finite variance but infinite third moment. It was shown that the sizes of the largest components, rescaled by $n^{-(\tau-2)/(\tau-1)}$, converge to hitting times of a *thinned Lévy process*. The latter is a special case of the general multiplicative coalescents studied by Aldous and Limic in [2] and [3]. We next discuss these results in more detail.

1.1 Inhomogeneous random graphs

In our random graph model, vertices have weights, and the edges are independent, with edge probabilities being approximately equal to the rescaled product of the weights of the two end vertices of the edge. While there are many different versions of such random graphs (see below), it will be convenient for us to work with the so-called Poissonian random graph or Norros-Reittu model [28]. To define the model, we consider the vertex set $[n] := \{1, 2, \ldots, n\}$ and suppose each vertex is assigned a weight, vertex i having weight w_i. Now, attach an edge between vertices i and j with probability

$$p_{ij} = 1 - \exp\Big(-\frac{w_i w_j}{\ell_n}\Big), \quad \text{where} \quad \ell_n = \sum_{i\in[n]} w_i. \tag{1.3}$$

Different edges are independent. In this model, the average degree of vertex i is close to w_i, thus incorporating inhomogeneity in the model.

There are many adaptations of this model, for which equivalent results hold. Indeed, the model considered here is a special case of the so-called *rank-1 inhomogeneous random graph* introduced in great generality by Bollobás, Janson and Riordan [11]. It is asymptotically equivalent with many related models, such as the *random graph with given prescribed degrees* or Chung-Lu model, where instead

$$p_{ij} = \max(w_i w_j/\ell_n, 1), \tag{1.4}$$

and which has been studied intensively by Chung and Lu (see [13, 14, 15, 16, 17]). A further adaptation is the *generalized random graph* introduced by Britton, Deijfen and Martin-Löf in [12], for which

$$p_{ij} = \frac{w_i w_j}{\ell_n + w_i w_j}. \tag{1.5}$$

See Janson [25] for conditions under which these random graphs are *asymptotically equivalent*, meaning that all events have asymptotically equal probabilities. As discussed in more detail in [22, Section 1.3], these conditions apply in the setting to be studied in this paper. Therefore, all results proved here also hold for these related rank-1 models.

Let the weight sequence $\boldsymbol{w} = (w_i)_{i\in[n]}$ be defined by

$$w_i = [1-F]^{-1}(i/n), \tag{1.6}$$

where F is a distribution function on $[0,\infty)$ for which we assume that there exists a $\tau \in (3,4)$ and $0 < c_F < \infty$ such that

$$\lim_{x\to\infty} x^{\tau-1}[1-F(x)] = c_F, \tag{1.7}$$

and where $[1-F]^{-1}$ is the generalized inverse function of $1-F$ defined, for $u \in (0,1)$, by

$$[1-F]^{-1}(u) = \inf\{s\colon [1-F](s) \leq u\}. \tag{1.8}$$

By convention, we set $[1-F]^{-1}(1) = 0$.

For the setting in (1.3) and (1.6), by [11, Theorem 3.13], the number of vertices with degree k, which we denote by N_k, satisfies

$$N_k/n \xrightarrow{\mathbb{P}} \mathbb{E}\Big[\mathrm{e}^{-W}\frac{W^k}{k!}\Big], \qquad k \geq 0, \tag{1.9}$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability, and where W has distribution function F appearing in (1.6). We recognize the limiting distribution as a so-called *mixed Poisson distribution with mixing distribution* F, i.e., conditionally on $W = w$, the distribution is Poisson with mean w. As discussed in more detail in [22], since a Poisson random variable with large parameter w is closely concentrated around its mean w, the tail behavior of the degrees in our random graph is close to that of the distribution F. As a result, when (1.7) holds, and with D_n the degree of a uniformly chosen vertex in $[n]$, $\limsup_{n\to\infty}\mathbb{E}[D_n^a] < \infty$ when $a < \tau-1$ and $\limsup_{n\to\infty}\mathbb{E}[D_n^a] = \infty$ when $a \geq \tau-1$. In particular, the degree of a uniformly chosen vertex in $[n]$ has finite second, but infinite third moment when (1.7) holds with $\tau \in (3,4)$.

Under the key assumption in (1.7),

$$[1-F]^{-1}(u) = \big(c_F/u\big)^{1/(\tau-1)}(1+o(1)), \quad u \downarrow 0, \tag{1.10}$$

and the third moment of the degrees tends to infinity, i.e., with $W \sim F$, we have $\mathbb{E}[W^3] = \infty$. Define

$$\nu = \mathbb{E}[W^2]/\mathbb{E}[W], \tag{1.11}$$

so that, again by (1.7), $\nu < \infty$. Then, by [11, Theorem 3.1] (see also [11, Section 16.4] for a detailed discussion on rank-1 inhomogeneous random graphs, of which our random graph is an example), when $\nu > 1$, there is one giant component of size proportional to n, while all other components are of smaller size $o(n)$, and when $\nu \leq 1$, the largest connected component contains a proportion of vertices that converges to zero in probability. Thus, the critical value of the model is $\nu = 1$. The main goal of this paper is to investigate what happens close to the critical point, i.e., when $\nu = 1$.

With the definition of the weights in (1.6), we shall write $\mathcal{G}_n^0(\boldsymbol{w})$ for the graph constructed with the probabilities in (1.3), while, for any fixed $\lambda \in \mathbb{R}$, we shall write $\mathcal{G}_n^\lambda(\boldsymbol{w})$ when we use the weight sequence

$$\boldsymbol{w}(\lambda) = (1+\lambda n^{-(\tau-3)/(\tau-1)})\boldsymbol{w}. \tag{1.12}$$

We shall assume that n is so large that $1+\lambda n^{-(\tau-3)/(\tau-1)} \geq 0$, so that $w_i(\lambda) \geq 0$ for all $i \in [n]$. When $\tau > 4$, so that $\mathbb{E}[W^3] < \infty$, it was shown in [6, 22, 30] that the scaling limit of the random graphs studied here are (apart from a trivial scaling constant) *equal* to the scaling limit of the ordered connected components in the Erdős-Rényi random graph in Theorem 1.1. When $\tau \in (3,4)$ the situation is entirely different, as discussed next.

Throughout this paper, we make use of the following standard notation. We let $\xrightarrow{d}$ denote convergence in distribution, and $\xrightarrow{\mathbb{P}}$ convergence in probability. For a sequence of random variables $(X_n)_{n\geq 1}$, we write $X_n = o_{\mathbb{P}}(b_n)$ when $|X_n|/b_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. For a non-negative function $n \mapsto g(n)$, we write $f(n) = O(g(n))$ when $|f(n)|/g(n)$ is uniformly bounded, and $f(n) = o(g(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$. Furthermore, we write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. Finally, we abbreviate

$$\alpha = 1/(\tau-1), \qquad \rho = (\tau-2)/(\tau-1), \qquad \eta = (\tau-3)/(\tau-1). \tag{1.13}$$

1.2 The scaling limit for $\tau \in (3,4)$

We next recall two key results that we recently established in [7].

Theorem 1.2 (Weak convergence of the ordered critical clusters for $\tau \in (3,4)$ [7])**.** *Fix the Norros-Reittu random graph with weights $\boldsymbol{w}(\lambda)$ defined in (1.6) and (1.12). Assume that $\nu = 1$ and that (1.7) holds. Then, for all $\lambda \in \mathbb{R}$,*

$$\big(|\mathcal{C}_{(1)}|n^{-\rho}, |\mathcal{C}_{(2)}|n^{-\rho}, \dots\big) \xrightarrow{d} (\gamma_1(\lambda), \gamma_2(\lambda), \dots), \tag{1.14}$$

in the product topology, for some non-degenerate limit $(\gamma_i(\lambda))_{i\geq 1}$.

In order to further specify the scaling limit $(\gamma_i(\lambda))_{i\geq 1}$, we need to introduce a non-negative continuous-time process $(\mathcal{S}_t)_{t\geq 0}$, referred to as a *thinned Lévy process*, and defined as

$$\mathcal{S}_t = b - abt + ct + \sum_{i=2}^{\infty} \frac{b}{i^\alpha}\Big[\mathcal{I}_i(t) - \frac{at}{i^\alpha}\Big], \tag{1.15}$$

where a, b, c have been identified in [7, Theorem 2.4] as $a = c_F^\alpha/\mathbb{E}[W]$, $b = c_F^\alpha$ and $c = \theta = \lambda + \zeta$ with $\zeta \in (-\infty, 0)$ the constant given in [7, (2.18)][1]. Further, here we use the notation

$$\mathcal{I}_i(t) = \mathbb{1}_{\{T_i \leq t\}}, \tag{1.16}$$

where $(T_i)_{i\geq 2}$ are independent exponential random variables with mean

$$\mathbb{E}[T_i] = i^\alpha/a. \tag{1.17}$$

Let $H_1(0)$ denote the first hitting time of 0 of the process $(\mathcal{S}_t)_{t\geq 0}$, i.e.,

$$H_1(0) = \inf\{t \geq 0 \colon \mathcal{S}_t = 0\}, \tag{1.18}$$

and $\mathcal{C}(1)$ the connected component to which vertex 1 (with the largest weight) belongs. We recall from [7, Theorem 2.1 and Proposition 3.7] that also $|\mathcal{C}(1)|n^{-\rho}$ converges in distribution:

Theorem 1.3 (Weak convergence of the cluster of vertex 1 for $\tau \in (3,4)$)**.** *Fix the Norros-Reittu random graph with weights $\boldsymbol{w}(\lambda)$ defined in* (1.6) and (1.12)*. Assume that $\nu = 1$ and that (1.7) holds. Then, for all $\lambda \in \mathbb{R}$,*

$$n^{-\rho}|\mathcal{C}(1)| \xrightarrow{d} H_1^a(0), \tag{1.19}$$

with $H_1^a(0)$ the hitting time of 0 of $(\mathcal{S}_t)_{t\geq 0}$ with $a = c_F^\alpha/\mathbb{E}[W]$, $b = c_F^\alpha$, $c = \theta$.

[1] There is a typo in [7, Theorem 2.4], in which $c = \theta - ab$ should read $c = \theta = \lambda + \zeta$.

By scaling, $H_1^a(0)/a$ for some a, b, c has the same distribution as the hitting time $H_1(0)$ obtained by taking $b' = a' = 1$, and $c' = c/(ab) = (\lambda + \zeta)/(ab)$. We shall reparametrize $a' = b' = 1$ and let

$$\mathcal{S}_t = 1 + \tilde{\beta} t + \sum_{i=2}^{\infty} c_i [\mathcal{I}_i(t) - c_i t], \tag{1.20}$$

where we set

$$\tilde{\beta} = \beta - 1 \qquad \text{with} \qquad \beta = c' = \theta/(ab) = (\lambda + \zeta)/(ab), \tag{1.21}$$

used the notation

$$c_i = i^{-\alpha}, \tag{1.22}$$

and where $\mathcal{I}_i(t)$ is defined in (1.16)–(1.17) with $a' = 1$.

1.3 Main results

In this section we state our three main theorems. The first theorem concerns the probability that $H_1^a(0) > u$ for some $u > 0$ large, where $H_1^a(0)$ is the weak limit of $n^{-\rho}|\mathcal{C}(1)|$ identified in Theorem 1.3. This is achieved by investigating the hitting time $H_1(0)$ of 0 of the process $(\mathcal{S}_t)_{t \geq 0}$ in (1.20).

Theorem 1.4 (Tail behavior scaling limit cluster vertex 1 for $\tau \in (3, 4)$)**.** *When $u \to \infty$, there exists $I > 0$ independent of $\tilde{\beta}$ and $A = A(\tilde{\beta})$ and $\kappa_{ij}(\tilde{\beta}) \in \mathbb{R}$ such that*

$$\mathbb{P}(H_1(0) > u) = \mathbb{P}(H_1^a(0) > au) = \frac{A}{u^{(\tau-1)/2}} \mathrm{e}^{-Iu^{\tau-1} + u^{\tau-1} \sum_{i+j \geq 1} \kappa_{ij} u^{-i(\tau-2)-j(\tau-3)}} (1 + o(1)). \tag{1.23}$$

The constants I, A and κ_{ij} are specified in Section 2. By scaling, these constants only depend on a, b through $c' = c/(ab) = (\lambda + \zeta)/(ab)$, any other dependence disappears since the law of $H_1(0)$ only depends on c'. Since $\tau \in (3, 4)$, the sum over i, j such that $i + j \geq 1$ is in fact *finite*, as we can ignore all terms for which $\tau - 1 - i(\tau - 2) - j(\tau - 3) \leq 0$.

We can even go one step further and study the optimal trajectory the process $t \mapsto \mathcal{S}_t$ takes in order to achieve the unlikely event that $H_1(0) > u$ when u is large. In order to describe this trajectory, we need to introduce some further notation. In the proof, it will be crucial to *tilt* the distribution, i.e., to investigate the measure $\tilde{\mathbb{P}}$ with Radon-Nikodym derivative $\mathrm{e}^{\theta u \mathcal{S}_u}/\mathbb{E}[\mathrm{e}^{\theta u \mathcal{S}_u}]$, for some appropriately chosen θ. The selection of an appropriate θ for the thinned Lévy process$(\mathcal{S}_t)_{t \geq 0}$ is quite subtle, and has been the main topic of our paper [1]. The main results from paper [1] are reported in Section 2, and will play an important role in the present analysis. We refer to below (2.12) for the definition of θ^* that appears in the description of the optimal trajectory that is identified in the following theorem:

Theorem 1.5 (Optimal trajectory)**.** *For $p \in [0, 1]$, define*

$$I_E(p) = (\tau - 1) \int_0^{\infty} \Big(\frac{\mathrm{e}^{\theta^* v}(1 - \mathrm{e}^{-pv})}{\mathrm{e}^{\theta^* v}(1 - \mathrm{e}^{-v}) + \mathrm{e}^{-v}} - pv \Big) \frac{dv}{v^{\tau-1}} \tag{1.24}$$

with θ^* as defined below (2.12). Then, for $u \to \infty$, for any $\varepsilon > 0$,

$$\mathbb{P}\Big(\sup_{p \in [0,1]} |\mathcal{S}_{pu} - u^{\tau-2} I_E(p)| \leq u^{\tau-2} \varepsilon \mid H_1(0) > u \Big) = 1 - o(1). \tag{1.25}$$

Combining Theorem 1.4 and Theorem 1.5, and showing that, for u large, the probability that $1 \in \mathcal{C}_{(1)}$ is overwhelmingly large, eventually leads to the following result for the largest cluster size (recall $\gamma_1(\lambda)$ from (1.14)):

Theorem 1.6 (Tail behavior scaling limit for $\tau \in (3, 4)$)**.** *When $u \to \infty$, there exists $I > 0$ independent of λ and $A = A(\lambda), \kappa_{i,j} = \kappa_{i,j}(\lambda)$ such that*

$$\mathbb{P}(\gamma_1(\lambda) > au) = \frac{A}{u^{(\tau-1)/2}} \mathrm{e}^{-Iu^{\tau-1} + u^{\tau-1} \sum_{i+j \geq 1} \kappa_{ij} u^{-i(\tau-2)-j(\tau-3)}} (1 + o(1)). \tag{1.26}$$

The constants I, A and κ_{ij} are equal to those in Theorem 1.4.

Brownian motion on a parabola. Note that substituting $\tau = 4$ into (1.26) yields $\frac{A}{u^{3/2}} \mathrm{e}^{-Iu^3+\kappa_{01}u^2+(\kappa_{10}+\kappa_{02})u}(1+o(1))$, which agrees with the result of Pittel in (1.2). This suggests a smooth transition from the case $\tau \in (3,4)$ to the case $\tau > 4$. We next further explore this relation.

Consider the process $(W_t^\lambda)_{t\geq 0} = (W_t + \lambda t - t^2/2)_{t\geq 0}$ with $(W_t)_{t\geq 0}$ a standard Wiener process as mentioned in Theorem 1.1. We now apply the technique of exponential change of measure to this process. First note that the moment generating function of W_u^λ can be computed as

$$\log \phi(u;\vartheta) \equiv \log \mathbb{E}[\mathrm{e}^{\vartheta u W_u^\lambda}] = \vartheta u(\lambda u - \tfrac{1}{2}u^2 + \tfrac{1}{2}\vartheta u^2) \tag{1.27}$$

and let θ_u^* be the solution of $\theta_u^* = \arg\min_\vartheta \log \phi(u;\vartheta)$, which is given by

$$\theta_u^* = \frac{1}{2} - \frac{\lambda}{u}. \tag{1.28}$$

The main term is

$$\phi(u) = \phi(u;\theta_u^*) = \mathbb{E}[\mathrm{e}^{\theta_u^* u W_u^\lambda}] = \mathrm{e}^{-\frac{1}{8}u^3 + \frac{1}{2}\lambda u^2 - \frac{1}{2}\lambda^2 u} = \mathrm{e}^{-\frac{1}{8}u(u-2\lambda)^2}. \tag{1.29}$$

Noting that

$$\mathbb{P}(\gamma_1(\lambda) > u) \leq \mathbb{P}(W_u^\lambda > 0) \leq \mathbb{P}(\mathrm{e}^{\theta_u^* u W_u^\lambda} > 1) \leq \mathbb{E}[\mathrm{e}^{\theta_u^* u W_u^\lambda}], \tag{1.30}$$

we see that this upper bound agrees to leading order with the result of Pittel in (1.2). In order to derive the full asymtptotics in (1.2), one can define the measure

$$\widetilde{\mathbb{P}}(E) = \phi(u)^{-1}\mathbb{E}[\mathrm{e}^{\theta_u^* u W_u^\lambda} \mathbb{1}_E], \tag{1.31}$$

rewrite

$$\mathbb{P}(\gamma_1(\lambda) > u) = \phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{\theta_u^* u W_u^\lambda} \mathbb{1}_{\{\gamma_1(\lambda)>u\}}], \tag{1.32}$$

and then deduce the asymptotics of the latter expectation in full detail. Our analysis will be based on this intuition, now applied to a more involved, so-called thinned Lévy, stochastic process.

2 Overview of the proofs

In this section, we give the overview of the proofs of Theorems 1.4-1.6. The point of departure for our proofs is the conjecture that $\mathbb{P}(H_1(0) > u) \approx \mathbb{P}(\mathcal{S}_u > 0)$ for large u. The event $\{H_1(0) > u\}$ obviously implies $\{\mathcal{S}_u > 0\}$, but because of the strong downward drift of the process $(\mathcal{S}_t)_{t\geq 0}$, it seems plausible that both events are roughly equivalent.

In [1] a detailed study was presented on the large deviations behavior of the process $(\mathcal{S}_t)_{t\geq 0}$. Using exponential tilting of measure the following two theorems were proved.

Theorem 2.1 (Exact asymptotics tail $\mathcal{S}_u$ [1, Theorem 1.1]). *There exists $I, D > 0$ and $\kappa_{ij} \in \mathbb{R}$ such that, as $u \to \infty$,*

$$\mathbb{P}(\mathcal{S}_u > 0) = \frac{D}{u^{(\tau-1)/2}} \mathrm{e}^{-Iu^{\tau-1} + u^{\tau-1}\sum_{i+j\geq 1}\kappa_{ij}u^{-i(\tau-2)-j(\tau-3)}}(1+o(1)). \tag{2.1}$$

Theorem 2.2 (Sample path large deviations [1, Theorem 1.2]). *There exists a function $p \mapsto I_E(p)$ on $[0,1]$ such that, for any $\varepsilon > 0$ and $p \in [0,1)$,*

$$\lim_{u\to\infty} \mathbb{P}\big(\big|\mathcal{S}_{pu} - u^{\tau-2}I_E(p)\big| \leq \varepsilon u^{\tau-2} \mid \mathcal{S}_u > 0\big) = 1. \tag{2.2}$$

In [1] it is explained that specific challenges arise in the identification of a tilted measure due to the power-law nature of $(\mathcal{S}_t)_{t\geq 0}$. General principles prescribe that the tilt should follow from a variational problem, but in the case of $(\mathcal{S}_t)_{t\geq 0}$ this involves a Riemann sum that is hard to control. In [1] this Riemann sum is approximated by its limiting integral, and it is proved that the tilt that follows from the corresponding approximate variational problem is sufficient to establish the large deviations results in Theorems 2.1 and 2.2. Details about this tilted measure are presented in Subsection 2.1.

It is clear that Theorems 2.1 and 2.2 for the event $\{\mathcal{S}_u > 0\}$ are the counterparts of Theorems 1.4 and 1.5 for $\{H_1(0) > u\}$. Let us now sketch how we make formal the conjecture that $\mathbb{P}(H_1(0) > u) \approx \mathbb{P}(\mathcal{S}_u > 0)$ for large u. We show that $\mathbb{P}(H_1(0) > u)$ has the same asymptotic behavior as $\mathbb{P}(\mathcal{S}_u > 0)$ in (2.1), with the same constants except for the constant D. Despite the similarity of this result, the proof method we shall use is entirely different from the exponential tilting in [1]. In order to establish the asymptotics for $\mathbb{P}(H_1(0) > u)$, we establish sample path large deviations, not conditioned on the event $\{\mathcal{S}_u > 0\}$, but on the event $\{H_1(0) > u\}$. This is much harder, since we have to investigate the probability that $\mathcal{S}_t > 0$ for *all* $t \in [0, u]$. In order to prove these strong sample-path properties, we first prove that $\mathcal{S}_t$ is close to its expected value for a finite, but large, number of t's, followed by a proof that the path cannot deviate much in the small time intervals between these times. Now here is our strategy for the proofs. We extend the conjecture $\mathbb{P}(H_1(0) > u) \approx \mathbb{P}(\mathcal{S}_u > 0)$ by a conjectured sample path behavior that says that, under the tilted measure, the typical sample path of $(\mathcal{S}_t)_{t\geq 0}$ that leads to the event $\{\mathcal{S}_u > 0\}$ remains positive and hence implies $\{H_1(0) > u\}$. To be more specific, we divide up this likely sample path into three parts: the early part, the middle part, and the end part. Our proof consists of treating each of these parts separately. We shall prove consecutively that with high probability the process:
(i) does not cross zero in the initial part of the trajectory ('no early hits');
(ii) is high up in the state space in the middle part of the trajectory, while experiencing small fluctuations, and therefore does not hit zero ('no middle ground');
(iii) is forced to remain positive until the very end.
In the last step, we have to be very careful, and it is in this step that it will turn out that the constant D arising in the asymptotics of $\mathbb{P}(\mathcal{S}_u > 0)$ in (2.1) is *different* from the constant A arising in the asymptotics of $\mathbb{P}(H_1(0) > u)$ in (1.23).

We next summarize the technique of exponential tilting developed in [1] for the thinned Lévy process $(\mathcal{S}_t)_{t\geq 0}$ with $\tau \in (3, 4)$, which allows us to give more details about how we shall establish the conjectured sample path behavior for each of the three parts described above.

2.1 Tilting and properties of the tilted process

All results presented in this subsection are proved in [1].

Exponential tilting. We use the notion of exponential tilting of measure in order to give a convenient description of the probability of interest as follows.

$$\mathbb{P}(\mathcal{S}_u > 0) = \phi(u; \vartheta)\widetilde{\mathbb{E}}_\vartheta[\mathrm{e}^{-\vartheta u \mathcal{S}_u} \mathbb{1}_{\{\mathcal{S}_u > 0\}}], \tag{2.3}$$

where ϑ is chosen later on. We define the measure $\widetilde{\mathbb{P}}_\vartheta$ with corresponding expectation $\widetilde{\mathbb{E}}_\vartheta$ by the equality, for every event E,

$$\widetilde{\mathbb{P}}_\vartheta(E) = \frac{1}{\phi(u; \vartheta)} \mathbb{E}_\vartheta[\mathrm{e}^{\vartheta u \mathcal{S}_u} \mathbb{1}_E], \tag{2.4}$$

where the normalizing constant $\phi(u; \vartheta)$ is defined as

$$\phi(u; \vartheta) = \mathbb{E}[\mathrm{e}^{\vartheta u \mathcal{S}_u}]. \tag{2.5}$$

Choosing a good ϑ is rather delicate, and we explain this in more detail now. By the independence of the indicators $(\mathcal{I}_i(u))_{i\geq 2}$, we obtain that

$$\begin{aligned}\phi(u;\vartheta) = \mathbb{E}[\mathrm{e}^{\vartheta u \mathcal{S}_u}] &= \mathrm{e}^{\vartheta u(1+\tilde{\beta}u)} \prod_{i=2}^{\infty} \mathrm{e}^{-\vartheta u^2 c_i^2}\Big(\mathrm{e}^{-c_i u} + \mathrm{e}^{\vartheta c_i u}(1-\mathrm{e}^{-c_i u})\Big) \\ &= \mathrm{e}^{\vartheta u(1+\tilde{\beta}u)} \mathrm{e}^{\sum_{i=2}^{\infty} f(i/u^{\tau-1};\vartheta)}\end{aligned} \tag{2.6}$$

with

$$f(x;\vartheta) = \log\big(1+\mathrm{e}^{-x^{-\alpha}}(\mathrm{e}^{-\vartheta x^{-\alpha}}-1)\big) + \vartheta x^{-\alpha} - \vartheta x^{-2\alpha}. \tag{2.7}$$

The function $x \mapsto f(x;\vartheta)$ is integrable at $x=0$ and at $x=\infty$, so the above sum can be approximated by the integral

$$\sum_{i=2}^{\infty} f(i/u^{\tau-1};\vartheta) = u^{\tau-1}\int_0^{\infty} f(x;\vartheta)dx + e_\vartheta(u) \equiv u^{\tau-1}\Lambda(\vartheta) + e_\vartheta(u), \tag{2.8}$$

for some error term $u \mapsto e_\vartheta(u)$ given by

$$e_\vartheta(u) = \vartheta\left\{u[\zeta(\alpha)-1] - u^2[\zeta(2\alpha)-1]\right\} + o_\vartheta(1), \tag{2.9}$$

where $\alpha = 1/(\tau-1)$ and the Riemann zeta functions $\zeta(\cdot)$ defined as

$$\zeta(s) = \lim_{N\to\infty}\Big\{\sum_{n=1}^{N} n^{-s} - \frac{N^{1-s}}{1-s} - \frac{1}{2}N^{-s}\Big\}, \quad \mathrm{Re}(s) > -1, s \neq 1, \tag{2.10}$$

where $\mathrm{Re}(s)$ denotes the real part of $s \in \mathbb{C}$. Equation (2.10) follows from Euler-Maclaurin summation [21, p. 333]. The error term in (2.9) converges to 0 uniformly for ϑ in compact sets bounded away from zero. This implies that

$$\phi(u;\vartheta) = \mathrm{e}^{u^{\tau-1}\Lambda(\vartheta)+\vartheta u(\zeta(\alpha)+(\tilde{\beta}-\zeta(2\alpha)+1)u)+o_\vartheta(1)}. \tag{2.11}$$

Let θ_u^* be the solution of

$$\theta_u^* = \arg\min_{\vartheta}\Big[\Lambda(\vartheta) + \vartheta u^{2-\tau}(\zeta(\alpha) + (\tilde{\beta}-\zeta(2\alpha)+1)u)\Big] \tag{2.12}$$

and let θ^* be the value of ϑ where $\vartheta \mapsto \Lambda(\vartheta)$ is minimal. It is not hard to see that $I \equiv -\Lambda(\theta^*) > 0$ and that θ^* is unique. In [1, Lemma 3.6], we have seen that $\theta_u^* = \theta^* + o(1)$. Further, $\theta^* > 0$ by [1, Lemma 3.5]. Define $\phi(u) = \phi(u;\theta_u^*)$. The next result investigates the main term $\phi(u)$:

Proposition 2.3 (Asymptotics of main term [1, Proposition 2.1])**.** *As $u \to \infty$, and with $I = -\min_{\vartheta\geq 0}\Lambda(\vartheta) > 0$, there exist $\kappa_{ij} \in \mathbb{R}$ such that*

$$\phi(u) = \mathbb{E}[\mathrm{e}^{\theta_u^* u \mathcal{S}_u}] = \mathrm{e}^{-Iu^{\tau-1}+u^{\tau-1}\sum_{i+j\geq 1}\kappa_{ij}u^{-i(\tau-2)-j(\tau-3)}}(1+o(1)). \tag{2.13}$$

Properties of the process under the tilted measure. From now on, we will take $\vartheta = \theta_u^*$, and we define $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_{\theta_u^*}$ with corresponding expectation $\widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}_{\theta_u^*}$. In what follows, we abbreviate $\theta = \theta_u^*$. Under this new measure, the rare event of $\mathcal{S}_u$ being positive becomes quite likely. To describe these results, let us introduce some notation. Recall from (1.24) that, for $p \in [0,1]$,

$$I_E(p) = (\tau-1)\int_0^{\infty}\Big(\frac{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-pv})}{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-v})+\mathrm{e}^{-v}} - pv\Big)\frac{dv}{v^{\tau-1}}, \tag{2.14}$$

where we take $\vartheta = \theta^*$, which turns out to be the limit of θ_u^* as $u \to \infty$ (see, e.g., [1, Lemma 3.6]). As we see in Theorem 2.2, the function $p \mapsto I_E(p)$ will serve to describe the asymptotic mean of the process $p \mapsto \mathcal{S}_{pu}$ conditionally on $\mathcal{S}_u > 0$. It is not hard to check that

$$I_E(0) = 0, \qquad \text{and } I_E(1) = 0, \tag{2.15}$$

the latter by definition of θ^*, since $0 = \Lambda'(\theta^*) = I_E(1)$. Finally,

$$I_E(p) > 0 \text{ for every } p \in (0,1), \tag{2.16}$$

and

$$I_E'(0) > 0 \text{ and } I_E'(1) < 0. \tag{2.17}$$

Lemma 2.4 (Expectation of $\mathcal{S}_t$ [1, Lemma 2.2])**.** *As $u \to \infty$,*
(a) $\widetilde{\mathbb{E}}[\mathcal{S}_t] = u^{\tau-2} I_E(t/u) + O(1 + t + t|\theta^* - \theta_u^*|u^{\tau-3})$ *uniformly in* $t \in [0, u]$.
(b) $\widetilde{\mathbb{E}}[\mathcal{S}_t - \mathcal{S}_u] = u^{\tau-2} I_E(t/u) + O(u - t + u^{-1} + |\theta^* - \theta_u^*|u^{\tau-2})$ *uniformly in* $t \in [u/2, u]$.
(c) $\widetilde{\mathbb{E}}[\mathcal{S}_t - \mathcal{S}_u] = u^{\tau-3} I_E'(1)(t-u)(1 + o(1)) + O(u^{-1})$ *when* $u - t = o(u)$.
(d) $u\widetilde{\mathbb{E}}[\mathcal{S}_u] = o(1)$ *when* $u \to \infty$.

We will also need some consequences of the asymptotic properties of $\widetilde{\mathbb{E}}[\mathcal{S}_t]$. This is stated in the following corollary:

Corollary 2.5. *As $u \to \infty$,*

(a) $\widetilde{\mathbb{E}}[\mathcal{S}_t] \geq \underline{c} t u^{\tau-3}$ *and* $\widetilde{\mathbb{E}}[\mathcal{S}_t] \leq \overline{c} t u^{\tau-3}$ *uniformly for* $t \in [\varepsilon, u/2]$, *where* $0 < \underline{c} < \overline{c} < \infty$;

(b) $\widetilde{\mathbb{E}}[\mathcal{S}_{u-t} - \mathcal{S}_u] \geq \underline{c} t u^{\tau-3}$ *and* $\widetilde{\mathbb{E}}[\mathcal{S}_{u-t} - \mathcal{S}_u] \leq \overline{c} t u^{\tau-3}$ *uniformly for* $t \in [T u^{-(\tau-2)}, u/2]$, *where* $0 < \underline{c} < \overline{c} < \infty$;

(c) $\widetilde{\mathbb{E}}[\mathcal{S}_t] = \widetilde{\mathbb{E}}[\mathcal{S}_{t_1}](1 + o(1))$ *for* $t \in [t_1, t_2]$ *and* $t_1 \in [\varepsilon, u/2]$ *and* $t_2 - t_1 = O(u^{-(\tau-2)})$;

(d) $\widetilde{\mathbb{E}}[\mathcal{S}_t] = \widetilde{\mathbb{E}}[\mathcal{S}_{t_1}](1 + o_T(1))$ *for* $t \in [t_1, t_2]$ *and* $t_1 \in [u/2, u - T u^{-(\tau-2)}]$, *where* $o_T(1)$ *denotes a quantity* $c(T, u)$ *such that* $\lim_{T\to\infty} \limsup_{u\to\infty} c(T, u) = 0$ *and* $t_2 - t_1 = O(u^{-(\tau-2)})$.

Proof. Part (a) for $t \in [\varepsilon, \varepsilon u]$ for $\varepsilon > 0$ sufficiently small follows from Lemma 2.4(a) together with the facts that $I_E(0) = 0$, $I_E'(0) > 0$, and that $1 + t + t|\theta^* - \theta_u^*|u^{\tau-3} = o(t u^{\tau-3})$. The fact that $I_E'(0) > 0$ also implies that $\underline{c}$ can be taken to be strictly positive. For $t \in [\varepsilon u, u/2]$, Part (a) follows from the fact that $I_E(p) > 0$ for all $p \in [\varepsilon, 1/2]$ and that $1 + t + t|\theta^* - \theta_u^*|u^{\tau-3} = o(u^{\tau-2})$.

Part (b) follows as Part (a), now using Lemma 2.4(b) together with the fact that $I_E(1) = 0$, $I_E'(1) < 0$.

Part (c) follows from Lemma 2.4(a), by subtracting the two terms. Note that the error term $O(1 + t_1 + t_1|\theta^* - \theta_u^*|u^{\tau-3})$ is $o(t_1 u^{\tau-3})$ since $t_1 \geq \varepsilon$, while $\widetilde{\mathbb{E}}[\mathcal{S}_{t_1}] = \Theta(t_1 u^{\tau-3})$ by Part (a) of this corollary. Further, note that

$$u^{\tau-2}[I_E(t/u) - I_E(t_1/u)] = O(u^{\tau-2} \max_{p\in[0,1]} |I_E'(p)|(t_2 - t_1)) = O(1), \tag{2.18}$$

which is $o(1)\widetilde{\mathbb{E}}[\mathcal{S}_{t_1}]$.
Part (d) follows again from Lemma 2.4(a) by subtracting the two terms. Note again that the error term $O(1 + t_1 + t_1|\theta^* - \theta_u^*|u^{\tau-3})$ is $o(t_1 u^{\tau-3})$, while $\widetilde{\mathbb{E}}[\mathcal{S}_{t_1}] = \Theta(t_1 u^{\tau-3})$ by part (b) of this corollary and Lemma 2.4(a). Further, note that

$$u^{\tau-2}[I_E(t/u) - I_E(t_1/u)] = O(u^{\tau-2} \max_{p\in[0,1]} |I_E'(p)|(t_2 - t_1)) = O_T(1), \tag{2.19}$$

which is $o_T(1)\widetilde{\mathbb{E}}[\mathcal{S}_{t_1}]$. □

The next lemma concerns the variance of the process. Define, for $p \in [0,1]$,

$$I_V(p) = (\tau-1)\int_0^\infty \frac{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-pv})}{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-v})+\mathrm{e}^{-v}}\Big(1-\frac{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-pv})}{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-v})+\mathrm{e}^{-v}}\Big)\frac{dv}{v^{\tau-2}} \tag{2.20}$$

and

$$J_V(p) = (\tau-1)\int_0^\infty \frac{\mathrm{e}^{\theta^* v}(\mathrm{e}^{-pv}-\mathrm{e}^{-v})}{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-v})+\mathrm{e}^{-v}}\Big(1-\frac{\mathrm{e}^{\theta^* v}(\mathrm{e}^{-pv}-\mathrm{e}^{-v})}{\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-v})+\mathrm{e}^{-v}}\Big)\frac{dv}{v^{\tau-2}}, \tag{2.21}$$

$$G_V(p) = (\tau-1)\int_0^\infty \frac{\mathrm{e}^{2\theta^* v}(1-\mathrm{e}^{-pv})(\mathrm{e}^{-pv}-\mathrm{e}^{-v})}{(\mathrm{e}^{\theta^* v}(1-\mathrm{e}^{-v})+\mathrm{e}^{-v})^2}\frac{dv}{v^{\tau-2}}. \tag{2.22}$$

Again, it is not hard to see that

$$0 < I_V(p) < \infty \text{ for every } p \in (0,1], \text{ while } I_V(0) = 0, \tag{2.23}$$

and

$$0 < J_V(p) < \infty \text{ for every } p \in [0,1), \text{ while } J_V(1) = 0. \tag{2.24}$$

Lemma 2.6 (Covariance structure of $\mathcal{S}_t$ [1, Lemma 2.3])**.** *As $u \to \infty$,*
(a) $\widetilde{\mathrm{Var}}[\mathcal{S}_t] = u^{\tau-3}I_V(t/u) + O(1 + t|\theta^* - \theta^*_u|u^{\tau-4})$ *uniformly in $t \in [0,u]$.*
(b) $\widetilde{\mathrm{Var}}[\mathcal{S}_t - \mathcal{S}_u] = u^{\tau-3}J_V(t/u) + O((u-t)u^{-1} + (u-t)|\theta^* - \theta^*_u|u^{\tau-4})$ *uniformly in $t \in [0,u]$.*
(c) $\widetilde{\mathrm{Cov}}[\mathcal{S}_t, \mathcal{S}_u - \mathcal{S}_t] = -u^{\tau-3}G_V(t/u) + O((u-t)u^{-1} + (u-t)|\theta^* - \theta^*_u|u^{\tau-4})$ *uniformly in $t \in [0,u]$.*

The next result bounds the Laplace transform of the couple $(\mathcal{S}_t, \mathcal{S}_u)$:

Proposition 2.7 (Joint moment generating function of $(\mathcal{S}_t, \mathcal{S}_u)$ [1, Proposition 2.4])**.** *(a) As $u \to \infty$,*

$$\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\lambda\frac{\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]}{\sqrt{I_V(t/u)u^{\tau-3}}}}\Big] = \mathrm{e}^{\frac12\lambda^2+\Theta}, \tag{2.25}$$

where $|\Theta| \le o_u(1)$ as $u \to \infty$ uniformly in $t \in [u/2,u]$ and λ in a compact set.
(b) Fix $\varepsilon > 0$ small. As $u \to \infty$, for any $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\lambda_1\frac{\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]}{\sqrt{I_V(t/u)u^{\tau-3}}}+\lambda_2\frac{\mathcal{S}_u-\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_u-\mathcal{S}_t]}{\sqrt{J_V(t/u)u^{\tau-3}}}}\Big] = \mathrm{e}^{\frac12\lambda_1^2+\frac12\lambda_2^2-\lambda_1\lambda_2\frac{G_V(t/u)}{I_V(t/u)J_V(t/u)}+\Theta}, \tag{2.26}$$

where $|\Theta| \le o_u(1) + O(t^{3(3-\tau)/2})$ uniformly in $t \in [\varepsilon, u - u^{-(\tau-5/2)}]$ and λ_1, λ_2 in a compact set.

By Proposition 2.7 and the fact that $u\widetilde{\mathbb{E}}[\mathcal{S}_u] = o(1)$ (see [1, Lemma 4.1]), $u^{-(\tau-3)/2}\mathcal{S}_u$ converges to a normal distribution with mean 0 and variance $I_V(1)$. We next extend this intuition by proving that the density of $\mathcal{S}_u$ close to zero behaves like $(2\pi I_V(1))^{-1/2}\,u^{-(\tau-3)/2}$:

Proposition 2.8 (Density of $\mathcal{S}_u$ near zero [1, Proposition 2.5])**.** *Uniformly in $s = o(u^{(\tau-3)/2})$, the density $\widetilde{f}_{\mathcal{S}_u}$ of $\mathcal{S}_u$ satisfies*

$$\widetilde{f}_{\mathcal{S}_u}(s) = Bu^{-(\tau-3)/2}(1+o(1)), \tag{2.27}$$

with $B = (2\pi I_V(1))^{-1/2}$ and $I_V(p)$ defined in (2.20). Moreover, $\widetilde{f}_{\mathcal{S}_t}(s)$ is uniformly bounded by a constant times $u^{-(\tau-3)/2}$ for all s, u and $t \in [u/2,u]$.

There are three more results from [1] that will be used in this paper. The first is a description of the distribution of the indicator processes $(\mathcal{I}_i(t))_{t\ge0}$ under the measure $\widetilde{\mathbb{P}}$. Since our indicator processes $(\mathcal{I}_i(t))_{t\ge0}$ are *independent*, this property also holds under the measure $\widetilde{\mathbb{P}}$:

Lemma 2.9 (Indicator processes under the tilted measure [1, Lemma 4.2])**.** *Under the measure* $\widetilde{\mathbb{P}}$*, the distribution of the indicator processes* $(\mathcal{I}_i(t))_{t\geq 0}$ *is that of independent indicator processes. More precisely,*

$$\mathcal{I}_i(t) = \mathbb{1}_{\{T_i \leq t\}}, \tag{2.28}$$

where $(T_i)_{i\geq 2}$ *are independent random variables with distribution*

$$\widetilde{\mathbb{P}}(T_i \leq t) = \begin{cases} \frac{e^{\theta c_i u}(1-e^{-c_i t})}{e^{\theta c_i u}(1-e^{-c_i u})+e^{-c_i u}} & \text{for } t \leq u; \\ \frac{e^{\theta c_i u}(1-e^{-c_i u})+(e^{-c_i u}-e^{-c_i t})}{e^{\theta c_i u}(1-e^{-c_i u})+e^{-c_i u}} & \text{for } t > u. \end{cases} \tag{2.29}$$

The second lemma describes what happens to the variances for small p or for p close to 1:

Lemma 2.10 (Asymptotic variance near extremities [1, Lemma 4.3(b)])**.** *As* $p \to 1$*,* $J_V(p) = -(1-p)J_V'(1)(1+o(1))$ *with* $J_V'(1) < 0$*, while, as* $p \to 0$*,*

$$I_V(p) = p^{\tau-3}\mathcal{I}_V(1+o(1)), \qquad \text{with} \qquad \mathcal{I}_V = (\tau-1)\int_0^\infty (1-e^{-y})e^{-y}\frac{dy}{y^{\tau-2}}. \tag{2.30}$$

Consequently, there exist $0 < \underline{c} < \bar{c} < \infty$ *such that, for every* $p \in [0,\varepsilon]$ *with* $\varepsilon > 0$ *sufficiently small,*

$$\underline{c}p^{\tau-3} \leq I_V(p) \leq \bar{c}p^{\tau-3}. \tag{2.31}$$

We finally rely on the following corollary that allows us to compute sums that we will encounter frequently:

Corollary 2.11 (Replacing sums by integrals in general [1, Corollary 3.3])**.** *For every* $a \in \mathbb{R}, a > \tau - 1$ *and* $b > 0$*, there exists a constant* $c(a,b)$ *such that*

$$\sum_{i=2}^{\infty} c_i^a e^{-bc_i u} = c(a,b)u^{\tau-a-1}(1+o(1)). \tag{2.32}$$

2.2 No early hits and middle ground

In this section, we prove that the tilted process is unlikely to hit 0 until a time that is very close to u. We start by investigating the early hits.

No early hits. In this step, we prove that it is unlikely that the process hits zero early on, i.e., in the first time interval $[0,\varepsilon]$ for some $\varepsilon > 0$ sufficiently small. In its statement, we write $0 \in \mathcal{S}_{[0,t]}$ for the event that $\{\mathcal{S}_s = 0\}$ for some $s \in [0,t]$, so that $\mathbb{P}(H_1(0) > u) = \mathbb{P}(0 \notin \mathcal{S}_{[0,u]})$.

Lemma 2.12 (No early hits)**.** *For every* $u \in [0,\infty)$*, as* $\varepsilon \downarrow 0$*,*

$$\mathbb{P}(0 \in \mathcal{S}_{[0,\varepsilon]}, \mathcal{S}_u > 0) = o_\varepsilon(1)\mathbb{P}(\mathcal{S}_u > 0), \tag{2.33}$$

where $o_\varepsilon(1)$ *denotes a function that converges to zero as* $\varepsilon \downarrow 0$*, uniformly in* u*.*

The proof of Lemma 2.12 follows from a straightforward application of the FKG-inequality for independent random variables (see [19], or [20, Theorem 2.4, p. 34]). The standard versions of the FKG-inequality hold for independent indicator random variables, and in our case we need it for independent exponentials. It is not hard to prove that the FKG-inequality we need holds by an approximation argument.

Proof. We note that the process $(\mathcal{S}_t)_{t\geq 0}$ is a deterministic function of the exponential random variables $(T_i)_{i\geq 2}$ (recall (1.15), (1.16) and (1.17)). Now, the event $\{0\in\mathcal{S}_{[0,\varepsilon]}\}$ is *increasing* in terms of the random variables $(T_i)_{i\geq 2}$ (use that $\mathcal{S}_t$ only has positive jumps). Here we say that an event A is increasing when, if A occurs for a realization $(t_i)_{i\geq 2}$ of $(T_i)_{i\geq 2}$, and if $(t_i')_{i\geq 2}$ is coordinatewise larger than $(t_i)_{i\geq 2}$, then A also occurs for $(t_i')_{i\geq 2}$. Clearly, the event $\{\mathcal{S}_u>0\}$ is *decreasing* (for a definition, change the role of t_i and t_i' in the definition of an increasing event), so that the FKG-inequality implies that these events are negatively correlated:

$$\mathbb{P}(0\in\mathcal{S}_{[0,\varepsilon]},\mathcal{S}_u>0)\leq \mathbb{P}(0\in\mathcal{S}_{[0,\varepsilon]})\mathbb{P}(\mathcal{S}_u>0). \tag{2.34}$$

We conclude the proof by noting that $\mathbb{P}(0\in\mathcal{S}_{[0,\varepsilon]})=o_\varepsilon(1)$ independently of u. □

The key to our proof of Theorem 1.4 will be to show that $\mathbb{P}(H_1(0)>u)=\Theta(\mathbb{P}(\mathcal{S}_u>0))$, so that Lemma 2.12 and the known asymptotics of $\mathbb{P}(\mathcal{S}_u>0)$ imply that it is unlikely to have an early hit of zero.

No middle ground. By (2.4) (recall that $\phi(u)=\phi(u;\theta)$ with $\theta=\theta_u^*$), Lemma 2.12 and Theorem 2.1,

$$\mathbb{P}(H_1(0)>u)=\phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[0,u]}>0\}}]=\phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[\varepsilon,u]}>0\}}]+\phi(u)u^{-(\tau-1)/2}o_\varepsilon(1). \tag{2.35}$$

For $M>0$ arbitrarily fixed, we split

$$\mathbb{P}(H_1(0)>u)=\phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[\varepsilon,u]}>0,\mathcal{S}_u\in[0,M/u]\}}]+\phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[\varepsilon,u]}>0,\mathcal{S}_u>M/u\}}]+\phi(u)u^{-(\tau-1)/2}o_\varepsilon(1). \tag{2.36}$$

By Proposition 2.8, we can bound

$$\begin{aligned}\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[\varepsilon,u]}>0,\mathcal{S}_u>M/u\}}]&\leq \widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_u>M/u\}}]\leq\int_{M/u}^\infty \mathrm{e}^{-\theta uv}f_{\mathcal{S}_u}(v)dv\\&\leq O(u^{-(\tau-3)/2})\int_{M/u}^\infty \mathrm{e}^{-\theta uv}dv=O(u^{-(\tau-1)/2})\mathrm{e}^{-\theta M}.\end{aligned} \tag{2.37}$$

As a result, we arrive at

$$\mathbb{P}(H_1(0)>u)=\phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[\varepsilon,u]}>0,\mathcal{S}_u\in[0,M/u]\}}]+\phi(u)u^{-(\tau-1)/2}o_M(1)+\phi(u)u^{-(\tau-1)/2}o_\varepsilon(1), \tag{2.38}$$

where $o_M(1)$ denotes a quantity $c(M,u)$ such that $\limsup_{M\to\infty}\limsup_{u\to\infty}c(M,u)=0$.

We continue to prove that the dominant contribution to the expectation of the right-hand side of (2.3) originates from paths that remain positive until time $u-t$ for $t=Tu^{-(\tau-2)}$.

Proposition 2.13 (No middle ground)**.** *Fix $\varepsilon>0$. For every $u\in[0,\infty)$ and $\varepsilon,M>0$ fixed,*

$$\widetilde{\mathbb{P}}(0\in\mathcal{S}_{[\varepsilon,u-Tu^{-(\tau-2)}]},\mathcal{S}_u\in[0,M/u])\leq o_T(1)u^{-(\tau-1)/2}, \tag{2.39}$$

where we recall that $o_T(1)$ denotes a quantity $c(T,u)$ such that $\lim_{T\to\infty}\limsup_{u\to\infty}c(T,u)=0$.

We prove Proposition 2.13 in Section 3.

By (2.38) and Proposition 2.13,

$$\mathbb{P}(H_1(0)>u)=\phi(u)\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[u-Tu^{-(\tau-2)},u]}>0\}}]+\phi(u)u^{-(\tau-1)/2}\big[o_\varepsilon(1)+o_M(1)+o_T(1)\big]. \tag{2.40}$$

Since ε, M and T are arbitrary, it now suffices to identify the asymptotics of the expectation appearing on the right-hand side of (2.40).

2.3 Remaining positive near the end

To prove Theorem 1.4, by Proposition 2.3 and equation (2.40), it suffices to prove that, with $\gamma = (\tau-1)/2$,

$$\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u \mathcal{S}_u} \mathbb{1}_{\{\mathcal{S}_{[u-t,u]}>0, \mathcal{S}_u \in [0,M/u]\}}] = (A + o_T(1))u^{-\gamma}(1+o(1)), \tag{2.41}$$

where $t = Tu^{-(\tau-2)}$. In the above expectation, we see two terms. The term $\mathrm{e}^{-\theta u \mathcal{S}_u}$ forces $\mathcal{S}_u$ to be small, more precisely, $\mathcal{S}_u = \Theta(1/u)$ for u large, while the term $\mathbb{1}_{\{\mathcal{S}_{[u-t,u]}>0\}}$ forces the path to remain positive until time u. We now study these two effects.

In order to investigate the probability that $\mathcal{S}_{[u-t,u]} > 0$, we proceed as follows. Let

$$\mathcal{J}(u) = \{j : \mathcal{I}_j(u) = 1\} \tag{2.42}$$

denote the set of indices for which $T_j \le u$. We condition on the set $\mathcal{J}(u)$. Note that $\mathcal{S}_u$ is measurable with respect to $\mathcal{J}(u)$. We now rewrite $\mathcal{S}_{u-t}$ in a convenient form. For this, recall (1.20) and write

$$\begin{aligned}\mathcal{S}_{u-t} &= \frac{t}{u} + \frac{u-t}{u}\mathcal{S}_u + \sum_{i=2}^{\infty} c_i[\mathcal{I}_i(u-t) - \frac{u-t}{u}\mathcal{I}_i(u)] \\ &= \frac{t}{u} + \frac{u-t}{u}\mathcal{S}_u - \sum_{i=2}^{\infty} c_i[\mathbb{1}_{\{T_i \in (u-t,u]\}} - \frac{t}{u}\mathcal{I}_i(u)].\end{aligned} \tag{2.43}$$

Thus, with

$$Q_u(t) \equiv u\mathcal{S}_{u-t} - t - (u-t)\mathcal{S}_u = -\sum_{i=2}^{\infty} c_i[u\mathbb{1}_{\{T_i\in(u-t,u]\}} - t\mathcal{I}_i(u)], \tag{2.44}$$

we have that $\mathcal{S}_{u-t} > 0$ precisely when $Q_u(t) > -t-(u-t)\mathcal{S}_u$. We rewrite

$$Q_u(t) = -\sum_{i\in\mathcal{J}(u)} c_i[u\mathbb{1}_{\{T_i\in(u-t,u]\}} - t]. \tag{2.45}$$

Note that, for any $t = o(u)$,

$$\begin{aligned}\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u\mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[u-t,u]}>0\}}] &= \frac{1}{u}\int_0^\infty \mathrm{e}^{-\theta v}\widetilde{\mathbb{P}}\big(\mathcal{S}_{[u-t,u]} > 0 \mid u\mathcal{S}_u = v\big)\widetilde{f}_{\mathcal{S}_u}(v/u)dv \\ &= \frac{1}{u}\int_0^\infty \mathrm{e}^{-\theta v}\widetilde{\mathbb{P}}\big(Q_u(s) > -v - s + sv/u\,\forall s\in[0,t] \mid u\mathcal{S}_u = v\big)\widetilde{f}_{\mathcal{S}_u}(v/u)dv.\end{aligned} \tag{2.46}$$

We aim to use dominated convergence on the above integral, and we start by proving pointwise convergence. By Proposition 2.8, $\widetilde{f}_{\mathcal{S}_u}(v/u) = Bu^{-(\tau-3)/2}(1+o(1))$ pointwise in v (in fact, even when $v = o(u^{(\tau-1)/2})$). This leads us to study, for all $v > 0$,

$$g_{u,t}(v) \equiv \widetilde{\mathbb{P}}\big(Q_u(s) > -v - s + sv/u\,\forall s\in[0,t] \mid u\mathcal{S}_u = v\big). \tag{2.47}$$

We split

$$Q_u(t) = A_u(t) - B_u(t), \tag{2.48}$$

where

$$B_u(t) \equiv u\sum_{i\in\mathcal{J}(u)} c_i[\mathbb{1}_{\{T_i\in(u-t,u]\}} - \widetilde{\mathbb{P}}(T_i > u-t \mid T_i \le u)], \qquad A_u(t) \equiv -\sum_{i\in\mathcal{J}(u)} c_i[u\widetilde{\mathbb{P}}(T_i > u-t \mid T_i\le u) - t]. \tag{2.49}$$

Thus, $(A_u(t))_{t\in[0,u]}$ is *deterministic* given $\mathcal{J}(u)$, while $(B_u(t))_{t\in[0,u]}$ is *random* given $\mathcal{J}(u)$. The main result for the near-end regime is the following proposition, which proves that $g_u(v)$ converges pointwise.

Proposition 2.14 (Weak conditional convergence of time-reversed process)**.**
(a) As $u \to \infty$, conditionally on $u\mathcal{S}_u = v$,

$$(A_u(tu^{-(\tau-2)}))_{t\geq 0} \xrightarrow{d} (\kappa t)_{t\geq 0}, \tag{2.50}$$

where $\kappa \in (0,\infty)$ is given by

$$\kappa = \int_0^\infty x^{-\alpha} \frac{\mathrm{e}^{\theta x^{-\alpha}}\mathrm{e}^{-x^{-\alpha}}}{\mathrm{e}^{\theta x^{-\alpha}}(1-\mathrm{e}^{-x^{-\alpha}}) + \mathrm{e}^{-x^{-\alpha}}} \left[\mathrm{e}^{x^{-\alpha}} - 1 - x^{-\alpha}\right] dx. \tag{2.51}$$

(b) As $u \to \infty$, conditionally on $u\mathcal{S}_u = v$,

$$(B_u(tu^{-(\tau-2)}))_{t\geq 0} \xrightarrow{d} (L_t)_{t\geq 0}, \tag{2.52}$$

where $(-L_t)_{t\geq 0}$ is a Lévy process with no positive jumps and with Laplace transform

$$\mathbb{E}[\mathrm{e}^{a(-L_s)}] = \mathrm{e}^{s\int_{-\infty}^0 (\mathrm{e}^{az}-1-az)\Pi(dz)}, \quad a \geq 0, \tag{2.53}$$

and characteristic measure

$$\Pi(dz) = (\tau-1)\frac{(-z)^{-(\tau-1)}\mathrm{e}^{-\theta z}}{\mathrm{e}^{-\theta z}(1-\mathrm{e}^{z}) + \mathrm{e}^{z}}\mathrm{e}^{z}dz. \tag{2.54}$$

Proposition 2.14 is proved in Section 5, and determines the precise constant A from (1.23), as we now explain in more detail.

We proceed by investigating some properties of the supremum of the Lévy process from (2.52) that we need later on. Note in particular that the distribution of L_s in (2.53) does not depend on v. With a slight abuse of notation, also the distribution of the limiting process $(L_s)_{s\geq 0}$ shall be denoted by $\mathbb{P}$.

Lemma 2.15 (Supremum of the Lévy process)**.** *Let $I_\infty \equiv \inf_{t\geq 0}(-L_t + \kappa t)$. Then*

$$\mathbb{P}(I_\infty \geq -v) = \mathcal{W}(v)/\mathcal{W}(\infty), \tag{2.55}$$

where $\mathcal{W}\colon [0,\infty) \to [0,\infty)$ is the unique continuous increasing function that has Laplace transform

$$\int_0^\infty \mathrm{e}^{-ax}\mathcal{W}(x)dx = \frac{1}{\psi(a)}, \quad a > \Psi(0), \tag{2.56}$$

where the Laplace exponent ψ is given by $\mathbb{E}[\mathrm{e}^{a(\kappa t - L_t)}] = \mathrm{e}^{t\psi(a)}$ and is computed in (2.57) below, while $\Psi(0)$ is the largest solution of the equation $\psi(a) = 0$, and $\mathcal{W}(\infty) = 1/\psi'(0) = 1/\kappa$ is a constant.

Proof. We rewrite (2.53) to see that $X_s \equiv -L_s + \kappa s$ is a Lévy process with no positive jumps and Laplace exponent

$$\begin{aligned}\psi(a) &= \kappa a + \int_{(-\infty,0)} (\mathrm{e}^{az} - 1 - az)\,\Pi(dz) \\ &= \beta' a + \int_{(-\infty,0)} \left(\mathrm{e}^{az} - 1 - az\mathbb{1}_{\{z>-1\}}\right)\Pi(dz)\end{aligned} \tag{2.57}$$

with

$$\beta' = \kappa + \int_{(-\infty,0)} (-z)\mathbb{1}_{\{z\leq -1\}}\Pi(dz) > 0 \tag{2.58}$$

as defined in [5, Section VII.1]. Indeed, recall from [5, Section VII.1] that $\mathbb{E}[\mathrm{e}^{aX_s}] = \mathrm{e}^{s\psi(a)}$ and note that our β' corresponds to a in [5]. Also note from (2.51) that $\kappa > 0$. Thus $\psi'(0+) = \kappa > 0$ and [5, Corollary

2(ii) in Section VII.1] yields that X_s drifts to ∞ (for a definition, see [5, Theorem 12(ii) in Section VI.3]). This in turn implies (see [5, Proof of Theorem 8, in Section VII.2])

$$\mathbb{P}(I_\infty \geq -v) = \mathcal{W}(v)/\mathcal{W}(\infty), \tag{2.59}$$

where $\mathcal{W}$ is given in the statement of [5, Theorem 8, in Section VII.2]. For the definition of Ψ see before [5, Theorem 1 of Section VII.1]. Also note from the second equation of the proof of [5, Proof of Theorem 8, in Section VII.2] that $\mathcal{W}(\infty) > 0$. To see that $\mathcal{W}(\infty) = 1/\psi(0)$, note that if $a \downarrow 0$,

$$\int_0^\infty \mathrm{e}^{-ax}\mathcal{W}(x)dx = \frac{\mathcal{W}(\infty)}{a(1+o(1))}. \tag{2.60}$$

Now, $\psi(0) = 0$, so that $1/\psi(a) = 1/(a\psi'(0))(1 + o(1))$ as $a \downarrow 0$, which identifies $\mathcal{W}(\infty) = 1/\psi'(0) = 1/\kappa$. □

By Proposition 2.14 and the continuity of $\mathcal{W}$ in Lemma 2.15, with $\mathcal{M}_T = \sup_{0\leq s\leq T}(L_s - \kappa s)$ for each $v \geq 0$ and for $t = Tu^{-(\tau-2)}$, for $u \to \infty$,

$$g_{u,t}(v) \to g_T(v) \equiv \mathbb{P}\big(\mathcal{M}_T \leq v\big). \tag{2.61}$$

Further, as $T \to \infty$,

$$g_T(v) \downarrow g(v) \equiv \mathbb{P}(\sup_{0\leq s<\infty}(L_s - \kappa s) \leq v) = \frac{\mathcal{W}(v)}{\mathcal{W}(\infty)}. \tag{2.62}$$

Now we are ready to complete the proofs of our main results.

2.4 Completion of the proofs

Completion of the proof of Theorem 1.4. We start by completing the proof of Theorem 1.4. Recall that it remains to prove (2.41) with $\gamma = (\tau - 1)/2$. By (2.46) and (2.47), we need to compute

$$\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u \mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[u-t,u]}>0,\mathcal{S}_u\in[0,M/u]\}}] = \frac{1}{u^{(\tau-1)/2}}\int_0^M \mathrm{e}^{-\theta v} g_{u,t}(v)\big[u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(v/u)\big]dv, \tag{2.63}$$

where $t = Tu^{-(\tau-2)}$. A similar problem was encountered in [1, Proof of Theorem 1.1], which is restated here as Theorem 2.1, apart from the fact that there the function $g_{u,t}(v)$ was absent.

We wish to use bounded convergence. For this, we note that $u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(v/u) \to B$ by Proposition 2.8 for each v (in fact, for all $v = o(u)$), while, by (2.61)–(2.62), $g_{u,t}(v) \to g_T(v)$, which, in turn, converges to $g(v)$ as $T \to \infty$. Further, since $g_{u,t}(v) \leq 1$ and $u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(v/u)$ is uniformly bounded (see Proposition 2.8), the integrand $\mathrm{e}^{-\theta v}g_{u,t}(v)\big[u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(v/u)\big]$ is uniformly bounded by a constant. Thus, by the Bounded Convergence Theorem,

$$\begin{aligned}\widetilde{\mathbb{E}}[\mathrm{e}^{-\theta u \mathcal{S}_u}\mathbb{1}_{\{\mathcal{S}_{[u-t,u]}>0,\mathcal{S}_u\in[0,M/u]\}}] &= \frac{B}{u^{(\tau-1)/2}}\int_0^M \mathrm{e}^{-\theta v} g_T(v)dv(1+o(1)) \\ &= \frac{B}{u^{(\tau-1)/2}}\int_0^M \mathrm{e}^{-\theta v} g(v)dv(1+o(1)+o_T(1)).\end{aligned} \tag{2.64}$$

This identifies (recall (2.41), (2.55), (2.56) and (2.61))

$$A = B\int_0^\infty \mathrm{e}^{-\theta v}g(v)dv = B\int_0^\infty \mathrm{e}^{-\theta v}\mathbb{P}\big(\mathcal{M} \leq v\big)dv = \frac{B}{\theta}\mathbb{E}[\mathrm{e}^{-\theta\mathcal{M}}] = \frac{B\psi'(0)}{\psi(\theta)}. \tag{2.65}$$

Since $D = B/\theta$ by [1, (7.4)] and $\mathbb{P}\big(\mathcal{M} \leq v\big) < 1$ for every v, we also immediately obtain that $A \in (0, D)$. and completes the proof of Theorem 1.4. □

Path properties: Proof of Theorem 1.5. We bound, using that $\{H_1(0) > u\} \subseteq \{\mathcal{S}_u > 0\}$,

$$\mathbb{P}\Big(\sup_{p\in[0,1]} |\mathcal{S}_{pu} - u^{\tau-2} I_E(p)| > \varepsilon u^{\tau-2} \mid H_1(0) > u\Big) \tag{2.66}$$
$$\leq \mathbb{P}\Big(\sup_{p\in[0,1]} |\mathcal{S}_{pu} - u^{\tau-2} I_E(p)| > \varepsilon u^{\tau-2} \mid \mathcal{S}_u > 0\Big) \frac{\mathbb{P}(\mathcal{S}_u > 0)}{\mathbb{P}(H_1(0) > u)}.$$

By Theorems 2.1 and 1.4, the ratio of probabilities converges to $D/A \in (0, \infty)$, while, by Theorem 2.2, the conditional probability converges to 0. This completes the proof of Theorem 1.5. □

Completion of the proof of Theorem 1.6. We finally complete the proof of Theorem 1.6 using Theorem 1.4 and recalling (1.21). Denote

$$\mathcal{S}_t^{(i)} = c_i + \tilde{\beta}_i t + \sum_{j=1:\, j\neq i}^{\infty} c_j [\mathcal{I}_j(t) - c_j t], \tag{2.67}$$

where $\tilde{\beta}_i = (\lambda + \zeta)/(ab) - c_i^2$ (see [7, Remark 3.9] and recall a', b', c' from above (1.20)). The intuition for the above formula is that

$$\mathcal{S}_t^{(i)} = \sum_{j\geq 1}^{\infty} c_j [\mathcal{I}_j(t) - c_j t], \tag{2.68}$$

where we slightly abuse notation to set $\mathcal{I}_i(0) = 1$ for the process $(\mathcal{S}_t^{(i)})_{t\geq 0}$. Since $(\mathcal{S}_t^{(i)})_{t\geq 0}$ describes the scaling limit of the exploration process of the cluster of vertex $i \geq 1$, while $\mathcal{I}_j(t)$ has the interpretation as the indicator that vertex j is found in the exploration before time t, it is reasonable to set $\mathcal{I}_i(0) = 1$ for $(\mathcal{S}_t^{(i)})_{t\geq 0}$. [2]

Define

$$H^{(i)}(0) = \inf\{t \geq 0 \colon \mathcal{S}_t^{(i)} = 0\}. \tag{2.69}$$

Then, $H_1(0) = H^{(1)}(0)$. Let $\mathcal{C}(i)$ be the connected component to which vertex i belongs, and let $\mathcal{C}_{\leq}(i)$ be the set $\mathcal{C}(i)$ if none of the vertices $j \in [i-1] = \{1, \ldots, i-1\}$ belongs to $\mathcal{C}(i)$, and the empty set $\varnothing$ otherwise. We know from [7, (3.78)] and the scaling explained around (1.20) that $n^{-\rho}|\mathcal{C}(i)| \xrightarrow{d} a \cdot H^{(i)}(0)$ for each $i \geq 1$ with $\rho = (\tau-2)/(\tau-1)$ (cf. (1.13)). Finally, denote

$$H_i(0) = \begin{cases} 0 & \text{if } \exists j < i \text{ such that } \mathcal{I}_j(H^{(i)}(0)) = 1; \\ H^{(i)}(0) & \text{otherwise.} \end{cases} \tag{2.70}$$

Then, by [7, (3.79)], $n^{-\rho}|\mathcal{C}_{\leq}(i)| \xrightarrow{d} a \cdot H_i(0)$. This provides us with the appropriate background to complete the proof of Theorem 1.6.

We start with the lower bound. By construction, $\gamma_1(\lambda) \geq a \cdot H_1(0)$ (see [7, Theorems 1.1 and 2.1] and recall that $\mathcal{C}_{(i)}$ denotes the i^{th}-largest connected component). Therefore,

$$\mathbb{P}(\gamma_1(\lambda) > au) \geq \mathbb{P}(H_1(0) > u), \tag{2.71}$$

and thus the lower bound follows from Theorem 1.4.

For the upper bound, we use that (cf. [7, Theorems 1.1])

$$\mathbb{P}(\gamma_1(\lambda) > au) = \lim_{n\to\infty} \mathbb{P}(\exists i \colon n^{-\rho}|\mathcal{C}_{\leq}(i)| \geq au) \leq \lim_{n\to\infty} \sum_{i\geq 1} \mathbb{P}(n^{-\rho}|\mathcal{C}_{\leq}(i)| \geq au). \tag{2.72}$$

[2] We take this opportunity to correct some typos in [7]. In [7, (3.76)], the term $-abti^{-\alpha} = -abtc_i$ should be replaced by $-abti^{-2\alpha} = -abtc_i^2$. This corresponds to the choice of $\tilde{\beta}_i = (\lambda+\zeta)/(ab) - c_i^2$ here. Further, in [7, (3.79)], the product over $j \in [i-1]$ should be over $q \in [i-1]$.

By the weak convergence of $n^{-\rho}|\mathcal{C}_{\leq}(i)|$ and the fact that there are with high probability only finitely many clusters that are larger than εn^{ρ} (as proved in [7, Theorem 1.6]),

$$\mathbb{P}(\gamma_1(\lambda) > au) \leq \mathbb{P}(H_1(0) > u) + \sum_{i\geq 2} \mathbb{P}(H_i(0) > u). \tag{2.73}$$

The first term is the main term, and we prove that $\sum_{i\geq 2} \mathbb{P}(H_i(0) > u) = o(\mathbb{P}(H_1(0) > u))$ now.

For this, we note that

$$\begin{aligned}\mathbb{P}(H_i(0) > u) &= \mathbb{P}(\mathcal{I}_j(u) = 0\,\forall j \in [i-1], \mathcal{S}^{(i)}_{[0,u]} > 0) \\ &= \mathbb{P}(\mathcal{I}_j(u) = 0\,\forall j \in [i-1])\mathbb{P}(\mathcal{S}^{(i)}_{[0,u]} > 0 \mid \mathcal{I}_j(u) = 0\,\forall j \in [i-1]).\end{aligned} \tag{2.74}$$

We can rewrite, on the event $\{\mathcal{I}_j(u) = 0\,\forall j \in [i-1]\}$,

$$\mathcal{S}^{(i)}_t = \tfrac{(\lambda+\zeta)}{ab}t + \sum_{j\geq i+1} c_j(\mathcal{I}_j(t) - c_j t) + c_i - \sum_{j=1}^{i} c_j^2 t \leq \tfrac{(\lambda+\zeta)}{ab}t + \sum_{j\geq i+1} c_j(\mathcal{I}_j(t) - c_j t) + c_1 - \sum_{j=1}^{i} c_j^2 t. \tag{2.75}$$

Therefore,

$$\begin{aligned}\mathbb{P}(\mathcal{S}^{(i)}_{[0,u]} > 0 \mid \mathcal{I}_j(u) = 0\,\forall j \in [i-1]) &\leq \mathbb{P}\Big(\tfrac{(\lambda+\zeta)}{ab}t + \sum_{j\geq i+1} c_j(\mathcal{I}_j(t) - c_j t) + c_1 - \sum_{j=1}^{i} c_j^2 t > 0\ \forall t \in [0,u]\Big) \\ &= \mathbb{P}\big(\mathcal{S}^{(1)}_{[0,u]} > 0 \mid \mathcal{I}_j(u) = 0\,\forall j \in [i]\setminus\{1\}\big).\end{aligned} \tag{2.76}$$

The event $\big\{\mathcal{I}_j(u) = 0 \forall j \in [i]\setminus\{1\}\big\}$ is decreasing (recall the notions used in the proof of Lemma 2.12) in the random variables $(T_i)_{i\geq 2}$, while the event $\{\mathcal{S}^{(1)}_{[0,u]} > 0\}$ is increasing. Thus, by the FKG-inequality,

$$\mathbb{P}\big(\mathcal{S}^{(1)}_{[0,u]} > 0 \mid \mathcal{I}_j(u) = 0\,\forall j \in [i]\setminus\{1\}\big) \leq \mathbb{P}(\mathcal{S}^{(1)}_{[0,u]} > 0) = \mathbb{P}(H_1(0) > u). \tag{2.77}$$

We can identify

$$\mathbb{P}(\mathcal{I}_j(u) = 0 \forall j \in [i-1]) = \mathrm{e}^{-\sum_{j=1}^{i-1} c_j u}. \tag{2.78}$$

Combining (2.73), (2.77)–(2.78) we arrive at

$$\mathbb{P}(\gamma_1(\lambda) > au) \leq \mathbb{P}(H_1(0) > u)\Big[1 + \sum_{i\geq 2} \mathrm{e}^{-\sum_{j=1}^{i-1} c_j u}\Big]. \tag{2.79}$$

Since $c_j = j^{-\alpha}$ with $\alpha \in (1/3, 1/2)$, $\sum_{j=1}^{i-1} c_j \geq (i-1)c_{i-1} = (i-1)^{1-\alpha}$. Therefore,

$$\sum_{i\geq 2} \mathrm{e}^{-\sum_{j=1}^{i-1} c_j u} = o(1). \tag{2.80}$$

This completes the proof of Theorem 1.6. □

3 No middle ground: Proof of Proposition 2.13

In this section, we show that the probability to hit zero in the time interval $[\varepsilon, u - Tu^{-(\tau-2)}]$, where T is a constant, becomes negligible as $T \to \infty$.

The strategy of proof is as follows. We start in Proposition 3.2 by investigating the value of $\mathcal{S}_t$ at some discrete times $(t_k)_{k\geq 1}$ in $[0, u]$ and show that with high probability $\mathcal{S}_t$ does not deviate far from its mean. Next, in Proposition 3.3, we show that it is unlikely for the process $(\mathcal{S}_t)_{t\geq 0}$ to make a substantial deviation in the interval $[t_k, t_{k+1}]$ from its value in t_k.

We start with a preparatory lemma that will allow us to give bounds on the asymptotic parameters appearing in the upcoming proofs:

Lemma 3.1 (Asymptotics of parameters)**.** *There exists $K \geq 1$ such that*

$$\widetilde{\mathbb{P}}\Big(\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u) \geq K u^{\tau-3}\Big) \leq C u^{-(\tau-1)}, \tag{3.1}$$

and, for all $|\lambda| \leq \delta u$ with $\delta > 0$ sufficiently small, there exists $K > 0$ such that

$$\widetilde{\mathbb{P}}\Big(\sum_{i=2}^{\infty} c_i [1 - \mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big) \geq K \lambda^2 u^{\tau-4}\Big) \leq C u^{-(\tau-1)}. \tag{3.2}$$

Proof. We use the second moment method. With Lemma 2.9 we compute that

$$\widetilde{\mathbb{E}}\big[\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u)\big] \leq \sum_{i=2}^{\infty} c_i^2 C(\theta)\big(1 - \mathrm{e}^{-c_i u}\big). \tag{3.3}$$

Split the sum into i with $c_i u \leq 1$ and $c_i u > 1$. For the first, we bound $1 - \mathrm{e}^{-c_i u} \leq O(1) c_i u$, for the latter, we bound $1 - \mathrm{e}^{-c_i u} \leq 1$, to obtain

$$\widetilde{\mathbb{E}}\big[\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u)\big] \leq O(1) \sum_{i\colon c_i u \leq 1} c_i^3 u + O(1) \sum_{i\colon c_i u > 1} c_i^2 = O(1) u^{\tau-3}(1 + o(1)), \tag{3.4}$$

the latter by an explicit computation using that $c_i = i^{1/(\tau-1)}$.

Further, with Corollary 2.11

$$\widetilde{\mathrm{Var}}\big(\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u)\big) \leq \sum_{i=2}^{\infty} c_i^4 (1 - \widetilde{\mathbb{P}}(T_i \leq u)) \leq C(\theta) \sum_{i=2}^{\infty} c_i^4 \mathrm{e}^{-c_i u} = O(1) u^{\tau-5}(1 + o(1)). \tag{3.5}$$

The Chebychev inequality now proves (3.1).

For (3.2), we again compute

$$\widetilde{\mathbb{E}}\Big[\sum_{i=2}^{\infty} c_i [1 - \mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big)\Big] \leq C(\theta) \sum_{i=2}^{\infty} c_i \mathrm{e}^{-c_i u/2} [\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i] = C(\theta) \sum_{i=2}^{\infty} c_i \mathrm{e}^{-c_i u/2} \mathrm{e}^{|\lambda| c_i} (\lambda c_i)^2/2. \tag{3.6}$$

Thus, for $|\lambda| \leq \delta u$ and again using Corollary 2.11, we obtain

$$\widetilde{\mathbb{E}}\Big[\sum_{i=2}^{\infty} c_i [1 - \mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big)\Big] = O(\lambda^2 u^{\tau-4}). \tag{3.7}$$

Further,

$$\begin{aligned} \widetilde{\mathrm{Var}}\Big(\sum_{i=2}^{\infty} c_i [1 - \mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big)\Big) &\leq C(\theta) \sum_{i=2}^{\infty} c_i^2 \mathrm{e}^{-c_i u/2} \big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big)^2 \\ &\leq C(\theta) |\lambda|^4 \sum_{i=2}^{\infty} c_i^6 \mathrm{e}^{-c_i u/2} \mathrm{e}^{2|\lambda| c_i} = O(|\lambda|^4 u^{\tau-7}). \end{aligned} \tag{3.8}$$

Again the claim follows from the Chebychev inequality. □

We continue to show that the probability for $\mathcal{S}_t$ to deviate far from its mean at some *discrete* times in the time interval $[\varepsilon, u - T u^{-(\tau-2)}]$ is small when T is large enough:

Proposition 3.2 (Probability to deviate far from mean at discrete times)**.** *Let $\eta > 0$ and $\delta_u = u^{-(\tau-2)}$. For any $\varepsilon > 0$ and $M > 0$,*

$$\limsup_{u\to\infty} u^{(\tau-1)/2}\widetilde{\mathbb{P}}\Big(\exists\, k \in \mathbb{N} \text{ s.t. } k\delta_u \in [\varepsilon, u - T\delta_u]\colon\ \big|\mathcal{S}_{k\delta_u} - \widetilde{\mathbb{E}}[\mathcal{S}_{k\delta_u}]\big| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_{k\delta_u}], \mathcal{S}_u \in [0, M/u]\Big) = o_T(1), \tag{3.9}$$

where we recall the definition of $o_T(1)$ from Proposition 2.13.

Proof. The proof is split between the cases $t \in [\varepsilon, u/2]$, $t \in [u/2, u-\varepsilon]$ and $t \in [u-\varepsilon, u - u^{-(\tau-2)}]$, where $\varepsilon > 0$ is some arbitrary constant.

Proof for $t \in [\varepsilon, u/2]$. We start by proving the proposition for $t \in [\varepsilon, u/2]$, for which we use Proposition 2.7 with $\lambda_1 = \pm 1$ and $\lambda_2 = 0$ to see that, for any $x > 0$,

$$\widetilde{\mathbb{P}}\Big(\Big|\frac{\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]}{\sqrt{I_V(t/u)u^{\tau-3}}}\Big| > x\Big) \le c\mathrm{e}^{-x}, \tag{3.10}$$

where we note that the e^{Θ} error term can be put inside the constant c since $|\Theta| \le o_u(1) + O(t^{3(\tau-3)})$ and $t \ge \varepsilon$ is strictly positive. By (2.31) in Lemma 2.10, $I_V(p) \le cp^{\tau-3}$ for all $p \in [0, 1/2]$. Applying this to $p = t/u$ yields

$$\widetilde{\mathbb{P}}\left(|\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| > cxt^{(\tau-3)/2}\right) \le c\mathrm{e}^{-x}. \tag{3.11}$$

By Corollary 2.5(a), we have $\widetilde{\mathbb{E}}[\mathcal{S}_t]/(tu^{\tau-3}) \in [\underline{c}, \overline{c}]$ for $t \in [\varepsilon, u/2]$ and some constants $\underline{c}, \overline{c} > 0$. Therefore, taking $x = a\eta t^{\frac{1}{2}(5-\tau)}u^{\tau-3}$ for some $a > 0$ chosen appropriately,

$$\widetilde{\mathbb{P}}\left(|\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_t]\right) \le c\mathrm{e}^{-a\eta t^{\frac{1}{2}(5-\tau)}u^{\tau-3}}. \tag{3.12}$$

We take $t = k\delta_u$ for $k\delta_u \in [\varepsilon, u/2]$, so that there are at most $u/\delta_u = u^{\tau-1}$possible values of k. Thus,

$$\widetilde{\mathbb{P}}\left(\exists t_k \in [\varepsilon, u/2]\colon |\mathcal{S}_{t_k} - \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\right) \le c(\epsilon)u^{\tau-1}\mathrm{e}^{-a\eta u^{(\tau-1)/2}}. \tag{3.13}$$

This proves the proposition for $k\delta_u \in [\varepsilon, u/2]$.

Proof for $t \in [u/2, u-\varepsilon]$. We continue by proving the proposition for $t \in [u/2, u-\varepsilon]$, for which we again use Proposition 2.7 with $\lambda_1 = \pm 1$ and $\lambda_2 = 0$ to see that, for any $x > 0$,

$$\widetilde{\mathbb{P}}\Big(\Big|\frac{\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]}{\sqrt{I_V(t/u)u^{\tau-3}}}\Big| > x\Big) \le c\mathrm{e}^{-x}. \tag{3.14}$$

By Lemma 2.10 and the fact that $I_V(p) > 0$ for every $p \in (0,1)$, we obtain that there exists a constant $c > 0$ such that $I_V(p) \ge c$ for all $p \in [1/2, 1-\varepsilon]$. Applying this to $p = t/u$ yields

$$\widetilde{\mathbb{P}}\left(|\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| > cxu^{(\tau-3)/2}\right) \le c\mathrm{e}^{-x}. \tag{3.15}$$

By Lemma 2.4(d) and Corollary 2.5(b), we have $\widetilde{\mathbb{E}}[\mathcal{S}_t]u^{-(\tau-2)} \in [\underline{c}, \overline{c}]$ for all $t \in [u/2, u-\varepsilon]$ and some constants $\underline{c} = \underline{c}(\varepsilon), \overline{c} = \overline{c}(\varepsilon) > 0$. Therefore, taking $x = a\eta u^{(\tau-1)/2}$ for some $a > 0$ chosen appropriately,

$$\widetilde{\mathbb{P}}\left(|\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_t]\right) \le c\mathrm{e}^{-a\eta u^{(\tau-1)/2}}. \tag{3.16}$$

We take $t = t_k = k\delta_u$ for $k\delta_u \in [u/2, u-\varepsilon]$, so that there are at most $u/\delta_u = u^{\tau-1}$ possible values of k. Thus,

$$\widetilde{\mathbb{P}}\left(\exists t_k \in [u/2, u-\varepsilon]\colon |\mathcal{S}_{t_k} - \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\right) \le c(\epsilon)u^{\tau-1}\mathrm{e}^{-a\eta u^{(\tau-1)/2}}. \tag{3.17}$$

This proves the proposition for $k\delta_u \in [u/2, u-\varepsilon]$.

Proof for $t \in [u-\varepsilon, u-Tu^{-(\tau-2)}]$: Rewrite. The proof for $t \in [u-\varepsilon, u-Tu^{-(\tau-2)}]$ is the hardest, and is split into three steps. We start by rewriting the event of interest. We define $s = u - t$ and investigate $\mathcal{S}_{u-s}$ in what follows, so that now $s \in [Tu^{-(\tau-2)}, \varepsilon]$.

Recall the definition of $Q_u(s)$ in (2.44),

$$Q_u(s) = u\mathcal{S}_{u-s} - s - (u-s)\mathcal{S}_u = -\sum_{i=2}^{\infty} c_i[u\mathbb{1}_{\{T_i \in (u-s,u]\}} - s\mathcal{I}_i(u)], \tag{3.18}$$

so that $|\mathcal{S}_{u-s} - \widetilde{\mathbb{E}}[\mathcal{S}_{u-s}]| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_{u-s}]$ precisely when

$$|Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s)] + (u-s)\big(\mathcal{S}_u - \widetilde{\mathbb{E}}[\mathcal{S}_u]\big)| > \eta u\widetilde{\mathbb{E}}[\mathcal{S}_{u-s}]. \tag{3.19}$$

When $\mathcal{S}_u \in [0, M/u]$ and using that $u\widetilde{\mathbb{E}}[\mathcal{S}_u] = o(1)$ by Lemma 2.4(d), we therefore obtain that if (3.19) holds, then

$$|Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s)]| > \eta u\widetilde{\mathbb{E}}[\mathcal{S}_{u-s}] - M + o(1). \tag{3.20}$$

By Lemma 2.4(d) and Corollary 2.5(b), we have that $\widetilde{\mathbb{E}}[\mathcal{S}_{u-s}] \geq csu^{\tau-3}$ for some $c > 0$. Therefore, $\eta u\widetilde{\mathbb{E}}[\mathcal{S}_{u-s}] \geq c\eta T$, so that, by taking $T = T(M)$ sufficiently large, we obtain that

$$\begin{aligned}&\widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon \big|\mathcal{S}_{u-s_k} - \widetilde{\mathbb{E}}[\mathcal{S}_{u-k}]\big| > \eta\widetilde{\mathbb{E}}[\mathcal{S}_{u-s_k}], \mathcal{S}_u \in [0, M/u]\Big) \\ &\quad\leq \widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta cs_k u^{\tau-2}, \mathcal{S}_u \in [0, M/u]\Big).\end{aligned} \tag{3.21}$$

We condition on $\mathcal{J}(u)$ from (2.42), and note that $\mathcal{S}_u$ is measurable w.r.t $\mathcal{J}(u)$ to obtain

$$\begin{aligned}&\widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta cs_k u^{\tau-2}, \mathcal{S}_u \in [0, M/u]\Big) \\ &\quad= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\{\mathcal{S}_u \in [0,M/u]\}}\widetilde{\mathbb{P}}\big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta cs_k u^{\tau-2} \mid \mathcal{J}(u)\big)\Big].\end{aligned} \tag{3.22}$$

This is the starting point of our analysis. We split, writing $\eta' = \eta/2$,

$$\begin{aligned}&\widetilde{\mathbb{P}}\big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta cs_k u^{\tau-2} \mid \mathcal{J}(u)\big) \\ &\quad\leq \widetilde{\mathbb{P}}\big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)]| > \eta' cs_k u^{\tau-2} \mid \mathcal{J}(u)\big) \\ &\qquad+ \mathbb{1}_{\{\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |\widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)] - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta' cs_k u^{\tau-2}\}}.\end{aligned} \tag{3.23}$$

We conclude using the union bound that

$$\begin{aligned}&\widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta cs_k u^{\tau-2}, \mathcal{S}_u \in [0, M/u]\Big) \\ &\quad\leq \sum_{k \colon s_k \in [Tu^{-(\tau-2)}, \varepsilon]} \widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{P}}\Big(|Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)]| > \eta' cs_k u^{\tau-2} \mid \mathcal{J}(u)\Big)\mathbb{1}_{\{\mathcal{S}_u \in [0,M/u]\}}\Big] \\ &\qquad+ \widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |\widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)] - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta' cs_k u^{\tau-2}\Big).\end{aligned} \tag{3.24}$$

We will bound both contributions separately, and start by setting the stage. We compute that

$$\begin{aligned}Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)] &= -\sum_{i=2}^{\infty} c_i u\big[\mathbb{1}_{\{T_i \in (u-s,u]\}} - \widetilde{\mathbb{P}}(T_i \in (u-s, u] \mid \mathcal{J}(u))\big] \\ &= -\sum_{i=2}^{\infty} c_i u\big[\mathbb{1}_{\{T_i \in (u-s,u]\}} - p_{i,u}(s)\big],\end{aligned} \tag{3.25}$$

where we abbreviate

$$p_{i,u}(s) = \widetilde{\mathbb{P}}(T_i \in (u-s, u] \mid \mathcal{J}(u)) = \widetilde{\mathbb{P}}(T_i \in (u-s, u] \mid i \in \mathcal{J}(u)). \tag{3.26}$$

It turns out that both contributions in (3.24) can be expressed in terms of $p_{i,u}(s)$, and we continue our analysis by studying this quantity in more detail.

Proof for $t \in [u-\varepsilon, u-Tu^{-(\tau-2)}]$: Analysis of $p_{i,u}(s)$. We next analyse the conditional probability $p_{i,u}(s)$. We compute (recall (1.22), (2.28) and (2.42))

$$p_{i,u}(s) = \widetilde{\mathbb{P}}(T_i \in (u-s,u] \mid i \in \mathcal{J}(u)) = \frac{\widetilde{\mathbb{P}}(T_i \in (u-s,u])}{\widetilde{\mathbb{P}}(T_i \leq u)} = \frac{\widetilde{\mathbb{P}}(T_i \leq u) - \widetilde{\mathbb{P}}(T_i \leq u-s)}{\widetilde{\mathbb{P}}(T_i \leq u)}. \tag{3.27}$$

Using the distribution of T_i formulated in Lemma 2.9, we obtain, for any $s \in [0,u]$,

$$\widetilde{\mathbb{P}}(T_i \leq u-s) = \frac{\mathrm{e}^{\theta c_i u}(1-\mathrm{e}^{-c_i(u-s)})}{\mathrm{e}^{\theta c_i u}(1-\mathrm{e}^{-c_i u}) + \mathrm{e}^{-c_i u}}, \tag{3.28}$$

so that

$$p_{i,u}(s) = \frac{\mathrm{e}^{\theta c_i u}(1-\mathrm{e}^{-c_i u}) - \mathrm{e}^{\theta c_i u}(1-\mathrm{e}^{-c_i(u-s)})}{\mathrm{e}^{\theta c_i u}(1-\mathrm{e}^{-c_i u})} = \frac{\mathrm{e}^{-c_i u}(\mathrm{e}^{c_i s}-1)}{1-\mathrm{e}^{-c_i u}} = \frac{\mathrm{e}^{c_i s}-1}{\mathrm{e}^{c_i u}-1}. \tag{3.29}$$

We start by bounding $p_{i,u}(s)$, for $s \in [0,\varepsilon]$, by

$$p_{i,u}(s) \leq O(s/u), \qquad \text{and} \qquad p_{i,u}(s) \leq O(c_i s)\mathrm{e}^{-c_i u}(c_i u \wedge 1)^{-1}. \tag{3.30}$$

Moreover, for u sufficiently large,

$$|up_{i,u}(s) - s| \leq s(c_i u \wedge 1). \tag{3.31}$$

Proof for $t \in [u-\varepsilon, u-Tu^{-(\tau-2)}]$: Completion first term (3.24). For the first term in (3.24), we use Markov's inequality in the form $\mathbb{P}(|X-\mathbb{E}[X]| > a) \leq a^{-4}\mathbb{E}[(X-\mathbb{E}[X])^4]$ to obtain

$$\widetilde{\mathbb{P}}\big(|Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)]| > \eta' csu^{\tau-2} \mid \mathcal{J}(u)\big) \leq (\eta' csu^{\tau-2})^{-4}\widetilde{\mathbb{E}}\big[(Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)])^4 \mid \mathcal{J}(u)\big], \tag{3.32}$$

and recall from (3.25) that

$$Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)] = -\sum_{i=2}^{\infty} c_i u\big[\mathbb{1}_{\{T_i \in (u-s,u]\}} - \widetilde{\mathbb{P}}(T_i \in (u-s,u] \mid \mathcal{J}(u))\big] = -\sum_{i=2}^{\infty} c_i u\big[\mathbb{1}_{\{T_i \in (u-s,u]\}} - p_{i,u}(s)\big]. \tag{3.33}$$

The summands are conditionally independent given $\mathcal{J}(u)$ and identically 0 when $\mathcal{I}_i(u) = 0$, so that

$$\begin{aligned}\widetilde{\mathbb{E}}\big[(Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)])^4 \mid \mathcal{J}(u)\big] &\leq \sum_{i\geq 2} c_i^4 u^4 p_{i,u}(s)\mathcal{I}_i(u) \\ &\quad + \sum_{i,j\geq 2\colon i\neq j} c_i^2 c_j^2 u^4 p_{i,u}(s)(1-p_{i,u}(s))\mathcal{I}_i(u) p_{j,u}(s)(1-p_{j,u}(s))\mathcal{I}_j(u).\end{aligned} \tag{3.34}$$

By the second bound in (3.30) and Corollary 2.11, the first term is at most

$$O(1)su^4 \sum_{i\geq 2} c_i^5 \mathrm{e}^{-c_i u}(c_i u \wedge 1)^{-1} \leq O(1)su^4 \sum_{i\geq 2} c_i^5 \mathrm{e}^{-c_i u}[1 + (c_i u)^{-1}] = O(su^{\tau-2}). \tag{3.35}$$

By (3.1) in Lemma 3.1, we may assume that $\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u) \leq Ku^{\tau-3}$, since the complement has a probability that is $o(u^{-(\tau-1)/2})$. Then, in a similar way, using the first bound in (3.30), the second term is at most

$$\Big(\sum_{i\geq 2} c_i^2 u^2 p_{i,u}(s)\mathcal{I}_i(u)\Big)^2 \leq O(1)\Big(s\sum_{i\geq 2} c_i^2 u \mathcal{I}_i(u)\Big)^2 = O\big((su^{\tau-2})^2\big). \tag{3.36}$$

As a result,

$$\widetilde{\mathbb{E}}\big[(Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)])^4 \mid \mathcal{J}(u)\big] \leq O(su^{\tau-2}) + O\big((su^{\tau-2})^2\big). \tag{3.37}$$

Since $s \geq Tu^{-(\tau-2)}$, this can be simplified to

$$\widetilde{\mathbb{E}}\big[(Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)])^4 \mid \mathcal{J}(u)\big] \leq O\big((su^{\tau-2})^2\big). \tag{3.38}$$

We conclude using (3.32) that, on the event that $\{\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u) \leq K u^{\tau-3}\}$,

$$\widetilde{\mathbb{P}}\big(|Q_u(s) - \widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)]| > \eta' c s u^{\tau-2} \mid \mathcal{J}(u)\big) \leq \frac{(csu^{\tau-2})^2}{(\eta' c s u^{\tau-2})^4} = O(\eta^{-4}(su^{\tau-2})^{-2}), \tag{3.39}$$

so that, also using that $\widetilde{\mathbb{P}}(\mathcal{S}_u \in [0, M/u]) = O(u^{-(\tau-1)/2})$ by Proposition 2.8,

$$\begin{aligned} u^{(\tau-1)/2} \widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{P}}\Big(|Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)]| > \eta' c s_k u^{\tau-2} \mid \mathcal{J}(u)\Big) \mathbb{1}_{\{\mathcal{S}_u \in [0, M/u]\}}\Big] \\ \leq O(\eta^{-4}(su^{\tau-2})^{-2}) u^{(\tau-1)/2} \widetilde{\mathbb{P}}(\mathcal{S}_u \in [0, M/u]) = O(\eta^{-4}(su^{\tau-2})^{-2}). \end{aligned} \tag{3.40}$$

This bound is true for any $s \in [Tu^{-(\tau-2)}, \varepsilon]$. Taking $s = s_k = ku^{-(\tau-2)}$ and summing out over $k \geq T$ leads to

$$\begin{aligned} u^{(\tau-1)/2} \sum_{k\colon s_k \in [Tu^{-(\tau-2)}, \varepsilon]} \widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{P}}\Big(|Q_u(s_k) - \widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)]| > \eta' c s_k u^{\tau-2} \mid \mathcal{J}(u)\Big) \mathbb{1}_{\{\mathcal{S}_u \in [0, M/u]\}}\Big] \\ \leq O(\eta^{-4}) \sum_{k \geq T} k^{-2} = O(\eta^{-4}/T) = o_T(1), \end{aligned} \tag{3.41}$$

when we take $T = T(\eta)$ sufficiently large, as required.

Proof for $t \in [u - \varepsilon, u - Tu^{-(\tau-2)}]$: Completion second term (3.24). For the second term in (3.24), we need to bound

$$\widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |\widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)] - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta' c s_k u^{\tau-2}\Big). \tag{3.42}$$

We compute using (3.18)

$$\widetilde{\mathbb{E}}[Q_u(s)] = -\sum_{i=2}^{\infty} c_i \big[u \widetilde{\mathbb{P}}(T_i \in (u-s, u]) - s \widetilde{\mathbb{P}}(i \in \mathcal{J}(u))\big], \tag{3.43}$$

while

$$\widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)] = -\sum_{i=2}^{\infty} c_i \mathcal{I}_i(u) \big[u \widetilde{\mathbb{P}}(T_i \in (u-s, u] \mid i \in \mathcal{J}(u)) - s\big]. \tag{3.44}$$

As a result, using (3.26),

$$\widetilde{\mathbb{E}}[Q_u(s) \mid \mathcal{J}(u)] - \widetilde{\mathbb{E}}[Q_u(s)] = -\sum_{i=2}^{\infty} c_i \big[\mathcal{I}_i(u) - \widetilde{\mathbb{P}}(i \in \mathcal{J}(u))\big][up_{i,u}(s) - s] =: sX + Y(s), \tag{3.45}$$

where with (3.29)

$$X = -\sum_{i=2}^{\infty} c_i \big[\mathcal{I}_i(u) - \widetilde{\mathbb{P}}(i \in \mathcal{J}(u))\big] \frac{1 + c_i u - \mathrm{e}^{c_i u}}{\mathrm{e}^{c_i u} - 1}, \qquad Y(s) = -u \sum_{i=2}^{\infty} c_i \big[\mathcal{I}_i(u) - \widetilde{\mathbb{P}}(i \in \mathcal{J}(u))\big] \frac{\mathrm{e}^{c_i s} - 1 - c_i s}{\mathrm{e}^{c_i u} - 1}. \tag{3.46}$$

As a result,

$$\begin{aligned} \widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |\widetilde{\mathbb{E}}[Q_u(s_k) \mid \mathcal{J}(u)] - \widetilde{\mathbb{E}}[Q_u(s_k)]| > \eta' c s_k u^{\tau-2}\Big) \\ \leq \widetilde{\mathbb{P}}\big(|X| \geq \eta' c u^{\tau-2}/2\big) + \widetilde{\mathbb{P}}\Big(\exists s_k \in [Tu^{-(\tau-2)}, \varepsilon] \colon |Y(s_k)| \geq \eta' c s_k u^{\tau-2}/2\Big). \end{aligned} \tag{3.47}$$

For both terms, we use the Chebychev inequality.

For X, as $\widetilde{\mathbb{E}}[X]=0$, this leads to

$$\widetilde{\mathbb{P}}\big(|X|\geq \eta' c u^{\tau-2}/2\big)\leq \frac{4}{(\eta' c u^{\tau-2})^2}\mathrm{Var}(X). \tag{3.48}$$

We use Lemma 2.9 to see that $\widetilde{\mathbb{P}}(i\in\mathcal{J}(u))=\frac{1-\mathrm{e}^{-c_iu}}{1-\mathrm{e}^{-c_iu}+\mathrm{e}^{-c_iu(1+\theta)}}$, so that

$$\widetilde{\mathbb{P}}(i\in\mathcal{J}(u))\widetilde{\mathbb{P}}(i\notin\mathcal{J}(u))=\widetilde{\mathbb{P}}(\mathcal{I}_i(u)=1)\widetilde{\mathbb{P}}(\mathcal{I}_i(u)=0)=\frac{(1-\mathrm{e}^{-c_iu})\mathrm{e}^{-c_iu(1+\theta)}}{(1-\mathrm{e}^{-c_iu}+\mathrm{e}^{-c_iu(1+\theta)})^2}\leq O(1)c_iu\mathrm{e}^{-c_iu(1+\theta)}, \tag{3.49}$$

since $1-\mathrm{e}^{-x}+\mathrm{e}^{-x(1+\theta)}$ is uniformly bounded from below away from 0 for all $x\geq 0$. We use this together with Corollary 2.11 to compute that

$$\begin{aligned}\mathrm{Var}(X)&=\sum_{i=2}^{\infty}c_i^2\widetilde{\mathbb{P}}(\mathcal{I}_i(u)=1)\widetilde{\mathbb{P}}(\mathcal{I}_i(u)=0)\Big(\frac{1+c_iu-\mathrm{e}^{c_iu}}{\mathrm{e}^{c_iu}-1}\Big)^2\\&\leq O(1)u\sum_{i=2}^{\infty}c_i^3\mathrm{e}^{-c_iu(1+\theta)}=O(1)u^{\tau-3}.\end{aligned} \tag{3.50}$$

Therefore,

$$\widetilde{\mathbb{P}}\big(|X|\geq \eta' c u^{\tau-2}/2\big)\leq O(1)(\eta')^{-2}u^{-2(\tau-2)}\mathrm{Var}(X)=O(1)(\eta')^{-2}u^{-(\tau-1)}=o(u^{-(\tau-1)/2}), \tag{3.51}$$

as required below.

For the term involving $Y(s)$, we start by using the union bound to obtain

$$\widetilde{\mathbb{P}}\Big(\exists s_k\in[Tu^{-(\tau-2)},\varepsilon]\colon |Y(s_k)|\geq \eta' c s_k u^{\tau-2}/2\Big)\leq \varepsilon u^{\tau-2}\max_{k\colon s_k\in[Tu^{-(\tau-2)},\varepsilon]}\widetilde{\mathbb{P}}\big(|Y(s_k)|\geq \eta' c s_k u^{\tau-2}/2\big). \tag{3.52}$$

Then, by the Chebychev inequality and as $\widetilde{\mathbb{E}}[Y(s_k)]=0$,

$$\widetilde{\mathbb{P}}\big(|Y(s_k)|\geq \eta' c s_k u^{\tau-2}/2\big)\leq \frac{4}{(\eta' c s_k u^{\tau-2})^2}\mathrm{Var}(Y(s_k)), \tag{3.53}$$

where, using (3.49), $\mathrm{e}^{c_is}-1-c_is=O(s^2c_i^2)$ and $\mathrm{e}^{c_iu}-1\geq c_iu$,

$$\begin{aligned}\mathrm{Var}(Y(s))&=u^2\sum_{i=2}^{\infty}c_i^2\widetilde{\mathbb{P}}(i\in\mathcal{J}(u))\widetilde{\mathbb{P}}(i\notin\mathcal{J}(u))\Big(\frac{\mathrm{e}^{c_is}-1-c_is}{\mathrm{e}^{c_iu}-1}\Big)^2\\&\leq O(s^4)\sum_{i=2}^{\infty}c_i^4\mathrm{e}^{-c_iu(1+\theta)}=O(s^4)O(u^{\tau-5}),\end{aligned} \tag{3.54}$$

where we used Corollary 2.11 in the last equality. Substituting this into (3.52) and (3.53), we arrive at

$$\begin{aligned}\widetilde{\mathbb{P}}\Big(\exists s_k\in[Tu^{-(\tau-2)},\varepsilon]\colon |Y(s_k)|\geq \eta' c s_k u^{\tau-2}/2\Big)&\leq \varepsilon u^{\tau-2}\max_{k\colon s_k\in[Tu^{-(\tau-2)},\varepsilon]}O(s_k^2u^{\tau-5}u^{-2(\tau-2)})(\eta')^{-2}\\&=O(\varepsilon^3/(\eta')^2)u^{-3}=o(u^{-(\tau-1)/2}),\end{aligned} \tag{3.55}$$

since $\tau\in(3,4)$. Combining (3.51) and (3.55) in (3.47) completes the proof. □

We now know that with high probability the process does not deviate much from its mean when observed at the discrete times $k\delta_u\in[\varepsilon,u-T\delta_u]$. We continue to show that this actually holds with high probability on the whole interval $[\varepsilon,u-T\delta_u]$. We complete the preparations for the proof of Proposition 2.13 by proving that it is unlikely for the process to deviate far from the mean for all times $t\in[\varepsilon,u-T\delta_u]$ simultaneously:

Proposition 3.3 (Probability to deviate far from mean at some time)**.** *For every $\eta > 0$ and $M > 0$,*

$$\limsup_{u\to\infty} u^{(\tau-1)/2}\widetilde{\mathbb{P}}\big(\exists\, t \in [\varepsilon, u - T\delta_u] \colon |\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t], \mathcal{S}_u \in [0, M/u]\big) = o_T(1). \tag{3.56}$$

Proof. Fix $T > 0$ and recall that $\delta_u = u^{-(\tau-2)}$. Let

$$\begin{aligned} E_u = \{|\mathcal{S}_{k\delta_u} - \widetilde{\mathbb{E}}[\mathcal{S}_{k\delta_u}]| \leq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{k\delta_u}] \ \forall k \text{ s.t. } k\delta_u \in [\varepsilon, u - T\delta_u]\} \\ \cap \Big\{ \sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big) \leq K\lambda^2 u^{\tau-4} \Big\} \cap \Big\{ \sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u) \leq K u^{\tau-3} \Big\}, \end{aligned} \tag{3.57}$$

where we take $\lambda = \delta u$ with $\delta > 0$ sufficiently small and $K \geq 1$ as in Lemma 3.1. We first give a bound on $\widetilde{\mathbb{P}}(E_u^c \cap \{\mathcal{S}_u \in [0, M/u]\})$. We apply (3.1) in Lemma 3.1 to obtain that

$$\widetilde{\mathbb{P}}(\sum_{i=2}^{\infty} c_i^2 \mathcal{I}_i(u) \geq K u^{\tau-3}) = O(u^{-(\tau-1)}) = o(u^{-(\tau-1)/2}), \tag{3.58}$$

which is contained in the error term in (3.56). Further, by (3.2) in Lemma 3.1

$$\widetilde{\mathbb{P}}(\sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big) \geq K\lambda^2 u^{\tau-4}) = O(u^{-(\tau-1)}) = o(u^{-(\tau-1)/2}). \tag{3.59}$$

Combined with Proposition 3.2, this ensures that

$$\limsup_{u\to\infty} u^{(\tau-1)/2}\widetilde{\mathbb{P}}(E_u^c \cap \{\mathcal{S}_u \in [0, M/u]\}) = o_T(1). \tag{3.60}$$

As a result, we are left to control the fluctuations of the process on any interval $I_k = [k\delta_u, (k+1)\delta_u]$. We use Boole's inequality to bound

$$\begin{aligned} &\widetilde{\mathbb{P}}(E_u, \exists\, t \in [\varepsilon, u - T\delta_u] \colon |\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]) \\ &\quad \leq \sum_{k \colon k\delta_u \in [\varepsilon, u - T\delta_u]} \widetilde{\mathbb{P}}\big(E_u, \exists\, t \in I_k \colon |\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t]| \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]\big). \end{aligned} \tag{3.61}$$

Let $t_k = k\delta_u$, so that $I_k = [t_k, t_{k+1}]$. We split the analysis into four cases, depending on whether $t_k \leq u/2$ or not, and on whether $\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]$ or $\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \leq -2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]$, which we refer to as 'large upper' and 'large lower' deviations, respectively.

Part 1: The case $t_k \leq u/2$ and a large upper deviation. We start by bounding the probability that there exists a $t \in I_k = [t_k, t_{k+1}]$, $\epsilon \leq t_k \leq u/2$ such that $\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]$. Using that $\widetilde{\mathbb{E}}[\mathcal{S}_t] = \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}](1 + o(1))$ for any $t \in I_k$ by Corollary 2.5(c), we bound

$$\widetilde{\mathbb{P}}(E_u, \exists\, t \in I_k \colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]) \leq \widetilde{\mathbb{P}}(\exists\, t \leq \delta_u \colon \mathcal{S}_{t+t_k} - \mathcal{S}_{t_k} \geq \frac{\eta}{2}\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]). \tag{3.62}$$

By (1.20),

$$\mathcal{S}_{t+t_k} - \mathcal{S}_{t_k} = \tilde{\beta} t + \sum_{i=2}^{\infty} c_i[\mathcal{I}_i(t + t_k) - \mathcal{I}_i(t_k) - c_i t], \tag{3.63}$$

which can be stochastically dominated by the process $\tilde{\beta} t + \mathcal{R}_t$ with $\mathcal{R}_t \equiv \sum_{i=2}^{\infty} c_i[N_i(t) - c_i t]$, where $(N_i(t))_{t\geq 0}$ is a Poisson process with rate $c_i t$. As a result,

$$\widetilde{\mathbb{P}}(E_u, \exists\, t \in I_k \colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]) \leq \widetilde{\mathbb{P}}(\exists\, t \leq \delta_u \colon \tilde{\beta} t + \mathcal{R}_t \geq \frac{\eta}{2}\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]). \tag{3.64}$$

Since $(\mathcal{R}_t)_{t\geq 0}$ is a finite-variance Lévy process, it is well-concentrated. In more detail, for $\lambda \in \mathbb{R}$, we define the exponential martingale

$$\mathcal{M}_t(\lambda) = \mathrm{e}^{\lambda \mathcal{R}_t - t\phi(\lambda)}, \qquad \text{where} \qquad \phi(\lambda) = \log \widetilde{\mathbb{E}}[\mathrm{e}^{\lambda \mathcal{R}_1}] = \sum_{i=2}^{\infty} c_i[\mathrm{e}^{c_i\lambda} - 1 - c_i\lambda]. \tag{3.65}$$

Then, for every $\lambda \geq 0$, using that $\phi(\lambda) \geq 0$ and by Doob's inequality,

$$\begin{aligned}\widetilde{\mathbb{P}}(\exists\, s \leq t \colon \tilde{\beta}s + \mathcal{R}_s \geq x) &\leq \widetilde{\mathbb{P}}(\exists\, s \leq t \colon \mathcal{M}_s(\lambda) \geq \mathrm{e}^{x\lambda - t\phi(\lambda) - t|\tilde{\beta}|\lambda}) \\ &\leq \mathrm{e}^{-2[x\lambda - t\phi(\lambda) - t|\tilde{\beta}|\lambda]}\widetilde{\mathbb{E}}[\mathcal{M}_t(\lambda)^2] = \mathrm{e}^{-[2x\lambda - t\phi(2\lambda) - 2t|\tilde{\beta}|\lambda]}.\end{aligned} \tag{3.66}$$

We apply this inequality to $x = \frac{\eta}{2}\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]$, $t = \delta_u$ and $\lambda = 1$, and Corollary 2.5(a) implies that $\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}] \geq c t_k u^{\tau-3}$ for $t_k = k\delta_u \in [\varepsilon, u/2]$. Therefore (using $t_k = k\delta_u$)

$$\widetilde{\mathbb{P}}(E_u, \exists\, t \in I_k \colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]) \leq (1 + o(1))\mathrm{e}^{-ck\delta_u u^{\tau-3}}, \tag{3.67}$$

which is small even when summed out over k as above.

Part 2: The case $t_k \leq u/2$ and a large lower deviation. We continue with bounding the probability that there exists a $t \in I_k = [t_k, t_{k+1}]$, $\epsilon \leq t_k \leq u/2$ such that $\mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \leq -2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]$, which is slightly more involved. Again using that $\widetilde{\mathbb{E}}[\mathcal{S}_t] = \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}](1 + o(1))$ for any $t \in I_k$ by Corollary 2.5(c), we bound

$$\widetilde{\mathbb{P}}(E_u, \exists\, t \in I_k \colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \leq -2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]) \leq \widetilde{\mathbb{P}}(\exists\, t \leq \delta_u \colon \mathcal{S}_{t+t_k} - \mathcal{S}_{t_k} \leq -\frac{\eta}{2}\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]). \tag{3.68}$$

Further,

$$\begin{aligned}\mathcal{S}_{t+t_k} - \mathcal{S}_{t_k} &= \tilde{\beta}t + \sum_{i=2}^{\infty} c_i[\mathcal{I}_i(t+t_k) - \mathcal{I}_i(t_k) - c_i t] \\ &= \tilde{\beta}t + \sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(t_k)]\big[\mathcal{I}_i(t+t_k) - \mathcal{I}_i(t_k) - c_i t\big] - t\sum_{i=2}^{\infty} c_i^2\mathcal{I}_i(t_k) \\ &= \tilde{\beta}t + \sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(t_k)][N_i(t) - c_i t] - \sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(t_k)][N_i(t) - \mathbb{1}_{\{N_i(t)\geq 1\}}] - t\sum_{i=2}^{\infty} c_i^2\mathcal{I}_i(t_k) \\ &\geq \tilde{\beta}t + \mathcal{R}'_t - \mathcal{D}_t - \delta_u\mathcal{E}_u,\end{aligned} \tag{3.69}$$

where we set

$$\mathcal{R}'_t = \sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(t_k)][N_i(t) - c_i t] \tag{3.70}$$

and

$$\mathcal{D}_t = \sum_{i=2}^{\infty} c_i[1 - \mathcal{I}_i(t_k)][N_i(t) - \mathbb{1}_{\{N_i(t)\geq 1\}}], \quad \mathcal{E}_u = \sum_{i=2}^{\infty} c_i^2\mathcal{I}_i(u). \tag{3.71}$$

Thus, conditionally on $(\mathcal{I}_i(t_k))_{i\geq 1}$, the process $(\mathcal{R}'_t)_{t\geq 0}$ is a Lévy process similar to the Lévy process investigated in Part 1 above, $\mathcal{D}_t$ is the contribution due to i for which $N_i(t) \geq 2$, while $\mathcal{E}_u$ yields an upper bound for the decrease in the drift of our process. We deal with the three terms one by one, starting with $(\mathcal{R}'_t)_{t\geq 0}$. As in the previous part,

$$\widetilde{\mathbb{P}}(\exists\, s \leq \delta_u \colon \tilde{\beta}s + \mathcal{R}'_s \leq -\frac{\eta}{4}\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]) \tag{3.72}$$

is small enough even when summed out over k such that $t_k \in [\varepsilon, u/2]$. This again follows by Doob's inequality and the bound that for any $\lambda \geq 0$, and with $\mathcal{F}_{t_k}$ the σ-algebra generated by $(\mathcal{S}_t)_{t\in[0,t_k]}$,

$$\widetilde{\mathbb{P}}(\exists\, s \leq t\colon \tilde{\beta}s + \mathcal{R}'_s \leq -x \mid \mathcal{F}_{t_k}) \leq \widetilde{\mathbb{P}}(\exists\, s \leq t\colon \mathcal{M}'_s(-\lambda) \geq \mathrm{e}^{x\lambda - t\phi'(-\lambda) - t|\tilde{\beta}|\lambda} \mid \mathcal{F}_{t_k}) \tag{3.73}$$
$$\leq \mathrm{e}^{-[2x\lambda - t\phi'(-\lambda) - t|\tilde{\beta}|\lambda]}\widetilde{\mathbb{E}}[\mathcal{M}'_t(-\lambda)^2 \mid \mathcal{F}_{t_k}] = \mathrm{e}^{-[2x\lambda - t\phi'(-2\lambda) - 2t|\tilde{\beta}|\lambda]},$$

where

$$\mathcal{M}'_t(\lambda) = \mathrm{e}^{\lambda\mathcal{R}'_t - t\phi'(\lambda)}, \qquad \text{with} \qquad \phi'(\lambda) = \log\widetilde{\mathbb{E}}[\mathrm{e}^{\lambda\mathcal{R}'_1} \mid \mathcal{F}_{t_k}]. \tag{3.74}$$

We compute that

$$\phi'(\lambda) = \sum_{i=2}^{\infty}\log\widetilde{\mathbb{E}}[\mathrm{e}^{\lambda c_i[1-\mathcal{I}_i(t_k)][N_i(1)-c_i]} \mid \mathcal{F}_{t_k}] = \sum_{i=2}^{\infty} c_i[1-\mathcal{I}_i(t_k)]\big(\mathrm{e}^{\lambda c_i} - 1 - \lambda c_i\big). \tag{3.75}$$

Now follow the same steps as in Part 1, using that $0 \leq \phi'(-2) \leq \text{const}$. By Lemma 3.1, the term $\delta_u\mathcal{E}_u$ is, with probability at least $1 - Cu^{-(\tau-1)}$ bounded by $\delta_u K u^{\tau-3} = K/u$, which is $o(\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}])$ as $\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}] \geq ct_k u^{\tau-3}$ for $t_k \in [\varepsilon, u/2]$ by Corollary 2.5(a). We continue to bound $\mathcal{D}_t$ by bounding

$$\widetilde{\mathbb{P}}(\exists\, t \leq \delta_u\colon \mathcal{D}_t \geq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]/4) = \widetilde{\mathbb{P}}(\mathcal{D}_{\delta_u} \geq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]/4), \tag{3.76}$$

since the process $t \mapsto \mathcal{D}_t$ is non-decreasing. By the Markov inequality,

$$\widetilde{\mathbb{P}}(\mathcal{D}_t \geq x) \leq x^{-1}\widetilde{\mathbb{E}}[\mathcal{D}_t] \leq x^{-1}\sum_{i=2}^{\infty} c_i\widetilde{\mathbb{E}}[N_i(t) - \mathbb{1}_{\{N_i(t)\geq 1\}}] \leq cx^{-1}\sum_{i=2}^{\infty} c_i(c_i t)^2 \leq ct^2/x. \tag{3.77}$$

Applying this to $x = \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]/4$ with $\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}] \geq ct_k u^{\tau-3}$ and $t = \delta_u = u^{-(\tau-2)}$ yields

$$\widetilde{\mathbb{P}}(\exists\, t \leq \delta_u\colon \mathcal{D}_t \geq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]/4) \leq cu^{-2(\tau-2)}(\eta t_k)^{-1}u^{-(\tau-3)} = ck^{-1}u^{-(2\tau-5)}. \tag{3.78}$$

When summing this out over k such that $t_k = k\delta_u \in [\varepsilon, u/2]$ we obtain a bound $c(\log u)u^{-(2\tau-5)} = o(u^{-(\tau-1)/2})$, since $(2\tau - 5) > (\tau-1)/2$ precisely when $\tau > 3$. This proves that

$$\sum_{k\colon k\delta_u\in[\varepsilon,u/2]}\widetilde{\mathbb{P}}(\mathcal{D}_{\delta_u} \geq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]/4) = o(u^{-(\tau-1)/2}), \tag{3.79}$$

as required. Collecting terms completes Part 2.

Part 3: The case $t_k \geq u/2$ and a large upper deviation. This proof is more subtle. We fix k such that $t_k \in [u/2, u - T\delta_u]$ and condition on $\mathcal{F}_{t_k}$, which is the σ-field generated by $(\mathcal{S}_t)_{t\leq t_k}$ to write (recall (3.57))

$$\widetilde{\mathbb{P}}\big(E_u, \mathcal{S}_u \in [0, M/u], \exists\, t \in I_k\colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]\big) \tag{3.80}$$
$$\leq \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\{|\mathcal{S}_{t_k} - \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]| \leq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}], \sum_{i=2}^{\infty} c_i[1-\mathcal{I}_i(u/2)]\left(\mathrm{e}^{\lambda c_i}-1-\lambda c_i\right) \leq K\lambda^2 u^{\tau-4}\}}\widetilde{\mathbb{P}}\big(\exists\, t \in I_k\colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t] \mid \mathcal{F}_{t_k}\big)\Big].$$

First observe that on $\{|S_{t_k} - \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]| \leq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\}$, we have

$$\widetilde{\mathbb{P}}\big(\exists\, t \in I_k\colon \mathcal{S}_t - \widetilde{\mathbb{E}}[\mathcal{S}_t] \geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t] \mid \mathcal{F}_{t_k}\big) \leq \widetilde{\mathbb{P}}\big(\exists\, t \leq \delta_u\colon \mathcal{S}_{t+t_k} - \mathcal{S}_{t_k} \geq \frac{\eta}{2}\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}] \mid \mathcal{F}_{t_k}\big) \tag{3.81}$$

by using that $\widetilde{\mathbb{E}}[\mathcal{S}_t] = \widetilde{\mathbb{E}}[\mathcal{S}_{t_k}](1 + o_T(1))$ for any $t \in I_k$ by Corollary 2.5(d). Similar to (3.69), we bound from above

$$\mathcal{S}_{t+t_k} - \mathcal{S}_{t_k} = \tilde{\beta}t + \sum_{i=2}^{\infty} c_i[1-\mathcal{I}_i(t_k)]\big[\mathcal{I}_i(t+t_k) - \mathcal{I}_i(t_k) - c_i t\big] - t\sum_{i=2}^{\infty} c_i^2\mathcal{I}_i(t_k) \tag{3.82}$$
$$\leq \tilde{\beta}t + \sum_{i=2}^{\infty} c_i[1-\mathcal{I}_i(t_k)][N_i(t) - c_i t] = \tilde{\beta}t + \mathcal{R}'_t,$$

where we note that $\mathcal{R}'_t$ is as in Part 2. Conditionally on $\mathcal{F}_{t_k}$, the process $(\mathcal{R}'_t)_{t\geq 0}$ is a Lévy process, and we use

$$\begin{aligned}\widetilde{\mathbb{P}}(\exists\, s\leq t\colon \tilde\beta s+\mathcal{R}'_s\geq x\mid \mathcal{F}_{t_k})&\leq \widetilde{\mathbb{P}}(\exists\, s\leq t\colon \mathcal{M}'_s(\lambda)\geq \mathrm{e}^{x\lambda-t\phi''(\lambda)-t|\tilde\beta|\lambda}\mid \mathcal{F}_{t_k})\\&\leq \mathrm{e}^{-2[x\lambda-t\phi''(\lambda)-t|\tilde\beta|\lambda]}\widetilde{\mathbb{E}}[\mathcal{M}'_t(\lambda)^2\mid \mathcal{F}_{t_k}]=\mathrm{e}^{-2x\lambda+t\phi''(2\lambda)+2t|\tilde\beta|\lambda},\end{aligned}\tag{3.83}$$

where we recall equations (3.74) and (3.75). Since $\mathrm{e}^{\lambda c_i}-1-\lambda c_i\geq 0$ for every $\lambda\in\mathbb{R}$, and since $1-\mathcal{I}_i(t_k)\leq 1-\mathcal{I}_i(u/2)$ for every $t_k\geq u/2$, a.s.

$$\phi'(\lambda)\leq \sum_{i=2}^{\infty}c_i[1-\mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i}-1-\lambda c_i\big).\tag{3.84}$$

On the event $\{\sum_{i=2}^{\infty}c_i[1-\mathcal{I}_i(u/2)]\big(\mathrm{e}^{\lambda c_i}-1-\lambda c_i\big)\leq K\lambda^2u^{\tau-4}\}$ (recall (3.80)), we have that $\phi'(\lambda)\leq K\lambda^2u^{\tau-4}$, so that we can further bound, choosing $\lambda=\delta u$ and $t=\delta_u=u^{-(\tau-2)}$,

$$\widetilde{\mathbb{P}}(\exists\, s\leq t\colon \tilde\beta s+\mathcal{R}'_s\geq x\mid \mathcal{F}_{t_k})\leq \mathrm{e}^{-2x\lambda+tK4\lambda^2u^{\tau-4}+2t|\tilde\beta|\lambda}\leq \mathrm{e}^{-2x\delta u+K4\delta^2+2u^{-(\tau-3)}|\tilde\beta|\delta}.\tag{3.85}$$

We take $x=\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]$ and note that Corollary 2.5(b) and Lemma 2.4(d) yield that $\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\geq c(u-t_k)u^{\tau-3}$ for $t_k\in[u/2,u-T\delta_u]$. Then,

$$\widetilde{\mathbb{P}}(\exists\, s\leq t\colon \tilde\beta s+\mathcal{R}'_s\geq x\mid \mathcal{F}_{t_k})\leq c\mathrm{e}^{-c\delta(u-t_k)u^{\tau-2}}.\tag{3.86}$$

Summing over k with $t_k=k\delta_u\in[u/2,u-T\delta_u]$ and $\delta_u=u^{-(\tau-2)}$, using Proposition 2.8 and $\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\leq c(u-t_k)u^{\tau-3}$ by Corollary 2.5(b) and Lemma 2.4(d) yields as an upper bound (recall also (3.80) and the definition of E_u from (3.57))

$$\begin{aligned}&\sum_{k\colon t_k\in[u/2,u-T\delta_u]}\widetilde{\mathbb{P}}\big(E_u,\mathcal{S}_u\in[0,M/u],\exists\, t\in I_k\colon \mathcal{S}_t-\widetilde{\mathbb{E}}[\mathcal{S}_t]\geq 2\eta\widetilde{\mathbb{E}}[\mathcal{S}_t]\big)\\&\leq c\sum_{k\colon t_k\in[u/2,u-T\delta_u]}\widetilde{\mathbb{P}}(|\mathcal{S}_{t_k}-\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]|\leq \eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}])\mathrm{e}^{-c\delta(u-k\delta_u)u^{\tau-2}}\\&\leq c\sum_{k\colon t_k\in[u/2,u-T\delta_u]}\eta\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\Big(\sup_w \widetilde{f}_{\mathcal{S}_{t_k}}(w)\Big)\mathrm{e}^{-c\delta(u-k\delta_u)u^{\tau-2}}\\&\leq cu^{-(\tau-3)/2}\sum_{k\colon t_k\in[u/2,u-T\delta_u]}C(u-k\delta_u)u^{\tau-3}\mathrm{e}^{-c\delta(u-k\delta_u)u^{\tau-2}}\\&\leq cu^{-(\tau-1)/2}\sum_{k\colon t_k\in[u/2,u-T\delta_u]}C(u-k\delta_u)u^{\tau-2}\mathrm{e}^{-c\delta(u-k\delta_u)u^{\tau-2}}\\&\leq cu^{-(\tau-1)/2}\mathrm{e}^{-c\delta T}=o_T(1)u^{-(\tau-1)/2},\end{aligned}\tag{3.87}$$

as required.

Part 4: The case $t_k\geq u/2$ and a large lower deviation. We again start from (3.69), and note that the bounds on $\mathcal{D}_t$ and $\delta_u\mathcal{E}_u$ proved in Part 2 still apply, now using that by Corollary 2.5(b) and Lemma 2.4(d) $\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]\geq c(u-t_k)u^{\tau-3}$ for $t_k\in[u/2,u-T\delta_u]$ with $\delta_u=u^{-(\tau-2)}$ below (3.77). We further use this estimate to replace the statement that $\delta_u\mathcal{E}_u$ is $o(\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}])$ from below (3.73) by $\delta_u\mathcal{E}_u\leq o_T(1)\widetilde{\mathbb{E}}[\mathcal{S}_{t_k}]$. The exponential martingale bound for $\mathcal{R}'_t$ performed in Part 3 can easily be adapted to deal with a large lower deviation as well. We omit further details. □

Proof of Proposition 2.13. The proof follows by combining Propositions 3.2 and 3.3. Indeed, choose $\eta=1/4$ and observe that $\widetilde{\mathbb{E}}[\mathcal{S}_t]>0$ on $[\varepsilon,u-T\delta_u]$, using that $\widetilde{\mathbb{E}}[\mathcal{S}_t]\geq ctu^{\tau-3}$ for $t\in[\varepsilon,u/2]$ and $\widetilde{\mathbb{E}}[\mathcal{S}_t]\geq c(u-t)u^{\tau-3}$ for all $t\in[u/2,u-T\delta_u]$ by Corollary 2.5(a),(b) and Lemma 2.4(d). □

4 Conditional expectations given $u\mathcal{S}_u = v$

A major difficulty in the proof of Proposition 2.14 is the fact that, while the summands in the definition of $Q_u(t)$ in (2.44) are independent, this property is lost due to the fact that we *condition on* $\mathcal{S}_u$. The following lemma allows us to deal with such expectations:

Lemma 4.1 (Conditional expectations given a continuous random variable)**.** *Let $G((\mathcal{S}_s)_{s\geq 0})$ be a functional of the process $(\mathcal{S}_s)_{s\geq 0}$ such that $G((\mathcal{S}_s)_{s\geq 0}) \geq 0$ $\widetilde{\mathbb{P}}$-a.s., and $0 < \widetilde{\mathbb{E}}[G((\mathcal{S}_s)_{s\geq 0})] < \infty$. Then, for every $w \in \mathbb{R}$,*

$$\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0}) \mid \mathcal{S}_u = w\big] = \frac{1}{\widetilde{f}_{\mathcal{S}_u}(w)} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kw} \widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0}) \mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big] \frac{dk}{2\pi}, \tag{4.1}$$

where i *denotes the imaginary unit.*

For $G((\mathcal{S}_s)_{s\geq 0}) = 1$, (4.1) is just the usual Fourier inversion theorem applied to the (continuous) random variable $\mathcal{S}_u$. The expectation $\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big]$ *factorizes* when $G((\mathcal{S}_s)_{s\geq 0})$ is of product form in the underlying random variables $(\mathcal{I}_i(s))_{s\geq 0}$. In our applications, $\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0}) \mid \mathcal{S}_u = w\big]$ will be close to constant in w. Then, in order to compute its asymptotics, it suffices to check that the computation in the proof of Proposition 2.8 is hardly affected by the presence of $G((\mathcal{S}_s)_{s\geq 0})$.

Proof. Define the measure $\widetilde{\mathbb{P}}^G$ by

$$\widetilde{\mathbb{P}}^G(E) = \frac{\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\mathbb{1}_E\big]}{\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\big]}. \tag{4.2}$$

Under the measure $\widetilde{\mathbb{P}}^G$, the random variable $\mathcal{S}_u$ is again continuous, since $0 < \widetilde{\mathbb{E}}[G((\mathcal{S}_s)_{s\geq 0})] < \infty$. Let $\widetilde{f}^G_{\mathcal{S}_u}$ denote the density of $\mathcal{S}_u$ under the measure $\widetilde{\mathbb{P}}^G$. Then, we obtain, by the Fourier inversion theorem applied to $\widetilde{\mathbb{P}}^G$, that

$$\widetilde{f}^G_{\mathcal{S}_u}(w) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kw} \widetilde{\mathbb{E}}^G\big[\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big] \frac{dk}{2\pi}. \tag{4.3}$$

Now, by (4.2),

$$\widetilde{f}^G_{\mathcal{S}_u}(w) = \frac{\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0}) \mid \mathcal{S}_u = w\big]}{\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\big]} \widetilde{f}_{\mathcal{S}_u}(w), \tag{4.4}$$

while

$$\widetilde{\mathbb{E}}^G\big[\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big] = \frac{\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big]}{\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\big]}. \tag{4.5}$$

Therefore, substituting both sides in (4.3) and multiplying through by $\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\big]$ proves the claim. □

Let $\widetilde{\mathbb{P}}_v$ denote $\widetilde{\mathbb{P}}$ conditionally on $u\mathcal{S}_u = v$, so that Lemma 4.1 implies that

$$\widetilde{\mathbb{E}}_v\big[G((\mathcal{S}_s)_{s\geq 0})\big] = \frac{1}{\widetilde{f}_{\mathcal{S}_u}(v/u)} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kv/u} \widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big] \frac{dk}{2\pi}. \tag{4.6}$$

In many cases, it shall prove to be convenient to rewrite the above using

$$\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0})\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\big] = \widetilde{\mathbb{E}}\Big[\mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\widetilde{\mathbb{E}}\big[G((\mathcal{S}_s)_{s\geq 0}) \mid \mathcal{J}(u)\big]\Big], \tag{4.7}$$

since the random variables $(T_i)_{i\in\mathcal{J}(u)}$ are, conditionally on $\mathcal{J}(u)$, independent with

$$\widetilde{\mathbb{P}}(T_i \leq u - t \mid T_i \leq u) = \frac{1 - \mathrm{e}^{-c_i(u-t)}}{1 - \mathrm{e}^{-c_i u}}. \tag{4.8}$$

In the following lemma, we investigate the effect on $\mathbb{P}(i \in \mathcal{J}(u))$ of conditioning on $\mathcal{S}_u = w$:

Lemma 4.2 (The set $\mathcal{J}(u)$ conditionally on $\mathcal{S}_u = w$)**.** *There exists a constant $d > 0$ such that for any i and $w = o(u^{(\tau-3)/2})$,*

$$\left|\widetilde{\mathbb{P}}(j \in \mathcal{J}(u) \mid \mathcal{S}_u = w) - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\right| \leq dc_i \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\widetilde{\mathbb{P}}(j \notin \mathcal{J}(u))u^{-(\tau-3)/2}. \tag{4.9}$$

Proof. By Lemma 4.1 (for the second term use $G \equiv 1$)

$$\begin{aligned}
&\left|\widetilde{\mathbb{P}}(j \in \mathcal{J}(u) \mid \mathcal{S}_u = w) - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\right| \qquad (4.10)\\
&= \frac{1}{\widetilde{f}_{\mathcal{S}_u}(w)} \left|\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kw} \widetilde{\mathbb{E}}\Big[\Big(\mathbb{1}_{\{j \in \mathcal{J}(u)\}} - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\Big) \mathrm{e}^{\mathrm{i}k\mathcal{S}_u}\Big] \frac{dk}{2\pi}\right|\\
&= \frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)} \left|\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ku^{-(\tau-3)/2}w} \widetilde{\mathbb{E}}\Big[\Big(\mathbb{1}_{\{j \in \mathcal{J}(u)\}} - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\Big) \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big] \frac{dk}{2\pi}\right|.
\end{aligned}$$

Recall Lemma 2.9. Under the measure $\widetilde{\mathbb{P}}$, the distribution of the indicator processes $(\mathcal{I}_j(t))_{t\geq 0}$ is that of independent indicator processes. Define $\mathcal{S}_u^{(j)} = \mathcal{S}_u - c_j(\mathcal{I}_j(u) - c_j u)$. By (1.20) and (2.42), the random variables $\mathcal{I}_j(u)$ and $\mathcal{S}_u^{(j)}$ are independent under $\widetilde{\mathbb{P}}$. This yields

$$\begin{aligned}
&\left|\widetilde{\mathbb{P}}(j \in \mathcal{J}(u) \mid \mathcal{S}_u = w) - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\right| \qquad (4.11)\\
&\leq \frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)} \int_{-\infty}^{+\infty} \left|\widetilde{\mathbb{E}}\Big[\Big(\mathbb{1}_{\{j \in \mathcal{J}(u)\}} - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\Big) \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j(\mathcal{I}_j(u)-c_j u)}\Big]\right| \left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u^{(j)}}\Big]\right| \frac{dk}{2\pi}\\
&= \frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)} \int_{-\infty}^{+\infty} \widetilde{\mathbb{P}}(j \notin \mathcal{J}(u))\widetilde{\mathbb{P}}(j \in \mathcal{J}(u)) \left|\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} - 1\right| \left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u^{(j)}}\Big]\right| \frac{dk}{2\pi}.
\end{aligned}$$

Next we claim that there exist constants C_1, C_2 such that for all $j \geq 2$

$$\left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u^{(j)}}\Big]\right| \leq C_1 \mathrm{e}^{-C_2|k|^{\tau-2}}. \tag{4.12}$$

Indeed, for $\mathcal{S}_u^{(j)}$ replaced by $\mathcal{S}_u$ the result was derived in the proof of Proposition 2.8 in [1]. To prove the same for $\mathcal{S}_u^{(j)}$ with $j \geq 2$ arbitrary, and following the approach in [1], we obtain for $\frac{k}{2\pi}u^{-(\tau-3)/2-1} \leq 1/8$ the bound

$$\begin{aligned}
\log\Big(\left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u^{(j)}}\Big]\right|\Big) &\leq -cu^{4-\tau}k^2 \sum_{i\geq 2: c_i<1/u, i\neq j} c_i^3 \leq -c_0|k|^{\tau-2} + cu^{4-\tau}k^2 c_j^3 \mathbb{1}_{\{c_j<1/u\}} \qquad (4.13)\\
&\leq -c_0|k|^{\tau-2} + cu^{-(\tau-1)}k^2 \leq -c_0|k|^{\tau-2} + c,
\end{aligned}$$

while for $y_k = 8\frac{k}{2\pi}u^{-(\tau-3)/2} > u$,

$$\begin{aligned}
\log\Big(\left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u^{(j)}}\Big]\right|\Big) &\leq -c_0|k|^{\tau-2} + cu^{4-\tau}k^2 c_j^3 \mathbb{1}_{\{c_j<1/y_k\}} \leq -c_0|k|^{\tau-2} + cu^{4-\tau}k^2 y_k^{-3} \qquad (4.14)\\
&\leq -c_0|k|^{\tau-2} + cu^{4-\tau}u^{(\tau-3)}u^{-1} = -c_0|k|^{\tau-2} + c.
\end{aligned}$$

Substituting (4.12) in (4.11) yields

$$\begin{aligned}
&\left|\widetilde{\mathbb{P}}(j \in \mathcal{J}(u) \mid \mathcal{S}_u = w) - \widetilde{\mathbb{P}}(j \in \mathcal{J}(u))\right| \qquad (4.15)\\
&\leq \frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)} \int_{-\infty}^{+\infty} \widetilde{\mathbb{P}}(i \notin \mathcal{J}(u))\widetilde{\mathbb{P}}(i \in \mathcal{J}(u)) \left|\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} - 1\right| C_1 \mathrm{e}^{-C_2|k|^{\tau-2}} \frac{dk}{2\pi}.
\end{aligned}$$

We further have

$$\left|\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} - 1\right| = \Big(2(1-\cos(ku^{-(\tau-3)/2}c_j))\Big)^{1/2} \leq \sqrt{2}|k|u^{-(\tau-3)/2}c_j, \tag{4.16}$$

which yields

$$\begin{aligned}
&\Big|\widetilde{\mathbb{P}}(j\in\mathcal{J}(u)\mid\mathcal{S}_u=w)-\widetilde{\mathbb{P}}(j\in\mathcal{J}(u))\Big| \qquad (4.17)\\
&\le C_3\frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)}\int_{-\infty}^{+\infty}\widetilde{\mathbb{P}}(i\notin\mathcal{J}(u))\widetilde{\mathbb{P}}(i\in\mathcal{J}(u))ku^{-(\tau-3)/2}c_j\mathrm{e}^{-C_2|k|^{\tau-2}}\frac{dk}{2\pi}\\
&=C_3\widetilde{\mathbb{P}}(i\notin\mathcal{J}(u))\widetilde{\mathbb{P}}(i\in\mathcal{J}(u))u^{-(\tau-3)/2}c_j\frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)}\int_{-\infty}^{+\infty}k\mathrm{e}^{-C_2|k|^{\tau-2}}\frac{dk}{2\pi}.
\end{aligned}$$

For $w=o(u^{(\tau-3)/2})$ and by Proposition 2.8, $u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(w)=B(1+o(1))$ uniformly in w and the claim in (i) follows. □

Corollary 4.3. *There exists a constant $C>0$ such that for any i and $w=o(u^{(\tau-3)/2})$,*

$$\widetilde{\mathbb{P}}(i\in\mathcal{J}(u)\mid\mathcal{S}_u=w)\le C(1\wedge c_iu). \qquad (4.18)$$

Proof. The bound by 1 is obvious. The bound by Cc_iu follows once we recall (2.29) and observe that for $c_j\le 1/u$, $\widetilde{\mathbb{P}}(T_j\le u)=\widetilde{\mathbb{P}}(j\in\mathcal{J}(u))\le C(\tau)c_ju$. Now use Lemma 4.2(i). □

5 The near-end ground: Proof of Proposition 2.14

In this section, we prove Proposition 2.14. The proof is divided into several key parts. In Section 5.1, we show convergence of the mean process A_u in Proposition 2.14(a). In Section 5.2, we prove the convergence of B_u in Proposition 2.14(b).

5.1 Convergence of the mean process A_u

Recall the definition of A_u from (2.49). By (4.8),

$$A_u(tu^{-(\tau-2)})=-\sum_{j\in\mathcal{J}(u)}c_j\left[u\frac{\mathrm{e}^{-c_j(u-tu^{-(\tau-2)})}-\mathrm{e}^{-c_ju}}{1-\mathrm{e}^{-c_ju}}-tu^{-(\tau-2)}\right]. \qquad (5.1)$$

We use that $|\mathrm{e}^x-1-x|\le\mathrm{e}^Dx^2/2$ for $0\le x\le D$ with $x=c_jtu^{-(\tau-2)}$, where for $0\le t\le T$, $c_jtu^{-(\tau-2)}\le tu^{-(\tau-2)}\le$ const., to obtain

$$A_u(tu^{-(\tau-2)})=-\sum_{j\in\mathcal{J}(u)}c_jtu^{-(\tau-2)}\left[c_ju\frac{\mathrm{e}^{-c_ju}}{1-\mathrm{e}^{-c_ju}}-1\right]+E_u(t) \qquad (5.2)$$

with an error term $E_u(t)$ bounded by

$$\begin{aligned}
|E_u(t)|&\le C\sum_{j\in\mathcal{J}(u)}\Big(c_jtu^{-(\tau-2)}\Big)^2c_ju\frac{\mathrm{e}^{-c_ju}}{1-\mathrm{e}^{-c_ju}} \qquad (5.3)\\
&\le CT^2u^{-2(\tau-2)}\Big[u\sum_{j\in\mathcal{J}(u)\colon c_j>1/u}c_j^3+\sum_{j\in\mathcal{J}(u)\colon c_j\le 1/u}c_j^2\Big],
\end{aligned}$$

uniformly in $t\le T$. Since $\sum_{j\ge 2}c_j^3<\infty$ and $u^{-2(\tau-2)+1}=u^{5-2\tau}=o(1)$, the first term vanishes. Further, by Corollary 4.3 with $w=v/u$,

$$u^{-2(\tau-2)}\widetilde{\mathbb{E}}_v\Big[\sum_{j\in\mathcal{J}(u)\colon c_j\le 1/u}c_j^2\Big]=u^{-2(\tau-2)}\sum_{j\in\mathcal{J}(u)\colon c_j\le 1/u}c_j^2\widetilde{\mathbb{P}}_v(j\in\mathcal{J}(u))\le u^{5-2\tau}\sum_{j\in\mathcal{J}(u)\colon c_j\le 1/u}c_j^3=o(1), \qquad (5.4)$$

so that also the second term is $o_{\widetilde{\mathbb{P}}_v}(1)$.

In the above proof, we see that it is useful to split a sum over $j \in \mathcal{J}(u)$ into $j \in \mathcal{J}(u)$ such that $c_j > 1/u$ and $j \in \mathcal{J}(u)$ such that $c_j \leq 1/u$. Then we use upper bounds similar to the ones in Corollary 4.3 to bound the arising sums. We will follow this strategy often below.

We further rewrite (5.2) into

$$A_u(tu^{-(\tau-2)}) = t \sum_{j\in\mathcal{J}(u)} q_j(u) + E_u(t) \qquad \text{with} \qquad q_j(u) \equiv u^{-(\tau-2)} c_j \mathrm{e}^{-c_j u} \frac{\mathrm{e}^{c_j u} - 1 - c_j u}{1-\mathrm{e}^{-c_j u}}. \tag{5.5}$$

Note that $0 \leq q_j(u) \leq 1$ for u big. Below, we will frequently rely on the bounds

$$q_j(u) \leq C(\tau) u^{-(\tau-2)} c_j (1 \wedge c_j u) \tag{5.6}$$

and, using (2.29) for $t = u$,

$$\widetilde{\mathbb{P}}(T_j \leq u) \leq C(\tau)(1 \wedge c_j u), \qquad 1 - \widetilde{\mathbb{P}}(T_j \leq u) \leq \mathrm{e}^{-c_j u(1+\theta)}. \tag{5.7}$$

By (5.3), to prove the claim of Proposition 2.14(a), it is enough to show that

$$\kappa_u \equiv \sum_{j\in\mathcal{J}(u)} q_j(u) = \sum_{j\geq 2} \mathcal{I}_j(u) q_j(u) \xrightarrow{\widetilde{\mathbb{P}}_v} \kappa. \tag{5.8}$$

For this, we compute the Laplace transform of κ_u under the measure $\widetilde{\mathbb{P}}_v$ using Lemma 4.1 and a change of variable. For $a \geq 0$,

$$\widetilde{\mathbb{E}}_v[\mathrm{e}^{-a\kappa_u}] = \frac{1}{u^{(\tau-3)/2} \widetilde{f}_{\mathcal{S}_u}(v/u)} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}kvu^{-(\tau-1)/2}} \widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u} \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] \frac{dk}{2\pi}. \tag{5.9}$$

By Proposition 2.8, for each $v > 0$, $u^{(\tau-3)/2} \widetilde{f}_{\mathcal{S}_u}(v/u) \to B$. We aim to use *dominated convergence* on the integral appearing in (5.9), for which we have to prove (a) pointwise convergence for each $k \in \mathbb{R}$; and (b) a uniform bound that is integrable. We start by proving pointwise convergence:

Lemma 5.1 (Pointwise convergence)**.** *For $a \geq 0$ arbitrary, $v = o(u^{(\tau-1)/2})$, and with κ_u as in (5.8),*

$$\mathrm{e}^{-\mathrm{i}kvu^{-(\tau-1)/2}} \widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u} \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] = \mathrm{e}^{-a\kappa} \mathrm{e}^{-I_V(1)k^2/2} + o(1). \tag{5.10}$$

Proof. Trivially, $\mathrm{e}^{-\mathrm{i}kvu^{-(\tau-1)/2}} \to 1$ pointwise when $v = o(u^{(\tau-1)/2})$. To compute $\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u} \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big]$, recall the definition of $\mathcal{S}_u$ from (1.20) and recall that the indicator processes $\mathcal{I}_j(t) = \mathbb{1}_{\{T_j \leq t\}}$ are independent under the measure $\widetilde{\mathbb{P}}$ (cf. Lemma 2.9), to see that

$$\begin{aligned}\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u} \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] &= \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2} + \tilde{\beta}\mathrm{i}ku^{-(\tau-5)/2}} \\ &\quad \times \prod_{j\geq 2} \mathrm{e}^{-\mathrm{i}ku^{-(\tau-5)/2} c_j^2} \Big(1 + \big(\mathrm{e}^{-aq_j(u) + \mathrm{i}ku^{-(\tau-3)/2} c_j} - 1\big) \widetilde{\mathbb{P}}(T_j \leq u)\Big).\end{aligned} \tag{5.11}$$

The remainder of the proof proceeds in three steps.

Step 1: Asymptotic factorization. We start by proving that

$$\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u} \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] = \mathrm{e}^{-a\widetilde{\mathbb{E}}[\kappa_u]} \widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] + o(1). \tag{5.12}$$

To this end, we first use

$$\Big|\prod_{j\geq 2} a_j - \prod_{j\geq 2} b_j\Big| \leq \sum_{j\geq 2} \prod_{j_1<j} |a_{j_1}||a_j - b_j|| \prod_{j_2>j} |b_{j_2}| \leq \sum_{j\geq 2} |a_j - b_j| \qquad \text{if } \sup_j(|a_j| \vee |b_j|) \leq 1, \tag{5.13}$$

to get (recall that $q_j(u) \geq 0$)

$$\Big|\prod_{j\geq 2} a_j - \prod_{j\geq 2} b_j\Big| = \Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] - \mathrm{e}^{-a\widetilde{\mathbb{E}}[\kappa_u]}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big]\Big| \tag{5.14}$$
$$= \Big|\prod_{j\geq 2}\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\mathcal{I}_j(u)q_j(u)+\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big] - \prod_{j\geq 2}\mathrm{e}^{-aq_j(u)\widetilde{\mathbb{P}}(T_j\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big]\Big|$$
$$\leq \sum_{j\geq 2}\Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\mathcal{I}_j(u)q_j(u)+\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big] - \mathrm{e}^{-aq_j(u)\widetilde{\mathbb{P}}(T_j\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big]\Big|$$
$$\equiv \sum_{j\geq 2}\Delta_j(-aq_j(u)),$$

where we abbreviate $q \equiv -aq_j(u) \leq 0$ such that

$$\Delta_j(q) = \Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathcal{I}_j(u)q+\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big] - \mathrm{e}^{q\widetilde{\mathbb{P}}(T_j\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big]\Big|. \tag{5.15}$$

To bound $\Delta_j(q)$, we write $\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} = 1 + (\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} - 1)$ and use the triangle inequality to bound each summand by

$$\Delta_j(q) = \Big|\Big(1 - \widetilde{\mathbb{P}}(T_j \leq u) + \mathrm{e}^{q}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j}\widetilde{\mathbb{P}}(T_j \leq u)\Big) \tag{5.16}$$
$$-\mathrm{e}^{q\widetilde{\mathbb{P}}(T_j\leq u)}\Big(1 - \widetilde{\mathbb{P}}(T_j \leq u) + \mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j}\widetilde{\mathbb{P}}(T_j \leq u)\Big)\Big|$$
$$\leq \Big|1 - \widetilde{\mathbb{P}}(T_j \leq u) + \mathrm{e}^{q}\widetilde{\mathbb{P}}(T_j \leq u) - \mathrm{e}^{q\widetilde{\mathbb{P}}(T_j\leq u)}\Big|$$
$$+ \Big|\mathrm{e}^{q} - \mathrm{e}^{q\widetilde{\mathbb{P}}(T_j\leq u)}\Big|\Big|\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} - 1\Big|\widetilde{\mathbb{P}}(T_j \leq u).$$

We can bound

$$\Big|\mathrm{e}^{q} - \mathrm{e}^{q\widetilde{\mathbb{P}}(T_j\leq u)}\Big| \leq |q|\mathrm{e}^{(q\vee 0)} \qquad \text{and} \qquad \Big|\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j} - 1\Big| \leq |k|u^{-(\tau-3)/2}c_j, \tag{5.17}$$

which gives a bound $|q|\mathrm{e}^{(q\vee 0)}|k|u^{-(\tau-3)/2}c_j\widetilde{\mathbb{P}}(T_j \leq u)$ on the last line of (5.16).

To bound the first line of (5.16), we use the error bounds $|\mathrm{e}^{-x} - 1 + x| \leq |x|^2$ for all $x \geq 0$ to all the exponential functions in it, to obtain

$$\Big|1 - \widetilde{\mathbb{P}}(T_j \leq u) + \mathrm{e}^{q}\widetilde{\mathbb{P}}(T_j \leq u) - \mathrm{e}^{q\widetilde{\mathbb{P}}(T_j\leq u)}\Big| \leq Cq^2\widetilde{\mathbb{P}}(T_j \leq u). \tag{5.18}$$

Together, this leads us to

$$\Delta_j(-aq_j(u)) = \Delta_j(q) \leq C|q|\mathrm{e}^{(q\vee 0)}\Big(|q| + |k|u^{-(\tau-3)/2}c_j\Big) \leq C(a)q_j(u)\Big(q_j(u) + |k|u^{-(\tau-3)/2}c_j\Big) \equiv \Xi_j. \tag{5.19}$$

To prove (5.12), by (5.14) and (5.19) it is enough to show that $\sum_{j\geq 2}\Xi_j = o(1)$. Consider the sum over $c_j > 1/u$ first. By (5.6),

$$\sum_{j\geq 2\colon c_j>1/u}\Xi_j \leq C\sum_{j\geq 2\colon c_j>1/u} u^{-(\tau-2)}c_j\Big(u^{-(\tau-2)}c_j + c_ju^{-(\tau-3)/2}\Big) \tag{5.20}$$
$$\leq C\sum_{j\geq 2\colon c_j>1/u} u^{-(\tau-2)}u^{-(\tau-3)/2}c_j^2 \leq C\sum_{j\geq 2\colon c_j>1/u} u^{-3(\tau-3)/2}c_j^3 = o(1),$$

where we have used that $\sum_j c_j^3 < \infty$ and $\tau > 3$ in the last equality. For $c_j \leq 1/u$ and by (5.6), we similarly get

$$\sum_{j\geq 2\colon c_j\leq 1/u}\Xi_j \leq C\sum_{j\geq 2\colon c_j\leq 1/u} u^{-(\tau-3)}c_j^2\Big(u^{-(\tau-3)}c_j^2 + c_ju^{-(\tau-3)/2}\Big) \leq C\sum_{j\geq 2\colon c_j\leq 1/u} u^{-3(\tau-3)/2}c_j^3 = o(1). \tag{5.21}$$

This completes the proof that $\sum_{j\geq 2}\Xi_j = o(1)$ and thus of the claim in (5.12). □

Step 2: The limit of $\widetilde{\mathbb{E}}[\kappa_u]$. We proceed by showing that $\lim_{u\to\infty}\widetilde{\mathbb{E}}[\kappa_u]=\kappa$ with $\kappa>0$ as in (2.51). By definition of κ_u in (5.8), $q_j(u)$ in (5.5) and $\widetilde{\mathbb{P}}(T_j\leq u)$ in (2.29),

$$\begin{aligned}\lim_{u\to\infty}\widetilde{\mathbb{E}}[\kappa_u] &= \lim_{u\to\infty}\sum_{j\geq 2} q_j(u)\widetilde{\mathbb{P}}(T_j\leq u) \qquad (5.22)\\ &= \lim_{u\to\infty} u^{-(\tau-1)}\sum_{j\geq 2} c_j u \mathrm{e}^{-c_j u}\frac{\mathrm{e}^{c_j u}-1-c_j u}{1-\mathrm{e}^{-c_j u}}\frac{\mathrm{e}^{\theta c_j u}(1-\mathrm{e}^{-c_j u})}{\mathrm{e}^{\theta c_j u}(1-\mathrm{e}^{-c_j u})+\mathrm{e}^{-c_j u}}\\ &= \lim_{\Delta\to 0^+}\Delta\sum_{j\geq 2} x_j^{-\alpha}\mathrm{e}^{-x_j^{-\alpha}}\left[\mathrm{e}^{x_j^{-\alpha}}-1-x_j^{-\alpha}\right]\frac{\mathrm{e}^{\theta x_j^{-\alpha}}}{\mathrm{e}^{\theta x_j^{-\alpha}}(1-\mathrm{e}^{-x_j^{-\alpha}})+\mathrm{e}^{-x_j^{-\alpha}}}\\ &= \int_0^\infty x^{-\alpha}\frac{\mathrm{e}^{\theta x^{-\alpha}}\mathrm{e}^{-x^{-\alpha}}}{\mathrm{e}^{\theta x^{-\alpha}}(1-\mathrm{e}^{-x^{-\alpha}})+\mathrm{e}^{-x^{-\alpha}}}\left[\mathrm{e}^{x^{-\alpha}}-1-x^{-\alpha}\right]dx,\end{aligned}$$

with $\Delta=u^{-(\tau-1)}$ and $x_j=j\Delta,\ j\geq 2$. Here we used that the integrand in the last line of (5.22) is continuous and integrable over $(0,\infty)$. Set $-x^{-\alpha}=z$ to get the representation (2.51) for κ. □

Step 3: Completion of the proof. By Proposition 2.7, we know that

$$\widetilde{\mathbb{E}}[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}]\to \mathrm{e}^{-k^2 I_V(1)/2}. \qquad (5.23)$$

Therefore, Steps 1-2 and (5.23) complete the proof of pointwise convergence in Lemma 5.1. □

To show that the dominated convergence theorem can be applied, it remains to show that the integrand in (5.9) has an integrable dominating function:

Lemma 5.2 (Domination by an integrable function)**.**

$$\int_{-\infty}^{\infty}\sup_{u\geq u_0}\left|\widetilde{\mathbb{E}}\big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big]\right|dk<\infty. \qquad (5.24)$$

Proof. By definition of $\mathcal{S}_u$ from (1.20) and the independence in Lemma 2.9,

$$\begin{aligned}\left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\right|^2 &\leq \prod_{j\geq 2}\left|\widetilde{\mathbb{P}}(T_j\leq u)\mathrm{e}^{-aq_j(u)}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j}+(1-\widetilde{\mathbb{P}}(T_j\leq u))\right|^2 \qquad (5.25)\\ &= \prod_{j\geq 2}\Big[1+\widetilde{\mathbb{P}}(T_j\leq u)^2(\mathrm{e}^{-2aq_j(u)}+1)\\ &\quad +2\widetilde{\mathbb{P}}(T_j\leq u)\cos(ku^{-(\tau-3)/2}c_j)\mathrm{e}^{-aq_j(u)}(1-\widetilde{\mathbb{P}}(T_j\leq u))-2\widetilde{\mathbb{P}}(T_j\leq u)\Big].\end{aligned}$$

We can rewrite each factor as

$$\begin{aligned}&1-2\widetilde{\mathbb{P}}(T_j\leq u)\Big\{\widetilde{\mathbb{P}}(T_j\leq u)(1-\mathrm{e}^{-2aq_j(u)})/2+(1-\cos(ku^{-(\tau-3)/2}c_j)\mathrm{e}^{-aq_j(u)})(1-\widetilde{\mathbb{P}}(T_j\leq u))\Big\} \qquad (5.26)\\ &\quad\leq 1-2\widetilde{\mathbb{P}}(T_j\leq u)(1-\cos(ku^{-(\tau-3)/2}c_j)\mathrm{e}^{-aq_j(u)})(1-\widetilde{\mathbb{P}}(T_j\leq u)),\end{aligned}$$

since $q_j(u)\geq 0$. We then use $\log(1+x)\leq x$ for $x\geq -1$ to obtain

$$\log\left(\left|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\right|^2\right)\leq\sum_{j\geq 2}2\widetilde{\mathbb{P}}(T_j\leq u)\big(\cos(ku^{-(\tau-3)/2}c_j)\mathrm{e}^{-aq_j(u)}-1\big)(1-\widetilde{\mathbb{P}}(T_j\leq u)). \qquad (5.27)$$

The latter equals

$$\sum_{j\geq 2}2\widetilde{\mathbb{P}}(T_j\leq u)(\cos(ku^{-(\tau-3)/2}c_j)-1)(1-\widetilde{\mathbb{P}}(T_j\leq u))+e_j(u), \qquad (5.28)$$

with an overall error term (using that $\sup_j q_j(u)$ is arbitrarily small for u big enough)

$$\sum_{j\geq 2}|e_j(u)|\leq C(a)\sum_{j\geq 2}\widetilde{\mathbb{P}}(T_j\leq u)q_j(u)(1-\widetilde{\mathbb{P}}(T_j\leq u)). \tag{5.29}$$

Applying (5.7), we get

$$\sum_{j\geq 2}|e_j(u)|\leq C(a)\Big\{\sum_{j\geq 2:c_j>1/u}u^{-(\tau-2)}c_j\mathrm{e}^{-c_ju(1+\theta)}+\sum_{j\geq 2:c_j\leq 1/u}c_juu^{-(\tau-2)}c_j^2u\Big\}\leq C(a), \tag{5.30}$$

where we have used the bounds

$$\sum_{i:c_i>1/u}u^{-(\tau-2)}c_j\mathrm{e}^{-c_ju(1+\theta)}\leq 1,\quad u^{-(\tau-4)}\sum_{i:c_i\leq 1/u}c_i^3\leq C(\tau), \tag{5.31}$$

whose proof is straightforward.

Together with (5.27) and (5.28), we obtain

$$\log\Big(\Big|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\Big|\Big)\leq\sum_{j\geq 2}\widetilde{\mathbb{P}}(T_j\leq u)(\cos(ku^{-(\tau-3)/2}c_j)-1)(1-\widetilde{\mathbb{P}}(T_j\leq u))+C(a). \tag{5.32}$$

As all summands are nonpositive we obtain together with (2.29)

$$\log\Big(\Big|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\Big|\Big)\leq C\sum_{j\geq 2:c_j\leq 1/u}c_ju(\cos(ku^{-(\tau-3)/2}c_j)-1)+C(a). \tag{5.33}$$

Following the proof of [1, Proposition 2.5, (6.7)-(6.10)], we obtain

$$\log\Big(\Big|\widetilde{\mathbb{E}}\Big[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\Big|\Big)\leq C_1-C_2|k|^{\tau-2} \tag{5.34}$$

and integrability of $|\widetilde{\mathbb{E}}[\mathrm{e}^{-a\kappa_u}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}]|$ against k uniformly in u follows. □

Completion of the proof of Proposition 2.14(a). By the dominated convergence theorem, Lemmas 5.1 and 5.2 complete the proof of Proposition 2.14(a). □

5.2 Convergence of the process B_u

In this section, we investigate the convergence of the B_u process and prove Proposition 2.14(b). Since the limit is a *random process*, this part is more involved than the previous section. We first note that

$$B_u(tu^{-(\tau-2)})=\sum_{i\in\mathcal{J}(u)}c_iu[\mathbb{1}_{\{T_i\in(u-tu^{-(\tau-2)},u]\}}-\widetilde{\mathbb{P}}(T_i>u-tu^{-(\tau-2)}\mid T_i\leq u)], \tag{5.35}$$

and the processes $(\mathbb{1}_{\{T_i\in(u-tu^{-(\tau-2)},u]\}})_{t\geq 0}$ are, conditionally on $\mathcal{J}(u)$, independent. Thus, $(B_u(tu^{-(\tau-2)}))_{t\geq 0}$ is, conditionally on $\mathcal{J}(u)$, a sum of (conditionally) independent processes having zero mean. We make crucial use of this observation, as well as the technique in Lemma 4.1, to compute expectations of various functionals of the process $(B_u(tu^{-(\tau-2)}))_{t\geq 0}$.

In order to prove the stated convergence in distribution, we follow the usual path of first proving weak convergence of the one-dimensional marginals, followed by the weak convergence of all finite-dimensional distributions, and complete the proof by showing tightness. We now discuss each of these steps in more detail.

5.2.1 Convergence of the one-dimensional marginal of B_u

We start by computing the one-dimensional marginal of $B_u(tu^{-(\tau-2)})$ (recall (5.35)) and show that it is consistent with the claimed Lévy process limit. We achieve this by computing the Laplace transform

$$\psi_{u,v}(a) = \widetilde{\mathbb{E}}_v\big[\mathrm{e}^{-aB_u(tu^{-(\tau-2)})}\big], \tag{5.36}$$

and proving that it converges to the Laplace transform of the claimed Lévy process limit at time t. The main result in this section is the following proposition:

Proposition 5.3 (One-time marginal of $B_u(tu^{-(\tau-2)})$)**.** *There exists a measure Π such that, for every $v, a > 0$ fixed and as $u \to \infty$,*

$$\psi_{v,u}(a) \to \mathrm{e}^{t\int_0^\infty (\mathrm{e}^{-az}-1+az)\Pi(dz)}, \tag{5.37}$$

which is the Laplace transform of a Lévy process $(L_s)_{s\geq 0}$ with non-negative jumps and characteristic measure Π

$$\Pi(dz) \equiv \mathrm{e}^{z}\frac{\mathrm{e}^{-\theta z}}{\mathrm{e}^{-\theta z}(1-\mathrm{e}^{z})+\mathrm{e}^{z}}(\tau-1)(-z)^{-(\tau-1)}dz. \tag{5.38}$$

Therefore, the one-dimensional marginals of the process $(B_u(su^{-(\tau-2)}))_{t\geq 0}$ converge to those of $(L_s)_{s\geq 0}$.

The remainder of this section is devoted to the proof of Proposition 5.3. As for A_u, we use Lemma 4.1 and a change of variables to rewrite

$$\begin{aligned}\psi_{v,u}(a) &\equiv \widetilde{\mathbb{E}}_v\Big[\mathrm{e}^{-aB_u(tu^{-(\tau-2)})}\Big] = \frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(v/u)}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}kvu^{-(\tau-1)/2}}\widetilde{\mathbb{E}}\big[\mathrm{e}^{-aB_u(tu^{-(\tau-2)})}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big]\frac{dk}{2\pi}\\ &= \frac{1}{u^{(\tau-3)/2}\widetilde{f}_{\mathcal{S}_u}(v/u)}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}kvu^{-(\tau-1)/2}}\widetilde{\mathbb{E}}\Big[\psi_{\mathcal{J}}(a)\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\frac{dk}{2\pi},\end{aligned} \tag{5.39}$$

where

$$\begin{aligned}\psi_{\mathcal{J}}(a) &\equiv \widetilde{\mathbb{E}}\Big[\mathrm{e}^{-aB_u(tu^{-(\tau-2)})} \mid \mathcal{J}(u)\Big] \qquad (5.40)\\ &= \prod_{j\in\mathcal{J}(u)}\mathrm{e}^{ac_ju\widetilde{\mathbb{P}}(T_j>u-tu^{-(\tau-2)}|T_j\leq u)}\Big(1+(\mathrm{e}^{-ac_ju}-1)\widetilde{\mathbb{P}}(T_j>u-tu^{-(\tau-2)} \mid T_j\leq u)\Big)\\ &= \prod_{j\in\mathcal{J}(u)}\mathrm{e}^{ac_jup^u_{j,t}}\Big(1+(\mathrm{e}^{-ac_ju}-1)p^u_{j,t}\Big),\end{aligned}$$

and where we abbreviate

$$p^u_{j,t} = \widetilde{\mathbb{P}}(T_j > u - tu^{-(\tau-2)} \mid T_j \leq u) = \frac{\mathrm{e}^{c_jtu^{-(\tau-2)}}-1}{1-\mathrm{e}^{-c_ju}}\mathrm{e}^{-c_ju}, \tag{5.41}$$

by (4.8). We again wish to use dominated convergence on the integral in (5.39).

We proceed along the lines of the proof of the convergence of the mean process A_u. Basically, in the proof below, we replace $-a\kappa_u$ in (5.9) (recall the definition of κ_u and $q_j(u)$ from (5.8) and (5.5)) by $\sum_{j\in\mathcal{J}(u)} r^u_{j,t}$, where we define

$$r^u_{j,t} \equiv (\mathrm{e}^{-ac_ju}-1+ac_ju)p^u_{j,t} = (\mathrm{e}^{-ac_ju}-1+ac_ju)\frac{\mathrm{e}^{c_jtu^{-(\tau-2)}}-1}{1-\mathrm{e}^{-c_ju}}\mathrm{e}^{-c_ju}. \tag{5.42}$$

In what follows, we frequently make use of the bounds

$$p^u_{j,t} \leq Ctu^{-(\tau-1)}(c_ju\mathrm{e}^{-c_ju}\wedge 1) \leq Ctu^{-(\tau-1)}, \tag{5.43}$$

and

$$r^u_{j,t} \leq C(a,T)c_ju^{-(\tau-2)}(1\wedge c_ju). \tag{5.44}$$

We again start by proving pointwise convergence:

Lemma 5.4 (Pointwise convergence revisited)**.** *For $a \geq 0$ arbitrary, $v = o(u^{(\tau-1)/2})$,*

$$\mathrm{e}^{-\mathrm{i}kvu^{-(\tau-1)/2}}\widetilde{\mathbb{E}}\big[\psi_{\mathcal{J}}(a)\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] = \mathrm{e}^{t\int_{-\infty}^0 (\mathrm{e}^{az}-1-az)\Pi(dz)}\mathrm{e}^{-I_V(1)k^2/2} + o(1). \tag{5.45}$$

Proof. The first factor on the left-hand side of (5.45) converges to 1. We identify the limit of the expectation in the following steps that mimic the pointwise convergence proof in Lemma 5.1. It will be convenient to split the asymptotic factorization in Step 1 of that proof into two parts, denoted by Steps 1(a) and 1(b). We start by showing that we can simplify $\psi_{\mathcal{J}}(a)$:

Step 1(a): Simplification of $\psi_{\mathcal{J}}(a)$. As a first step towards the identification of the pointwise limit, we show that we can simplify the expectation in (5.45) as follows:

$$\widetilde{\mathbb{E}}\Big[\Big|\psi_{\mathcal{J}}(a) - \mathrm{e}^{\sum_{j\in\mathcal{J}(u)} r_{j,t}^u}\Big|\Big] = o(1). \tag{5.46}$$

To prove (5.46), we denote the difference in (5.46) by

$$E_u(t) = \prod_{j\in\mathcal{J}(u)} \mathrm{e}^{ac_jup_{j,t}^u}\left|\prod_{j\in\mathcal{J}(u)}\Big(1+(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u\Big) - \prod_{j\in\mathcal{J}(u)}\mathrm{e}^{(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u}\right|, \tag{5.47}$$

so that

$$\Big|\widetilde{\mathbb{E}}\Big[\Big|\psi_{\mathcal{J}}(a) - \mathrm{e}^{\sum_{j\in\mathcal{J}(u)} r_{j,t}^u}\Big|\Big]\Big| \leq \widetilde{\mathbb{E}}[E_u(t)]. \tag{5.48}$$

Using the first line of (5.13) and applying the error bound $|\mathrm{e}^x - (1+x)| \leq |x|^2$ for $|x| \leq 1$ to the differences $|a_j - b_j|$, the error of the approximation can be bounded by

$$E_u(t) \leq C\prod_{j\in\mathcal{J}(u)}\mathrm{e}^{ac_jup_{j,t}^u}\sum_{j\in\mathcal{J}(u)}\prod_{i\in\mathcal{J}(u),i<j}\Big(1+(\mathrm{e}^{-ac_iu}-1)p_{i,t}^u\Big)\left[(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u\right]^2\prod_{i\in\mathcal{J}(u),i>j}\mathrm{e}^{(\mathrm{e}^{-ac_iu}-1)p_{i,t}^u}. \tag{5.49}$$

Next use that $1-x\leq \mathrm{e}^{-x}$ for $x\geq 0$ to obtain as a further bound to the above

$$C\sum_{j\in\mathcal{J}(u)}\mathrm{e}^{ac_jup_{j,t}^u}\prod_{i\in\mathcal{J}(u)\setminus\{j\}}\mathrm{e}^{(\mathrm{e}^{-ac_iu}-1+ac_iu)p_{i,t}^u}\left[(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u\right]^2. \tag{5.50}$$

For $t \leq T$ with $T>0$ fixed, we further have by (5.43) that $\mathrm{e}^{ac_jup_{j,t}^u} \leq C(a,T)$. Together with $\mathrm{e}^{-x}-1+x\geq 0$ for $x\geq 0$, we obtain

$$E_u(t) \leq C(a,T)\prod_{i\in\mathcal{J}(u)}\mathrm{e}^{(\mathrm{e}^{-ac_iu}-1+ac_iu)p_{i,t}^u}\sum_{j\in\mathcal{J}(u)}\left[(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u\right]^2. \tag{5.51}$$

The bound $\mathrm{e}^{-x}-1+x \leq x^2/2$, $\forall x\geq 0$ yields

$$E_u(t) \leq C(a,T)\mathrm{e}^{\frac{a^2}{2}\sum_{i\in\mathcal{J}(u)}(c_iu)^2p_{i,t}^u}\sum_{j\in\mathcal{J}(u)}\left[(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u\right]^2. \tag{5.52}$$

We first bound the sum in (5.52). With $1-\mathrm{e}^{-x}\leq x$ for $x\geq 0$ and by (5.43) we obtain

$$\sum_{j\in\mathcal{J}(u)}\left[(\mathrm{e}^{-ac_ju}-1)p_{j,t}^u\right]^2 \leq \sum_{j\in\mathcal{J}(u)}(ac_ju)^2\left(p_{j,t}^u\right)^2 \leq C(a,T)u^{-2(\tau-3)-1}\Big\{C + \sum_{j\in\mathcal{J}(u):\, c_j\leq 1/u} c_j^2u^{-1}\Big\}. \tag{5.53}$$

This yields as an upper bound for (5.46) (recall (5.48)),

$$\begin{aligned}\widetilde{\mathbb{E}}[E_u(t)] &\leq C(a,T)u^{-2(\tau-3)-1}\Big\{\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\frac{a^2}{2}\sum_{i\in\mathcal{J}(u)}(c_iu)^2p^u_{i,t}}\Big] \\ &\qquad + \sum_{j\colon c_j\leq 1/u} c_j^2u^{-1}\widetilde{\mathbb{E}}\Big[\mathbb{1}_{\{j\in\mathcal{J}(u)\}}\mathrm{e}^{\frac{a^2}{2}(c_ju)^2p^u_{j,t}}\Big]\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\frac{a^2}{2}\sum_{i\in\mathcal{J}(u)\setminus\{j\}}(c_iu)^2p^u_{i,t}}\Big]\Big\} \\ &\leq C(a,T)u^{-2(\tau-3)-1}\{1+\sum_{j\colon c_j\leq 1/u}c_j^3\}\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\frac{a^2}{2}\sum_{i\in\mathcal{J}(u)}(c_iu)^2p^u_{i,t}}\Big],\end{aligned} \tag{5.54}$$

where we have used (5.7) in the last line.

The claim (5.46) follows once we show that $\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\frac{a^2}{2}\sum_{i\in\mathcal{J}(u)}(c_iu)^2p^u_{i,t}}\Big]$ is bounded. To prove this, consider first the sum over $c_i > 1/u$ only. By (5.43) and (5.31),

$$\sum_{i\in\mathcal{J}(u):c_i>1/u}(c_iu)^2p^u_{i,t} \leq C\sum_{i\in\mathcal{J}(u):c_i>1/u}(c_iu)^2c_itu^{-(\tau-2)}\mathrm{e}^{-c_iu} \leq C\sum_{i:c_i>1/u}tu^{-(\tau-1)} \leq C(T). \tag{5.55}$$

Using (5.43) once more, it remains to show the boundedness of

$$\begin{aligned}\widetilde{\mathbb{E}}\Big[\mathrm{e}^{\frac{a^2}{2}\sum_{i\in\mathcal{J}(u):c_i\leq 1/u}(c_iu)^2p^u_{i,t}}\Big] &\leq \widetilde{\mathbb{E}}\Big[\mathrm{e}^{C(a)\sum_{i\in\mathcal{J}(u):c_i\leq 1/u}(c_iu)^2tu^{-(\tau-1)}}\Big] \\ &= \prod_{c_i\leq 1/u}\Big(\widetilde{\mathbb{P}}(T_i\leq u)\mathrm{e}^{C(a)(c_iu)^2tu^{-(\tau-1)}} + (1-\widetilde{\mathbb{P}}(T_i\leq u))\Big),\end{aligned} \tag{5.56}$$

which is equivalent to bounding

$$\sum_{c_i\leq 1/u}\log\Big(\widetilde{\mathbb{P}}(T_i\leq u)\mathrm{e}^{C(a)(c_iu)^2tu^{-(\tau-1)}} + (1-\widetilde{\mathbb{P}}(T_i\leq u))\Big) \leq \sum_{c_i\leq 1/u}\widetilde{\mathbb{P}}(T_i\leq u)\Big(\mathrm{e}^{C(a)(c_iu)^2tu^{-(\tau-1)}}-1\Big) \tag{5.57}$$

appropriately. Here we used that $\log(1+x)\leq x$ for $x\geq 0$. Next bound $\widetilde{\mathbb{P}}(T_i\leq u)\leq Cc_iu$ in the above to obtain that for $c_i\leq 1/u$ we have $C(a)(c_iu)^2tu^{-(\tau-1)}\leq C(a,T)u^{-(\tau-1)}\leq \log(2)$ for u big enough. Hence we can use that $\mathrm{e}^x-1\leq 2x$ for $0\leq x\leq\log(2)$ and thus get as a further upper bound to (5.57)

$$C(a,T)\sum_{c_i\leq 1/u}c_iu(c_iu)^2u^{-(\tau-1)} \leq C(a,T). \tag{5.58}$$

The last inequality follows from (5.31). This completes the proof of (5.46). □

Step 1(b): Asymptotic factorization. We next show that

$$\widetilde{\mathbb{E}}\big[\mathrm{e}^{\sum_{j\in\mathcal{J}(u)}r^u_{j,t}}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] = \mathrm{e}^{\widetilde{\mathbb{E}}[\sum_{j\in\mathcal{J}(u)}r^u_{j,t}]}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big]+o(1). \tag{5.59}$$

To prove (5.59), we note that, by the definition of $r^u_{j,t}$ in (5.42),

$$\begin{aligned}\bar{E}_u(t) &\equiv \Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{\sum_{j\in\mathcal{J}(u)}r^u_{j,t}}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] - \mathrm{e}^{\widetilde{\mathbb{E}}[\sum_{j\in\mathcal{J}(u)}r^u_{j,t}]}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big]\Big| \\ &= \Big|\prod_{j\geq 2}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathcal{I}_j(u)r^u_{j,t}+\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big] - \prod_{j\geq 2}\mathrm{e}^{r^u_{j,t}\widetilde{\mathbb{P}}(T_j\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big]\Big|.\end{aligned} \tag{5.60}$$

As in the calculations of the Laplace transform of A_u in (5.14), we now apply (5.13). Note that here we cannot apply the second bound of (5.13) as $\sup_j(|a_j| \vee |b_j|)$ is not bounded by 1 (recall that $r^u_{j,t} \geq 0$). Instead, we get

$$\begin{aligned} \bar{E}_u(t) \leq \sum_{j\geq 2} \prod_{2\leq j_1\leq j-1} &\Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathcal{I}_{j_1}(u) r^u_{j_1,t} + \mathrm{i}ku^{-(\tau-3)/2}c_{j_1}\mathcal{I}_{j_1}(u)}\big]\Big| \\ &\times \Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathcal{I}_j(u) r^u_{j,t} + \mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big] - \mathrm{e}^{r^u_{j,t}\widetilde{\mathbb{P}}(T_j\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big]\Big| \\ &\times \prod_{j_2\geq j+1}\Big|\mathrm{e}^{r^u_{j_2,t}\widetilde{\mathbb{P}}(T_{j_2}\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_{j_2}\mathcal{I}_{j_2}(u)}\big]\Big|. \end{aligned} \tag{5.61}$$

We proceed to prove that the first and the third product are bounded by constants. Indeed, we can bound the third product using (5.7) by

$$\prod_{j_2\geq j+1} \mathrm{e}^{r^u_{j_2,t}\widetilde{\mathbb{P}}(T_{j_2}\leq u)} \leq \mathrm{e}^{C\sum_{j\geq 2} r^u_{j,t}(c_j u\wedge 1)}, \tag{5.62}$$

where, by (5.44) and (5.31),

$$\sum_{j\geq 2} r^u_{j,t}(c_j u \wedge 1) \leq C. \tag{5.63}$$

For the first product in (5.61), we obtain as an upper bound

$$\begin{aligned} \prod_{j\geq 2}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathcal{I}_j(u) r^u_{j,t}}\big] &= \prod_{j\geq 2}\Big(\widetilde{\mathbb{P}}(T_j\leq u)\mathrm{e}^{r^u_{j,t}} + (1-\widetilde{\mathbb{P}}(T_j\leq u))\Big) \\ &= \prod_{j\geq 2}\Big(1+\widetilde{\mathbb{P}}(T_j\leq u)(\mathrm{e}^{r^u_{j,t}}-1)\Big) \leq \mathrm{e}^{\sum_{j\geq 2}\widetilde{\mathbb{P}}(T_j\leq u)(\mathrm{e}^{r^u_{j,t}}-1)}. \end{aligned} \tag{5.64}$$

As $r^u_{j,t}$ is uniformly bounded for u big enough the above is again bounded by (5.63).

Hence, it suffices to bound the middle part of (5.61), that is, it remains to show that

$$\sum_{j\geq 2}\Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathcal{I}_j(u) r^u_{j,t} + \mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big] - \mathrm{e}^{r^u_{j,t}\widetilde{\mathbb{P}}(T_j\leq u)}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}c_j\mathcal{I}_j(u)}\big]\Big| = \Delta_j(r^u_{j,t}) = o(1), \tag{5.65}$$

where we recall the definition of $\Delta_j(q)$ in (5.15). By (5.19),

$$\Delta_j(r^u_{j,t}) \leq C r^u_{j,t}\mathrm{e}^{r^u_{j,t}}(r^u_{j,t} + |k|u^{-(\tau-3)/2}c_j) \tag{5.66}$$

for $u = u(k)$ big enough. The bound on $r^u_{j,t}$ in (5.44) is equal to $C(a,T)$ times the bounds on $q_j(u)$ in (5.6). The remaining calculations for A_u in (5.19)-(5.21) therefore directly carry over, so that (5.65) follows. □

Step 2: The limit of $\mathbb{E}[\sum_{j\in\mathcal{J}(u)} r^u_{j,t}]$. In this step, we identify the limit of $\mathbb{E}[\sum_{j\in\mathcal{J}(u)} r^u_{j,t}]$. For this, we use that by definition of $r^u_{j,t}$ in (5.42), that of $p^u_{j,t}$ in (5.41), and (2.29) with $t = u$,

$$\begin{aligned} \widetilde{\mathbb{E}}\Big[\sum_{j\in\mathcal{J}(u)} r^u_{j,t}\Big] &= \sum_{j\geq 2}(\mathrm{e}^{-ac_j u} - 1 + ac_j u)\frac{\mathrm{e}^{c_j t u^{-(\tau-2)}}-1}{1-\mathrm{e}^{-c_j u}}\mathrm{e}^{-c_j u}\frac{\mathrm{e}^{\theta c_j u}(1-\mathrm{e}^{-c_j u})}{\mathrm{e}^{\theta c_j u}(1-\mathrm{e}^{-c_j u})+\mathrm{e}^{-c_j u}} \\ &= tu^{-(\tau-1)}\sum_{j\geq 2}(\mathrm{e}^{-ac_j u} - 1 + ac_j u)\frac{\mathrm{e}^{c_j u t u^{-(\tau-1)}}-1}{tu^{-(\tau-1)}}\mathrm{e}^{-c_j u}\frac{\mathrm{e}^{\theta c_j u}}{\mathrm{e}^{\theta c_j u}(1-\mathrm{e}^{-c_j u})+\mathrm{e}^{-c_j u}} \\ &\overset{u\to\infty}{\to} t\int_0^\infty (\mathrm{e}^{-ax^{-\alpha}} - 1 + ax^{-\alpha})x^{-\alpha}\mathrm{e}^{-x^{-\alpha}}\frac{\mathrm{e}^{\theta x^{-\alpha}}}{\mathrm{e}^{\theta x^{-\alpha}}(1-\mathrm{e}^{-x^{-\alpha}})+\mathrm{e}^{-x^{-\alpha}}}dx. \end{aligned} \tag{5.67}$$

The convergence of the sum to the integral follows as in (5.22). Next set $-x^{-\alpha} = z$ to get

$$\lim_{u\to\infty} \frac{1}{t}\widetilde{\mathbb{E}}\Big[\sum_{j\in\mathcal{J}(u)} r_{j,t}^u\Big] = \int_{-\infty}^{0} (\mathrm{e}^{az} - 1 - az)\Pi(dz), \tag{5.68}$$

with $\Pi(dz)$ as in (5.38), respectively, (2.54). For Π to be the Lévy measure of a real-valued Lévy process with no positive jumps as in [5, Section V.1], by the Lévy-Khintchine formula in [5, Section 0.2 and Theorem 1 in Section I.1], we have to check that Π is a measure on $(-\infty, 0)$ that satisfies $\int \Pi(dz)(1\wedge z^2) < \infty$. Indeed, close to 0, $z^2\Pi(dz)$ behaves like $(\tau-1)z^{-(\tau-3)}dz$, which is integrable at 0 and for $z \to \infty$, $\Pi(dz)$ behaves like $\mathrm{e}^{-z}(\tau-1)z^{-(\tau-1)}dz$, whose integral is finite for all $n \in \mathbb{N}$. □

Step 3: Completion of the proof. The convergence of $\widetilde{\mathbb{E}}\big[\mathrm{e}^{iku^{-(\tau-3)/2}\mathcal{S}_u}\big] \to \mathrm{e}^{-k^2 I_V(1)/2}$ is already proved in (5.23). Therefore, Steps 1(a)-1(b) and 2, together with (5.23), complete the proof of pointwise convergence in Lemma 5.4. □

To show that the dominated convergence theorem can be applied, it again remains to show that the integrand has an integrable dominating function:

Lemma 5.5 (Domination by an integrable function)**.**

$$\int_{-\infty}^{\infty} \sup_{u\geq u_0} \Big|\widetilde{\mathbb{E}}\big[\psi_{\mathcal{J}}(a)\mathrm{e}^{iku^{-(\tau-3)/2}\mathcal{S}_u}\big]\Big|\, dk < \infty. \tag{5.69}$$

Proof. This follows in a similar way as in the proof of Lemma 5.2. We compute

$$\begin{aligned}\Big|\widetilde{\mathbb{E}}\big[\psi_{\mathcal{J}}(a)\mathrm{e}^{iku^{-(\tau-3)/2}\mathcal{S}_u}\big]\Big|^2 &= \Big|\widetilde{\mathbb{E}}\big[\mathrm{e}^{iku^{-(\tau-3)/2}\mathcal{S}_u} \prod_{j\in\mathcal{J}(u)} \mathrm{e}^{ac_j u p_{j,t}^u}\big(1 + (\mathrm{e}^{-ac_j u} - 1)p_{j,t}^u\big)\big]\Big|^2 \\ &= \prod_{j\geq 2}\Big|1 - \widetilde{\mathbb{P}}(T_j \leq u) + \mathrm{e}^{iku^{-(\tau-3)/2}c_j}\widetilde{\mathbb{P}}(T_j \leq u)\mathrm{e}^{ac_j u p_{j,t}^u}\big(1 + (\mathrm{e}^{-ac_j u} - 1)p_{j,t}^u\big)\Big|^2.\end{aligned} \tag{5.70}$$

This is identical to the bound appearing in (5.25), apart from the fact that the term $e^{-aq_j(u)}$ in (5.25) is replaced with $b_{j,t}(u) = \mathrm{e}^{ac_j u p_{j,t}^u}\big(1 + (\mathrm{e}^{-ac_j u} - 1)p_{j,t}^u\big)$ in the above. Proceeding as in (5.25) to (5.28), we finally obtain

$$\begin{aligned}&\log\Big|\widetilde{\mathbb{E}}\Big[\psi_{\mathcal{J}}(a)\mathrm{e}^{iku^{-(\tau-3)/2}\mathcal{S}_u}\Big]\Big|^2 \\ &\leq \sum_{j\geq 2} 2\widetilde{\mathbb{P}}(T_j \leq u)\Big[\widetilde{\mathbb{P}}(T_j \leq u)((b_{j,t}(u))^2 - 1)/2 + \big(\cos(ku^{-(\tau-3)/2}c_j)b_{j,t}(u) - 1\big)(1 - \widetilde{\mathbb{P}}(T_j \leq u)),\end{aligned} \tag{5.71}$$

where the additional first term in comparison to (5.27) arises because $b_{j,t}(u) \leq 1$ no longer holds. Indeed, since $\mathrm{e}^{xp}(1 + (\mathrm{e}^{-x} - 1)p) \geq 1$ for $x \geq 0$ and $p \in [0, 1]$, we have that $b_{j,t}(u) \geq 1$. Further,

$$b_{j,t}(u) = \mathrm{e}^{ac_j u p_{j,t}^u}\big(1 + (\mathrm{e}^{-ac_j u} - 1)p_{j,t}^u\big) \leq \mathrm{e}^{(\mathrm{e}^{-ac_j u} - 1 + ac_j u)p_{j,t}^u} = \mathrm{e}^{r_{j,t}(u)}. \tag{5.72}$$

The first part of the sum in (5.71) can, by (5.72) and since $\mathrm{e}^x - 1 \leq 2x$ for $0 \leq x \leq \log(2)$, be bounded by

$$\sum_{j\geq 2} \widetilde{\mathbb{P}}(T_j \leq u)^2(\mathrm{e}^{2r_{j,t}^u} - 1) \leq C(a, T)\sum_{j\geq 2}(1 \wedge c_j u)^2 r_{j,t}^u. \tag{5.73}$$

Now we can apply (5.44) and (5.31) to get as a further bound

$$C(a, T)\Big\{\sum_{j\geq 2: c_j > 1/u} u^{-(\tau-1)} + \sum_{j\geq 2: c_j \leq 1/u} c_j^3 u^{-(\tau-4)}\Big\} \leq C. \tag{5.74}$$

For the second part of the sum (5.71), we proceed as in (5.27)-(5.34) to split it as

$$
\begin{aligned}
&-\sum_{j\geq 2} 2\widetilde{\mathbb{P}}(T_j \leq u)\widetilde{\mathbb{P}}(T_j \leq u)\big(1-\cos(ku^{-(\tau-3)/2}c_j)\big)(1-\widetilde{\mathbb{P}}(T_j \leq u)) \\
&+\sum_{j\geq 2} \widetilde{\mathbb{P}}(T_j \leq u)\cos(ku^{-(\tau-3)/2}c_j)(b_{j,t}(u)-1)(1-\widetilde{\mathbb{P}}(T_j \leq u)).
\end{aligned}
\tag{5.75}
$$

By a second order Taylor expansion and the fact that $r_{j,t}(u)$ is bounded, there exists a constant C such that $b_{j,t}(u)-1 \leq Cr_{j,t}(u)$. Now we can proceed as in (5.27)-(5.34), where we again take advantage of being able to dominate the bounds on $r_{j,t}^u$ in (5.44) by the bounds on $q_j(u)$ in (5.6). Integrability of $|\widetilde{\mathbb{E}}[\psi_{\mathcal{J}}(a)\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}]|$ against k follows. □

Proof of Proposition 5.3. The claim follows from Lemmas 5.4, 5.5 and the dominated convergence theorem. □

5.2.2 Convergence of the finite-dimensional distributions of B_u

In this section, the convergence of the one-dimensional marginals of the process $(B_u(tu^{-(\tau-2)}))_{t\geq 0}$ gets extended to convergence of its finite-dimensional distributions. In the same way as above, it can be shown that, for $0 < t_1 \cdots < t_n$, the increments $\big(B_u(t_iu^{-(\tau-2)}) - B_u(t_{i-1}u^{-(\tau-2)})\big)_{i=1}^n$ (where, by convention, $t_0 = 0$) converge in distribution, under $\widetilde{\mathbb{P}}_v$, to *independent* Lévy random variables with the correct distribution.

In what follows, we only outline some minor changes in the proof. Instead of (5.40), we fix $n \in \mathbb{N}$, $\vec{a} \in (\mathbb{R}^+)^n$ and $0 = t_0 < t_1 < \cdots < t_n \leq T$ and consider

$$
\begin{aligned}
\psi_{\mathcal{J}}(\vec{a}) &\equiv \widetilde{\mathbb{E}}\Big[\mathrm{e}^{-\sum_{k=1}^n a_k\big(B_u(t_ku^{-(\tau-2)})-B_u(t_{k-1}u^{-(\tau-2)})\big)} \mid \mathcal{J}(u)\Big] \\
&= \prod_{j\in\mathcal{J}(u)} \mathrm{e}^{c_ju\sum_{k=1}^n a_kp^u_{j,t_{k-1},t_k}}\Big(1+\sum_{k=1}^n(\mathrm{e}^{-a_kc_ju}-1)p^u_{j,t_{k-1},t_k}\Big)
\end{aligned}
\tag{5.76}
$$

with (the two-point analogue to (5.43))

$$
p^u_{j,s,t} \equiv \widetilde{\mathbb{P}}(T_j \in (u-tu^{-(\tau-2)}, u-su^{-(\tau-2)}] \mid T_j \leq u) = \mathrm{e}^{-c_ju}\frac{\mathrm{e}^{c_jtu^{-(\tau-2)}}-\mathrm{e}^{c_jsu^{-(\tau-2)}}}{1-\mathrm{e}^{-c_ju}} \tag{5.77}
$$

for $0 \leq s \leq t \leq T$, using (4.8). Then, clearly, (5.43) is replaced with

$$
p^u_{j,s,t} \leq C(t-s)u^{-(\tau-1)}(\mathrm{e}^{-c_ju}c_ju \wedge 1). \tag{5.78}
$$

We follow Steps 1(a)-(b) to Step 3 in the proof of convergence of the one-time marginal.

Similarly to Step 1(a), one can show that

$$
\widetilde{\mathbb{E}}\Big[\Big|\psi_{\mathcal{J}}(\vec{a}) - \prod_{j\in\mathcal{J}(u)} \mathrm{e}^{\sum_{k=1}^n(\mathrm{e}^{-a_kc_ju}-1+a_kc_ju)p^u_{j,t_{k-1},t_k}}\Big|\Big] = o(1). \tag{5.79}
$$

We then continue to reason as from (5.39) onwards, where $r^u_{j,t}$ in (5.42) gets replaced by

$$
r^u_{j,\vec{t}} \equiv \sum_{k=1}^n(\mathrm{e}^{-a_kc_ju}-1+a_kc_ju)p^u_{j,t_{k-1},t_k}. \tag{5.80}
$$

The remaining calculations are analogous to the one-dimensional case. The asymptotic factorization in Step 1(b) is replaced with

$$
\widetilde{\mathbb{E}}\big[\mathrm{e}^{\sum_{j\in\mathcal{J}(u)} r^u_{j,\vec{t}}}\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] = \mathrm{e}^{\widetilde{\mathbb{E}}[\sum_{j\in\mathcal{J}(u)} r^u_{j,\vec{t}}]}\widetilde{\mathbb{E}}\big[\mathrm{e}^{\mathrm{i}ku^{-(\tau-3)/2}\mathcal{S}_u}\big] + o(1) \tag{5.81}
$$

and we calculate the limit of $\widetilde{\mathbb{E}}[\sum_{j\in\mathcal{J}(u)} r^u_{j,\vec{t}}]$ in a similar way as in Step 2 in the previous subsection as

$$\widetilde{\mathbb{E}}\Big[\sum_{j\in\mathcal{J}(u)} r^u_{j,\vec{t}}\Big] = \sum_{j\geq 2}\sum_{k=1}^{n}(\mathrm{e}^{-a_k c_j u} - 1 + a_k c_j u)\frac{\mathrm{e}^{c_j t_k u^{-(\tau-2)}} - \mathrm{e}^{c_j t_{k-1} u^{-(\tau-2)}}}{1-\mathrm{e}^{-c_j u}}\mathrm{e}^{-c_j u}\widetilde{\mathbb{P}}(j\in\mathcal{J}(u)) \tag{5.82}$$

$$\stackrel{u\to\infty}{\to} \int_0^\infty \sum_{k=1}^{n}(\mathrm{e}^{-a_k x^{-\alpha}} - 1 + a_k x^{-\alpha})x^{-\alpha}(t_k - t_{k-1})\mathrm{e}^{-x^{-\alpha}}\frac{\mathrm{e}^{\theta x^{-\alpha}}}{\mathrm{e}^{\theta x^{-\alpha}}(1-\mathrm{e}^{-x^{-\alpha}}) + \mathrm{e}^{-x^{-\alpha}}}dx$$

$$= \int_{-\infty}^0 \sum_{k=1}^{n}(\mathrm{e}^{a_k z} - 1 - a_k z)(t_k - t_{k-1})\Pi(dz).$$

Finally, we note that

$$\mathrm{e}^{\widetilde{\mathbb{E}}[\sum_{j\in\mathcal{J}(u)} r^u_{j,\vec{t}}]} = \exp\left[\int_{-\infty}^0 \sum_{k=1}^{n}(\mathrm{e}^{a_k z} - 1 - a_k z)(t_k - t_{k-1})\Pi(dz)\right] = \mathbb{E}\Big[\mathrm{e}^{\sum_{k=1}^n a_k(-(L_{t_k} - L_{t_{k-1}}))}\Big], \tag{5.83}$$

where we have used that, by definition, Lévy processes have independent stationary increments. This completes the convergence of the finite-dimensional distributions of $(B_u(tu^{-(\tau-2)}))_{t\geq 0}$. □

5.2.3 Tightness of B_u

We next turn to *tightness* of the process $(B_u(tu^{-(\tau-2)}))_{t\geq 0}$. For this, we use the following tightness criterion:

Proposition 5.6 (Tightness criterion [8, Theorem 15.6 and the comment following it])**.** *The sequence $\{X_n\}$ is tight in $D([0,T],\mathbb{R}^d)$ if the limiting process X has a.s. no discontinuity at $t = T$ and there exist constants $C > 0$, $r > 0$ and $a > 1$ such that for $0 \leq t_1 < t_2 < t_3 \leq T$ and for all n,*

$$\mathbb{E}\Big[|X_n(t_2) - X_n(t_1)|^r\,|X_n(t_3) - X_n(t_2)|^r\Big] \leq C|t_3 - t_1|^a. \tag{5.84}$$

Let

$$V^{(u)}(t) = B_u(tu^{-(\tau-2)}) = \sum_{i\in\mathcal{J}(u)} c_i u[\mathbb{1}_{\{T_i\in(u-tu^{-(\tau-2)},u]\}} - \widetilde{\mathbb{P}}(T_i > u - tu^{-(\tau-2)} \mid T_i \leq u)]. \tag{5.85}$$

We show tightness of $V^{(u)}(t)$ given $u\mathcal{S}_u = v$. In what follows, we therefore bound

$$\widetilde{\mathbb{E}}_v[(V^{(u)}(t_2) - V^{(u)}(t_1))^2(V^{(u)}(t_3) - V^{(u)}(t_2))^2] \tag{5.86}$$
$$= \widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{E}}\big[(V^{(u)}(t_2) - V^{(u)}(t_1))^2(V^{(u)}(t_3) - V^{(u)}(t_2))^2 \mid \mathcal{J}(u)\big] \mid u\mathcal{S}_u = v\Big].$$

First observe that with

$$\mathcal{I}^u_i(s,t) \equiv \mathbb{1}_{\{T_i\in(u-tu^{-(\tau-2)},u-su^{-(\tau-2)}]\}} \tag{5.87}$$

we have $p^u_{i,s,t} = \widetilde{\mathbb{E}}[\mathcal{I}^u_i(s,t) \mid T_i \leq u]$ (recall (5.77)) and

$$\widetilde{\mathbb{E}}[(V^{(u)}(t_2) - V^{(u)}(t_1))^2(V^{(u)}(t_3) - V^{(u)}(t_2))^2 \mid \mathcal{J}(u)] \tag{5.88}$$
$$= \widetilde{\mathbb{E}}\Big[\prod_{n\in\{1,2\}}\Big(\sum_{i\in\mathcal{J}(u)} c_i u\left[\mathcal{I}^u_i(t_n,t_{n+1}) - p^u_{i,t_n,t_{n+1}}\right]\Big)^2 \mid \mathcal{J}(u)\Big].$$

By the conditional independence of the processes conditional on $\mathcal{J}(u)$ (recall comment preceding (4.8)), and as we subtract their respective expectations, we obtain

$$
\begin{aligned}
&\widetilde{\mathbb{E}}[(V^{(u)}(t_2)-V^{(u)}(t_1))^2(V^{(u)}(t_3)-V^{(u)}(t_2))^2 \mid \mathcal{J}(u)] \qquad (5.89)\\
&=\widetilde{\mathbb{E}}\Big[\sum_{i\in\mathcal{J}(u)}(c_iu)^4\prod_{n\in\{1,2\}}\Big(\mathcal{I}_i^u(t_n,t_{n+1})-p^u_{i,t_n,t_{n+1}}\Big)^2 \mid \mathcal{J}(u)\Big]\\
&\quad+\widetilde{\mathbb{E}}\Big[\sum_{i\in\mathcal{J}(u)}\sum_{j\in\mathcal{J}(u)\setminus\{i\}}(c_iu)^2(c_ju)^2\left(\mathcal{I}_i^u(t_1,t_2)-p^u_{i,t_1,t_2}\right)^2\left(\mathcal{I}_j^u(t_2,t_3)-p^u_{j,t_2,t_3}\right)^2 \mid \mathcal{J}(u)\Big]\\
&\quad+2\widetilde{\mathbb{E}}\Big[\sum_{i\in\mathcal{J}(u)}\sum_{j\in\mathcal{J}(u)\setminus\{i\}}(c_iu)^2(c_ju)^2\prod_{n\in\{1,2\}}\Big(\Big(\mathcal{I}_i^u(t_n,t_{n+1})-p^u_{i,t_n,t_{n+1}}\Big)\Big(\mathcal{I}_j^u(t_n,t_{n+1})-p^u_{j,t_n,t_{n+1}}\Big)\Big) \mid \mathcal{J}(u)\Big].
\end{aligned}
$$

We can bound this from above by

$$
C\Big\{\sum_{i\in\mathcal{J}(u)}(c_iu)^4p^u_{i,t_1,t_2}p^u_{i,t_2,t_3}+\prod_{n\in\{1,2\}}\Big(\sum_{i\in\mathcal{J}(u)}(c_iu)^2p^u_{i,t_n,t_{n+1}}\Big)\Big\}. \qquad (5.90)
$$

By (5.78),

$$
\begin{aligned}
&\widetilde{\mathbb{E}}[(V^{(u)}(t_2)-V^{(u)}(t_1))^2(V^{(u)}(t_3)-V^{(u)}(t_2))^2 \mid \mathcal{J}(u)] \qquad (5.91)\\
&\leq C(t_2-t_1)(t_3-t_2)\Big\{\sum_{i\in\mathcal{J}(u)}(c_iu)^4u^{-2(\tau-1)}(\mathrm{e}^{-c_iu}c_iu\wedge 1)^2+\Big(\sum_{i\in\mathcal{J}(u)}(c_iu)^2u^{-(\tau-1)}(\mathrm{e}^{-c_iu}c_iu\wedge 1)\Big)^2\Big\}.
\end{aligned}
$$

For the first sum, note that $(c_iu)^4u^{-2(\tau-1)}=c_i^4u^{-2(\tau-3)}$, so that its sum is order $o(1)$ as $\sum_i c_i^3<\infty$ and $\tau>3$. For the second sum in (5.91), we note that the sum over i such that $c_i>1/u$ is clearly bounded, since it is bounded by

$$
\sum_{i\colon c_i>1/u}(c_iu)u^{-(\tau-1)}\mathrm{e}^{-c_iu}, \qquad (5.92)
$$

which converges to a constant as $u\to\infty$ since it is a Riemann approximation to a finite integral. For the contributions due to $c_i\leq 1/u$, we bound its expectation as

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_v\Big[\Big(\sum_{i\in\mathcal{J}(u)\colon c_i\leq 1/u}(c_iu)^2u^{-(\tau-1)}\Big)^2\Big] \qquad (5.93)\\
&\leq\sum_{i\neq j,c_i\leq 1/u,c_j\leq 1/u}(c_iu)^2(c_ju)^2u^{-2(\tau-1)}c_iuc_ju+\sum_{i\in\mathcal{J}(u)\colon c_i\leq 1/u}(c_iu)^4u^{-2(\tau-1)}c_iu\\
&\leq\Big(\sum_{i\colon c_i\leq 1/u}(c_iu)^3u^{-(\tau-1)}\Big)^2+\sum_{i\colon c_i\leq 1/u}c_i^3\leq C,
\end{aligned}
$$

by (5.31). Hence, we get with (5.86) and (5.91)-(5.93),

$$
\widetilde{\mathbb{E}}_v[(V^{(u)}(t_2)-V^{(u)}(t_1))^2(V^{(u)}(t_3)-V^{(u)}(t_2))^2]\leq C(t_2-t_1)(t_3-t_2)\leq C(t_3-t_1)^2, \qquad (5.94)
$$

as required. □

5.2.4 Completion of the proof of Proposition 2.14(b)

The convergence of the finite-dimensional distributions together with tightness yields $(B_u(tu^{-(\tau-2)}))_{t\geq 0}\xrightarrow{d}(L_t)_{t\geq 0}$ by [8, Theorem 5.1].

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