Torsional rigidity for regions with a Brownian boundary

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Abstract

Let $\mathbb{T}^m$ be the m-dimensional unit torus, $m \in \mathbb{N}$. The torsional rigidity of an open set $\Omega \subset \mathbb{T}^m$ is the integral with respect to Lebesgue measure over all starting points $x \in \Omega$ of the expected lifetime in Ω of a Brownian motion starting at x. In this paper we consider $\Omega = \mathbb{T}^m \backslash \beta[0,t]$, the complement of the path $\beta[0,t]$ of an independent Brownian motion up to time t. We compute the leading order asymptotic behaviour of the expectation of the torsional rigidity in the limit as $t \to \infty$. For $m = 2$ the main contribution comes from the components in $\mathbb{T}^2 \backslash \beta[0,t]$ whose inradius is comparable to the largest inradius, while for $m = 3$ most of $\mathbb{T}^3 \backslash \beta[0,t]$ contributes. A similar result holds for $m \geq 4$ after the Brownian path is replaced by a shrinking Wiener sausage $W_{r(t)}[0,t]$ of radius $r(t) = o(t^{-1/(m-2)})$, provided the shrinking is slow enough to ensure that the torsional rigidity tends to zero. Asymptotic properties of the capacity of $\beta[0,t]$ in $\mathbb{R}^3$ and $W_1[0,t]$ in $\mathbb{R}^m$, $m \geq 4$, play a central role throughout the paper. Our results contribute to a better understanding of the geometry of the complement of Brownian motion on $\mathbb{T}^m$, which has received a lot of attention in the literature in past years.

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1 Main results and discussion

Section 1.1 defines our object of interest, the torsional rigidity of the complement of Brownian motion on the unit torus, Section 1.2 states our main theorems for its asymptotic scaling, Section 1.4 places these theorems in their proper context and makes a link with its principal Dirichlet eigenvalue, while Section 1.3 identifies the asymptotic scaling of the capacity of the Wiener sausage that serves as a key ingredient.

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1.1 Torsional rigidity

Let $(\tilde{\beta}(s), s \geq 0; \tilde{\mathbb{P}}_x, x \in \mathbb{T}^m)$ be Brownian motion on the m-dimensional unit torus $\mathbb{T}^m$, i.e., the Markov process whose generator is the Laplacian on $\mathbb{T}^m$. Given an open set $\Omega \subset \mathbb{T}^m$, the exit time of Ω is

$$\tilde{\tau}_\Omega = \inf\{s \geq 0\colon\ \tilde{\beta}(s) \notin \Omega\}. \tag{1.1}$$

The expected exit time given $\tilde{\beta}(0) = x$ is

$$w_\Omega(x) = \tilde{\mathbb{E}}_x(\tilde{\tau}_\Omega), \qquad x \in \mathbb{T}^m. \tag{1.2}$$

The *torsional rigidity* of Ω is defined as

$$\mathcal{T}(\Omega) = \int_\Omega dx\ w_\Omega(x). \tag{1.3}$$

The torsional rigidity of a cross section of a beam shows up in the computation of the angular change when a beam of a given length and a given modulus of rigidity is exposed to a twisting moment [3], [20]. It also arises in the calculation of the heat content of sets with time-dependent boundary conditions [5], in the definition of gamma convergence [9], and in the study of minimal submanifolds [17].

Figure 1: Simulation of $\beta[0,t]$ for $t = 15$ and $m = 2$. The Brownian path $\beta[0,t]$ is black, its complement $\mathcal{B}(t) = \mathbb{T}^m \backslash \beta[0,t]$ is white.

Let $(\beta(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{T}^m)$ be a second independent Brownian motion on $\mathbb{T}^m$. The object of interest in this paper is the random set (see Fig. 1.1)

$$\mathcal{B}(t) = \mathbb{T}^m \backslash \beta[0,t]. \tag{1.4}$$

In particular, we are interested in the *expected torsional rigidity of* $\mathcal{B}(t)$:

$$\spadesuit(t) = \mathbb{E}_0\big(\mathcal{T}\big(\mathcal{B}(t)\big)\big), \qquad t \geq 0. \tag{1.5}$$

Since $|\mathbb{T}^m| = 1$ and $|\beta[0,t]| = 0$, the torsional rigidity is the expected time needed by $\tilde{\beta}$ to hit $\beta[0,t]$ averaged over all starting points in $\mathbb{T}^m$.

The case $m = 1$ is uninteresting. For $m = 2$, as t gets large the set $\mathcal{B}(t)$ decomposes into a large number of disjoint small components (see Fig. 1.1), while for $m \geq 3$ it remains connected. As shown in [14], in the latter case $\mathcal{B}(t)$ consists of "lakes" connected by "narrow channels", so that we may think of it as a *porous medium*. Below we identify the asymptotic behaviour of $\spadesuit(t)$ as $t \to \infty$ when $m = 2, 3$.

For $m \geq 4$ we have $♠(t) = \infty$ for all $t \geq 0$ because Brownian motion is polar. To get a non-trivial scaling, the Brownian path must be thickened to a *shrinking Wiener sausage*

$$W_{r(t)}[0,t] = \big\{x \in \mathbb{T}^m \colon\, d_t(x) \leq r(t)\big\}, \qquad t > 0, \tag{1.6}$$

where $r\colon\, (0,\infty) \to (0,\infty)$ is such that $\lim_{t\to\infty} t^{1/(m-2)} r(t) = 0$. This choice of shrinking is appropriate because for $m \geq 3$ typical regions in $\mathcal{B}(t)$ have a size of order $t^{-1/(m-2)}$ (see [11] and [14]). The object of interest is the random set

$$\mathcal{B}_{r(t)}(t) = \mathbb{T}^m \backslash W_{r(t)}[0,t], \tag{1.7}$$

in particular, the *expected torsional rigidity of* $\mathcal{B}_{r(t)}(t)$:

$$♠_{r(t)}(t) = \mathbb{E}_0\big(\mathcal{T}\big(\mathcal{B}_{r(t)}(t)\big)\big), \qquad t > 0. \tag{1.8}$$

Below we identify the asymptotic behaviour of $♠_{r(t)}(t)$ as $t \to \infty$ for $m \geq 4$ subject to a condition under which $r(t)$ does not decay too fast.

1.2 Asymptotic scaling of expected torsional rigidity

Theorems 1.1–1.3 below are our main results for the scaling of $♠(t)$ and $♠_{r(t)}(t)$ as $t \to \infty$. In what follows we write $f \asymp g$ when $0 < c \leq f(t)/g(t) \leq C < \infty$ for t large enough.

Theorem 1.1 *If $m = 2$, then*

$$♠(t) \asymp t^{1/4}\, e^{-4(\pi t)^{1/2}}, \qquad t \to \infty. \tag{1.9}$$

Theorem 1.2 *If $m = 3$, then*

$$♠(t) = [1 + o(1)]\, \frac{2}{t^2}\, \mathbb{E}_0\left(\frac{1}{\mathrm{cap}\,(\beta[0,1])^2}\right), \qquad t \to \infty, \tag{1.10}$$

where $\mathrm{cap}\,(\beta[0,1])$ *is the Newtonian capacity of* $\beta[0,1]$ *in* $\mathbb{R}^3$*. All inverse moments of* $\mathrm{cap}\,(\beta[0,1])$ *are finite.*

Theorem 1.3 *If $m \geq 4$ and*

$$\lim_{t\to\infty} t^{1/(m-2)} r(t) = 0, \qquad \begin{cases} m = 4\colon & \lim_{t\to\infty} \frac{t}{\log^3 t}\, \frac{1}{\log(1/r(t))} = \infty, \\ m \geq 5\colon & \lim_{t\to\infty} \frac{t}{\log^3 t}\, r(t)^{m-4} = \infty, \end{cases} \tag{1.11}$$

then, subject to (1.13)–(1.14) *below,*

$$♠_{r(t)}(t) = [1 + o(1)]\, \frac{1}{\kappa_m\, t^{2/(m-2)}}\, \mathbb{E}_0\left(\frac{1}{\mathrm{cap}\,(W_{\varepsilon(t)}[0,1])}\right), \qquad t \to \infty, \tag{1.12}$$

where $\varepsilon(t) = t^{1/(m-2)} r(t)$ *and* $\mathrm{cap}\,(W_\varepsilon[0,1])$ *is the Newtonian capacity of* $W_\varepsilon[0,1]$ *in* $\mathbb{R}^m$*. All inverse moments of* $\mathrm{cap}\,(W_\varepsilon[0,1])$ *are finite for all* $\varepsilon > 0$.

We expect that similar results hold when $\mathbb{T}^m$ is replaced by a smooth m-dimensional compact connected Riemannian manifold without boundary. We further expect that the torsional rigidity satisfies a strong law of large numbers for $m \geq 3$ but not for $m = 2$.

1.3 Asymptotic scaling of capacity of Wiener sausage

A key ingredient in the proof of Theorem 1.3 is the following scaling behaviour of the capacity of the Wiener sausage for $m \geq 4$. Let

$$\mathcal{C}(t) = \begin{cases} \dfrac{\log t}{t} \operatorname{cap}(W_1[0,t]), & m = 4, \\ \dfrac{1}{t} \operatorname{cap}(W_1[0,t]), & m \geq 5. \end{cases} \tag{1.13}$$

Then there exist constants $c_m \in (0,\infty)$, $m \geq 4$, such that

$$\mathcal{C}(t) = [1+o(1)]\, c_m \quad \text{in } \mathbb{P}_0\text{-probability as } t \to \infty. \tag{1.14}$$

In Section 7 we prove (1.14) for $m \geq 5$ with the help of subadditivity. For $m = 4$, however, (1.14) is presently an hypothesis we plan to investigate in a separate paper.

1.4 Discussion

We refer the reader to [14] and [4] for an overview on what is known about the geometry of the complement of Brownian motion on the unit torus.

1. Theorems 1.1 and 1.2 identify the scaling of the expected torsional rigidity in low dimensions. This scaling may be viewed in the following context. Let $d(x,y)$ denote the distance between $x,y \in \mathbb{T}^m$. The distance of x to $\beta[0,t]$ is denoted by

$$d_t(x) = \min_{y \in \beta[0,t]} d(x,y). \tag{1.15}$$

The *inradius* of $\mathcal{B}(t)$ is the random variable ρ_t defined by

$$\rho_t = \max_{x \in \mathbb{T}^m} d_t(x). \tag{1.16}$$

A detailed analysis of ρ_t and related quantities was given in [12], [4] for $m = 2$ and in [11], [14] for $m \geq 3$. In [7] it was shown that for $m = 2$,

$$\mathbb{E}_0(\rho_t) = e^{-(\pi t)^{1/2}[1+o(1)]}, \qquad t \to \infty, \tag{1.17}$$

while for $m \geq 3$,

$$\mathbb{E}_0(\rho_t) = [1+o(1)] \left(\frac{m}{(m-2)\kappa_m} \frac{\log t}{t} \right)^{1/(m-2)}, \qquad t \to \infty, \tag{1.18}$$

where κ_m is the Newtonian capacity of the ball with radius 1 in $\mathbb{R}^m$,

$$\kappa_m = 4\pi^{m/2} \Big/ \Gamma\left(\frac{m-2}{2}\right). \tag{1.19}$$

A ball of radius r in $\mathbb{T}^m$ with r sufficiently small has a torsional rigidity proportional to r^{m+2}. Theorem 1.1 and (1.17) show that $\log \spadesuit(t) = -[1+o(1)]\, 4(\pi t)^{1/2} = [1+o(1)] \log \mathbb{E}_0(\rho_t)^4$ for $m = 2$, while Theorem 1.2 and (1.18) show that $\spadesuit(t) \asymp t^{-2} \gg \mathbb{E}_0(\rho_t)^5$ for $m = 3$. Thus, for $m = 2$ the main contribution to the asymptotic behaviour of $\log \spadesuit(t)$ comes from the components in $\mathcal{B}(t)$ that have a size of order ρ_t (which are atypical; see [12] and [4]), while for $m = 3$ the main contribution to the asymptotic behaviour of $\spadesuit(t)$ comes from regions in $\mathcal{B}(t)$ that have a size of order t^{-1} (which are typical; see [11] and [14]), i.e., most of $\mathcal{B}(t)$ contributes.

2. For $m = 2$ it is shown in [4] that

$$\rho_t = t^{-1/8+o(1)}\, e^{-(\pi t)^{1/2}} \quad \text{in } \mathbb{P}_0\text{-probability}, \qquad t \to \infty, \tag{1.20}$$

which is a considerable sharpening of (1.17). The proof is long and difficult. Combining (1.20) with what we found in Theorem 1.1, we get the relation

$$\spadesuit(t) \asymp t^{3/4+o(1)}\, \mathbb{E}_0(\rho_t)^4, \tag{1.21}$$

provided (1.20) also holds in mean (which is expected but has not been proved). Clearly, $\spadesuit(t)$ is not dominated by the largest component in $\mathcal{B}(t)$ alone: smaller components contribute too as long as they have a comparable size. Apparently, the number of such components is at least of order $t^{3/4+o(1)}$.

3. Theorem 1.3 identifies the scaling of the expected torsional rigidity in high dimensions. Via the scaling relation in distribution

$$\mathrm{cap}\,(W_\varepsilon[0,1]) = \mathrm{cap}\,(\varepsilon W_1[0,\varepsilon^{-2}]) = \varepsilon^{m-2}\mathrm{cap}\,(W_1[0,\varepsilon^{-2}]), \qquad \varepsilon > 0, \tag{1.22}$$

it follows from (1.13)–(1.14) that $\mathrm{cap}\,(W_\varepsilon[0,1]) = [1+o(1)]\,c_m\varepsilon^{m-4}$ in $\mathbb{P}_0$-probability as $\varepsilon \downarrow 0$ when $m \geq 5$. In that case Theorem 1.3 yields the asymptotics

$$\spadesuit_{r(t)}(t) = [1+o(1)]\,\frac{1}{\kappa_m c_m\, t\, r(t)^{m-4}}, \qquad t \to \infty. \tag{1.23}$$

It also follows from (1.13)–(1.14) that $\mathrm{cap}\,(W_\varepsilon[0,1]) = [1+o(1)]\,c_4/2\log(1/\varepsilon)$ in $\mathbb{P}_0$-probability as $\varepsilon \downarrow 0$ when $m = 4$. In that case Theorem 1.3 yields the asymptotics

$$\spadesuit_{r(t)}(t) = [1+o(1)]\,\frac{2\log(1/t^{1/2}r(t))}{\kappa_4 c_4\, t}, \qquad t \to \infty. \tag{1.24}$$

By the second half of (1.11), both (1.23) and (1.24) correspond to the regime where $\spadesuit_{r(t)}(t) = o(1/\log^3 t)$. We have not attempted to improve this to $o(1)$.

4. We did not investigate the regime for $m \geq 4$ where $r(t)$ decays so fast that $\spadesuit_{r(t)}(t)$ diverges as $t \to \infty$. In that regime, the Brownian motion $\tilde\beta$ in (1.1)–(1.3) runs around $\mathbb{T}^m$ many times before it hits $W_{r(t)}[0,t]$, and the growth of $\spadesuit_{r(t)}(t)$ depends on the global rather than the local properties of $W_{r(t)}[0,t]$.

5. We will see in Section 2 that various geometric quantities, such as inradius, principal Dirichlet eigenvalue, square-integrated distance function and torsional rigidity, are closely related. In Section 6 we will give a quick proof of the following inequality relating the torsional rigidity to

$$\lambda_1\big(\mathcal{B}(t)\big), \quad \lambda_1\big(\mathcal{B}_{r(t)}(t)\big), \tag{1.25}$$

the principal Dirichlet eigenvalue of $\mathcal{B}(t)$ for $m = 2,3$ and $\mathcal{B}_{r(t)}(t)$ for $m \geq 4$.

Theorem 1.4 *(a) If $m = 2,3$, then for t large enough,*

$$\mathbb{E}_0\big(\lambda_1\big(\mathcal{B}(t)\big)\big) \geq \spadesuit(t)^{-2/(m+2)}. \tag{1.26}$$

(b) If $m \geq 4$ and $\lim_{t\to\infty} \spadesuit_{r(t)}(t) = 0$, then for t large enough,

$$\mathbb{E}_0\big(\lambda_1\big(\mathcal{B}_{r(t)}(t)\big)\big) \geq \spadesuit_{r(t)}(t)^{-2/(m+2)}. \tag{1.27}$$

Combining the result for $m = 2$ with what we found in Theorem 1.1, we obtain

$$\mathbb{E}_0\big(\lambda_1\big(\mathcal{B}(t)\big)\big) \succeq t^{-1/8}\, e^{2(\pi t)^{1/2}}, \tag{1.28}$$

where $f \succeq g$ means that $f(t)/g(t) \geq c > 0$ for t large enough. In [7] we conjectured that $\log \mathbb{E}_0(\lambda_1(\mathcal{B}(t))) = [1+o(1)]\,2(\pi t)^{1/2}$, which fits the lower bound in (1.28). However, a better estimate than (1.28) is possible. Namely, in Section 2 we will see that $\lambda_1(\mathcal{B}(t)) \asymp 1/\rho_t^2$, and so Jensen gives the lower bound

$\mathbb{E}_0(\lambda_1(\mathcal{B}(t)) \geq 1/\mathbb{E}_0(\rho_t)^2$. Assuming that the scaling in (1.20) also holds in mean (which is expected but has not been proved), we get

$$\mathbb{E}_0\big(\lambda_1\big(\mathcal{B}(t)\big)\big) \succeq t^{1/4+o(1)}\, e^{2(\pi t)^{1/2}}, \tag{1.29}$$

which is better than (1.28) by a factor $t^{3/8+o(1)}$. Presumably (1.29) captures the correct scaling behaviour.

Outline. The remainder of this paper is organized as follows. In Section 2 we recall some analytical facts about the torsional rigidity. In Sections 3–5 we prove Theorems 1.1–1.3, respectively. The proof of Theorem 1.4 is given in Section 6, while the proof of the scaling in (1.13)–(1.14) for $m \geq 5$ is given in Section 7.

2 Analytical facts for the torsional rigidity

Let M be an m-dimensional Riemannian manifold without boundary that is both geodesically and stochastically complete. In most this paper we focus on the case where M is the m-dimensional unit torus $\mathbb{T}^m$. However, the results mentioned below hold in greater generality. In Section 2.1 we give the spectral representation of the Dirichlet heat kernel, in Section 2.2 we derive certain a priori estimates on the torsional rigidity that will be needed later on.

2.1 Dirichlet heat kernel

For an open set $\Omega \subset M$ with boundary $\partial\Omega$, we denote the Dirichlet heat kernel by $p_\Omega(x, y; t)$, $x, y \in \Omega$, $t > 0$. The integral defined by

$$u_\Omega(x;t) = \int_\Omega p_\Omega(x, y; t)\, dy, \tag{2.1}$$

is the unique weak solution of the boundary value problem

$$\begin{aligned} \tfrac{\partial u(x;t)}{\partial t} &= \Delta u(x;t), & x \in \Omega,\ t > 0, \\ u(x;0) &= 1, & x \in \Omega, \\ u(x;t) &= 0, & x \in \partial\Omega,\ t \geq 0, \end{aligned} \tag{2.2}$$

where the latter boundary condition holds at all regular points of $\partial\Omega$. The interpretation of (2.2) is that $u_\Omega(x;t)$ is the temperature at point x at time t when the initial temperature in Ω is 1 and the temperature of $\partial\Omega$ is kept at 0. The *heat content* of Ω at time t is defined as

$$H_\Omega(t) = \int_\Omega u_\Omega(x;t)\, dx. \tag{2.3}$$

Since the Dirichlet heat kernel is non-negative and monotone in Ω, we have that

$$0 \leq p_\Omega(x, y; t) \leq p_M(x, y; t). \tag{2.4}$$

By the stochastic completeness of M, we have that

$$0 \leq u_\Omega(x;t) \leq \int_M p_M(x, y; t)\, dy = 1. \tag{2.5}$$

In particular, if Ω has finite measure $|\Omega|$, then

$$0 \leq H_\Omega(t) \leq |\Omega|. \tag{2.6}$$

In that case we also have an eigenfunction expansion for the Dirichlet heat kernel in terms of the Dirichlet eigenvalues $0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots$ and a corresponding orthonormal set of eigenfunctions $\varphi_1, \varphi_2, \cdots$ in $L^2(\Omega)$:

$$p_\Omega(x, y; t) = \sum_{j\in\mathbb{N}} e^{-t\lambda_j(\Omega)} \varphi_j(x)\varphi_j(y). \tag{2.7}$$

Since $|\Omega| < \infty$, we have by Cauchy-Schwarz that $\int_\Omega dx\, |\varphi_j(x)| \le |\Omega|^{1/2}$. Hence the φ_j are in $L^1(\Omega)$, and so in $L^p(\Omega)$ for all $1 \le p \le 2$. Below we show that they are in fact in $L^\infty(\Omega)$, which by Hölder's inequality implies that they are in $L^p(\Omega)$ for all $1 \le p \le \infty$.

2.2 A priori estimates

Lemma 2.1 below provides an upper bound on the Dirichlet eigenfunctions in terms of the Dirichlet eigenvalues. Lemma 2.2 below states upper and lower bounds on the torsional rigidity that will be needed later on.

Lemma 2.1 *Suppose that* $\sup_{x\in M} p(x, x; t) < \infty$ *for all* $t > 0$. *Then*

$$\|\varphi_j\|^2_{L^\infty(\Omega)} \le e \sup_{x\in M} p_M(x, x; \lambda_j(\Omega)^{-1}), \qquad j \in \mathbb{N}. \tag{2.8}$$

Proof. Since $|\Omega| < \infty$, the spectrum of the Dirichlet Laplacian on Ω is discrete. By (2.7) and the domain monotonicity of the Dirichlet heat kernel [10], we have that

$$\varphi_j(x)^2 \le e\, p_\Omega(x, x; \lambda_j(\Omega)^{-1}) \le e\, p_M(x, x; \lambda_j(\Omega)^{-1}). \tag{2.9}$$

Taking first the supremum over $x \in M$ in the right-hand side of (2.9) and subsequently in the left-hand side of (2.9), we get (2.8). ∎

It follows from Parseval's formula that

$$H_\Omega(t) = \sum_{j\in\mathbb{N}} e^{-t\lambda_j(\Omega)} \left(\int_\Omega \varphi_j(x)\, dx\right)^2 \le e^{-t\lambda_1(\Omega)} \sum_{j\in\mathbb{N}} \left(\int_\Omega \varphi_j(x)\, dx\right)^2 = e^{-t\lambda_1(\Omega)}|\Omega|. \tag{2.10}$$

The unique weak solution of the boundary value problem

$$\begin{aligned} -\Delta w(x) &= 1, \quad x \in \Omega, \\ w(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2.11}$$

is given by

$$w_\Omega(x) = \int_0^\infty u_\Omega(x; t)\, dt. \tag{2.12}$$

The *torsional rigidity* of Ω is defined by

$$\mathcal{T}(\Omega) = \int_\Omega w_\Omega(x)\, dx. \tag{2.13}$$

It follows from (2.3), (2.12)–(2.13) and Fubini's theorem that

$$\mathcal{T}(\Omega) = \int_0^\infty H_\Omega(t)\, dt, \tag{2.14}$$

i.e., *the torsional rigidity is the integral of the heat content.* Let

$$\delta_\Omega(x) = \min_{y\in\mathbb{R}^m\setminus\Omega} d(x, y) \tag{2.15}$$

denote the distance of $x \in \Omega$ to $\mathbb{R}^m\setminus\Omega$.

Lemma 2.2 *(a) Let M be a Riemannian manifold that is both geodesically and stochastically complete. Let Ω be an open subset of M with $|\Omega| < \infty$. Then*

$$\mathcal{T}(\Omega) \le \lambda_1(\Omega)^{-1}|\Omega|. \tag{2.16}$$

(b) Suppose that M and Ω satisfy the hypotheses in (a). Then

$$\mathcal{T}(\Omega) \ge \lambda_1(\Omega)^{-1}\|\varphi_1\|^{-2}_{L^\infty(\Omega)}. \tag{2.17}$$

(c) Let $\Omega \subset \mathbb{R}^m$. Then

$$\mathcal{T}(\Omega) \ge \frac{1}{2m}\int_\Omega \delta_\Omega(x)^2\,dx. \tag{2.18}$$

(d) Let $\Omega \subset \mathbb{R}^2$ be simply connected and $\delta_\Omega \in L^2(\Omega)$. Then

$$\mathcal{T}(\Omega) \le 16\int_\Omega \delta_\Omega(x)^2\,dx. \tag{2.19}$$

(e) Let $\Omega \subset \mathbb{T}^m$. Then Ω can be embedded in $\mathbb{R}^m$ if and only if $\max_{i=1}^m |x_i - y_i| \le \frac{1}{2}$ for all $x = (x_1, \dots, x_m) \in \Omega$ and $y = (y_1, \dots, y_m) \in \Omega$. If $\Omega \subset \mathbb{T}^2$ can be embedded in $\mathbb{R}^2$, then

$$\frac{1}{4}\int_\Omega \delta_\Omega(x)^2\,dx \le \mathcal{T}(\Omega) \le 16\int_\Omega \delta_\Omega(x)^2\,dx. \tag{2.20}$$

Proof. (a) Since $|\Omega| < \infty$, the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ is discrete. Hence (2.16) follows by integrating (2.10) over $t \in [0, \infty)$.
(b) By the first identity in (2.10) and (2.14), we have that

$$\mathcal{T}(\Omega) \ge \int_0^\infty e^{-t\lambda_1(\Omega)}\,dt \left(\int_\Omega \varphi_1(x)\,dx\right)^2 = \lambda_1(\Omega)^{-1}\left(\int_\Omega \varphi_1(x)\,dx\right)^2. \tag{2.21}$$

By Lemma 2.1, we have that $\|\varphi_1\|_{L^\infty(\Omega)} < \infty$, and so

$$1 = \int_\Omega \varphi_1(x)^2\,dx \le \|\varphi_1\|_{L^\infty(\Omega)}\int_\Omega |\varphi_1(x)|\,dx. \tag{2.22}$$

Inequality (2.17) follows from (2.21)–(2.22) and the fact that φ_1 does not change sign.
(c) For every $x \in \Omega$ the open ball $B_{\delta_\Omega(x)}(x)$ with centre x and radius $\delta(x)$ is contained in Ω. Therefore, by domain monotonicity, the expected exit time satisfies $w_\Omega(y) \ge w_{B_{\delta(x)}(x)}(y)$. Hence, by (2.11),

$$w_\Omega(y) \ge w_{B_{\delta(x)}(x)}(y) = \frac{\delta_\Omega(x)^2 - |x-y|^2}{2m}, \qquad |y-x| \le \delta_\Omega(x). \tag{2.23}$$

Choose $y = x$, integrate over $x \in \Omega$ and use (2.13), to get the claim.
(d) It is well known that the Dirichlet Laplacian on a simply connected proper subset of $\mathbb{R}^2$ satisfies a strong Hardy inequality:

$$\int_\Omega |\nabla v(x)|^2\,dx \ge \frac{1}{16}\int_\Omega \frac{v(x)^2}{\delta_\Omega(x)^2}\,dx \qquad \forall\, v \in C_c^\infty(\Omega). \tag{2.24}$$

Theorem 1.5 in [8] implies (2.19).
(e) Recall that the metric on $\mathbb{T}^m$ is given by

$$d(x,y) = \left(\sum_{i=1}^m \min\left\{|x_i - y_i|, 1 - |x_i - y_i|\right\}^2\right)^{1/2}. \tag{2.25}$$

Note that $\mathrm{diam}(\mathbb{T}^m) = \frac{1}{2}\sqrt{m}$ because $\min\{|x_i - y_i|, 1 - |x_i - y_i|\} \le \frac{1}{2}$. If $|x_i - y_i| \le \frac{1}{2}$ for all i, then $d(x,y) = |x-y|$. Next, suppose that $d(x,y) = |x-y|$. Then $\sum_{i=1}^m \min\{|x_i - y_i|, 1 - |x_i - y_i|\}^2 = \sum_{i=1}^m |x_i - y_i|^2$. Let $I = \{i\colon |x_i - y_i| > \frac{1}{2}\}$. Then $\sum_{i \in I}(1 - 2|x_i - y_i|) = 0$. We therefore conclude that $I = \emptyset$. Finally, (2.20) follows from (2.18) for $m = 2$ and (2.19). ∎

3 Torsional rigidity for $m = 2$

In Section 3.1 we show that the inverse of the principal Dirichlet eigenvalue of $\mathcal{B}(1) = \mathbb{T}^2 \backslash \beta[0,1]$ has a finite exponential moment. In Section 3.2 we use this result to prove Theorem 1.1.

3.1 Exponential moment of the inverse principal Dirichlet eigenvalue

Lemma 3.1 *There exists a $c > 0$ such that*

$$\mathbb{E}_0\left(\exp\left[\frac{c}{\lambda_1(\mathcal{B}(1))}\right]\right) < \infty. \tag{3.1}$$

Proof. Let $\mathrm{cap}\,(A)$ denote the logarithmic capacity of a measurable set $A \subset \mathbb{R}^2$. It is well known (see [16]) that if $\mathrm{cap}\,(A) > 0$ and ϵA is a homothety of A by a factor ϵ, then

$$\mathrm{cap}\,(\epsilon A) = \frac{2\pi}{\log(1/\varepsilon)}\,[1 + o(1)], \qquad \varepsilon \downarrow 0, \tag{3.2}$$

and

$$\lambda_1(\mathbb{T}^2 \backslash \epsilon A) = \frac{2\pi}{\log(1/\varepsilon)}\,[1 + o(1)], \qquad \varepsilon \downarrow 0. \tag{3.3}$$

In particular, if L_ε is a straight line segment of length ε, then there exists a $c' \in (0, \infty)$ such that

$$\lambda_1(\mathbb{T}^2 \backslash L_\varepsilon) \geq \frac{c'}{\log(1/\varepsilon)}, \qquad 0 < \varepsilon \leq \tfrac{1}{2}. \tag{3.4}$$

Since $\mathrm{cap}\,(\beta[0,1]) \geq \mathrm{cap}\,(L_{|\beta(1)|}) \geq \mathrm{cap}\,(L_{(\frac{1}{2} \wedge |\beta(1)|)})$, we get

$$\begin{aligned} &\mathbb{E}_0\left(\exp\left[\frac{c}{\lambda_1(\mathcal{B}(1))}\right]\right) \leq \mathbb{E}_0\left((\tfrac{1}{2} \wedge |\beta(1)|)^{-c/c'}\right) \\ &\leq (\tfrac{1}{2})^{-c/c'} + \mathbb{E}_0\left(|\beta(1)|^{-c/c'}\right) = (\tfrac{1}{2})^{-c/c'} + \int_{\mathbb{R}^2} |x|^{-c/c'}\,\frac{1}{4\pi}\,e^{-|x|^2/4}\,dx, \end{aligned} \tag{3.5}$$

which is finite when $c/c' < 2$. ∎

3.2 Proof of Theorem 1.1

Proof. The proof comes in 6 Steps, and is based on Lemmas 3.2–3.5 below. We use the following abbreviations (recall (1.15) and (1.25)):

$$D_t^2 = \int_{\mathbb{T}^2} d_t(x)^2\,dx, \qquad \lambda_t = \lambda_1(\mathcal{B}(t)). \tag{3.6}$$

1. Note that $\beta[0,t]$ is a closed subset of $\mathbb{T}^2$ a.s. Hence $\mathcal{B}(t)$ is open and its components are open and countable. Let $\{\Omega_1(t), \Omega_2(t), \cdots\}$ enumerate these components. Let

$$\phi_i(t) = \sup_{x,y \in \Omega_i(t)} d(x,y), \tag{3.7}$$

and abbreviate

$$\mathcal{I}_u(t) = \{i \in \mathbb{N}\colon\ \phi_i(t) \leq u\}, \quad \mathcal{E}_u(t) = \left\{\sup_{i \in \mathbb{N}} \phi_i(t) > u\right\}, \qquad u \in (0,1). \tag{3.8}$$

It follows from the proof of Lemma 2.2(d) that if $i \in \mathcal{I}_{1/2}(t)$, then $\Omega_i(t)$ can be isometrically embedded in $\mathbb{R}^2$. Since $\beta[0,t]$ is continuous a.s., each $\Omega_i(t)$ is simply connected. Since the torsional rigidity is additive on disjoint sets we have that

$$\mathcal{T}(\mathcal{B}(t)) = \sum_{i\in\mathbb{N}} \mathcal{T}(\Omega_i(t)) = \sum_{i\in\mathcal{I}_{1/2}(t)} \mathcal{T}(\Omega_i(t)) + \sum_{i\notin\mathcal{I}_{1/2}(t)} \mathcal{T}(\Omega_i(t)). \tag{3.9}$$

2. The first term in the right-hand side of (3.9) is estimated from above by Lemma 2.2(c). This gives (recall (2.15))

$$\sum_{i\in\mathcal{I}_{1/2}(t)} \mathcal{T}(\Omega_i(t)) \le 16 \sum_{i\in\mathcal{I}_{1/4}(t)} \int_{\Omega_i(t)} \delta_{\Omega_i(t)}(x)^2\,dx \le 16 \sum_{i\in\mathbb{N}} \int_{\Omega_i(t)} \delta_{\Omega_i(t)}(x)^2\,dx = 16 D_t^2. \tag{3.10}$$

The second term in the right-hand side of (3.9) is estimated from above by Lemma 2.2(a). This gives

$$\sum_{i\notin\mathcal{I}_{1/2}(t)} \mathcal{T}(\Omega_i(t)) \le \sum_{i\notin\mathcal{I}_{1/2}(t)} \lambda_t^{-1}\,|\Omega_i(t)| \le 1_{\mathcal{E}_{1/2}(t)}\,\lambda_t^{-1} \sum_{i\in\mathbb{N}} |\Omega_i(t)| = 1_{\mathcal{E}_{1/2}(t)}\,\lambda_t^{-1}. \tag{3.11}$$

By Cauchy-Schwarz, this term contributes to $\spadesuit(t)$ at most

$$\mathbb{E}_0\left(1_{\mathcal{E}_{1/2}(t)}\,\lambda_t^{-1}\right) \le \left(\mathbb{P}_0(\mathcal{E}_{1/2}(t))\right)^{1/2} \left(\mathbb{E}_0\left(\lambda_t^{-2}\right)\right)^{1/2}. \tag{3.12}$$

To bound the probability in the right-hand side of (3.12) from above, we let $\{Q_1,\cdots,Q_N\}$, $N=10^4$, be any open disjoint collection of squares in $\mathbb{T}^2$, each with area 10^{-4} and not containing 0. Furthermore, we let $\bar{Q}_{N,\epsilon}$ be the open ϵ-neighbourhood of the union of the boundaries of these squares with $\epsilon = 10^{-3}$. Then $\beta[0,1]$ starting at 0 has a positive probability $p' = p'(N,\epsilon)$ of making a closed loop around each of these squares and staying inside $\bar{Q}_{N,\epsilon}$. Translating $\{Q_1,\cdots,Q_N\}$ such that these squares do not contain $\beta(1)$, we find that $\beta[1,2]$ starting at $\beta(1)$ has a positive probability p' of making a closed loop around each of these translated squares and staying inside $\bar{Q}_{N,\epsilon}+\beta(1)$. Continuing this way, by induction we find that the probability of $\beta[0,t]$ not making any of these closed translated loops is at most $(1-p')^{\lfloor t\rfloor}$, where $\lfloor\cdot\rfloor$ denotes the integer part. Hence $\mathbb{P}_0(\sup_{i\in\mathbb{N}}\phi_i(t) > \frac12) \le (1-p')^{\lfloor t\rfloor}$, and so

$$\mathbb{P}_0(\mathcal{E}_{1/2}(t)) \le e^{-pt}, \qquad t\ge 2, \tag{3.13}$$

for some $p>0$. We conclude that

$$\spadesuit(t) \le 16\,\mathbb{E}_0\left(D_t^2\right) + e^{-pt/2}\Big(\mathbb{E}_0(\lambda_t^{-2})\Big)^{1/2}, \qquad t\ge 2. \tag{3.14}$$

Since $t\mapsto\lambda_t$ is non-decreasing, Lemma 3.1 implies that the second term decays exponentially fast in t and therefore is harmless for the upper bound in (1.9).

3. To derive a lower bound for $\spadesuit(t)$, we note that by Lemma 2.2(d) we have

$$\begin{aligned}
\mathcal{T}(\mathcal{B}(t)) &= \sum_{i\in\mathbb{N}} \mathcal{T}(\Omega_i(t)) \ge \sum_{i\in\mathcal{I}_{1/2}(t)} \mathcal{T}(\Omega_i(t))\\
&\ge \tfrac14 \sum_{i\in\mathcal{I}_{1/2}(t)} \int_{\Omega_i(t)} \delta_{\Omega_i(t)}(x)^2\,dx \ge \tfrac14 \sum_{i\in\mathbb{N}} \int_{\Omega_i(t)} \delta_{\Omega_i(t)}(x)^2\,dx - \tfrac14 \sum_{i\notin\mathcal{I}_{1/2}(t)} \int_{\Omega_i(t)} \delta_{\Omega_i(t)}(x)^2\,dx\\
&\ge \tfrac14 D_t^2 - \tfrac14 \sum_{i\notin\mathcal{I}_{1/2}(t)} 1_{\mathcal{E}_{1/4}(t)} \int_{\Omega_i(t)} \delta_{\Omega_i(t)}(x)^2\,dx \ge \tfrac14 D_t^2 - \tfrac18 1_{\mathcal{E}_{1/4}(t)},
\end{aligned} \tag{3.15}$$

where in the last inequality we use that $\delta_{\Omega_i(t)}(x) \le \mathrm{diam}(\mathbb{T}^2) = \frac12\sqrt{2}$ and $|\mathbb{T}^2|=1$. We conclude by (3.13) that

$$\spadesuit(t) \ge \tfrac14 \mathbb{E}_0\left(D_t^2\right) - e^{-pt}, \qquad t\ge 2. \tag{3.16}$$

The second term is again harmless for the lower bound in (1.9).

4. The estimates in (3.14) and (3.16) show that $\spadesuit(t) \asymp \mathbb{E}_0(D_t^2)$ up to exponentially small error terms. In order to obtain the leading order asymptotic behaviour of $\mathbb{E}_0(D_t^2)$, we make a dyadic partition of $\mathbb{T}^2$ into squares as follows. Partition $\mathbb{T}^2$ into four 1-squares of area $\frac{1}{4}$ each, and proceed by induction to partition each k-square into four $(k+1)$-squares. So, $\mathbb{T}^2$ is partitioned into 2^{2k} k-squares. We define a k-square to be *good* when the path $\beta[0,t]$ does not hit this square, but does hit the unique $(k-1)$-square to which it belongs. Clearly, if x belongs to a good k-square, then $\mathrm{dist}(x,\beta[0,t]) \leq (2\sqrt{2})2^{-k}$. Hence, as the area of each k-square is 2^{-2k}, we get

$$\mathbb{E}\left(D_t^2\right) \leq 8 \sum_{k\in\mathbb{N}} 2^{-4k}\, \mathbb{E}\left(\#\text{ good } k\text{-squares}\right), \tag{3.17}$$

where we write $\mathbb{E} = \int_{\mathbb{T}^2} dx\, \mathbb{E}_x$, which is the same as $\mathbb{E}_0$ for the quantity under consideration, by translation invariance. To estimate the right-hand side of (3.17) we need three lemmas.

Lemma 3.2 *For $k \in \mathbb{N}$, let $p_k(t) = \mathbb{P}(\beta[0,t] \cap S_k) = \emptyset)$, where S_k is any of the k-squares. Then*

$$p_k(t) \leq e^{-t\lambda_1(\mathbb{T}^2\backslash S_k)}. \tag{3.18}$$

Proof. Let $p_{\mathbb{T}^2\backslash S_k}(x,y;t)$ be the Dirichlet heat kernel for $\mathbb{T}^2\backslash S_k$. By the eigenfunction expansion in (2.7), we have that

$$\begin{aligned} p_k(t) &= \int_{\mathbb{T}^2\backslash S_k} dx \int_{\mathbb{T}^2\backslash S_k} dy\; p_{\mathbb{T}^2\backslash S_k}(x,y;t) = \int_{\mathbb{T}^2\backslash S_k} dx \int_{\mathbb{T}^2\backslash S_k} dy\; \sum_{j\in\mathbb{N}} e^{-t\lambda_j(\mathbb{T}^2\backslash S_k)} \varphi_j(x)\varphi_j(y) \\ &\leq e^{-t\lambda_1(\mathbb{T}^2\backslash S_k)} \sum_{j\in\mathbb{N}} \left(\int_{\mathbb{T}^2\backslash S_k} dx\, \varphi_j(x)\right)^2 = e^{-t\lambda_1(\mathbb{T}^2\backslash S_k)} |\mathbb{T}^2\backslash S_k| \leq e^{-t\lambda_1(\mathbb{T}^2\backslash S_k)}, \end{aligned} \tag{3.19}$$

where we use Parseval's identity in the last equality. ∎

Lemma 3.3 *There exists a $C < \infty$ such that, for all $k \in \mathbb{N}$,*

$$\left|\lambda_1(\mathbb{T}^2\backslash S_k) - \frac{2\pi}{k\log 2}\right| \leq \frac{C}{k^2}. \tag{3.20}$$

Proof. By [18, Theorem 1] we have that, for any disc $D_\epsilon \subset \mathbb{T}^2$ with radius ϵ,

$$\lambda_1(\mathbb{T}^2\backslash D_\epsilon) = \frac{2\pi}{\log(1/\epsilon)} + O\big([\log(1/\epsilon)]^{-2}\big), \qquad \epsilon \downarrow 0. \tag{3.21}$$

This implies, by monotonicity and continuity of $\epsilon \mapsto \lambda_1(\mathbb{T}^2\backslash D_\epsilon)$, the existence of a $C' < \infty$ such that

$$\left|\lambda_1(\mathbb{T}^2\backslash D_\epsilon) - \frac{2\pi}{\log(1/\epsilon)}\right| \leq C'[\log(1/\epsilon)]^{-2}, \qquad 0 < \epsilon \leq \tfrac{1}{2}. \tag{3.22}$$

For $S_k \subset \mathbb{T}^2$ there exist two discs D_1 and D_2, with the same centre and radii 2^{-k-1} and $2^{-k-1}\sqrt{2}$, such that $D_1 \subset S_k \subset D_2$. Hence $\lambda_1(D_2) \leq \lambda_1(S_k) \leq \lambda_1(D_1)$, and (3.20) follows by applying (3.22) to $\epsilon = 2^{-k-1}$ and $\epsilon = 2^{-k-1}\sqrt{2}$, respectively. ∎

Lemma 3.4

$$\int_{\mathbb{T}^2} dx\; \mathbb{P}_x(S_k \text{ is a good } k\text{-square}) = p_k(t) - p_{k-1}(t). \tag{3.23}$$

Proof. Let E_k be the event that S_k is not hit. Since S_k is a good k-square if and only if the event $E_k \cap E_{k-1}^c$ occurs, the lemma follows because $E_{k-1} \subset E_k$. ∎

5. We are now ready to estimate $\mathbb{E}(D_t^2)$. By (3.17) and Lemma 3.4,

$$\mathbb{E}\left(D_t^2\right) \leq 8 \sum_{k\in\mathbb{N}} 2^{-2k} \int_{\mathbb{T}^2} dx\ \mathbb{P}_x(S_k \text{ is a good } k\text{-square}) = 8 \sum_{k\in\mathbb{N}} 2^{-2k}[p_k(t) - p_{k-1}(t)] = 6 \sum_{k\in\mathbb{N}} 2^{-2k} p_k(t), \tag{3.24}$$

where $p_0(t) = 0$. In order to bound this sum from above we consider the contributions coming from $k = 1, \ldots K$ and $k = K+1, \ldots, \lfloor \frac{1}{4} t^{1/2} \rfloor$ and $k > \lfloor \frac{1}{4} t^{1/2} \rfloor$, respectively, where $\lfloor \cdot \rfloor$ denotes the integer part, and we choose

$$K = \lfloor (C \log 2)/\pi \rfloor \tag{3.25}$$

with C the constant in (3.20). Since

$$\sum_{k=1}^{K} 2^{-2k} p_k(t) \leq \sum_{k=1}^{K} 2^{-2k} p_K(t) \leq e^{-t\lambda_1(\mathbb{T}^2 \setminus S_K)}, \tag{3.26}$$

the first contribution is exponentially small in t. For $k = K+1, \ldots, \lfloor \frac{1}{4} t^{1/2} \rfloor$ we have $C/k^2 \leq \pi/k \log 2$, and hence by Lemmas 3.2–3.3,

$$\sum_{k=K+1}^{\lfloor \frac{1}{4} t^{1/2} \rfloor} 2^{-2k} p_k(t) \leq \sum_{k=K+1}^{\lfloor \frac{1}{4} t^{1/2} \rfloor} 2^{-2k} e^{-\frac{\pi t}{k \log 2}} \leq \sum_{k=K+1}^{\lfloor \frac{1}{4} t^{1/2} \rfloor} 2^{-2k} e^{-\frac{4\pi t^{1/2}}{\log 2}} = O(e^{-4\pi t^{1/2}}), \tag{3.27}$$

and so the second contribution is $o(t^{1/4} e^{-4(\pi t)^{1/2}})$. Finally, for $k > \lfloor \frac{1}{4} t^{1/2} \rfloor$ we have $e^{Ct/k^2} \leq e^{16C}$, and hence

$$\sum_{k > \lfloor \frac{1}{4} t^{1/2} \rfloor} 2^{-2k}\, p_k(t) \leq e^{16C} \sum_{k > \lfloor \frac{1}{4} t^{1/2} \rfloor} e^{-2k \log 2 - \frac{2\pi t}{k \log 2}}. \tag{3.28}$$

The summand is increasing for $1 \leq k \leq (\pi t)^{1/2}/\log 2$ and decreasing for $k \geq (\pi t)^{1/2}/\log 2$. Moreover, it is bounded from above by $e^{-4(\pi t)^{1/2}}$. We conclude that for $t \to \infty$,

$$\begin{aligned} \sum_{k > \lfloor \frac{1}{4} t^{1/2} \rfloor} e^{-2k \log 2 - \frac{2\pi t}{k \log 2}} &\leq 2\, e^{-4(\pi t)^{1/2}} + \int_{[0,\infty)} dk\, e^{-2k \log 2 - \frac{2\pi t}{k \log 2}} \\ &= 2\, e^{-4(\pi t)^{1/2}} + \frac{(4\pi t)^{1/2}}{\log 2} K_1\big(4(\pi t)^{1/2}\big) = \frac{\pi^{3/4}}{\sqrt{2} \log 2}\, t^{1/4}\, e^{-4(\pi t)^{1/2}} [1 + o(1)], \end{aligned} \tag{3.29}$$

where we use formula 3.324.1 from [15] and formula 9.7.2 from [1]. Putting the estimates in (3.14) and (3.24)–(3.29) together, we obtain that

$$\spadesuit(t) \leq \frac{96 \pi^{3/4}\, e^{16C}}{\sqrt{2} \log 2}\, t^{1/4}\, e^{-4(\pi t)^{1/2}} [1 + o(1)]. \tag{3.30}$$

This is the desired upper bound in (1.9).

6. To obtain a lower bound for $\mathbb{E}(D_t^2)$, we consider a good k-square. This square contains a square with the same centre, parallel sides and area 2^{-2k-2}. The distance from this square to $\beta[0,t]$ is bounded from below by 2^{-k-2}. Hence

$$\begin{aligned} \mathbb{E}\left(D_t^2\right) &\geq \tfrac{1}{16} \sum_{k\in\mathbb{N}} 2^{-2k} \int_{\mathbb{T}^2} dx\ \mathbb{P}_x(S_k \text{is a good } k\text{-square}) \\ &= \tfrac{1}{16} \sum_{k\in\mathbb{N}} 2^{-2k}\, [p_k(t) - p_{k-1}(t)] = \tfrac{3}{64} \sum_{k\in\mathbb{N}} 2^{-2k}\, p_k(t), \end{aligned} \tag{3.31}$$

since $p_0(t) = 0$. The following lemma provides a lower bound for the right-hand side of (3.31).

Lemma 3.5 *There exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,*

$$p_k(t) \geq \tfrac{1}{4} e^{-t\lambda_1(\mathbb{T}^2 \backslash S_k)}. \tag{3.32}$$

Proof. By the eigenfunction expansion in (2.7) we have that

$$\begin{aligned} p_k(t) &= \int_{\mathbb{T}^2 \backslash S_k} dx \int_{\mathbb{T}^2 \backslash S_k} dy \sum_{j \in \mathbb{N}} e^{-t\lambda_j(\mathbb{T}^2 \backslash S_k)} \varphi_j(x) \varphi_j(y) \\ &\geq e^{-t\lambda_1(\mathbb{T}^2 \backslash S_k)} \left(\int_{\mathbb{T}^2 \backslash S_k} dx\, \varphi_1(x) \right)^2. \end{aligned} \tag{3.33}$$

By the results of [18], $\|\varphi_1 - 1\|_{L^2(\mathbb{T}^2 \backslash S_k)} \to 0$ as $k \to \infty$. This implies that $|\int_{\mathbb{T}^2 \backslash S_k} dx\, \varphi_1(x)| \geq \frac{1}{2}$ for k sufficiently large. ∎

Combining (3.20), (3.24), (3.31) and Lemma 3.5, we have that

$$\mathbb{E}\left(D_t^2\right) \geq \tfrac{3}{256} \sum_{\{k \in \mathbb{N}:\ k \geq k_0\}} e^{-2k \log 2 - \frac{2\pi t}{k \log 2} - \frac{Ct}{k^2}}. \tag{3.34}$$

Now let t be such that $\pi t / \log 2 > k_0$. Then

$$\begin{aligned} \mathbb{E}\left(D_t^2\right) &\geq \tfrac{3}{256} \sum_{\left\{k \in \mathbb{N}:\ k \geq \frac{(\pi t)^{1/2}}{\log 2}\right\}} e^{-2k \log 2 - \frac{2\pi t}{k \log 2} - \frac{Ct}{k^2}} \\ &\geq \tfrac{3}{256}\, e^{-C} \sum_{\left\{k \in \mathbb{N}:\ k \geq \frac{(\pi t)^{1/2}}{\log 2}\right\}} e^{-2k \log 2 - \frac{2\pi t}{k \log 2}}. \end{aligned} \tag{3.35}$$

Because the summand is strictly decreasing in k, we can replace the sum over k by an integral with a minor correction. This gives

$$\mathbb{E}\left(D_t^2\right) \geq \tfrac{3}{256}\, e^{-C} \left(\int_{\frac{(\pi t)^{1/2}}{\log 2}}^{\infty} dk\, e^{-2k \log 2 - \frac{2\pi t}{k \log 2}} - e^{-4(\pi t)^{1/2}} \right). \tag{3.36}$$

We have

$$\begin{aligned} &\int_{\frac{(\pi t)^{1/2}}{\log 2}}^{\infty} dk\, e^{-2k \log 2 - \frac{2\pi t}{k \log 2}} = \frac{(\pi t)^{1/2}}{\log 2} \int_1^{\infty} dx\, e^{-2(\pi t)^{1/2}(x + \frac{1}{x})} \geq \frac{(\pi t)^{1/2}}{\log 4} \int_0^{\infty} dx\, e^{-2(\pi t)^{1/2}(x + \frac{1}{x})} \\ &= \frac{(\pi t)^{1/2}}{\log 2} K_1\big(4(\pi t)^{1/2}\big) = \frac{\pi^{3/4}}{2^{3/2} \log 2} t^{1/4} e^{-4(\pi t)^{1/2}} [1 + o(1)], \end{aligned} \tag{3.37}$$

where we use once more formulas 3.324.1 from [15] and 9.7.2 from [1]. Combining (3.16), (3.36) and (3.37), we get

$$♠(t) \geq \frac{3\pi^{3/4}\, e^{-C}}{2^{23/2} \log 2}\, t^{1/4}\, e^{-4(\pi t)^{1/2}} [1 + o(1)]. \tag{3.38}$$

This is the desired lower bound in (1.9). ∎

4 Torsional rigidity for $m = 3$

It is well known that cap $(\beta[0,1])$ has a strictly positive Newton capacity when $m = 3$. In Section 4.1 we show that the inverse of the capacity of $\beta[0,1]$ on $\mathbb{R}^3$ has a finite exponential moment. In Section 4.2 we show that for every closed set $K \subset \mathbb{T}^3$ that has a small enough diameter the principal Dirichlet eigenvalue of $\mathbb{T}^3 \backslash K$ is bounded from below by a constant times the capacity of K. (The same is true for $m \geq 4$, a fact that will be needed in Section 5.) In Section 4.3 we use these results to prove Theorem 1.2.

4.1 Exponential moment of the inverse capacity

Lemma 4.1 *Let $m = 3$. Then there exists a $c > 0$ such that*

$$\mathbb{E}\left(\exp\left[\frac{c}{\mathrm{cap}\,(\beta[0,1])}\right]\right) < \infty. \tag{4.1}$$

Proof. We use the fact that, for any compact set $A \subset \mathbb{R}^3$,

$$\frac{1}{\mathrm{cap}\,(A)} = \inf\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\mu(dx)\mu(dy)}{4\pi\,|x-y|}\colon\ \mu \text{ is a probability measure on } A\right]. \tag{4.2}$$

As test probability measure we choose the sojourn measure of $\beta[0,t]$, i.e.,

$$\mu_{\beta[0,1]}(C) = \int_0^1 1_C(\beta(t))\,dt, \qquad C \subset \mathbb{R}^3, \tag{4.3}$$

for which

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\mu_{\beta[0,1]}(dx)\mu_{\beta[0,1]}(dy)}{4\pi\,|x-y|} = \int_0^1 ds \int_0^1 dt\, \frac{1}{4\pi\,|\beta(s)-\beta(t)|}. \tag{4.4}$$

It therefore suffices to prove that

$$\mathbb{E}_0\left(\exp\left[c\int_0^1 ds\int_0^1 dt\,\frac{1}{|\beta(s)-\beta(t)|}\right]\right) < \infty \tag{4.5}$$

for small enough $c > 0$. A proof of this fact is hidden in [13]. For the convenience of the reader we write it out here.

By Cauchy-Schwarz and Jensen, we have

$$\begin{aligned}
&\mathbb{E}_0\left(\exp\left[c\int_0^1 ds\int_0^1 dt\,\frac{1}{|\beta(s)-\beta(t)|}\right]\right) \le \mathbb{E}_0\left(\exp\left[2c\int_0^1 ds\int_s^1 dt\,\frac{1}{|\beta(s)-\beta(t)|}\right]\right)\\
&\le \mathbb{E}_0\left(\exp\left[2c\int_0^1 ds\int_s^{1+s} dt\,\frac{1}{|\beta(s)-\beta(t)|}\right]\right) \le \int_0^1 ds\,\mathbb{E}_0\left(\exp\left[2c\int_s^{1+s} dt\,\frac{1}{|\beta(s)-\beta(t)|}\right]\right)\\
&= \mathbb{E}_0\left(\exp\left[2c\int_0^1 dt\,\frac{1}{|\beta(t)|}\right]\right).
\end{aligned} \tag{4.6}$$

It therefore suffices to prove that the right-hand side is finite for small enough $c > 0$. Expanding the exponent, we get

$$\begin{aligned}
\mathbb{E}_0\left(\exp\left[2c\int_0^1 dt\,\frac{1}{|\beta(t)|}\right]\right) &= \sum_{k\in\mathbb{N}_0}\frac{(2c)^k}{k!}\,\mathbb{E}_0\left(\left[\int_0^1 dt\,\frac{1}{|\beta(t)|}\right]^k\right)\\
&= \sum_{k\in\mathbb{N}_0}(2c)^k\int_{0\le t_1<\cdots<t_k\le 1}\mathbb{E}_0\left(\frac{1}{|\beta(t_1)|\times\cdots\times|\beta(t_k)|}\right)dt_1\times\cdots\times dt_k.
\end{aligned} \tag{4.7}$$

The integrand equals

$$\mathbb{E}_0\left(\frac{1}{|\beta(t_1)|\times\cdots\times|\beta(t_{k-1})|}\,\mathbb{E}_0\left(\frac{1}{|\beta(t_{k-1})+[\beta(t_k)-\beta(t_{k-1})]|}\,\middle|\,\mathcal{F}_{t_{k-1}}\right)\right), \tag{4.8}$$

where $\mathcal{F}_t$ is the sigma-algebra of β up to time t. However,

$$\begin{aligned}
&\mathbb{E}_0\left(\frac{1}{|\beta(t_{k-1})+[\beta(t_k)-\beta(t_{k-1})]|}\,\middle|\,\mathcal{F}_{t_{k-1}}\right) = \mathbb{E}_0\left(\frac{1}{|x+\sqrt{t_k-t_{k-1}}\beta(1)|}\right)\Bigg|_{x=\beta(t_{k-1})}\\
&\le \sup_{x\in\mathbb{R}^3}\mathbb{E}_0\left(\frac{1}{|x+\sqrt{t_k-t_{k-1}}\beta(1)|}\right) \le \mathbb{E}_0\left(\frac{1}{|\sqrt{t_k-t_{k-1}}\beta(1)|}\right) \le \frac{\gamma}{\sqrt{t_k-t_{k-1}}}
\end{aligned} \tag{4.9}$$

with $\gamma = \mathbb{E}_0(|\beta(1)|^{-1}) < \infty$, where in the second inequality we use that $|x + \beta(1)|$ is stochastically larger than $|\beta(1)|$ for any $x \neq 0$. Iterating (4.8)–(4.9), we get

$$\mathbb{E}_0\left(\frac{1}{|\beta(t_1)| \times \cdots \times |\beta(t_k)|}\right) \leq \gamma^k \prod_{i=1}^{k} \frac{1}{\sqrt{t_i - t_{i-1}}}, \tag{4.10}$$

where $t_0 = 0$. Hence

$$\begin{aligned}\mathbb{E}_0\left(\exp\left[2c\int_0^1 dt\, \frac{1}{|\beta(t)|}\right]\right) &= \sum_{k\in\mathbb{N}_0} (2c)^k\gamma^k \int_{0\leq t_1<\cdots<t_k\leq 1} \frac{dt_1}{\sqrt{t_1}} \times \cdots \times \frac{dt_k}{\sqrt{t_k - t_{k-1}}}\\ &\leq \sum_{k\in\mathbb{N}_0} (2c)^k\gamma^k \left(\int_0^1 dt\, \frac{1}{\sqrt{t}}\right)^k = \sum_{k\in\mathbb{N}_0} (4c)^k\gamma^k, \end{aligned}\tag{4.11}$$

which is finite for $c < 1/4\gamma$. ∎

4.2 Principal Dirichlet eigenvalue and capacity

Lemma 4.2 *Let $m \geq 3$, and let K be a closed subset of $\mathbb{T}^m$ with* $\mathrm{diam}(K) \leq \frac{1}{2}$. *Then*

$$\lambda_1(\mathbb{T}^m\backslash K) \geq k_m\, \mathrm{cap}\,(K), \tag{4.12}$$

where

$$k_m = \int_0^1 ds\, (4\pi s)^{-m/2}\, e^{-m/4s}, \tag{4.13}$$

and $\mathrm{cap}\,(K)$ *is the Newtonian capacity of K embedded in $\mathbb{R}^m$.*

Proof. Since $\mathrm{diam}(K) \leq \frac{1}{2}$, K can be embedded in $\mathbb{R}^m$ by Lemma 2.2(e). We let $K \subset [-\frac{1}{2}, \frac{1}{2})^m \subset \mathbb{R}^m$, identify $[-\frac{1}{2}, \frac{1}{2})^m$ with $\mathbb{T}^m$, and define $\tilde{K} \subset \mathbb{R}^m$ by $\tilde{K} = \cup_{k\in\mathbb{Z}^m}\{k+K\}$. Let φ_1 be the first eigenfunction on $\mathbb{T}^m\backslash K$ with Dirichlet boundary conditions on K, and let $\lambda_1(\mathbb{T}^m\backslash K)$ be the corresponding first Dirichlet eigenvalue. Then

$$e^{-t\lambda_1(\mathbb{T}^m\backslash K)}\varphi_1(x) = \int_{\mathbb{T}^m\backslash K} dy\, p_{\mathbb{T}^m\backslash K}(x, y; t)\, \varphi_1(y). \tag{4.14}$$

Integrating both sides of this identity over $x \in \mathbb{T}^m\backslash K$, we get

$$e^{-t\lambda_1(\mathbb{T}^m\backslash K)} \int_{\mathbb{T}^m\backslash K} dx\, \varphi_1(x) = \int_{\mathbb{T}^m\backslash K} dx\, \varphi_1(x) - \int_{\mathbb{T}^m\backslash K} dy\, \mathbb{P}_y(T_K \leq t)\, \varphi_1(y), \tag{4.15}$$

where T_K is the first hitting time of K by Brownian motion on $\mathbb{T}^m$. It follows that for any $t > 0$,

$$\begin{aligned}\lambda_1(\mathbb{T}^m\backslash K) &= -\frac{1}{t}\log\left(1 - \frac{\int_{\mathbb{T}^m\backslash K} dy\, \mathbb{P}_y(T_K \leq t)\, \varphi_1(y)}{\int_{\mathbb{T}^m\backslash K} dy\, \varphi_1(y)}\right)\\ &\geq \frac{1}{t}\frac{\int_{\mathbb{T}^m\backslash K} dy\, \mathbb{P}_y(T_K \leq t)\, \varphi_1(y)}{\int_{\mathbb{T}^m\backslash K} dy\, \varphi_1(y)} \geq \frac{1}{t}\inf_{y\in\mathbb{T}^m} \mathbb{P}_y(T_K \leq t), \end{aligned}\tag{4.16}$$

where we use the inequality $-\log(1 - z) \geq z$, $z \in [0, 1)$. Let $\tilde{\beta}$ be Brownian motion on $\mathbb{R}^m$, and let $\tilde{T}_{\tilde{K}}$ be the first hitting time of $\tilde{K}$ by $\tilde{\beta}$. Then

$$\mathbb{P}_y(T_K \leq t) = \tilde{\mathbb{P}}_y(\tilde{T}_{\tilde{K}} \leq t) \geq \tilde{\mathbb{P}}_y(\tilde{T}_K \leq t) \geq \tilde{\mathbb{P}}_y(\tilde{L}_K \leq t), \tag{4.17}$$

where $\tilde{L}_K$ is the last exit time from K by $\tilde{\beta}$. Let μ_K denote the equilibrium measure on K in $\mathbb{R}^m$. Then (see [20])

$$\tilde{\mathbb{P}}_y(\tilde{L}_K \le t) = \int_K \mu_K(dz) \int_0^t ds\,(4\pi s)^{-m/2}\, e^{-|z-y|^2/4s}. \tag{4.18}$$

By (4.17)–(4.18),

$$\inf_{y\in\mathbb{T}^m} \mathbb{P}_y(T_K \le t) = \inf_{y\in\left[-\frac{1}{2},\frac{1}{2}\right)^m} \tilde{\mathbb{P}}_y(\tilde{T}_{\tilde{K}} \le t) \ge \inf_{y\in\left[-\frac{1}{2},\frac{1}{2}\right)^m} \int_K \mu_K(dz) \int_0^t ds\,(4\pi s)^{-m/2} e^{-|z-y|^2/4s}. \tag{4.19}$$

But $|z-y| \le \sqrt{m}$ for $z \in K$ and $y \in [-\frac{1}{2},\frac{1}{2})^m$. Hence the right-hand side of (4.19) is bounded from below by $\mathrm{cap}\,(K) \int_0^t ds\,(4\pi s)^{-m/2}\, e^{-m/4s}$. We now get the claim by choosing $t=1$ in (4.16). ∎

We note that if $m=3$ and $K = B_\epsilon \subset \mathbb{T}^3$ is a closed ball with radius ϵ, then $\lambda_1(\mathbb{T}^3\backslash B_\epsilon) = \mathrm{cap}\,(B_\epsilon)[1+o(1)]$ as $\epsilon \downarrow 0$ (see [16]). In that case, since $k_3 = 0.0101\cdots$, we see that the constant in (4.12) is off by a large factor.

4.3 Proof of Theorem 1.2

Proof. Write, recalling (1.1)–(1.5) and using Fubini's theorem,

$$♠(t) = (\mathbb{E}_0 \otimes \tilde{\mathbb{E}})\big(\tilde{\tau}_{\mathbb{T}^3\backslash\beta[0,t]}\big) = (\mathbb{E}_0 \otimes \tilde{\mathbb{E}}) \left(\int_0^\infty ds\, 1_{\{\tilde{\tau}_{\mathbb{T}^3\backslash\beta[0,t]}>s\}}\right) = (\mathbb{E}_0 \otimes \tilde{\mathbb{E}}) \left(\int_0^\infty ds\, 1_{\{\tilde{\beta}[0,s]\cap\beta[0,t]=\emptyset\}}\right), \tag{4.20}$$

where $\tilde{\mathbb{E}}$ denotes expectation over $\tilde{\beta}$ with $\tilde{\beta}(0)$ drawn uniformly from $\mathbb{T}^3$. By symmetry, we may replace $\mathbb{E}_0 \otimes \tilde{\mathbb{E}}$ by $\tilde{\mathbb{E}}_0 \otimes \mathbb{E}$. The proof comes in 7 Steps.

1. Pick $\eta\colon (0,\infty) \to (0,\infty)$ such that

$$\lim_{t\to\infty} \eta(t)\log t = 0, \qquad \lim_{t\to\infty} \frac{t\sqrt{\eta(t)}}{\log^2 t} = \infty. \tag{4.21}$$

We begin by showing that the integral over $s \in [\eta(t),\infty)$ decays faster than any negative power of t and therefore is negligible. Indeed, for any $K(t) \in [\eta(t),\infty)$ we have, by the spectral decomposition in (2.7),

$$(\tilde{\mathbb{E}}_0 \otimes \mathbb{E}) \left(\int_{\eta(t)}^{K(t)} ds\, 1_{\{\tilde{\beta}[0,s]\cap\beta[0,t]=\emptyset\}}\right) \le \tilde{\mathbb{E}}_0 \left(\int_{\eta(t)}^{K(t)} ds\, e^{-t\lambda_1(\mathbb{T}^3\backslash\tilde{\beta}[0,s])}\right). \tag{4.22}$$

By Lemma 4.2, $\lambda_1(\mathbb{T}^3\backslash A) \ge c_3\,\mathrm{cap}\,(A)$ for every closed set $A \subset B_{1/4}(0) \subset \mathbb{T}^3$ (in the lower bound we interpret A as a subset of $\mathbb{R}^3$). Hence the right-hand side of (4.22) is bounded from above by

$$K(t)\, \tilde{\mathbb{E}}_0 \left(e^{-c_3 t\,\mathrm{cap}\,(\tilde{\beta}[0,\eta(t)]\cap B_{1/4}(0))}\right), \tag{4.23}$$

where we use that $\mathrm{cap}\,(\tilde{\beta}[0,s]) \ge \mathrm{cap}\,(\tilde{\beta}[0,\eta(t)])$ for $s \ge \eta(t)$. In Step 2 we show that $\mathbb{P}_0(\tilde{\beta}[0,\eta(t)] \subsetneq B_{1/4}(0))$ decays faster than any negative power of t. Hence we may replace $\mathrm{cap}\,(\tilde{\beta}[0,\eta(t)]\cap B_{1/4}(0))$ by $\mathrm{cap}\,(\tilde{\beta}[0,\eta(t)]$ in (4.23) at the cost of a negligible error term $o(t^{-2})$. Next, we note that $\mathrm{cap}\,(\tilde{\beta}[0,\eta(t)]$ is equal to $\sqrt{\eta(t)}\,\mathrm{cap}\,(\tilde{\beta}[0,1])$ in distribution. Moreover, since $au+bu^{-1} \ge 2\sqrt{ab}$ for all $a,b,u \in (0,\infty)$, we have, for any $c>0$,

$$\begin{aligned} e^{-c_3 t\sqrt{\eta(t)}\,\mathrm{cap}\,(\tilde{\beta}[0,1])} &= e^{-c_3 t\sqrt{\eta(t)}\,\mathrm{cap}\,(\tilde{\beta}[0,1]) - c\,\mathrm{cap}\,(\tilde{\beta}[0,1])^{-1}}\, e^{c\,\mathrm{cap}\,(\tilde{\beta}[0,1])^{-1}} \\ &\le e^{-2\sqrt{c_3 c t\sqrt{\eta(t)}}}\, e^{c\,\mathrm{cap}\,(\tilde{\beta}[0,1])^{-1}}. \end{aligned} \tag{4.24}$$

By Lemma 4.1, we therefore have

$$\tilde{\mathbb{E}}_0\left(e^{-c_3 t\,\mathrm{cap}\,(\tilde\beta[0,\eta(t)])}\right) \le C\,e^{-2\sqrt{c_3 c t\sqrt{\eta(t)}}} + o(t^{-2}) \tag{4.25}$$

for some $C<\infty$ and $c>0$ small enough. Hence (4.23) is $O(K(t)^{-1})$ when we pick

$$K(t) = e^{\sqrt{c_3 c t\sqrt{\eta(t)}}}. \tag{4.26}$$

The second half of (4.21) ensures that $K(t)$ grows faster than any positive power of t, and so we conclude that the integral in the left-hand side of (4.22) is $o(t^{-2})$. To estimate

$$(\tilde{\mathbb{E}}_0\otimes\mathbb{E})\left(\int_{K(t)}^{\infty} ds\,1_{\{\tilde\beta[0,s]\cap\beta[0,t]=\emptyset\}}\right) \tag{4.27}$$

we reverse the roles of $\tilde\beta$ and $\tilde\beta$, and do the same estimate using that $\mathrm{cap}\,(\beta[0,t]) \ge \mathrm{cap}\,(\beta[0,\eta(t)])$ for $t\in[\eta(t),\infty)$. This leads to

$$\begin{aligned}(\tilde{\mathbb{E}}_0\otimes\mathbb{E})\left(\int_{K(t)}^{\infty} ds\,1_{\{\tilde\beta[0,s]\cap\beta[0,t]=\emptyset\}}\right) &\le C\int_{K(t)}^{\infty} ds\,e^{-2\sqrt{c_3 c s\sqrt{\eta(t)}}} + o(t^{-2})\\ &= [1+o(1)]\,C\sqrt{\frac{K(t)}{c_3 c\sqrt{\eta(t)}}}\,e^{-2\sqrt{c_3 c K(t)\sqrt{\eta(t)}}} + o(t^{-2}),\end{aligned} \tag{4.28}$$

in which the first term is even much smaller than $o(t^{-2})$.

2. The probability that $\tilde\beta$ leaves the ball of radius $\tilde\eta(t) = (M(t)\,\eta(t)\log t)^{1/2}$ prior to time $\eta(t)$ decays faster than any negative power of t when $\lim_{t\to\infty} M(t)=\infty$. Indeed, by Lévy's maximal inequality ([22, Theorem 3.6.5]),

$$\begin{aligned}&\tilde{\mathbb{P}}_0\big(\exists\, s\in[0,\eta(t)]:\ \tilde\beta[0,s]\notin B_{\tilde\eta(t)}(0)\big) \le 2\,\tilde{\mathbb{P}}_0\big(\tilde\beta(\eta(t))\notin B_{\tilde\eta(t)}(0)\big)\\ &\quad= O(\exp\big[-\tfrac18\tilde\eta^2(t)/\eta(t)\big]) = O(\exp[-\tfrac18 M(t)\log t]) = O(t^{-\frac18 M(t)}) = o(t^{-2}).\end{aligned} \tag{4.29}$$

Hence, with a negligible error we may restrict the expectation in the right-hand side of (4.20) to the event

$$\mathcal{E}_t = \{\tilde\beta[0,\eta(t)] \subset B_{\tilde\eta(t)}(0)\}. \tag{4.30}$$

The first half of (4.21) guarantees that $\lim_{t\to\infty}\tilde\eta(t)=0$ for some choice of $M(t)$ with $\lim_{t\to\infty}M(t)=\infty$.

3. Fix $0<\delta<\frac18$, and consider the successive excursions of β between the boundaries of the balls $B_{1/4}(0)$ and $B_\delta(0)$, i.e., put $\sigma_0=\inf\{u\ge0:\ \beta(u)\in\partial B_{1/4}(0)\}$ and, for $k\in\mathbb{N}$,

$$\begin{aligned}\bar\sigma_k &= \inf\{u\ge\sigma_{k-1}:\ \beta(u)\in\partial B_\delta(0)\},\\ \sigma_k &= \inf\{u\ge\bar\sigma_k:\ \beta(u)\in\partial B_{1/4}(0)\}.\end{aligned} \tag{4.31}$$

For $k\in\mathbb{N}$, let $\beta_k=\beta([\sigma_{k-1},\sigma_k])$ denote the k-th excursion from $\partial B_{1/4}(0)$ to $\partial B_\delta(0)$ and back. Let $\bar X_k=\beta(\bar\sigma_k)$ denote the location where this excursion first hits $\partial B_\delta(0)$. Clearly, under the law $\mathbb{P}$, $(\bar\sigma_k-\sigma_{k-1},\sigma_k-\bar\sigma_k,\bar X_k)_{k\in\mathbb{N}}$ is a uniformly ergodic Markov chain on $(0,\infty)^2\times\mathbb{T}^3$. Let

$$N_\delta(t) = \sup\{k\in\mathbb{N}:\ \sigma_k\le t\} \tag{4.32}$$

be the number of completed excursions prior to time t. By the renewal theorem, we have

$$\lim_{t\to\infty} t^{-1}\mathbb{E}(N_\delta(t)) = \frac{1}{e_\delta+e'_\delta},\qquad e_\delta=\mathbb{E}(\bar\sigma_1-\sigma_0),\quad e'_\delta=\mathbb{E}(\sigma_1-\bar\sigma_1). \tag{4.33}$$

Moreover, for every $\delta' > 0$ there exists a $C_\delta(\delta') > 0$ such that

$$\mathbb{P}\Big(t^{-1}|N_\delta(t) - (e_\delta + e'_\delta)^{-1}| \geq \delta'\Big) \leq e^{-C_\delta(\delta')t}, \qquad t \geq 0. \tag{4.34}$$

4. Fix $\tilde\beta[0,\eta(t)] \subset B_{\tilde\eta(t)}(0)$. For $s \in [0,\eta(t)]$ and $N \in \mathbb{N}$, the probability that the first N excursions do not hit $\tilde\beta[0,s]$ equals

$$\Pi\big(N;\tilde\beta[0,s]\big) = \mathbb{E}\left(\prod_{k=1}^{N} 1_{\{\tilde\beta[0,s]\cap\beta_k=\emptyset\}}\right) = \mathbb{E}\left(\mathbb{E}\left(\prod_{k=1}^{N}\big[1 - p\big(\bar X_k, \bar X_{k+1}; \tilde\beta[0,s]\big)\big] \;\middle|\; \mathcal{F}_{N+1}\right)\right), \tag{4.35}$$

where $\mathcal{F}_{N+1}$ is the sigma-algebra generated by $\bar X_k$, $1 \leq k \leq N+1$, and

$$p\big(x,y;\tilde\beta[0,s]\big) = \mathbb{P}_x^y\left(\sigma_{\tilde\beta[0,s]} < \infty\right), \qquad x,y \in \partial B_\delta(0), \tag{4.36}$$

is the probability that a Brownian motion, starting from $x \in \partial B_\delta(0)$ and conditioned to re-enter $B_\delta(0)$ at $y \in \partial B_\delta(0)$ after it has exited $B_{1/4}(0)$, hits $\tilde\beta[0,s]$. The following lemma gives a sharp estimate of $p(x,y;\tilde\beta[0,s])$.

Lemma 4.3 *If $\tilde\eta(t) \leq \frac{1}{2}\delta$ and $\tilde\beta[0,\eta(t)] \subset B_{\tilde\eta(t)}(0)$, then*

$$p\big(x,y;\tilde\beta[0,s]\big) = [1+O(\delta)]\left\{(\kappa_3\delta)^{-1}\mathrm{cap}\,(\tilde\beta[0,s]) + O(\delta^{-2})\,\tilde\eta^2(t)\right\}, \qquad \delta \downarrow 0, \tag{4.37}$$

for all $x,y \in \partial B_\delta(0)$ and $s \in [0,\eta(t)]$.

Proof. We begin by showing that if $\tilde\eta(t) \leq \frac{1}{2}\delta$, then

$$\big|\mathbb{P}_x\big(\sigma_{\tilde\beta[0,s]} < \infty\big) - (\kappa_3\delta)^{-1}\mathrm{cap}\,(\tilde\beta[0,s])\big| \leq 2\delta^{-2}\tilde\eta^2(t) \tag{4.38}$$

for all $x \in \partial B_\delta(0)$ and $\tilde\beta[0,s] \subset B_{\tilde\eta(t)}(0)$. Indeed, for any compact set $K \subset \mathbb{R}^3$, we have

$$\mathrm{cap}\,(K) = \int_K \mu_K(dy), \qquad \mathbb{P}_x(\sigma_K < \infty) = \int_K \frac{\mu_K(dy)}{\kappa_3|x-y|}, \qquad x \in K, \tag{4.39}$$

where μ_K is the equilibrium measure on K (see [23], [20], [24]). If $|x| = \delta$ and $|y| \leq \frac{1}{2}\delta$, then $||x-y|^{-1} - |x|^{-1}| \leq 2\delta^{-2}|y|$. Hence (4.39) yields the estimate $|\mathbb{P}_x(\sigma_K < \infty) - (\kappa_3\delta)^{-1}\mathrm{cap}\,(K)| \leq 2\kappa_3^{-1}\delta^{-2}\tilde\eta(t)\mathrm{cap}\,(K)$, provided $K \subset B_{\tilde\eta(t)}(0)$. In that case $\mathrm{cap}\,(K) \leq \mathrm{cap}\,(B_{\tilde\eta(t)}(0)) = \kappa_3\tilde\eta(t)$, and the claim in (4.38) follows. Furthermore, since $\mathbb{P}_a(\sigma_{B_\delta(0)} < \infty) = \kappa_3(4\delta)$ for all $a \in B_{1/4}(0)$, we have

$$0 \leq \mathbb{P}_x\big(\sigma_{\tilde\beta[0,s]} < \infty\big) - \inf_{y\in\partial B_\delta(0)} p\big(x,y;\tilde\beta[0,s]\big) \leq \kappa_3(4\delta) \sup_{y,z\in\partial B_\delta(0)} p\big(y,z;\tilde\beta[0,s]\big) \qquad \forall\, x \in \partial B_\delta(0). \tag{4.40}$$

Hence (4.38) implies (4.37). ∎

5. Recalling (4.32), we have

$$1_{\{\tilde\beta[0,s]\cap\beta[0,\sigma_0]=\emptyset\}} \prod_{k=1}^{N_\delta(t)+1} 1_{\{\tilde\beta[0,s]\cap\beta_k=\emptyset\}} \leq 1_{\{\tilde\beta[0,s]\cap\beta[0,t]=\emptyset\}} \leq \prod_{k=1}^{N_\delta(t)} 1_{\{\tilde\beta[0,s]\cap\beta_k=\emptyset\}}. \tag{4.41}$$

In terms of the probability defined in (4.35), and with the help of the large deviation estimate in (4.34), this sandwich gives us, on the event $\mathcal{E}_t$,

$$\mathbb{E}\left(\int_0^{\eta(t)} ds\, 1_{\{\tilde\beta[0,s]\cap\beta[0,t]=\emptyset\}}\right) = O\left(\eta(t)\,e^{-C_\delta(\delta')t}\right) + [1+o_t(1)]\,\Pi\Big([1+o_t(1)](e_\delta+e'_\delta)^{-1}t;\tilde\beta[0,s]\Big), \tag{4.42}$$

where the error terms $o_t(1)$ tend to zero as $t \to \infty$ (here we use that $\lim_{t\to\infty} \mathbb{P}(B_{\tilde\eta(t)}(0) \cap \beta[0,\sigma_0]) = 1$).

6. Combining the estimates in Steps 1–5, and using that $\mathrm{cap}\,(\tilde\beta[0,s])$ equals $\mathrm{cap}\,(\tilde\beta[0,1])\sqrt{s}$ in distribution under $\tilde{\mathbb{P}}_0$, we get

$$\begin{aligned}\spadesuit(t) &= o(t^{-2}) + [1+o_t(1)]\,\tilde{\mathbb{E}}_0\left(\int_0^{\eta(t)} ds\, e^{-A_\delta(t)\sqrt{s}}\right)\\ &= o(t^{-2}) + [1+o_t(1)]\,\tilde{\mathbb{E}}_0\left(\frac{2}{A_\delta(t)^2}\,\left\{1+e^{-A_\delta(t)\sqrt{\eta(t)}}\,[A_\delta(t)\sqrt{\eta(t)}-1]\right\}\right)\end{aligned} \tag{4.43}$$

with

$$t^{-1}A_\delta(t) = [1+O(\delta)]\,[1+o_t(1)]\,(e_\delta+e'_\delta)^{-1}\,(\kappa_3\delta)^{-1}\mathrm{cap}\,(\tilde\beta[0,1]), \qquad t\to\infty. \tag{4.44}$$

The term between braces in (4.43) is bounded and tends to one in $\tilde{\mathbb{P}}_0$-probability as $t \to \infty$ because of the first half of (4.21). Therefore (4.43)–(4.44) lead us, for fixed δ, to

$$\lim_{t\to\infty} t^2\spadesuit(t) = \lim_{t\to\infty} t^2\,\tilde{\mathbb{E}}_0\left(\frac{2}{A_\delta(t)^2}\right) = [1+O(\delta)]\,2(\kappa_3\delta)^2(e_\delta+e'_\delta)^2\,\tilde{\mathbb{E}}_0\left(\frac{1}{\mathrm{cap}\,(\tilde\beta[0,1])^2}\right), \tag{4.45}$$

where we use Lemma 4.1, which also implies that the expectation in the right-hand side is finite.

7. Finally, letting $\delta \downarrow 0$ and using that

$$\lim_{\delta\downarrow 0} \delta e_\delta = 1/\kappa_3, \qquad \lim_{\delta\downarrow 0} e'_\delta = \mathbb{E}_0(\tau_{B_{1/4}(0)}) < \infty, \tag{4.46}$$

we arrive at

$$\lim_{t\to\infty} t^2\spadesuit(t) = 2\,\tilde{\mathbb{E}}_0\left(\frac{1}{\mathrm{cap}\,(\tilde\beta[0,1])^2}\right). \tag{4.47}$$

This proves the claim in (1.10). ∎

5 Torsional rigidity for $m \geq 4$

The same estimates as in the proof of Theorem 1.2 for $m = 3$ in Section 4.3 can be used to prove Theorem 1.3 for $m \geq 4$ after we replace $\tilde\beta[0,s]$ by $\tilde W_{r(t)}[0,s]$. The details are explained in Sections 5.1–5.2.

5.1 Proof of Theorem 1.3 for $m \geq 5$

Proof. In the proof we assume that

$$\lim_{t\to\infty} t^{1/(m-2)} r(t) = 0, \qquad \lim_{t\to\infty} \frac{t}{\log^3 t}\, r(t)^{m-4} = \infty. \tag{5.1}$$

1-2. The estimates in Steps 1–2 are sharp enough to produce a negligible error term $o(t^{-2/(m-2)})$ when (4.21) is replaced by

$$\lim_{t\to\infty} \eta(t)\log t = 0, \qquad \lim_{t\to\infty} \frac{t\, r(t)^{m-4}\,\eta(t)}{\log^2 t} = \infty, \tag{5.2}$$

where we note that by the second half of (5.1) there exists a choice of $\eta(t)$ satisfying (5.2). Indeed, the analogues of (4.22)–(4.23) give (recall that Lemma 4.2 also holds for $m \geq 4$)

$$(\tilde{\mathbb{E}}_0 \otimes \mathbb{E})\left(\int_{\eta(t)}^{K(t)} ds\, 1_{\{\tilde W_{r(t)}[0,s]\cap\beta[0,t]=\emptyset\}}\right) \leq K(t)\,\tilde{\mathbb{E}}_0\left(e^{-c_m t\,\mathrm{cap}\,(\tilde W_{r(t)}[0,\eta(t)]\cap B_{1/4}(0))}\right), \tag{5.3}$$

where we use that $\mathrm{cap}\,(\tilde W_{r(t)}[0,s]) \geq \mathrm{cap}\,(\tilde W_{r(t)}[0,\eta(t)])$ for $s \geq \eta(t)$. The estimate in Step 2 shows that, because of the first half of (5.2), $\mathbb{P}_0(\mathcal{E}_t^c)$ with $\mathcal{E}_t$ defined in (4.30) decays faster than any negative power of t, so that we can remove the intersection with $B_{1/4}(0)$ at the expense of a negligible error term. Since $t\,\mathrm{cap}\,(\tilde W_{r(t)}[0,\eta(t)])$ equals $t\,r(t)^{m-2}\mathrm{cap}\,(\tilde W_1[0,\eta(t)/r(t)^2])$ in distribution under $\tilde{\mathbb{P}}_0$, we obtain that

$$\tilde{\mathbb{E}}_0\left(e^{-c_m t\,\mathrm{cap}\,(\tilde W_{r(t)}[0,\eta(t)])}\right) = \tilde{\mathbb{E}}_0\left(e^{-c_m\, t\,r(t)^{m-2}\,\mathrm{cap}\,(\tilde W_1[0,\eta(t)/r(t)^2])}\right). \tag{5.4}$$

Via an estimate similar as in (4.24) with c replaced by $c\eta(t)/r(t)^2$, we obtain, with the help of Lemma 7.1 below (which is the analogue of Lemma 4.1 and is proved in Section 7.1),

$$\tilde{\mathbb{E}}_0\left(e^{-c_m\, t\,r(t)^{m-2}\,\mathrm{cap}\,(\tilde W_1[0,\eta(t)/r(t)^2])}\right) \leq C\,e^{-2\sqrt{c_m c\, t\, r(t)^{m-2}\,\eta(t)/r(t)^2}} + o(t^{-2/(m-2)}). \tag{5.5}$$

Hence the right-hand side of (5.3) is $O(K(t)^{-1})$ when we pick

$$K(t) = e^{\sqrt{c_m c\, t\, r(t)^{m-2}\,\eta(t)/r(t)^2}}. \tag{5.6}$$

The second half of (5.2) ensures that $K(t)$ grows faster than any positive power of t, and so (5.3) is negligible. The contribution

$$(\tilde{\mathbb{E}}_0 \otimes \mathbb{E})\left(\int_{K(t)}^\infty ds\, 1_{\{\tilde W_{r(t)}[0,s]\cap\beta[0,t]=\emptyset\}}\right) \tag{5.7}$$

can again be estimated in a similar way by reversing the roles of β and $\tilde\beta$. This leads to a term that is even much smaller.

3-5. Step 3 is unaltered. In Step 4 the term δ^{-1} is to be replaced by $\delta^{-(m-2)}$, because in (4.39) the term $1/\kappa_3|x-y|$ is to be replaced by $1/\kappa_m|x-y|^{m-2}$. Step 5 is unaltered.

6-7. In Step 6 we use that $\mathrm{cap}\,(\tilde W_{r(t)}[0,s])$ equals $s^{(m-2)/2}\mathrm{cap}\,(\tilde W_{r(t)/\sqrt{s}}[0,1])$ in distribution under $\tilde{\mathbb{P}}_0$. This gives

$$♠_{r(t)}(t) = I_1(t) + o(t^{-2/(m-2)}), \tag{5.8}$$

where

$$I_1(t) = \tilde{\mathbb{E}}_0\left(\int_0^{\eta(t)} ds\, e^{-A_\delta(t,s)\,s^{(m-2)/2}}\right) \tag{5.9}$$

with

$$A_\delta(t,s) = [1+O(\delta)]\,[1+o_t(1)]\,(e_\delta+e'_\delta)^{-1}\,t\,\delta^{-(m-2)}\,\mathrm{cap}\,(\tilde W_{r(t)/\sqrt{s}}[0,1]). \tag{5.10}$$

With the change of variable $u = t^{1/(m-2)}\sqrt{s}$, the integral becomes

$$I_1(t) = t^{-2/(m-2)}I_2(t), \tag{5.11}$$

where

$$I_2(t) = \tilde{\mathbb{E}}_0\left(2\int_0^{t^{1/(m-2)}\sqrt{\eta(t)}} du\,u\,e^{-A'_\delta(t,u)\,u^{m-2}}\right) \tag{5.12}$$

with (recall (1.6))

$$A'_\delta(t,u) = [1+O(\delta)]\,[1+o_t(1)]\,(e_\delta+e'_\delta)^{-1}\,\delta^{-(m-2)}\,\mathrm{cap}\,(\tilde W_{\epsilon(t)/u}[0,1]), \tag{5.13}$$

where $\varepsilon(t) = t^{1/(m-2)}r(t)$. Now, (1.22) tells us that

$$\mathrm{cap}\,(\tilde W_{\epsilon(t)/u}[0,1]) = [1+o(1)]\,u^{-(m-4)}\mathrm{cap}\,(\tilde W_{\epsilon(t)}[0,1]) \quad \text{in } \mathbb{P}_0\text{-probability as } t\to\infty \tag{5.14}$$

for every $u \in (0,\infty)$ and $m \geq 5$, where we use that $\varepsilon(t) = o(1)$ by the first half of (5.1). Therefore with the help of (5.2) and dominated convergence, we find that

$$I_2(t) = [1+o(1)]\,\tilde{\mathbb{E}}_0\left(2\int_0^\infty du\,u\,e^{-A''_\delta(t)\,u^2}\right) = [1+o(1)]\,\tilde{\mathbb{E}}_0\left(\frac{1}{A''_\delta(t)}\right), \qquad t\to\infty, \tag{5.15}$$

with

$$A''_\delta(t) = [1+O(\delta)]\,[1+o_t(1)]\,(e_\delta+e'_\delta)^{-1}\,\delta^{-(m-2)}\,\mathrm{cap}\,(\tilde{W}_{\epsilon(t)}[0,1]). \tag{5.16}$$

In Step 7 the first line in (4.46) is replaced by the statement that $\lim_{\delta\downarrow 0}\delta^{m-2}e_\delta = 1/\kappa_m$. Combining (5.8), (5.11) and (5.15), and letting $\delta\downarrow 0$, we get the scaling in (1.12). ∎

5.2 Proof of Theorem 1.3 for $m = 4$

Proof. In the proof we assume that

$$\lim_{t\to\infty} t^{1/(m-2)} r(t) = 0, \qquad \lim_{t\to\infty}\frac{t}{\log^3 t}\,\frac{1}{\log(1/r(t))} = \infty. \tag{5.17}$$

1-2. The estimates in Steps 1–2 are sharp enough to produce a negligible error term $o(t^{-2/(m-2)})$ when (4.21) is replaced by

$$\lim_{t\to\infty}\eta(t)\log t = 0, \qquad \lim_{t\to\infty}\frac{t}{\log^2 t}\,\frac{\eta(t)}{\log(\eta(t)/r(t)^2)} = \infty, \tag{5.18}$$

where we note that by the second half of (5.17) there exists a choice of $\eta(t)$ satisfying (5.18). The estimate uses (4.24) with c replaced by $c(\eta(t)/r(t)^2)/\log(\eta(t)/r(t)^2)$, and also Lemma 7.1 below (which is the analogue of Lemma 4.1 and is proved in Section 7.1).

3-5. These steps are unaltered.

6-7. These steps are unaltered: (1.13)–1.14 tell us that

$$\mathrm{cap}\,(\tilde{W}_{\epsilon(t)/u}[0,1]) = [1+o(1)]\,\mathrm{cap}\,(\tilde{W}_{\epsilon(t)}[0,1]) \quad \text{in } \mathbb{P}_0\text{-probability as } t\to\infty \tag{5.19}$$

for every $u \in (0,\infty)$, where we use that $\varepsilon(t) = t^{1/(m-2)}r(t) = o(1)$ by the first half of (5.17). This is used in (5.12)–(5.13) to get (5.15)–(5.16) with $m = 4$. ∎

6 Proof of Theorem 1.4

Proof. By a direct calculation via the Fourier transform, we have that the Dirichlet heat kernel on $\mathbb{T}^m$ is given by (recall the notation in Section 2.1)

$$p_{\mathbb{T}^m}(x,y;s) = (4\pi s)^{-m/2}\sum_{\lambda\in(2\pi\mathbb{Z})^m} e^{-|x-y-\lambda|^2/4s}, \tag{6.1}$$

where $|x-y-\lambda| = d(x-y,\lambda)$. It follows that

$$p_{\mathbb{T}^m}(x,x;s) = (4\pi s)^{-m/2}\sum_{\lambda\in\mathbb{Z}^m} e^{-\pi^2|\lambda|^2/s}. \tag{6.2}$$

By translation invariance, $p_{\mathbb{T}^m}(x,x;s)$ is independent of x, and we will denote it by $\pi(s)$. By the eigenfunction expansion in (2.7) with $M = \mathbb{T}^m$ and $\Omega = \mathcal{B}(t) = \mathbb{T}^m\backslash\beta[0,t]$, and by the monotonicity of the Dirichlet heat kernel, we have for $s > 0$,

$$e^{-s\lambda_t}\varphi(x)^2 \leq p_{\mathcal{B}(t)}(x,x;s) \leq \pi(s), \tag{6.3}$$

where we abbreviate $\lambda_t = \lambda_1(\mathcal{B}(t))$ as in (3.6). Taking the supremum over x, we obtain

$$\|\varphi_1\|_{L^\infty(\mathcal{B}(t))}^{-2} \geq \pi(s)^{-1} e^{-s\lambda_t}. \tag{6.4}$$

By Lemma 2.2(b) we have, for $s > 0$,

$$\mathcal{T}(\mathcal{B}(t)) \geq \lambda_t^{-1}\, \pi(s)^{-1} e^{-s\lambda_t}. \tag{6.5}$$

Since $q \mapsto q^{-1} e^{-sq}$ is convex for every $s > 0$, Jensen gives that

$$\spadesuit(t) \geq \pi(s)^{-1} \mathbb{E}_0(\lambda_t)^{-1} e^{-s\mathbb{E}_0(\lambda_t)}. \tag{6.6}$$

For $s = 1$ this reads

$$\mathbb{E}_0(\lambda_t)\, e^{\mathbb{E}_0(\lambda_t)} \geq \pi(1)^{-1} \spadesuit(t)^{-1}. \tag{6.7}$$

Since the right-hand side of (6.7) increases to infinity as $t \to \infty$, there exists $t_0 < \infty$ such that $\mathbb{E}_0(\lambda_t) \geq 1$ for $t \geq t_0$. We now put

$$s_t = \mathbb{E}_0(\lambda_t)^{-1} \tag{6.8}$$

and note that $s_t \leq 1$ for $t \geq t_0$. By (6.2) and (6.6), we find that, for $t \geq t_0$,

$$\begin{aligned} \spadesuit(t) &\geq e^{-1}\pi(s_t)^{-1} s_t = (4\pi)^{m/2} e^{-1}\, s_t^{(2+m)/2} \left(\sum_{\lambda \in \mathbb{Z}} e^{-\pi^2|\lambda|^2/s_t} \right)^{-m} \\ &\geq (4\pi)^{m/2} e^{-1}\, s_t^{(2+m)/2} \left(\sum_{\lambda \in \mathbb{Z}} e^{-\pi^2|\lambda|^2} \right)^{-2} \geq s_t^{(2+m)/2}. \end{aligned} \tag{6.9}$$

We conclude that, for $t \geq t_0$,

$$\mathbb{E}_0(\lambda_t) \geq \spadesuit(t)^{-2/(m+2)}. \tag{6.10}$$

∎

7 Capacity of Wiener sausage for $m \geq 4$

In Section 7.1 we derive the analogue of Lemma 4.1, showing that the inverse of $\mathcal{C}(t)$ for $m \geq 4$ defined in (1.13) has a finite exponential moment uniformly in $t \geq 2$. In Section 7.2 we prove (1.13)–(1.14) for $m \geq 5$.

7.1 Exponential moment of the inverse capacity

Lemma 7.1 *Let $m \geq 4$. Then there exists a $c > 0$ such that*

$$\sup_{t \geq 2} \mathbb{E}_0\left(\exp\left[\frac{c}{\mathcal{C}(t)} \right] \right) < \infty. \tag{7.1}$$

Proof. The proof is similar to that of Lemma 4.1. For any compact set $A \subset \mathbb{R}^m$, we use the representation (compare with (4.2))

$$\frac{1}{\mathrm{cap}\,(A)} = \inf \left[\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\mu(dx)\mu(dy)}{\kappa_m\, |x-y|^{m-2}} \colon\ \mu \text{ is a probability measure on } A \right]. \tag{7.2}$$

As test probability measure we choose the sojourn measure of $W_1[0,t]$, namely,

$$\mu_{W_1[0,t]} = \frac{1}{t} \int_0^t \nu_{\beta(s)}\, ds \quad \text{with} \quad \nu_z(dx) = \frac{1}{\omega_m} 1_{B_1(z)}(x)\, dx, \quad z \in \mathbb{R}^m, \tag{7.3}$$

where $\omega_m = |B_1(0)|$. Since μ has support in $W_1[0,t]$, we have

$$\frac{1}{\mathrm{cap}\,(W_1[0,t])} \leq \frac{1}{\kappa_m \omega_m^2 t^2} \int_0^t du \int_0^t dv \int_{B_1(0)} dx \int_{B_1(0)} dy \; \frac{1}{|\beta(u)+x-\beta(v)-y|^{m-2}}. \tag{7.4}$$

Moreover, there exists a $C = C(m) > 0$ such that for all u and v,

$$\int_{B_1(0)} dx \int_{B_1(0)} dy \; \frac{1}{|\beta(u)+x-\beta(v)-y|^{m-2}} \leq \frac{C}{|\beta(u)-\beta(v)|^{m-2} \vee 1}. \tag{7.5}$$

7.1.1 $m \geq 5$

Abbreviate $\bar{c} = c\,C/\kappa_m \omega_m^2$, and estimate

$$\begin{aligned} \exp\left[\frac{c}{\mathcal{C}(t)}\right] &\leq \exp\left[\frac{\bar{c}}{t}\int_0^t du \int_0^t dv \; \frac{1}{|\beta(u)-\beta(v)|^{m-2} \vee 1}\right] \\ &\leq \frac{1}{t}\int_0^t du \; \exp\left[\bar{c}\int_0^t dv \; \frac{1}{|\beta(u)-\beta(v)|^{m-2} \vee 1}\right] \\ &\leq \frac{1}{t}\int_0^t du \; \exp\left[\bar{c}\int_{\mathbb{R}} dv \; \frac{1}{|\beta(u)-\beta(v)|^{m-2} \vee 1}\right]. \end{aligned} \tag{7.6}$$

Taking the expectation and using the translation invariance of Brownian motion, we obtain the t-independent bound

$$\begin{aligned} \mathbb{E}_0\left(\exp\left[\frac{c}{\mathcal{C}(t)}\right]\right) &\leq \mathbb{E}_0\left(\exp\left[\bar{c}\int_{\mathbb{R}} dv \; \frac{1}{|\beta(v)|^{m-2} \vee 1}\right]\right) \\ &\leq \mathbb{E}_0\left(\exp\left[2\bar{c}\int_0^\infty dv \; \frac{1}{|\beta(v)|^{m-2} \vee 1}\right]\right), \end{aligned} \tag{7.7}$$

and so it remains to show that the right-hand side is finite for c small enough. Arguing in the same way as in the proof of Lemma 4.1, we obtain

$$\begin{aligned} &\mathbb{E}_0\left(\exp\left[2\bar{c}\int_0^\infty dv \; \frac{1}{|\beta(v)|^{m-2} \vee 1}\right]\right) \\ &\leq \sum_{k\in\mathbb{N}_0} (2\bar{c})^k \, \mathbb{E}_0\left(\int_{0\leq v_1<\cdots<v_k<\infty} \prod_{i=1}^k \frac{dv_i}{|\beta(v_i)|^{m-2} \vee 1}\right) \\ &\leq \sum_{k\in\mathbb{N}_0} (2\bar{c})^k \left[\int_0^\infty dv \; \mathbb{E}_0\left(\frac{1}{|\beta(v)|^{m-2} \vee 1}\right)\right]^k. \end{aligned} \tag{7.8}$$

Therefore it remains to prove the finiteness of the integral. That that end, we estimate

$$\begin{aligned} \int_0^\infty dv \; \mathbb{E}_0\left(\frac{1}{|\beta(v)|^{m-2} \vee 1}\right) &\leq 1 + \int_1^\infty dv \; \mathbb{E}_0\left(|\beta(v)|^{-(m-2)} \wedge 1\right) \\ &\leq 1 + \int_1^\infty dv \; \mathbb{E}_0\left(|\beta(v)|^{-(m-2)}\right) \\ &= 1 + \mathbb{E}_0\left(|\beta(1)|^{-(m-2)}\right) \int_1^\infty dv \, v^{-(m-2)/2} < \infty, \end{aligned} \tag{7.9}$$

where the last inequality holds because $m \geq 5$.

7.1.2 $m = 4$

Abbreviate $\bar{c} = c\,C/\kappa_4\omega_4^2$, and replace (7.6) by

$$\exp\left[\frac{c}{\mathcal{C}(t)}\right] \leq \frac{1}{t}\int_0^t du\, \exp\left[\frac{\bar{c}}{\log t}\int_{u-t}^{u+t} dv\, \frac{1}{|\beta(u)-\beta(v)|^2 \vee 1}\right] \tag{7.10}$$

and (7.7) by

$$\mathbb{E}_0\left(\exp\left[\frac{c}{\mathcal{C}(t)}\right]\right) \leq \mathbb{E}_0\left(\exp\left[\frac{2\bar{c}}{\log t}\int_0^t dv\, \frac{1}{|\beta(v)|^2 \vee 1}\right]\right) \tag{7.11}$$

and (7.8) by

$$\mathbb{E}_0\left(\exp\left[\frac{2\bar{c}}{\log t}\int_0^t dv\, \frac{1}{|\beta(v)|^2 \vee 1}\right]\right) \leq \sum_{k\in\mathbb{N}_0}\left(\frac{2\bar{c}}{\log t}\right)^k \left[\int_0^t dv\, \mathbb{E}_0\left(\frac{1}{|\beta(v)|^2 \vee 1}\right)\right]^k \tag{7.12}$$

and (7.9) by

$$\int_0^t dv\, \mathbb{E}_0\left(\frac{1}{|\beta(v)|^2 \vee 1}\right) = 1 + \mathbb{E}_0\left(|\beta(1)|^{-2}\right)\int_1^t dv\, v^{-1} \leq c' \log t, \tag{7.13}$$

for some $c' \in (0,\infty)$. ∎

7.2 Scaling of the capacity

The proof for $m \geq 5$ uses subadditivity. The proof for $m = 4$ is much more complicated and is addressed in a future paper.

Note that capacity is subadditive: $\text{cap}\,(W_1[0,s+t]) \leq \text{cap}\,(W_1[0,s]) + \text{cap}\,(W_1[s,s+t])$ for all $s,t \geq 0$. Hence, Kingman's subadditive ergodic theorem yields that

$$\lim_{t\to\infty} t^{-1}\text{cap}\,(W_1[0,t]) = \bar{c}_m \quad \beta - a.s. \tag{7.14}$$

for some $\bar{c}_m \geq 0$. We therefore get the claim with $c_m = \bar{c}_m$, provided we show that $\bar{c}_m > 0$.

In view of (7.2), we can get a lower bound on capacity by choosing a test probability measure. We again choose the sojourn measure of $W_1[0,t]$ in (7.3). This gives

$$\frac{t}{\text{cap}\,(W_1[0,t])} \leq t\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{\mu_{W_1[0,t]}(dx)\mu_{W_1[0,t]}(dy)}{\kappa_m|x-y|^{m-2}} = \frac{1}{t}\int_0^t du\int_0^t dv\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\nu_{\beta(u)}(dx)\nu_{\beta(v)}(dy)}{\kappa_m|x-y|^{m-2}}. \tag{7.15}$$

Now, there exists a $C < \infty$ such that

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\nu_a(dx)\nu_b(dy)}{\kappa_m|x-y|^{m-2}} \leq \frac{C}{|a-b|^{m-2}\vee 1} \qquad \forall\, a,b \in \mathbb{R}^m. \tag{7.16}$$

Hence

$$\frac{t}{\text{cap}\,(W_1[0,t])} \leq \frac{1}{t}\int_0^t du\int_0^t dv\, \frac{C}{|\beta(u)-\beta(v)|^{m-2}\vee 1}. \tag{7.17}$$

To prove that $\bar{c}_m > 0$ it suffices to show that the right-hand side has a finite expectation. To that end, we estimate

$$\frac{1}{t}\int_0^t du\int_0^t dv\, \mathbb{E}_0\left(\frac{1}{|\beta(u)-\beta(v)|^{m-2}\vee 1}\right) \leq 2\int_0^t dv\, \mathbb{E}_0\left(\frac{1}{|\beta(v)|^{m-2}\vee 1}\right) \tag{7.18}$$

and note that, as shown in (7.9), the integral converges as $t \to \infty$ when $m \geq 5$.

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