EURANDOM PREPRINT SERIES

2014-009

July 9, 2014

The scaling limits of the non critical strip wetting model

J. Sohier
ISSN 1389-2355

The scaling limits of the non critical strip wetting model.

J.Sohier*

June 13, 2014

Abstract

The strip wetting model is defined by giving a (continuous space) one dimensionnal random walk S a reward β each time it hits the strip $\mathbb{R}^+ \times [0, a]$ (where a is a given positive parameter), which plays the role of a defect line. We show that this model exhibits a phase transition between a delocalized regime ($\beta < \beta_c^a$) and a localized one ($\beta > \beta_c^a$), where the critical point $\beta_c^a > 0$ depends on S and on a. In this paper we give a precise pathwise description of the transition, extracting the full scaling limits of the model. Our approach is based on Markov renewal theory.

Keywords: scaling limits for physical systems, fluctuation theory for random walks, Markov renewal theory .

Mathematics subject classification (2000): 60K15, 60K20, 60K05, 82B27, 60K35, 60F17.

1 Introduction and main results

1.1 Definition of the models

We consider $(S_n)_{n\geq 0}$ a random walk such that $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$ where the X_i's are i.i.d. and X_1 has a density $h(\cdot)$ with respect to the Lebesgue measure. We denote by $\mathbf{P}$ the law of S, and by $\mathbf{P}_x$ the law of the same process starting from x. We will assume that $h(\cdot)$ is continuous and bounded on $\mathbb{R}$, that $h(\cdot)$ is positive in a neighborhood of the origin, that $\mathbf{E}[X] = 0$ and that $\mathbf{E}[X^2] =: \sigma^2 \in (0, \infty)$. We fix $a > 0$ in the sequel.

The fact that h is continuous and positive in the neighborhood of the origin entails that

$$n_0 := \inf_{n\in\mathbb{N}} \{(\mathbf{P}[S_n > a], \mathbf{P}[-S_n > a]) \in (0,1)^2\} < \infty. \qquad (1) \quad \texttt{hypn}$$

*TU/e

We will assume that $n_0 = 1$ (and thus that $(\mathbf{P}[-S_1 > a], \mathbf{P}[S_1 > a]) \in (0,1)^2$). We stress that our work could be extended to the generic $n_0 \geq 2$ case, althought this should lead to some specific technical difficulties.

For N a positive integer, we consider the event $\mathcal{C}_N := \{S_1 \geq 0, \ldots, S_N \geq 0\}$. We define the probability law (the *free wetting model in a strip*) $\mathbf{P}^f_{N,a,\beta}$ on $\mathbb{R}^N$ by

$$\frac{d\mathbf{P}^f_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z^f_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^{N} \mathbf{1}_{S_k \in [0,a]}\right) \mathbf{1}_{\mathcal{C}_N} \tag{2}$$

where $N \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $Z^f_{N,a,\beta}$ is the normalization constant usually called the partition function of the system. The second model we define is the *constrained counterpart* of the above, that is

$$\frac{d\mathbf{P}^c_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z^c_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^{N} \mathbf{1}_{S_k \in [0,a]}\right) \mathbf{1}_{\mathcal{C}_N} \mathbf{1}_{S_N \in [0,a]}. \tag{3}$$

Note in particular that

$$\mathbf{P}^c_{N,a,\beta} = \mathbf{P}^f_{N,a,\beta}\left[\,\cdot\,|S_N \in [0,a]\right], \tag{4}$$

that $\mathbf{P}^f_{N,a,0}$ is the law of $(S_1, \ldots, S_N)$ under the constraint $\mathcal{C}_N := \{S_1 \geq 0, \ldots, S_N \geq 0\}$ and that $\mathbf{P}^c_{N,a,0}$ is the law of the same vector under the additional constraint $S_N \in [0,a]$.

$\mathbf{P}_{N,a,\beta}$ is a $(1+1)-$dimensional model for a linear chain of length N which is attracted or repelled to a defect *strip* $[0,\infty) \times [0,a]$. By $(1+1)-$dimensional, we mean that the configurations of the linear chain are described by the trajectories $(i, S_i)_{i \leq N}$ of the walk, so that we are dealing with directed models. The strength of this interaction with the strip is tuned by the parameter β. Regarding the terminology, note that the use of the term *wetting* has become customary to describe the positivity constraint $\mathcal{C}_N$ and refers to the interpretation of the field as an effective model for the interface of separation between a liquid above a wall and a gas, see [DGZ05].

It is an interesting problem to understand when the reward β is strong enough to pin the chain near the defect strip, a phenomenon that we will call *localization*, and what are the macroscopic effects of the reward on the system. In this paper, we choose to characterize these effects through the scaling limits of the laws $\mathbf{P}^c_{N,a,\beta}$ and $\mathbf{P}^f_{N,a,\beta}$. More precisely, we first show the existence of a critical point $\beta^c_a > 0$ depending on a, and then we solve the *full scaling limits* of the system in the case where $\beta \neq \beta^c_a$.

We point out that these questions have been answered in depth in the case of the standard wetting model, that is formally in the $a = 0$ case, and that extending these results to our setup is an open problem which has been raised by Giacomin ([Gia07, Chapter 3]).

1.2 The free energy.

A standard way to define localization for our models is by looking at the Laplace asymptotic behavior of the partition function $Z^c_{N,a,\beta}$ as $N \to \infty$. More precisely, we define the free energy $F^a(\beta)$ by

$$F^a(\beta) := \lim_{N\to\infty} \frac{1}{N} \log\left(Z^f_{N,a,\beta}\right) \tag{5}$$

where the existence of the limit will follow as a by-product of our approach.

The basic observation is the fact that the free energy is non-negative. The following inequality holds:

$$\begin{aligned} Z^f_{N,a,\beta} &\geq \mathbf{E}\left[\exp\left(\beta\sum_{k=1}^{N}\mathbf{1}_{S_k\in[0,a]}\right)\mathbf{1}_{S_k>a,k=1\ldots N}\right] \\ &\geq \mathbf{P}\left[S_j > a, j = N_0 \ldots, N\right]. \end{aligned} \tag{6}$$

Integrating over S_1, one gets:

$$\mathbf{P}[S_j > a, j = 1\ldots, N] \geq \int_{(a,\infty)} h(t)\mathbf{P}_t\left[S_1 > a, \ldots, S_{N-1} > a\right] dt. \tag{7}$$

We prove then in Lemma 2.2 below that for fixed M, the quantity $N^{1/2}\mathbf{P}_t\left[S_1 > a, \ldots, S_{N-1} > a\right] \in [c, c']$ for every N and every $t \in [a, M]$ where c, c' are positive constants. Thus:

$$Z^f_{N,a,\beta} \geq \frac{c}{N^{1/2}} \int_{[a,M]} h^{\star N_0}(t)dt. \tag{8}$$

Therefore $F^a(\beta) \geq 0$ for every β. Since the lower bound has been obtained by ignoring the contribution of the paths that touch the strip, one is led to the following:

Definition 1.1 *For $g \in \{c, f\}$, the model $\{\mathbf{P}^g_{N,a,\beta}\}$ is said to be localized if $F^a(\beta) > 0$.*

It is standard that $F^a(\cdot)$ is a convex increasing function, and in particular it is a continuous function as long as it is finite. Therefore, there exists a critical value $\beta^a_c \in \mathbb{R}$ such that the strip wetting model is localized for $\beta > \beta^a_c$.

1.3 Value of the critical point for the (p, q) random walk.

valcr

We point out that the effect of adding a layer of width $a > 0$ does affect the value of the critical point in a drastic way with respect to the standard homogeneous pinning.

To illustrate this issue, we first stress that our techniques can be applied in the discrete setup without too much effort. Let us then consider the simplest case of a discrete (p, q) random walk S (that is a symmetric random walk with increments in $\{-1, 0, 1\}$ such that $\mathbf{P}[S_1 = 1] = p = \frac{1-\mathbf{P}[S_1=0]}{2} = \frac{1-q}{2}$ with $p \in (0, 1/2)$) and $a \in \mathbb{Z}^+$. A byproduct of our characterization of the critical point of Part 3 (see in particular (40)) is the fact that in this case, $e^{-\beta_c^a}$ is equal to the spectral radius of the matrix M_a which is a tridiagonal $(a+1) \times (a+1)$ matrix defined by

$$M_a := \begin{bmatrix} q & p & 0 & 0 & \dots \\ p & q & p & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & p & q & p \\ & \dots & 0 & p & \frac{1+q}{2} \end{bmatrix}.$$

In the particular cases where $a \in \{1, 2\}$ (recall that in the homogeneous pinning case, that is $a = 0$, it is shown in [Gia07][Chapter 2], that $\beta_c^0 = -\log(1-p)$), this characterization leads to the equality

$$\beta_c^1 = -\log\left(1 - \frac{3-\sqrt{5}}{2}p\right). \tag{9}$$

In particular, it is an interesting phenomenon that this explicit critical point satisfies the strict (intuition matching) inequality $\beta_c^0 > \beta_c^1$.

Another consequence of this characterization is the fact that in this particular case, one gets directly the convergence $\beta_c^a \to 0$ as $a \to \infty$.

1.4 Scaling limits.

We define the map $X^N : \mathbb{R}^N \mapsto C([0, 1])$:

$$X_t^N(x) := \frac{x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} + (Nt - \lfloor Nt \rfloor)\frac{x_{\lfloor Nt \rfloor + 1} - x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}}; t \in [0, 1] \tag{10}$$

X

where $\lfloor Nt \rfloor$ denotes the integer part of Nt. Note that $X_t^N(x)$ is the linear interpolation of the process $\{x_{\lfloor Nt \rfloor}/\sigma N^{1/2}\}_{t \in \mathbb{N}/N \cap [0,1]}$. Then we define the measures

$$Q^c_{N,a,\beta} := \mathbf{P}^c_{N,a,\beta} \circ (X^N)^{-1} \tag{11}$$

and in an analogous way $Q^f_{N,a,\beta}$. These measures are defined on $C([0,1])$ the space of real continuous functions defined on $[0,1]$. We consider the following standard processes:

- ⋆ the Brownian motion $(B_t)_{t\in[0,1]}$.
- ⋆ the Brownian meander $(m_t)_{t\in[0,1]}$ which is the Brownian motion conditioned to stay positive on $[0,1]$.
- ⋆ the normalized Brownian excursion $(e_t)_{t\in[0,1]}$ which is the brownian bridge conditioned to stay positive on $[0,1]$.

Our main result is the following:

MAIN **Theorem 1.2** *Both the free and the constrained models undergo a wetting transition at $\beta = \beta_c^a$. More precisely:*

1. *in the subcritical regime, that is if $\beta < \beta_c^a$, then*
   - *$(Q^c_{N,a,\beta})_N$ converges weakly in $C([0,1])$ to the law of e.*
   - *$(Q^f_{N,a,\beta})_N$ converges weakly in $C([0,1])$ to the law of m.*
2. *in the supercritical regime, that is if $\beta > \beta_c^a$, then both $(Q^c_{N,a,\beta})_N$ and $(Q^f_{N,a,\beta})_N$ converge in $C([0,1])$ to the measure concentrated on the constant function taking value zero.*

The following result shows that in the subcritical phase, the dry region reduces to a finite number of points all being at a finite distance from zero in the free case, from zero and from N in the constrained case.

TRLO **Theorem 1.3** *For $\beta < \beta_c^a$, the following convergences hold:*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \mathbf{P}^f_{N,\beta,a}\left[\max \mathcal{A} \geq L\right] = 0 \tag{12}$$

and

$$\lim_{L\to\infty}\limsup_{N\to\infty} \mathbf{P}^c_{N,\beta,a}\left[\max(\mathcal{A}\cap[1,N/2])\geq L\right]=0$$
$$\lim_{L\to\infty}\limsup_{N\to\infty} \mathbf{P}^c_{N,\beta,a}\left[\min(\mathcal{A}\cap[N/2,N])\leq N-L\right]=0. \tag{13}$$

The proof of our main result mainly focuses on the delocalized phase. For the localized phase, we show a Markov renewal theorem which seems to be new in this particular (continuous space) setup, from which one deduces in a standard way the asymptotic behavior of the partition functions both in the free and in the constrained case, thus obtaining the convergence of the process (without rescaling) towards the law of a positive recurrent irreducible Markov chain (see [Gia07][Chapter 3] for the corresponding result in the standard homogeneous case). From this convergence result, we immediately deduce Theorem 1.2 in the localized phase.

A common feature shared by the strip wetting model and the classical homogeneous one is the fact that the measure $\mathbf{P}_{N,a,\beta}$ exhibits a remarkable decoupling between the contact level set $\mathcal{I}_N := \{i \leq N, S_i \in [0,a]\}$ and the excursions of S between two consecutive contact points. More precisely, conditioning on $I_N = \{t_1,\dots,t_k\}$ and on $(S_{t_1},\dots,S_{t_k})$, the *bulk* excursions $e_i = \{e_i(n)\}_n := \left\{\{S_{t_i+n}\}_{0\leq n\leq t_{i+1}-t_i}\right\}$ are independent under $\mathbf{P}_{N,a,\beta}$ and are distributed like the walk $(S', \mathbf{P}_{S_{t_i}})$ conditioned on the event $\left\{S'_{t_{i+1}-t_i} = S_{t_{i+1}}, S'_{t_i+j} > a, j\in\{1,\dots,t_{i+1}-t_i-1\}\right\}$. It is therefore clear that to extract scaling limits on $\mathbf{P}_{N,a,\beta}$, one has to combine good control over the law of the contact set $\mathcal{I}_N$ and suitable asymptotics properties of the excursions. This decoupling turns out to be the starting point of our proofs, see (61) and (60) for details.

Theorem 1.2 characterizes the Brownian scaling of the model when $\beta\neq\beta_c^a$. Infinite scaling results like Theorem 1.2 have been proved in different contexts involving polymer measures. The first mathematical paper dealing with such an issue is [IY01] where the authors proved an analogous convergence in the homogeneous pinning model for the case where S is a symmetric random walk with increments taking values in $\{-1,0,1\}$. Their results have been strongly generalized in [DGZ05] where the same assumptions are made on S as in this paper, and a further generalization of their results in the case where S is in the domain of attraction of the standard normal law has been obtained in [CGZ06].

Analogous results have also been obtained in [CGZ07] in the case of inhomogeneous, but periodic pinning models, and more recently in [CD09] in the case where the interaction is of *Laplacian* type.

What is left by our analysis is the critical case. There are some specific issues related to this case, and we point out that the scaling limit of these laws at the critical point have been solved recently (although in a weak sense) in [Soh13]. Note also that related models with a different characterization of the large scale limits have been considered recently [Fun08]. Finally, a closely related pinning model in continuous time has been considered and resolved in [CKMV09]; we stress however that their techniques are very peculiar to the continuous time setup.

1.5 Organization of the paper

The core of our approach is a precise pathwise description of the law $\mathbf{P}^c_{N,a,\beta}$ based on Markov renewal theory. We stress the importance of Markov renewal theory to derive fundamental results on the large scale behavior of the system. The other techniques we use are local limit estimates issued from the theory of fluctuation for random walks, an infinite version of the Perron Frobenius Theorem and two closely related scaling limit theorems. Let us describe more in detail the content of this paper:

- in section 2, we recall some fluctuation theory for random walk which will be of basic importance; more precisely, we first recall some recent local limit estimates for random walks conditioned to stay non negative, and we give the tails of the return probability to the strip for large N which have been proved in [Soh13].

- in section 3 we show that the law $\mathbf{P}^c_{N,a,\beta}$ admits a description in terms of a Markov renewal process. More precisely, we show that the set of contact points with the strip under $\mathbf{P}^c_{N,a,\beta}$ is distributed according to the law of a Markov renewal process conditioned to hit the strip at time N. This representation implies in particular a very useful expression for the partition function $Z^c_{N,a,\beta}$ which will be the key to our main results.

- in sections 4, we make use of Markov renewal theorems in the finite mean case and of a uniform equivalence result in the infinite mean case; these theorems imply estimates on $Z^c_{N,a,\beta}$. Deducing the asymptotic behavior of $Z^f_{N,a,\beta}$ in both phases is then a standard procedure.

- section 5 is devoted to the proof of Theorems 1.3 and 1.2. These proofs are carried out exploiting the asymptotic estimates deduced from the Markov renewal representation combined with powerful limit theorems which have been obtained in [Shi83] (for the free case) and much more recently in [CC13] (for the constrained case).

2 Some preliminary facts

2.1 Some recurrent notations and terminology

For a_n, b_n two positive sequences, we will write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. More generally, for $a_n(x)$ a positive sequence depending on a parameter $x \in \Delta$ where Δ is a subset of $\mathbb{R}^d, d \geq 1$, $\alpha \in \mathbb{R}$ and $b(\cdot)$ a measurable function on Δ, we will often say that the equivalence

$$a_n(x) \sim \frac{b(x)}{n^\alpha} \tag{14}$$

holds *uniformly for x in Δ* if the following holds:

$$\lim_{n\to\infty} \sup_{x\in\Delta} |n^\alpha a_n(x) - b(x)| = 0. \tag{15}$$

In this paper, we will deal with kernels of two kind. Kernels of the first kind are just σ-finite kernels on $\mathbb{R}$, that is functions $A : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \mapsto \mathbb{R}^+$ where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field of $\mathbb{R}$ and such that for each $x \in \mathbb{R}$, $A_{x,\cdot}$ is a σ-finite measure on $\mathbb{R}$ and $A_{\cdot,F}$ is a Borel function for every $F \in \mathcal{B}(\mathbb{R})$. Given two such kernels A and B, their composition is denoted by $(A \circ B)_{x,dy} := \int_{z\in\mathbb{R}} A_{x,dz} B_{z,dy}$ and of course $A^{\circ k}_{x,dy}$ denotes the k-fold composition of A with itself where $A^{\circ 0}_{x,dy} := \delta_x(dy)$.

The second kind of kernels is obtained by letting a kernel of the first kind depend on a further parameter $n \in \mathbb{Z}^+$, that is we consider objects of the form $A_{x,dy}(n)$ with $x, y \in \mathbb{R}, n \in \mathbb{Z}^+$. Given two such kernels $A_{x,dy}(n), B_{x,dy}(n)$ we define their convolution

$$(A * B)_{x,dy}(n) := \sum_{m=0}^{n} (A(m) \circ B(n-m))_{x,dy} = \sum_{m=0}^{n} \int_{\mathbb{R}} A_{x,dz}(m) B_{z,dy}(n-m), \tag{16}$$

and the k-fold convolution of the kernel A with itself will be denoted by $A^{*k}_{x,dy}$ where by definition $A^{*0}_{x,dy} := \delta_x(dy)\mathbf{1}_{n=0}$. Finally given two kernels $A_{x,dy}(n)$ and $B_{x,dy}$ and a positive sequence a_n, we will write

$$A_{x,dy}(n) \sim \frac{B_{x,dy}}{a_n} \tag{17}$$

to mean $A_{x,F}(n) \sim \frac{B_{x,F}}{a_n}$ for every $x \in \mathbb{R}$ and for every bounded set $F \subset \mathbb{R}$.

Natural and useful examples of the above kernels are the partition functions; namely, for $x, y \in [0,a] \times \mathbb{R}^+$, we define:

$$Z^c_{N,a,\beta}(x,dy) := \mathbf{E}_x\left[\exp\left(\beta\sum_{k=1}^{N}\mathbf{1}_{S_k\in[0,a]}\right)\mathbf{1}_{S_k\geq 0,k=1\dots N}\mathbf{1}_{S_N\in dy}\mathbf{1}_{y\in[0,a]}\right], \tag{18}$$

and its free counterpart

$$Z^f_{N,a,\beta}(x,dy) := \mathbf{E}_x\left[\exp\left(\beta\sum_{k=1}^{N}\mathbf{1}_{S_k\in[0,a]}\right)\mathbf{1}_{S_k\geq 0,k=1\dots N}\mathbf{1}_{S_N\in dy}\right]. \tag{19}$$

2.2 Markov renewal and random walk fluctuation theory.

MRSS

Let us introduce the following transition kernel:

$$\begin{aligned} &F_{x,dy}(n) := \mathbf{P}_x[S_1 > a, S_2 > a, \dots, S_{n-1} > a, S_n \in dy]\mathbf{1}_{x,y\in[0,a]} \text{ if } n \geq 2, \\ &F_{x,dy}(1) := h(y-x)\mathbf{1}_{x,y\in[0,a]}dy. \end{aligned} \tag{20}$$

We write $f_{x,y}(n)$ for the density of $F_{x,dy}(n)$ with respect to the Lebesgue measure. Also, for notational convenience, in the sequel we will use the following notation:

$$\frac{1}{dx}\mathbf{P}[B, S_k \in dx] =: \mathbf{P}[B, S_k = x]. \tag{21}$$

We denote by $(\tau_n)_{n\geq 0}$ the times of return to $[0,a]$ of S, that is $\tau_0 := 0$ and, for $n \geq 1$, $\tau_n := \inf\{k > \tau_{n-1} | S_k \in [0,a]\}$. Note that $(\tau_n)_{n\geq 0}$ is not a true renewal process. Introducing the process $(J_n)_{n\geq 0}$ where $J_n := S_{\tau_n}$, the process τ is a so called *Markov renewal process* whose modulating chain is the Markov chain J. The topic of Markov renewal theory is a classical one, a well known reference is [Asm03].

We finally denote by l_N the cardinality of $\{k \leq N | S_k \in [0,a]\}$. With these notations, we can write the joint law of $(l_N, (\tau_n)_{n\leq l_N}, (J_n)_{n\leq l_N})$ under $\mathbf{P}^c_{N,a,\beta}$ under the following form:

$$\begin{aligned} &\mathbf{P}^c_{N,a,\beta}[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1, \dots, k] \\ &\qquad = \frac{e^{\beta k}}{Z^c_{N,a,\beta}} F_{0,dy_1}(t_1)F_{y_1,dy_2}(t_2-t_1)\dots F_{y_k,dy_{k-1}}(N-t_{k-1}) \end{aligned} \tag{22}$$

HPP

where $k \in \mathbb{N}, 0 < t_1 < \ldots < t_k = N$ and $(y_i)_{i=1,\ldots,k} \in \mathbb{R}^k$.

It is then clear that getting asymptotic estimates on the partition functions $Z^c_{N,a,\beta}$ (and thus $Z^f_{N,a,\beta}$), requires an accurate control on the asymptotic behavior of $F_{\cdot,\cdot}(n)$ for large n.

To achieve this, we collect some basic facts about random walk fluctuation theory.

For n an integer, we denote by T_n the *nth ladder epoch*; that is $T_0 := 0$ and, for $n \geq 1$, $T_n := \inf\{k \geq T_{n-1}, S_k > S_{T_{n-1}}\}$. We also introduce the so-called *ascending ladder heights* $(H_n)_{n\geq 0}$, which, for $k \geq 1$, are given by $H_k := S_{T_k}$. Note that the process (T, H) is a bivariate renewal process on $(\mathbb{R}^+)^2$. In a similar way, we write (T^-, H^-) for the strict descending ladder variables process, which is defined by $(T_0^-, H_0^-) := (0, 0)$ and

$$T_n^- := \inf\{k \geq T_{n-1}, S_k < S_{T_{n-1}}\} \quad \text{and} \quad H_k^- := -S_{t_k^-}. \tag{23}$$

Let us consider the renewal function $U(\cdot)$ associated to the ascending ladder heights process:

$$U(x) := \sum_{k=0}^{\infty} \mathbf{P}[H_k \leq x] = \mathbf{E}[\mathcal{N}_x] = \int_0^x \sum_{m=0}^{\infty} u(m, y) dy \tag{24}$$

where $\mathcal{N}_x$ is the cardinality of $\{k \geq 0, H_k \leq x\}$ and $u(m, y) := \frac{1}{dy}\mathbf{P}[\exists k \geq 0, T_k = m, H_k \in dy]$ is the renewal mass function associated to (T, H). It follows in particular from this definition that $U(\cdot)$ is a subadditive increasing function, and in our context it is also continuous. Note also that $U(0) = 1$. We denote by $V(x)$ the analogous quantity for the process H^-, and by $v(m, y)$ the renewal mass function associated to the descending renewal (T^-, H^-).

The following local limit estimates have been proved recently (Do3 [Don10] and CaCha [CC13]):

fluctu **Lemma 2.1** *Uniformly on sequences x_n, y_n such that $x_n \vee y_n = o(\sqrt{n})$, the following equivalences hold:*

$$\mathbf{P}_{x_n}[S_1 \geq 0, \ldots, S_n \geq 0] \sim V(x_n)\mathbf{P}[T_1^- > n] \sim \frac{V(x_n)}{\sqrt{2\pi}\sigma\sqrt{n}} \tag{25}$$ pacons

and

$$\mathbf{P}_{x_n}[S_1 \geq 0, \ldots, S_n \geq 0, S_n = y_n] \sim \frac{V(x_n)U(y_n)}{n}\mathbf{P}[S_n = y_n]. \tag{26}$$

Note that making use of Gnedenko's classical local limit theorem, for sequences $(x_n), (y_n)$

satisfying the same assumptions as in Lemma 2.1, one gets the equivalence

$$\mathbf{P}_{x_n}[S_1 \geq 0, \ldots, S_n \geq 0, S_n = y_n] \sim \frac{V(x_n)U(y_n)}{\sigma\sqrt{2\pi}n^{3/2}}. \qquad (27)$$ const

A consequence of Lemma 2.1 is the following result:

stayabove **Lemma 2.2** *For any $x \in [0,a]$, one has the following convergence:*

$$\mathbf{P}_x[S_1 > a, \ldots, S_n > a] \sim \frac{\mathbf{P}[H_1 \geq a-x]}{\sqrt{2\pi}\sigma n^{1/2}}. \qquad (28)$$

Proof of Lemma 2.2 We integrate over S_1 to get:

$$\begin{aligned}
\mathbf{P}_x[S_1 > a, \ldots, S_n > a] &= \int_{u\in[a,\infty)} \mathbf{P}_x[S_1 = u, \ldots, S_n > a]du \\
&= \int_{u\in[a,\infty)} h(u-x)\mathbf{P}_{u-a}[S_1 > 0, \ldots, S_{n-1} > 0]du \\
&= \int_{u\in[a,n^{1/4}]} h(u-x)\mathbf{P}_{u-a}[S_1 > 0, \ldots, S_{n-1} > 0]du \\
&+ \int_{u\in[n^{1/4},\infty)} h(u-x)\mathbf{P}_{u-a}[S_1 > 0, \ldots, S_{n-1} > 0]du.
\end{aligned} \qquad (29)$$

For the second term in the right hand side of the above equalities, we immediately get that, for any n large enough:

$$\begin{aligned}
&\int_{u\in[n^{1/4},\infty)} h(u-x)\mathbf{P}_{u-a}[S_1 > 0, \ldots, S_{n_1} > 0]du & (30)\\
&\leq \int_{u\in[n^{1/4}-a,\infty)} h(u)du & (31)\\
&\leq \frac{4}{n^{1/2}} \int_{u\in[n^{1/4}/2,\infty)} u^2 h(u)du, & (32)\\
& & (33)
\end{aligned}$$

and since $\mathbf{E}[X^2] < \infty$, it immediately follows that this term is $o(n^{-1/2})$. On the other hand, making use of Lemma 2.1, we get that

$$\int_{u\in[a,n^{1/4}]} h(u-x)\mathbf{P}_{u-a}[S_1 > 0, \ldots, S_{n_1} > 0]du \sim \int_{u\in[a,n^{1/4}]} h(u-x)\frac{V(u-a)}{\sqrt{2\pi}\sigma\sqrt{n}}du. \qquad (34)$$

Then we recall that, using duality arguments (see for example [Soh13][Proof of Theorem 3.1] for a proof), one can show that

$$\int_{u\in[a,\infty)} h(u-x)V(u-a)du = \mathbf{P}[H_1 \geq a-x], \tag{35}$$

from which we finally deduce Lemma stayabove 2.2. □

We define the following function:

$$\Phi_a(x,y) := \frac{\mathbf{P}[H_1^- \geq a-y]\mathbf{P}[H_1 \geq a-x]}{\sigma\sqrt{2\pi}}\ \mathbf{1}_{x,y\in[0,a]}. \tag{36}$$ DefPh

By using similar techniques as the ones we just developed for the proof of Lemma stayabove 2.2, in Soh1 [Soh13] the author showed the following result, which will be the cornerstone of our approach:

Pr **Lemma 2.3** *The following equivalence holds uniformly on* $(x,y)\in[0,a]^2$*:*

$$n^{3/2} f_{x,y}(n) \sim \Phi_a(x,y). \tag{37}$$ EQC

Since Φ_a is bounded on $[0,a]^2$, a trivial consequence of the above result is the fact that the left hand side in (EQC 37) is dominated by a multiple of its right hand side.

3 An infinite dimensional problem

MRT

3.1 Defining the free energy

In this section, we define the free energy in a way that allows us to make use of the Markov renewal structure we pointed at in the previous part. For $\lambda \geq 0$, we introduce the following kernel:

$$B^\lambda_{x,dy} := \sum_{n=1}^{\infty} e^{-\lambda n} F_{x,dy}(n) \tag{38}$$

and the associated integral operator

$$(B^\lambda h)(x) := \int_{[0,a]} B^\lambda_{x,dy} h(y). \tag{39}$$

Making use of the asymptotics (EQC 37), one can show as in CD1 [CD08][Lemma 4.1] that for any $\lambda \geq 0, B^\lambda_{x,dy}$ is a compact operator on the Hilbert space $L^2([0,a])$. Using this fact, we introduce $\delta^a(\lambda)$, the spectral radius of the operator B^λ, which is an isolated and simple eigenvalue of $B^{\lambda}_{x,dy}$ (see Theorem 1 in [Zer87]). The function $\delta^a(\cdot)$ is non-increasing, continuous on $[0,\infty)$ and analytic on $(0,\infty)$ because the operator $B^{\lambda}_{x,dy}$ has these properties. The analyticity and the fact that $\delta^a(\cdot)$ is not constant (as $\delta^a(\lambda)\to 0$ as $\lambda\to\infty$) force $\delta^a(\cdot)$ to be strictly decreasing.

We denote by $(\delta^a)^{-1}(\cdot)$ its inverse function, defined on $(0,\delta^a(0)]$. We now define β^a_c and $F^a(\beta)$ by:

$$\beta^a_c := -\log(\delta^a(0)), \;\; F^a(\beta) := (\delta^a)^{-1}(\exp(\beta)) \text{ if } \beta \geq \beta^a_c \text{ and } 0 \text{ otherwise.} \tag{40}$$

DefFEE

Note that this definition entails in particular the analyticity of $F(\cdot)$ on $\mathbb{R}\setminus\{\beta^a_c\}$. Also, one deduces from this expression the specific values of β^a_c obtained in Part (1.3).

Of course it is not clear *a priori* that the quantity we define in (40) actually coincides with the classical definition of the free energy, that is the limit of the quantity $\frac{1}{N}\log\left(Z^f_{N,a,\beta}\right)$. We will show in the next parts that this is the case. Indeed, this definition will entail a representation for the constrained partition function of the system which is explicited in section 3.2, and this representation in turn will provide estimates on the partition function of the system in both phases. These estimates will finally validate the coherence of the definition given in equation (40).

3.2 A useful representation for $Z^c_{N,a,\beta}$

UREP

The fact that $b^{F^a(\beta)}(x,y) > 0$ for every $(x,y)\in[0,a]$ implies the uniqueness (up to a multiplication by a positive constant) and the positivity almost everywhere of the right (respectively the left) Perron Frobenius eigenfunctions $v_\beta(\cdot)$ (respectively $w_\beta(\cdot)$) of $B^{F^a(\beta)}_{x,dy}$. We refer to [CD08][Section 4.2] for more details. In particular, one can show that the function $v_\beta(\cdot)$ is positive everywhere (not only almost everywhere); hence we can define the kernel

$$K^\beta_{x,dy}(n) := e^\beta F_{x,dy}(n) e^{-F^a(\beta)n}\frac{v_\beta(y)}{v_\beta(x)}, \tag{41}$$

and it is easy to check that

$$\int_{y\in\mathbb{R}}\sum_{n\in\mathbb{N}} K^\beta_{x,dy}(n) = 1 \wedge \frac{e^\beta}{e^{\beta^a_c}}. \tag{42}$$

INVMP

Then we define the law $\mathcal{P}_\beta$ under which the joint process $(\tau_k, J_k)_{k\geq 0}$ is an inhomogeneous Markov chain (defective if $\beta < \beta_c^a$) on $\mathbb{Z}^+ \times [0,a]$ by:

$$\mathcal{P}_\beta\left[(\tau_{k+1}, J_{k+1}) \in (\{n\}, dy) | (\tau_k, J_k) = (m,x)\right] := K_{x,dy}(n-m). \tag{43}$$ Def

The sequence $(\tau_k)_{k\geq 0}$ is a Markov renewal, the process $(J_i)_{i\geq 0}$ being its modulating chain. We then have the following property, whose proof is contained in ([22]):

RMRP **Proposition 3.1** *For any $N \in \mathbb{N}$, the vector $(l_N, (\tau_n)_{n\leq l_N}, (J_n)_{n\leq l_N})$ has the same law under $\mathbf{P}^c_{N,a,\beta}$ as under the conditional law $\mathcal{P}_\beta(\cdot|\mathcal{A}_N)$ where $\mathcal{A}_N := \{\exists k \in \mathbb{N} | \tau_k = N\}$. Equivalently:*

$$\begin{aligned} &\mathbf{P}^c_{N,a,\beta}\left[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1,\dots,k\right] \\ &\qquad = \mathcal{P}_\beta\left[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1,\dots,k | \mathcal{A}_N\right]. \end{aligned} \tag{44}$$

Proposition 3.1 shows in particular that the partition $Z^c_{N,a,\beta}$ can be interpreted as the Green function associated to the Markov renewal τ, that is $Z^c_{N,a,\beta} = \mathcal{P}_\beta[N \in \tau]$ and more generally for $x, y \in [0,a]$, $Z^c_{N,a,\beta}(x,dy) = \mathcal{P}_\beta[\exists k, \tau_k = N, J_k \in dy | J_0 = x]$. Equivalently, we have the equality

$$Z^c_{N,a,\beta} = \exp(F^a(\beta)N) \int_{[0,a]} \frac{v_\beta(0)}{v_\beta(y)} \sum_{k\geq 0} (K^\beta)^{*k}_{0,dy}(N) \tag{45}$$ Rep

which is a consequence of the more general equality:

$$Z^c_{N,a,\beta}(x,dy) = \exp(F^a(\beta)N) \frac{v_\beta(x)}{v_\beta(y)} \sum_{k\geq 0} (K^\beta)^{*k}_{x,dy}(N) \tag{46}$$ FUN

which holds for $x, y \in [0,a]$.

4 The localized phase

TLP

4.1 The key Markov renewal theorem

Let $\beta > \beta_c^a$. In this case, the two functions $w_\beta(\cdot)$ and $v_\beta(\cdot)$ are uniquely defined up to a multiplicative constant, and we use this degree of freedom to fix $\int_{\mathbb{R}} v_\beta(x) w_\beta(x) \mathbf{1}_{x\in[0,a]} = 1$. Thus the measure μ_β defined by

$$\mu_\beta(dx) := v_\beta(x) w_\beta(x) \mathbf{1}_{x\in[0,a]} dx \tag{47}$$

is a probability measure. One easily checks that for $\beta > \beta_c^a$, the probability μ_β is invariant for the kernel $\sum_{n\geq 1} K^\beta_{x,dy}(n)$, and hence for the Markov process (J_n).

To get estimates on the partition function of the Markov renewal process, we are led to show an analogous to the classical Markov renewal theorem (which can be found in As [Asm03]) in the case where the state space of the J_i's is not countable. Surprisingly enough, the author has not been able to find a proof of such a natural result in the litterature, so that we prove it by making use of the ergodic properties of the *forward Markov chain* naturally linked to the Markovian renewal.

We state the main result of this part:

FTCC **Lemma 4.1** *In the localized regime, for $x \in [0,a]$ the following convergence holds in total variation norm:*

$$\lim_{N\to\infty} \mathcal{P}_\beta \left[\exists k \in \mathbb{N}, \tau_k = N, J_N \in dy | J_0 = x\right] = \frac{\mu_\beta(dy)}{\int_{[0,a]^2} \mu_\beta(du) \sum_{n\geq 0} n K^\beta_{u,dy}(n)}. \tag{48}$$

Eq111

Proof of Lemma FTCC *4.1* We consider the Markov process $(A_k, J'_k)_{k\geq 0}$ on $\mathbb{N}\times[0,a]$ whose transitions are given by:

$$P_\beta\left[A_j = k, J'_j \in dy | A_{j-1} = l, J'_{j-1} = x\right] := \delta_{k,l-1}\delta_x(dy) \tag{49}$$

if $l \geq 2$ (where $\delta_x(\cdot)$ is the Dirac measure concentrated on $\{x\}$) and by

$$P_\beta\left[A_j = k, J'_j \in dy | A_{j-1} = 1, J'_{j-1} = x\right] := K^\beta_{x,dy}(k). \tag{50}$$

Note that this Markov chain is nothing but the well known *forward recurrence chain* associated to the Markov renewal. In words, A_i denotes the time one has to wait from time i until the next renewal happens (that is $A_i = \inf\{k > i, \exists j, \tau_j = k\} - i$), the Markov chain J' containing the last location of its modulating chain.

We introduce the probability measure on $\mathbb{N}\times[0,a]$ defined by :

$$\Pi^\beta(i, dy) := \frac{1}{\int_{[0,a]^2} \mu_\beta(dx) \sum_{k\geq 1} k K^\beta_{x,du}(k)} \int_0^a \mu_\beta(dx) \sum_{j\geq i} K^\beta_{x,dy}(j). \tag{51}$$

Note that $\int_{[0,a]^2} \mu_\beta(dx) \sum_{k\geq 1} k K^\beta_{x,du}(k) < \infty$ since $\beta > \beta_c^a$, so that in particular $\Pi^\beta(\cdot,\cdot)$ is non degenerate. $\Pi^\beta(\cdot,\cdot)$ is the invariant probability of the Markov process $(A_k, J'_k)_{k\geq 0}$.

Indeed, for all $(i, y) \in \mathbb{N} \times [0, a]$, we check:

$$\Pi^\beta P_\beta(i, dy) = \Pi^\beta(i+1, dy) + \int_0^a \Pi^\beta(1, dx) K^\beta_{x,dy}(i) = \Pi^\beta(i, dy) \tag{52}$$ OEIY

where in the second equality we used the fact that $\mu_\beta(\cdot)$ is the invariant probability for the Markov process $(J_k)_{k\geq 1}$ (noting that $\Pi^\beta(1, dx)$ is a multiple of $\mu_\beta(dx)$, this is exactly saying that $\int_0^a \mu_\beta(dx) K^\beta_{x,dy}(i) = \mu_\beta(dy)$, which is the second part of equation (OEIY 52)).

We note that, making use of the positivity of μ_β on $[0, a] \times \mathbb{N}$, the Markov chain (A, J') satisfies the hypothesis of the classical ergodic Theorem, so that

$$|| \int \lambda(dx) P^n_\beta(x, \cdot) - \Pi^\beta || \to 0 \tag{53}$$

as $n \to \infty$, where $|| \cdot ||$ denotes the total variation norm on $\mathbb{N} \times [0, a]$ and $\lambda(\cdot)$ any initial distribution. This implies in particular that, as $j \to \infty$, the following convergence holds in total variation norm :

$$P_\beta\left[A_j = 1, J'_j \in dy | J_0 = x\right] \to \frac{\mu_\beta(dy)}{\int_{[0,a]^2} \mu_\beta(du) \sum_{k\geq 1} k K^\beta_{u,dy}(k)} =: \frac{\mu_\beta(dy)}{C_\beta} \tag{54}$$ nu

and since $P_\beta\left[A_j = 1, J'_j \in dx\right] = \mathcal{P}_\beta\left[\exists k \in \mathbb{N}, \tau_k = j, J_k \in dx\right]$, the proof is complete. □

4.2 Asymptotic of the partition functions

In the localized phase, this result provides an estimate for $Z^c_{N,a,\beta}$.

delest **Theorem 4.2** *For $\beta > \beta^a_c$, for every $x \in [0, a], y \in \mathbb{R}^+$, as $N \to \infty$, one has the convergence :*

$$Z^c_{N,a,\beta}(x, dy) \sim \frac{v_\beta(x) v_\beta(y)}{C_\beta} \exp(F^a(\beta) N) dy \tag{55}$$

where for a fixed $x \in [0, a]$, the convergence of $Z^c_{N,a,\beta}(x, dy) \exp(-F^a(\beta)N)$ towards $\frac{v_\beta(x)v_\beta(y)}{C_\beta} dy$ holds in total variation norm.

These estimates imply in particular that there exist two positive constants $C^a(\beta)$ and $C^a_f(\beta)$ such that, :

1. $Z^c_{N,a,\beta} \sim C^a(\beta) \exp(F^a(\beta) N)$
2. $Z^f_{N,a,\beta} \sim C^a_f(\beta) \exp(F^a(\beta) N)$.

As we noted right after Theorem 1.2, this result readily implies a much finer result on the scaling limits of the system than the one of Theorem 1.2.

Proof of Theorem 4.2 Combining identity (46) and Lemma 4.1, we get :

$$Z^c_{N,a,\beta}(x,dy) \sim \exp(F^a(\beta)N)\frac{v_\beta(x)}{v_\beta(y)}\frac{\mu_\beta(dy)}{\int_{[0,a]^2}\mu_\beta(dx)\sum_{k\geq 1}kK^\beta_{x,dy}(k)}. \tag{56}$$

The free case follows; we have the relation:

$$Z^f_{N,a,\beta} = e^{F^a(\beta)N}\sum_{t=0}^{N} Z_{N-t,a,\beta}(dx)e^{-F^a(\beta)(N-t)}\mathbf{P}_x[S_1 > a,\dots,S_t > a]e^{-F^a(\beta)t}. \tag{57}$$

Using the total variation convergence part of Lemma 4.1, this entails:

$$Z^f_{N,a,\beta} \sim C_a(\beta)e^{F^a(\beta)N}\sum_{t=0}^{\infty}e^{-F^a(\beta)t}\int_{[0,a]}\frac{\mu_\beta(dx)}{C_\beta}\mathbf{P}_x[\tau_1 > t+1]. \tag{58}$$

□

5 The delocalized phase

TDP

5.1 Some results borrowed from the standard homogeneous wetting.

We stress that we can adapt in a straightforward way some of the techniques borrowed from different papers on the topic of scaling limits linked to polymer models to our case of interest. We first mention that, combining Proposition 7.2 in [CD08], the considerations on Markov renewal processes of Part 3 (see in particular (45)) and the asymptotics of Lemma 2.3, we can prove the following asymptotics on the partition functions in the delocalized phase:

esti **Proposition 5.1** *For $\beta < \beta^a_c$,as $N \to \infty$, we have the following:*

1. $Z^c_{N,a,\beta}(x,dy) \sim C'^a(\beta)/N^{3/2}\Phi_a(x,y)dy$
2. $Z^f_{N,a,\beta}(x,dy) \sim C'^a_f(\beta)/N^{1/2}\Phi_a(x,y)dy$

where $C'^a(\beta)$ and $C'^a_f(\beta)$ are positive constants depending on β.

From Proposition 5.1, we can describe the set of contact points in the subcritical regime. Namely, we introduce a probability law $p^f_{\beta,N}(\cdot,\cdot)$ on $V_N\times(\mathbb{R}^+)^N$ and a probability law $p^c_{\beta,N}(\cdot,\cdot)$ on $V_{N-1}\times(\mathbb{R}^+)^N$ which are defined by:

$$p^f_{\beta,N}(A,dx) := \frac{1}{Z^f_{N,a,\beta}} e^{\beta|A|} \prod_{j=1}^{|A|} F_{x_{t_{j-1}},dx_{t_j}}(t_j - t_{j-1}) \mathbf{1}_{x_{t_j}\in[0,a],\forall j\in\{0,\dots,|A|\}} \tag{59}$$

and

$$p^c_{\beta,N}(A,dx) := \frac{1}{Z^c_{N,a,\beta}} e^{\beta(|A|+1)} \prod_{j=1}^{|A|+1} F_{x_{t_{j-1}},dx_{t_j}}(t_j - t_{j-1}) \mathbf{1}_{x_{t_j}\in[0,a],\forall j\in\{0,\dots,|A|\}} \tag{60}$$ ten

where $t_0 := 0$, $t_{|A|+1} := N$, $x_0 := 0$ and $A := \{t_1 < t_2 < \ldots < t_{|A|}\}$.

These laws are linked to the laws $\mathbf{P}^f_{N,a,\beta}$ and $\mathbf{P}^c_{N,a,\beta}$ in the following way; in the free case, we can write

$$\mathbf{P}^f_{N,a,\beta}(dx) = \sum_{A\subset V_N} \int_{[0,a]^{|A|}} p^f_{\beta,N}(A,dy)\mathbb{P}_{A,y}(dx) \tag{61}$$ nine

where $\mathbb{P}_{A,y}(\cdot)$ is the law of $(S_1,\ldots,S_N)$ conditioned on the event $\mathcal{E}_{N,A,y}$ which is defined by:

$$\mathcal{E}_{N,A,y} := \Big\{S_i = y_i; i\in A\cup\{0\}\Big\} \cap \Big\{S_i > a, i\notin A\Big\}. \tag{62}$$ epscond

In the same way, for the constrained case, one readily realizes that

$$\mathbf{P}^c_{N,a,\beta}(dx) = \sum_{A\subset V_N} \int_{[0,a]^{|A|+1}} p^c_{\beta,N}(A,dy)\mathbb{P}^c_{A,y}(dx) \tag{63}$$

where for $y\in(\mathbb{R}^+)^N$, $\mathbb{P}_{A,y}(\cdot)$ is the law of $(S_1,\ldots,S_N)$ conditioned on the event $\mathcal{E}_{N,A,y}$ which is defined by:

$$\mathcal{E}_{N,A,y} := \Big\{S_i = y_i; i\in A\cup\{0\}\cup\{N\}\Big\} \cap \Big\{S_i > a, i\notin A\cup\{N\}\Big\}. \tag{64}$$

For $A\in V_N$, we write $\mathcal{L}(A) := \max A$ and $R(A) := \min((A\cap[N/2,N])\cup\{N\})$. The following lemma implies Lemma TRLO 1.3:

MLL **Lemma 5.2** *For $\beta < \beta^a_c$, the following estimate holds:*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^N} p^f_{\beta,N}(\mathcal{L}(A)\geq L, x) = 0. \tag{65}$$ end

The corresponding estimates in the constrained case read:

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^N} p^c_{\beta,N}(L(A) \geq L, x) = 0 \tag{66}$$

and

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^N} p^c_{\beta,N}(R(A) \leq N - L, x) = 0. \tag{67}$$ env

The proof of these convergences follows by making use of the equivalences from Proposition 5.1 and goes along the same lines as the proof of [DGZ05][Proposition 7].

IVLSR

5.2 Scaling limits in the subcritical regime

The goal of this part is to prove Theorem 1.2. We fix $\beta < \beta_c^a$. We treat the free case and the constrained case separately, the techniques we use for each case are quite similar but the constrained case is technically more involved.

The main idea of both proofs is the same. Combining the estimates on the contact set of Lemma 1.3 and the representations of equations (61) and (60), we consider the trajectories whose contacts with the strip are very close to $\{0\}$ (and from the endpoint in the constrained case). After integrating over the first step after the last contact with the strip and making use of Markov's property, the remaining process is simply the random walk conditioned to stay over the strip (and to come back to it close to the endpoint for the constrained case). Finally, in the free case, the convergence towards brownian meander will be a consequence of a result due to Shimura [Shi83] and of a recent result due to Caravenna and Chaumont [CC13] for the constrained one.

We define $\tau_{(-\infty,0)} := \inf\{j \geq 0, S_j < 0\}$.

The free case The main tool in the first part of the proof of Theorem 1.2 will be the following result which has been proved in [Shi83, Example 4.1]:

ShiTT **Theorem 5.3** *Let x_N be a positive sequence such that $x_N N^{-1/2} \to 0$ as $N \to \infty$. One has the following functional limit convergence:*

$$\mathbf{P}_{x_N}\Big[\ \cdot\ \Big|\tau_{(-\infty,0)} > N\Big] \circ \Big(X^N\Big)^{-1} \Longrightarrow m(\cdot). \tag{68}$$

For clarity, we summarize the steps of the proof of Theorem 1.2 in the next key lemma; then we show that we may apply Lemma 5.4 to our setup, and finally we go to its proof.

KEYLEm **Lemma 5.4** *Let l_N be the random variable $\mathcal{L}(A)$ under $\mathbf{P}^f_{N,a,\beta}$ and let L be a positive integer. Assume the following assumptions hold:*

1. *for any $\varepsilon > 0$, one has*

$$\lim_{N\to\infty} Q^f_{N,a,\beta}\left[\sup_{t\in[0,\frac{l_N}{N}]} \omega_t \geq \varepsilon\right] = 0. \tag{69}$$

2. *for every $A \subset V_N$ such that $\mathcal{L}(A) = L$ and for every $x \in \mathbb{R}^N$, if X follows the law $\mathbb{P}_{A,x}$, one has the convergence in law:*

$$\left(\sqrt{1-\frac{L}{N}}X^N_{\frac{L}{N}+t(1-\frac{L}{N})}\right)_{t\in[0,1]} \Rightarrow m \tag{70}$$

where m denotes the law of the brownian meander.

Then one has the weak convergence

$$Q^f_{N,a,\beta} \Rightarrow m. \tag{71}$$

Proof of Theorem 1.2 for the free case

First point of Lemma 5.4. We write :

$$Q^f_{N,a,\beta}\left[\sup_{t\in[0,\frac{l_N}{N}]} \omega_t \geq \varepsilon\right] = \mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geq \varepsilon\sigma\sqrt{N}; l_N > L\right] \tag{72}$$

$$+\mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geq \varepsilon\sigma\sqrt{N}; l_N \leq L\right],$$

so that choosing L_0 large enough and making use of Lemma 1.3, for any fixed $\eta > 0$, we can get the following bound which holds for every N and for every $L \geq L_0$:

$$\left|Q^f_{N,a,\beta}\left[\sup_{t\in[0,\frac{l_N}{N}]} \omega_t \geq \varepsilon\right] - \mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geq \varepsilon\sigma\sqrt{N}; l_N \leq L\right]\right| \leq \eta/2. \tag{73}$$

Then we note that:

$$\mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geq \varepsilon\sigma\sqrt{N}; l_N \leq L\right] = \frac{\mathbf{E}\left[\mathbf{1}_{\max_{j=1,\dots,l_N} S_j \geq \varepsilon\sigma\sqrt{N}} e^{\beta\sum_{i=1}^N \mathbf{1}_{S_i\in[0,a]}} \mathbf{1}_{l_N\leq L}\mathbf{1}_{T_1^->N}\right]}{Z^f_{N,a,\beta}} \tag{74}$$

so that using the estimates on $Z^f_{N,a,\beta}$ from Proposition 5.1 and Lemma 2.1, we get that there exists a constant $\mathcal{C} > 0$ such that :

$$\mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geq \varepsilon\sigma\sqrt{N}; l_N \leq L\right] \leq \mathcal{C}N^{1/2}e^{\beta L}\mathbf{E}\left[\mathbf{1}_{\max_{j=1,\dots,L} S_j \geq \varepsilon\sigma\sqrt{N}}\mathbf{1}_{l_N\leq L}\mathbf{1}_{T_1^- > N}\right]$$
$$\leq \mathcal{C}e^{\beta L}\mathbf{P}\left[\max_{j=1,\dots,L} S_j \geq \varepsilon\sigma\sqrt{N}|T_1^- > N\right]. \tag{75}$$

Then we consider some $\gamma > 0$. As soon as N is large enough, we have of course the bound:

$$\mathbf{P}\left[\max_{j=1,\dots,L} S_j \geq \varepsilon\sigma\sqrt{N}\Big|T_1^- > N\right] \leq \mathbf{P}\left[\max_{j=1,\dots,\gamma N} S_j \geq \varepsilon\sigma\sqrt{N}\Big|T_1^- > N\right]. \tag{76}$$

We can rewrite the right hand side above as:

$$\mathbf{P}\left[\sup_{t\in[0,\gamma]} S^N_t \geq \varepsilon\Big|T_1^- > N\right]. \tag{77}$$

where S^N is the image of $(S_j)_{j\leq N}$ under the map X^N. Making use of Theorem 5.3, for every bounded continuous function $\Phi(\cdot)$ on $C([0,1])$, one has the convergence

$$\mathbf{E}\Big[\Phi\left((S^N_t)_{t\in[0,1]}\right)\Big|T_1^- > N\Big] \to m(\Phi). \tag{78}$$

Finally, we note that the set of discontinuities of the functional

$$\begin{array}{ccc}\mathcal{C}([0,1],\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \mathbf{1}_{\sup_{[0,\gamma]} f \geq \varepsilon}\end{array} \tag{79}$$

is of null m-measure, so that by the continuous mapping theorem (see [Bil68]) the quantity (77) converges towards $m(\sup_{t\in[0,\gamma]}\omega_t > \varepsilon)$ which can be made arbitrarily small when γ is chosen accordingly. □

The second point of Lemma 5.4 is fulfilled.

We first prove the second statement in the case where $A = \emptyset$. Let $\varepsilon > 0$. We consider a Lipschitz bounded functional Φ on $\mathcal{C}([0,1],\mathbb{R})$, that is such that there exist two positive constants c_1 and c_2 verifying that for every $f, g \in \mathcal{C}([0,1], \mathbb{R})$, one has:

$$|\Phi(f)| < c_1 \quad \text{and} \quad |\Phi(f) - \Phi(g)| \leq c_2 ||f - g||_\infty. \tag{80}$$

Here, the event $\mathcal{E}_{N,A,x}$ appearing in (62) is the event $\{S_1 > a, \ldots, S_N > a\}$; conditioning on S_1 and using Markov's property, one gets:

$$\begin{aligned} &\mathbb{E}_A^f\Big[\Phi\left((X_t^N)_{t\in[0,1]}\right)\Big] \\ &\quad = \int_a^\infty \frac{\mathbf{E}\Big[\Phi\left(X^N(t, S_2, \ldots, S_N)\right), S_1 = t, S_2 > a, \ldots, S_N > a\Big]}{\mathbf{P}[S_1 > a, \ldots, S_N > a]} dt. \end{aligned} \tag{81}$$

Then we use the Markov property and the invariance by translation of S to get that for any $t \geq a$:

$$\begin{aligned} &\mathbf{E}\Big[\Phi\left(X^N(t, S_2, \ldots, S_N)\right), S_1 = t, S_2 > a, \ldots, S_N > a\Big] \\ &\quad = h(t)\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t, S_1 + a, \ldots, S_{N-1} + a)\right), \tau_{(-\infty,0)} > N - 1\Big]. \end{aligned} \tag{82}$$

For any $(x_1, \ldots, x_{N-1}) \in (\mathbb{R}^+)^{N-1}$ and $t \in [a, N^{1/4}]$, one has

$$\begin{aligned} &\Big|\Phi\left(X^N(t, x_1 + a, \ldots, x_{N-1} + a)\right) - \Phi\left(X^{N-1}(x_1, \ldots, x_{N-1})\right)\Big| \\ &\quad \leq \frac{c_2 \sup_{j=1,\ldots,N-1} |x_j - x_{j-1}|}{\sqrt{N}} + c_2 \frac{a + N^{1/4}}{\sqrt{N}}. \end{aligned} \tag{83}$$

Theorem 5.3 implies that for $t \in (a, N^{1/4}), \mathbf{E}_{t-a}\Big[\ \cdot\ \Big|\tau_{(-\infty,0)} > N - 1\Big] \circ \Big(X^{N-1}\Big)^{-1}$ converges towards $m(\cdot)$. In particular, using the tightness criterion of Kolmogorov, this implies the fact that $\frac{\sup_{j=1,\ldots,N-1} |S_j - S_{j-1}|}{\sqrt{N}} + \frac{a+N^{1/4}}{\sqrt{N}} =: \mathcal{Y}_N^a$ converges towards zero in probability when $(S_j)_{j\leq N}$ is distributed according to $\mathbf{E}_{t-a}\Big[\ \cdot\ \Big|\tau_{(-\infty,0)} > N - 1\Big]$.

Thus, one has:

$$\begin{aligned} &\Big|\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t, S_1 + a, \ldots, S_{N-1} + a)\right) \Big|\tau_{(-\infty,0)} > N - 1\Big] \\ &\qquad - \mathbf{E}_{t-a}\Big[\Phi\left(X^{N-1}(S_1, \ldots, S_{N-1})\right) \Big|\tau_{(-\infty,0)} > N - 1\Big]\Big| \end{aligned} \tag{84}$$

$$
\begin{aligned}
&\le \mathbf{E}_{t-a}\Big[\Big|\Phi\left(X^N(t, S_1+a,\dots,S_{N-1}+a)\right) \\
&\qquad\qquad - \Phi\left(X^{N-1}(S_1,\dots,S_{N-1})\right)\Big|\mathbf{1}_{\mathcal{Y}_N^a>\varepsilon}\Big|\tau_{(-\infty,0)}>N-1\Big] \\
&+ \mathbf{E}_{t-a}\Big[\Big|\Phi\left(X^N(t, S_1+a,\dots,S_{N-1}+a)\right) \\
&\qquad\qquad - \Phi\left(X^{N-1}(S_1,\dots,S_{N-1})\right)\Big|\mathbf{1}_{\mathcal{Y}_N^a\le\varepsilon}\Big|\tau_{(-\infty,0)}>N-1\Big] \\
&\qquad\qquad \le 2c_1\mathbf{P}_{t-a}\Big[\mathcal{Y}_N^a>\varepsilon\Big|\tau_{(-\infty,0)}>N-1\Big]+c_2\varepsilon
\end{aligned}
$$

where in the last inequality we made use of (83). We finally choose N_0 large enough such that the last term above is smaller than say ε.

We then rewrite (81) as

$$
\begin{aligned}
&\mathbb{E}_A^f\Big[\Phi\left((X_t^N)_{t\in[0,1]}\right)\Big] \\
&= \int_a^{N^{1/4}} h(t)\frac{\mathbf{P}_{t-a}[\tau_{(-\infty,0)}>N-1]}{\mathbf{P}[S_1>a,\dots,S_N>a]}\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t,S_1+a,\dots,S_{N-1}+a)\right)\Big|\tau_{(-\infty,0)}>N-1\Big]dt \\
&+ \int_{N^{1/4}}^{\infty} h(t)\frac{\mathbf{P}_{t-a}[\tau_{(-\infty,0)}>N-1]}{\mathbf{P}[S_1>a,\dots,S_N>a]}\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t,S_1+a,\dots,S_{N-1}+a)\right)\Big|\tau_{(-\infty,0)}>N-1\Big]dt.
\end{aligned}
\tag{85}
$$

Combining (84), Theorem 5.3 and the dominated convergence theorem (in particular we use the fact that $\int_a^\infty h(t)\frac{\mathbf{P}_{t-a}[\tau_{(-\infty,0)}>N-1]}{\mathbf{P}[S_1>a,\dots,S_N>a]}dt=1$ for every N), we get that the first term in the right hand side of the above equality converges as $N\to\infty$ towards $m(\Phi)$. On the other hand, the second term is smaller than

$$
c_2\int_{N^{1/4}}^{\infty}\frac{t^2h(t)}{N^{1/2}\mathbf{P}[S_1>a,\dots,S_N>a]}dt. \tag{86}
$$

By Lemma 2.2, the sequence $(N^{1/2}\mathbf{P}[S_1>a,\dots,S_N>a])_N$ converges towards a positive limit as $N\to\infty$.

Then we make use of this convergence and of the fact that $\mathbf{E}[X^2]<\infty$ to get that the term in (86) vanishes as $N\to\infty$; hence the $A=\emptyset$ case is resolved.

For a generic $A\subset V_N$ and $x\in(\mathbb{R}^+)^N$, we make use of the Markov property and of what we just proved. More precisely, since for such A, we have $A\cap[0,L]=A$, using

Markov's property, for every function $H : \mathbb{R}^{N-L} \to \mathbb{R}$, we get the equality

$$\begin{aligned}\mathbf{E}\left[H\left((S_L,\dots,S_N)\right),\mathcal{E}_{N,A,x}\right] &= \mathbf{E}\left[H\left((S_L,\dots,S_N)\right),\mathcal{E}_{L,A\cap[0,L],x}\cap\{S_{L+1}>a,\dots,S_N>a\}\right] \\ &= \mathbf{P}\left[\mathcal{E}_{L,A\cap[0,L],x}\right]\mathbf{P}_{x_{|A|}}\left[H(S_0,\dots,S_{N-L})S_1>a,\dots,S_{N-L}>a\right].\end{aligned} \tag{87}$$

From this we deduce:

$$\begin{aligned}&\mathbb{E}_{A,x}\left[\Phi\left(\left(\sqrt{1-\frac{L}{N}}S^N_{\frac{L}{N}+t(1-\frac{L}{N})}\right)_{t\in[0,1]}\right)\right] \\ &\qquad = \mathbb{E}_{x_{|A|}}\left[\Phi\left((S^{N-L}_t)_{t\in[0,1]}\right)\Big| S_1>a,\dots,S_{N-L}>a\right]\end{aligned} \tag{88}$$

Finally, we note that for all $x\in[0,a]$, one has the equality:

$$\mathbf{E}_x\left[\Phi\left(S^{N-L}_t)_{t\in[0,1]}\right)\Big| S_1>a,\dots,S_{N-L}>a\right] = \mathbb{E}_{\{0\},x}\left[\Phi\left(S^{N-L}_t)_{t\in[0,1]}\right)\right]. \tag{89}$$

We already proved that the right hand side in the above equality converges towards $m(\Phi)$ in the particular case $x=0$. Getting the same convergence for any $x\in[0,a]$ works in the same way, and hence we get the second point of Lemma KEYLEm 5.4. □

5.2.1 Proof of Lemma KEYLEm 5.4

We consider $\varepsilon,\eta>0$, L_0 a positive integer and Φ a continuous function on $C([0,1],\mathbb{R})$. We write:

$$\begin{aligned}&Q^f_{N,a,\beta}\Big[\Phi(\omega)\Big] \\ &\quad = \sum_{l=0}^{L_0}\sum_{A\subset V_N;\mathcal{L}(A)=l}\int_{[0,a]^{|A|}} p^f_{\beta,N}(A,dx)\,\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] + Q^f_{N,a,\beta}\Big[\Phi(\omega)\mathbf{1}_{\mathcal{L}(A)>L_0}\Big].\end{aligned} \tag{90}$$

Fin

Then we note that for each A and x appearing in the right hand side above, the convergence $\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] \to m(\Phi)$ holds. We note L for the quantity $\mathcal{L}(A)$ and for notational convenience we write $f_N(t) := L/N + t(1-L/N)$ and $g_N(t) := \frac{(t-L/N)}{1-L/N}$ its inverse (and we set $f_0(t)=g_0(t)=t$).

We first note that for every $n>0$, for every $t_1<t_2<\ldots<t_n\in[0,1]^n$ and for every

continuous bounded function $F : [0,1]^n \to \mathbb{R}$, one has the convergence

$$\mathbb{P}_{A,x}\left[F(X^N_{t_1}, X^N_{t_2}, \ldots, X^N_{t_n})\right] \to m\left[F(\omega_{t_1}, \ldots, \omega_{t_n})\right]. \tag{91}$$

Indeed, making use of the second assumption of Lemma 5.4, we get:

$$\Bigg|\mathbb{P}_{A,x}\Big[F\left(X^N_{t_1}, X^N_{t_2}, \ldots, X^N_{t_n}\right)\Big] - \mathbb{P}_{A,x}\left[F\left(\sqrt{1-\frac{L}{N}}X^N_{f_N(t_1)}, \ldots, \sqrt{1-\frac{L}{N}}X^N_{f_N(t_n)}\right)\right]\Bigg| \to 0 \tag{92}$$

as $N \to \infty$ by dominated convergence because $F(\cdot)$ is continuous and bounded; as the convergence of the second term above towards $m(F(\omega_{t_1}, \ldots, \omega_{t_n}))$ is the hypothesis, the finite dimensional convergence is proven.

We are left with proving the tightness of the sequence X^N under the law $\mathbb{P}_{A,x}$ for $A \subset V_N$ such that $\mathcal{L}(A) \le L_0$. For this, for $\eta > 0$ and for a continuous function f on $[0,1] \to \mathbb{R}^+$ verifying $\sup_{t\in[0,\eta]} f(t) \le \varepsilon$, we introduce its η-cut counterpart $f^{(\eta)}$; namely, $f^{(\eta)}(x) = \frac{xf(\eta)}{\eta}\mathbf{1}_{x\in[0,\eta]} + f(x)\mathbf{1}_{x\ge\eta}$. Clearly, we have $||f^{(\eta)} - f||_\infty \le \varepsilon$.

Using standard properties of the brownian motion, for C large enough, one has $m(\mathcal{B}_C) \ge 1-\varepsilon$ where

$$\mathcal{B}_C := \Big\{f \in C([0,1],\mathbb{R}), \sup_{x,y\in[0,1]} |f(x)-f(y)| \le C|x-y|^{2/3}\Big\}. \tag{93}$$

Therefore for such a C and for N large enough, we have:

$$\mathbb{P}_{A,x}\left[\left(\sqrt{1-\frac{L}{N}}X^N_{f_N(t)}\right)_{t\in[0,1]} \in \mathcal{B}_C\right] \ge 1-2\eta. \tag{94}$$

Now we are ready to prove the Kolmogorov criterion for X^N under the law $\mathbb{P}_{A,x}$. We have to show that given $\delta > 0$, there exists N_0 such that:

$$\mathbb{P}_{A,x}\Big[\sup_{s,t,|s-t|\le\delta} |X^N_s - X^N_t| \ge \varepsilon\Big] \le \eta,\ N \ge N_0. \tag{95}$$ AMPBB

Using the first hypothesis of Lemma 5.4, we can restrict ourselves to show (95) by replacing X^N by its L/N-cut counterpart, which we denote by $\tilde{X}^N$. As the modulus of continuity of $\tilde{X}^N$ is under control on $[0, L/N]$, we have to show that there exists $\delta > 0$ such that for N large enough, one has:

$$\mathbb{P}_{A,x}\Big[\sup_{s,t>L/N,|s-t|\le\delta} |\tilde{X}^N_s - \tilde{X}^N_t| \ge \varepsilon\Big] \le \eta. \tag{96}$$

A

Now we write:

$$\Big|\tilde{X}^N_s - \tilde{X}^N_t\Big| \le \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_{f_N(t)}\Big| + \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_s\Big| + \Big|\tilde{X}^N_{f_N(t)} - \tilde{X}^N_t\Big| \tag{97}$$

so that, for every $\delta > 0$,

$$\begin{aligned} &\mathbb{P}_{A,x}\Big[\sup_{s,t>L/N,|s-t|\le\delta} \Big|\tilde{X}^N_s - \tilde{X}^N_t| \ge \varepsilon\Big] \\ &\quad \le \mathbb{P}_{A,x}\Big[\sup_{|s-t|\le\delta} \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_{f_N(t)}\Big| \ge \varepsilon/3\Big] + 2\mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]} \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_s\Big| \ge \varepsilon/3\Big]. \end{aligned} \tag{98}$$

The first term in the right hand side of the above inequality can be made smaller than $\eta/2$ for δ small enough as soon as N is large enough using the second hypothesis of Lemma KEYLEm 5.4. For the left hand side, we get

$$\begin{aligned} &\mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]} \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_s\Big| \ge \varepsilon/3\Big] \\ &\quad = \mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]} \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_{g_N(f_N(s))}\Big| \ge \varepsilon/3; \Big(\tilde{X}^N_{f_N(t)}\Big)_{t\in[0,1]} \in \mathcal{B}_C\Big] \\ &\qquad + \mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]} \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_{g_N(f_N(s))}\Big| \ge \varepsilon/3; \Big(\tilde{X}^N_{f_N(t)}\Big)_{t\in[0,1]} \in \mathcal{B}_C^c\Big]. \end{aligned} \tag{99}$$

UNDERNIER

The last term of equation (UNDERNIER 99) above can be made smaller than $\eta/3$ for N large enough since $\mathcal{B}_C$ is an m continuity set (that is a set whose boundary is of null m measure) and by using the Porte-Manteau theorem, which states that in this case

$$\mathbb{P}_{A,x}\Big[(\tilde{X}^N_{f_N(t)})_{t\in[0,1]} \in \mathcal{B}_C\Big] \to m(\mathcal{B}_C) \tag{100}$$

as $N \to \infty$.

Finally, for $(\tilde{X}^N_{f_N(t)})_{t\in[0,1]} \in \mathcal{B}_C$, we have

$$\sup_{s\in[0,1]} \Big|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_{g_N(f_N(s))}\Big| \le C \sup_{s\in[0,1]} \Big|s - g_N(s)\Big|^{2/3} \tag{101}$$

and $\sup_{s\in[0,1]} \left|s - g_N(s)\right|^{2/3} \leq (L/N)^{2/3}$. Thus for N large enough, one has:

$$\mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]} \left|\tilde{X}^N_{f_N(s)} - \tilde{X}^N_{g_N(f_N(s))}\right| \geq \varepsilon/3; \left(\tilde{X}^N_{f_N(t)}\right)_{t\in[0,1]} \in \mathcal{B}_C\Big] \leq \eta/3 \tag{102}$$

which proves (96). Thus we have shown that $\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] \to m(\Phi)$.

Now we make use of equation (90) and the triangle equality to get that

$$\begin{aligned}
&\left|Q^f_{N,a,\beta}\Big[\Phi(\omega)\Big] - m(\Phi)\right| \\
&\quad\leq \sum_{l=0}^{L_0} \sum_{A\subset V_N;\mathcal{L}(A)=l} \int_{[0,a]^{|A|}} p^f_{\beta,N}(A,x) \left|\mathbb{P}_{A,x}\Big[\Phi\left(X^N\right)\right) - m(\Phi)\Big| \\
&\qquad + m(|\Phi|)Q^f_{N,a,\beta}\Big[\mathbf{1}_{\mathcal{L}(A)>L_0}\Big] + Q^f_{N,a,\beta}\Big[\Big|\Phi(\omega)\Big|\mathbf{1}_{\mathcal{L}(A)>L_0}\Big].
\end{aligned} \tag{103}$$

As $\Phi(\cdot)$ is bounded, recalling that

$$\sum_{l=0}^{\infty} \sum_{A\subset V_N;\mathcal{L}(A)=l} \int_{[0,a]^{|A|}} p^f_{\beta,N}(A,x) = 1, \tag{104}$$

using dominated convergence and the fact that $\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] \to m(\Phi)$), we are done by considering L_0 large enough and by using Lemma 1.3. □

5.3 The constrained case

The strategy in this part is similar to the one of the preceeding section, and we choose to skip some of the proofs for lightness. We first mention that the analogous of Shimura's result has been recently shown for the normalized excursion in [CC13][Corollary 2.5].

Theorem 5.5 *Let x_N and y_N two positive sequences such that both $x_N/\sqrt{N}$ and $y_N/\sqrt{N}$ vanish as $N \to \infty$. One has the following weak convergence:*

$$\mathbf{P}_{x_N}\Big[\ \cdot\ \Big| S_N = y_N, \tau_{(-\infty,0)} > N\Big] \circ \left(X^N\right)^{-1} \Rightarrow e(\cdot). \tag{105}$$

Like we did in the free case, we first give a technical lemma which immediately implies the convergence in the constrained case of Theorem 1.2.

Lemma 5.6 *Let (l_N, r_N) denote the random variables $(L(A), R(A))$ under $\mathbf{P}^c_{N,a,\beta}$. Assume the following holds:*

1. *For any $\varepsilon > 0$, one has*

$$\lim_{N\to\infty} Q^c_{N,a,\beta}\left[\sup_{t\in[0,l_N/N]\cup[r_N/N,1]} w_t \geq \varepsilon\right] = 0. \tag{106}$$

2. *For every $A \subset V_{N-1}$ such that $(L(A), R(A)) = (L, R)$ and for every $x \in \mathbb{R}^N$, if X follows the law $\mathbb{P}^c_{A,x}$, one has the convergence in law:*

$$\left(\sqrt{\frac{R-L}{N}}X^N_{\frac{L}{N}+t(\frac{R-L}{N})}\right)_{t\in[0,1]} \Rightarrow e. \tag{107}$$

Then one has the second convergence of Theorem 1.2.

The proof of Lemma 5.6 closely follows the one of Lemma 5.4, so that we choose to skip it.

5.3.1 Proof of Theorem 1.2 in the constrained case.

We show that the hypothesis of Lemma 5.6 are fulfilled.

The first point of Lemma 5.6 is fulfilled. Combining the equivalence:

$$\mathbf{P}[S_N \in [0,a]; S_j > 0, j \leq N] \sim \frac{\int_0^a U(u)du}{\sqrt{2\pi}\sigma N^{3/2}} \tag{108}$$

which follows from (27) and the asymptotics on $Z^c_{N,\beta,a}$ in Proposition 5.1, the proof of this point goes very much along the same lines as in the constrained case by using standard facts on the normalized excursion instead of the meander, so that once again we choose to skip it.

The second point of Lemma 5.6 is fulfilled. Here we make use of Theorem 5.5 in a crucial way. We first treat the $A = \emptyset$ case. Once again we consider $\varepsilon > 0$ and Φ a Lipschitz bounded functional on $\mathcal{C}([0,1], \mathbb{R})$ verifying the same properties as in (80). We write:

$$\mathbb{E}^c_{A,x}\Big[\Phi\left(X^N_t\right)_{t\in[0,1]}\Big]$$
$$= \int_{t,t'\in[a,\infty)^2, u\in[0,a]} \frac{\mathbf{E}\Big[\Phi\left(X^N(t,S_2,\dots,S_{N-2},t',u)\right), S_1=t, S_2>a,\dots,S_{N-2}>a, S_{N-1}=t', S_N=u\Big]}{\mathbf{P}\Big[S_1>a,\dots,S_{N-1}>a, S_N\in[0,a]\Big]} dt dt' du. \tag{109}$$

asplit

For $t,t'\geq a, u\in[0,a]$, we use twice Markov's property to get:

$$\mathbf{E}\Big[\Phi\left(X^N(t,S_2,\dots,S_{N-2},t',u)\right), S_1=t, S_2>a,\dots,S_{N-2}>a, S_{N-1}=t', S_N=u\Big]$$
$$= h(t)\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t,S_1+a,\dots,S_{N-2}+a,t',u)\right), \tau_{(-\infty,0)>N-2}, S_{N-2}=t'\Big]h(u-t') \tag{110}$$

$$= h(t)h(u-t')\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t,S_1+a,\dots,S_{N-2}+a,t',u)\right)|\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\Big]$$
$$\times \mathbf{P}_{t-a}\left[\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\right]. \tag{111}$$

As in (83), for any $(x_1,\dots,x_{N-2})\in(\mathbb{R}^+)^{N-2}, (t,t')\in(a,N^{1/4})^2, u\in[0,a]$, we have:

$$\Big|\Phi\left(X^N(t,x_1+a,\dots,x_{N-2}+a,t',u)\right) - \Phi\left(X^{N-2}(x_1,\dots,x_{N-2})\right)\Big|$$
$$\leq \frac{c_2\sup_{j=1,\dots,N-2}|x_j-x_{j-1}|}{\sqrt{N}} + c_2\frac{a+2N^{1/4}}{\sqrt{N-2}}. \tag{112}$$

Ko1

Since Theorem 5.5 asserts that

$$\mathbf{E}_{t-a}\Big[\Phi\left(X^{N-2}(S_1,\dots,S_{N-2})\right)|\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\Big] \to e(\Phi), \tag{113}$$

we can deduce from (112) as in the free case that

$$\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t,S_1+a,\dots,S_{N-2}+a,t',u)\right)|\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\Big] \to e(\Phi). \tag{114}$$

exc

We first recall that a consequence of Lemma 2.3 is the fact that

$$N^{3/2}\mathbf{P}\Big[S_1>a,\dots,S_{N-1}>a, S_N\in[0,a]\Big] \to \int_{[0,a]^2}\Phi_a(x,y)dxdy > 0 \tag{115}$$

cccon

as $N \to \infty$. We split the integral over $[a,\infty)^2$ appearing in (109):

$$\mathbb{E}^c_{A,x}\Big[\Phi\left(X^N_t\right)_{t\in[0,1]}\Big] = \int_{u\in[0,a]}\left(\int_{(t,t')\in\mathcal{D}_1^N}\dots + \int_{(t,t')\in\mathcal{D}_2^N}\dots\right) \tag{116}$$

where, given $\varepsilon > 0$ and $C > 0$, we defined

$$\begin{aligned}\mathcal{D}_1^N &= \{(t,t')\in[a,\varepsilon\sqrt{N}]^2\},\\ \mathcal{D}_2^N &= \{(t,t')\in\mathbb{R}^+, t\vee t' \geq \varepsilon\sqrt{N}\}.\end{aligned} \tag{117}$$

As we proceeded in the free case, making use of the equivalence (27) and of the convergence (114), we deduce that

$$\begin{aligned}&\lim_{\varepsilon\to0^+}\lim_{N\to\infty}\int_{u\in[0,a]}\int_{(t,t')\in\mathcal{D}_1^N} h(t)h(u-t')\frac{\mathbf{P}_{t-a}\left[\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\right]}{\mathbf{P}\Big[S_1>a,\dots,S_{N-1}>a, S_N\in[0,a]\Big]}\\ &\quad\times\mathbf{E}_{t-a}\Big[\Phi\left(X^N(t,S_1+a,\dots,S_{N-2}+a,t',u)\right)|\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\Big]dtdt'du\\ &\quad\to e(\Phi).\end{aligned} \tag{118}$$

Φ being bounded, we are left with showing that

$$\lim_{\varepsilon\to0^+}\lim_{N\to\infty}\int_{u\in[0,a]}\int_{(t,t')\in\mathcal{D}_2^N} h(t)h(u-t')\frac{\mathbf{P}_{t-a}\left[\tau_{(-\infty,0)>N-2}, S_{N-2}=t'\right]}{\mathbf{P}\Big[S_1>a,\dots,S_{N-1}>a, S_N\in[0,a]\Big]}dtdt'du = 0. \tag{119}$$

lastconv

We show (119) pointwise for $u\in[0,a]$ (and indeed we just show it for $u=0$), the general case follows by dominated convergence arguments.

Since $h(\cdot)$ is bounded, by Gnedenko's local limit theorem, we have

$$\sup_{n\in\mathbb{N}}\sup_{t\in\mathbb{R}}\sqrt{n}\mathbf{P}[S_n=t]<\infty. \tag{120}$$

Recalling (115), we are then left with showing that

$$\lim_{\varepsilon\to0^+}\lim_{N\to\infty} N\int_{\mathcal{D}_2^N} h(t)h(t')dtdt' = 0. \tag{121}$$

d2

Since

$$\int_{\mathcal{D}_2^N} h(t)h(t')dtdt' \leq \frac{1}{\varepsilon^2 N}\int_{\varepsilon\sqrt{N}}^{\infty} v^2 h(v)dv, \tag{122}$$

recalling that $\mathbf{E}[X^2]<\infty$, we immediately get (121).

To conclude the proof of Theorem MAIN 1.2, we are left with dealing with the generic case $A \subset V_N$, which is done similarly to the free case.

□

Acknowledgement: Most of the results shown here were obtained during the PhD thesis of the author, and he is very grateful to Giambattista Giacomin and Francesco Caravenna for several enlightening discussions about this paper.

References

As [Asm03] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

Bil [Bil68] P. Billingsley. *Convergence of probability measures.* John Wiley & Sons Inc., New York, 1968.

CaCha [CC13] F. Caravenna and L. Chaumont. An invariance principle for random walk bridges conditioned to stay positive. *Electron. J. Probab.*, 18:1–32, 2013.

CD1 [CD08] F. Caravenna and J.-D. Deuschel. Pinning and wetting transition for $(1+1)$-dimensional fields with Laplacian interaction. *Ann. Probab.*, 36(6):2388–2433, 2008.

CD2 [CD09] F. Caravenna and J.-D. Deuschel. Scaling limits of $(1+1)$-dimensional pinning models with Laplacian interaction. *Ann. Probab.*, 37(3):903–945, 2009.

CGZ [CGZ06] F. Caravenna, G. Giacomin, and L. Zambotti. Sharp asymptotic behavior for wetting models in $(1+1)$-dimension. *Electron. J. Probab.*, 11:no. 14, 345–362 (electronic), 2006.

CGZ1 [CGZ07] F. Caravenna, G. Giacomin, and L. Zambotti. Infinite volume limits of polymer chains with periodic charges. *Markov Process. Related Fields*, 13(4):697–730, 2007.

CrKMV [CKMV09] M. Cranston, L. Koralov, S. Molchanov, and B. Vainberg. Continuous model for homopolymers. *J. Funct. Anal.*, 256(8):2656–2696, 2009.

DGZ [DGZ05] J.-D. Deuschel, G. Giacomin, and L. Zambotti. Scaling limits of equilibrium wetting models in (1 + 1)-dimension. *Probab. Theory Related Fields*, 132(4):471–500, 2005.

Do3 [Don10] R.A. Doney. Local behavior of first passage probabilities. *Probab. Theory Related Fields*, 5:299–315, 2010.

FuN [Fun08] T. Funaki. A scaling limit for weakly pinned Gaussian random walks. In *Proceedings of RIMS Workshop on Stochastic Analysis and Applications*, RIMS Kôkyûroku Bessatsu, B6, pages 97–109. Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

GB [Gia07] G. Giacomin. *Random polymer models.* Imperial College Press, London, 2007.

IY [IY01] Y. Isozaki and N. Yoshida. Weakly pinned random walk on the wall: pathwise descriptions of the phase transition. *Stochastic Process. Appl.*, 96(2):261–284, 2001.

Shi [Shi83] M. Shimura. A class of conditional limit theorems related to ruin problem. *Ann. Probab.*, 11(1):40–45, 1983.

Soh1 [Soh13] J. Sohier. The scaling limits of a heavy tailed markov renewal process. *Ann. Inst. H. Poincaré Probab. Statist.*, 2:483–505, 2013.

Ze [Zer87] M. Zerner. Quelques propriétés spectrales des opérateurs positifs. *J. Funct. Anal.*, 72(2):381–417, 1987.