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A blood bank model with perishable blood and demand impatience

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Abstract: We consider a stochastic model for a blood bank, in which amounts of blood are offered and demanded according to independent compound Poisson processes. Blood is perishable, i.e., blood can only be kept in storage for a limited amount of time. Furthermore, demand for blood is impatient, i.e., a demand for blood may be cancelled if it cannot be satisfied soon enough. For a range of perishability functions and demand impatience functions, we derive the steady-state distributions of the amount of blood X_b kept in storage, and of the amount of demand for blood X_d (at any point in time, at most one of these quantities is positive). Under certain conditions we also obtain the fluid and diffusion limits of the blood inventory process, showing in particular that the diffusion limit process is an Ornstein-Uhlenbeck process.

1. Introduction

This paper is devoted to the study of a stochastic blood bank model in which amounts of blood are offered and demanded according to stochastic processes, and in which blood is perishable (i.e., blood can only be kept for a limited amount of time) and demand for blood is impatient (i.e., a demand for blood may be canceled if it cannot be satisfied soon enough). Let us first provide some background, and subsequently sketch the blood bank model in some more detail.

One of the major issues in securing blood supply to patients worldwide is to provide blood of the best achievable quality, in the needed quantities. In most countries blood, which is collected as whole blood units from human donors, is separated into different components which are subsequently stored under different storage conditions according to their biological characteristics, functions and respective expiration dates. Blood units and components are ordered by local hospital blood banks (LBB) from the Central Blood Bank (CBB) respectively) according to their operational needs. The CBB has to run its inventory and supply according to these requests and to the need to keep sufficient stock for immediate release in emergency situations. It also has to perform

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tests to determine the unit's blood type and to detect the presence of various pathogens which are able to cause transfusion-transmitted diseases, such as Hepatitis B (HBV), Hepatitis C (HCV), Human Immunodeficiency Virus (HIV) and Syphilis; see, e.g., Steiner et al. [16].

Blood consists of several components: Red Blood Cells (RBC), plasma and platelets. In addition there are 8 blood groups (types): O^+, O^-, A^+, A^-, B^+ ,B^-, AB^+, AB^- ($-$ means Rh negative) where the interrelationship between the transfusion issuing policies among the 8 types is quite intricate. It turns out that each of the negative types can satisfy the corresponding $+$ type, but not vice versa. Blood components are perishable as RBC can be used for only 35 to 42 days and platelets for only 5 days (plasma, however, can be frozen and kept for one year). Accordingly, if RBC and particularly platelets are not used for blood transfusion within their expiration dates then they perish.

In most developed countries demand requirements of about $50,000$ blood donations are needed per one million persons per year. About 95% of these donations are aggregated by CBB's and the remaining 5% by LBB's. Blood units stored at the CBB are usually ordered by LBB's for planned elective surgeries. However, as it happens rather frequently, elective surgeries turn out to become emergency ones due to various conditions of the patient involved. In such cases, hospitals use their own local blood banks to supply the demand, and they cancel the required demand from the CBB; this is what we refer to as demand impatience. A good review on supply chain management appears in Beliën and Forcé [4] and the references cited there in. Other relevant studies are Ghandforoush and Sen [9] and Stanger et al. [15].

In this paper we consider the analysis of blood perishability and demand impatience, concentrating on only one blood type. We do this by considering the stochastic inventory processes $\{X_b(t), t \geq 0\}$, with $X_b(t)$ the amount of blood kept in storage at time t, and $\{X_d(t), t \geq 0\}$, with $X_d(t)$ the amount of demand for blood (the shortage) at time t. If $X_b(t) > 0$ then $X_d(t) = 0$, and if $X_d(t) > 0$ then $X_b(t) = 0$. We assume that amounts of blood arrive according to a Poisson process, and that requests for blood arrive according to another, independent, Poisson process. The delivered and requested amounts of blood are assumed to be random variables. We represent the perishability of blood by letting the amount of blood, when positive, decrease in a state-dependent way: if the amount is v, then the decrement rate is $\xi_b v + \alpha_b$. The ξ_b factor is motivated by the fact that a large amount of blood present suggests that some of the blood has been present for quite a while – and hence there is a relatively high perishability rate when much blood is in inventory. The α_b factor provides additional modelling flexibility. One can in this way represent the blood perishability more accurately; but the α_b term could also, e.g., represent a fluid demand rate of individuals or organizations, which contact the CBB directly, and that is only satisfied when there is inventory. Similarly, we represent the demand impatience by a decrement rate $\xi_d v + \alpha_d$. The ξ_d factor is motivated by the following fact. When there is a large shortage (demand) of blood, there are probably many patients waiting for blood, so many patients that might become impatient (i.e., they could recover, or die, or become in need of emergency surgery) leading to a cancellation of the required demand from the CBB. Again, the α_d factor provides additional modelling flexibility; it not only allows us to represent demand impatience more accurately, but it could also, e.g., represent additional donations of individuals in times of blood shortage. To keep things simple, in the remainder of the paper, we shall refer to the $\xi_b v + \alpha_b$ term as the blood perishability rate, and to the $\xi_d v + \alpha_d$ term as the demand impatience rate.

The inclusion of both the perishability factor $\xi_b v + \alpha_b$ and the demand impatience factor $\xi_d v + \alpha_d$ makes the analysis of the ensuing model mathematically quite challenging, but leads to a very general model that contains many well-known models as special cases. Our two-sided stochastic process, with both jumps upward and jumps downward, and with the rather general slope factors $\xi v + \alpha$, could represent a quite large class of stochastic phenomena. It should for example be noted that this model is a two-sided generalization of the well-known shot-noise model that describes certain physical phenomena (cf. [10]). In some of our calculations we remove either the ξ factors or the α factors, and this results in easier calculations and more explicit results.

Our main results are: (i) Determination of the steady-state distributions of the amounts of blood and of demand in inventory; in particular, we present a detailed analysis of the case in which the delivered and requested amounts of blood are both exponentially distributed. (ii) Expressions for mean amounts of blood and demand in storage, and for the probability of not being able to satisfy demand. (iii) We obtain the fluid and diffusion limits of the blood inventory process, showing in particular that the diffusion limit process is an Ornstein-Uhlenbeck process.

The paper is organized as follows: Section 2 presents a detailed model description. A global analysis of the densities of demand and of blood amount in storage is contained in Section 3. A detailed analysis of the case of exponentially distributed delivered and requested blood amounts, when $\alpha_b = \alpha_d = 0$ (i.e., pure proportionality) is provided in Section 4, while Section 5 considers the case of positive α_b and α_d. Section 6 is devoted to the case $\xi_b = \xi_d = 0$. The fluid and diffusion scalings are discussed in Section 7, and in Section 8 we present numerical results for certain performance measures like mean net amount of blood and the probability that there is a shortage of blood. These results indicate, a.o., that the probability that there is a shortage of blood can be accurately approximated via a Normal approximation, based on the Ornstein-Uhlenbeck process appearance in the diffusion scaling. Section 9 contains some conclusions and suggestions for further research.

2. Model description

We consider the following highly simplified model of a blood bank, restricting ourselves to only one type of blood.

Blood amounts arrive according to a Poisson process with rate λ_b. The amounts which successively arrive are independent, identically distributed random variables $B_1, B_2, \dots$ with distribution $F_b(\cdot)$; $\bar{F}_b(x) = 1 - F_b(x)$.

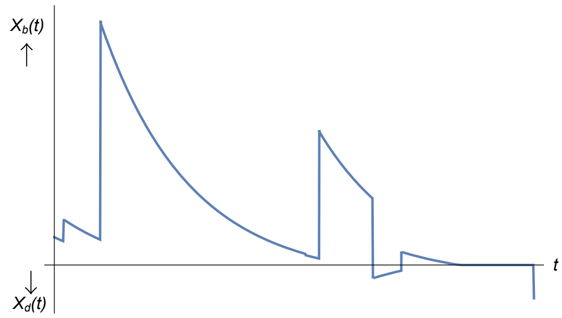


Fig 1: Sample path of net amount of blood available as a function of time.

Demands for blood arrive according to a Poisson process with rate λ_d. The successive demand amounts are independent, identically distributed random variables $D_1, D_2, \ldots$ with distribution $F_d(\cdot)$; $\bar{F}_d(x) = 1 - F_d(x)$. We view these amounts as continuous quantities, measured in, e.g., liters.
If there is enough blood for a demand, then that demand is immediately satisfied. If there is some blood, but not enough to fully satisfy a demand, then that demand is partially satisfied, using all the available blood; the remainder of the demand may be satisfied later.
Blood has a finite expiration date. We make the assumption that if the total amount of blood present is $x > 0$, then blood is discarded – because of its finite expiration date – at a rate $\xi_b x + \alpha_b$, so linear in x.
Blood demands have a finite patience. We make the assumption that if the total amount of demand present is $x > 0$, then demand disappears – because of its finite patience – at a rate $\xi_d x + \alpha_d$, so linear in x.

Notice that *either* the total amount of blood present, *or* the total amount of demands, is zero, *or* both are zero; they cannot be both positive. Hence we can easily in one figure depict the two-sided process $\{X(t), t \geq 0\} = \{(X_b(t), X_d(t)), t \geq 0\}$ of total blood and total demand amounts present at any time t, as we have done in Figure 1. For our purposes, we are mainly interested in the characteristics of the process described above in stationarity. Let us denote by X_d the steady-state total amount of demand and by X_b the steady-state total amount of blood present, with corresponding density functions $f(\cdot)$ and $g(\cdot)$, respectively. Notice that these are defective densities; we have $\int_{0+}^{\infty} f(v)\mathrm{d}v = \pi_d = \mathbb{P}(\text{demand} > 0)$ and $\int_{0+}^{\infty} g(v)\mathrm{d}v = \pi_b = \mathbb{P}(\text{blood} > 0)$. If $\alpha_b = \alpha_d = 0$, then neither X_b nor X_d has probability mass at zero, and $\pi_b + \pi_d = 1$ (when there is only a very small amount x present, the "decay" rate $\xi_b x$ or $\xi_d x$ is very small). However, if α_b and/or α_d is positive, then there is a positive probability π_0 of being in 0.

When ξ_d and ξ_b are positive, existence of these steady-state densities is obvious; otherwise, the conditions for the existence of the steady-state distributions requires some discussion, cf. Section 6. In the next section we try to determine

$f(\cdot)$ and $g(\cdot)$, using the LCT (Level Crossing Technique).

3. Analysis of the densities of demand and of blood amount

In this section we present a global approach towards determining $f(\cdot)$ and $g(\cdot)$ in the most general form of our model. At the end of the section, we also consider a few other important performance measures, viz., π_0 and the mean length of time during which, uninterruptedly, there is a positive amount of blood (resp. demand). The latter could be viewed as the busy period of the X_b process (resp. of the X_d process).

First we consider the density $g(\cdot)$ of the amount of blood. We equate the rate at which some positive blood level v is upcrossed and downcrossed, respectively. LCT leads to the following integral equation: for $v > 0$,

$$\begin{aligned} & \lambda_b \int_0^v g(y)\bar{F}_b(v-y)\mathrm{d}y + \lambda_b \int_0^\infty f(y)\bar{F}_b(v+y)\mathrm{d}y + \pi_o\lambda_b\bar{F}_b(v) \\ = \quad & \lambda_d \int_v^\infty g(y)\bar{F}_d(y-v)\mathrm{d}y + (\xi_b v + \alpha_b)g(v). \end{aligned} \tag{3.1}$$

Here the three terms in the lefthand side represent the rate of crossing level v from below; the first term corresponds to a jump from a blood inventory level between 0 and v, whereas the second term corresponds to a jump from a shortage level, and the third term corresponds to a jump from level 0. The two terms in the righthand side represent the rate of crossing level v from above; the first term corresponds to a jump from above v, and the second term to a smooth crossing.

Next we consider the density $f(\cdot)$ of the amount of demand (shortage). We equate the rate at which some positive demand level v is upcrossed and downcrossed, respectively. LCT leads to the following integral equation: for $v > 0$,

$$\begin{aligned} & \lambda_d \int_0^v f(y)\bar{F}_d(v-y)\mathrm{d}y + \lambda_d \int_0^\infty g(y)\bar{F}_d(v+y)\mathrm{d}y + \pi_0\lambda_d\bar{F}_d(v) \\ = \quad & \lambda_b \int_v^\infty f(y)\bar{F}_b(y-v)\mathrm{d}y + (\xi_d v + \alpha_d)f(v). \end{aligned} \tag{3.2}$$

It should be noted that these two, coupled, equations are symmetric (swap f and g, and the b and d parameters).

Let us introduce the following Laplace Transforms (LT):

$$\phi(s) := \int_0^\infty \mathrm{e}^{-sy} f(y)\mathrm{d}y, \tag{3.3}$$

$$\gamma(s) := \int_0^\infty \mathrm{e}^{-sy} g(y)\mathrm{d}y. \tag{3.4}$$

We now take Laplace transforms in (3.2), considering its five terms and calling them T_1, T_2, T_3, T_4 and T_5, successively. We have:

$$T_1 = \lambda_d \int_{v=0}^\infty \mathrm{e}^{-sv} \int_{y=0}^v f(y)\bar{F}_d(v-y)\mathrm{d}y\mathrm{d}v = \lambda_d\phi(s)\frac{1-\mathbb{E}[\mathrm{e}^{-sD}]}{s}, \tag{3.5}$$

$$T_2 = \lambda_d \int_{v=0}^{\infty} \mathrm{e}^{-sv} \int_{y=0}^{\infty} g(y)\bar{F}_d(v+y)\mathrm{d}y\mathrm{d}v$$
$$= \lambda_d \int_{y=0}^{\infty} \mathrm{e}^{sy} g(y) \int_{z=y}^{\infty} \mathrm{e}^{-sz}\bar{F}_d(z)\mathrm{d}z\mathrm{d}y, \tag{3.6}$$

$$T_3 = \pi_0\lambda_d \int_{0}^{\infty} \mathrm{e}^{-sy}\bar{F}_d(y)\mathrm{d}y, \tag{3.7}$$

$$T_4 = \lambda_b \int_{v=0}^{\infty} \mathrm{e}^{-sv} \int_{y=v}^{\infty} f(y)\bar{F}_b(y-v)\mathrm{d}y\mathrm{d}v$$
$$= \lambda_b \int_{y=0}^{\infty} \mathrm{e}^{-sy} f(y) \int_{z=0}^{y} \mathrm{e}^{sz}\bar{F}_b(z)\mathrm{d}z\mathrm{d}y, \tag{3.8}$$

$$T_5 = \xi_d \int_{v=0}^{\infty} v\mathrm{e}^{-sv} f(v)\mathrm{d}v + \alpha_d\phi(s) = -\xi_d\phi'(s) + \alpha_d\phi(s). \tag{3.9}$$

3.1. The case of Coxian jumps

We shall now analyse these equations for the case that $F_b(\cdot)$ and $F_d(\cdot)$ are Coxian distributions. More specifically, if X_i, $i = 1, 2, \dots, K$ are independent, exponentially distributed random variables, and $\mathbb{E}[X_i] = \frac{1}{\beta_i}$, $i = 1, 2, \dots, K$, then a Coxian amount of blood B can be represented as:

$$B = \sum_{j=1}^{i} X_j \text{ with probability } p_i \prod_{j=1}^{i-1}(1-p_j), \quad i = 1, 2, \dots, K. \tag{3.10}$$

The class of Coxian distributions lies dense in the class of all distributions of nonnegative random variables (cf. [2], Section III.4); hence one can approximate $F_b(\cdot)$ arbitrarily closely by a Coxian distribution. In the above case, it is easily verified that one can represent $\bar{F}_b(x)$ as follows:

$$\bar{F}_b(x) = \mathbb{P}(B > x) = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1}(1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j}\mathrm{e}^{-\beta_j x}, \tag{3.11}$$

if all β_j are different. If two β_j coincide, then a term with $x\mathrm{e}^{-\beta_j x}$ (Erlang-2) must be added. We leave this to the reader, but in Remark 2 below we outline how Erlang terms can be handled in solving the integral equations (3.2) and (3.1).

Taking Laplace transforms in those integral equations, we shall encounter the following terms:

$$\int_{z=0}^{y} \mathrm{e}^{sz}\bar{F}_b(z)\mathrm{d}z = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1}(1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j}\frac{1}{\beta_j - s}(1 - \mathrm{e}^{(s-\beta_j)y}), \tag{3.12}$$

$$\int_{z=y}^{\infty} \mathrm{e}^{-sz} \bar{F}_b(z) \mathrm{d}z = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j + s} \mathrm{e}^{-(s+\beta_j)y}, \tag{3.13}$$

$$\mathbb{E}[\mathrm{e}^{-sB}] = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{\beta_j}{\beta_j + s}, \tag{3.14}$$

and hence

$$\frac{1-\mathbb{E}[\mathrm{e}^{-sB}]}{s} = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j + s}. \tag{3.15}$$

Their counterparts for the case that $F_d(\cdot)$ is Coxian, being given by

$$\bar{F}_d(x) = \mathbb{P}(D > x) = \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1} (1-q_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \mathrm{e}^{-\delta_j x}, \tag{3.16}$$

are readily obtained by replacing K by L, p_i by q_i and β_i by δ_i everywhere.

Combining (3.2) with (3.5)-(3.9), and using (3.12) and the counterparts of (3.13) and (3.15) for $\bar{F}_d(\cdot)$, we find:

$$\begin{aligned}
& \lambda_d \phi(s) \sum_{i=1}^{K} q_i \prod_{h=1}^{i-1} (1-q_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \frac{1}{\delta_j + s} \\
+ \quad & \lambda_d \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1} (1-q_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \frac{1}{\delta_j + s} [\gamma(\delta_j) + \pi_0] \\
= \quad & \lambda_b \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j - s} (\phi(s) - \phi(\beta_j)) \\
- \quad & \xi_d \phi'(s) + \alpha_d \phi(s).
\end{aligned} \tag{3.17}$$

When $\xi_d > 0$, this equation can be written in the following form:

$$\phi'(s) = A_H(s)\phi(s) + A_I(s), \tag{3.18}$$

with the homogeneous term $A_H(s)$ being given by

$$\begin{aligned}
A_H(s) \quad := \quad & -\frac{1}{\xi_d} \Bigg[\lambda_d \sum_{i=1}^{K} q_i \prod_{h=1}^{i-1} (1-q_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \frac{1}{\delta_j + s} \\
& - \quad \lambda_b \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j - s} - \alpha_d \Bigg],
\end{aligned} \tag{3.19}$$

and the inhomogeneous term $A_I(s)$ being given by

$$\begin{aligned} A_I(s) \quad &:= \quad -\frac{1}{\xi_d}\Bigg[\lambda_d \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1}(1-q_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l-\delta_j}\frac{1}{\delta_j+s}[\gamma(\delta_j)+\pi_0] \\ &+ \quad \lambda_b \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1}(1-p_h) \sum_{j=1}^{i} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l-\beta_j}\frac{1}{\beta_j-s}\phi(\beta_j)\Bigg]. \qquad (3.20) \end{aligned}$$

The solution of (3.18) is given by the following expression:

$$\phi(s) = \phi(0)\mathrm{e}^{\int_0^s A_H(z)\mathrm{d}z} + \int_0^s A_I(u)\mathrm{e}^{\int_u^s A_H(z)\mathrm{d}z}\mathrm{d}u, \quad s \geq 0. \qquad (3.21)$$

$\gamma(s)$ is given by a mirror expression, where $\phi(0)$ is replaced by $\gamma(0)$ and where $A_H(s)$ and $A_I(s)$ are replaced by expressions in which K and L are swapped, and p and q, and β_i and δ_i. It should be noticed, though, that $\phi(0)$, $\gamma(0)$ and π_0 still have to be determined. Furthermore, it should be noticed that $A_H(s)$ and $A_I(s)$ have singularities at $s = \beta_1, \dots, \beta_K$. These singularities are removable, but handling Formula (3.21) clearly requires some care. Instead of working out the details, we shall in the next section focus on the case of exponentially distributed amounts of blood and demand – so $K = L = 1$. For that case we shall not only work out the solution of the differential equation for $\phi(s)$ in detail, including the determination of the missing constants, but we shall also present a different way of handling equations (3.2) and (3.1), without resorting to Laplace transforms. We refer to [7] for a detailed discussion of first-order linear differential equations with singularities like (3.18); in [7] such equations arise in the study of another two-sided stochastic process, representing a queueing model with additional inventory capacity.

Remark 1. *If $\xi_d = 0$, then $\phi(s)$ is obtained from (3.17) in a standard manner; see also Remark 7 for the special case that $K = 1$.*

Remark 2. *We now outline how (3.13) and (3.14) change when the B_i have an Erlang-$(l+1,\beta)$ distribution, and when the D_i have an Erlang-$(k+1,\delta)$ distribution; (3.12) and (3.15) do not change (but of course $\mathbb{E}[\mathrm{e}^{-sD}]$ changes). Firstly,*

$$\int_{z=0}^{y} \mathrm{e}^{sz}\bar{F}_b(z)\mathrm{d}z = \sum_{j=0}^{l} \frac{\beta^j}{(\beta-s)^{j+1}}\left[1 - \sum_{i=0}^{j} \mathrm{e}^{-(\beta-s)y}\frac{((\beta-s)y)^i}{i!}\right]. \qquad (3.22)$$

Term T_4 now becomes:

$$\begin{aligned} T_4 &= \lambda_b \int_{v=0}^{\infty} \mathrm{e}^{-sv} \int_{y=v}^{\infty} f(y)\bar{F}_b(y-v)\mathrm{d}y\mathrm{d}v \\ &= \lambda_b \sum_{j=0}^{l} \frac{\beta^j}{(\beta-s)^{j+1}}\left[\phi(s) - \sum_{i=0}^{j}\frac{(\beta-s)^i}{i!}\int_{y=0}^{\infty} y^i \mathrm{e}^{-\beta y} f(y)\mathrm{d}y\right]. \qquad (3.23) \end{aligned}$$

It should be noted that $s = \beta$ *is a removable singularity. E.g., for* $l = 0$ *one has* $T_4 = \lambda_b \frac{\phi(s)-\phi(\beta)}{\beta - s}$.
Secondly,

$$\int_{z=y}^{\infty} \mathrm{e}^{-sz} \bar{F}_b(z) \mathrm{d}z = \sum_{j=0}^{k} \frac{\delta^j}{(s+\delta)^{j+1}} \sum_{i=0}^{j} \mathrm{e}^{-(s+\delta)y} \frac{((s+\delta)y)^i}{i!}. \tag{3.24}$$

Term T_2 *now becomes:*

$$\begin{aligned} T_2 &= \lambda_d \int_{v=0}^{\infty} \mathrm{e}^{-sv} \int_{y=0}^{\infty} g(y) \bar{F}_d(v+y) \mathrm{d}y \, \mathrm{d}v \\ &= \lambda_d \sum_{j=0}^{k} \frac{\delta^j}{(s+\delta)^{j+1}} \sum_{i=0}^{j} \frac{(s+\delta)^i}{i!} \int_{y=0}^{\infty} y^i \mathrm{e}^{-\delta y} g(y) \mathrm{d}y. \end{aligned} \tag{3.25}$$

It is readily seen that the resulting counterpart of (3.17) can again be written in the form (3.18), and hence the solution is formally still given by (3.21).

3.2. *A few simple performance measures*

In this subsection we first relate π_0 to the densities $f(\cdot)$ and $g(\cdot)$ (cf. Proposition 1), and subsequently we express the mean length of time during which there is, uninterruptedly, a positive amount of blood present, into $f(\cdot)$, $g(\cdot)$ and π_0; we do the same for the mean length of time during which there is, uninterruptedly, a positive demand (Proposition 2).

Proposition 1. *Let* π_0 *be the steady-state atom probability of the zero period. Then*

$$\pi_0 = \frac{\alpha_d f(0) + \alpha_b g(0)}{\lambda_d + \lambda_b}.$$

Proof. Substitute $v = 0$ in (3.1) and (3.2) and take the sum. The result is obtained after several steps of elementary algebra. □

The result introduced in the Proposition above is very intuitive. By LCT $\alpha_d f(0) + \alpha_b g(0)$ is the rate at which level 0 is reached (i.e., the process will now really stay at 0 for a while), so that $[\alpha_d f(0) + \alpha_b g(0)]^{-1}$ is the expected length of time between two successive times level 0 is reached by the fluid. More precisely, the (*zero periods* , *non-zero periods*) generate an alternating renewal process whose expected cycle length is $[\alpha_d f(0) + \alpha_b g(0)]^{-1}$. The expected length of the zero period is $[\lambda_d + \lambda_b]^{-1}$, since the end of the zero period is terminated at the moment of the next jump. But the jump process is a Poisson process with rate $\lambda_d + \lambda_b$. Now the renewal reward theorem simply says that

$$\pi_0 = \frac{E[\text{zero period}]}{E[\text{cycle}]}.$$

In preparation of the next proposition, for the process $\mathbf{X} = \{X(t) : t \geq 0\}$ we define a modified process $\mathbf{X_m} = \{X_m(t) : t \geq 0\}$ where $\mathbf{X_m}$ is constructed by deleting the *zero periods (only the zero periods, not the emptiness periods)* from $\mathbf{X}$ and gluing together the *non-zero periods*. The modified process is $\mathbf{X_m}$ such that $X_m(t) = X_d(t)\mathbf{1}_{\{X_d(t)>0\}} + X_b(t)\mathbf{1}_{\{X_b(t)>0\}}$ where by definition of the model $\{X_d(t) > 0\} \Rightarrow \{X_b(t) = 0\}$ and $\{X_b(t) > 0\} \Rightarrow \{X_d(t) = 0\}$.

Proposition 2. *Let B_b and I_b be the generic non-emptiness period and the emptiness period, respectively, of the inventory system. Similarly, let B_d and I_d be the generic non-emptiness period and the emptiness period, respectively, of the demand process. Then*

$$(i)\begin{cases} \mathbb{E}B_b = \frac{1-\pi_0}{\alpha_b g(0)+\lambda_d \int_0^\infty \bar{F}_d(y)g(y)dy}, \\ \\ \mathbb{E}B_d = \frac{1-\pi_0}{\alpha_d f(0)+\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy}. \\ \end{cases}$$

And

$$(ii)\begin{cases} \mathbb{E}I_b = \frac{1}{\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy+\lambda_b\pi_0} - \mathbb{E}B_b, \\ \mathbb{E}I_d = \frac{1}{\lambda_d \int_0^\infty \bar{F}_d(y)g(y)dy+\lambda_d\pi_0} - \mathbb{E}B_d. \end{cases}$$

Proof. (i) Consider the non-emptiness period of the inventory system. The steady state densities of the inventory system and the demand process of $\mathbf{X_m}$ are given by

$$g_m(x) = \frac{g(x)}{1-\pi_0}, \qquad f_m(x) = \frac{f(x)}{1-\pi_0},$$

respectively. At the end of the non-emptiness period of the inventory system there are two disjoint ways (disjoint events) to downcross level $0+$. Either level 0 is downcrossed by a negative jump or level $0+$ is reached by the fluid reduction (both in $\mathbf{X_m}$). The rate of the first event is $\lambda_d \int_0^\infty \bar{F}_d(y)g_m(y)dy$ and by LCT the rate of the second event is $\alpha_b g_m(0)$. Since the events are disjoint, the rate of downcrossings of level $0+$ is $\lambda_d \int_0^\infty \bar{F}_d(y)g_m(y)dy + \alpha_b g_m(0)$. That means that the expected length of the non-emptiness period is given by

$$\left[\lambda_d \int_0^\infty \bar{F}_d(y)g_m(y)dy + \alpha_b g_m(0)\right]^{-1}. \tag{3.26}$$

Thus

$$\mathbb{E}B_b = \frac{1-\pi_0}{\alpha_b g(0) + \lambda_d \int_0^\infty \bar{F}_d(y)g(y)dy}.$$

Now $\mathbb{E}B_d$ is obtained by complete symmetry.

(ii) Define a *cycle* in the real process $\mathbf{X}$ (not the modified process $\mathbf{X}_m$) as the time between two upcrossings of level $0+$. By definition, the emptiness period plus the non-emptiness period is a cycle in $\mathbf{X}$. That means that the expected length of the emptiness period is the expected length of the cycle minus the expected length of the non-emptiness period. The non-emptiness period in $\mathbf{X}$ and in $\mathbf{X}_m$ are identical and the length of the expected cycle is

$[\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0]^{-1}$, since, $\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0$ is the rate of the upcrossings of level $0+$. We obtain

$$\mathbb{E}I_b + \mathbb{E}B_b = \frac{1}{\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0},$$

yielding $\mathbb{E}I_b$. $\mathbb{E}I_d$ is obtained by symmetry. □

4. The exponential case

We assume in this section that $\bar{F}_b(x) = \mathrm{e}^{-\mu_b x}$ and $\bar{F}_d(x) = \mathrm{e}^{-\mu_d x}$. Moreover, we take $\alpha_b = \alpha_d = 0$. Under these assumptions, we can not only work out (3.21) in detail, but we shall also solve (3.2) and (3.1) without using Laplace transforms. In the next section, we shall allow α_b and α_d to be positive, still requiring exponentially distributed amounts of blood deliveries and demands.

Equations (3.2) and (3.1) reduce to:

$$\lambda_d \int_0^v f(y)\mathrm{e}^{-\mu_d(v-y)}\mathrm{d}y + \lambda_d \mathrm{e}^{-\mu_d v} \int_0^\infty g(y)\mathrm{e}^{-\mu_d y}\mathrm{d}y = \lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d v f(v), \quad v > 0; \tag{4.1}$$

$$\lambda_b \int_0^v g(y)\mathrm{e}^{-\mu_b(v-y)}\mathrm{d}y + \lambda_b \mathrm{e}^{-\mu_b v} \int_0^\infty f(y)\mathrm{e}^{-\mu_b y}\mathrm{d}y = \lambda_d \int_v^\infty g(y)\mathrm{e}^{-\mu_d(y-v)}\mathrm{d}y + \xi_b v g(v), \quad v > 0. \tag{4.2}$$

In addition, the level crossing identity for level $v = 0$ gives the condition (take $v = 0$ in either (4.1) or (4.2)):

$$\lambda_b \int_0^\infty f(y)\mathrm{e}^{-\mu_b y}\mathrm{d}y = \lambda_d \int_0^\infty g(y)\mathrm{e}^{-\mu_d y}\mathrm{d}y. \tag{4.3}$$

Taking LT in (4.1) and (4.2) yields:

$$\lambda_d \frac{\phi(s)}{\mu_d + s} + \lambda_d \frac{\gamma(\mu_d)}{\mu_d + s} = \lambda_b \frac{\phi(s) - \phi(\mu_b)}{\mu_b - s} - \xi_d \phi'(s), \tag{4.4}$$

$$\lambda_b \frac{\gamma(s)}{\mu_b + s} + \lambda_b \frac{\phi(\mu_b)}{\mu_b + s} = \lambda_d \frac{\gamma(s) - \gamma(\mu_d)}{\mu_d - s} - \xi_b \gamma'(s). \tag{4.5}$$

Hence we have two inhomogeneous first order differential equations in the LTs $\phi(s)$ and $\gamma(s)$:

$$\phi'(s) = \phi(s)\left[\frac{\lambda_b}{\xi_d}\frac{1}{\mu_b - s} - \frac{\lambda_d}{\xi_d}\frac{1}{\mu_d + s}\right] - \frac{\lambda_b}{\xi_d}\frac{\phi(\mu_b)}{\mu_b - s} - \frac{\lambda_d}{\xi_d}\frac{\gamma(\mu_d)}{\mu_d + s}, \tag{4.6}$$

$$\gamma'(s) = \gamma(s)\left[\frac{\lambda_d}{\xi_b}\frac{1}{\mu_d - s} - \frac{\lambda_b}{\xi_b}\frac{1}{\mu_b + s}\right] - \frac{\lambda_d}{\xi_b}\frac{\gamma(\mu_d)}{\mu_d - s} - \frac{\lambda_b}{\xi_b}\frac{\phi(\mu_b)}{\mu_b + s}. \tag{4.7}$$

A standard approach yields:

$$\begin{aligned} \phi(s) &= \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\xi_d}} \left(\frac{\mu_d}{\mu_d + s}\right)^{\frac{\lambda_d}{\xi_d}} \Bigg[\phi(0) \\ &- \frac{\lambda_d}{\xi_d}\gamma(\mu_d)\int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_d}} \left(\frac{\mu_d + z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_d} - 1} \frac{\mathrm{d}z}{\mu_d} \\ &- \frac{\lambda_b}{\xi_d}\phi(\mu_b)\int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_d} - 1} \left(\frac{\mu_d + z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_d}} \frac{\mathrm{d}z}{\mu_b} \Bigg]. \end{aligned} \tag{4.8}$$

Similarly,

$$\begin{aligned} \gamma(s) &= \left(\frac{\mu_d}{\mu_d - s}\right)^{\frac{\lambda_d}{\xi_b}} \left(\frac{\mu_b}{\mu_b + s}\right)^{\frac{\lambda_b}{\xi_b}} \Bigg[\gamma(0) \\ &- \frac{\lambda_b}{\xi_b}\phi(\mu_b)\int_0^s \left(\frac{\mu_d - z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_b}} \left(\frac{\mu_b + z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_b} - 1} \frac{\mathrm{d}z}{\mu_b} \\ &- \frac{\lambda_d}{\xi_b}\gamma(\mu_d)\int_0^s \left(\frac{\mu_d - z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_b} - 1} \left(\frac{\mu_b + z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_b}} \frac{\mathrm{d}z}{\mu_d} \Bigg] \end{aligned} \tag{4.9}$$

Notice that the exponents in the above integrals have powers which are larger than -1 (e.g., $\frac{\lambda_d}{\xi_d} - 1$), so that these integrals do not lead to singularities. We still need to determine the two constants $\phi(0) = \pi_d$ and $\gamma(0) = \pi_b$. Together with $\phi(\mu_b)$ and $\gamma(\mu_d)$, we have four unknowns. We determine these unknowns using the following four equations: (i) From (4.3), we get $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$, while (ii) $\pi_d + \pi_b = 1$. Finally, we take (iii) $s = \mu_b$ in (4.8) and (iv) $s = \mu_d$ in (4.9).

Notice that the identity $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$ allows us to reduce the two integrals in (4.8) to one integral (and similarly in (4.9)):

$$\begin{aligned} \phi(s) &= \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\xi_d}} \left(\frac{\mu_d}{\mu_d + s}\right)^{\frac{\lambda_d}{\xi_d}} \Bigg[\phi(0) \\ &- \frac{\lambda_d}{\xi_d}\gamma(\mu_d)\frac{\mu_b + \mu_d}{\mu_b\mu_d}\int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_d} - 1} \left(\frac{\mu_d + z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_d} - 1} \mathrm{d}z \Bigg]. \end{aligned} \tag{4.10}$$

Remark 3. *If $\lambda_b = 0$ then we have a known queueing model or shot-noise model with state-dependent service rate; cf. Keilson & Mermin [10] and Bekker et al. [3] for the so-called shot noise model.*

Remark 4. *The case $\lambda_d = \xi_d$ is special. Formula (4.8) now reduces to*

$$\begin{aligned} \phi(s) &= \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\lambda_d}} \frac{\mu_d}{\mu_d + s} \Big[\phi(0) \\ &- \gamma(\mu_d) \int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\lambda_d}} \frac{\mathrm{d}z}{\mu_d} \\ &- \frac{\lambda_b}{\lambda_d}\phi(\mu_b) \int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\lambda_d}-1} \frac{\mu_d + z}{\mu_d}\frac{\mathrm{d}z}{\mu_b} \Big]. \end{aligned} \tag{4.11}$$

Both integrals are easily evaluated (rewrite, in the last integral, $\mu_d + z = \mu_d + \mu_b - (\mu_b - z)$). We find:

$$\begin{aligned} \phi(s) = &\left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\lambda_d}} \frac{\mu_d}{\mu_d + s} \Big[\phi(0) \\ &+ \frac{\gamma(\mu_d)}{\mu_d}\frac{\lambda_d}{\lambda_b + \lambda_d}\mu_b - \phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d} - \frac{\phi(\mu_b)}{\mu_d}\frac{\lambda_b}{\lambda_b + \lambda_d}\mu_b\Big] \\ &+ \frac{\mu_d}{\mu_d + s}\left[\frac{\gamma(\mu_d)}{\mu_d}\frac{\lambda_d(\mu_b - s)}{\lambda_b + \lambda_d} + \phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d} - \frac{\phi(\mu_b)}{\mu_d}\frac{\lambda_b}{\lambda_b + \lambda_d}(\mu_b - s)\right]. \end{aligned} \tag{4.12}$$

Now observe, cf. (4.3), that $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$. Hence, in both lines of the above formula, two terms cancel. Moreover, $\phi(s)$ should be analytic for $s = \mu_b$, yielding

$$\phi(0) = \phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d}. \tag{4.13}$$

Finally we obtain:

$$\phi(s) = \frac{\mu_d}{\mu_d + s}\phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d} = \phi(0)\frac{\mu_d}{\mu_d + s} = \pi_d\frac{\mu_d}{\mu_d + s}, \tag{4.14}$$

and hence

$$f(x) = \pi_d\mu_d \mathrm{e}^{-\mu_d x}, \quad x > 0; \tag{4.15}$$

the shortage (amount of demand present) appears to be exponentially distributed when $\lambda_d = \xi_d$.
It should be noticed that, if $\lambda_d = \xi_d$, then the first and last term of (4.1) are equal when (4.15) holds; and using (4.3) it is also readily verified that the second and third term of (4.1) are equal. The constant π_d will in general still depend on the parameters $\lambda_d = \xi_d$, λ_b, μ_b and ξ_b.
We end this remark with the observation that in the one-sided shot-noise process (so $\lambda_b = 0$), Bekker et al. [3] also observe that $\lambda_d = \xi_d$ results in an exponential density.

4.1. *A direct approach to the exponential case*

In this subsection we shall determine $f(\cdot)$ *without* using Laplace transforms. Notice that, once $f(\cdot)$ has been determined, $g(\cdot)$ follows by swapping parameters (symmetry).

Differentiate (4.1) w.r.t. v:

$$\lambda_d f(v) - \mu_d \left[\lambda_d \int_0^v f(y)\mathrm{e}^{-\mu_d(v-y)}\mathrm{d}y + \lambda_d \int_0^\infty g(y)\mathrm{e}^{-\mu_d(v+y)}\mathrm{d}y\right]$$
$$= -\lambda_b f(v) + \lambda_b\mu_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d f(v) + \xi_d v f'(v). \quad (4.16)$$

Using (4.1) once more, now to replace the term between square brackets in (4.16), we get:

$$\begin{aligned} \xi_d v f'(v) &= (\lambda_d + \lambda_b - \xi_d) f(v) \\ &- \mu_d \left(\lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d v f(v)\right) \\ &- \mu_b\lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y, \end{aligned} \quad (4.17)$$

and once more differentiating w.r.t. v then gives:

$$\xi_d v f''(v) + \xi_d f'(v) - (\lambda_d + \lambda_b - \xi_d - \mu_d \xi_d v) f'(v)$$
$$= -\mu_d\xi_d f(v) + (\mu_b + \mu_d)\lambda_b f(v) - \mu_b(\mu_b + \mu_d)\lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y. \quad (4.18)$$

The integral that appears in (4.18) can be eliminated by using (4.17), and we thus finally obtain the following second order homogeneous differential equation:

$$\begin{aligned} &\xi_d v f''(v) + (2\xi_d - \lambda_d - \lambda_b + \mu_d\xi_d v - \mu_b\xi_d v)\, f'(v) + \\ &\quad + (\mu_d\xi_d - \mu_b\xi_d - \mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\xi_d v)\, f(v) = 0. \end{aligned} \quad (4.19)$$

As additional equations we have $\int_0^\infty f(v)\mathrm{d}v = \pi_d$, and the continuity condition (4.3); this amounts to a level crossing identity at 0. (4.19) is a known type of second order differential equation.
We can rewrite (4.19) as follows:

$$v f''(v) + (A + Bv) f'(v) + (C + Dv) f(v) = 0, \quad (4.20)$$

where

$$A = 2 - \frac{\lambda_b + \lambda_d}{\xi_d}, \quad B = \mu_d - \mu_b, \quad C = \mu_d - \mu_b + \frac{\lambda_d\mu_b - \lambda_b\mu_d}{\xi_d}, \quad D = -\mu_b\mu_d. \quad (4.21)$$

Note that we divided both sides of equation (4.19) by ξ_d here. We will try to transform the differential equation into one of which the solution is easily derived. In order to do so, we first guess f to be of the form $f(v) = e^{\beta v} h(v)$, where β is a constant and h another real-valued function. Substituting this into (4.20) gives

$$v h''(v) + [(2\beta + B)v + A]\, h'(v) + \left[(\beta^2 + B\beta + D)v + A\beta + C\right] h(v) = 0. \quad (4.22)$$

Next, we would like to choose β such that $\beta^2 + B\beta + D = 0$, that is

$$\beta = \frac{-B \pm \sqrt{B^2 - 4D}}{2}, \tag{4.23}$$

which equals either $-\mu_d$ or μ_b. Since the solution of (4.20) we are looking for is a density, and necessarily $f(v) = e^{\beta v} h(v) \to 0$ as $v \to \infty$, we set β equal to the negative root $-\mu_d$. Lastly, we apply a change of variable, $x = \delta v$, and $h(v) = w(x)$, so that (4.22) is transformed into

$$xw''(x) + \left[(2\beta + B)\delta^{-1}x + A\right] w'(x) + \delta^{-1}\left[A\beta + C\right] w(x) = 0. \tag{4.24}$$

By choosing $(2\beta + B)\delta^{-1} = -1$, i.e.

$$\delta = -(2\beta + B) = \mu_b + \mu_d, \tag{4.25}$$

we obtain

$$xw''(x) + [A - x]w'(x) + \delta^{-1}\left[A\beta + C\right] w(x) = 0, \tag{4.26}$$

which is known as Kummer's equation, $xw''(x) + (b - x)w'(x) - aw(x) = 0$, see [14], with parameters

$$a = -\delta^{-1}\left[A\beta + C\right] = 1 - \frac{\lambda_d}{\xi_d},$$
$$b = A = 2 - \frac{\lambda_b + \lambda_d}{\xi_d}.$$

Kummer's equation has two linearly independent solutions, namely $w(x) = M(a, b, x)$, where M is Kummer's hypergeometric function, also denoted by ${}_1F_1(a, b, x)$, and $U(a, b, x)$, Tricomi's hypergeometric function. These are defined as, see [14, Eq. (1.3.1)],

$$M(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n, \tag{4.27}$$

$$U(a, b, x) = \frac{\Gamma(b-1)}{\Gamma(1 + a - b)}\, M(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)}\, x^{1-b}\, M(1 + a - b, 2 - b, x), \tag{4.28}$$

where $(.)_n$ is the Pochhammer symbol, which is used to represent $(y)_n = y \cdot (y + 1) \cdot \ldots \cdot (y + n - 1)$. We can therefore deduce that $f(v)$ is of the form

$$e^{\beta v}\left[c_1\, M(a, b, \delta v) + c_2\, U(a, b, \delta v)\right], \tag{4.29}$$

or

$$e^{-\mu_d v}\Big[c_1 M\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d)v\right) + + c_2 U\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d)v\right)\Big], \tag{4.30}$$

where c_1 and c_2 are constants. From [14, p. 60], we have

$$M(a,b,x) \sim \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}, \qquad \text{as } x \to \infty. \tag{4.31}$$

Hence,

$$\begin{aligned} e^{-\mu_d v} M\left(1 - \frac{\lambda_d}{\xi_d}, 2 - \frac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d) v\right) \\ \sim \frac{\Gamma(2 - \frac{\lambda_b + \lambda_d}{\xi_d})}{\Gamma(1 - \frac{\lambda_d}{\xi_d})} e^{\mu_b v} \left((\mu_b + \mu_d) v\right)^{\lambda_b/\xi_d - 1} \to \infty, \end{aligned} \tag{4.32}$$

for all $\mu_b > 0$, which leads us to conclude $c_1 = 0$. We deduce c_2 by exploiting the restriction that

$$\int_0^\infty f(v)\, dv = \pi_d, \tag{4.33}$$

where π_d is the probability of positive demand. Hence

$$\pi_d c_2^{-1} = \int_0^\infty e^{-\mu_d v} U\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d) v\right) dv. \tag{4.34}$$

By slightly transforming [14, (3.2.51)], we find

$$c_2^{-1} = \frac{1}{\pi_d} \frac{\Gamma\left(\frac{\lambda_b + \lambda_d}{\xi_d}\right)}{\Gamma\left(1 + \frac{\lambda_b}{\xi_d}\right)} {}_2F_1\left(1 - \tfrac{\lambda_d}{\xi_d}, 1, 1 + \tfrac{\lambda_b}{\xi_d}, -\tfrac{\mu_b}{\mu_d}\right), \tag{4.35}$$

Altogether, this yields:

Proposition 1.

$$f(v) = \pi_d \frac{\Gamma\left(1 + \frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b + \lambda_b}{\xi_d}\right)} \frac{e^{-\mu_d v}\, U\left(1 - \frac{\lambda_d}{\xi_d}, 2 - \frac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d) v\right)}{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)} \tag{4.36}$$

with corresponding Laplace transform, again by [14, Eq. (3.2.51)],

$$\phi(s) = \pi_d \frac{\mu_d}{\mu_d + s} \frac{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, \frac{s - \mu_b}{s + \mu_d}\right)}{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)}. \tag{4.37}$$

By exchanging parameters, a similar expression can be formulated for the density and LT of the amount of blood present.

Last, we obtain expressions for π_d and π_b. These follow immediately by a level-crossing argument around the point $v = 0$:

$$\lambda_b \int_0^\infty f(v) \bar{F}_b(v) dv = \lambda_d \int_0^\infty g(v) \bar{F}_d(v) dv. \tag{4.38}$$

In the exponential case, this becomes (see already (4.3)):

$$\lambda_b \int_0^\infty f(v) e^{-\mu_b v} dv = \lambda_d \int_0^\infty g(v) e^{-\mu_d v} dv. \tag{4.39}$$

An equivalent form of (4.39) is $\lambda_b \phi(\mu_b) = \lambda_d \gamma(\mu_d)$, or by filling in $s = \mu_b$ in (4.37),

$$\frac{\mu_d}{\mu_b + \mu_d} \frac{\pi_d \, \lambda_b}{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)} = \frac{\mu_b}{\mu_b + \mu_d} \frac{\pi_b \, \lambda_d}{{}_2F_1\left(1 - \frac{\lambda_b}{\xi_b}, 1, 1 + \frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)}, \tag{4.40}$$

where we used that ${}_2F_1(a, b, c, 0) = 0$. Using $\pi_b + \pi_d = 1$, we solve (4.40) and obtain

$$\pi_d = \frac{\lambda_d \mu_b \, {}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)}{\lambda_d \mu_b \, {}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) + \lambda_b \mu_d \, {}_2F_1\left(1 - \frac{\lambda_b}{\xi_b}, 1, 1 + \frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)}. \tag{4.41}$$

By substituting this result into both (4.36) and (4.37),

$$f(v) = \bar{C}^{-1} \frac{\Gamma\left(1 + \frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b + \lambda_d}{\xi_d}\right)} \lambda_d \mu_b \, e^{-\mu_d v} \, U\left(1 - \frac{\lambda_d}{\xi_d}, 2 - \frac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d) v\right), \tag{4.42}$$

$$\phi(s) = \bar{C}^{-1} \frac{\lambda_d \mu_b \, \mu_d}{\mu_d + s} \, {}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, \frac{s - \mu_b}{s + \mu_d}\right), \tag{4.43}$$

where

$$\bar{C} = \lambda_d \mu_b \, {}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) + \lambda_b \mu_d \, {}_2F_1\left(1 - \frac{\lambda_b}{\xi_b}, 1, 1 + \frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right). \tag{4.44}$$

Similar expressions can be formulated for the density $g(v)$ of the amount of blood present, its LT $\gamma(s)$ and the probability π_b of being positive.

Remark 5. *By applying the Pfaff transformation*

$$ {}_2F_1(a, b, c, z) = (1 - z)^{-b} \, {}_2F_1\left(c - a, b, c, \frac{z}{1 - z}\right), \tag{4.45}$$

we may reformulate

$$ {}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) = \frac{\mu_d}{\mu_b + \mu_d} \, {}_2F_1\left(\frac{\lambda_b + \lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b}{\mu_b + \mu_d}\right), \tag{4.46}$$

so that

$$\pi_d = \frac{\lambda_d \, {}_2F_1\left(\frac{\lambda_b + \lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b}{\mu_b + \mu_d}\right)}{\lambda_d \, {}_2F_1\left(\frac{\lambda_b + \lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b}{\mu_b + \mu_d}\right) + \lambda_b \, {}_2F_1\left(\frac{\lambda_b + \lambda_d}{\xi_b}, 1, \frac{\lambda_d}{\xi_b}, \frac{\mu_d}{\mu_b + \mu_d}\right)}. \tag{4.47}$$

By also transforming the hypergeometric term in the numerator of (4.36)*, we get an equivalent form of* (4.42)*, namely*

$$f(v) = \bar{C}_{\text{alt}}^{-1} \frac{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_b}{\xi_d}\right)} \frac{\lambda_b \mu_b(\mu_b+\mu_d)}{\mu_d} \, e^{-\mu_d v} \, U\left(1-\frac{\lambda_d}{\xi_d}, 2-\frac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b+\mu_d)v\right), \tag{4.48}$$

with

$$\bar{C}_{\text{alt}} = \lambda_d \, {}_2F_1\left(\frac{\lambda_b+\lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b}{\mu_b+\mu_d}\right) + \lambda_b \, {}_2F_1\left(\frac{\lambda_b+\lambda_d}{\xi_b}, 1, \frac{\lambda_d}{\xi_b}, \frac{\mu_d}{\mu_b+\mu_d}\right). \tag{4.49}$$

As a consequence, (4.43) is given by

$$\phi(s) = \pi_d \frac{{}_2F_1\left(\frac{\lambda_b+\lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b-s}{\mu_b+\mu_d}\right)}{{}_2F_1\left(\frac{\lambda_b+\lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b}{\mu_b+\mu_d}\right)} = \bar{C}_{\text{alt}}^{-1} \, \lambda_d \, {}_2F_1\left(\frac{\lambda_b+\lambda_d}{\xi_d}, 1, \frac{\lambda_b}{\xi_d}, \frac{\mu_b-s}{\mu_b+\mu_d}\right) \tag{4.50}$$

By close inspection of these derived density functions, we make a few additional remarks.

First, it can be verified, e.g. nummerically, that the expressions in (4.8) and (4.43) coincide.

Secondly, the confluent hypergeometric function $U(a,b,z)$ has limiting form as $z \to 0$,

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \frac{\Gamma(b-1)}{\Gamma(a)} \, z^{1-b} + O(z^{2-b}), \qquad b \le 2, \tag{4.51}$$

see [11, Subsec. 13.2]. Note that $b < 2$ for all parameter settings. This expression shows that $U(a,b,z)$ has a singularity at $z = 0$ if Re$(b) > 1$, which in our case translates to $f(v)$ and $g(v)$ being analytic at $v = 0$ if $\lambda_b + \lambda_d > \xi_d$ and $\lambda_b + \lambda_d > \xi_b$, respectively. Assuming $\lambda_b + \lambda_d > \max\{\xi_b, \xi_d\}$, (4.51) also implies that

$$\begin{aligned} \lim_{v\to 0} f(v) &= \bar{C}^{-1} \frac{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}\right)} \lambda_d \mu_b \cdot \frac{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}-1\right)}{\Gamma\left(\frac{\lambda_b}{\xi_d}\right)} \\ &= \bar{C}^{-1} \frac{\frac{\lambda_b}{\xi_d}}{\frac{\lambda_b+\lambda_d}{\xi_d}-1} \lambda_d \mu_b = \bar{C}^{-1} \frac{\lambda_b \lambda_d \mu_b \mu_d}{\lambda_b+\lambda_d-\xi_d}. \end{aligned} \tag{4.52}$$

Similarly,

$$\lim_{v\to 0} g(v) = \bar{C}^{-1} \frac{\lambda_b \lambda_d \mu_b \mu_d}{\lambda_b+\lambda_d-\xi_b}. \tag{4.53}$$

By equating these two expressions, we conclude that $\lim_{v\to 0} f(v) = \lim_{v\to 0} g(v) < \infty$, i.e. the overall density function is continuous at $v = 0$ if and only if

$\xi_b = \xi_d$.

The asymptotic behavior of U is given by [14, p. 60],

$$U(a,b,z) \sim z^{-a}, \qquad z \to \infty, \tag{4.54}$$

which implies that the density function will decay as

$$f(v) \sim C^* \, e^{-\mu_d v} \, v^{\lambda_d/\xi_d - 1}, \qquad v \to \infty, \tag{4.55}$$

for some constant C^*. Note that for $\lambda_d = \xi_d$ the asymptotic behaviour is consistent with (4.15). Let us take a closer look at this case $\lambda_d = \xi_d$. Formula (4.43) reduces to

$$\phi(s) = \bar{C}^{-1} \, \lambda_d \mu_b \, \frac{\mu_d}{\mu_d + s} = \pi_d \, \frac{\mu_d}{\mu_d + s}, \tag{4.56}$$

where we used that ${}_2F_1(0,a,b,z) = 1$ for all a, b, z. Hence, conditioned on being positive, the amount of demand present is exponentially distributed with parameter μ_d, regardless of the values of $\lambda_d = \xi_d$, as well as λ_b, ξ_b, and μ_b. Note that we also could have deduced this from (4.37) by observing ${}_2F_1(0,1,a,x) = 1$ for all $a, x \in \mathbb{R}$, so that

$$\phi(s) = \pi_d \, \frac{\mu_d}{\mu_d + s}. \tag{4.57}$$

This is in accordance with what we found in (4.15) with the approach based on Laplace transforms.

A second special case arises when the process is symmetric, that is, $\lambda_b = \lambda_d = \lambda$, $\mu_b = \mu_d = \mu$ and $\xi_b = \xi_d = \xi$. Obviously, we get $\pi_b = \pi_d = \frac{1}{2}$ due to the symmetry. If we define $\eta := \lambda/\xi$,

$$\begin{aligned} f(v) &= \frac{\Gamma(1+\eta)\, \mu e^{-\mu v}\, U\left(1-\eta, 2(1-\eta), 2\mu v\right)}{2\,\Gamma(2\eta)_2F_1\left(2\eta, 1, 1+\eta, \frac{1}{2}\right)} \\ &= \frac{\Gamma(1+\eta)}{2\,\Gamma(2\eta)_2F_1\left(2\eta, 1, 1+\eta, \frac{1}{2}\right)} \, \frac{\mu}{2\sqrt{\pi}} \, (2\mu v)^{\eta - \frac{1}{2}} \, K_{\frac{1}{2}-\eta}(\mu v), \end{aligned} \tag{4.58}$$

where $K_\alpha(\cdot)$ is the modified Bessel function of the second kind, see [11, Eq. (13.6.10)].

Based on the LT in (4.43), we can directly derive a couple of characteristics of the process.

Corollary 1. *The expected amount of demand (blood) present, given that it is positive equals*

$$\mathbb{E}[X_d | X_d > 0] = \frac{1}{\xi_d} \left[\frac{\lambda_d}{\mu_d} - \frac{\lambda_b}{\mu_b} + \frac{\lambda_b}{\mu_b} \frac{1}{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)} \right], \tag{4.59}$$

$$\mathbb{E}[X_b | X_b > 0] = \frac{1}{\xi_b} \left[\frac{\lambda_b}{\mu_b} - \frac{\lambda_d}{\mu_d} + \frac{\lambda_d}{\mu_d} \frac{1}{{}_2F_1\left(1 - \frac{\lambda_b}{\xi_b}, 1, 1 + \frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)} \right]. \tag{4.60}$$

Accordingly, the expected net amount of blood present equals

$$\mathbb{E}Q = \left(\frac{\lambda_b}{\mu_b} - \frac{\lambda_d}{\mu_d}\right)\left(\frac{\pi_b}{\xi_b} + \frac{\pi_d}{\xi_d}\right) + \frac{\lambda_b\lambda_d}{\bar{C}}\left(\frac{1}{\xi_b} - \frac{1}{\xi_d}\right). \tag{4.61}$$

Proof. The expression in (4.59) can be derived through differentiation of (4.43). However, we choose to use the implicit equation in (4.6) with $s = 0$ and the fact that $\phi(0) = \pi_d$ to obtain

$$\begin{aligned}\mathbb{E}[X_d|X_d > 0] &= -\frac{\phi'(0)}{\phi(0)} = \frac{1}{\xi_d}\left[\frac{\lambda_d}{\mu_d} - \frac{\lambda_b}{\mu_b} + \frac{\lambda_b}{\mu_b}\frac{\phi(\mu_b)}{\phi(0)} + \frac{\lambda_d}{\mu_d}\frac{\gamma(\mu_d)}{\phi(0)}\right] \\ &= \frac{1}{\xi_d}\left[\frac{\lambda_d}{\mu_d} - \frac{\lambda_b}{\mu_b} + \left(\frac{1}{\mu_b} + \frac{1}{\mu_d}\right)\frac{\lambda_b\phi(\mu_b)}{\varphi(0)}\right] \\ &= \frac{1}{\xi_d}\left[-m + \frac{\mu_b + \mu_d}{\mu_b\mu_d}\frac{\lambda_b\phi(\mu_b)}{\phi(0)}\right].\end{aligned}$$

Here we defined $m := \frac{\lambda_b}{\mu_b} - \frac{\lambda_d}{\mu_d}$, which can be regarded as the mean net amount of blood brought into the system per time unit, and $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$. From (4.43), we know

$$\frac{\phi(\mu_b)}{\phi(0)} = \frac{\mu_d}{\mu_b + \mu_d}\frac{1}{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)}, \tag{4.62}$$

so that

$$\mathbb{E}[X_d|X_d > 0] = \frac{1}{\xi_d}\left[-m + \frac{\lambda_b}{\mu_b}\,{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)^{-1}\right]. \tag{4.63}$$

The expression for $\mathbb{E}[X_b|X_b > 0]$ follows directly. Furthermore,

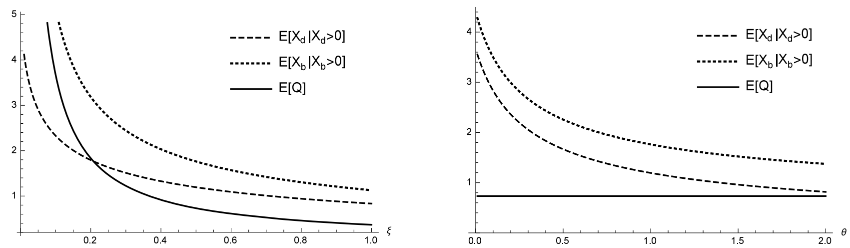
$$\begin{aligned}\mathbb{E}[Q] &= \pi_b\mathbb{E}[X_b|X_b > 0] + \pi_d\mathbb{E}[-X_d|X_d > 0] \\ &= m\left[\frac{\pi_b}{\xi_b} + \frac{\pi_d}{\xi_d}\right] + \frac{\lambda_d}{\mu_d\,\xi_b}\frac{\pi_b}{{}_2F_1\left(1 - \frac{\lambda_b}{\xi_b}, 1, 1 + \frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)} \\ &\quad - \frac{\lambda_b}{\mu_b\,\xi_d}\frac{\pi_d}{{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)}.\end{aligned}$$

Note that $\pi_d\,{}_2F_1\left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)^{-1} = \lambda_d\mu_b\bar{C}^{-1}$. Hence,

$$\mathbb{E}[Q] = m\left[\frac{\pi_b}{\xi_b} + \frac{\pi_d}{\xi_d}\right] + \frac{\lambda_b\lambda_d}{\bar{C}}\left(\frac{1}{\xi_b} - \frac{1}{\xi_d}\right),$$

which completes the proof. □

Remark 6. *Note that if $\xi_b = \xi_d = \xi$, we get $\mathbb{E}[Q] = m(\pi_b + \pi_d)/\xi = m/\xi$. The expression in (4.59) contains no ξ_b. Indeed, while the value of ξ_b influences the probability that $X_d > 0$, it does not influence the mean of X_d given that $X_d > 0$.*



(a) (b)

Fig 2: Expected mean amount of blood, demand, and net blood present.

In Figure 2 we plotted the behavior of the three performance metrics in Corollary 1 while keeping m fixed. In Figure 2.A we set $\lambda_b = 1.2$, $\lambda_d = 1$, $\mu_b = 1$, $\mu_d = 1.2$, so that $m = 11/30$ and vary $\xi_b = \xi_d = \xi$ between 0 and 1. In Figure 2.B we fix $\xi_b = \xi_d = 0.5$ and take $\lambda_b = 1.2\theta$, $\lambda_d = \theta$, $\mu_b = \theta$, $\mu_d = 1.2\theta$, so that still $m = 11/30$, and vary θ.

Furthermore, using the PASTA property of the Poisson process, we get $\mathbb{P}(\text{demand immediately satisfied}) = \int_0^\infty g(u)(1-\mathrm{e}^{-\mu_d u})\mathrm{d}u = \pi_b - \gamma(\mu_d)$, which is by equivalent form of (4.43) for the LST $\gamma(\cdot)$, equal to

$$\bar{C}^{-1}\lambda_b\mu_d\left({}_2F_1\left(1-\tfrac{\lambda_d}{\xi_d},1,1+\tfrac{\lambda_b}{\xi_d},-\tfrac{\mu_b}{\mu_d}\right)-\frac{\mu_b}{\mu_b+\mu_d}\right).$$

5. A variant

In this section we return to the case that the expiration rate of blood and the patience rate of demand are not purely proportional to the amount of blood and of demand, respectively. Accordingly, we generalize the expiration rate of blood from $\xi_b v$ to $\xi_b v + \alpha_b$, and the patience rate of demand from $\xi_d v$ to $\xi_d v + \alpha_d$. We do stick to the assumptions of exponentially distributed amounts of demand and blood, though. Hence this is a generalization of the results in Section 4.1. Equations (4.16)-(4.19) change into:

$$\begin{aligned}\lambda_d f(v) - \mu_d\left[\lambda_d\int_0^v f(y)\mathrm{e}^{-\mu_d(v-y)}\mathrm{d}y + \lambda_d\int_0^\infty g(y)\mathrm{e}^{-\mu_d(v+y)}\mathrm{d}y + \pi_0\lambda_d\mathrm{e}^{-\mu_d v}\right] \\ = -\lambda_b f(v) + \lambda_b\mu_b\int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d f(v) + (\xi_d v+\alpha_d)f'(v), \quad (5.1)\end{aligned}$$

$$(\xi_d v + \alpha_d) f'(v) = (\lambda_d + \lambda_b - \xi_d) f(v) - \mu_d \left(\lambda_b \int_v^\infty f(y) \mathrm{e}^{-\mu_b (y-v)} \mathrm{d}y + (\xi_d v + \alpha_d) f(v) \right) - \mu_b \lambda_b \int_v^\infty f(y) \mathrm{e}^{-\mu_b (y-v)} \mathrm{d}y, \tag{5.2}$$

$$(\xi_d v + \alpha_d) f''(v) + \xi_d f'(v) - (\lambda_d + \lambda_b - \xi_d - \mu_d(\xi_d v + \alpha_d)) f'(v) = -\mu_d \xi_d f(v) + (\mu_b + \mu_d) \lambda_b f(v) - \mu_b (\mu_b + \mu_d) \lambda_b \int_v^\infty f(y) \mathrm{e}^{-\mu_b (y-v)} \mathrm{d}y, \tag{5.3}$$

$$(\xi_d v + \alpha_d) f''(v) + (2\xi_d - \lambda_d - \lambda_b + \mu_d(\xi_d v + \alpha_d) - \mu_b(\xi_d v + \alpha_d)) f'(v) + (\mu_d \xi_d - \mu_b \xi_d - \mu_d \lambda_b + \mu_b \lambda_d - \mu_b \mu_d (\xi_d v + \alpha_d)) f(v) = 0. \tag{5.4}$$

Solution to differential equation (5.4)

We attempt to solve differential equation (5.4). Let us introduce a new variable z such that $\xi_d z = \xi_d v + \alpha_d$. Hence

$$z(v) = v + \frac{\alpha_d}{\xi_d}, \qquad v(z) = z - \frac{\alpha_d}{\xi_d}. \tag{5.5}$$

Then (5.4) can be rewritten as

$$\xi_d z f''(z - \alpha_d/\xi_d) + (2\xi_d - \lambda_d - \lambda_b + (\mu_d - \mu_b)\xi_d z) f'(z - \alpha_d/\xi_d) + (\mu_d \xi_d - \mu_b \xi_d - \mu_d \lambda_b + \mu_b \lambda_d - \mu_b \mu_d (\xi_d z)) f(z - \alpha_d/\xi_d) = 0, \tag{5.6}$$

which almost resembles Equation (4.19). If we furthermore define $\hat{f}(z) = f(z - \alpha_d/\xi_d)$ and note that $\hat{f}'(z) = f'(z - \alpha_d/\xi_d)$ and $\hat{f}''(z) = f''(z - \alpha_d/\xi_d)$, (5.6) becomes

$$\xi_d z \hat{f}''(z) + (2\xi_d - \lambda_d - \lambda_b + (\mu_d - \mu_b)\xi_d z) \hat{f}'(z) + (\mu_d \xi_d - \mu_b \xi_d - \mu_d \lambda_b + \mu_b \lambda_d - \mu_b \mu_d (\xi_d z)) \hat{f}(z) = 0, \tag{5.7}$$

which is exactly the same as (4.19). From (4.30), we already know that the only feasible solution to (5.7) is of the form

$$\hat{f}(z) = C^{-1} e^{-\mu_d z} \, U\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d) z\right), \tag{5.8}$$

with $z > 0$, so that

$$f(z) = \hat{f}(z + \alpha_d/\xi_d) = C^{-1} \, \pi_d \, e^{-\mu_d (z + \alpha_d/\xi_d)} \, U\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d)(z + \alpha_d/\xi_d)\right), \tag{5.9}$$

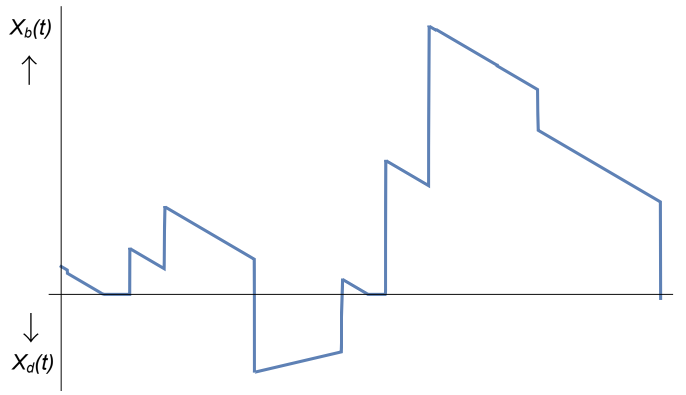


Fig 3: Sample path of the net amount of blood present if $\xi_b = \xi_d = 0$.

for some positive constant C. Computing this constant C in this case proves more tedious, because we are now after the values of the integral

$$\int_{\alpha_d/\xi_d}^{\infty} e^{-\mu_d z} U\left(1 - \frac{\lambda_d}{\xi_d}, 2 - \frac{\lambda_b+\lambda_d}{\xi_d}, (\mu_d + \mu_b)z\right) dz. \tag{5.10}$$

We had an expression for the Laplace transform of $t^k U(a, b, t)$ at hand in the previous case, but finding an expression for this integral has so far been unsuccessful. Another issue that arises for this setting is the observation that the event $\{v = 0\}$ will have positive probability, due to the additional decay (of both blood and demand) that does not slow down for v close to 0. Therefore, we have the additional equation $\pi_d + \pi_b + \pi_0 = 1$, where $\pi_0 > 0$ denotes the probability of the event "no inventory, no demand".

6. Another variant

In this section we assume that the expiration rate of blood and the patience rate of demand are constant; so we take $\xi_b = \xi_d = 0$. A visualization of a possible sample path is depicted in Figure 3.

We again restrict ourselves to the case of exponentially distributed amounts of demand and of blood deliveries. We now need to impose stability conditions. In the case of positive demand, the drift is towards zero if $\lambda_d \mathbb{E}D < \alpha_d + \lambda_b \mathbb{E}B$, while in the case of a positive amount of blood, the drift is towards zero if $\lambda_b \mathbb{E}B < \alpha_b + \lambda_d \mathbb{E}D$. If these two conditions are violated, either the amount of demand or the amount of blood present increases without bound (see also Remark 8 below). In this case, (5.4) reduces to

$$\alpha_d f''(v) + (-\lambda_d - \lambda_b + \mu_d\alpha_d - \mu_b\alpha_d) f'(v) + (-\mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\alpha_d) f(v) = 0. \tag{6.1}$$

Hence $f(\cdot)$ is a mixture of two exponential terms: $f(v) = R_+ \mathrm{e}^{-x_+ v} + R_- \mathrm{e}^{-x_- v}$, where x_+ and x_- are the positive and negative root of the equation

$$\alpha_d x^2 - (\mu_d \alpha_d - \mu_b \alpha_d - \lambda_d - \lambda_b)x + (-\mu_d \lambda_b + \mu_b \lambda_d - \mu_b \mu_d \alpha_d) = 0. \tag{6.2}$$

Notice that the last term in the lefthand side of (6.2) is negative if the stability condition $\lambda_d \mathbb{E}D < \alpha_d + \lambda_b \mathbb{E}B$ holds, i.e., if $\mu_b \lambda_d < \mu_d \lambda_b + \mu_b \mu_d \alpha_d$, thus guaranteeing that the product of the two roots x_+ and x_- is negative, and hence that there is a positive and a negative root. One should subsequently observe that R_- must be zero to have a probability density. Hence $f(v)$ is simply (a constant times) an exponential; similarly for $g(v)$. In addition, the steady-state amounts of demand and of blood have an atom at 0 (since ξ_d and ξ_b are no longer zero, the demand and blood processes can reach 0).

Remark 7. *Alternatively, we could have taken Laplace transforms; (4.4) now changes into (only its last term has to be adapted):*

$$\lambda_d \frac{\phi(s)}{\mu_d + s} + \lambda_d \frac{\gamma(\mu_d)}{\mu_d + s} = \lambda_b \frac{\phi(s) - \phi(\mu_b)}{\mu_b - s} + \alpha_d \phi(s), \tag{6.3}$$

or

$$\phi(s) = \frac{\lambda_d \gamma(\mu_d)(\mu_b - s) + \lambda_b \phi(\mu_b)(\mu_d + s)}{\alpha_d(\mu_d + s)(\mu_b - s) - \lambda_d(\mu_b - s) + \lambda_b(\mu_d + s)}. \tag{6.4}$$

Now notice that the zeros of the denominator of (6.4) are exactly $s = x_-$ and $s = x_+$, and that the numerator should be zero for $s = x_+$. Inversion then again gives an exponential density $f(v)$, $v > 0$; similarly for $g(v)$.

Remark 8. *Interestingly, the model of this section is closely related to the model with workload removal that is considered in [5]. There an $M/G/1$ queue is studied with the extra feature that, at Poisson epochs, a stochastic amount of work is removed. In the $M/M/1$ case with removal of exponential amounts of work (cf. Section 5.1 of [5]), one has the model of the present section when we concentrate on the amount of demand present. One difference with the model in [5] is that, when the workload in that model has become zero, the work becomes positive at rate λ_d, whereas in the present model the amount of blood can become positive (so zero demand is present) and the amount of demand does not have to become positive when demands arrive (because they are immediately satisfied; cf. Fig. 3). So the atom at zero is in the present model larger than in the model of [5]. In our model a positive demand level may be reached from below zero (by a jump, i.e., a demand arriving at an epoch that there is some, but not enough, blood present). The memoryless property of the exponential demand requirement distribution implies that this jump results in a demand level that is exp(μ_d), just as if the initial demand level had been zero. In the case of non-exponential demand requirements, our model becomes equivalent with an $M/G/1$ queue with exponential amounts of work removed, and with the special feature that the first service requirement of a busy period has a different distribution. Lemmas 4.1 and 4.2 of [5] present the stability condition of that $M/G/1$ queue with work removal; it amounts to $\lambda_d \mathbb{E}D < \alpha_d + \lambda_b \mathbb{E}B$, which indeed is one of the two stability conditions of the present demand/blood model.*

Finally we observe that Formula (5.1) of [5] coincides with (6.2) (take $\alpha_d = 1$, $\lambda_d = \lambda_+$, $\lambda_b = \lambda_-$, $\mu_d = 1/\beta$ and $\mu_b = 1/\gamma$).

7. Scaling limits

We finally study the model with $\alpha_b = \alpha_d = 0$ from an asymptotic perspective, by obtaining the fluid and diffusion limits of the blood inventory process. That is, we will create a sequence of processes, indexed by $n = 1, 2, ...$, in which we let the rates of blood and demand arrivals grow large. If we then scale the process in a proper manner, we are able to deduce a non-degenerate limiting process, that provides insight in the overall behavior of the arrival volume when the system grows large. Furthermore, only the first two moments of the blood and demand distributions are needed.

7.1. Identification of the limiting process

First, we introduce some additional notation. Let $X_b(t)$ and $X_d(t)$ denote the amount of blood and demand, respectively, at time $t > 0$. Let

$$Q(t) := X_b(t) - X_d(t), \tag{7.1}$$

be the net amount of blood available at time t. Remember that $X_b(t), X_d(t) \geq 0$, and $X_b(t) > 0$ *or* $X_d(t) > 0$ for all t, since $\alpha_d = \alpha_b = 0$. Let $N_b(t)$, $N_d(t)$ be the two independent Poisson processes counting the number of arrivals of blood and demand, respectively. Then the following integral representation holds for $Q(t)$,

$$Q(t) = Q(0) - \xi_b \int_0^t Q^+(s)\,ds + \xi_d \int_0^t Q^-(s)\,ds + \sum_{i=1}^{N_b(t)} B_i - \sum_{i=1}^{N_d(t)} D_i, \tag{7.2}$$

where $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$. For the sake of exhibition we will concentrate on the case $\xi_b = \xi_d =: \xi$. Our analysis can be extended to the general case, but involves more technicalities which do not contribute to the insight provided by the limiting process. A sketch of this generalization is given at the end of this section. Define

$$X(t) = \sum_{i=1}^{N_b(t)} B_i - \sum_{i=1}^{N_d(t)} D_i, \tag{7.3}$$

so that (7.2) reduces to

$$Q(t) = Q(0) - \xi \int_0^t Q(s)\,ds + X(t). \tag{7.4}$$

The first step in the definition of the sequence of processes under investigation is defining the asymptotic scheme we are interested in. As mentioned above, we intend to let the arrival rates grow to infinity. Therefore, in the n^{th} process

$Q_n(t)$, we replace the rates of the arrival processes by $n\lambda_b$ and $n\lambda_d$. This induces Poisson processes $N_b^{(n)}(t)$ and $N_d^{(n)}(t)$ with arrival rates $n\lambda_b$ and $n\lambda_d$, respectively. However, we have

$$N_b^{(n)}(t) \stackrel{d}{=} N_b(nt) \qquad \text{and} \qquad N_d^{(n)}(t) \stackrel{d}{=} N_d(nt), \tag{7.5}$$

so that the term $X(t)$ in (7.2) in this asymptotic scheme can be replaced by

$$X_n(t) = \sum_{i=1}^{N_b(nt)} B_i - \sum_{i=1}^{N_d(nt)} D_i. \tag{7.6}$$

The first step in our analysis is obtaining the fluid limit of the process. Bearing in mind application of the Functional Strong Law of Large Numbers (FSLLN), we scale the process as $\bar{Q}_n(t) = \frac{Q_n(t)}{n}$, so that with (7.4)

$$\bar{Q}_n(t) = \bar{Q}_n(0) - \xi \int_0^t \bar{Q}_n(s)\, ds + \bar{X}_n(t), \tag{7.7}$$

where $\bar{X}_n(t) = X_n(t)/n$. Let us rewrite (7.7) a bit, by observing

$$\mathbb{E}\bar{X}_n(t) = \frac{1}{n}\left[\mathbb{E}N_b(nt)\mathbb{E}B - \mathbb{E}N_d(nt)\mathbb{E}D\right] = \lambda_b \mathbb{E}Bt - \lambda_d \mathbb{E}Dt. \tag{7.8}$$

Setting $m := \lambda_b \mathbb{E}B - \lambda_d \mathbb{E}D$,

$$\bar{Q}_n(t) = \bar{Q}_n(0) - \xi \int_0^t \left(\bar{Q}_n(s) - \frac{m}{\xi}\right) ds + \bar{Y}_n(t), \tag{7.9}$$

where $\bar{Y}_n(t) : \bar{X}_n(t) - mt$ is now a centralized process.

The essential step in establishing a result on the convergence of $\bar{Q}_n$ is the application of [12, Thm 4.1]. Let $D[0,\infty)$ be the space of all one-dimensional real-valued functions defined on $[0,\infty)$, endowed with the usual J_1-Skorohod topology. Then, we know that the integral representation

$$x(t) = y(t) + \int_0^t u(x(s))\, ds, \tag{7.10}$$

with boundary condition $u(0) = u_0$, has a unique solution, given that u is a Lipschitz function. This implies that the function $H_u : D[0,\infty) \to D[0,\infty)$, which maps the function $u(\cdot)$ into the solution $x(\cdot)$ of (7.10), is well-defined. Furthermore, [12, Thm 4.1] tells us that the mapping H_u is continuous. Back to our case, we set $u(x) = -\xi x + m$, to be able to write $\bar{Q}_n = H_u\left(\bar{Y}_n + \bar{Q}_n(0)\right)$. Since u is clearly Lipschitz continuous, the mapping H_u is indeed continuous, which allows us to state the next result.

Proposition 2 (Fluid limit). *Let $\mathbb{E}[B]$, $\mathbb{E}[D] < \infty$ and $\bar{Q}_n(0) = Q_n(0)/n \to q(0) \in \mathbb{R}$, as $n \to \infty$. Then for $n \to \infty$,*

$$\bar{Q}_n \Rightarrow q, \tag{7.11}$$

where $\Rightarrow$ denotes convergence in distribution and

$$q(t) = \frac{m}{\xi} + \left(q(0) - \frac{m}{\xi}\right) e^{-\xi t}. \tag{7.12}$$

Proof. First, we concentrate on the process $\bar{Y}_n$. Observe that, by the elementary renewal theorem for renewal reward processes, see e.g. [13, Thm 3.6.1], we have

$$\frac{1}{nt} \sum_{i=1}^{N_b(nt)} B_i \overset{a.s.}{\Rightarrow} \lambda_b \mathbb{E}[B], \qquad \frac{1}{nt} \sum_{i=1}^{N_d(nt)} D_i \overset{a.s.}{\Rightarrow} \lambda_d \mathbb{E}[D], \tag{7.13}$$

for $n \to \infty$ and for all $t > 0$. Hence, $\bar{X}_n(t) \overset{a.s.}{\Rightarrow} \lambda_b \mathbb{E}[B]t - \lambda_d \mathbb{E}[D]t = mt$. By definition of $\bar{Y}_n$ and the assumption of convergence of $\bar{Q}(0)$, this implies

$$\bar{Y}_n + \bar{Q}_n \overset{a.s.}{\Rightarrow} q(0) \tag{7.14}$$

as $n \to \infty$. Next, note $\bar{Q}_n = H_u\left(\bar{Y}_n + \bar{Q}_n(0)\right)$. Due to Lipschitz continuity of u, H_u constitutes a continuous mapping, and hence we can apply the Continuous Mapping Theorem (CMT), to find

$$\bar{Q}_n = H_u\left(\bar{Y}_n + \bar{Q}_n(0)\right) \Rightarrow H_u\left(q(0)\right) \equiv q, \tag{7.15}$$

where $q(\cdot)$ is the solution of

$$q(t) = q(0) + \int_0^t u(q(s))\, ds = q(0) - \xi \int_0^t \left(q(s) - \frac{m}{\xi}\right) ds. \tag{7.16}$$

The unique solution of this integral equation is given in (7.11). □

According to Proposition 2, the fluid limit approaches $\mathbb{E}Q = \frac{m}{\xi}$ exponentially fast. To obtain an expression for the *diffusion limit* of the process, we analyze the fluctuations of the process around the fluid limit in (7.11), again by scaling the process in a proper manner. First, we subtract $q(t)$ on both sides of (7.9), and multiply by $\sqrt{n}$:

$$\sqrt{n}\left(\bar{Q}_n(t) - q(t)\right) = \sqrt{n}\left(\bar{Q}_n(0) - q(0)\right) - \xi \int_0^t \sqrt{n}\left(\bar{Q}_n(s) - q(s)\right) ds + \sqrt{n}\,\bar{Y}_n(t). \tag{7.17}$$

Let $\hat{Q}_n \equiv \sqrt{n}\left(\bar{Q}_n - q\right)$ and $\hat{Y}_n \equiv \sqrt{n}\,\bar{Y}_n$, then this reduces to

$$\hat{Q}_n(t) = \hat{Q}_n(0) - \xi \int_0^t \hat{Q}_n(s)\, ds + \hat{Y}_n(t). \tag{7.18}$$

Again the term $\hat{Y}_n(t)$ needs special attention.

Lemma 1. *Let $\mathbb{E}B$, $\mathbb{E}D$, $\mathbb{E}[B^2], \mathbb{E}[D^2] < \infty$. Then $\hat{Y}_n \Rightarrow \sigma W$ as $n \to \infty$, where $\sigma^2 := \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$ and W is a standard Brownian motion.*

Proof. Recall that

$$\hat{Y}_n(t) = \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{N_b(nt)} B_i - \lambda_b \mathbb{E}[B]t\right) - \left(\frac{1}{n}\sum_{i=1}^{N_d(nt)} D_i - \lambda_d \mathbb{E}[D]t\right)\right]. \quad (7.19)$$

By the Functional Central Limit Theorem (FCLT) for renewal-reward processes given in [17, Thm 7.4.1], the process

$$\hat{Y}_n^b(t) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{N_b(nt)} B_i - \lambda_b \mathbb{E}[B]t\right), \quad (7.20)$$

converges weakly to $\sigma_b W_b$, where W_b is a standard Brownian motion, and

$$\sigma_b^2 = \lambda_b \operatorname{Var} B + \lambda_b (E[B])^2 = \lambda_b \mathbb{E}[B^2]. \quad (7.21)$$

Similarly $\hat{Y}_n^d(t) \Rightarrow \sigma_d W_d$, with the obvious parameter switches and W_d is standard Brownian motion. Since the processes $\hat{Y}_n^b$ and $\hat{Y}_n^d$ are independent, so are their limits, and

$$\hat{Y}_n \Rightarrow \sqrt{\lambda_b \mathbb{E}[B^2]}\, W_b + \sqrt{\lambda_d \mathbb{E}[D^2]}\, W_d \stackrel{d}{=} \sqrt{\lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]} W, \quad (7.22)$$

for $n \to \infty$ and W a standard Brownian motion. □

Now, we are ready to prove the diffusion counterpart of Proposition 2.

Proposition 3 (Diffusion limit). *Let $\mathbb{E}B, \mathbb{E}D, \mathbb{E}[B^2], \mathbb{E}[D^2] < \infty$. If $\hat{Q}_n(0) \to \hat{Q}(0)$, then $\hat{Q}_n \Rightarrow \hat{Q}$ as $n \to \infty$, where $\hat{Q}$ satisfies the integral equation*

$$\hat{Q}(t) = \hat{Q}(0) - \xi \int_0^t \hat{Q}(s)\, ds + \sigma W(t). \quad (7.23)$$

In other words, $\hat{Q}$ is an Ornstein-Uhlenbeck diffusion process with infinitesimal mean ξ and infinitesimal variance $\sigma^2 := \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$.

Proof. We again rely on the result that the mapping H_u as in the proof of Proposition 2 is continuous if u is Lipschitz continuous. Here, we set $u(x) = -\xi x$ which again clearly satisfies this condition. We have $\hat{Q}_n \equiv H_u(\hat{Q}_n(0) + \hat{Y}_n)$. From Lemma 1, we know

$$\hat{Q}_n(0) + \hat{Y}_n \Rightarrow \hat{Q}(0) + \sigma W, \quad (7.24)$$

for $n \to \infty$. As a consequence of the CMT, we conclude

$$\hat{Q}_n = H_u\left(\hat{Q}_n(0) + \hat{Y}_n\right) \Rightarrow H_u\left(\hat{Q}(0) + \sigma W\right) \equiv \hat{Q}, \quad (7.25)$$

where $\hat{Q}$ solves (7.23). □

7.2. Generalization for $\xi_b \neq \xi_d$

We now sketch the scaling approach towards fluid and diffusion limits for the general case in which ξ_b may differ from ξ_d. In case $\xi_b \neq \xi_d$, the integral equation for $\bar{Q}_n$ as in (7.7) becomes

$$\begin{aligned}\bar{Q}_n(t) &= \bar{Q}_n(0) + \int_0^t (-\xi_b \bar{Q}_n^+(s) + \xi_d \bar{Q}_n^-(s) - m)\, ds + \bar{Y}_n(t) \\ &= \bar{Q}_n(0) - \int_0^t (\left[\xi_b 1_{\{\bar{Q}_n(s)\geq 0\}} + \xi_d 1_{\{\bar{Q}_n(s)<0\}}\right] \bar{Q}_n(s) + m)\, ds + \bar{Y}_n(t),\end{aligned} \tag{7.26}$$

where $\bar{Y}_n(t)$ is defined as before. Note that $\hat{Q}_n \equiv H_u(\bar{Q}_n(0) + \bar{Y}_n)$, where we now have

$$u(x) = -\left[\xi_b 1_{\{x\geq 0\}} + \xi_d 1_{\{x<0\}}\right] x + m, \tag{7.27}$$

which is still Lipschitz continuous. Therefore by the same reasoning of the proof of Proposition 2, we obtain the fluid limit $\bar{Q}_n \overset{a.s.}{\Rightarrow} q$, where q is the solution of

$$q(t) = q(0) - \int_0^t (\left[\xi_b 1_{\{q(s)\geq 0\}} + \xi_d 1_{\{q(s)<0\}}\right] q(s) - m)\, ds. \tag{7.28}$$

The solution to this integral equation is more elaborate than (7.11) and depends on the sign of m and $q(0)$. Assuming $m \geq 0$, one can check that,

$$q(t) = \frac{m}{\xi_b} + \left(q(0) - \frac{m}{\xi_b}\right) e^{-\xi_b t}, \qquad \text{if } q(0) \geq 0, \tag{7.29}$$

$$q(t) = \begin{cases} \frac{m}{\xi_d} + \left(q(0) - \frac{m}{\xi_d}\right) e^{-\xi_d t}, & \text{if } 0 \leq t < t_d^*, \\ \frac{m}{\xi_b}\left(1 - e^{-\xi_b (t - t_d^*)}\right), & \text{if } t \geq t_d^*, \end{cases} \qquad \text{if } q(0) < 0, \tag{7.30}$$

where

$$t_d^* = -\frac{1}{\xi_d} \log\left(\frac{m/\xi_d}{m/\xi_d - q(0)}\right). \tag{7.31}$$

If $m < 0$,

$$q(t) = \frac{m}{\xi_d} + \left(q(0) - \frac{m}{\xi_d}\right) e^{-\xi_d t}, \qquad \text{if } q(0) \leq 0, \tag{7.32}$$

$$q(t) = \begin{cases} \frac{m}{\xi_b} + \left(q(0) - \frac{m}{\xi_b}\right) e^{-\xi_b t}, & \text{if } 0 \leq t < t_b^*, \\ \frac{m}{\xi_d}\left(1 - e^{-\xi_d (t - t_b^*)}\right), & \text{if } t \geq t_b^*, \end{cases} \qquad \text{if } q(0) > 0, \tag{7.33}$$

where

$$t_b^* = -\frac{1}{\xi_b} \log\left(\frac{m/\xi_b}{m/\xi_b - q(0)}\right). \tag{7.34}$$

Note that the equilibrium of the fluid limit also depends on the sign of m:

$$\lim_{t\to\infty} q(t) = \begin{cases} m/\xi_b, & \text{if } m \geq 0, \\ m/\xi_d, & \text{if } m < 0. \end{cases} \tag{7.35}$$

In the remainder, without loss of generality $m \geq 0$. Furthermore, set $q(0) = m/\xi_b$ so that $q \equiv m/\xi_b$. Subtracting $q(t)$ on both sides of (7.26) yields,

$$\begin{aligned}\left(\bar{Q}_n(t) - q(t)\right) &= \left(\bar{Q}_n(0) - q(0)\right) \\ &\quad - \int_0^t \left[\xi_b 1_{\{Q(s)\geq 0\}} + \xi_d 1_{\{Q(s)<0\}}\right] \bar{Q}_n(s) - \xi_b\, q(s)\, ds + \bar{Y}_n(t) \qquad (7.36) \\ &= \left(\bar{Q}_n(0) - q(0)\right) - \int_0^t \xi_b \left(\bar{Q}_n(s) - q(s)\right)\, ds \\ &\quad + \int_0^t 1_{\{\bar{Q}_n(s)<0\}}(\xi_b - \xi_d)\bar{Q}_n(s)\, ds + \bar{Y}_n(t). \qquad (7.37)\end{aligned}$$

Let $\hat{Q}_n(t) = \sqrt{n}\left(\bar{Q}_n(t) - q(t)\right)$. Then

$$\hat{Q}_n(t) = \hat{Q}_n(0) - \xi_b \int_0^t \hat{Q}_n(s)\, ds + \int_0^t 1_{\{\bar{Q}_n(s)<0\}}(\xi_b - \xi_d)\bar{Q}_n(s)\, ds + \hat{Y}_n(t) \quad (7.38)$$

Now, we argue non-rigorously that the one-but-last term vanishes as $n \to \infty$. Namely, by defining the function $G : D[0,\infty) \to D[0,\infty)$ by the integration operator:

$$G(u) = \int_0^t 1_{\{u(s)<0\}}(\xi_b - \xi_d)u(s)\, ds, \qquad (7.39)$$

this term can be expressed as $G(\bar{Q}_n)$. Hence by the fact that $\hat{Q}_n \overset{a.s.}{\Rightarrow} m/\xi_b$ and the CMT we see $G(\hat{Q}_n) \Rightarrow 0$.

Under this claim, we deduce by the approach of Proposition 3, that if $\hat{Q}_n \Rightarrow \hat{Q}$ for $n \to \infty$, then $\hat{Q}$ satisfies the stochastic integral equation

$$\hat{Q}(t) = \hat{Q}(0) - \xi_b \int_0^t \hat{Q}(s)\, ds + \sigma W(t), \qquad (7.40)$$

which implies that $\hat{Q}$ is an Ornstein-Uhlenbeck process with infinitesimal mean ξ_b and variance $\sigma^2 := \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$.
The result that the scaled process converges to an Ornstein-Uhlenbeck process can be intuitively justified by the so-called *mean-reverting* behavior of the original process. That is, the further the process is away from its mean, the greater the drift towards that equilibrium. This is the defining feature of the OU diffusion process. The decay rates ξ_b and ξ_d are responsible for the original process being 'forced' towards 0 and therefore the similarities should not be surprising. However, note that in the diffusion limit Q_n has drift ξ_b (cq. ξ_d) towards m/ξ_b (cq. m/ξ_b, if $m > 0$ (cq. < 0) at *any* position of the process. This implies that if $Q_n \in (0, m)$, it has an upward drift equal to ξ_b, which is at first sight counter-intuitive.

8. Numerical evaluation

This section is devoted to the numerical evaluation of the density functions $f(\cdot)$ (amount of shortage) and $g(\cdot)$ (amount of blood present) found in Proposition 1, as well as the performance characteristics resulting from these in Corollary 1. In particular, we examine the approximation of the density of the net blood present that follows from the diffusion limit that we obtained in Section 7. We shall restrict ourselves to the case of exponentially distributed amounts of blood requirements and blood deliveries.

The diffusion limit in Proposition 3 promises to describe the process' distribution accurately for large blood and demand arrival rates. Therefore, we stick to the framework of Section 7. Namely, we denote by $Q_n(t)$ the net amount of blood present at time t for the process with arrival rate $n\lambda_b$ and $n\lambda_d$, where λ_b and λ_d are fixed quantities. We fix μ_b, μ_d, ξ_b and ξ_d as $n \to \infty$. Let Q_n be the stationary counterpart of $Q_n(t)$. From Proposition 1 we have an exact expression for the density of Q_n, namely $h_n(v)$, defined as

$$h_n(v) = \begin{cases} f_n(-v), & \text{if } v < 0, \\ g_n(v), & \text{if } v \geq 0, \end{cases} \tag{8.1}$$

where f_n and g_n are the density functions of the amount of demand and blood present, respectively, where the subscript indicates that all arrival rates are multiplied by n.

On the other hand, Proposition 3 tells us that $\hat{Q}_n$ is approximately an Ornstein-Uhlenbeck process with long-term mean 0, drift $-\xi^*$ and infinitesimal variance σ^2, where

$$\xi^* = \begin{cases} \xi_b, & \text{if } m \geq 0, \\ \xi_d, & \text{if } m < 0, \end{cases} \qquad m = \frac{\lambda_b}{\mu_b} - \frac{\lambda_d}{\mu_d}, \qquad \sigma^2 = \frac{2\lambda_b}{\mu_b^2} + \frac{2\lambda_d}{\mu_d^2}. \tag{8.2}$$

The stationary distribution of this process is a normal distribution with mean 0 and variance $\sigma^2/(2\xi^*)$. Hence, we have

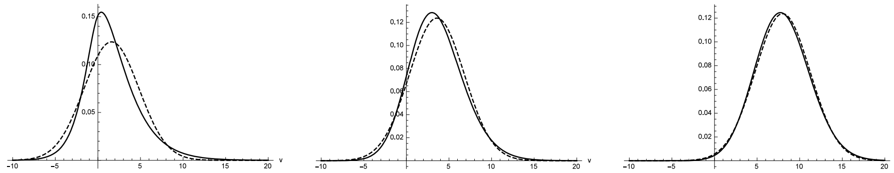
$$\frac{Q_n - n\,m/\xi^*}{\sqrt{n}} \approx N\left(0, \frac{\sigma}{\sqrt{2\xi^*}}\right), \tag{8.3}$$

for n large, and we may approximate

$$\frac{Q_n}{\sqrt{n}} \approx N\left(\frac{\sqrt{n}m}{\xi^*}, \frac{\sigma}{\sqrt{2\xi^*}}\right). \tag{8.4}$$

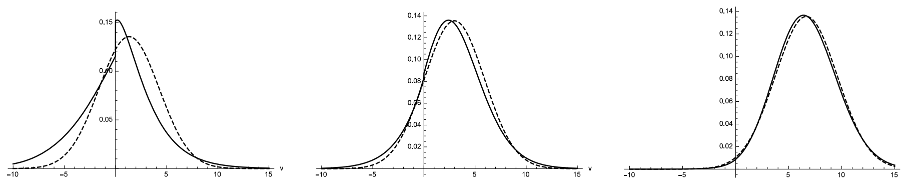
Let $\varphi_n(v)$ denote the density of this normal random variable. The exact density of $Q_n/\sqrt{n}$ is given by $\hat{h}_n(v) := \sqrt{n}h_n(\sqrt{n}v)$. We choose to scale Q_n by $\sqrt{n}$ and thereby keep the variance of the limit fixed as $n \to \infty$, to be able to do a fair visual comparison.

In Figure 4 we depict the two densities for $\lambda_b = 1$, $\lambda_d = 1.2$, $\mu_b = 0.5$, $\mu_d = 1$,



(a) $n = 1$ (b) $n = 5$ (c) $n = 25$

Fig 4: Density function of $Q_n/\sqrt{n}$ (black) and the Gaussian limiting density function (dashed) for $\xi_b = \xi_d$.



(a) $n = 1$ (b) $n = 5$ (c) $n = 25$

Fig 5: Density function of $Q_n/\sqrt{n}$ (black) and the Gaussian limiting density (dashed) for $\xi_b \neq \xi_d$.

$\xi_b = \xi_d = 0.5$. Figure 5 shows the density function for the same parameters, except $\xi_b = 0.6$ and $\xi_d = 0.1$.

First, observe that for these parameter settings, the Gaussian approximation is already quite accurate for $n = 5$. For $n = 25$, the plots almost align. In Figure 5 we clearly see a discontinuity of the exact density at $v = 0$, as we already found by the left and right limits in (4.52) and (4.53). Note however, that as $n \to \infty$ the jump at $v = 0$ vanishes. Also, as n grows, a shift towards the right (in this case, because $m > 0$) occurs, so that $\pi_d \to 0$ and $\pi_b \to 1$ as we predicted in Section 7.

The fluid limit of Q_n gives us an approximation for its mean. Namely, since $Q_n/n \overset{a.s.}{\Rightarrow} m/\xi_*$, we have

$$\mathbb{E}Q_n \approx n\, m/\xi^*. \tag{8.5}$$

The exact value is given in Corollary 1. The diffusion limit also provides an approximation for π_d (and π_b) as n grows large. Namely

$$\begin{aligned}\pi_d = \mathbb{P}(Q_n \le 0) &= \mathbb{P}\left(\frac{Q_n - n\,m/\xi^*}{\sqrt{n}} \le -\frac{\sqrt{n}m}{\xi^*}\right)\\ &\approx \mathbb{P}\left(N(0, \sigma/\sqrt{2\xi^*})) \le -\frac{\sqrt{n}m}{\xi^*}\right) = \Phi\left(-\sqrt{\frac{2\,n}{\xi^*}}\frac{m}{\sigma}\right),\end{aligned} \tag{8.6}$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. Note that this ap-

	$\xi_b = \xi_d = 0.5$			$\xi_b = 0.6,\ \xi_d = 0.1$		
n	$\mathbb{E}Q_n/n$	π_d	(8.6)	$\mathbb{E}Q_n/n$	π_d	(8.6)
1	1.6	0.31977	0.30990	0.21442	0.43755	0.32531
2	1.6	0.24092	0.24145	0.80897	0.33806	0.26092
5	1.6	0.12478	0.13363	1.19738	0.18271	0.15559
10	1.6	0.04978	0.05833	1.30135	0.07873	0.07604
20	1.6	0.00958	0.01325	1.32941	0.01827	0.02141
50	1.6	0.00010	0.00023	1.33334	0.00036	0.00068

TABLE 1

Gaussian approximations of performance metrics compared against true values.

proximation confirms that $\pi_d \to 0$ if $m < 0$, and $\pi_d \to 1$ if $m > 0$.

We numerically test these approximations against the true values for the two parameter sets as before in Table 1. Note that the expected (scaled) means for Q_n approximated by the diffusion limit equal $8/5$ and $4/3$, respectively.

We see that in case $\xi_b = \xi_d$, the approximation for the mean is exact, as we saw in Corollary 1, while for $\xi_b \neq \xi_d$, the value of $\mathbb{E}Q_n/n$ converges to the limit $4/3$ rather quickly. The approximation for positive demand performs better for the first scenario. We can blame this on the discontinuity of the density function of Q_n if $\xi_b \neq \xi_d$.

9. Conclusions and suggestions for further research

In this paper we have studied a stochastic model for a blood bank. We have presented a global approach to the model in its full generality, and we have obtained very detailed exact expressions for the densities of amount of inventory and amount of demand (shortage) in special cases (exponential amounts of donated and requested blood; and either $\xi_b = \xi_d = 0$ or $\alpha_b = \alpha_d = 0$). Moreover, we have shown how an appropriate scaling, for the model in full generality, leads to an Ornstein-Uhlenbeck diffusion process, which can be used as a tool to obtain simple yet accurate approximations for some key performance measures.

Our model is a two-sided model, in the sense that we simultaneously consider the amount of blood in inventory and the amount of demand (shortage), one of the two at any time being zero. Such two-sided processes arise in many different settings, and thus are of considerable interest. The present setting is reminiscent of an organ transplantation problem, where there is either a queue of persons waiting to receive an organ, or a queue of donor organs. The perishability/impatience aspect features there, too [6]. A quite different setting is that of insurance risk. We refer to Albrecher and Lautscham [1] who extend the classical Cramér-Lundberg insurance risk model by allowing the capital of an insurance company to become negative – a situation that is usually indicated by "ruin" in the insurance literature. Their process thus becomes two-sided. The capital might become positive again; however, at a rate $\omega(x)$ when the capital has a negative value $-x$, bankruptcy is declared and the process ends. Interestingly, similar special functions (like hypergeometric functions) play a role in [1] and in the present study.

Our results are restricted to one type of blood. It would be very interesting to extend the analysis to multiple types of blood. Another important extension would be to use our results to facilitate the decision process that is faced by the CBB on a daily basis: Which amounts of blood, and of which types, should today be sent to the local blood banks (hospitals)? Knowing that, e.g., blood types O^-, A^-, B^-, AB^- can satisfy the corresponding $+$ type (but not vice versa), one may try to optimize the blood allocation process on the basis of actual amounts of blood present.

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