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On a class of reflected AR(1) processes

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Abstract

In this paper, we study a reflected AR(1) process, i.e., a process $(Z_n)_n$ obeying the recursion $Z_{n+1} = \max\{aZ_n + X_n, 0\}$, with $(X_n)_n$ a sequence of i.i.d. random variables. We find explicit results for the distribution of Z_n (in terms of transforms) in case X_n can be written as $Y_n - B_n$, with $(B_n)_n$ being a sequence of independent random variables which are all $\exp(\lambda)$ distributed, and $(Y_n)_n$ i.i.d.; when $|a| < 1$ we can also perform the corresponding stationary analysis. Extensions are possible to the case that $(B_n)_n$ are of phase-type. Under a heavy-traffic scaling, it is shown that the process converges to a reflected Ornstein-Uhlenbeck process; the corresponding steady-state distribution converges to the distribution of a Normal random variable conditioned on being positive.

KEYWORDS. Reflected processes ⋆ queueing ⋆ scaling limits

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1 Introduction

The analysis of stochastic recursions forms an important area of study within applied probability. In particular, due to the fact that their underlying recursions are of an extremely simple form, together with their wide applicability in fields such as economics, engineering, and biology [6, 11], models of the *autoregressive* type have been studied extensively in the literature, with a special focus on the simple subclass of *first order autoregressive models* or, more simply, *AR(1) models*. In such models, the update rule

$$Z_{n+1} = aZ_n + X_n, \quad n = 0, 1, \ldots,$$

is considered, for a sequence of i.i.d. generally distributed random variables $(X_n)_{n=0,1,\ldots}$ and a scalar $a \in \mathbb{R}$, with $Z_0 = z$ being given.

In many practical situations, however, the quantities Z_n can attain non-negative values only, thus providing motivation for studying the alternative recursion

$$Z_{n+1} = \max\{aZ_n + X_n, 0\}, \quad n = 0, 1, \ldots. \tag{1}$$

We refer to (1) as a *reflected AR(1) process* and the main objective of this paper is to identify the distribution of Z_n in (1) and (assuming $|a| < 1$) of its stationary counterpart Z_∞. To the best of our knowledge, this paper is the first to successfully study these quantities. We also note that the specific case of $a = 1$ corresponds to the well-studied sequence of customer waiting times in a GI/G/1 queue [1, 8], while the case of $a = -1$ was investigated extensively by Vlasiou [14] in the context of carousel models. Related literature on iterated random functions includes e.g. [5, 9, 10, 16].

In this paper, we specialize the setup above to the case in which X_n may be written as $Y_n - B_n$, with $(B_n)_n$ being a sequence of independent random variables which are all $\exp(\lambda)$ distributed, and $(Y_n)_n$ i.i.d. (and in addition independent of $(B_n)_n$) with distribution function $F_Y(\cdot)$ and Laplace-Stieltjes transform (LST) $\varphi_Y(\cdot)$. Our contributions are the following. First, in Section 2 we determine the distribution of Z_n for each $n = 0, 1, \ldots$ in terms of a 'double transform' (corresponding to the Laplace-Stieltjes transform of Z_n at a geometrically distributed time epoch). Second, in Section 3.1 we show that under a particular scaling of the model parameters, the reflected AR(1) process converges to a *reflected Ornstein-Uhlenbeck process*; which has been studied in considerable detail [12, 15]. Third, in Section 3.2 we prove that under the same scaling, Z_∞ converges to a random variable having a truncated Normal distribution (i.e., a Normal distribution conditioned on being non-negative). This requires a separate argument from the process level convergence as the scaling limit and the steady-state limit do not necessarily commute. Finally, in Section 4 we complete the paper by highlighting a connection between the distribution at each time epoch of the reflected AR(1) process $(Z_n)_n$ and the first passage time distribution of a corresponding unreflected AR(1) process.

2 Transform of transient and stationary distribution

The main goal of this section concerns the identification of the distribution of Z_n and its stationary counterpart Z_∞, in terms of transforms. We do this primarily relying on Wiener-Hopf theory, providing us with a relation between the transform evaluated in s and in as, which can then be iterated to yield an expression for the transform under consideration in terms of infinite sums and products. Various ramifications are included as well, covering e.g. the case that the $(B_n)_{n=0,1,\ldots}$ are from more general classes of distributions.

2.1 Transient distribution

Starting point of our analysis is an expression involving the transforms

$$Z_z(r,s) := \sum_{n=0}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0=z], \quad U_z(r,s) := \sum_{n=0}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-s\min\{aZ_n+X_n,0\}}|Z_0=z].$$

Observe that, with $x^- := \min\{0,x\}$ and $x^+ := \max\{0,x\}$, we have $1+\mathrm{e}^x = \mathrm{e}^{x^+}+\mathrm{e}^{x^-}$, and hence, for $n\in\mathbb{N}$, with $W_n := -\min\{aZ_n+X_n,0\}$,

$$\mathrm{e}^{-sZ_{n+1}} = \mathrm{e}^{-s(aZ_n+X_n)} + 1 - \mathrm{e}^{sW_n}.$$

Taking expectations and realizing that Z_n and X_n are independent, this leads us to

$$\mathbb{E}[\mathrm{e}^{-sZ_{n+1}}|Z_0=z] = \mathbb{E}[\mathrm{e}^{-sX}]\mathbb{E}[\mathrm{e}^{-saZ_n}|Z_0=z] + 1 - \mathbb{E}[\mathrm{e}^{sW_n}|Z_0=z],$$

which trivially leads to the identity

$$Z_z(r,s) - r\varphi_Y(s)\frac{\lambda}{\lambda-s}Z_z(r,as) = \mathrm{e}^{-sz} + \frac{r}{1-r} - rU_z(r,s), \quad \mathrm{Re}\ s=0. \tag{2}$$

Multiplying both sides of (2) by $\lambda - s$, one obtains

$$(\lambda-s)\left(Z_z(r,s)-\mathrm{e}^{-sz}\right) - r\lambda\varphi_Y(s)Z_z(r,as) = (\lambda-s)\left(\frac{r}{1-r} - r\,U_z(r,s)\right), \quad \mathrm{Re}\ s=0. \tag{3}$$

The primary objective now is to determine both $Z_z(r,s)$ and $U_z(r,s)$. We do this by formulating and solving a Wiener-Hopf boundary value problem, cf. Cohen [7].

To this end we first make the following observations:

- The left hand side of (3) is analytic in $\mathrm{Re}\ s>0$, and continuous in $\mathrm{Re}\ s\geq 0$.
- The right hand side of (3) is analytic in $\mathrm{Re}\ s<0$, and continuous in $\mathrm{Re}\ s\leq 0$.
- $Z_z(r,s)$ is for $\mathrm{Re}\ s\geq 0$ bounded by $\sum_{n=0}^{\infty} r^n = (1-r)^{-1}$, and hence the left hand side of (3) behaves at most as a linear function in s for large s, $\mathrm{Re}\ s>0$.
- $U_z(r,s)$ is for $\mathrm{Re}\ s\leq 0$ bounded by $\sum_{n=0}^{\infty} r^n = (1-r)^{-1}$, and hence the right hand side of (3) behaves at most as a linear function in s for large s, $\mathrm{Re}\ s<0$.

Liouville's theorem [13] now implies that both sides of (3), in their respective half-planes, are equal to the same linear function in s, i.e.,

$$(\lambda - s)\left(Z_z(r,s) - \mathrm{e}^{-sz}\right) - r\lambda\varphi_Y(s)Z_z(r,as) = C_{0,z}(r) + sC_{1,z}(r), \quad \mathrm{Re}\, s \geq 0, \tag{4}$$

and

$$(\lambda - s)\left(\frac{r}{1-r} - rU_z(r,s)\right) = C_{0,z}(r) + sC_{1,z}(r), \quad \mathrm{Re}\, s \leq 0, \tag{5}$$

with $C_{0,z}(r)$ and $C_{1,z}(r)$ two functions of r which still have to be determined.
Taking $s = 0$ in either (4) or (5) immediately yields that $C_{0,z}(r) = 0$. Determining $C_{1,z}(r)$ is considerably more complicated. Before we determine $C_{1,z}(r)$, we first further explore its relation to the distribution of the Z_n. In the first place, one sees from the definition of $Z_z(r,s)$ that

$$Z_z(r,s) - \mathrm{e}^{-sz} \to \sum_{n=1}^{\infty} r^n \mathbb{P}(Z_n = 0 | Z_0 = z), \quad s \to \infty.$$

Combined with (4), this implies that

$$C_{1,z}(r) = -\sum_{n=1}^{\infty} r^n \mathbb{P}(Z_n = 0 | Z_0 = z). \tag{6}$$

In the second place, one sees from the definition of $U_z(r,s)$ that

$$U_z(r,s) \to \sum_{n=0}^{\infty} r^n \mathbb{P}(\min\{aZ_n + X_n, 0\} = 0 | Z_0 = z), \quad s \to -\infty.$$

Notice that $\mathbb{P}(\min\{aZ_n + X_n, 0\} = 0 | Z_0 = z) = 1 - \mathbb{P}(Z_{n+1} = 0 | Z_0 = z)$. Hence, upon combining this with (5) we obtain that, in agreement with (6),

$$C_{1,z}(r) = -\frac{r}{1-r} + r\sum_{n=0}^{\infty} r^n (1 - \mathbb{P}(Z_{n+1} = 0 | Z_0 = z)) = -\sum_{n=1}^{\infty} r^n \mathbb{P}(Z_n = 0 | Z_0 = z). \tag{7}$$

Thirdly, by substituting $s = \lambda$ in (4), we obtain

$$C_{1,z}(r) = -r\varphi_Y(\lambda)Z_z(r, a\lambda). \tag{8}$$

In the sequel we use the identity (8), in combination with another relation, to determine $C_{1,z}(r)$; at this moment we only confirm that (8) is in agreement with (6), by showing that the coefficients of r^n in both expressions agree for each n. To do this, observe that, for $n = 1, 2, \ldots$,

$$\begin{aligned} \mathbb{P}(Z_n = 0 | Z_0 = z) &= \mathbb{P}(aZ_{n-1} + X_{n-1} < 0 | Z_0 = z) = \mathbb{P}(aZ_{n-1} + Y_{n-1} < B_{n-1} | Z_0 = z) \\ &= \varphi_Y(\lambda)\mathbb{E}[\mathrm{e}^{-a\lambda Z_{n-1}} | Z_0 = z], \end{aligned} \tag{9}$$

where the last equality is a consequence of the fact that B_{n-1} is exponential and hence memoryless.

Remark 2.1 By differentiating (4) with respect to s and subsequently taking $s = 0$, one may derive yet another expression for $C_{1,z}(r)$:

$$C_{1,z}(r) = \frac{r}{1-r}(\lambda \mathbb{E}Y - 1) + (r\lambda a - \lambda)\sum_{n=0}^{\infty} r^n \mathbb{E}[Z_n | Z_0 = z]. \tag{10}$$

The above indicates that knowledge of $C_{1,z}(r)$ immediately gives specific information about the distribution of the Z_n. We shall next determine $C_{1,z}(r)$, and hence $Z_z(r,s)$ and $U_z(r,s)$. To this end, we decompose (4) as

$$Z_z(r,s) = K(r,s)Z_z(r,as) + L_z(r,s), \quad \mathrm{Re}\, s \geq 0,$$

introducing the two functions

$$K(r,s) := r\frac{\lambda}{\lambda - s}\varphi_Y(s), \quad L_z(r,s) := \frac{sC_{1,z}(r)}{\lambda - s} + \mathrm{e}^{-sz}. \tag{11}$$

Iteration of this equation yields

$$\begin{aligned} Z_z(r,s) &= L_z(r,s) + K(r,s)[L_z(r,as) + K(r,as)Z_z(r,a^2 s)] \\ &= L_z(r,s) + K(r,s)L_z(r,as) + K(r,s)K(r,as)Z_z(r,a^2 s) \\ &= \dots \\ &= \sum_{n=0}^{\infty} L_z(r,a^n s)\prod_{j=0}^{n-1} K(r,a^j s), \end{aligned} \tag{12}$$

following the convention that an empty product is defined to be one. Notice that, for fixed r with $|r| < 1$, the d'Alembert test (or: ratio test) shows that the infinite series converges, as the ratio of two successive terms tends to r.
Inserting the definitions given in (11), we thus obtain from (12)

$$Z_z(r,s) = \sum_{n=0}^{\infty}\left(\frac{a^n s\, C_{1,z}(r)}{\lambda - a^n s} + \mathrm{e}^{-a^n sz}\right) r^n \prod_{j=0}^{n-1}\frac{\lambda}{\lambda - a^j s}\varphi_Y(a^j s). \tag{13}$$

Since we now focus on determining $C_{1,z}(r)$, we rewrite (13) as follows:

$$Z_z(r,s) = \sum_{n=0}^{\infty}\mathrm{e}^{-a^n sz} r^n \prod_{j=0}^{n-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s} + C_{1,z}(r)\sum_{n=0}^{\infty}\frac{a^n s}{\lambda - a^n s} r^n \prod_{j=0}^{n-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s}. \tag{14}$$

Plugging in $s = a\lambda$ leads to

$$Z_z(r,a\lambda) = \sum_{n=0}^{\infty}\mathrm{e}^{-a^{n+1}\lambda z} r^n \prod_{j=0}^{n-1}\frac{\varphi_Y(a^{j+1}\lambda)}{1 - a^{j+1}} + C_{1,z}(r)\sum_{n=0}^{\infty}\frac{a^{n+1}}{1 - a^{n+1}} r^n \prod_{j=0}^{n-1}\frac{\varphi_Y(a^{j+1}\lambda)}{1 - a^{j+1}}. \tag{15}$$

Combining this relation with (8) facilitates the identification of $C_{1,z}(r)$. Introducing

$$g(a,n) := \prod_{j=0}^{n-1}\frac{\varphi_Y(a^{j+1}\lambda)}{1 - a^{j+1}},$$

we find

$$C_{1,z}(r)\left(1+\varphi_Y(\lambda)\sum_{n=0}^{\infty}\frac{(ar)^{n+1}}{1-a^{n+1}}g(a,n)\right)=-\varphi_Y(\lambda)\sum_{n=0}^{\infty}\mathrm{e}^{-a^{n+1}\lambda z}r^{n+1}g(a,n), \tag{16}$$

and hence

$$C_{1,z}(r)=-\frac{\varphi_Y(\lambda)\sum_{n=0}^{\infty}\mathrm{e}^{-a^{n+1}\lambda z}r^{n+1}g(a,n)}{1+\varphi_Y(\lambda)\sum_{n=0}^{\infty}\frac{(ar)^{n+1}}{1-a^{n+1}}g(a,n)}. \tag{17}$$

Substitution in (14) finally gives the following result.

Theorem 2.2 *With $C_{1,z}(r)$ given by (17),*

$$Z_z(r,s) \;=\; \sum_{n=0}^{\infty}\left(\mathrm{e}^{-a^n sz}r^n+C_{1,z}(r)\frac{a^n s}{\lambda-a^n s}r^n\right)\prod_{j=0}^{n-1}\frac{\lambda}{\lambda-a^j s}\varphi_Y(a^j s). \tag{18}$$

Remark 2.3 In the special case $a=0$ one obtains from (17) that

$$C_{1,z}(r)=-\frac{r}{1-r}\varphi_Y(\lambda).$$

This is seen to be in agreement with (7) since, for $a=0$ and $n=1,2,\dots,$

$$\mathbb{P}(Z_n=0|Z_0=z)=\mathbb{P}(Y_{n-1}<B_{n-1})=\varphi_Y(\lambda).$$

Remark 2.4 In the classical case M/G/1 case, $a=1$ that is, the identity (4) reduces to

$$[\lambda-s-r\lambda\varphi_Y(s)]Z_z(r,s)-(\lambda-s)\mathrm{e}^{-sz}=s\,C_{1,z}(r). \tag{19}$$

$C_{1,z}(r)$ now is obtained by observing that $\lambda-s-r\lambda\varphi_Y(s)$ has a unique zero $s=s(r)$ in the right half plane, which should also be a zero of $(\lambda-s)\mathrm{e}^{-sz}-sC_{1,z}(r)$.

Remark 2.5 It follows from (14), by taking the coefficient of r^n, that, for $n=0,1,\dots$:

$$\mathbb{E}[\mathrm{e}^{-sZ_n}|Z_0=z]=\mathrm{e}^{-a^n sz}\prod_{j=0}^{n-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda-a^j s}-\sum_{m=0}^{n-1}\mathbb{P}(Z_{n-m}=0|Z_0=z)\frac{a^m s}{\lambda-a^m s}\prod_{j=0}^{m-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda-a^j s}. \tag{20}$$

Remark 2.6 In (18), it looks as if $Z_z(r,s)$ has singularities in $s=\lambda/a^j$, $j=0,1,\dots$, but it can be seen that these are removable singularities. Let us show this for $s=\lambda$. First of all, we had already observed in (4) that substitution of $s=\lambda$ gives (8), and the correctness of (8) was also verified in a direct probabilistic manner. Secondly, we can rewrite (13) as follows, to isolate the singularity for $s = \lambda$:

$$\begin{aligned} Z_z(r,s) &= \left(\frac{s\,C_{1,z}(r)}{\lambda - s} + \mathrm{e}^{-sz}\right) + \sum_{n=1}^{\infty}\left(\frac{a^n s\,C_1(r)}{\lambda - a^n s} + \mathrm{e}^{-a^n sz}\right) r^n \left(\frac{\lambda}{\lambda - s}\varphi_Y(s)\right)\prod_{j=1}^{n-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s} \\ &= \mathrm{e}^{-sz} + \frac{1}{\lambda - s}\left(s\,C_{1,z}(r) + \lambda\varphi_Y(s)\sum_{n=1}^{\infty}\left(\frac{a^n s\,C_1(r)}{\lambda - a^n s} + \mathrm{e}^{-a^n sz}\right) r^n \prod_{j=1}^{n-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s}\right) \\ &= \mathrm{e}^{-sz} + \frac{1}{\lambda - s}\left[s\,C_{1,z}(r) + r\lambda\varphi_Y(s)Z_z(r,as)\right]. \end{aligned} \quad (21)$$

The term between square brackets in the last line indeed becomes zero for $s = \lambda$ (cf. (8)), confirming that $s = \lambda$ is not a pole of $Z_z(r,s)$. Hence the same holds for the expression for $Z_z(r,s)$ in (18). One may subsequently use (4) to show that $Z_z(r,s)$ has no singularity in $s = \lambda/a$, and hence also not in $s = \lambda/a^2$, etc.

We now briefly consider an extension in which $X_n = Y_n - B_n$, with $B_1, B_2, \dots$ still i.i.d. but not necessarily exponentially distributed; we now allow the B_i to be a sum of k independent, exponentially distributed random variables, with rates $\lambda_1, \dots, \lambda_k$ (this is the so-called *hypo-exponential* distribution). As is readily verified, (3) now changes into

$$\prod_{i=1}^{k}(\lambda_i - s)Z_z(r,s) - r\prod_{i=1}^{k}\lambda_i\varphi_Y(s)Z_z(r,as) = \prod_{i=1}^{k}(\lambda_i - s)\left(\frac{1}{1-r} - r\,U(r,s)\right), \quad \mathrm{Re}\, s = 0. \quad (22)$$

Liouville's theorem now yields (cf. (4)):

$$\prod_{i=1}^{k}(\lambda_i - s)Z_z(r,s) - r\prod_{i=1}^{k}\lambda_i\varphi_Y(s)Z_z(r,as) = \sum_{i=0}^{k} s^i C_i(r), \quad \mathrm{Re}\, s \geq 0. \quad (23)$$

Substitution of $s = 0$ readily gives $C_0(r) = \prod_{i=1}^{k}\lambda_i$. Formula (12) still holds, but with an obvious adaptation of the functions $K(r,s)$ and $L_z(r,s)$ as they were defined in (11). The remaining k unknown functions $C_1(r), \dots, C_k(r)$ are obtained by performing the following three steps:

- Substitute $s = a\lambda_i$ for $i = 1, \dots, k$ into the new version of (12), thus linearly expressing $Z_z(r, a\lambda_i)$ into $C_1(r), \dots, C_k(r)$ for $i = 1, \dots, k$.
- Substitute $s = \lambda_i$ for $i = 1, \dots, k$ into (23), thus linearly expressing $Z_z(r, a\lambda_i)$ into $C_1(r), \dots, C_k(r)$ in another way.
- Finally eliminate all $Z_z(r, a\lambda_i)$ from the former k equations using the latter k equations, and then solve the resulting set of k linear equations in $C_1(r), \dots, C_k(r)$.

If some of the λ_i coincide, the usual adaptation should be made: one should also differentiate (23) with respect to s and substitute $s = \lambda_i$ (more precisely: if the multiplicity of λ_i is d, one has to differentiate (23) $d-1$ times, substitute $s=\lambda_i$ in each of them, and solve the resulting equations).

If the distribution of $B_1, B_2, \ldots$ is *hyperexponential* (i.e., with probability p_i sampled from an exponential distribution with rate λ_i, where the p_i sum to 1), then one can set up a procedure very similar to the one we developed for the hypo-exponential distribution.

2.2 Stationary distribution

Our next goal is to identify the Laplace transform of the steady-state counterpart Z_∞. There are at least two ways of obtaining this.

- Consider the relation $Z_\infty \stackrel{\rm d}{=} \max\{aZ_\infty + X, 0)\}$, leading to

$$\mathbb{E}[{\rm e}^{-sZ_\infty}] - \varphi_Y(s)\frac{\lambda}{\lambda - s}\mathbb{E}[{\rm e}^{-asZ_\infty}] = 1 - \mathbb{E}[{\rm e}^{-s\min(aZ_\infty + X, 0)}]. \tag{24}$$

  Again use Wiener-Hopf factorization and Liouville's theorem, to arrive at an equation in which $\mathbb{E}[{\rm e}^{-sZ_\infty}]$ is expressed into $\mathbb{E}[{\rm e}^{-asZ_\infty}]$. As before, that equation is subsequently solved by iteration.

- Apply an Abel theorem to Theorem 2.2, to obtain $\mathbb{E}[{\rm e}^{-sZ_\infty}] = \lim_{r\to 1}(1-r)Z_z(r,s)$. Such an Abel theorem states that

$$\lim_{r\to 1}(1-r)\sum_{n=0}^{\infty} g_n r^n = g_\infty,$$

  if $g_n \to g_\infty$. The first sum in the right hand side of (18) can indeed be seen as a quantity of the form $\sum g_n r^n$, and those g_n converge to

$$\prod_{j=0}^{\infty}\frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s}.$$

  The second sum in the right hand side of (18) has the structure $\sum_{n=0}^{\infty} b_n r^n \sum_{n=0}^{\infty} c_n r^n$, where $b_n \to b_\infty \neq 0$ and $c_n \to 0$ while $\sum_{n=0}^{\infty} c_n$ converges. It is clear from (7) that $\lim_{r\to 1}(1-r)C_{1,z}(r) = -\mathbb{P}(Z_\infty = 0)$ (which is then our b_∞), and thus the second sum in the right hand side of (18) becomes

$$-\mathbb{P}(Z_\infty = 0)\sum_{m=0}^{\infty}\frac{a^m s}{\lambda - a^m s}\prod_{j=0}^{m-1}\frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s}.$$

With $\varphi_X(\cdot)$ denoting the LST of $X = Y - B$, i.e.,

$$\varphi_X(s) := \varphi_Y(s)\frac{\lambda}{\lambda - s},$$

we thus find the following result.

Theorem 2.7 *The stationary LST is given by*

$$\begin{aligned}\varphi_Z(s) := \mathbb{E}\mathrm{e}^{-sZ_\infty} &= \prod_{j=0}^{\infty} \frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s} - \mathbb{P}(Z_\infty = 0) \sum_{m=0}^{\infty} \frac{a^m s}{\lambda - a^m s} \prod_{j=0}^{m-1} \frac{\lambda\varphi_Y(a^j s)}{\lambda - a^j s} \\ &= \prod_{j=0}^{\infty} \varphi_X(a^j s) - \frac{\varphi_Y(\lambda)\varphi_Z(a\lambda)}{\lambda} \sum_{m=0}^{\infty} \frac{a^m s}{\varphi_Y(a^m s)} \prod_{j=0}^{m} \varphi_X(a^j s),\end{aligned}$$

where

$$\varphi_Z(a\lambda) = \prod_{j=0}^{\infty} \varphi_X(a^{j+1}\lambda) \Bigg/ \left(1 + \frac{\varphi_Y(\lambda)}{\lambda} \sum_{m=0}^{\infty} \frac{a^{m+1}\lambda}{\varphi_Y(a^{m+1}\lambda)} \prod_{j=0}^{m} \varphi_X(a^{j+1}\lambda)\right).$$

Remark 2.8 We can extend the above analysis to the case in which successive X_n still are i.i.d., but Y_n and B_n are dependent in a specific way, viz., (Y_n, B_n) has a bivariate matrix-exponential distribution, as introduced in [4]. Badila et al. [2] present an exact analysis of the waiting time process in an ordinary single server queue (i.e., $a = 1$) in which the n-th service time and subsequent interarrival time have such a bivariate matrix-exponential distribution. One can combine their Wiener-Hopf factorization approach with the iteration approach followed above.

3 Heavy-traffic scaling limit

In this section we impose a heavy-traffic scaling on the reflected AR(1) process and prove that the resulting heavy-traffic approximation is a reflected Ornstein-Uhlenbeck (OU) process. In addition, we show that the corresponding stationary distribution is truncated Normal.

3.1 Transient convergence

For each $N \in \{1, 2, \ldots\}$, let $Z^{(N)} \equiv (Z_n^{(N)})_n$ be a reflected AR(1) process, as introduced in the introduction. We impose a heavy-traffic scaling in which the increments $(X_n^{(N)})_n$ of $Z^{(N)}$ are such that

$$\mathbb{E}\, X_n^{(N)} = \frac{\gamma}{\sqrt{N}}, \quad \mathbb{V}\mathrm{ar}\, X_n^{(N)} = \frac{v}{N},$$

where $\gamma \in \mathbb{R}$ and $v > 0$. We also set $a_N := 1 - \alpha/N$ for $\alpha \in \mathbb{R}$.

Now let $D([0,\infty),\mathbb{R})$ denote the space of càdlàg functions on $[0,\infty)$ taking values in $\mathbb{R}$. For each $x \in D([0,\infty),\mathbb{R})$ with $x(0) \geq 0$ and $b \in \mathbb{R}$, let $z \in D([0,\infty),\mathbb{R})$ satisfy the integral equation

$$z(t) = x(t) - b\int_0^t z(s)\mathrm{d}s + \ell(t), \quad t \geq 0,$$

where $\ell \equiv \{\ell(t), t \geq 0\} \in D([0,\infty),\mathbb{R})$ is a non-decreasing function such that $\ell(0) = 0$ and

$$\int_0^\infty 1\{z(s) > 0\}\mathrm{d}\ell(s) = 0.$$

It then follows by [12, Prop. 2] that there exists a unique such z, denoted by $\Phi(x)$ and, moreover, the map $\Phi : D([0,\infty),\mathbb{R}) \mapsto D([0,\infty),\mathbb{R})$ is Lipschitz continuous with respect to the Skorokhod-J_1 topology [3].

Our main result of this section is the following weak convergence.

Proposition 3.1 *If $Z_0^{(N)} \Rightarrow Z_0^{(\infty)}$ as $N \to \infty$, then*

$$Z_{\lfloor N\cdot \rfloor}^{(N)} \Rightarrow \Phi(Z_0^{(\infty)} + \sqrt{v}B)$$

as $N \to \infty$, where B is a standard Brownian motion, independent of $Z_0^{(\infty)}$.

Remark 3.2 The process $\Phi(Z_0^{(\infty)} + \sqrt{v}B)$ is commonly referred to in the literature as a *reflected Ornstein-Uhlenbeck* (OU) process. Characteristics of its transient and stationary distribution are well known. In Section 3.2 below we address the issue of approximating the steady-state behavior of reflected AR(1) processes by that of reflected OUs.

Proof (of Prop. 3.1). Our proof of Proposition 3.1 is as follows. By (1), in combination with the definition $W_n = -\min(aZ_n + X_n, 0)$ for $n = 0, 1, \ldots$ in Section 2, we may write

$$Z_{n+1} = aZ_n + X_n + W_n, \ \ n = 0, 1, ...$$

As a consequence, $Z_{n+1} - Z_n = (a-1)Z_n + X_n + W_n$ for $n = 0, 1, \ldots$ and so summing over both sides of this equality we obtain that

$$Z_n \ \ = \ \ Z_0 + S_{n-1} + (a-1)\sum_{k=0}^{n-1} Z_k + L_{n-1}, \ n = 0, 1, \ldots,$$

where $S_n := \sum_{k=0}^{n} X_k$ and $L_n := \sum_{k=0}^{n} W_k$ for $n = 0, 1, \ldots$, adopting the convention that an empty sum is equal to 0 (i.e., $S_{-1} = L_{-1} = 0$).
Now parameterizing the above by $N = 1, 2, \ldots$, and reindexing by $\lfloor Nt \rfloor, \ t \geq 0$, it follows after elementary algebra that

$$Z_{\lfloor Nt \rfloor}^{(N)} = Z_0^{(N)} + S_{\lfloor Nt \rfloor - 1}^{(N)} + \varepsilon_{\lfloor Nt \rfloor}^{(N)} - \alpha \int_0^t Z_{\lfloor Ns \rfloor}^{(N)} \mathrm{d}s + L_{\lfloor Nt \rfloor - 1}^{(N)}, \ t \geq 0,$$

where

$$\varepsilon_{\lfloor Nt \rfloor}^{(N)} := \alpha \left(\int_0^t Z_{\lfloor Ns \rfloor}^{(N)} \mathrm{d}s - \frac{1}{N} \sum_{k=0}^{\lfloor Nt \rfloor - 1} Z_k^{(N)} \right), \ t \geq 0.$$

Now note that

$$L^{(N)} \equiv \{L_{\lfloor Nt \rfloor}^{(N)} - 1, t \geq 0\}$$

is non-decreasing, with $L_0^{(N)} = 0$. Moreover, since $1\{W_n^{(N)} > 0\}1\{Z_{n+1}^{(N)} > 0\} = 0$ for any $n \in \mathbb{N}$, it follows that

$$\int_0^\infty 1\{Z_{\lfloor Ns \rfloor}^{(N)} > 0\} \mathrm{d}L_{\lfloor Ns \rfloor - 1}^{(N)} = 0.$$

Hence, since clearly $Z^{(N)}_{\lfloor Nt\rfloor} \geq 0$ for $t \geq 0$, it follows by (1) and [12, Prop. 2] that we may write

$$Z^{(N)}_{\lfloor N\cdot\rfloor} \quad = \quad \Phi\left(Z^{(N)}_0 + S^{(N)}_{\lfloor N\cdot\rfloor-1} + \varepsilon^{(N)}_{\lfloor N\cdot\rfloor}\right),$$

where the map $\Phi : D([0,\infty),\mathbb{R}) \mapsto D([0,\infty),\mathbb{R})$ is Lipschitz continuous with respect to the Skorokhod-J_1 topology. Hence, by the continuous mapping theorem [3], in order to complete the proof it suffices to show that

$$Z^{(N)}_0 + S^{(N)}_{\lfloor N\cdot\rfloor-1} + \varepsilon^{(N)}_{\lfloor N\cdot\rfloor} \quad \Rightarrow \quad Z^{(\infty)}_0 + \sqrt{v}B$$

as $N \to \infty$, where B is a standard Brownian motion, independent of $Z^{(\infty)}_0$.
By the assumed independence of $Z^{(N)}_0$ and the sequence $\{X^{(N)}_n, n \geq 0\}$ it follows by the functional central limit theorem [3] that

$$Z^{(N)}_0 + S^{(N)}_{\lfloor N\cdot\rfloor-1} \quad \Rightarrow \quad Z^{(\infty)}_0 + \sqrt{v}B$$

as $N \to \infty$. It therefore remains to show that $\varepsilon^{(N)}_{\lfloor N\cdot\rfloor} \Rightarrow 0$ as $N \to \infty$.
To this end, first note that, for $t \geq 0$

$$\int_0^t Z^{(N)}_{\lfloor Ns\rfloor}\mathrm{d}s - \frac{1}{N}\sum_{k=0}^{\lfloor Nt\rfloor-1} Z^{(N)}_k = \frac{1}{N}\left((nt - \lfloor nt\rfloor)Z^{(N)}_{\lfloor Nt\rfloor} + (\lceil nt\rceil - nt)Z^{(N)}_{\lfloor Nt\rfloor-1}\right),$$

where we set $Z^{(N)}_{-1} = 0$. Thus for fixed $T \geq 0$, we obtain that

$$\sup_{0\leq t\leq T}\left|\varepsilon^{(N)}_{\lfloor Nt\rfloor}\right| \leq \frac{2\alpha}{N}\sup_{0\leq k\leq\lfloor Nt\rfloor} Z^{(N)}_k.$$

However, note that by (1) we have the bound

$$Z^{(N)}_{n+1} \leq a_N Z^{(N)}_n + \left|X^{(N)}_n\right|,\ n = 0, 1, \ldots,$$

from which we obtain that

$$Z^{(N)}_{n+1} \leq \sum_{k=0}^{n} a_N^k \left|X^{(N)}_{n-k}\right| + a_N^{n+1} Z^{(N)}_0.$$

However, as $a_N^{\lfloor Nt\rfloor} \to e^{-\alpha t}$ as $N \to \infty$, it follows that for sufficiently large N,

$$\frac{2\alpha}{N}\sup_{0\leq k\leq\lfloor NT\rfloor} Z^{(N)}_k \leq e^{\min(\gamma,0)T}\,\frac{2\alpha}{N}\left(\sum_{k=0}^{\lfloor NT\rfloor}\left|X^{(N)}_k\right| + Z^{(N)}_0\right) \Rightarrow 0$$

as $N \to \infty$, since $\mathbb{E}\,X^{(N)}_k = \gamma/\sqrt{N}$. We conclude that

$$\sup_{0\leq t\leq T}|\varepsilon^{(N)}_{\lfloor Nt\rfloor}| \Rightarrow 0$$

as $N \to \infty$, which implies that $\varepsilon^{(N)}_{\lfloor N\cdot\rfloor} \Rightarrow 0$ as $N \to \infty$, which completes the proof. □

3.2 Stationary convergence

In this section we consider the steady-state workload Z_∞ under the heavy-traffic scaling; to stress the dependence on N, we write $Z_\infty^{(N)}$. As before, the increments $X_n^{(N)}$, are such that $\mathbb{E}\, X_n^{(N)} = \gamma/\sqrt{N}$ for some $\gamma \in \mathbb{R}$, and $\mathbb{V}\mathrm{ar}\, X_n^{(N)} = v/N > 0$; here $X_n^{(N)} := Y_n^{(N)} - B_n$, with as before B_n being i.i.d. exponentially distributed with mean λ^{-1}. To ensure the existence of a stationary distribution, we assume $\alpha > 0$.

The main claim of this section is the following.

Proposition 3.3 *As $N \to \infty$, $Z_\infty^{(N)}/\sqrt{N}$ converges in distribution to a Normal random variable with mean γ/α and variance $v/(2\alpha)$, conditioned on being positive.*

Proof. We establish this result by considering, under the scaling introduced above, the behavior of the Laplace transform $\varphi_Z(s/\sqrt{N})$ as $N \to \infty$, as identified in Theorem 2.7. We here write $\varphi_Z(s/\sqrt{N}) := T_1(s) - T_1(s)T_2(s)$, with

$$T_1(s) \equiv T_1^{(N)}(s) := \prod_{k=0}^{\infty} \varphi_{X^{(N)}} \left(\frac{s a_N^k}{\sqrt{N}} \right), \quad T_2(s) \equiv T_2^{(N)}(s) := \xi(N) \sum_{\ell=0}^{\infty} \frac{\eta_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta_{N,\ell}(s),$$

where

$$\Delta_{N,\ell}(s) := \prod_{k=\ell+1}^{\infty} \zeta_N\left(\eta_{N,k}(s)\right), \quad \eta_{N,\ell}(s) := \frac{s a_N^\ell}{\sqrt{N}}, \quad \zeta_N(s) = \frac{1}{\varphi_{X^{(N)}}(s)};$$

in addition, with $\varphi_Z(\lambda a_N)$ following by normalization,

$$\xi(N) := \frac{\varphi_Y(\lambda)\varphi_Z(\lambda a_N)}{\lambda}.$$

The proof of the asymptotic Normality consists of three steps.

STEP 1. We first study the asymptotic behavior of $T_1^{(N)}(s)$, i.e.,

$$\prod_{k=0}^{\infty} \varphi_{X^{(N)}} \left(\frac{s}{\sqrt{N}} \left(1 - \frac{\alpha}{N}\right)^k \right) = \prod_{k=0}^{\infty} \mathbb{E} \exp\left(-\frac{s}{\sqrt{N}} \left(1 - \frac{\alpha}{N}\right)^k X_k^{(N)} \right)$$

as N grows large. To this end, we first write $X_k^{(N)} := \bar{X}_k^{(N)} + \gamma/\sqrt{N}$, where $\mathbb{E}\bar{X}_k^{(N)} = 0$, so that the expression in the previous display reads

$$\prod_{k=0}^{\infty} \mathbb{E} \exp\left(-\frac{s}{\sqrt{N}} \left(1 - \frac{\alpha}{N}\right)^k \bar{X}_k^{(N)} \right) \exp\left(-\sum_{k=0}^{\infty} \frac{s}{\sqrt{N}} \left(1 - \frac{\alpha}{N}\right)^k \frac{\gamma}{\sqrt{N}} \right).$$

It is immediate that

$$\sum_{k=0}^{\infty} \frac{s}{\sqrt{N}} \left(1 - \frac{\alpha}{N}\right)^k \frac{\gamma}{\sqrt{N}} \to \frac{s\gamma}{\alpha},$$

as $N \to \infty$. In addition,

$$\log \prod_{k=0}^{\infty} \mathbb{E} \exp\left(-\frac{s}{\sqrt{N}}\left(1-\frac{\alpha}{N}\right)^k \bar{X}_k^{(N)}\right)$$
$$= \sum_{k=0}^{\infty} \log\left(1+\frac{s^2}{2N}\left(1-\frac{\alpha}{N}\right)^{2k} v - \frac{s^3}{6N\sqrt{N}}\left(1-\frac{\alpha}{N}\right)^{3k} w + \cdots\right),$$

with v as defined above, and w some constant. Observe that, as $N \to \infty$,

$$\sum_{k=0}^{\infty} \frac{s^2}{2N}\left(1-\frac{\alpha}{N}\right)^{2k} v \to \frac{s^2 v}{4\alpha}, \quad \sum_{k=0}^{\infty} \frac{s^3}{6N\sqrt{N}}\left(1-\frac{\alpha}{N}\right)^{3k} w \to 0.$$

We conclude that, as $N \to \infty$,

$$\prod_{k=0}^{\infty} \varphi_{X^{(N)}}\left(\frac{s}{\sqrt{N}}\left(1-\frac{\alpha}{N}\right)^k\right) \to \exp\left(-\frac{s\gamma}{\alpha}+\frac{s^2 v}{4\alpha}\right) =: \Gamma(s),$$

corresponding to the Laplace transform of a Normal density with mean γ/α and variance $v/(2\alpha)$.

STEP 2. Our aim is to prove convergence to the Laplace transform of a Normal random variable with mean γ/α and variance $v/(2\alpha)$, conditioned on being positive. The numerator of this expression can be written as

$$\psi(s) := \int_0^\infty e^{-sx} \frac{1}{\sqrt{\pi v/\alpha}} \exp\left(-\frac{(x-\gamma/\alpha)^2}{v/\alpha}\right) \mathrm{d}x,$$

whereas the denominator equals $\psi(0)$. It is a matter of a direct computation to verify that

$$\psi(s) = \Gamma(s) \int_0^\infty \frac{1}{\sqrt{\pi v/\alpha}} \exp\left(-\frac{(x-(\gamma-\frac{1}{2}sv)/\alpha)^2}{v/\alpha}\right) \mathrm{d}x.$$

With self-evident notation, we can conclude that this limiting Laplace transform can be interpreted as

$$\psi(s) = \Gamma(s)\, \mathbb{P}\left(\mathcal{N}\left(\frac{\gamma-\frac{1}{2}sv}{\alpha}, \frac{v}{2\alpha}\right) > 0\right).$$

In the first step we have already established that $T_1^{(N)}(s)$ converges to $\Gamma(s)$ as $N \to \infty$. Recalling the definitions of $T_1^{(N)}(s)$ and $T_2^{(N)}(s)$, it is now directly seen that it is left to show that

$$T_2^{(N)}(s) = \xi(N) \sum_{\ell=0}^{\infty} \frac{\eta_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta_N(s) \to \frac{\mathbb{P}\left(\mathcal{N}\left(0, \frac{v}{2\alpha}\right) \in \left[-\frac{\gamma}{\alpha}, -\frac{\gamma-\frac{1}{2}sv}{\alpha}\right]\right)}{\mathbb{P}\left(\mathcal{N}\left(0, \frac{v}{2\alpha}\right) > -\frac{\gamma}{\alpha}\right)}. \tag{25}$$

STEP 3. We prove (25) by first showing that, for a function $\bar{\xi}(N)$, as $n \to \infty$,

$$\bar{\xi}(N) \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{\ell=0}^{\infty} \frac{\eta_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta_{N,\ell}(s) \right) \to f_{\mathcal{N}(0,v/\alpha)} \left(-\frac{\gamma - sv}{\alpha} \right), \tag{26}$$

with $f_{\mathcal{N}(\mu,\sigma^2)}(\cdot)$ denoting the density of a Normal random variable with mean μ and variance σ^2. To this end, we observe that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \frac{\eta_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta_{N,\ell}(s) \quad = \quad & \frac{\eta'_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta_{n,\ell}(s) - \frac{\eta_{N,\ell}(s)}{\varphi_Y^2\left(\eta_{N,\ell}(s)\right)} \varphi'_Y(\eta_{N,\ell}(s)) \eta'_{N,\ell}(s) \Delta_{N,\ell}(s) \\ & + \frac{\eta_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta'_{N,\ell}(s). \end{aligned} \tag{27}$$

Let us consider the three terms in the right hand side separately. Observe that, relying on computations similar to the ones used when proving $T_1^{(N)}(s) \to \Gamma(s)$,

$$\Delta_{N,\ell}(s) \sim \exp\left(\frac{s\gamma}{\alpha} \left(1 - \frac{\alpha}{N}\right)^{\ell+1} - \frac{s^2 v}{4\alpha} \left(1 - \frac{\alpha}{N}\right)^{2(\ell+1)} \right) \to \Delta(s) := \exp\left(\frac{s\gamma}{\alpha} - \frac{s^2 v}{4\alpha} \right).$$

Furthermore,

$$\Delta'_{N,\ell}(s) = \left(\sum_{k=\ell+1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s} \zeta_N\left(\eta_{N,k}(s)\right) \Big/ \zeta_N\left(\eta_{N,k}(s)\right) \right) \Delta_{n,\ell}(s).$$

Based on the above, it is now readily verified that the first term in (27) is proportional to $1/\sqrt{N}$, while the others behave like $1/N$. As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{\ell=0}^{\infty} \frac{\eta_{N,\ell}(s)}{\varphi_Y\left(\eta_{N,\ell}(s)\right)} \Delta_{N,\ell}(s) \right) \sim \sum_{\ell=0}^{\infty} \frac{1}{\sqrt{N}} \left(1 - \frac{\alpha}{N}\right)^{\ell} \Delta(s) \sim \frac{\sqrt{N}}{\alpha} \Delta(s).$$

Observe that we have established (26), as $\Delta(s)$ is proportional to the desired density. Due to Scheffé's lemma [17], it now follows that, for some function $\bar{\xi}(N)$ and some constant κ, as $N \to \infty$,

$$T_2^{(N)}(s) \sim \frac{1}{\bar{\xi}(N)} \left(\mathbb{P}\left(\mathcal{N}\left(0, \frac{v}{2\alpha}\right) < -\frac{\gamma - \frac{1}{2}sv}{\alpha} \right) - \kappa \right).$$

Now realize that

$$\frac{T_1^{(N)}(s) - \mathbb{E}\, \mathrm{e}^{-sZ_\infty^{(N)}/\sqrt{N}}}{T_1^{(N)}(s)} = T_2^{(N)}(s). \tag{28}$$

- As $\mathbb{E}\exp(-sZ_\infty^{(N)}/\sqrt{N})$ is the Laplace transform of a nonnegative random variable, it tends to 0 as $s \to \infty$. Because of the fact that $T_1^{(N)}(s) \to \Gamma(s)$, in combination with $\Gamma(s) \to \infty$ as $s \to \infty$, it follows from (28) that $\bar{\xi}(N) \to 1 - \kappa$ as $N \to \infty$.
- Due to $T_1^{(N)}(0) = 1$, the left hand side of (28) equals 0 when $s = 0$, and hence

$$\kappa = \mathbb{P}\left(\mathcal{N}\left(0, \frac{v}{2\alpha}\right) < -\frac{\gamma}{\alpha} \right).$$

Conclude that (25) follows. □

4 Connection with first passage times of AR(1) processes

In this section we show that the transient distribution of the reflected AR(1) process can be translated into distributional properties of first passage times across a geometric barrier of an associated *non-reflected* AR(1) process. More precisely, we show that the probability that Z_n exceeds some constant (given that $Z_0 = 0$) coincides with the probability that an associated non-reflected AR(1) process exceeds a geometric barrier.

The stated connection may be obtained as follows. First note that recursively applying (1) we arrive at (an empty sum being zero)

$$Z_{n+1} = \max\left\{a^{n+1}Z_0 + \sum_{j=0}^{n} a^{n-j}X_j, \max_{1\leq k\leq n+1}\sum_{j=k}^{n} a^{n-j}X_j\right\}, \; n = 0, 1, \ldots$$

However, observe that the above implies the equality in distribution

$$Z_{n+1} \stackrel{\mathrm{d}}{=} \max\left\{a^{n+1}Z_0 + \sum_{j=0}^{n} a^{j}X_j, \max_{-1\leq k\leq n-1}\sum_{j=0}^{k} a^{j}X_j\right\}, \; n = 0, 1, \ldots$$

Now reindexing time, the above also yields

$$Z_{n} \stackrel{\mathrm{d}}{=} \max\left\{a^{n}Z_0 + a^{-1}\sum_{j=1}^{n} a^{j}X_j, \max_{0\leq k\leq n-1} a^{-1}\sum_{j=1}^{k} a^{j}X_j\right\}, \; n = 1, 2, \ldots$$

From now on suppose that $Z_0 = 0$, in which case the above reduces to, for $n = 1, 2, \ldots,$

$$Z_n \stackrel{\mathrm{d}}{=} \max_{0\leq k\leq n} a^{-1}\sum_{j=1}^{k} a^{j}X_j \stackrel{\mathrm{d}}{=} \max_{0\leq k\leq n} a^{k-1}\sum_{j=1}^{k}\left(\frac{1}{a}\right)^{k-j}X_j.$$

However, for any $v \geq 0$, we clearly have that the following two events are equivalent:

$$\left\{\max_{0\leq k\leq n} a^{k-1}\sum_{j=1}^{k}\left(\frac{1}{a}\right)^{k-j}X_j \geq v\right\} = \left\{\inf\left\{k \geq 1 : \sum_{j=1}^{k}\left(\frac{1}{a}\right)^{k-j}X_j \geq \left(\frac{1}{a}\right)^{k-1}v\right\} \leq n\right\}$$

and as a consequence

$$\mathbb{P}(Z_n \geq v|Z_0 = 0) = \mathbb{P}\left(\inf\left\{k \geq 1 : \sum_{j=1}^{k}\left(\frac{1}{a}\right)^{k-j}X_j \geq \left(\frac{1}{a}\right)^{k-1}v\right\} \leq n\right).$$

Note that the process

$$\left(\sum_{j=1}^{k}\left(\frac{1}{a}\right)^{k-j}X_j\right)_k$$

corresponds to the unreflected AR(1) process $V_{n+1} = (1/a)V_n + X_n$ with $V_0 = 0$. We have thus found an interpretation of the distribution of Z_n in terms of the first passage time of an unreflected AR(1) process across a geometric barrier. For $a = 1$ we recover a well-known distributional identity: the waiting time of the n-th customer in a GI/G/1 queue (given it starts empty at time 0) has the same law as the running maximum (after n increments) of the unreflected process.

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