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A reinsurance risk model with a threshold coverage policy – The Gerber-Shiu penalty function

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Abstract

We consider a Cramér-Lundberg insurance risk process with the added feature of reinsurance. If an arriving claim finds the reserve below a certain threshold γ, or if it would bring the reserve below that level, then a reinsurer pays part of the claim. Using fluctuation theory and the theory of scale functions of spectrally negative Lévy processes, we derive expressions for the Laplace transform of the time to ruin and of the joint distribution of the deficit at ruin and the surplus before ruin. We specify these results in much more detail for the threshold set-up with proportional reinsurance.

1 Introduction

Let $X(t)$ be the surplus at time t of the classical *Cramér-Lundberg* risk process,

$$X(t) = u + ct - \sum_{i=1}^{N_t} Z_i \tag{1.1}$$

In this model the company earns premium at a fixed rate c, the claim arrival process $\{N_t : t \geq 0\}$ is a Poisson process at rate λ, $\{Z_i : i = 1, 2, \cdots\}$ are the successive claim amounts indexed by their appearance, and are i.i.d. positive random variables and $u = X(0)$.

In models like this it is of interest to study the distribution of the time to ruin, the joint distribution of the time to ruin, the deficit at ruin and the surplus before ruin. For a comprehensive overview of the state of the art of the classsical Cramér-Lundberg model see the book of Asmussen and Albrecher (2010).

In the last decade the classical Cramér-Lundberg model in (1.1) was modified to capture dividend payments to the shareholders. Under the threshold dividend policy, dividends at rate $\tilde{c} < c$ are paid whenever the reserve is above a threshold γ. This process has a 'bent' at γ and it is called refracted Lévy risk process; see *Gerber and Shiu* (2006), *Zhang et al.* (2006), *Dickson and Drekic* (2006), *Lin and Pavlova* (2006). *Wan* (2007) considered the more general model where the compound Poisson risk model is perturbed by a Brownian motion.

Lately, *Kyprianou and Loeffen* (2010) considered such a state dependent premium rate model for the general spectrally negative Lévy risk process. They used fluctuation theory and the theory of scale functions for spectrally negative Lévy processes to obtain the Laplace transform of the exit time, of the time to ruin, and of the joint probability for the surplus before and at ruin. In order to reduce risk, the insurer insures part of the risk. The insurer pays a premium to the reinsurer, while the reinsurer pays a part of each claim. Motivated by the threshold dividend policy, we consider the reinsurance threshold policy; the insurer pays a constant premium to the reinsurer and the reinsurer pays part of the claim that falls below a threshold γ.

We apply similar methods as in *Kyprianou and Loeffen* (2010) to obtain similar quantities of interest. Assume that the company has a reinsurance contract. To describe it define a function $I(x)$, where $I(x) < x$, and $I(x)$ is non-decreasing in x, $I(0) = 0$. The reinsurance pays part of the claim when the claim is below a given threshold γ. Let $\tilde{I}(x, y)$ denote the part that the insurer pays for a claim of size x occurring when the reserve level is y. The reinsurer pays $x - \tilde{I}(x, y)$. $\tilde{I}(x, y)$ is given by:

$$\tilde{I}(x,y) = \begin{cases} x & \text{if } y > \gamma, x \leq y - \gamma \\ y - \gamma + I(x - (y - \gamma)) & \text{if } y > \gamma, x \geq y - \gamma \\ I(x) & \text{if } y \leq \gamma \end{cases} \tag{1.2}$$

Examples for $I(x)$ are $I(x) = \min(a, x)$ for a given constant a, and $I(x) = \alpha x$, $0 < \alpha < 1$.

Throughout we will not specify the reserve level y in $\tilde{I}(x, y)$ but it will be clear from the context. We consider the following risk process: The premium rate, the claim arrival process and the claim amounts are as per (1.1). When a claim of size x arrival finds the reserve below γ the insurer pays only $I(x)$. When a claim of size x finds the reserve at level $y > \gamma$ the insurer pays $y-\gamma+I(x-(y-\gamma))$, i.e. he pays only part of the claim that falls below γ. We denote by U_t the reserve level at time t under this policy.

In a companion paper (Boxma et al. 2015) we analyze a risk process with state dependent premium rate and state dependent claim payments assuming a barrier dividend policy. Under this policy all the premium income is paid as dividends when the reserve level is bigger than a barrier b. In that paper we applied different tools to find the distribution of the deficit at ruin and the amount of dividends until ruin. In the present paper we consider a special case of state dependent claim payments and consider the expected discounted time to ruin and the joint distribution of the deficit at ruin and the reserve just before ruin.

The paper is organized as follows. Section 2 introduces some notations and a few identities related to exit times of spectrally negative Lévy processes which play a crucial role in the remainder of the paper. Section 3 presents expressions for the Laplace transform of the exit time from an upper barrier, the time to ruin and the joint probability for the surplus before and at ruin for general $I(x)$. In Section 4 these results are specified in much more detail for the case that $I(x) = \alpha x$.

2 Notations

Denote by X_1 a compound Poisson risk process with drift c (premium rate), and i.i.d. claims with distribution F_1 arriving according to a Poisson process at rate λ. Similarly X_0 is a compound

Poisson process with drift c (premium rate), and i.i.d. claims with distribution F_0 arriving according to a Poisson process at rate λ, where $F_0(x) = F_1(I^{-1}(x))$, where $I^{-1}(y) = \inf\{x : I(x) \geq y\}$. Let

$$\psi_i(s) = \mathbb{E}[e^{sX_i(1)}], \quad i = 0, 1$$

and let

$$\Phi_i(v) = \sup\{y \geq 0 : \psi_i(y) = v\}$$

Definition 2.1. *For a given spectrally negative Lévy process X, with Laplace exponent ψ, and $q \geq 0$, there is a unique q scale function associated with X, $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for $x < 0$, and on $(0, \infty)$, $W^{(q)}$ is the unique continuous function with Laplace transform*

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}. \tag{2.1}$$

We will denote $W^{(0)}$ by W. Consider a spectrally negative Lévy process X, with q-scale function $W^{(q)}$. For $b > 0$ let

$$\tau_b^+ = \inf\{t > 0 : X(t) \geq b\}$$

and

$$\tau_a^- = \inf\{t > 0 : X(t) < a\}.$$

For the modified process U define

$$\kappa_b^+ = \inf\{t > 0 : U_t \geq b\},$$

and

$$\kappa_a^- = \inf\{t > 0 : U_t < a\}.$$

We denote by $\mathbb{P}_x$ and by $\mathbb{E}_x$ the conditional probability and expectation given that $X(0) = x$. In the sequel we will apply the following identities, (Kyprianou (2010), Chapter 8):

Theorem 2.1. *Let X be a spectrally negative Lévy process. Then*

(i) *For $q \geq 0$, and $x \leq b$*

$$\mathbb{E}_x[e^{-q\tau_b^+} 1_{\tau_b^+ < \tau_0^-}] = \frac{W^{(q)}(x)}{W^{(q)}(b)}. \tag{2.2}$$

(ii) *Let $b > 0$, $x \in [0, b]$, $q \geq 0$ then*

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_b^+ \wedge \tau_0^-)dt = \left(\frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y) \right) dy. \tag{2.3}$$

3 The Laplace transform of the time to ruin-General $I(x)$

To derive the Laplace transform of the time to ruin we need to obtain some quantities. Let $b > \gamma$, and let $B \subset \mathbb{R}$. For $0 \leq x \leq b$ let

$$V^{(q)}(x, \gamma, b, B) = \int_{t=0}^\infty e^{-qt} \mathbb{P}_x(U_t \in B, t < \kappa_0^- \wedge \kappa_b^+)dt$$

$V^{(q)}(x,\gamma,b,B)$ is the discounted time that the process U_t spends in B, given that $U_0 = x$. Let $\gamma \leq x < b$, then

$$
\begin{aligned}
V^{(q)}(x,\gamma,b,B) &= \int_{t=0}^{\infty} e^{-qt} \int_{y\in B\cap[\gamma,b]} \mathbb{P}_{x-\gamma}(X_1(t)\in dy-\gamma, t<\tau_0^-\wedge\tau_{b-\gamma}^+)dydt && (3.1)\\
&+ \int_{t=0}^{\infty} e^{-qt} \int_{y=\gamma}^{b} \int_{\theta\geq y-\gamma} \mathbb{P}_{x-\gamma}(X_1(t)\in dy-\gamma, t<\tau_0^-\wedge\tau_{b-\gamma}^+)\lambda dF_1(\theta)dydt\cdot && (3.2)\\
&\quad \left(\int_{s=0}^{\infty}\int_{z\in B\cap[0,\gamma)} e^{-qs}\mathbb{P}_{(\gamma-I(\theta-(y-\gamma))}(X_0(s)\in dz, s<\tau_0^-\wedge\tau_\gamma^+)ds \right. && (3.3)\\
&+ \left. \mathbb{E}_{\gamma-I(\theta-(y-\gamma))}(e^{-q\tau_\gamma^+}, 1_{\tau_\gamma^+<\tau_0^-})V^{(q)}(\gamma,\gamma,b,B)\right) && (3.4)
\end{aligned}
$$

(3.1) describes the discounted time that U spends in $B\cap[\gamma,b]$ before it down-crosses the level γ. (3.2) multiplied by (3.3) is the discounted time that U spends in $B\cap[0,\gamma)$ from the moment it down-crosses γ until it exits $[0,\gamma)$. Similarly, (3.2) multiplied by (3.4) is the expected discounted time that U spends in B from the moment the process first hits γ after the first down-crossing of the level γ.

Using the scale function as in (2.3), and with $W_i^{(q)}(x)$ the scale function associated with X_i, $i=0,1$, we obtain that:

$$
\begin{aligned}
V^{(q)}(x,\gamma,b,B) &= \int_{y\in B\cap[\gamma,b]} \left(\frac{W_1^{(q)}(x-\gamma)W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)} - W_1^{(q)}(x-y)\right)dy\\
&+ \int_{y=0}^{b-\gamma}\int_{\theta\geq y}\left(\frac{W_1^{(q)}(x-\gamma)W_1^{(q)}(b-\gamma-y)}{W_1^{(q)}(b-\gamma)} - W_1^{(q)}(x-\gamma-y)\right)\lambda dF_1(\theta)dy\cdot\\
&\quad \left(\int_{z\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(\gamma-I(\theta-y))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(\gamma-I(\theta-y)-z)\right)dz\right.\\
&+ \left.\frac{W_0^{(q)}(\gamma-I(\theta-y))}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,b,B)\right), \quad \gamma\leq x<b \qquad (3.5)
\end{aligned}
$$

Similarly, for $x<\gamma$:

$$
\begin{aligned}
V^{(q)}(x,\gamma,b,B) &= \int_{t=0}^{\infty} e^{-qt}\int_{y\in B\cap[0,\gamma)} \mathbb{P}_x(X_0(t)\in dy, 1_{t<\tau_0^-\wedge\tau_\gamma^+})dt\\
&+ \int_{t=0}^{\infty} e^{-qt}\int_{y\in[0,\gamma)} \mathbb{P}_x(X_0(t)\in dy, 1_{\tau_\gamma^+<\tau_0^-})dtV^{(q)}(\gamma,\gamma,b,B)\\
&= \int_{y\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(x)W_0^{(q)}(\gamma-y)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(x-y)\right)dy\\
&+ \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,b,B) \qquad (3.6)
\end{aligned}
$$

To obtain $V^{(q)}(\gamma,\gamma,b,B)$ we substitute $x=\gamma$ in (3.5) and solve the associated equation.

$$\begin{aligned} V^{(q)}(\gamma,\gamma,b,B) &= \frac{1}{1-\int_{y=\gamma}^{b}\int_{\theta\geq y-\gamma}\left(\frac{W_1^{(q)}(0)W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)}\right)\lambda dF_1(\theta)\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))}{W_0^{(q)}(\gamma)}dy}\cdot \\ &\quad \left[\int_{y\in B\cap[\gamma,b)}\left(\frac{W_1^{(q)}(0)W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)}\right)dy\right. \\ &+ \left(\int_{y=\gamma}^{b}\int_{\theta\geq y-\gamma}\left(\frac{W_1^{(q)}(0)W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)}\right)\lambda dF_1(\theta)dy\cdot\right. \\ &\quad \left.\left.\cdot\int_{z\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(\gamma-I(\theta-(y-\gamma)))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)}-W_0^{(q)}(\gamma-I(\theta+\gamma-y)-z)\right)dz\right)\right] \end{aligned} \tag{3.7}$$

Next we obtain

$$V^{(q)}(x,\gamma,B)=\int_{t=0}^{\infty}e^{-qt}\mathbb{P}_x(U_t\in B,t<\kappa_0^-)dt=\lim_{b\to\infty}V^{(q)}(x,\gamma,b,B)$$

For a Lévy process $X(t)$ with Lévy exponent ψ, adapted to a σ-field $\mathcal{F}_t$ let

$$M_t(\beta)=e^{\beta X(t)-\psi(\beta)t}$$

be the Wald martingale associated with X. Define the measure $\mathbb{P}^\beta$ as follows: For $\mathcal{A}\in\mathcal{F}_t$ define

$$\mathbb{P}^\beta(\mathcal{A})=\mathbb{E}[e^{M_t(\beta)}1_{\mathcal{A}}]$$

Denote by $W_{(\beta)}^{(q)}$ the scale function associated with X under $\mathbb{P}^\beta$. Apply (Kyprianou 2010) Chapter 8.2, or eq. (53) in Kuznetsov et al. (2013):

$$W^{(q)}(x)=e^{\Phi(q)x}W_{(\Phi(q))}(x)=e^{\Phi(q)x}\frac{1}{\psi'_{\Phi(q)}(0^+)}\mathbb{P}_x^{\Phi(q)}(\underline{X}(\infty)\geq 0), \tag{3.8}$$

where $\underline{X}(t)=\inf_{s\leq t}X(s)$. Under the measure $\mathbb{P}_x^{\Phi(q)}$, X drifts to ∞. Applying (3.8) to X_1, we get that for $\gamma<y<b$

$$\frac{W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)}=\frac{e^{\Phi_1(q)(b-y)}\mathbb{P}_{b-y}^{\Phi_1(q)}(\underline{X}_1(\infty)\geq 0)}{e^{\Phi_1(q)(b-\gamma)}\mathbb{P}_{b-\gamma}^{\Phi_1(q)}(\underline{X}_1(\infty)\geq 0)}\underset{b\to\infty}{\longrightarrow}e^{-\Phi_1(q)(y-\gamma)} \tag{3.9}$$

Thus, taking the limit as $b\to\infty$ in (3.5) we get that for $x>\gamma$,

$$\begin{aligned} V^{(q)}(x,\gamma,B) &= \int_{y\in B\cap[\gamma,\infty)}\left(W_1^{(q)}(x-\gamma)e^{-\Phi_1(q)(y-\gamma)}-W_1^{(q)}(x-y)\right)dy \\ &+ \int_{y=\gamma}^{\infty}\int_{\theta\geq y-\gamma}\left(W_1^{(q)}(x-\gamma)e^{-\Phi_1(q)(y-\gamma)}-W_1^{(q)}(x-y)\right)\lambda dF_1(\theta)dy\cdot \\ &\quad \left(\int_{z\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)}-W_0^{(q)}(\gamma-I(\theta+\gamma-y)-z)\right)dz\right. \\ &+ \left.\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,B)\right) \end{aligned} \tag{3.10}$$

For $x < \gamma$, we obtain a similar expression as in (3.6), by replacing $V^{(q)}(x, \gamma, b, B)$ and $V^{(q)}(\gamma, \gamma, b, B)$ by $V^{(q)}(x, \gamma, B)$ and $V^{(q)}(\gamma, \gamma, B)$ respectively.

$V^{(q)}(\gamma, \gamma, B)$ is obtained by substituting $x = \gamma$ in (3.10) .

$$
\begin{aligned}
V^{(q)}(\gamma,\gamma,B) &= \frac{1}{1-\int_{y=\gamma}^{\infty}\int_{\theta\geq y-\gamma} W_1^{(q)}(0)e^{-\Phi_1(q)(y-\gamma)}\lambda dF_1(\theta)\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))}{W_0^{(q)}(\gamma)}dy}\cdot \\
&\quad \left[\int_{y\in B\cap(\gamma,\infty]} W_1^{(q)}(0)e^{-\Phi_1(q)(y-\gamma)}dy\right. \\
&+ \left(\int_{y=\gamma}^{\infty}\int_{\theta\geq y-\gamma} W_1^{(q)}(0)e^{-\Phi_1(q)(y-\gamma)}\lambda dF_1(\theta)dy\cdot\right. \qquad (3.11)\\
&\quad \left.\left.\cdot\int_{z\in B\cap(0,\gamma]}\left(\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(\gamma-I(\theta+\gamma-y)-z)\right)dz\right)\right]
\end{aligned}
$$

Introduce $\mathcal{E}_q$, an Exponentially(q) distributed random variable. The Laplace transform of the time to ruin is:

$$
\begin{aligned}
&\mathbb{E}_x[e^{-q\kappa_0^-}1_{\kappa_0^-<\infty}] = \mathbb{P}_x[\mathcal{E}_q > \kappa_0^-] \\
&= 1-\mathbb{P}_x[\mathcal{E}_q \leq \kappa_0^-] = q\int_0^{\infty} e^{-qt}\mathbb{P}_x(U_t\in[0,\infty))dt \\
&= 1-qV^{(q)}(x,\gamma,[0,\infty))
\end{aligned}
$$

Let $h(x, y)$ be a bounded function. The Gerber-Shiu penalty function reflects the discounted cost of the insurer at the time of ruin, κ_0^-, as a function of the surplus just before ruin $U_{\kappa_{0-}^-}$ and the deficit at ruin $|U_{\kappa_0^-}|$, assuming a discount factor q.

$$
\begin{aligned}
m(x,q) &= \mathbb{E}_x[e^{-q\kappa_0^-}h(U_{\kappa_{0-}^-},|U_{\kappa_0^-}|)] \\
&= \int_{y=0}^{\gamma} V^{(q)}(x,\gamma,dy)\int_{\theta\geq y}\lambda dF_0(\theta)h(y,\theta-y)d\theta \\
&+ \int_{y=\gamma}^{\infty} V^{(q)}(x,\gamma,dy)\int_{\theta\geq y, I(\theta-(y-\gamma))>\gamma}\lambda dF_1(\theta)h(y,I(\theta-(y-\gamma))-\gamma)d\theta
\end{aligned}
$$

4 $I(x) = \alpha x$

In this section we consider the case where $I(x) = \alpha x$, $0 < \alpha < 1$. In this case we will obtain simpler expressions, especially for equations (3.5)-(3.11). We will get expressions involving only one integral instead of two. We apply a similar approach as in Kyprianou and Loeffen (2010) (See also Chapter 8 in Kyprianou (2013)). Throughout we put index 1 for quantities related to the risk process $X_1(t)$ with premium rate c and claim distribution F_1, and 0 for quantities related to the risk process $X_0(t)$ with premium rate c and claim distribution F_0, where $F_0(x) = F_1(x/\alpha)$.

This section is organized as follows. Section 4.1 considers the exit time from an upper threshold b before getting ruined, and establishes a key identity which is applied in the remainder of this section. In Section 4.2 we obtain an expression for the discounted time that $U_t \in B$ before exiting

$[0, b]$, where $B \subset \mathbb{R}$, and in Section 4.3 we obtain a similar expression for the case that $b = \infty$. Section 4.4 presents the Laplace transform of the time to ruin, and Section 4.5 the ruin probability. Section 4.6 presents an expression for the Gerber-Shiu penalty function and the joint probability of the surplus before and at ruin.

4.1 Exit time for U

Let $\gamma < b$. We first obtain an expression for $p(x, \gamma, b, q)$, where

$$p(x, \gamma, b, q) = \mathbb{E}_x(e^{-q\kappa_b^+} \mathbf{1}_{\{\kappa_b^+ < \kappa_0^-\}} | U_0 = x).$$

Let $\gamma \leq x < b$. By (2.2) and (2.3):

$$\begin{aligned} p(x, \gamma, b, q) &= \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} \\ &+ \int_{y=0}^{b-\gamma} \int_{\theta \geq y} \left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-\gamma-y) - W_1^{(q)}(x-\gamma-y) \right) \\ &\times \frac{W_0^{(q)}(\gamma - \alpha(\theta - y))}{W_0^{(q)}(\gamma)} \lambda dF_1(\theta) dy \, p(\gamma, \gamma, b, q) \end{aligned} \tag{4.1}$$

For $0 < x \leq \gamma$,

$$p(x, \gamma, b, q) = \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q)$$

Since $W_1^{(q)}(0) = 1/c > 0$, see e.g. Section 8 in Kyprianou (2006), substituting $x = \gamma$ in (4.1), we obtain after some calculations that:

$$p(\gamma, \gamma, b, q) = W_0^{(q)}(\gamma) \left(W_1^{(q)}(b-\gamma) W_0^{(q)}(\gamma) c - \int_{y=0}^{b-\gamma} \int_{\theta \geq y} W_1^{(q)}(b-\gamma-y) W_0^{(q)}(\gamma - \alpha(\theta - y)) \lambda dF_1(\theta) \, dy \right)^{-1} \tag{4.2}$$

For $x \geq \gamma$ let

$$A(x) = \int_{y=0}^{x-\gamma} \int_{\theta \geq y} W_1^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma - \alpha(\theta - y)) \lambda dF_1(\theta) \, dy \tag{4.3}$$

We define $A(x) = 0$ for $x < \gamma$. We want to find a simpler expression for $A(x)$, $x \geq \gamma$. We apply similar techniques as in Kyprianou and Loeffen (2010) (see also p. 72 in Kyprianou (2013)) for a risk process where the premium rate decreases when the process up-crosses a given threshold. Consider

the Laplace transform with respect to x of $A(x)$:

$$\begin{aligned}
&\int_{x=\gamma}^{\infty} e^{-sx} \int_{y=0}^{x-\gamma} \int_{\theta\geq y} W_1^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma-\alpha(\theta-y))\lambda dF_1(\theta)\, dy dx \\
= \;& e^{-s\gamma} \int_{x=\gamma}^{\infty} e^{-s(x-\gamma)} \int_{y=0}^{x-\gamma} \int_{\theta\geq y} W_1^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma-\alpha(\theta-y))\lambda dF_1(\theta) dy\, dx \\
= \;& e^{-s\gamma} \int_{x=0}^{\infty} e^{-sx} \int_{y=0}^{x} \int_{\theta\geq y} W_1^{(q)}(x-y) W_0^{(q)}(\gamma-\alpha(\theta-y))\lambda dF_1(\theta) dy\, dx \qquad (4.4) \\
= \;& e^{-s\gamma} \int_{y=0}^{\infty} e^{-sy} \int_{\theta\geq y} W_0^{(q)}(\gamma-\alpha(\theta-y))\lambda dF_1(\theta) \int_{x=y}^{\infty} e^{-s(x-y)} W_1^{(q)}(x-y) dx\, dy \\
= \;& \frac{e^{-s\gamma}}{\psi_1(s)-q} \int_{y=0}^{\infty} e^{-sy} \int_{\theta\geq y} W_0^{(q)}(\gamma-\alpha(\theta-y))\lambda dF_1(\theta) dy \qquad (4.5)
\end{aligned}$$

where in the last equality we applied the identity:

$$\int_0^{\infty} e^{-sx} W_1^{(q)}(x) dx = \frac{1}{\psi_1(s)-q}$$

Consider now the double integral in the last line of (4.5).

$$\begin{aligned}
&\int_{y=0}^{\infty} e^{-sy} \int_{\theta\geq y} W_0^{(q)}(\gamma-\alpha(\theta-y)) dF_1(\theta) dy \\
= \;& \int_{y=0}^{\infty} e^{-sy} \int_{\theta\geq y} W_0^{(q)}(\gamma-\alpha(\theta-y)) dF_0(\alpha\theta) dy \\
= \;& \int_{y=0}^{\infty} e^{-sy} \int_{\theta\geq \alpha y} W_0^{(q)}(\gamma-\theta+\alpha y) dF_0(\theta) dy \\
= \;& \frac{1}{\alpha} \int_{y=0}^{\infty} e^{-sy/\alpha} \int_{\theta\geq y} W_0^{(q)}(\gamma-\theta+y) dF_0(\theta) dy \qquad (4.6)
\end{aligned}$$

To obtain the last expression assume that the process behaves as X_0, i.e. the claim amount is always distributed as F_0, (i.e. $\alpha = 1$). In this case $\tau_b^+ = \kappa_b^+, \tau_0^- = \kappa_0^-$. Then,

$$\mathbb{E}(e^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \tau_0^-\}} | X_0(0) = \gamma) = \frac{W_0^{(q)}(\gamma)}{W_0^{(q)}(b)}. \qquad (4.7)$$

By (4.2) (with X_1 behaving now exactly as X_0) and (4.7) we obtain the following equality

$$\frac{W_0^{(q)}(\gamma)}{W_0^{(q)}(b)} = W_0^{(q)}(\gamma) \left(W_0^{(q)}(b-\gamma) W_0^{(q)}(\gamma) c - \int_{y=0}^{b-\gamma} \int_{\theta\geq y} W_0^{(q)}(b-\gamma-y) W_0^{(q)}(\gamma-(\theta-y))\lambda dF_0(\theta)\, dy \right)^{-1} \qquad (4.8)$$

Since b is any threshold such that $b > \gamma$, replacing b by x, $x > \gamma$ in (4.8) we obtain the following equality:

$$\begin{aligned}
&\int_{y=0}^{x-\gamma} \int_{\theta\geq y} W_0^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma-(\theta-y))\lambda dF_0(\theta)\, dy \\
= \;& c W_0^{(q)}(x-\gamma) W_0^{(q)}(\gamma) - W_0^{(q)}(x) \qquad (4.9)
\end{aligned}$$

Taking Laplace transform from the two sides of (4.9) we obtain that :

$$\begin{aligned} & \int_{x=\gamma}^{\infty} e^{-sx} \int_{y=0}^{x-\gamma} \int_{\theta \geq y} W_0^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma-(\theta-y)) \lambda dF_0(\theta)\, dy \\ = \; & c W_0^{(q)}(\gamma) \int_{x=\gamma}^{\infty} e^{-sx} W_0^{(q)}(x-\gamma) dx - \int_{x=\gamma}^{\infty} e^{-sx} W_0^{(q)}(x) dx \end{aligned} \tag{4.10}$$

Repeating the same calculations as in (4.5) (replacing F_1 by F_0 and $W_1^{(q)}$ by $W_0^{(q)}$), the lefthand side of (4.10) equals

$$\begin{aligned} & e^{-s\gamma} \int_{x=0}^{\infty} e^{-sx} \int_{y=0}^{x} \int_{\theta \geq y} W_0^{(q)}(x-y) W_0^{(q)}(\gamma-(\theta-y)) \lambda dF_0(\theta)\, dy \\ = \; & e^{-s\gamma} \int_{y=0}^{\infty} e^{-sy} \int_{\theta \geq y} W_0^{(q)}(\gamma-(\theta-y)) \lambda dF_0(\theta) \int_{x=y}^{\infty} e^{-s(x-y)} W_0^{(q)}(x-y) dx\, dy \\ = \; & \frac{e^{-s\gamma}}{\psi_0(s)-q} \int_{y=0}^{\infty} e^{-sy} \int_{\theta \geq y} W_0^{(q)}(\gamma-(\theta-y)) \lambda dF_0(\theta) dy \end{aligned} \tag{4.11}$$

The righthand side of (4.10) equals:

$$\begin{aligned} & c W_0^{(q)}(\gamma) \int_{x=\gamma}^{\infty} e^{-sx} W_0^{(q)}(x-\gamma) dx - \int_{x=\gamma}^{\infty} e^{-sx} W_0^{(q)}(x) dx \\ = \; & c W_0^{(q)}(\gamma) \frac{e^{-s\gamma}}{\psi_0(s)-q} - \mathcal{L}_\gamma W_0^{(q)}[s] \end{aligned} \tag{4.12}$$

where

$$\mathcal{L}_\gamma W_0^{(q)}[s] = \int_{x=\gamma}^{\infty} e^{-sx} W_0^{(q)}(x) dx$$

Thus, (4.10), (4.11) and (4.12) yield:

$$\begin{aligned} & \int_{y=0}^{\infty} e^{-sy} \int_{\theta \geq y} W_0^{(q)}(\gamma-(\theta-y)) \lambda dF_0(\theta) dy \\ = \; & c W_0^{(q)}(\gamma) - e^{\gamma s}(\psi_0(s)-q) \mathcal{L}_\gamma W_0^{(q)}[s] \end{aligned} \tag{4.13}$$

(4.5), (4.6), and (4.13) yield that the Laplace transform of $A(x)$ is

$$\frac{1}{\alpha} \frac{e^{-s\gamma}}{\psi_1(s)-q} (c W_0^{(q)}(\gamma) - e^{\gamma s/\alpha}(\psi_0(s/\alpha)-q) \mathcal{L}_\gamma W_0^{(q)}[s/\alpha]) \tag{4.14}$$

Let $\delta = \frac{c}{\alpha} - c$, then it is easy to see that

$$\psi_1(s) = \psi_0(s/\alpha) - \delta s. \tag{4.15}$$

Thus the Laplace transform of $A(x)$ can be written as follows:

$$\frac{1}{\alpha} \frac{e^{-s\gamma}}{\psi_1(s)-q} c W_0^{(q)}(\gamma) - \frac{1}{\alpha} e^{\gamma \frac{s}{\alpha}(1-\alpha)} \mathcal{L}_\gamma W_0^{(q)}[s/\alpha] - \frac{1}{\psi_1(s)-q} \frac{\delta s}{\alpha} e^{\gamma \frac{s}{\alpha}(1-\alpha)} \mathcal{L}_\gamma W_0^{(q)}[s/\alpha]) \tag{4.16}$$

To invert the Laplace transform notice that:

$$\frac{e^{-s\gamma}}{\psi_1(s)-q} = \int_\gamma^\infty e^{-sx}W_1^{(q)}(x-\gamma)dx \tag{4.17}$$

$$\frac{1}{\alpha}e^{\gamma s(\frac{1}{\alpha}-1)}\mathcal{L}_\gamma W_0^{(q)}[s/\alpha] = \int_\gamma^\infty e^{-sx}W_0^{(q)}(\alpha x+\gamma(1-\alpha))dx \tag{4.18}$$

$$\frac{1}{\alpha}\frac{se^{\gamma s(\frac{1}{\alpha}-1)}}{\psi_1(s)-q}\mathcal{L}_\gamma W_0^{(q)}[s/\alpha] = \frac{e^{-s\gamma}}{\psi_1(s)-q}W_0^{(q)}(\gamma)+\int_{x=\gamma}^\infty e^{-sx}\int_{y=\gamma}^x W_1^{(q)}(x-y)\alpha W_0^{(q)'}(\alpha y+\gamma(1-\alpha))dydx, \tag{4.19}$$

where

$$W_0^{(q)'}(x)=\frac{d}{dx}W_0^{(q)}(x)$$

Thus,

$$\alpha W_0^{(q)'}(\alpha y+\gamma(1-\alpha))=\frac{d}{dy}W_0^{(q)}(\alpha y+\gamma(1-\alpha))$$

(4.19) is derived as follows:

$$\begin{aligned}
&\int_{x=\gamma}^\infty e^{-sx}\int_{y=\gamma}^x W_1^{(q)}(x-y)\alpha W_0^{(q)'}(\alpha y+\gamma(1-\alpha))dydx\\
=\;&\alpha\int_{y=\gamma}^\infty e^{-sy}W_0^{(q)'}(\alpha y+\gamma(1-\alpha))\int_{x=y}^\infty e^{-s(x-y)}W_1^{(q)}(x-y)dydx\\
=\;&\frac{1}{\psi_1(s)-q}\left[-e^{-s\gamma}W_0^{(q)}(\gamma)+s\int_{y=\gamma}^\infty e^{-sy}W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy\right]\\
=\;&\frac{1}{\psi_1(s)-q}\left[-e^{-s\gamma}W_0^{(q)}(\gamma)+\frac{1}{\alpha}se^{\gamma s(\frac{1}{\alpha}-1)}\mathcal{L}_\gamma W_0^{(q)}[s/\alpha]\right]
\end{aligned}$$

(4.3), (4.5), (4.6) and (4.9) yield:

$$A(x)=cW_0^{(q)}(\gamma)W_1^{(q)}(x-\gamma)-W_0^{(q)}(\alpha x+\gamma(1-\alpha))-\alpha\delta\int_\gamma^x W_1^{(q)}(x-y)W_0^{(q)'}(\alpha y+\gamma(1-\alpha))dy \tag{4.20}$$

For $x\geq\gamma$ define

$$w_\alpha^{(q)}(x,z)=W_0^{(q)}(\alpha x-z+\gamma(1-\alpha))+\alpha\delta\int_\gamma^x W_1^{(q)}(x-y)W_0^{(q)'}(\alpha y-z+\gamma(1-\alpha))dy \tag{4.21}$$

and for $x<\gamma$ define

$$w_\alpha^{(q)}(x,z)=W_0^{(q)}(x-z). \tag{4.22}$$

In this subsection we need only $w_\alpha^{(q)}(x,0)$, later we will use $w_\alpha^{(q)}(x,z)$. For $x>\gamma$

$$A(x)=cW_0^{(q)}(\gamma)W_1^{(q)}(x-\gamma)-w_\alpha^{(q)}(x,0).$$

Substituting (4.20) in (4.1) and (4.2), we obtain for $\gamma \le x \le b$ that

$$\begin{aligned}
& p(x,\gamma,b,q) = \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} + \\
& \int_0^{b-\gamma}\int_{\theta\ge y}\left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}W_1^{(q)}(b-\gamma-y) - W_1^{(q)}(x-\gamma-y)\right) \\
\times\ & \frac{W_0^{(q)}(\gamma-\alpha(\theta-y))}{W_0^{(q)}(\gamma)}\lambda dF_1(\theta)dy\, p(\gamma,\gamma,b,q) \\
=\ & \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} + \left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}\frac{A(b)}{W_0^{(q)}(\gamma)} - \frac{A(x)}{W_0^{(q)}(\gamma)}\right) p(\gamma,\gamma,b,q) \\
=\ & \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} + \left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}A(b) - A(x)\right)\frac{1}{W_1^{(q)}(b-\gamma)W_0^{(q)}(\gamma)c - A(b)}
\end{aligned}$$

Thus, for $x \ge \gamma$

$$\begin{aligned}
& p(x,\gamma,b,q) \\
=\ & \frac{1}{W_1^{(q)}(b-\gamma)w_\alpha^{(q)}(b,0)} \\
\times\ & \Big(W_1^{(q)}(x-\gamma)w_\alpha^{(q)}(b,0) + W_1^{(q)}(x-\gamma)W_1^{(q)}(b-\gamma)W_0^{(q)}(\gamma)c - W_1^{(q)}(x-\gamma)w_\alpha^{(q)}(b,0) \\
-\ & (W_1^{(q)}(x-\gamma)W_1^{(q)}(b-\gamma)W_0^{(q)}(\gamma)c - W_1^{(q)}(b-\gamma)w_\alpha^{(q)}(x,0)\Big) = \frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}
\end{aligned}$$

Notice that

$$p(\gamma,\gamma,b,q) = \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)} = \frac{w_\alpha^{(q)}(\gamma,0)}{w_\alpha^{(q)}(b,0)}$$

Thus, for $x < \gamma$

$$p(x,\gamma,b,q) = \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}p(\gamma,\gamma,b,q) = \frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}.$$

Hence we can, for all $x \le b$, write $p(x,\gamma,b,q) = \frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}$.

4.2 $V^{(q)}(x,\gamma,b,B)$

We obtain for $B \subset (0,b)$ the discounted time the process is in B before exiting $(0,b)$. Define

$$V^{(q)}(x,\gamma,b,B) = \int_0^\infty e^{-qt}\mathbb{P}_x(U_t \in B, t < \kappa_0^- \wedge \kappa_b^+)dt$$

First we consider the case $\gamma \le x < b$. By (3.5)

$$\begin{aligned} V^{(q)}(x,\gamma,b,B) &= \int_{y\in B\cap[\gamma,b)} \left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) - W_1^{(q)}(x-y) \right) dy \\ &+ \int_{y=0}^{b-\gamma} \int_{\theta\ge y} \left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-\gamma-y) - W_1^{(q)}(x-\gamma-y) \right) \lambda dF_1(\theta) \cdot \\ &\quad \left\{ \int_{z\in[0,\gamma)\cap B} \left(\frac{W_0^{(q)}(\gamma-\alpha(\theta-y))}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma-z) - W_0^{(q)}(\gamma-z-\alpha(\theta-y)) \right) dz \right. \\ &+ \left. \frac{W_0^{(q)}(\gamma-\alpha(\theta-y))}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma,\gamma,b,B) \right\} dy \end{aligned}$$

Notice that for $x \ge \gamma$:

$$\int_{y=0}^{x-\gamma} \int_{\theta\ge y} W_1^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma-\alpha(\theta-y)) \lambda dF_1(\theta) dy = A(x) = cW_0^{(q)}(\gamma) W_1^{(q)}(x-\gamma) - w_\alpha^{(q)}(x,0).$$

To obtain

$$A(x,z) = \int_{y=0}^{x-\gamma} \int_{\theta\ge y} W_1^{(q)}(x-\gamma-y) W_0^{(q)}(\gamma-z-\alpha(\theta-y)) \lambda dF_1(\theta) dy,$$

substitute $\gamma_z = \gamma - z, x_z = x - z$ in (4.3). Similar to the calculation of $A(x)$ in (4.20),

$$A(x,z) = cW_0^{(q)}(\gamma-z) W_1^{(q)}(x-\gamma) - w_\alpha^{(q)}(x,z)$$

where, cf. (4.21), for $x \ge \gamma$:

$$w_\alpha^{(q)}(x,z) = W_0^{(q)}(\alpha(x-z) + (\gamma-z)(1-\alpha)) + \alpha\delta \int_{\gamma-z}^{x-z} W_1^{(q)}(x-z-y) W_0^{(q)\prime}(\alpha y + (1-\alpha)(\gamma-z)) dy \tag{4.23}$$

After some algebra,

$$w_\alpha^{(q)}(x,z) = W_0^{(q)}(\alpha x - z + \gamma(1-\alpha)) + \alpha\delta \int_{\gamma}^{x} W_1^{(q)}(x-y) W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha)) dy. \tag{4.24}$$

Thus, for $\gamma \le x \le b$ we obtain

$$\begin{aligned} V^{(q)}(x,\gamma,b,B) &= \int_{y\in B\cap[\gamma,b)} \left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) - W_1^{(q)}(x-y) \right) dy \\ &+ \int_{z\in(0,\gamma)\cap B} \left(\frac{W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)} \left(-\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,0) + w_\alpha^{(q)}(x,0) \right) \right. \\ &+ \left. \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,z) - w_\alpha^{(q)}(x,z) \right) dz \\ &+ \left(w_\alpha^{(q)}(x,0) - \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,0) \right) \frac{V^{(q)}(\gamma,\gamma,b,B)}{W_0^{(q)}(\gamma)} \end{aligned} \tag{4.25}$$

Substituting $x = \gamma$ we obtain a relation for $V^{(q)}(\gamma, \gamma, b, B)$

$$\begin{aligned} V^{(q)}(\gamma,\gamma,b,B) &= \int_{y\in B\cap[\gamma,b)} \left(\frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) \right) dy \\ &+ \int_{z\in[0,\gamma)\cap B} \left(\frac{W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)} \left(-\frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,0) + w_\alpha^{(q)}(\gamma,0) \right) \right. \\ &+ \left. \frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,z) - w_\alpha^{(q)}(\gamma,z) \right) dz \\ &+ \left(w_\alpha^{(q)}(\gamma,0) - \frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,0) \right) \frac{V^{(q)}(\gamma,\gamma,b,B)}{W_0^{(q)}(\gamma)} \end{aligned}$$

Since $w_\alpha^{(q)}(\gamma, z) = W_0^{(q)}(\gamma - z)$, we obtain that

$$\begin{aligned} \frac{W_1^{(q)}(0) w_\alpha^{(q)}(b,0)}{W_1^{(q)}(b-\gamma)} \frac{V^{(q)}(\gamma,\gamma,b,B)}{W_0^{(q)}(\gamma)} &= \int_{y\in B\cap[\gamma,b)} \frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) dy \\ &+ \int_{z\in(0,\gamma)\cap B} \left(W_0^{(q)}(\gamma-z) \left(1 - \frac{W_1^{(q)}(0) w_\alpha^{(q)}(b,0)}{W_1^{(q)}(b-\gamma) W_0^{(q)}(\gamma)} \right) \right. \\ &+ \left. \frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,z) - W_0^{(q)}(\gamma-z) \right) dz \end{aligned}$$

and, noticing that the two $W_0^{(q)}(\gamma - z)$ terms cancel in the righthand side

$$\begin{aligned} V^{(q)}(\gamma,\gamma,b,B) &= \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)} \int_{y\in B\cap[\gamma,b)} W_1^{(q)}(b-y) dy \\ &+ \int_{z\in(0,\gamma)\cap B} \left(-W_0^{(q)}(\gamma-z) + \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)} w_\alpha^{(q)}(b,z) \right) dz \end{aligned}$$

Substituting $V^{(q)}(\gamma, \gamma, b, B)$ back into (4.25) we obtain that:

$$
\begin{aligned}
V^{(q)}(x,\gamma,b,B) &= \int_{y\in B\cap[\gamma,b)}\left(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}W_1^{(q)}(b-y)-W_1^{(q)}(x-y)\right)dy\\
&+ \left(\frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}-\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}\right)\int_{y\in B\cap[\gamma,b)}W_1^{(q)}(b-y)dy\\
&+ \int_{z\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)}\left(-\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}w_\alpha^{(q)}(b,0)+w_\alpha^{(q)}(x,0)\right)\right.\\
&+ \left.\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)}w_\alpha^{(q)}(b,z)-w_\alpha^{(q)}(x,z)\right)dz\\
&+ \left(\frac{w_\alpha^{(q)}(x,0)}{W_0^{(q)}(\gamma)}-\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)W_0^{(q)}(\gamma)}w_\alpha^{(q)}(b,0)\right)\cdot\\
&\quad\int_{z\in B\cap[0,\gamma)}\left(-W_0^{(q)}(\gamma-z)+\frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)}w_\alpha^{(q)}(b,z)\right)dz
\end{aligned}
$$

Thus, for $\gamma\le x<b$,

$$
\begin{aligned}
V^{(q)}(x,\gamma,b,B) &= \int_{y\in B\cap[\gamma,b)}\left(\frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}W_1^{(q)}(b-y)-W_1^{(q)}(x-y)\right)dy\\
&+ \int_{y\in B\cap[0,\gamma)}\left(\frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}w_\alpha^{(q)}(b,y)-w_\alpha^{(q)}(x,y)\right)dy \qquad (4.26)
\end{aligned}
$$

For $x<\gamma$ we obtain from (3.6)

$$
\begin{aligned}
V^{q)}(x,\gamma,b,B) &= \int_{y\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}W_0^{(q)}(\gamma-y)-W_0^{(q)}(x-y)\right)dy\\
&+ \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,b,B)\\
&= \int_{y\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}W_0^{(q)}(\gamma-y)-W_0^{(q)}(x-y)\right)dy\\
&+ \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}\left(\frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)}\int_{y\in B\cap[\gamma,b)}W_1^{(q)}(b-y)dy\right.\\
&+ \left.\int_{y\in B\cap[0,\gamma]}(-W_0^{(q)}(\gamma-y)+\frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)}w_\alpha^{(q)}(b,y))dy\right)\\
&= \int_{y\in B\cap[0,\gamma)}\left(\frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b,0)}w_\alpha^{(q)}(b,y)-W_0^{(q)}(x-y)\right)dy\\
&+ \frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b,0)}\int_{y\in B\cap[\gamma,b)}W_1^{(q)}(b-y)dy \qquad (4.27)
\end{aligned}
$$

Since for $x<\gamma$, cf. (4.22), $w_\alpha^{(q)}(x,z)=W_0^{(q)}(x-z)$, cf. (4.22), we get a similar expression to (4.26).

4.3 $b \to \infty$

In this subsection we obtain

$$V^{(q)}(x,\gamma,B) = \mathbb{E}_x\left[\int_{t=0}^{\infty} e^{-qt} 1_{(U_t \in B, t<\kappa_0^-)}\right] \tag{4.28}$$

By letting $b \to \infty$ in $V^{(q)}(x,\gamma,b,B)$:

$$V^{(q)}(x,\gamma,B) = \lim_{b\to\infty} V^{(q)}(x,\gamma,b,B)$$

Applying the first equality in (3.8) to (4.23) we obtain

$$\begin{aligned}
\frac{w_\alpha^{(q)}(b,z)}{W_1^{(q)}(b)} &= \frac{W_0^{(q)}(\alpha b - z + \gamma(1-\alpha)) + \delta\alpha\int_\gamma^b W_1^{(q)}(b-y)W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha))dy}{W_1^{(q)}(b)} \\
&= \frac{e^{\Phi_0(q)(\alpha b - z + \gamma(1-\alpha))} W_{(\Phi_0(q))}(\alpha b - z + \gamma(1-\alpha))}{e^{\Phi_1(q)b} W_{(\Phi_1(q))}(b)} \qquad (4.29) \\
&+ \delta\alpha\int_\gamma^b \frac{e^{\Phi_1(q)(b-y)} W_{(\Phi_1(q))}(b-y)}{e^{\Phi_1(q)b} W_{(\Phi_1(q))}(b)} W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha))dy \qquad (4.30)
\end{aligned}$$

Since $X_0(1) > X_1(1)$, $\psi_0(s) > \psi_1(s)$, and $\Phi_0(q) < \Phi_1(q)$. By Kuznetsov et al. (2013) Section 3.1, $W_{(\Phi(q))}(+\infty) = 1/\psi'(\Phi(q))$.

Thus

$$e^{(\Phi_0(q)\alpha - \Phi_1(q))b} \underset{b\to\infty}{\longrightarrow} 0$$

and the limit as $b \to \infty$ of the expression in (4.29) is 0. The limit

$$\frac{\int_\gamma^b W_1^{(q)}(b-y)W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha))dy}{W_1^{(q)}(b)} \underset{b\to\infty}{\longrightarrow} \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha))dy \tag{4.31}$$

For $z = 0$ in (4.31) define:

$$A = \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha))dy \tag{4.32}$$

Applying (3.9), (4.29)-(4.32)

$$\frac{W_1^{(q)}(b-y)}{w_\alpha^{(q)}(b,0)} = \frac{\frac{W_1^{(q)}(b-y)}{W_1^{(q)}(b)}}{\frac{w_\alpha^{(q)}(b,0)}{W_1^{q)}(b)}} \underset{b\to\infty}{\longrightarrow} \frac{e^{-\Phi_1(q)y}}{\delta\alpha A}, \tag{4.33}$$

and

$$\frac{w_\alpha^{(q)}(b-y)}{w_\alpha^{(q)}(b,0)} = \frac{\frac{w_\alpha^{(q)}(b-y)}{W_1^{(q)}(b)}}{\frac{w_\alpha^{(q)}(b,0)}{W_1^{q)}(b)}} \underset{b\to\infty}{\longrightarrow} \frac{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha))dy}{A} \tag{4.34}$$

Let $x > \gamma$. Letting $b \to \infty$ in (4.26) we obtain by (4.33) and (4.34):

$$\begin{aligned} V^{(q)}(x,\gamma,B) &= \int_{y\in B\cap[\gamma,\infty)} \left(\frac{w_\alpha^{(q)}(x,0)}{\delta\alpha A} e^{-\Phi_1(q)y} - W_1^{(q)}(x-y) \right) dy \\ &+ \int_{y\in B\cap[0,\gamma)} \left(\frac{w_\alpha^{(q)}(x,0)}{A} \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q')}(\alpha z - y + \gamma(1-\alpha))dz - w_\alpha^{(q)}(x,y) \right) dy \end{aligned} \tag{4.35}$$

Similarly, letting $b \to \infty$ in (4.27) and applying (4.33) and (4.34) we obtain for $0 < x < \gamma$ that:

$$\begin{aligned} V^{(q)}(x,\gamma,B) &= \int_{y\in B\cap[\gamma,\infty)} \frac{W_0^{(q)}(x)}{\delta\alpha A} e^{-\Phi_1(q)y} dy \\ &+ \int_{y\in B\cap(0,\gamma)} \left(\frac{W_0^{(q)}(x)}{A} \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q')}(\alpha z - y + \gamma(1-\alpha))dz - W_0^{(q)}(x-y) \right) dy \end{aligned} \tag{4.36}$$

Since for $x < \gamma$, $w_\alpha^{(q)}(x,z) = W_0^{(q)}(x-z)$, (4.36) can be written similarly to (4.35).

Thus we can write for all $x \geq 0$:

$$\begin{aligned} V^{(q)}(x,\gamma,dy) &= \left(\frac{w_\alpha^{(q)}(x,0)}{\delta\alpha A} e^{-\Phi_1(q)y} - W_1^{(q)}(x-y) \right) 1_{y\in[\gamma,\infty)} dy \\ &+ \left(\frac{w_\alpha^{(q)}(x,0)}{A} \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)'}(\alpha z - y + \gamma(1-\alpha))dz - w_\alpha^{(q)}(x,y) \right) 1_{y\in[0,\gamma)} dy \end{aligned} \tag{4.37}$$

4.4 The Laplace transform of the time to ruin

Let $\mathcal{E}(q)$ be an Exponentially distributed random variable with parameter q.

$$\begin{aligned} &\mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^-<\infty}] = \mathbb{P}_x[\mathcal{E}(q) > \kappa_0^-] = 1 - \mathbb{P}_x[U_s > 0, s < \mathcal{E}(q)] \\ &= 1 - q\int_0^\infty e^{-qt} 1_{(U_s\in(0,\infty), 0<s<t)} dt = 1 - q\int_{y=0}^{\infty} V^{(q)}(x,\gamma,dy) \end{aligned} \tag{4.38}$$

In the last equality of (4.38) we applied (4.28). To find the last integral we have to integrate (4.37) between 0 and ∞. Notice that

$$\begin{aligned} &\int_{y=0}^{\gamma}\int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)'}(\alpha z - y + \gamma(1-\alpha))dzdy \\ &= \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} \int_{y=0}^{\gamma} W_0^{(q)'}(\alpha z - y + \gamma(1-\alpha))dydz \\ &= \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)}(\alpha z + \gamma(1-\alpha))dz - \int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)}(\alpha(z-\gamma))dy \end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
&\int_{z=\gamma}^{\infty} e^{-\Phi_1(q)z} W_0^{(q)}(\alpha(z-\gamma))dz \\
= \;& \frac{e^{-\Phi_1(q)\gamma}}{\alpha} \int_{z=0}^{\infty} e^{-\frac{\Phi_1(q)}{\alpha}z} W_0^{(q)}(z)dz \\
= \;& \frac{e^{-\Phi_1(q)\gamma}}{\alpha(\psi_0(\Phi_1(q)/\alpha)-q)} = \frac{e^{-\Phi_1(q)\gamma}}{\alpha\delta\Phi_1(q)} \qquad (4.40)
\end{aligned}$$

We applied (4.15) in the last equality. Thus, by (4.37), (4.39) and (4.40),

$$\begin{aligned}
& 1 - q\int_{t=0}^{\infty} e^{-qt} 1_{(U_s\in(0,\infty),0<s<t)}dt = 1 - q\int_{y=0}^{\infty} V^{(q)}(x,\gamma,dy) \\
= \;& 1 - q\left[w_\alpha^{(q)}(x,0)\frac{e^{-\Phi_1(q)\gamma}}{\Phi_1(q)\delta\alpha A} - \int_{\gamma}^{\infty} W_1^{(q)}(x-z)dz + \frac{w_\alpha^{(q)}(x,0)\int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{A}\right. \\
- \;& \left. \frac{w_\alpha^{(q)}(x,0)e^{-\Phi_1(q)\gamma}}{\Phi_1(q)A\delta\alpha} - \int_{z=0}^{\gamma} w_\alpha^{(q)}(x,z)dz\right] \qquad (4.41)
\end{aligned}$$

Let $x>\gamma$. Substituting (4.21) for $w_\alpha^{(q)}(x,z)$ in (4.41) and adding and subtracting $\int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)dy$ (to obtain a simpler expression for (4.41)) we obtain that:

$$\begin{aligned}
& \mathbb{E}_x[e^{-q\kappa_0^-}1_{\kappa_0^-<\infty}] = 1 - q\left[-\int_{\gamma}^{\infty} W_1^{(q)}(x-z)dz + \frac{w_\alpha^{(q)}(x,0)\int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{A}\right. \\
& \left. -\int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))dz - \alpha\delta\int_{y=\gamma}^{x} W_1^{(q)}(x-y)\int_{z=0}^{\gamma} W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha)dzdy\right] \\
= \;& 1 - q\left[-\int_0^{\alpha x+\gamma(1-\alpha))} W_0^{(q)}(y)dy - \int_{\gamma}^{\infty} W_1^{(q)}(x-z)dz + \frac{w_\alpha^{(q)}(x,0)\int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{A}\right. \\
& + \int_0^{\alpha x+\gamma(1-\alpha))} W_0^{(q)}(y)dy - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))dz \\
& \left. -\alpha\delta\int_{y=\gamma}^{x} W_1^{(q)}(x-y)\int_{z=0}^{\gamma} W_0^{(q)'}(\alpha y - z + \gamma(1-\alpha))dzdy\right] \\
= \;& 1 - q\left[-\int_0^{\alpha x+\gamma(1-\alpha))} W_0^{(q)}(y)dy - \int_{\gamma}^{\infty} W_1^{(q)}(x-z)dz + \frac{w_\alpha^{(q)}(x,0)\int_{\gamma}^{\infty} e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{A}\right. \\
& + \int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)dy - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))dz \\
& \left. -\alpha\delta\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy + \alpha\delta\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))dy\right]
\end{aligned}$$

Thus, for $x > \gamma$,

$$\mathbb{E}_x[e^{-q\kappa_0^-}1_{\kappa_0^-<\infty}] = 1 + q\int_0^{\alpha x+\gamma(1-\alpha))} W_0^{(q)}(y)dy - q\frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty e^{-\Phi_1(q)y}W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{A}$$
$$+q\alpha\delta\int_{y=\gamma}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy + q\mathcal{C}(x),$$

where

$$\begin{aligned}\mathcal{C}(x) &= \int_\gamma^\infty W_1^{(q)}(x-z)dz - \int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)dy \\ &+ \int_{z=0}^\gamma W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))dz - \alpha\delta\int_{y=\gamma}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))dy\end{aligned}$$

Note that

$$\int_\gamma^\infty W_1^{(q)}(x-z)dz = \int_0^{x-\gamma} W_1^{(q)}(y)dy,$$
$$\int_{z=0}^\gamma W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))dz = \int_{y=\alpha(x-\gamma)}^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)dy,$$
$$\int_{y=\gamma}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))dy = \int_{y=0}^{x-\gamma} W_1^{(q)}(x-\gamma-y)W_0^{(q)}(\alpha y)dy.$$

We thus obtain a simpler expression for $\mathcal{C}(x)$:

$$\mathcal{C}(x) = \int_0^{x-\gamma} W_1^{(q)}(y)dy - \int_0^{\alpha(x-\gamma)} W_0^{(q)}(y)dy - \alpha\delta\int_{y=0}^{x-\gamma} W_1^{(q)}(x-\gamma-y)W_0^{(q)}(\alpha y)dy$$

Thus $\mathcal{C}(x) = g(x-\gamma)$, where

$$g(x) = \int_0^x W_1^{(q)}(y)dy - \int_0^{\alpha x} W_0^{(q)}(y)dy - \alpha\delta\int_{y=0}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha y)dy$$

Taking Laplace transform $\int_0^\infty e^{-sx}g(x)dx$ we obtain that

$$\int_0^\infty e^{-sx}g(x)dx = \frac{1}{s(\psi_1(s)-q)} - \frac{1}{s(\psi_0(s/\alpha)-q)} - \frac{\delta}{(\psi_0(s/\alpha)-q)(\psi_1(s)-q)}$$

Applying (4.15) we conclude that the last expression equals 0. Thus $g(x) = 0$ for $x \geq \gamma$, and

$$\mathbb{E}_x[e^{-q\kappa_0^-}1_{\kappa_0^-<\infty}] = 1 + q\int_0^{\alpha x+\gamma(1-\alpha))} W_0^{(q)}(y)dy + q\alpha\delta\int_{y=\gamma}^x W_1^{(q)}(x-y)W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy$$
$$-q\frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty e^{-\Phi_1(q)y}W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{\int_\gamma^\infty e^{-\Phi_1(q)y}W_0^{(q)'}(\alpha y+\gamma(1-\alpha))dy} \tag{4.42}$$

For $x < \gamma$ we obtain from (4.41) or directly from (4.37) that

$$\mathbb{E}_x[e^{-q\kappa_0^-}1_{\kappa_0^-<\infty}] = 1 - q\frac{W_0^{(q)}(x)\int_\gamma^\infty e^{-\Phi_1(q)y}W_0^{(q)}(\alpha y+\gamma(1-\alpha))dy}{A} + q\int_{z=0}^\gamma W_0^{(q)}(x-z)dz \tag{4.43}$$

4.5 Ruin probability

To obtain the ruin probability, take the limit of (4.42) for $x \geq \gamma$, or of (4.43) for $x < \gamma$ as $q \to 0$. For $x \geq \gamma$, the limit of the last two terms in the first line of (4.42) is 0.

To calculate the limit of the last expression in (4.42) notice that:

$$
\begin{aligned}
& q \frac{w_\alpha^{(q)}(x,0) \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y + \gamma(1-\alpha))dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha))dy} \\
&= q w_\alpha^{(q)}(x,0) \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \frac{\int_0^\infty e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y)dy - \int_0^\gamma e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y)dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha))dy} \\
&= w_\alpha^{(q)}(x,0) \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \frac{\frac{q}{\psi_0(\Phi_1(q)/\alpha) - q} - q \int_0^\gamma e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y)dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha))dy} \\
&= w_\alpha^{(q)}(x,0) \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \frac{\frac{q}{\delta \Phi_1(q)} - q \int_0^\gamma e^{-\Phi_1(q)y/\alpha} W_0^{(q)}(y)dy}{\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha))dy},
\end{aligned} \tag{4.44}
$$

where we applied (4.15) in the last line of (4.44).

If $\psi_1'(0) < 0$ then $\Phi_1(0) > 0$, the limit of the last expression is 0 and thus the ruin probability is 1. If $\psi_1'(0) \geq 0$ then $\Phi_1(0) = 0$. In this case,

$$
\lim_{q \to 0} \frac{q}{\Phi_1(q)} = \psi_1'(0) = \frac{\psi_0'(0)}{\alpha} - \delta. \tag{4.45}
$$

Next, we obtain a simpler expression for the denominator in (4.44).

$$
\begin{aligned}
& \int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)'}(\alpha y + \gamma(1-\alpha))dy = \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \int_\gamma^\infty e^{-\frac{\Phi_1(q)}{\alpha} y} W_0^{(q)'}(y)dy \\
&= \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \Big(-W_0^{(q)}(\gamma) e^{-\frac{\Phi_1(q)}{\alpha}\gamma} \\
&+ \frac{\Phi_1(q)}{\alpha} \int_0^\infty e^{-\frac{\Phi_1(q)}{\alpha} y} W_0^{(q)}(y)dy - \frac{\Phi_1(q)}{\alpha} \int_0^\gamma e^{-\frac{\Phi_1(q)}{\alpha} y} W_0^{(q)}(y)dy \Big) \\
&= \frac{1}{\alpha} e^{\Phi_1(q)\gamma(1-\alpha)/\alpha} \Big(-W_0^{(q)}(\gamma) e^{-\frac{\Phi_1(q)}{\alpha}\gamma} \\
&+ \frac{\Phi_1(q)}{\alpha} \frac{1}{\delta \Phi_1(q)} - \frac{\Phi_1(q)}{\alpha} \int_0^\gamma e^{-\frac{\Phi_1(q)}{\alpha} y} W_0^{(q)}(y)dy \Big)
\end{aligned}
$$

The limit of the denominator of (4.44) as $q \to 0$ hence is

$$
\frac{1}{\alpha}(-W_0(\gamma) + \frac{1}{\alpha\delta}) \tag{4.46}
$$

Applying (4.44), (4.45),(4.21) and (4.46) we obtain that the ruin probability for $x > \gamma$ is,

$$
\begin{aligned}
& \mathbb{P}_x(\kappa_0^- < \infty) = \lim_{q \to 0} \mathbb{E}_x[e^{-q\kappa_0^-} 1_{\kappa_0^- < \infty}] \\
&= 1 - \left(W_0(\alpha x + \gamma(1-\alpha)) + \alpha\delta \int_\gamma^x W_1(x-y) W_0'(\alpha y + \gamma(1-\alpha))dy \right) \frac{\alpha \psi_1'(0)}{1 - \alpha\delta W_0(\gamma)}
\end{aligned}
$$

Similarly, applying (4.22) for $x < \gamma$ we obtain that

$$\begin{aligned}\mathbb{P}_x(\kappa_0^- < \infty) &= \lim_{q\to 0}\mathbb{E}_x[e^{-q\kappa_0^-}1_{\kappa_0^-<\infty}] \\ &= 1 - W_0(x)\frac{\alpha\psi_1'(0)}{1-\alpha\delta W_0(\gamma)}\end{aligned}$$

4.6 Gerber-Shiu penalty function

Let $-U_{\kappa_0^-}$ be the deficit at ruin and $U_{\kappa_{0-}^-}$ the surplus just before ruin. We want to obtain

$$m(x,q) = \mathbb{E}_x[e^{-q\kappa_{0-}}h(U_{\kappa_{0-}^-}, |U_{\kappa_{0^-}}|)|U_0 = x],$$

where $h(x,y)$ is a nonnegative function – the penalty function. By (4.37):

$$\begin{aligned} m(x,q) &= \int_{y=0}^{\gamma} V^{(q)}(x,\gamma,dy)\int_{z=0}^{\infty} h(y,z)\lambda dF_0(y+z)dy \\ + &\int_{y=\gamma}^{\infty} V^{(q)}(x,\gamma,dy)\int_{z=0}^{\infty} h(y,z)\lambda dF_1(y-\gamma+\frac{\gamma+z}{\alpha})dy \\ &= \int_{y=0}^{\gamma}\left(\frac{w_\alpha^{(q)}(x,0)}{A}\int_{s=\gamma}^{\infty} e^{-\Phi_1(q)s}W_0^{(q')}(\alpha s - y + \gamma(1-\alpha))ds - w_\alpha^{(q)}(x,y)\right)\int_{z=0}^{\infty} h(y,z)\lambda dF_0(y+z)dy \\ &+ \int_{\gamma}^{\infty}\left(\frac{w_\alpha^{(q)}(x,0)}{\delta\alpha A}e^{-\Phi_1(q)y} - W_1^{(q)}(x-y)\right)\int_{z=0}^{\infty}\lambda dF_1(y-\gamma+\frac{\gamma+z}{\alpha})h(y,z)dy \end{aligned} \tag{4.47}$$

An application of the Gerber-Shiu penalty function is the derivation of the joint distribution of the reserve just before ruin and the deficit at ruin when $\psi_1'(0) \geq 0$, i.e. when $\Phi_1(0) = 0$.

Let $A, B \subset (0,\infty)$, and let $h^*(y,z) = 1_{(y\in A, z\in B)}$

In order to obtain $\mathbb{P}_x(U_{\kappa_{0-}^-} \in A, |U_{\kappa_0^-}| \in B)$, we substitute h^* in (4.47) and take the limit as $q \to 0$.

Notice that

$$\begin{aligned} &\int_{s=\gamma}^{\infty} e^{-\Phi_1(q)s}W_0^{(q)'}(\alpha s - y + \gamma(1-\alpha))ds \\ = &-e^{-\Phi_1(q)\gamma}W_0^{(q)}(\gamma - y) + \frac{\Phi_1(q)}{\alpha}e^{-\Phi_1(q)\frac{y-\gamma(1-\alpha)}{\alpha}}\int_{\gamma-y}^{\infty} e^{-\frac{\Phi_1(q)s}{\alpha}}W_0^{(q)}(s)ds \\ = &-e^{-\Phi_1(q)\gamma}W_0^{(q)}(\gamma - y) + \frac{\Phi_1(q)}{\alpha}e^{-\Phi_1(q)\frac{y-\gamma(1-\alpha)}{\alpha}}\int_{0}^{\infty} e^{-\frac{\Phi_1(q)s}{\alpha}}W_0^{(q)}(s)ds \\ - &\frac{\Phi_1(q)}{\alpha}e^{-\Phi_1(q)\frac{y-\gamma(1-\alpha)}{\alpha}}\int_{0}^{\gamma-y} e^{-\frac{\Phi_1(q)s}{\alpha}}W_0^{(q)}(s)ds \\ = &-e^{-\Phi_1(q)\gamma}W_0^{(q)}(\gamma - y) + e^{-\Phi_1(q)\frac{y-\gamma(1-\alpha)}{\alpha}}\frac{1}{\alpha\delta} \\ - &\frac{\Phi_1(q)}{\alpha}e^{-\Phi_1(q)\frac{y-\gamma(1-\alpha)}{\alpha}}\int_{0}^{\gamma-y} e^{-\frac{\Phi_1(q)s}{\alpha}}W_0^{(q)}(s)ds \end{aligned} \tag{4.48}$$

In the third term of the last equality we applied (2.1) and (4.15).

The limit as $q \to 0$ in (4.48) is

$$\frac{1 - \alpha\delta W_0(\gamma - y)}{\alpha\delta} \tag{4.49}$$

Similarly the limit as $q \to 0$ of A in (4.32) is

$$\frac{1 - \alpha\delta W_0(\gamma)}{\alpha\delta} \tag{4.50}$$

Denote by $w_\alpha^{(0)}(x, y) = w_\alpha(x, y)$, then we obtain

$$\mathbb{P}_x(U_{\kappa_{0^-}^-} \in A, |U_{\kappa_0^-}| \in B) = \int_{y \in B \cap (\gamma, \infty)} \left(\frac{w_\alpha(x, 0)}{1 - \alpha\delta W_0(\gamma)} - W_1(x - y) \right) \Pi_1(y - \gamma + \frac{\gamma + B}{\alpha}) dy$$
$$w_\alpha(x, 0) \int_{y \in B \cap (0, \gamma)} \left(\frac{1 - \alpha\delta W_0(\gamma - y)}{1 - \alpha\delta W_0(\gamma)} - w_\alpha(x, y) \right) \Pi_0(y + B) dy$$

where $\Pi_i(C)$ is the Lévy measure of the set C, in the compound Poisson case $\Pi_i(C) = \lambda F_i(C)$ and $F_i(C) = \mathbb{P}(Z^i \in C)$ and Z^i has distribution F_i, $i = 0, 1$.

5 Conclusions

In this paper we studied a compound Poisson risk process where the claims are "refracted" – i.e. only a part of the claim is paid when the reserve is under γ. We obtained expressions for the Laplace transform of the exit time from an upper barrier, the time to ruin and the joint probability for the surplus before and at ruin, for a general function $I(x)$ as defined in the Introduction. We obtained relatively simple expressions for the special but important case that $I(x) = \alpha x$. In this case the results are in the same flavor as for spectrally negative refracted Lévy processes, where the premium income rate when the reserve is above γ is αc, as was studied by Kyprianou and Loeffen (2010).

We conclude by mentioning two topics for further research. (i) A similar analysis might be applied when generalizing the compound Poisson process to a spectrally negative bounded variation Lévy risk process. (ii) It would be worthwhile to consider $I(x) = \min(a, x)$ where a is a positive constant.

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