EURANDOM PREPRINT SERIES

2015-009

March 25, 2015

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ISSN 1389-2355

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Abstract

In this paper we analyze the transient behavior of the workload process in a Lévy input queue. We are interested in the value of the workload process at a random epoch; this epoch is distributed as the sum of independent exponential random variables. We consider both cases of spectrally one-sided Lévy input processes, for which we succeed in deriving explicit results. As an application we approximate the mean and the Laplace transform of the workload process after a deterministic time.

Keywords: Queueing ∘ Lévy processes ∘ fluctuation theory ∘ spectrally one-sided input ∘ transient analysis

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The research of N. Starreveld and M. Mandjes is partly funded by the NWO Gravitation project NETWORKS, grant number 024.002.003.

1 Introduction

This paper studies the transient workload in a queue fed by a Lévy input process $X = \{X_t\}_{t\geq 0}$; here the workload process, in the sequel denoted by $\{\mathcal{Q}_t\}_{t\geq 0}$, is defined as the reflection of X at zero. This workload process can be constructed from the input process X as the (unique) solution of the so-called Skorokhod problem, [6, 13, 14]. It turns out that the process $\mathcal{Q}$ follows from X through

$$\mathcal{Q}_t = X_t + \max\{\mathcal{Q}_0, \mathcal{L}_t\},$$

where

$$\mathcal{L}_t := \sup_{0\leq s\leq t} -X_s = -\inf_{0\leq s\leq t} X_s.$$

The process $\{\mathcal{L}_t\}_{t\geq 0}$ is often referred to as *local time (at zero)* or *regulator process* [9].

As mentioned above, we are interested in the transient behavior of the workload process. In queueing theory transient analysis is a classical topic that is treated in various standard textbooks; see e.g. [2, 5, 12]. Typically, transient analysis is important in situations where the time horizon considered is relatively short, so that it cannot be ensured that the system is 'close to stationarity'. In addition, transient results are useful in cases that the net-input process changes over time; it for instance facilitates the analysis of systems with time-varying demand as well as the assessment of the impact of specific workload control mechanisms. In general, transient analysis allows us to assess the impact of the initial state $\mathcal{Q}_0$.

The main contribution of this paper is the generalization of the existing results on the transient behavior of the workload process $\{\mathcal{Q}_t\}_{t\geq 0}$. We consider n exponentially distributed random variables $T_1, \dots, T_n$ with parameters $q_1, \dots, q_n$ and we analyze the joint behavior of the vector

$$(\mathcal{Q}_{T_1}, \mathcal{Q}_{T_1+T_2}, \dots, \mathcal{Q}_{T_1+\ldots+T_n}),$$

with a specific focus on $\mathcal{Q}_{T_1+\ldots+T_n}$. It is noted that this also directly yields $\mathcal{Q}_T$ when T follows a *Coxian* distribution; see Section 6 for some additional background on this claim. This observation is particularly useful owing to the fact that any distribution on the positive half line can be approximated arbitrarily closely by a sequence of Coxian distributions (in the sense of convergence; see e.g. [2, Section III.4]). For the case of a spectrally positive input process, the results are given in terms of the Laplace-Stieltjes transform (LST), whereas we find an expression for the associated density for the spectrally negative case.

Apart from the general results obtained, a second contribution lies in the reasoning behind our proofs. More specifically, our proofs reveal that the above formulas obey an elegant and simple tree structure. The transient workload behavior consists of 2^n terms that can be recursively evaluated. We prove our results by induction; given that we know the expression for the quantity under consideration at $n-1$ exponential epochs, we derive the expression at n exponential epochs. In this induction step, from $n-1$ to n that is, it can be seen how each term produces two offsprings, thus giving insight into the underlying structure. The idea behind the proofs yields a *mechanism* to address questions related to transient analysis at random epochs, which may help in obtaining a deeper understanding of the behavior of the underlying continuous-time queueing system.

Transient analysis of queueing systems started with the analysis of the waiting times in the M/M/1 queue [12]. In [3, 16] the authors analyze the Laplace-Stieltjes transform of the waiting time process in the M/G/1 queue. The argument used there is also applied in [6], so as to derive Theorem 2.1 below for the case of a compound Poisson input process. The transient analysis of Lévy driven queues is of a much more recent date; see e.g. [2, 6, 8] for results on the workload process in a Lévy-driven queue at an exponential epoch (which are briefly summarized in Section 2). As a direct application the authors of [4] study clearing models, where special attention is paid to clearings at exponential epochs (relying on results on the workload at an exponential epoch in an M/G/1 setting).

Concerning the structure of the paper, in Section 2 we present our notation, as well as the preliminaries that are needed in order to prove our results. In Section 3 we present the main results of the paper, which are Thms. 3.1 and 3.2. We support the final results with intuitive arguments based on a tree structure; the proofs can also be interpreted along those lines. In Section 4 we present results obtained in numerical experiments. Section 5 contains the proof of Thm. 3.1. The proof of Thm. 3.2 can be found in the extended version [15]; it follows the line we follow in the proof of Thm. 3.1. In all sections, the spectrally positive and spectrally negative cases are treated separately. Finally, Section 6 contains conclusions and a brief discussion.

2 Model, Notation, and Preliminaries

In this section we present the workload at an exponential epoch for queues with spectrally positive (Section 2.1) and spectrally negative (Section 2.2) Lévy input processes. These results are heavily relied upon throughout the paper, and in addition serve as a benchmark. In passing, we also introduce our notation.

2.1 Spectrally positive Lévy processes

As mentioned, the building block of this paper is a Lévy process $X = \{X_t\}_{t\geq 0}$. In case X is a spectrally positive process, henceforth denoted by $X \in \mathscr{S}_+$, the *Laplace exponent* $\phi(\alpha) := \log \mathbb{E}\, e^{-\alpha X_1}$ is well defined for all $\alpha \geq 0$. By applying Hölder's inequality we get that $\phi(\cdot)$ is convex on $[0, +\infty)$ with slope $\phi'(0) = -\mathbb{E}\, X_1$ at the origin. In general, the inverse function $\psi(\cdot)$ is not well defined and we work with the right inverse

$$\psi(q) := \sup\{\alpha \geq 0 : \phi(\alpha) = q\}.$$

For the case the drift of our driving process X is negative we observe that $\psi'(0) = -\mathbb{E}\, X_1 > 0$ and thus $\phi(\cdot)$ is increasing on $[0, +\infty)$. In this case the inverse function $\psi(\cdot)$ is well defined.
Our interest is in the transient behavior of the *workload process* $\{\mathcal{Q}_t\}_{t\geq 0}$. We consider an exponentially distributed random variable T with parameter q (sampled independently from the Lévy input process) and focus on the transform $\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_T}$, where $\alpha \geq 0$ and x denotes the initial workload. In this case the transform $\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_T}$ is explicitly known, and is given in the following theorem [6, 8, 16].

Theorem 2.1. *Let $X \in \mathscr{S}_+$ and let T be exponentially distributed with parameter q, independently of X. For $\alpha \geq 0$, $x \geq 0$,*

$$\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_T} = \int_0^\infty q e^{-qt}\, \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_t} \mathrm{d}t = \frac{q}{q - \phi(\alpha)} \left(e^{-\alpha x} - \frac{\alpha}{\psi(q)} e^{-\psi(q) x} \right).$$

Using Laplace inversion techniques [1], information about the process can then be inferred from the LST as it uniquely determines the distribution of $\mathcal{Q}_t$, for each t and any initial workload x.

2.2 Spectrally negative Lévy processes

For a spectrally negative Lévy process X, henceforth denoted by $X \in \mathscr{S}_-$, we define the *cumulant* $\Phi(\beta) := \log \mathbb{E}\, e^{\beta X_1}$. This function is well-defined and finite for all $\beta \geq 0$, exactly because there are no positive jumps. We observe that $\Phi(\cdot)$ has slope $\Phi'(0) = \mathbb{E}\, X_1$ at the origin, thus $\Phi(\beta)$ in general is no bijection on $[0, +\infty)$. We define the right inverse through

$$\Psi(q) := \sup\{\beta \geq 0 : \Phi(\beta) = q\}.$$

When working with spectrally negative Lévy processes, the so-called q-scale functions, $W^{(q)}(\cdot)$ and $Z^{(q)}(\cdot)$ play a crucial role, particularly when studying the fluctuation properties of the reflected process [6, 11]. For $q \geq 0$, let $W^{(q)}(x)$, for $x \geq 0$, be a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\Phi(\beta) - q}, \qquad \beta > \Psi(q); \tag{2.1}$$

we let $W^{(q)}(x)$ equal 0 for $x < 0$. From [9, Th. 8.1.(i)] it follows that such a function exists. Having defined the function $W^{(q)}(\cdot)$, we define the function $Z^{(q)}(\cdot)$ as

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y. \tag{2.2}$$

We immediately see the importance of the q-scale function in the density of the *workload process* at an exponential epoch, given in the following theorem [6, Section 4.2], which is originally due to Pistorius [10].

Theorem 2.2. *Let $X \in \mathscr{S}_-$ and let T be exponentially distributed with parameter q, independently of X. For $\alpha \geq 0$, $x \geq 0$ and $\beta > 0$,*

$$\mathbb{P}_x(\mathcal{Q}_T \in \mathrm{d}y) = \left(e^{-\Psi(q)y} \Psi(q) Z^{(q)}(x) - q W^{(q)}(x - y)\right) \mathrm{d}y,$$

and

$$\int_0^\infty e^{-\beta x}\, \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_T} \mathrm{d}x = \frac{1}{\beta} \left(\frac{\Psi(q)}{\Psi(q) + \alpha} + \frac{q}{\Phi(\beta) - q} \frac{\Psi(q) - \beta}{\Psi(q) + \alpha} \frac{\alpha}{\alpha + \beta} \right).$$

The result on the LST of $\mathcal{Q}_T$ in the above theorem follows for $\beta > \Psi(q)$ by a direct computation from the density of $\mathcal{Q}_T$; by a standard analytic continuation argument the resulting expression then holds for any $\beta > 0$.

3 Main Results

In this section we present our main results, viz. Thms. 3.1 and 3.2. In both subsections we first derive the workload behavior at two exponential epochs as this clearly demonstrates how the various terms appear. Then we elaborate on the mechanism for obtaining the workload at n exponential epochs, yielding an intuitively appealing tree structure. The proofs can be interpreted along those lines. The proof of Thm. 3.1 is given in full detail in Section 5.

3.1 Spectrally positive case

Suppose we have a spectrally positive Lévy process X. We want to describe the behavior of the workload process $\{\mathcal{Q}_t\}_{t \geq 0}$ at consecutive exponential epochs. We do this by considering exponentially distributed random variables $T_1, \ldots, T_n$ with distinct parameters $q_1, \ldots, q_n$ and calculate, for

$\alpha_i \geq 0$ and some initial workload $x \geq 0$, the joint Laplace transform given by

$$\mathbb{E}_x \, e^{-\alpha_1 \, \mathcal{Q}_{T_1} \, -\alpha_2 \, \mathcal{Q}_{T_1+T_2} \, +\ldots+\alpha_n \, \mathcal{Q}_{T_1+\ldots+T_n}}. \tag{3.1}$$

It is instructive to first illustrate how to derive an expression for the joint transform at two exponential epochs, i.e for $\mathbb{E}_x \, e^{-\alpha_1 \, \mathcal{Q}_{T_1} \, -\alpha_2 \, \mathcal{Q}_{T_1+T_2}}$. From Thm. 2.1 we have an expression for the transform $\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T}$. Consider now two exponentially distributed random variables with parameters q_1, q_2. Then, conditioning on $\mathcal{Q}_{T_1}$ in combination with applying Thm. 2.1 twice, yields

$$\begin{aligned}
\mathbb{E}_x \, e^{-\alpha_1 \, \mathcal{Q}_{T_1} \, -\alpha_2 \, \mathcal{Q}_{T_1+T_2}} &= \int_0^\infty e^{-\alpha_1 y} \, \mathbb{E}_y \, e^{-\alpha_2 \, \mathcal{Q}_{T_2}} \, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y) \\
&= \int_0^\infty e^{-\alpha_1 y} \left(\frac{q_2}{q_2 - \phi(\alpha_2)} \left(e^{-\alpha_2 y} - \frac{\alpha_2}{\psi(q_2)} e^{-\psi(q_2) y} \right) \right) \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y) \\
&= \frac{q_2}{q_2 - \phi(\alpha_2)} \left(\mathbb{E}_x \, e^{-(\alpha_1+\alpha_2) \, \mathcal{Q}_{T_1}} - \frac{\alpha_2}{\psi(q_2)} \, \mathbb{E}_x \, e^{-(\alpha_1+\psi(q_2)) \, \mathcal{Q}_{T_1}} \right) \\
&= \frac{q_2}{q_2 - \phi(\alpha_2)} \left(\frac{q_1}{q_1 - \phi(\alpha_1+\alpha_2)} \left(e^{-(\alpha_1+\alpha_2)x} - \frac{\alpha_1+\alpha_2}{\psi(q_1)} e^{-\psi(q_1)x} \right) \right. \\
&\quad \left. - \frac{\alpha_2}{\psi(q_2)} \frac{q_1}{q_1 - \phi(\alpha_1+\psi(q_2))} \left(e^{-(\alpha_1+\psi(q_2))x} - \frac{\alpha_1+\psi(q_2)}{\psi(q_1)} e^{-\psi(q_1)x} \right) \right).
\end{aligned} \tag{3.2}$$

We see that by conditioning on the value of the workload at the first exponential epoch we can derive the transform at two exponential epochs. The above reasoning rests on the property that the process $\{\mathcal{Q}_t\}_{t \geq 0}$ is a Markov process.
Some special attention is needed for the case $\alpha_1 = 0$ and $q_1 = q_2$, i.e., when T has an Erlang-2 distribution. From the last term in (3.2) we see that an additional limiting argument is required. A straightforward application of 'l'Hôpital' then yields the expression for $\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T}$ as in [6, Section 4.1].

The main idea for the case of n exponentially distributed random variables T_i is very similar: condition on the workload at the first exponential epoch, thus obtaining

$$\mathbb{E}_x \, e^{-\alpha_1 \, \mathcal{Q}_{T_1} \, -\alpha_2 \, \mathcal{Q}_{T_1+T_2} \, -\ldots-\alpha_n \, \mathcal{Q}_{T_1+\ldots+T_n}} = \int_0^\infty e^{-\alpha_1 y} \, \mathbb{E}_y \, e^{-\alpha_2 \, \mathcal{Q}_{T_2} \, -\ldots-\alpha_n \, \mathcal{Q}_{T_2+\ldots+T_n}} \, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y). \tag{3.3}$$

Eqn. (3.3) is used in combination with Thm. 2.1 to determine the transform at n exponential epochs given the joint transform at $n-1$ epochs. At this point, it is useful to understand how the coefficients in the exponential terms of the transform appear; this is illustrated in Fig. 1 below. Specifically, due to the integration in (3.3) and Thm. 2.1, it follows that each term produces two new terms when an exponential epoch is added (i.e., when moving from $n-1$ to n exponential epochs), such that the transform at n exponential epochs consists of 2^n exponential terms. We observe that in the expression for n random variables the first term is, for every n, $\exp[-(\alpha_1 + \ldots + \alpha_n)x]$ (multiplied by some coefficient). The exponents $\exp[-(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))x]$, where $l = 1, \ldots, n$, produce one exponential term of higher order $\exp[-(\alpha_1 + \ldots + \alpha_l + \psi(q_{l+1}))x]$ as well as one term corresponding to $\exp[-\psi(q_1)x]$; it is seen that the latter terms always appear at the 'even positions'. This mechanism is depicted in the tree diagram in Fig. 1, where row n shows the 2^n factors when we have n exponentially distributed random variables $T_1, \ldots, T_n$. For ease we only write the exponent at every node, hence the node $\alpha_1 + \psi(q_2)$ represents the term corresponding to $\exp[-(\alpha_1 + \psi(q_2))x]$ (multiplied by some coefficient). In every row, the factors are counted from the left.

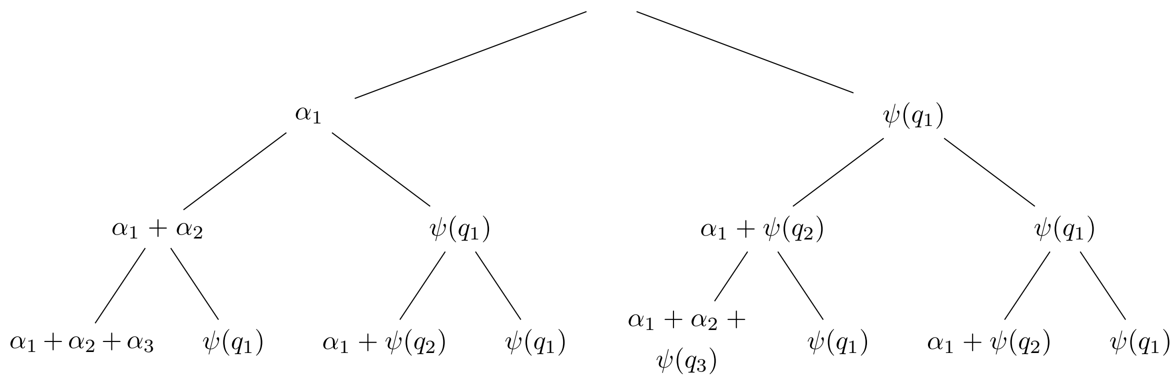

Figure 1: The exponents in (3.1) at every step

We observe that the entire tree consists of subtrees starting from a node $\psi(q_1)$ (apart from the first element of every row). Suppose we have the element $\exp[-(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))x]$ in the n-th row. This originates from a subtree generated by an initial node $\psi(q_1)$ that is $l-1$ rows higher in the tree. This follows from the fact that if we start from the node $\psi(q_1)$ we have to move $l-1$ times down and left in order to reach the node $\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l)$. So the node $\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l)$ in the n-th row belongs to a subtree spanned from the node $\psi(q_1)$ in the $(n-l+1)$-th row. For the ordering of terms, we assume that this initial node is at position $2j$ for some $j = 1, 2, \ldots, 2^{n-l}$; we recall here that the nodes $\psi(q_1)$ are located at the even positions of each row. Since the node is at position $2j$ there are $2j-1$ nodes in front of it. At every step downwards in the tree, the number of terms doubles since every term will give two new terms after using Thm. 2.1. Since we go down $l-1$ rows, those $2j-1$ nodes will produce in total $(2j-1)2^{l-1} = 2^l j - 2^{l-1}$ nodes. Hence, we see that the element $\exp[-(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))x]$ in the n-th row is at the position $2^l j - 2^{l-1} + 1$. The numbering of the coefficients is based on this ordering.

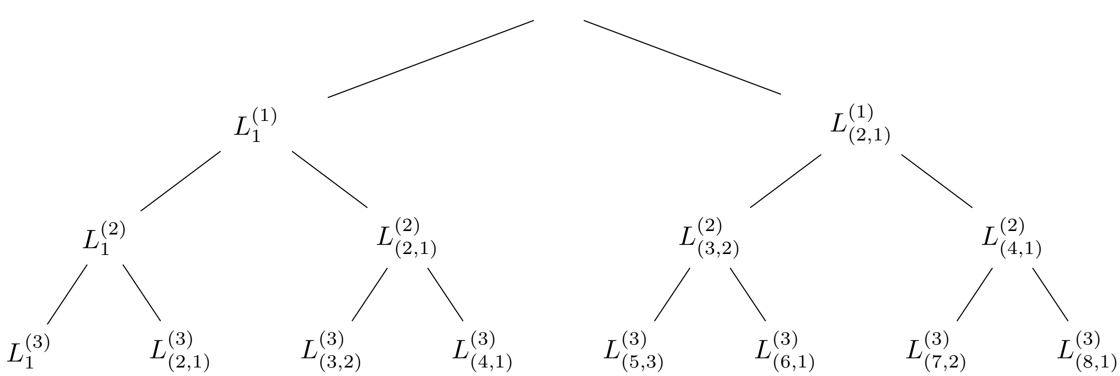

Figure 2: The coefficients of the exponential terms

In Fig. 2, we use the following notation:

- $L_1^{(n)}$ denotes the coefficient of the term $\exp[-(\alpha_1+\ldots+\alpha_n)x]$;
- $L_{(2j,1)}^{(n)}$ denote the coefficients of $\exp[-\psi(q_1)x]$ (where $j=1,2,\ldots,2^{n-1}$);
- $L_{(2^lj-2^{l-1}+1,l)}^{(n)}$ denote the coefficients of $\exp[-(\alpha_1+\ldots+\alpha_{l-1}+\psi(q_l))x]$ (where $l=2,3,\ldots,n$ and $j=1,2,\ldots,2^{n-l}$).

We note here that the superscript (n) in these factors corresponds to the number of exponential random variables considered (or, equivalently, in which row of the tree we are). We now proceed to the main result for the case of a spectrally positive input process.

Theorem 3.1. *Suppose we have n independent exponentially distributed random variables $T_1,\ldots,T_n$ with distinct parameters $q_1,...,q_n$. Then, for $\alpha_i\geq 0$ and $x\geq 0$, we have*

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1}-\alpha_2\, \mathcal{Q}_{T_1+T_2}-\ldots-\alpha_n\, \mathcal{Q}_{T_1+T_2+\ldots+T_n}}=\prod_{i=1}^{n}\frac{q_i}{q_i-\phi(\alpha_i+\ldots+\alpha_n)}e^{-(\alpha_1+\ldots+\alpha_n)x}$$
$$+\sum_{l=1}^{n}\sum_{j=1}^{2^{n-l}}L_{(2^lj-2^{l-1}+1,l)}^{(n)}(\bar{q},\bar{\alpha})e^{-(\alpha_1+\ldots+\alpha_{l-1}+\psi(q_l))x}, \tag{3.4}$$

where the coefficients $L_{(2^lj-2^{l-1}+1,l)}^{(n)}$ are defined below in Definition 3.1.

The vectors $\bar{q}=(q_1,\ldots,q_n)$ and $\bar{\alpha}=(\alpha_1,\ldots,\alpha_n)$ are here explicitly included, so as to show the dependence of the coefficients on the q's and α's. Later on these vectors are omitted to keep the notation concise.

Definition 3.1. *For $l=1,...,n$ and $j=1,\ldots,2^{n-l}$ we have*

$$L_{(2^lj-2^{l-1}+1,l)}^{(n)}(\bar{q},\bar{\alpha})=c^{(2^lj-2^{l-1}+1,n)}\prod_{i=1}^{n}\frac{q_i}{q_i-\phi(\alpha_i+d^{(i,2^lj-2^{l-1}+1)})}\prod_{i=l}^{n}\frac{\alpha_i+d^{(i,2^lj-2^{l-1}+1)}}{d^{(i-1,2^lj-2^{l-1}+1)}}.$$

where the $c^{(2^lj-2^{l-1}+1,n)}$ are given below in Lemma 3.1, $d^{(n,2^lj-2^{l-1}+1)}=0$ and the $d^{(i,2^lj-2^{l-1}+1)}$, for $i=1,2,\ldots,n-1$, are given through

$$d^{(i,2^lj-2^{l-1}+1)}=\begin{cases}\alpha_{i+1}+d^{(i+1,2^lj-2^{l-1}+1)} & \text{for } \left\lceil\frac{2^lj-2^{l-1}+1}{2^i}\right\rceil \text{ odd,}\\ \psi(q_{i+1}) & \text{for } \left\lceil\frac{2^lj-2^{l-1}+1}{2^i}\right\rceil \text{ even.}\end{cases}$$

Remark 1. The terms $d^{(i,j)}$ are given from a recursive formula. The fact that this recursion is well defined, follows because the last term equals zero (i.e., $d^{(n,j)}=0$ for all j's).

Lemma 3.1. *Consider $j=1,2,...,2^n$ and take the binary representation of $j-1$, i.e., $j-1=b_02^0+b_12^1+\ldots+b_{n-1}2^{n-1}$. Then, for $c^{(j,n)}$ (or, equivalently, the sign of the j-th element in the n-th row of the tree presented above) we have*

$$c^{(j,n)}=(-1)^{\mathrm{Par}\{b_0,\ldots,b_{n-1}\}},$$

where $\mathrm{Par}\{b_0,\ldots,b_{n-1}\}$ is 0 if the number of 1's in the binary expansion of $j-1$ is even and 1 if it is odd.

Similar to the Erlang-2 situation (i.e., $n = 2$ and $q_1 = q_2$), the case in which some of the q_i's are the same has to be treated separately. For instance, n successive applications of 'l'Hôpital' lead to an expression for $\mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_T}$, when T has an Erlang-n distribution.

3.2 Spectrally negative case

In this subsection we concentrate on the case of a spectrally negative input process X. The joint workload density has a structure that is very similar to that observed for the LST in the spectrally positive case. Due to the strong Markov property the joint density can be decomposed into

$$\mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y_1; \; \cdots \; ; \mathcal{Q}_{T_1+\ldots+T_n} \in \mathrm{d}y_n) = \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y_1) \; \cdots \; \mathbb{P}_{y_{n-1}}(\mathcal{Q}_{T_n} \in \mathrm{d}y_n).$$

That is, the joint density is simply the product of densities at single exponential epochs, as given in Thm. 2.2. Henceforth, we focus on the density of the workload process at consecutive exponential epochs, i.e.,

$$\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots+T_n} \in \mathrm{d}y), \qquad y \geq 0. \tag{3.5}$$

First we illustrate how to obtain an expression for the density $\mathbb{P}_x(\mathcal{Q}_{T_1+T_2} \in \mathrm{d}y)$ for some initial workload $x \geq 0$ and $y > 0$. From Thm. 2.2 we have an expression for the density $\mathbb{P}_x(\mathcal{Q}_T \in \mathrm{d}y)$. Consider now two exponentially distributed random variables T_1, T_2 with distinct parameters q_1, q_2. Conditioning on $\mathcal{Q}_{T_1}$ and applying Thm. 2.2 twice yields

$$\begin{aligned}
\mathbb{P}_x(\mathcal{Q}_{T_1+T_2} \in \mathrm{d}y) &= \int_{z=0}^{\infty} \mathbb{P}_z(\mathcal{Q}_{T_2} \in \mathrm{d}y) \, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}z) \\
&= \int_0^{\infty} \left(-q_1 W^{(q_1)}(x-z) + \Psi(q_1) e^{-\Psi(q_1) z} Z^{(q_1)}(x) \right) \mathbb{P}_z(\mathcal{Q}_{T_2} \in \mathrm{d}y) \mathrm{d}z \\
&= \Bigg[q_1 q_2 \int_0^{\infty} W^{(q_1)}(x-z) W^{(q_2)}(z-y) \mathrm{d}z \\
&\quad - q_1 \Psi(q_2) e^{-\Psi(q_2) y} \int_0^{\infty} W^{(q_1)}(x-z) Z^{(q_2)}(z) \mathrm{d}z \\
&\quad - q_2 \Psi(q_1) Z^{(q_1)}(x) \int_0^{\infty} e^{-\Psi(q_1) z} W^{(q_2)}(z-y) \mathrm{d}z \\
&\quad + \Psi(q_1) \Psi(q_2) e^{-\Psi(q_2) y} Z^{(q_1)}(x) \int_0^{\infty} e^{-\Psi(q_1) z} Z^{(q_2)}(z) \mathrm{d}z \Bigg] \mathrm{d}y.
\end{aligned} \tag{3.6}$$

After some standard calculus and using the definition of the q-scale functions we find the expression

$$\begin{aligned}
\mathbb{P}_x(Q_{T_1+T_2} \in \mathrm{d}y) = \Bigg[& q_1 q_2 \left(W^{(q_2)} \star W^{(q_1)} \right)(x-y) - \Psi(q_2) q_1 e^{-\Psi(q_2) y} \left(Z^{(q_2)} \star W^{(q_1)} \right)(x) \\
& - \Psi(q_1) e^{-\Psi(q_1) y} \frac{q_2}{q_1 - q_2} Z^{(q_1)}(x) + \Psi(q_2) e^{-\Psi(q_2) y} \frac{q_1}{q_1 - q_2} Z^{(q_1)}(x) \Bigg] \mathrm{d}y.
\end{aligned} \tag{3.7}$$

Again the case $q_1 = q_2$ has to treated separately, by using l'Hôpital's rule; the detailed computations corresponding to this case can be found in [6, Section 4.2].

We see that by conditioning on the value of the workload at the first exponential epoch we can derive the transform at two exponential epochs. As a next step our aim is to find an expression for (3.5) for an arbitrary $n > 0$ and for exponentially distributed random variables T_i with parameter q_i ($i = 1, \ldots, n$). Conditioning on the workload at the first $n-1$ exponential epochs yields

$$\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots+T_n} \in \mathrm{d}y) = \int_{z=0}^{\infty} \mathbb{P}_x(\mathcal{Q}_{T_1+\ldots+T_{n-1}} \in \mathrm{d}z)\, \mathbb{P}_z(Q_{T_n} \in \mathrm{d}y). \tag{3.8}$$

For the case of a spectrally positive input process (which was the topic of the previous subsection) one should condition on the value at the first exponential epoch, which allows the use of the induction hypothesis, but one needs to adjust the indices appropriately as the first exponential random variable is actually T_2. For the spectrally negative case, however, conditioning on the value of $T_1 + \ldots + T_{n-1}$ (and not only on T_1) allows us to circumvent this technicality.
Moreover, the transition from step $n-1$ to n can again be represented by using an elegant tree structure that is similar to the one developed for the spectrally positive case. The expression for the density at $n-1$ exponential epochs has 2^{n-1} terms and each term produces two new terms when integrated with the density $\mathbb{P}_z(Q_{T_n} \in \mathrm{d}y)$ (with respect to z). We also notice that in the expression for n exponentially distributed random variables the first term is always of the form $\left(W^{(q_n)} \star \ldots \star W^{(q_1)}\right)(x-y)$ while the other terms are of the form $\left(Z^{(q_l)} \star W^{(q_{l-1})} \star \ldots \star W^{(q_1)}\right)(x)$, for $l = 1, 2, \ldots, n$, multiplied by some coefficients that in general are functions of y. The underlying mechanism is illustrated in Fig. 3. In this tree the node $W^{(q_n)}(x-y)$ denotes the term $\left(W^{(q_n)} \star \ldots \star W^{(q_1)}\right)(x-y)$ while the nodes $Z^{(q_l)}(x)$, for $l = 1, \ldots, n$, denote the terms $\left(Z^{(q_l)} \star W^{(q_{l-1})} \star \ldots \star W^{(q_1)}\right)(x)$. We see that at every row, say row k for ease, a new subtree with root $\left(Z^{(q_k)} \star W^{(q_{k-1})} \star \ldots \star W^{(q_1)}\right)(x)$ is created. These terms do not change as we move downwards in the tree since they only depend on the initial workload x and do not take part in the integrations, similar to those carried out in (3.6). Their coefficients change though, by a mechanism that is identified in the proof of our result.

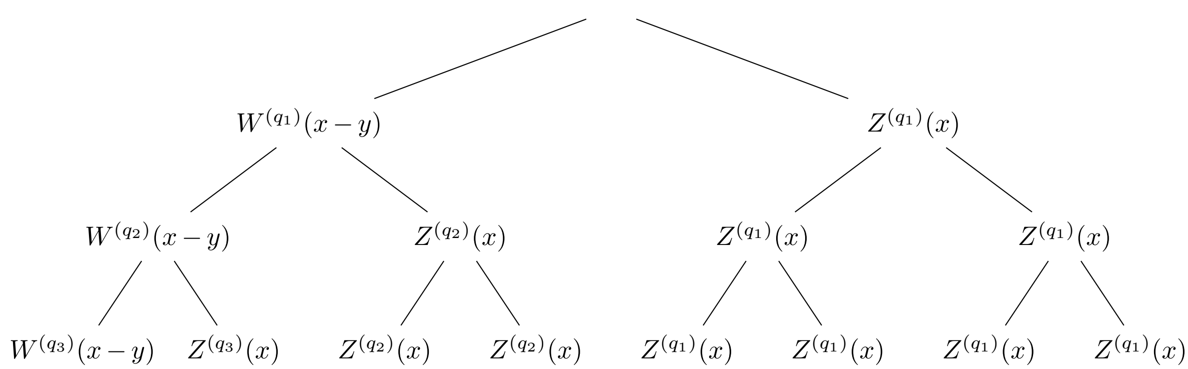

Figure 3: The convolution terms at every step

We now proceed with the main result for the spectrally negative case.

Theorem 3.2. *Suppose we have n independent exponentially distributed random variables $T_1, \ldots, T_n$ with distinct parameters $q_1, \ldots, q_n$. The density of $\mathcal{Q}_{T_1+\ldots+T_n}$, given that $\mathcal{Q}_0 = x$, is given by*

$$\mathbb{P}_x(\mathcal{Q}_{T_1+\ldots+T_n} \in \mathrm{d}y) = \Bigg[(-1)^n \prod_{i=1}^{n} q_i \cdot \left(W^{(q_n)} \star \ldots \star W^{(q_1)}\right)(x-y)$$
$$+ \sum_{l=1}^{n} \sum_{j=1}^{2^{n-l}} L^{(n)}_{(2^l j - 2^{l-1}+1, l)}(y) \left(Z^{(q_l)} \star W^{(q_{l-1})} \star \ldots \star W^{(q_1)}\right)(x)\Bigg] \mathrm{d}y,$$

where the coefficients $L^{(n)}_{(2^l j - 2^{l-1}+1, l)}(y)$ *are given in Definition 3.2.*

Definition 3.2. *For* $l = 1, \ldots, n$ *and* $j = 1, \ldots, 2^{n-l}$*, we have the following expression*

$$L^{(n)}_{(2^l j - 2^{l-1}+1, l)}(y) = c^{(2^l j - 2^{l-1}+1, n)} \Psi(q_{m(j,l)}) e^{-\Psi(q_{m(j,l)})y} \prod_{\substack{i=1,\\ i \neq m(j,l)}}^{n} q_i \prod_{i=l}^{m(j,l)} \frac{1}{q_i - q_{i+1}} \prod_{i=m(j,l)+1}^{n-1} \frac{1}{q_{m(j,l)} - q_{i+1}},$$

where $m(j,l) = \min\{k \in \mathbb{N} : \lceil \frac{2^l j - 2^{l-1}+1}{2^k} \rceil = 1\}$*. The terms* $c^{(2^l j - 2^{l-1}+1, n)}$ *are given below in Lemma 3.2.*

Lemma 3.2. *Consider* $j = 1, 2, \ldots, 2^n$ *and take the binary representation of* $2^n - j$*,* $2^n - j = \beta_0 \cdot 2^0 + \ldots + \beta_{n-1} \cdot 2^{n-1}$*. Then, for* $c^{(j,n)}$ *(or, equivalently, the sign of the* j*-th element in the* n*-th row of the tree presented above) we have the following formula*

$$c^{(j,n)} = (-1)^{\mathrm{Par}\{\beta_0, \beta_1, \ldots, \beta_{n-1}\}},$$

where $\mathrm{Par}\{\beta_0, \ldots, \beta_{n-1}\}$ *is* 0 *if the number of 1's in the binary expansion of* $2^n - j$ *is even and 1 if it is odd.*

Using the result obtained in Thm. 3.2 we can find an expression for the transform with respect to the initial workload as well; again analytic continuation is used to obtain the result for any $\beta > 0$.

Corollary 3.1. *For* $\alpha > 0$*,* $\beta > 0$ *and for* n *independent exponentially distributed random variables* $T_1, \ldots, T_n$ *with distinct parameters* $q_1, \ldots, q_n$*, we have*

$$\int_0^\infty e^{-\beta x} \, \mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_{T_1+\ldots+T_n}} \mathrm{d}x = c^{(1,n)} \prod_{i=1}^{n} q_i \frac{1}{\alpha + \beta} \prod_{i=1}^{n} \frac{1}{\Phi(\beta) - q_i}$$
$$+ \sum_{l=1}^{n} \sum_{j=1}^{2^{n-l}} c^{(2^l j - 2^{l-1}+1, n)} \prod_{\substack{i=1,\\ i \neq m(j,l)}}^{n} q_i \prod_{i=l}^{m(j,l)} \frac{1}{q_i - q_{i+1}} \prod_{i=m(j,l)+1}^{n-1} \frac{1}{q_{m(j,l)} - q_{i+1}} \cdot \frac{\Psi(q_{m(j,l)})}{\alpha + \Psi(q_{m(j,l)})} \prod_{i=1}^{l} \frac{1}{\Phi(\beta) - q_i} \frac{\Phi(\beta)}{\beta},$$

where $m(j,l)$ *and* $c^{(2^l j - 2^{l-1}+1, n)}$ *are given in Definition 3.2 and Lemma 3.2.*

The case in which some of the q_i's are the same should be treated separately again. For instance, the density of $\mathbb{P}_x(\mathcal{Q}_T \in \mathrm{d}y)$ for T having an Erlang-n distribution follows after n applications of l'Hôpital's rule.

4 Numerical Calculations

In this section we present numerical illustrations of the transient workload behavior. We consider examples corresponding to the spectrally positive case (noting that the spectrally negative case can be dealt with similarly). The expression found in Thm. 3.1 is, from an algorithmic standpoint, highly attractive; the only drawback is that for every n we have to compute 2^n terms, thus increasing the computation time significantly at every step. In our illustrations, we consider the impact of n, i.e., the number of exponential variables. We also comment on ways to determine the workload distribution at a fixed (deterministic, that is) time; the mean idea there, as we point out in more detail below, is to approximate a deterministic epoch t by the sum of exponentially distributed random variables with appropriately chosen parameters.

We focus on two specific Lévy processes: Brownian motion and the Gamma process. For the case of Brownian motion, the input process X is a Brownian motion with a drift, henceforth denoted by $X \in \mathbb{B}\mathrm{m}(d, \sigma^2)$. Then, $\phi(\alpha) := \log \mathbb{E}\, e^{-\alpha X_1} = -\alpha d + \alpha^2\sigma^2/2$, and the right inverse function is

$$\psi(q) = \frac{d + \sqrt{d^2 + 2\sigma^2 q}}{\sigma^2}.$$

For *reflected Brownian motion* (i.e., the workload of a queue with Brownian motion as input) there is an explicit expression for the conditional distribution $\mathbb{P}(\mathcal{Q}_t \leq y |\, \mathcal{Q}_0 = x)$, for $y > 0$, see e.g. [7, Section 1.6]. It is a matter of straightforward calculus to use this formula to find an expression for the transform

$$\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_t} = \int_0^\infty e^{-\alpha y}\, \mathbb{P}(\mathcal{Q}_t \in \mathrm{d}y |\, \mathcal{Q}_0 = x).$$

This result is used to evaluate the performance of our procedure in case a fixed time t is approximated by the sum of exponentials.

The *Gamma* process is characterized by the Lévy-Khintchine triplet (d, σ^2, Π), where $\sigma^2 = 0$; the Lévy measure is given, for some $\beta, \gamma > 0$, by $\Pi(\mathrm{d}x) = (\beta/x)\, e^{-\gamma x}$, for $x > 0$, and the drift is $d = \int_0^1 x\Pi(\mathrm{d}x)$. From the definition of the Lévy measure we see that the Gamma process is a spectrally positive process with a.s. non-decreasing sample paths. We also add a negative drift such that the Laplace transform is equal to

$$\phi(\alpha) := \log \mathbb{E}\, e^{-\alpha X_1} = \beta \log\left(\frac{\gamma}{\gamma + \alpha}\right) + \rho\alpha,$$

where $\rho > \beta/\gamma$ in case of a negative drift $d = (\beta/\gamma) - \rho$. For the Gamma process with parameters $\gamma, \beta > 0$ and a drift $(\beta/\gamma) - \rho$ we use the notation $\mathbb{G}(\gamma, \beta, \rho)$. If the input is a Gamma process there is no explicit expression for the transform $\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_t}$ in contrast with the case of a Brownian input.

Suppose now that we wish to characterize the distribution of $\mathcal{Q}_t$ for a deterministic t. The idea is that we can approximate t by a sum of, say n, independent exponential random variables. An optimal choice of the parameters q_i then follows from solving the following constrained optimization problem:

$$\min \mathrm{Var}(T_1 + ... + T_n) = \min \sum_{i=1}^n \frac{1}{q_i^2} \qquad \text{s.t.} \quad \sum_{i=1}^n \mathbb{E}\, T_i = t.$$

This constrained optimization problem has solution $q_1 = \ldots = q_n = n/t$. A complicating factor is that in Thm. 3.1 the parameters q_i should be chosen distinct. To remedy this, we propose to impose a small perturbation of the optimal q_i's such that they are distinct:

$$\frac{1}{q_i} = \frac{t}{n}(1 + \alpha_i), \tag{4.1}$$

Table 1: Numerical approximations for $X \in \mathbb{B}\mathrm{m}(-1,1)$, $x=0$ and $t=1$

	$n=1$	$n=4$	$n=6$	$n=7$	$n=8$	exact value	relative error
$\alpha=0.1$	0,9647	0,96064	0,96021	0,96005	0,96001	0,95914	-0,09 %
$\alpha=0.2$	0,9318	0,92410	0,92327	0,92299	0,92300	0,92128	-0,19 %
$\alpha=0.3$	0,9011	0,89008	0,88892	0,88851	0,88836	0,88611	-0,25 %
$\alpha=0.4$	0,8723	0,85836	0,85688	0,85638	0,85608	0,85338	-0,32 %
$\alpha=0.5$	0,8453	0,82870	0,82696	0,82637	0,82590	0,82285	-0,37 %
$\alpha=0.6$	0,8199	0,80094	0,79896	0,79828	0,79786	0,79432	-0,44 %
$\alpha=0.7$	0,7960	0,77488	0,77270	0,77196	0,77237	0,76760	-0,62 %
$\alpha=0.8$	0,7735	0,75040	0,74803	0,74723	0,74625	0,74254	-0,50 %
$\alpha=0.9$	0,7522	0,72735	0,72482	0,72397	0,72415	0,71900	-0,71 %
$\alpha=1$	0,7321	0,70562	0,70295	0,70205	0,70205	0,69684	-0,74 %

where the α_i are suitably chosen *small* numbers that sum up to 0. In the two tables that follow we present the numerical results obtained from calculating the expression in Thm. 3.1 for the case $X \in \mathbb{B}\mathrm{m}(-1,1)$ (Table 1) and for the case $X \in \mathbb{G}(1,1,2)$ (Table 2). Here, we consider the situation of $x=0$ and $t=1$. The parameters q_i are chosen according to (4.1) with, if n is even, the α_i's given by

$$\alpha_i = \begin{cases} 0.01 \cdot i & \text{if } i = 1, \dots, \frac{n}{2} \\ -0.01 \cdot i & \text{if } i = \frac{n}{2}+1, \dots, n. \end{cases}$$

(If n is odd we choose $\alpha_{n+1/2}=0$ and the rest as indicated above.) In the first table, $X \in \mathbb{B}\mathrm{m}(-1,1)$, we take $n = 1,4,6,7,8$ and compare our approximations with the exact values obtained from $\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_t}$ for different values of α. In the last column we present the relative errors between the exact value and the approximation value for $n=8$.

It should be realized that the numerical procedure has its limitations. First, from the expression in Thm. 3.1 we see that at every step we have to compute 2^n terms, which complicates the computation for n large. We also see that when the parameters q_i are 'almost equal' i.e., α_i in (4.1) is small) we add and subtract terms that are large in absolute value (as the denominators featuring in the result of Thm. 3.1 are close to zero), which potentially causes instability. Our numerical tests show that the choice of the parameters q_i influence the numerical stability; for the parameters indicated in (4.1) the results begin to deviate for $n > 9$ due to numerical issues. From Table 1, corresponding to the case of a Brownian input process (for which we can compare with exact results), we see that for $n=8$ our relative error is below 1%. For the case of a Gamma input process, we verified that the transform converges to the steady-state workload as given by the *generalized Pollaczek-Khintchine* formula [6, Thm. 3.2].

As a second application of Thm. 3.1 we use the results obtained in Tables 1 and 2 to approximate the value of $\mathbb{E}_x\, \mathcal{Q}_t$ for the two cases $X \in \mathbb{B}\mathrm{m}(-1,1)$ and $X \in \mathbb{G}(1,1,2)$, essentially relying on numerical differentiation. By considering an α sufficiently small and an n sufficiently large we use the

Table 2: Numerical approximations for $X \in \mathbb{G}(1,1,2)$, $x = 0$ and $t = 1$

	$n=1$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$\alpha = 0.1$	0,97582	0,99046	0,99037	0,99032	0,99028	0,99026
$\alpha = 0.2$	0,95527	0,98148	0,98130	0,98121	0,98112	0,98108
$\alpha = 0.3$	0,93754	0,97300	0,97275	0,97261	0,97249	0,97243
$\alpha = 0.4$	0,92205	0,96499	0,96465	0,96448	0,96432	0,96425
$\alpha = 0.5$	0,90838	0,95739	0,95699	0,95678	0,95659	0,95662
$\alpha = 0.6$	0,89621	0,95018	0,94972	0,94948	0,94925	0,94936
$\alpha = 0.7$	0,88530	0,94333	0,94281	0,94254	0,94228	0,94201
$\alpha = 0.8$	0,87543	0,93681	0,93623	0,93593	0,93565	0,93565
$\alpha = 0.9$	0,86647	0,93060	0,92996	0,92964	0,92933	0,92885
$\alpha = 1$	0,85828	0,92467	0,92398	0,92363	0,92330	0,92273

approximation

$$\mathbb{E}_x \, \mathcal{Q}_t \sim \frac{1 - \mathbb{E}_x \, e^{-\alpha \, \mathcal{Q}_{T_1+\ldots+T_n}}}{\alpha}.$$

We present our findings for the cases $X \in \mathbb{B}\mathrm{m}(-1,1)$ and $X \in \mathbb{G}(1,1,2)$, respectively, displaying the qualitative behavior of $\mathbb{E}_x \, \mathcal{Q}_t$ as a function of time for various values of x. For the mean value of the stationary workload we know that $\mathbb{E}\,\mathcal{Q} = \phi''(0)/(2\phi'(0))$, as follows directly from the generalized Pollaczek-Khintchine formula.

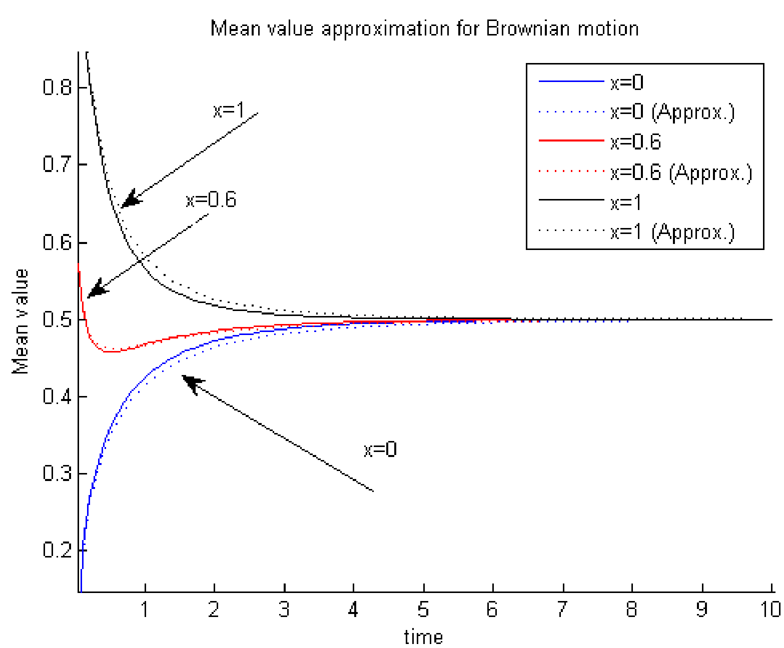

Figure 4: Mean value approximation with $n = 7$

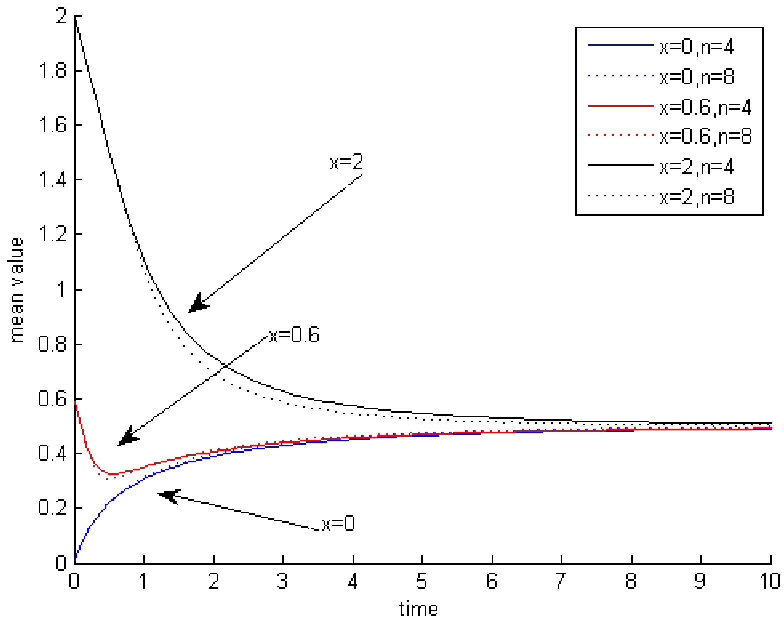

Figure 5: Mean value for Gamma process

In Figs. 4 and 5 we observe three different scenarios corresponding to different values of the initial workload. When the initial workload is 0, the mean workload increases and converges to the mean value of the steady-state workload. This follows directly, as, for any Lévy input process, $\mathcal{Q}_t \stackrel{\mathrm{d}}{=} \sup_{0\le s\le t} X_s$ when the initial workload is 0, implying that $\mathbb{E}_0\, \mathcal{Q}_t$ is increasing over time. When the initial workload is slightly above the steady-state workload, it is interesting to notice that $\mathbb{E}\, \mathcal{Q}_t$ first decreases below the steady-state version, and then converges from below. For higher initial workloads, $\mathbb{E}\, \mathcal{Q}_t$ is always decreasing and converges to the steady-state value from above.

5 Proofs

In this section we prove in full detail Thm. 3.1; the proof of Thm. 3.2 can be found in the extended version [15]. For both cases of spectrally one-sided processes, we first show an auxiliary lemma relating to the signs of each term. The main results are then proved using induction.

5.1 Proof of Theorem 3.1

Before deriving the main result, we first prove Lemma 3.1, which gives the sequence of the 2^n signs that appear in the expression of the transform at a time epoch corresponding to the sum of n exponentially distributed random variables. From Thm. 2.1 we see that for $n = 1$ the signs of the coefficients are $+, -$. For $n = 2$ and from Eqn. (3.2) we see that the signs are $+, -, -, +$ (where it is noted that we use the ordering of the terms presented in Fig. 2). Since we know how the terms are produced when we go from the step with n exponential times to the step with $n+1$ exponential random variables (see Section 3.1) we see that the signs at every step can be represented again by a tree graph. In this tree, row n again consists of 2^n nodes and, starting from the left, the nodes represent the sign of every factor when the expression is written as in Eqn. (3.4).

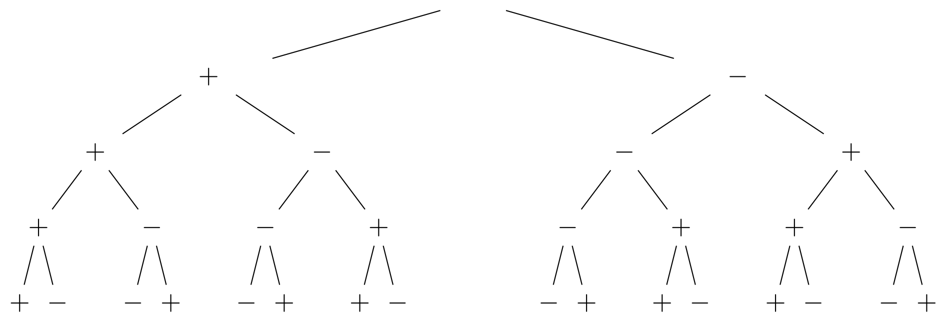

Figure 6: The sequence of the signs at every step

We see that row $n+1$ can be derived from row n when substituting every $+$ in row n by the pair $+, -$, and every $-$ by the pair $-, +$. We can understand why this holds by looking at the expression in Thm. 2.1 and the mechanism analyzed in Section 3.1. Denote by $c^{(j,n)}$ the sign of the j-th element in the n-th row in the above tree. Then $c^{(j,n)}$, for $j = 1, 2, \ldots, 2^n$, corresponds to the sign of the j-th coefficient when considering n exponentially distributed in Eqn. (3.4).

Remark 2. We observe that, because of symmetry, for the signs of the k-th row it holds that, for $j = 1, \ldots, 2^{k-1}$,

$$c^{(j,k)} = -c^{(j+2^{k-1},k)}.$$

Hence the signs j and $j + 2^{k-1}$ in the k-th row will always be opposite.

Proof of Lemma 3.1. We prove the lemma by induction on the number of exponentially distributed random variables.

(i) For $n = 1$ we have two nodes and this case corresponds to the signs of the expression derived in Thm. 2.1 for one exponentially distributed random variable T. We have that $c^{(1,1)} = +1$ and $c^{(2,1)} = -1$. Then we need the binary expansions of 0 and 1 which have no 1's and one 1, respectively. We see that $c^{(1,1)} = (-1)^0 = 1$ and $c^{(2,1)} = (-1)^1 = -1$.

(ii) We assume that the lemma holds for $n = k$. Hence, for $j = 1, \ldots, 2^k$, we have

$$c^{(j,k)} = (-1)^{\mathrm{Par}\{b_0, \ldots, b_{k-1}\}}.$$

Here we make the following observation. In the tree presented above, consider an arbitrary row n. The 2^n signs of that row and the first 2^n signs of the $(n+1)$-th row are the same.

Now consider the $(k+1)$-th row. Using the observation above and the induction hypothesis, the lemma holds for the first 2^k signs of this $(k+1)$-th row. Hence, we need to prove this statement only for $j = 2^k + 1, \ldots, 2^{k+1}$.

For $j = 1, 2, \ldots, 2^k$ we have

$$c^{(j,k+1)} = (-1)^{\mathrm{Par}\{b_0,\ldots,b_{k-1}\}} = (-1)^{\mathrm{Par}\{b_0,\ldots,b_{k-1},0\}}$$

where $j - 1 = b_0 + b_1 \cdot 2 + \ldots + b_{k-1} \cdot 2^{k-1} + 0 \cdot 2^k$. Consider now the element $j' = j + 2^k$. From Remark 2 we know that $c^{(j',k+1)} = -c^{(j,k+1)}$. We also know that the binary expansion of j' has one more 1 than the binary expansion of j since we add 2^k, i.e., $j' - 1 = j - 1 + 2^k = b_0 + \ldots + b_{k-1} \cdot 2^{k-1} + 1 \cdot 2^k$, which shows that $(-1)^{\mathrm{Par}\{b_0,\ldots,b_{k-1},1\}} = -(-1)^{\mathrm{Par}\{b_0,\ldots,b_{k-1}\}}$ leading to

$$c^{(j,k+1)} = (-1)^{\mathrm{Par}\{b_0,\ldots,b_k\}},$$

for all $j = 1, 2, \ldots, 2^{k+1}$.

□

Before proceeding with the proof of Theorem 3.1 we present some general remarks which are used in the proofs of Thms. 3.1 and 3.2.

Remark 3. For $l = 2, \ldots, n$ and $j = 1, \ldots, 2^{n-l}$ we observe the following

(a) $2^l j - 2^{l-1} + 1$ is an odd number.

(b) For all $i = 1, 2, \ldots, l-2$,

$$\left\lceil \frac{2^l j - 2^{l-1} + 1}{2^i} \right\rceil = \left\lceil 2^{l-1-i}(2j-1) + \frac{1}{2^i} \right\rceil = 2^{l-i-1}(2j-1) + 1,$$

which is always an odd number. In addition,

$$\left\lceil \frac{2^l j - 2^{l-1} + 1}{2^{l-1}} \right\rceil = 2j$$

is an even number.

(c)

$$\left\lceil \frac{2^l j - 2^{l-1} + 1}{2^l} \right\rceil = \left\lceil j - \frac{1}{2} + \frac{1}{2^l} \right\rceil = j$$

(d) For $i = 0, 1, \ldots$

$$\left\lceil \frac{2^l j - 2^{l-1} + 1}{2^{l+i}} \right\rceil = \left\lceil \frac{j}{2^i} - \frac{1}{2^{i+1}} + \frac{1}{2^{l+i}} \right\rceil = \left\lceil \frac{j}{2^i} \right\rceil.$$

Proof of Theorem 3.1. We use induction on the number of exponential random variables $T_1, \ldots, T_n$. For the proof it is sufficient to start with $n = 1$ (where it can be readily checked that Thm. 3.1 holds for $n = 1$), but the case $n = 2$ is more instructive. The joint transform for $n = 2$ can be found in Eqn. (3.2). First of all, when $n = 2$ we have in total $2^2 = 4$ terms. We see that the even terms correspond to $\exp[-\psi(q_1)x]$, and the third term corresponds to $\exp[-(\alpha_1 + \psi(q_2))x]$. According to (3.4) the coefficient of $\exp[-(\alpha_1 + \alpha_2)x]$ must be equal to

$$\frac{q_2}{q_2 - \phi(\alpha_2)} \frac{q_1}{q_1 - \phi(\alpha_1 + \alpha_2)},$$

following directly from (3.2). We have two coefficients corresponding to $\exp[-\psi(q_1)]$, which according to (3.4) should be equal to $L^{(2)}_{(2,1)}$ and $L^{(2)}_{(4,1)}$. Using Definition 3.1, we find the following expressions

$$L^{(2)}_{(2,1)} = -\prod_{i=1}^{2} \frac{q_i}{q_i - \phi(\alpha_i + d^{(i,2)})} \prod_{i=1}^{2} \frac{\alpha_i + d^{(i,2)}}{d^{(i-1,2)}} = -\frac{q_2}{q_2 - \phi(\alpha_2)} \frac{q_1}{q_1 - \phi(\alpha_1 + \alpha_2)} \frac{\alpha_1 + \alpha_2}{\psi(q_1)},$$

as $d^{(0,2)} = \psi(q_1)$, $d^{(1,2)} = \alpha_2$, and $d^{(2,2)} = 0$. Moreover,

$$L^{(2)}_{(4,1)} = \prod_{i=1}^{2} \frac{q_i}{q_i - \phi(\alpha_i + d^{(i,4)})} \prod_{i=1}^{2} \frac{\alpha_i + d^{(i,4)}}{d^{(i-1,4)}},$$

where we see from the table for the factors $d^{(i,j)}$ (see Definition 3.1) that $d^{(0,4)} = \psi(q_1)$, $d^{(1,4)} = \psi(q_2)$, and $d^{(2,4)} = 0$. This leads to the following result

$$L^{(2)}_{(4,1)} = \frac{q_2}{q_2 - \phi(\alpha_2)} \frac{\alpha_2}{\psi(q_2)} \frac{q_1}{q_1 - \phi(\alpha_1 + \psi(q_2))} \frac{\alpha_1 + \psi(q_2)}{\psi(q_1)}.$$

For the last term, the coefficient of $e^{-(\alpha_1 + \psi(q_2))x}$, we get

$$L^{(2)}_{(3,2)} = -\frac{q_1}{q_1 - \phi(\alpha_1 + \psi(q_2))} \frac{q_2}{q_2 - \phi(\alpha_2)} \prod_{i=2}^{2} \frac{\alpha_i + d^{(i,3)}}{d^{(i-1,3)}}.$$

Since $d^{(1,3)} = \psi(q_2)$ and $d^{(2,3)} = 0$, this agrees with Eqn. (3.4), and thus the results holds for $n = 2$.

We now assume that our formula holds for $n = k - 1$. Hence we have that

$$\mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1} - \ldots - \alpha_{k-1}\, \mathcal{Q}_{T_1 + \ldots + T_{k-1}}} = \prod_{i=1}^{k-1} \frac{q_i}{q_i - \phi(\alpha_i + \ldots + \alpha_{k-1})} e^{-(\alpha_1 + \ldots + \alpha_{k-1})x}$$
$$+ \sum_{l=1}^{k-1} \sum_{j=1}^{2^{k-l-1}} L^{(k-1)}_{(2^l j - 2^{l-1} + 1, l)} e^{-(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))x}, \tag{5.1}$$

where the coefficients $L^{(k-1)}_{(2^l j - 2^{l-1}+1,l)}$ are given by Definition 3.1 for $n = k - 1$ and the signs of all the factors are given by Lemma 3.1. In the induction step we prove this theorem for $n = k$ given that it holds for $n = k - 1$. The expression for $n = k$ is derived from calculating the integral

$$\mathcal{L} := \mathbb{E}_x\, e^{-\alpha_1\, \mathcal{Q}_{T_1} - \ldots - \alpha_k\, \mathcal{Q}_{T_1 + \ldots + T_k}} = \int_0^\infty e^{-\alpha_1 y}\, \mathbb{E}_y\, e^{-\alpha_2\, \mathcal{Q}_{T_2} - \ldots - \alpha_k\, \mathcal{Q}_{T_2 + \ldots + T_k}}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y),$$

where the expectation in the integral is known by the induction hypothesis. Here we see that we must raise all indices in (5.1) by one when we do the calculations because we start from time T_2 with parameter q_2 instead of from T_1. Combining the above with (5.1), we obtain

$$\mathcal{L} = \prod_{i=1}^{k-1} \frac{q_{i+1}}{q_{i+1} - \phi(\alpha_{i+1} + \ldots + \alpha_k)} \int_0^\infty e^{-(\alpha_1 + \ldots + \alpha_k)y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y)$$
$$+ \sum_{l=1}^{k-1} \sum_{j=1}^{2^{k-1-l}} L^{(k-1)}_{(2^l j - 2^{l-1}+1,l)} \int_0^\infty e^{-(\alpha_1 + \ldots + \alpha_l + \psi(q_{l+1}))y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y) =: \mathcal{I} + \mathcal{II} \tag{5.2}$$

The two integrals in (5.2) can be computed using Thm. 2.1. Each integral gives two new terms, corresponding to a move down and left for the first term, and down and right for the second term in the trees presented in Figs. 1 and 2. The exponents are easily observed after an application of Thm. 2.1. Therefore, below we primarily focus on the coefficients. When considering such integrals the two terms obtained are referred to as the first and second term and are denoted by adding a 1 or 2 as indices to $\mathcal{I}$ and $\mathcal{II}$. We now successively consider (the coefficients of) $\mathcal{I}_1$, $\mathcal{II}_1$, $\mathcal{I}_2$, and $\mathcal{II}_2$.

∘ *Coefficient of* $\mathcal{I}_1$. The coefficient of $\exp[-(\alpha_1 + ... + \alpha_k)x]$ is found, using Thm. 2.1, from the first term of the integral

$$\int_0^\infty e^{-\alpha_1 y} \prod_{i=1}^{k-1} \frac{q_{i+1}}{q_{i+1} - \phi(\alpha_{i+1} + ... + \alpha_k)} e^{-(\alpha_2 + ... + \alpha_k)y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y),$$

which is

$$\prod_{i=2}^{k} \frac{q_i}{q_i - \phi(\alpha_i + ... + \alpha_k)} \frac{q_1}{q_1 - \phi(\alpha_1 + ... + \alpha_k)} = \prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + ... + \alpha_k)}. \tag{5.3}$$

This corresponds to the coefficient of the first term in Thm. 3.1.

∘ *Coefficient of* $\mathcal{II}_1$. For $l = 2, 3, \ldots, k$ it is seen that the terms $L^{(k)}_{(2^l j - 2^{l-1}+1, l)}$ for $j = 1, 2, \ldots, 2^{k-l}$ can be derived from the terms $L^{(k-1)}_{(2^{l-1} j - 2^{l-2}+1, l-1)}$ by taking the first term of the integrals (this corresponds to a move down and left when we look at the tree in Fig. 2):

$$\int_0^\infty L^{(k-1)}_{(2^{l-1} j - 2^{l-2}+1, l-1)} e^{-\alpha_1 y} e^{-(\alpha_2 + ... + \alpha_{l-1} + \psi(q_l))y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y). \tag{5.4}$$

From Thm. 2.1 we obtain

$$\begin{aligned}
L^{(k)}_{(2^l j - 2^{l-1}+1, l)} &= L^{(k-1)}_{(2^{l-1} j - 2^{l-2}+1, l-1)} \cdot \frac{q_1}{q_1 - \phi(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))} \\
&= c^{(2^{l-1} j - 2^{l-2}+1, k-1)} \cdot \prod_{i=1}^{k-1} \frac{q_{i+1}}{q_{i+1} - \phi(\alpha_{i+1} + \bar{d}^{(i+1, 2^{l-1} j - 2^{l-2}+1)})} \cdot \\
&\quad \prod_{i=l-1}^{k-1} \frac{\alpha_{i+1} + \bar{d}^{(i+1, 2^{l-1} j - 2^{l-2}+1)}}{\bar{d}^{(i, 2^{l-1} j - 2^{l-2}+1)}} \cdot \frac{q_1}{q_1 - \phi(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))} \\
&= c^{(2^{l-1} j - 2^{l-2}+1, k-1)} \cdot \prod_{i=2}^{k} \frac{q_i}{q_i - \phi(\alpha_i + \bar{d}^{(i, 2^{l-1} j - 2^{l-2}+1)})} \cdot \\
&\quad \prod_{i=l}^{k} \frac{\alpha_i + \bar{d}^{(i, 2^{l-1} j - 2^{l-2}+1)}}{\bar{d}^{(i-1, 2^{l-1} j - 2^{l-2}+1)}} \frac{q_1}{q_1 - \phi(\alpha_1 + \ldots + \alpha_{l-1} + \psi(q_l))},
\end{aligned}$$

where $j = 1, 2, \ldots, 2^{k-l}$; here $\bar{d}^{(i, 2^{l-1} j - 2^{l-2}+1)}$ is given by

$$\bar{d}^{(i, 2^{l-1} j - 2^{l-2}+1)} = \begin{cases} \alpha_{i+1} + \bar{d}^{(i+1, 2^{l-1} j - 2^{l-2}+1)} & \text{if } \left\lceil \frac{2^{l-1} j - 2^{l-2}+1}{2^{i-1}} \right\rceil \text{ is odd,} \\ \psi(q_{i+1}) & \text{if } \left\lceil \frac{2^{l-1} j - 2^{l-2}+1}{2^{i-1}} \right\rceil \text{ is even.} \end{cases}$$

This table follows from Definition 3.1 and the observation that the factor $\bar{d}^{(i,2^{l-1}j-2^{l-2}+1)}$ initially was the factor added to the term α_{i-1} (this is why we use the notation $\bar{d}$ for these terms); this is due to the fact that in (5.4) all indices are raised by one. In order to bring this into the form of Definition 3.1 we observe the following:

(a) Concerning the signs we have the relation $c^{(2^{l-1}j-2^{l-2}+1,k-1)} = c^{(2^l j-2^{l-1}+1,k)}$ for all $l = 2, ..., k$ and $j = 1, ..., 2^{k-l}$. We see this as follows. From Lemma 3.1 we see it is sufficient to show that the numbers $2^{l-1}j - 2^{l-2}$ and $2^l j - 2^{l-1}$ have the same parity. But this holds as $2^l j - 2^{l-1} = 2(2^{l-1}j - 2^{l-2})$. Intuitively we can see this from the tree graph in Fig. 6; every time we move down and left the sign is always the same.

(b) Concerning the labeling of the terms, using the fact that

$$\left\lceil \frac{2^{l-1}j - 2^{l-2} + 1}{2^{i-1}} \right\rceil = \left\lceil \frac{2^l j - 2^{l-1} + 1}{2^i} \right\rceil$$

(which we obtain from Remark 3) we obtain

$$d^{(i,2^l j-2^{l-1}+1)} = \bar{d}^{(i,2^{l-1}j-2^{l-2}+1)}. \tag{5.5}$$

(c) From the four properties in Remark 3 we see that $d^{(1,2^l j-2^{l-1}+1)} = \alpha_2 + \ldots + \alpha_{l-1} + \psi(q_l)$ and

$$\frac{\alpha_i + d^{(i,2^l j-2^{l-1}+1)}}{d^{(i-1,2^l j-2^{l-1}+1)}} = 1,$$

for all $i = 1, 2, \ldots, l-1$.

The arguments in (a)-(c) show that, for $l = 2, 3, \ldots, n$, $j = 1, 2, \ldots, 2^{k-l}$,

$$L^{(k)}_{(2^l j-2^{l-1}+1,l)} = c^{(j,k)} \cdot \prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + d^{(i,2^l j-2^{l-1}+1)})} \cdot \prod_{i=l}^{k} \frac{\alpha_i + d^{(i,2^l j-2^{l-1}+1)}}{d^{(i-1,2^l j-2^{l-1}+1)}},$$

where the $d^{(i,2^l j-2^{l-1}+1)}$ are given by the table in Definition 3.1.

◦ *Coefficient of $\mathcal{I}_2$.* For the terms $L^{(k)}_{(2j,1)}$, $j = 1, 2, \ldots, 2^{k-1}$ (i.e., the coefficients of $\exp[-\psi(q_1)x]$ for k exponentially distributed random variables) we observe that these are given from all terms in the previous step, one from each (this corresponds to moving down and right in the tree graph in Fig. 1 or Fig. 2). The first term, $L^{(k)}_{(2,1)}$ results from the integration

$$\int_0^\infty \prod_{i=1}^{k-1} \frac{q_{i+1}}{q_{i+1} - \phi(\alpha_{i+1} + \ldots + \alpha_k)} e^{-(\alpha_1+\ldots+\alpha_k)y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y),$$

which leads to

$$L^{(k)}_{(2,1)} = -\prod_{i=2}^{k} \frac{q_i}{q_i - \phi(\alpha_i + \ldots + \alpha_k)} \frac{q_1}{q_1 + \phi(\alpha_1 + \ldots + \alpha_k)} \frac{\alpha_1 + \ldots + \alpha_k}{\psi(q_1)}.$$

Since $l = 1$ and $j = 1$, we have for $i = 1, 2, \ldots, k$

$$\left\lceil \frac{2}{2^i} \right\rceil = 1,$$

showing that $d^{(i,2)} = \sum_{s=i+1}^{k} \alpha_s$. Furthermore, we see that for all $i = 2, 3, \dots, k$

$$\frac{\alpha_i + d^{(i,1)}}{d^{(i-1,2)}} = 1,$$

and, hence, we get

$$\prod_{i=1}^{k} \frac{\alpha_i + d^{(i,2)}}{d^{(i-1,2)}} = \frac{\alpha_1 + d^{(1,2)}}{d^{(0,2)}} = \frac{\alpha_1 + \ldots + \alpha_k}{\psi(q_1)}.$$

By using these facts, it follows that

$$L^{(k)}_{(2,1)} = -\prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + d^{(i,2)})} \prod_{i=1}^{k} \frac{\alpha_i + d^{(i,2)}}{d^{(i-1,2)}}, \tag{5.6}$$

corresponding to Definition 3.1 and Lemma 3.1.

∘ *Coefficient of* $\mathcal{II}_2$. In general, the terms $L^{(k)}_{(2^{l+1}j-2^l+2,1)}$, for $l = 1, 2, \dots, k-1$ and $j = 1, 2, \dots, 2^{k-1-l}$, are derived from the integrals

$$\int_0^\infty L^{(k-1)}_{(2^l j - 2^{l-1}+1, l)} e^{-\alpha_1 y} e^{-(\alpha_2 + \ldots + \alpha_l + \psi(q_{l+1}))y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y). \tag{5.7}$$

Consider the terms $L^{(k)}_{(2^{l+1}j-2^l+2,1)}$ for $l = 1, \dots, k-1$ and $j = 1, \dots, 2^{k-1-l}$. From the integral in (5.7) we obtain, for $l = 1, \dots, k-2$ and $j = 1, 2, \dots, 2^{k-2-l}$, that

$$\begin{aligned}
L^{(k)}_{(2^{l+1}j-2^l+2,1)} =& -c^{(2^l j-2^{l-1}+1,k-1)} \cdot \prod_{i=1}^{k-1} \frac{q_{i+1}}{q_{i+1} - \phi(\alpha_{i+1} + \bar{d}^{(i+1,2^l j-2^{l-1}+1)})} \\
&\cdot \prod_{i=l}^{k-1} \frac{\alpha_{i+1} + \bar{d}^{(i+1,2^l j-2^{l-1}+1)}}{\bar{d}^{(i,2^l j-2^{l-1}+1)}} \cdot \frac{q_1}{q_1 - \phi(\alpha_1 + ... + \alpha_l + \psi(q_{l+1}))} \cdot \frac{\alpha_1 + ... + \alpha_l + \psi(q_{l+1})}{\psi(q_1)} \\
=& -c^{(2^l j-2^{l-1}+1,k-1)} \cdot \prod_{i=2}^{k} \frac{q_i}{q_i - \phi(\alpha_i + \bar{d}^{(i,2^l j-2^{l-1}+1)})} \\
&\cdot \prod_{i=l+1}^{k} \frac{\alpha_i + \bar{d}^{(i,2^l j-2^{l-1}+1)}}{\bar{d}^{(i-1,2^l j-2^{l-1}+1)}} \cdot \frac{q_1}{q_1 - \phi(\alpha_1 + \ldots + \alpha_l + \psi(q_{l+1}))} \cdot \frac{\alpha_1 + \ldots + \alpha_l + \psi(q_{l+1})}{\psi(q_1)},
\end{aligned}$$

where the factors $\bar{d}^{(i,2^l j-2^{l-1}+1)}$ are given by

$$\bar{d}^{(i,2^l j-2^{l-1}+1)} = \begin{cases} \alpha_{i+1} + \bar{d}^{(i+1,2^l j-2^{l-1}+1)} & \text{if } \left\lceil \frac{2^l j-2^{l-1}+1}{2^{i-1}} \right\rceil \text{ is odd,} \\ \psi(q_{i+1}) & \text{if } \left\lceil \frac{2^l j-2^{l-1}+1}{2^{i-1}} \right\rceil \text{ is even.} \end{cases}$$

Using the same observation as in (5.5), it is found, for $j = 1, \dots, 2^{k-l-1}$ and $i = l+1, ..., k$, that

$$d^{(i,2^{l+1}j-2^l+2)} = \bar{d}^{(i,2^l j-2^{l-1}+1)}.$$

From Remark 3 (a)-(c) we see that

$$d^{(1,2^{l+1}j-2^l+2)} = \alpha_2 + \ldots + \alpha_l + \psi(q_{l+1}), \quad d^{(0,2^{l+1}j-2^l+2)} = \psi(q_1)$$

and, for $i = 2, 3, \dots, l$,

$$\frac{\alpha_i + d^{(i,2^{l+1}j-2^l+2)}}{d^{(i-1,2^{l+1}j-2^l+2)}} = 1.$$

These observations allow us to write $L^{(k)}_{(2^{l+1}j-2^l+2,1)}$ as follows

$$L^{(k)}_{(2^{l+1}j-2^l+2,1)} = -c^{(2^l j-2^{l-1}+1,k-1)} \cdot \prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + d^{(i,2^{l+1}j-2^l+2)})} \cdot \prod_{i=1}^{k} \frac{\alpha_i + d^{(i,2^{l+1}j-2^l+2)}}{d^{(i-1,2^{l+1}j-2^l+2)}}.$$

Concerning the signs, we obtain the relation $c^{(2^{l+1}j-2^l+2,k)} = -c^{(2^l j-2^{l-1}+1,k-1)}$ since the numbers $2^{l+1}j - 2^l + 1 = 2(2^l j - 2^{l-1}) + 1$ and $2^l j - 2^{l-1}$ have opposite parities. This final expression agrees with those presented in Thm. 3.1.

Now, we combine the above results to complete the proof. Using the coefficients of $\mathcal{I}_1$ and $\mathcal{I}_2$, i.e., (5.3) and (5.6), we can rewrite (5.2) to

$$\begin{aligned}\mathcal{L} =& \prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + \dots + \alpha_k)} e^{-(\alpha_1+\dots+\alpha_k)x} - \prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + d^{(i,2)})} \prod_{i=1}^{k} \frac{\alpha_i + d^{(i,2)}}{d^{(i-1,2)}} e^{-\psi(q_1)x} \\ &+ \sum_{l=2}^{k} \sum_{j=1}^{2^{k-l}} L^{(k-1)}_{(2^{l-1}j-2^{l-2}+1,l-1)} \int_0^\infty e^{-(\alpha_1+\dots+\alpha_{l-1}+\psi(q_l))y}\, \mathbb{P}_x(\mathcal{Q}_{T_1} \in \mathrm{d}y).\end{aligned}$$

Using the definition of $L^{(k)}_{(2,1)}$, in conjunction with the coefficients $\mathcal{II}_1$ and $\mathcal{II}_2$ and Definition 3.1, the above expression can be written as

$$\begin{aligned}\mathcal{L} =& \prod_{i=1}^{k} \frac{q_i}{q_i - \phi(\alpha_i + \dots + \alpha_k)} e^{-(\alpha_1+\dots+\alpha_k)x} - L^{(k)}_{(2,1)} e^{-\psi(q_1)x} \\ &+ \sum_{l=2}^{k} \sum_{j=1}^{2^{k-l}} L^{(k)}_{(2^l j-2^{l-1}+1,l)} e^{-(\alpha_1+\dots+\alpha_{l-1}+\psi(q_l))x} + \sum_{l=2}^{k} \sum_{j=1}^{2^{k-l}} L^{(k)}_{(2^l j-2^{l-1}+2,1)} e^{-\psi(q_1)x}.\end{aligned}$$

It remains to write the last sum in the desired form. This double sum has in total $2^{k-1} - 1$ terms, and we observe that for $l = 2, \dots, k$ and $j = 1, \dots, 2^{k-l}$, $2^l j - 2^{l-1} + 2$ defines a partition of the even numbers $4, 6, \dots, 2^k$ into $k - 1$ classes each one containing 2^{k-l} numbers. Relabeling the terms with only one subscript, we can write this double sum as

$$\sum_{l=2}^{k} \sum_{j=1}^{2^{k-l}} L^{(k)}_{(2^l j-2^{l-1}+2,1)} e^{-\psi(q_1)x} = \sum_{i=2}^{2^{k-1}} L^{(k)}_{(2i,1)} e^{-\psi(q_1)x},$$

where $i = 2^l j - 2^{l-1} + 2$ for $l = 2, \dots, k$ and $j = 1, \dots, 2^{k-l}$. From the above it follows that $\mathcal{L}$ can be written as the expression in Thm. 3.1. This completes the proof.

□

6 Conclusion and Discussion

In this paper we have analyzed the transient behavior of spectrally one-sided Lévy-driven queues. We considered the joint behavior of $\mathcal{Q}_{T_1}, \mathcal{Q}_{T_1+T_2}, \dots, \mathcal{Q}_{T_1+\dots+T_n}$ where T_i is exponentially distributed

with parameter q_i, and we specifically focused on $\mathcal{Q}_{T_1+\ldots+T_n}$. From the main results it follows that this transient behavior obeys an elegant and appealing tree structure. Interestingly, some numerical illustrations showed that $\mathbb{E}_x\, \mathcal{Q}_t$ is first decreasing in t and then converges to the steady-state workload from below in case x is chosen 'slightly' above the stationary workload.
We have restricted ourselves to analyzing $\mathcal{Q}_T$ with T distributed as the sum of n independent exponential random variables, but our result is readily extended to that of $\mathcal{Q}_T$ with T obeying a *Coxian* distribution. This is a particularly useful fact, as any distribution on the positive half line can be approximated arbitrarily closely by a sequence of Coxian distributions, see e.g. [2, Section III.4]. In more detail, the analysis looks as follows. Consider the situation that T follows a Coxian distribution with n phases; we let the length of phase i be drawn from an exponential distribution with parameter q_i, and we let the probability of moving from phase i to $i+1$ be p_i (with the convention that $p_n = 0$). Then, for the spectrally-positive case,

$$\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_T} = \sum_{k=1}^{n}(1-p_k)\prod_{i=1}^{k-1} p_i \cdot \mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+\ldots+T_k}},$$

where $\mathbb{E}_x\, e^{-\alpha\, \mathcal{Q}_{T_1+\ldots+T_k}}$ is as obtained in Thm. 3.1. The density in the spectrally-negative case follows by a similar argument.
To conclude, we like to mention some topics that are of interest for future investigation. Although the class of Coxian distributions for the epoch T is sufficiently rich, it might of interest to study the behavior of $\mathcal{Q}_T$ if T has a general phase-type distribution. Specifically, we did not explicitly derive the results in case some parameters q_i are identical. This follows as a direct application of l'Hôpital's rule, but the expressions tend to become cumbersome. Another open question concerns the transient behavior for spectrally two-sided Lévy processes. Finally, we expect that the transient analysis presented here may be applicable in inference procedures, to estimate the queue's Lévy input process from a finite number of successive workload observations.

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