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Abstract

We consider gated polling systems with two special features: (i) retrials, and (ii) glue or reservation periods. When a type-i customer arrives, or retries, during a glue period of station i, it will be served in the next visit period of the server to that station. Customers arriving at station i in any other period join the orbit of that station and retry after an exponentially distributed time. Such polling systems can be used to study the performance of certain switches in optical communication systems.

For the case of exponentially distributed glue periods, we present an algorithm to obtain the moments of the number of customers in each station. For generally distributed glue periods, we consider the distribution of the total workload in the system, using it to derive a pseudo conservation law which in its turn is used to obtain accurate approximations of the individual mean waiting times. We also consider the problem of choosing the lengths of the glue periods, under a constraint on the total glue period per cycle, so as to minimize a weighted sum of the mean waiting times.

Keywords: Polling system, Retrials, Glue periods

1 Introduction

This paper is devoted to the performance analysis of a class of single server queueing systems with multiple customer types. Our motivation is twofold: (i) to obtain insight into the performance of certain switches in optical communication systems, and (ii) to obtain insight into the effect of having particular reservation periods, windows of opportunity during which a customer can make a reservation for service. Our class of queueing systems combines several features, viz., polling, retrials, and the new feature of so-called glue periods or reservation periods. These will first be discussed separately, while their relation to optical switching will also be outlined.

Polling systems are queueing models in which a single server, alternatingly, visits a finite number of, say, N queues (or stations) in some prescribed order. Polling systems have been extensively studied in the literature. For example, various different service disciplines (rules which describe the server's behaviour while visiting a queue) have been considered, both for models with and without switchover times between queues. We refer to Takagi [18, 19] and Vishnevskii and Semenova [20] for literature reviews and to Boon, van der Mei and Winands [4], Levy and Sidi [13] and Takagi [17] for overviews of the applicability of polling systems.

Switches in communication systems form an important application area of polling systems. Here, packets must be routed from source to destination, passing through a series of links and nodes. In copper-based transmission links, packets from various sources are time-multiplexed, and this may be modelled by a polling system. In recent years optical networking has become very important, because optical fibers offer major advantages with respect to copper cables: huge bandwidth, ultra-low losses and an extra dimension, viz., a choice of wavelengths.

When one wants to model the performance of an optical switch by a polling system [14, 16], one is faced with the following difficulty. Buffering of optical packets is not easy, as photons can not wait. Whenever there is a need to buffer photons, they are sent into a local fiber loop, thus providing a small delay to the photons without losing or displacing them. If, at the completion of the loop, a photon still needs to be buffered, it is again sent into the fiber delay loop, etc. From a queueing theoretic perspective, this raises the need to add the feature of *retrial queue* to a polling system: instead of having a queueing system with one server and N ordinary queues, it has one server and N retrial queues. Retrial queues have received much attention in the literature, see, e.g., the books by Falin and Templeton [9] and by Artalejo and Gomez-Corral [3], but they have hardly been studied in the setting of polling models. Langaris [10, 11, 12] has pioneered the study of polling models with retrial queues. However, for our purpose - the performance analysis of optical switches - his assumptions about the service discipline of the server at the various queues are not suitable.

A third important feature in the present paper is that of so-called glue or reservation periods. Just before the server arrives at a station there is some glue period. Customers (both new arrivals and retrying customers) arriving at the station during this glue period "stick" and will be served during the visit of the server. Customers arriving at the station in any other period join the orbit of that station immediately and will retry after an exponentially distributed time. One motivation for studying glue periods is the following. A sophisticated technology that one might try to add to the use of fiber delay loops in optical networking is *varying the speed of light by changing the refractive index of the fiber loop*, cf. [15]. Using a higher refractive index in a small part of the loop one can achieve 'slow light', which implies slowing down the packets. This feature is in our model incorporated as glue periods, where we slow down the packets arriving at the end of the fiber loop just before the server arrives, so that they do not have to retry but get served during the subsequent visit period. Not restricting ourselves to optical networks, one can also interpret a glue period as a reservation period, i.e., a period in which customers can make a reservation at a station for service in the subsequent visit period of that station. In our model, the reservation period immediately precedes the visit period, and could be seen as the last part of a switchover period.

A first attempt to study a queueing model which combines retrials and glue periods is [7], which mainly focuses on the case of a single server and a single station, but also outlines how that analysis can be extended to the case of two stations. In [1] an N-station polling model with retrials and with constant glue periods is considered, for the case of gated service discipline at all stations. The gated discipline is an important discipline in polling systems; it implicates that the server, when visiting a station, serves exactly those customers which were present upon his arrival. The steady-state joint station size (i.e., the number of customers in each station) distribution was derived in [1], both at an arbitrary epoch and at beginnings of switchover, glue and visit periods. In the current paper we present an algorithm to obtain the moments of the station size for the case of exponentially distributed glue periods. Using Little's law, that also gives mean sojourn times. Thereafter for each individual station we allow generally distributed glue periods and we focus our attention on other performance measures next to station sizes. In particular, we consider the steady-state distribution of the total workload in the system, which leads us to a pseudo conservation law, i.e., an exact expression for a weighted sum of the mean waiting times at all stations. We use that pseudo conservation law to derive an accurate approximation for the individual mean waiting times. We further consider the problem of choosing the lengths of the glue periods, given the total glue period in a cycle, so as to minimize a weighted sum of the mean waiting times.

The rest of the paper is organized as follows. In Section 2 we present a model description. Section 3 contains a detailed analysis for generating functions and moments of station sizes at different time epochs when the glue periods are exponentially distributed. We also present a numerical example, which in particular provides insight into the behavior of the polling system in the case of long glue periods. In Section 4 we derive a pseudo conservation law for a system with generally distributed glue periods. Subsequently we use this pseudo conservation law for deriving an approximation for the mean waiting times at all stations. In its turn, this approximation is used to minimize weighted sums of the mean waiting times by optimally choosing the lengths of the glue periods, given the total glue period per cycle. Finally, Section 5 lists some topics for further research.

2 The model

We consider a single server cyclic polling system with retrials and so-called glue periods. This model was first introduced in [7] for a single station vacation model and a two-station model with switchover times. Further in [1] this was extended to an N-station model with switchover times. In both papers, the model was studied for deterministic glue periods. We index the stations by i, $i = 1, \dots, N$, in the order of server movement. For ease of presentation, all references to station indices greater than N or less than 1 are implicitly assumed to be modulo N. Customers arrive at station i according to a Poisson process with rate λ_i; they are called type-i customers, $i = 1, \cdots, N$. The overall arrival rate is denoted by $\lambda = \lambda_1 + \cdots + \lambda_N$. The service times at station i are independent and identically distributed (i.i.d.) random variables with a generic random variable B_i, $i = 1, \cdots, N$. Let $\tilde{B}_i(s) = \mathbb{E}[e^{-sB_i}]$ be the Laplace Stieltjes transform (LST) of the service time distribution at station i. The switchover times from station i to station $i+1$ are i.i.d. random variables with a generic random variable S_i. Let $\tilde{S}_i(s) = \mathbb{E}[e^{-sS_i}]$ be the LST of the switchover time from station i to station $i+1$, $i = 1, \cdots, N$. The interarrival times, the service times, and the switchover times are assumed to be mutually independent. After a switch of the server to station i, there is a glue period for collecting retrying customers (which will be followed by the visit period of the server to station i). We assume that the successive glue periods at station i are i.i.d. random variables with a generic random variable G_i. Let $\tilde{G}_i(s) = \mathbb{E}[e^{-sG_i}]$ be the LST of the glue period distribution at station i.

Each station consists of an orbit and a queue. When customers (both new arrivals and retrying customers) arrive at station i during a glue period, they stick and wait in the queue to get served during the visit of the server to that station. When customers arrive at station i in any other period, they join the orbit of station i and will retry after a random amount of time. The inter-retrial time of each customer in the orbit of station i is exponentially distributed with mean ν_i^{-1} and is independent of all other processes.

A single server cyclically moves from one station to another serving the glued customers at each of the stations. The service discipline at all stations is gated. During the service period of station i, the server serves all glued customers in the queue of station i, i.e., all type-i customers waiting at the end of the glue period (but none of those in orbit, and neither any new arrivals).

Let $(X_1^{(i)}, X_2^{(i)}, \cdots, X_N^{(i)})$ denote the vector of numbers of customers of type 1 to type N in the system (hence in the orbit) at the start of a glue period of station i, $i = 1, \cdots, N$, in steady state. Further, let $(Y_1^{(i)}, Y_2^{(i)}, \cdots, Y_N^{(i)})$ denote the vector of numbers of customers of type 1 to type N in the system at the start of a visit period at station i, $i = 1, \cdots, N$, in steady state. We distinguish between those who are queueing (glued) and those who are in the orbit of station i: We write $Y_i^{(i)} = Y_i^{(iq)} + Y_i^{(io)}$, $i = 1, \cdots, N$, where q denotes in the queue and o denotes in the orbit. Finally, let $(Z_1^{(i)}, Z_2^{(i)}, \cdots, Z_N^{(i)})$ denote the vector of numbers of customers of type 1 to type N in the system (hence in the orbit) at the start of a switchover from station i to station $i+1$, $i = 1, \cdots, N$, in steady state.

The utilization of the server at station i, ρ_i, is defined by $\rho_i = \lambda_i \mathbb{E}[B_i]$ and the total utilization of the server ρ is given by $\rho = \sum_{i=1}^{N} \rho_i$. It can be shown that a necessary and sufficient condition for stability of this polling system is $\rho < 1$. We hence assume that $\rho < 1$.

The cycle length of station i, $i = 1, \cdots, N$ is defined as the time between two successive arrivals of the server at this station. The mean cycle length, $\mathbb{E}[C]$, is independent of the station involved (and the service discipline) and is given by

$$\mathbb{E}[C] = \frac{\sum_{i=1}^{N} (\mathbb{E}[G_i] + \mathbb{E}[S_i])}{1 - \rho}, \tag{2.1}$$

which can be derived as follows: Since the probability of the server being idle (in steady state) is $1 - \rho$, and this equals $\frac{\sum_{i=1}^{N} (\mathbb{E}[G_i] + \mathbb{E}[S_i])}{\mathbb{E}[C]}$ by the theory of regenerative processes, we have (2.1).

3 The polling system with retrials and exponential glue periods

In [1] the authors calculated the generating functions and the mean values of the number of customers at different time epochs when the glue periods are deterministic. In this section we assume that the glue periods are *exponentially* distributed with mean $\mathbb{E}[G_i] = 1/\gamma_i$, $i = 1, \cdots, N$. We will derive a set of partial differential equations for the joint generating function of the station size (i.e., the number of customers in each station) and then obtain a system of linear equations for the first and the second moments of the station size. We also provide an iterative algorithm for solving the system of linear equations.

Observe that the generating function for the vector of numbers of arrivals at station 1 to station N during the service time of a type-i customer, B_i, is $\beta_i(\mathbf{z}) := \tilde{B}_i(\sum_{j=1}^{N} \lambda_j (1 - z_j))$ for $\mathbf{z} = (z_1, z_2, \cdots, z_N)$. Similarly, the generating function for the vector of numbers of arrivals at station 1 to station N during a switchover time from station i to station $i+1$, S_i, is $\sigma_i(\mathbf{z}) := \tilde{S}_i(\sum_{j=1}^{N} \lambda_j (1 - z_j))$.

3.1 Station size analysis at embedded time points

In this subsection we study the steady-state joint distribution and the mean of the numbers of customers in the system at the start of a glue period, visit period and switchover period. Let us define the following joint generating functions of the number of customers in each station at the start of a glue period, visit period and switchover period:

$$\begin{aligned}\tilde{R}_g^{(i)}(\mathbf{z}) &= \mathbb{E}[z_1^{X_1^{(i)}} z_2^{X_2^{(i)}} \cdots z_N^{X_N^{(i)}}],\\ \tilde{R}_v^{(i)}(\mathbf{z}, w) &= \mathbb{E}[z_1^{Y_1^{(i)}} z_2^{Y_2^{(i)}} \cdots z_i^{Y_i^{(io)}} \cdots z_N^{Y_N^{(i)}} w^{Y_i^{(iq)}}],\\ \tilde{R}_s^{(i)}(\mathbf{z}) &= \mathbb{E}[z_1^{Z_1^{(i)}} z_2^{Z_2^{(i)}} \cdots z_N^{Z_N^{(i)}}],\end{aligned}$$

for $\mathbf{z} = (z_1, z_2, \cdots, z_N)$ with $|z_i| \leq 1$, $i = 1, \ldots, N$, and $|w| \leq 1$.

Let $M_i^o(t)$ represent the number of customers in the orbit of station i, $i = 1, \cdots, N$ and $\Upsilon(t)$ the number of glued customers, at time t. Further, let τ_j be the time at which an arbitrary glue period starts at station j, $j = 1, \cdots, N$. Note that $M_i^o(\tau_j) = X_i^{(j)}$. We define

$$\begin{aligned}\phi_i(\mathbf{z}; w; t) &= \mathbb{E}[z_1^{M_1^o(\tau_i+t)} \cdots z_N^{M_N^o(\tau_i+t)} w^{\Upsilon(\tau_i+t)} \mathbb{1}_{\{G_i>t\}}],\\ \phi_i(\mathbf{z}, w) &= \int_0^\infty \phi_i(\boldsymbol{z}; w; t) dt.\end{aligned}$$

Then all the generating functions for the numbers of customers in steady state described above, can be expressed in terms of $\phi_i(\mathbf{z}, w)$, as shown below in Proposition 1.

Proposition 1. *The generating functions $\tilde{R}_v^{(i)}(\mathbf{z}, w), \tilde{R}_s^{(i)}(\mathbf{z})$ and $\tilde{R}_g^{(i)}(\mathbf{z})$ satisfy the following:*

$$\tilde{R}_v^{(i)}(\mathbf{z}, w) = \gamma_i \phi_i(\mathbf{z}, w), \tag{3.1}$$

$$\tilde{R}_s^{(i)}(\mathbf{z}) = \gamma_i \phi_i(\mathbf{z}, \beta_i(\mathbf{z})), \tag{3.2}$$

$$\tilde{R}_g^{(i)}(\mathbf{z}) = \gamma_{i-1} \sigma_{i-1}(\mathbf{z}) \phi_{i-1}(\mathbf{z}, \beta_{i-1}(\mathbf{z})). \tag{3.3}$$

Proof. Equation (3.1) is obtained as follows: By the law of total expectation,

$$\begin{aligned}
\tilde{R}_v^{(i)}(\mathbf{z}, w) &= \int_0^\infty \mathbb{E}[z_1^{M_1^o(\tau_i+t)} \dots z_N^{M_N^o(\tau_i+t)} w^{\Upsilon(\tau_i+t)} \mid G_i > t] \gamma_i e^{-\gamma_i t} dt \\
&= \int_0^\infty \mathbb{E}[z_1^{M_1^o(\tau_i+t)} \dots z_N^{M_N^o(\tau_i+t)} w^{\Upsilon(\tau_i+t)} \mathbb{1}_{\{G_i > t\}}] \gamma_i dt \\
&= \gamma_i \phi_i(\mathbf{z}, w).
\end{aligned}$$

To obtain (3.2), observe that the customers at the end of a visit period are the customers in the orbit at the beginning of that visit plus the customers who arrive during the service times of the glued customers at the beginning of that visit. Hence

$$\begin{aligned}
\tilde{R}_s^{(i)}(\mathbf{z}) &= \mathbb{E}[z_1^{Y_1^{(i)}} z_2^{Y_2^{(i)}} \cdots z_i^{Y_i^{(io)}} \cdots z_N^{Y_N^{(i)}} [\beta_i(z_1, z_2, \dots, z_N)]^{Y_i^{(iq)}}] \\
&= \tilde{R}_v^{(i)}(\mathbf{z}, \beta_i(\mathbf{z})) \\
&= \gamma_i \phi_i(\mathbf{z}, \beta_i(\mathbf{z})).
\end{aligned}$$

Also, to obtain (3.3), observe that the customers at the end of a switchover from station $i-1$ to station i are the customers in the orbit at the beginning of that switchover plus the customers who arrived during that switchover period. Hence

$$\tilde{R}_g^{(i)}(\mathbf{z}) = \tilde{R}_s^{(i-1)}(\mathbf{z}) \sigma_{i-1}(\mathbf{z}),$$

from which and (3.2) we get (3.3). □

We have the following result for the generating functions $\phi_i(\mathbf{z}, w)$.

Theorem 1. *The generating functions $\phi_i(\mathbf{z}, w)$, $i = 1, \dots, N$, satisfy the following equation:*

$$\begin{aligned}
&\nu_i(w - z_i) \frac{\partial}{\partial z_i} \phi_i(\mathbf{z}, w) - \Big(\sum_{j=1, j \neq i}^N (\lambda_j(1 - z_j)) + \lambda_i(1 - w) + \gamma_i \Big) \phi_i(\mathbf{z}, w) \\
&\quad + \gamma_{i-1} \phi_{i-1}(\mathbf{z}, \beta_{i-1}(\mathbf{z})) \sigma_{i-1}(\mathbf{z}) = 0. \qquad (3.4)
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\phi_i(\mathbf{z}; w; t+\Delta t) \\
=\ & \mathbb{E}\big[z_1^{M_1^o(\tau_i+t+\Delta t)} \dots z_N^{M_N^o(\tau_i+t+\Delta t)} w^{\Upsilon(\tau_i+t+\Delta t)} \mathbb{1}_{\{G_i > t+\Delta t\}}\big] \\
=\ & \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{P}(M_1^o(\tau_i+t) = n_1, \dots, M_N^o(\tau_i+t) = n_N, \Upsilon(\tau_i+t) = k, G_i > t) \\
& \quad \times \mathbb{E}\big[z_1^{M_1^o(\tau_i+t+\Delta t)} \dots z_N^{M_N^o(\tau_i+t+\Delta t)} w^{\Upsilon(\tau_i+t+\Delta t)} \mathbb{1}_{\{G_i > t+\Delta t\}} \big| M_1^o(\tau_i+t) = n_1, \dots, \\
& \qquad M_N^o(\tau_i+t) = n_N, \Upsilon(\tau_i+t) = k, G_i > t\big] \\
=\ & \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{P}(M_1^o(\tau_i+t) = n_1, \dots, M_N^o(\tau_i+t) = n_N, \Upsilon(\tau_i+t) = k, G_i > t) \\
& \quad \times z_1^{n_1} \dots z_{i-1}^{n_{i-1}} z_{i+1}^{n_{i+1}} \dots z_N^{n_N} w^k ((1-e^{-\nu_i \Delta t})w + e^{-\nu_i \Delta t} z_i)^{n_i} e^{-(\sum_{j=1, j\neq i}^N (\lambda_j(1-z_j)) + \lambda_i(1-w))\Delta t} e^{-\gamma_i \Delta t} \\
=\ & e^{-\left(\sum_{j=1, j\neq i}^N (\lambda_j(1-z_j)) + \lambda_i(1-w) + \gamma_i\right)\Delta t} \phi_i(z_1, \cdots, z_{i-1}, z_i + (1-e^{-\nu_i \Delta t})(w - z_i), z_{i+1}, \cdots, z_N; w; t).
\end{aligned}$$

Thus, we have

$$\frac{\partial}{\partial t}\phi_i(\mathbf{z}; w; t) = \nu_i(w - z_i)\frac{\partial}{\partial z_i}\phi_i(\mathbf{z}; w; t) - \Big(\sum_{j=1, j\neq i}^N (\lambda_j(1-z_j)) + \lambda_i(1-w) + \gamma_i\Big)\phi_i(\mathbf{z}; w; t).$$

Since $\phi_i(\mathbf{z}; w; 0) = \mathbb{E}[z_1^{X_1^{(i)}} z_2^{X_2^{(i)}} \cdots z_N^{X_N^{(i)}}] = \tilde{R}_g^{(i)}(\mathbf{z}) = \gamma_{i-1}\sigma_{i-1}(\mathbf{z})\phi_{i-1}(\mathbf{z}, \beta_{i-1}(\mathbf{z}))$ and $\phi_i(\mathbf{z}; w; \infty) = 0$, integrating the above equation with respect to t from 0 to ∞ yields

$$\begin{aligned}
-\gamma_{i-1}\sigma_{i-1}(\mathbf{z})\phi_{i-1}(\mathbf{z}, \beta_{i-1}(\mathbf{z})) =& \nu_i(w - z_i)\frac{\partial}{\partial z_i}\phi_i(\mathbf{z}, w) \\
& - \Big(\sum_{j=1, j\neq i}^N \lambda_j(1-z_j) + \lambda_i(1-w) + \gamma_i\Big)\phi_i(\mathbf{z}, w).
\end{aligned}$$

This completes the proof. □

We now calculate the mean value of the station sizes at embedded time points using the differential equation (3.4). For an N-tuple $\boldsymbol{l} = (l_1, \dots, l_N)$ of nonnegative integers, we define

$$|\boldsymbol{l}| = l_1 + \cdots + l_N, \quad \boldsymbol{l}! = l_1! l_2! \cdots l_N!,$$

and $\mathbf{z}^{\boldsymbol{l}} = z_1^{l_1} z_2^{l_2} \cdots z_N^{l_N}$. With this notation, we define the following scaled moment:

$$\Phi_i^{(\boldsymbol{l},m)} = \frac{1}{\boldsymbol{l}! m!} \frac{\partial^{|\boldsymbol{l}|+m}}{\partial \mathbf{z}^{\boldsymbol{l}} \partial w^m} \phi_i(\mathbf{z}, w)\Big|_{\mathbf{z}=\mathbf{1}-, w=1-},$$

where $\partial \mathbf{z}^{\boldsymbol{l}} = \partial z_1^{l_1} \cdots \partial z_N^{l_N}$, and $\mathbf{1}$ is the N-dimensional row vector with all its components equal to one. The first scaled moments of $\phi_i(\mathbf{z}, w)$, $i = 1, 2, \cdots, N$, can be obtained from the following theorem.

Theorem 2. *We have*

(i) $\Phi_i^{(\mathbf{0},0)} = \frac{1}{\gamma_i}$, $i = 1, \cdots, N$.

(ii) $\Phi_i^{(\mathbf{1}_j,0)}$ *and* $\Phi_i^{(\mathbf{0},1)}$, $0 \le i, j \le N$, *are given by the following recursion: for* $j = 1, \cdots, N$,

$$\Phi_j^{(\mathbf{0},1)} = \frac{\lambda_j}{\gamma_j}\mathbb{E}[C], \tag{3.5}$$

$$\Phi_j^{(\mathbf{1}_j,0)} = \frac{\lambda_j}{\nu_j}\Big(\mathbb{E}[C] - \frac{1}{\gamma_j}\Big), \tag{3.6}$$

$$\Phi_i^{(\mathbf{1}_j,0)} = \frac{\gamma_{i+1}}{\gamma_i}\Phi_{i+1}^{(\mathbf{1}_j,0)} + \frac{\lambda_j}{\gamma_i}\Big((\delta_{i,j-1} - \rho_i)\mathbb{E}[C] - \frac{1}{\gamma_{i+1}} - \mathbb{E}[S_i]\Big), \quad i = j-1, j-2, \cdots, j-N+1, \tag{3.7}$$

where $\mathbf{0}$ *is the* N*-dimensional row vector with all its elements equal to zero,* $\mathbf{1}_j$ *is the* N*-dimensional row vector whose* j*th element is one and all other elements are zero, and* δ_{ij} *is the Kronecker delta. Note that if* i *is nonpositive in (3.7), then it is interpreted as* $i + N$.

Proof. Taking the partial derivative of Equation (3.4) with respect to z_j and putting $\mathbf{z} = \mathbf{1}-, w = 1-$, we have

$$-\nu_i\delta_{ij}\Phi_i^{(\mathbf{1}_i,0)} + \frac{(1-\delta_{ij})\lambda_j}{\gamma_i} - \gamma_i\Phi_i^{(\mathbf{1}_j,0)} + \gamma_{i-1}\Phi_{i-1}^{(\mathbf{1}_j,0)} + \gamma_{i-1}\lambda_j\mathbb{E}[B_{i-1}]\Phi_{i-1}^{(\mathbf{0},1)} + \lambda_j\mathbb{E}[S_{i-1}] = 0. \tag{3.8}$$

Taking the partial derivative of Equation (3.4) with respect to w and putting $\mathbf{z} = \mathbf{1}-, w = 1-$ yields

$$\nu_i\Phi_i^{(\mathbf{1}_i,0)} + \frac{\lambda_i}{\gamma_i} - \gamma_i\Phi_i^{(\mathbf{0},1)} = 0. \tag{3.9}$$

Summing (3.8) over $i = 1, \cdots, N$, we have

$$-\nu_j\Phi_j^{(\mathbf{1}_j,0)} + \lambda_j\sum_{i\neq j}\frac{1}{\gamma_i} + \lambda_j\sum_{i=1}^{N}\gamma_i\mathbb{E}[B_i]\Phi_i^{(\mathbf{0},1)} + \lambda_j\sum_{i=1}^{N}\mathbb{E}[S_i] = 0. \tag{3.10}$$

Adding (3.9) and (3.10) and multiplying the resulting equation by $\mathbb{E}[B_j]$ yields

$$\rho_j\sum_{i=1}^{N}\Big(\frac{1}{\gamma_i} + \mathbb{E}[S_i]\Big) - \gamma_j\mathbb{E}[B_j]\Phi_j^{(\mathbf{0},1)} + \rho_j\sum_{i=1}^{N}\gamma_i\mathbb{E}[B_i]\Phi_i^{(\mathbf{0},1)} = 0,$$

and summing this over $j = 1, \cdots, N$ gives

$$\sum_{i=1}^{N}\gamma_i\mathbb{E}[B_i]\Phi_i^{(\mathbf{0},1)} = \rho\mathbb{E}[C], \tag{3.11}$$

where we have used (2.1). Plugging (3.11) into (3.10) leads to

$$\Phi_j^{(\mathbf{1}_j,0)} = \frac{\lambda_j}{\nu_j}\Big(\mathbb{E}[C] - \frac{1}{\gamma_j}\Big),$$

which is (3.6). Inserting this equation into (3.9) yields (3.5). When $i = j$ in Equation (3.8), we have

$$\Phi_{j-1}^{(\mathbf{1}_j,0)} = \frac{\gamma_j}{\gamma_{j-1}}\Phi_j^{(\mathbf{1}_j,0)} + \frac{\lambda_j}{\gamma_{j-1}}\Big((1-\rho_{j-1})\mathbb{E}[C] - \frac{1}{\gamma_j} - \mathbb{E}[S_{j-1}]\Big). \tag{3.12}$$

On the other hand, when $i \neq j$, i.e., $i = j-1, j-2, \cdots, j-N+1$, in Equation (3.8), we have

$$\Phi_{i-1}^{(\mathbf{1}_j,0)} = \frac{\gamma_i}{\gamma_{i-1}}\Phi_i^{(\mathbf{1}_j,0)} + \frac{\lambda_j}{\gamma_{i-1}}\Big(-\rho_{i-1}\mathbb{E}[C] - \frac{1}{\gamma_i} - \mathbb{E}[S_{i-1}]\Big). \tag{3.13}$$

Finally, (3.7) follows from (3.12)-(3.13). □

Next, we calculate $\Phi_i^{(\boldsymbol{l},m)}$ for $|\boldsymbol{l}| + m \geq 2$. Equation (3.4) can be written as

$$\big(\nu_i(w-1) - \nu_i(z_i-1)\big)\frac{\partial}{\partial z_i}\phi_i(\mathbf{z},w) + \Big(\lambda_i(w-1) + \sum_{j=1,j\neq i}^{N}\lambda_j(z_j-1) - \gamma_i\Big)\phi_i(\mathbf{z},w)$$

$$+ \gamma_{i-1}\phi_{i-1}(\mathbf{z},\beta_{i-1}(\mathbf{z}))\sigma_{i-1}(\mathbf{z}) = 0.$$

From this we get

$$\begin{aligned}(\gamma_i + l_i\nu_i)\Phi_i^{(\boldsymbol{l},m)} =& \mathbb{1}_{\{m\geq 1\}}(l_i+1)\nu_i\Phi_i^{(\boldsymbol{l}+\mathbf{1}_i,m-1)} + \mathbb{1}_{\{m\geq 1\}}\lambda_i\Phi_i^{(\boldsymbol{l},m-1)} \\ &+ \sum_{j\neq i}\mathbb{1}_{\{l_j\geq 1\}}\lambda_j\Phi_i^{(\boldsymbol{l}-\mathbf{1}_j,m)} + \mathbb{1}_{\{m=0\}}\gamma_{i-1}\sum_{\boldsymbol{l}'\leq\boldsymbol{l}}\sum_{k=0}^{|\boldsymbol{l}-\boldsymbol{l}'|}\Phi_{i-1}^{(\boldsymbol{l}',k)}\Gamma_{i-1,k}^{(\boldsymbol{l}-\boldsymbol{l}')},\end{aligned} \tag{3.14}$$

where $\Gamma_{i,m}^{(\boldsymbol{l})} = \frac{1}{\boldsymbol{l}!}\frac{\partial^{|\boldsymbol{l}|}}{\partial\mathbf{z}^{\boldsymbol{l}}}\big((\beta_i(\mathbf{z})-1)^m\sigma_i(\mathbf{z})\big)\big|_{\mathbf{z}=\mathbf{1}-}$ and the inequality $\boldsymbol{l}' \leq \boldsymbol{l}$ is interpreted componentwise. Therefore, from (3.14) we have the following proposition.

Proposition 2. *For* $|\boldsymbol{l}| + m \geq 2$,

$$\begin{aligned}\Phi_i^{(\boldsymbol{l},m)} =& \frac{\mathbb{1}_{\{m\geq 1\}}\lambda_i}{\gamma_i + l_i\nu_i}\Phi_i^{(\boldsymbol{l},m-1)} + \sum_{j\neq i}\frac{\mathbb{1}_{\{l_j\geq 1\}}\lambda_j}{\gamma_i + l_i\mu_i}\Phi_i^{(\boldsymbol{l}-\mathbf{1}_j,m)} + \frac{\mathbb{1}_{\{m=0\}}\gamma_{i-1}}{\gamma_i + l_i\nu_i}\sum_{\boldsymbol{l}'\leq\boldsymbol{l}}\sum_{k=0}^{|\boldsymbol{l}-\boldsymbol{l}'|-1}\Phi_{i-1}^{(\boldsymbol{l}',k)}\Gamma_{i-1,k}^{(\boldsymbol{l}-\boldsymbol{l}')} \\ &+ \frac{\mathbb{1}_{\{m\geq 1\}}(l_i+1)\nu_i}{\gamma_i + l_i\nu_i}\Phi_i^{(\boldsymbol{l}+\mathbf{1}_i,m-1)} + \frac{\mathbb{1}_{\{m=0\}}\gamma_{i-1}}{\gamma_i + l_i\nu_i}\sum_{\boldsymbol{l}'\leq\boldsymbol{l}}\Phi_{i-1}^{(\boldsymbol{l}',|\boldsymbol{l}-\boldsymbol{l}'|)}\Gamma_{i-1,|\boldsymbol{l}-\boldsymbol{l}'|}^{(\boldsymbol{l}-\boldsymbol{l}')}.\end{aligned} \tag{3.15}$$

We note that (3.15) is a system of linear equations for $\Phi_i^{(\boldsymbol{l},m)}$. This system of linear equations can be solved by the Gaussian elimination method. However, we will use an iterative method to solve the system of linear equations (3.15). In the following theorem the iterative algorithm is presented and the convergence of iteration is guaranteed.

Theorem 3. *For $i, \boldsymbol{l}, m, n$ with $i = 1, \cdots, N$, $|\boldsymbol{l}| + m = k$, $n = 0, 1, \cdots$, define $\Phi_i^{(\boldsymbol{l},m)}(n)$ as follows:*

$$
\begin{aligned}
\Phi_i^{(\boldsymbol{l},m)}(0) =& 0, \\
\Phi_i^{(\boldsymbol{l},m)}(n) =& \frac{\mathbb{1}_{\{m\geq 1\}}\lambda_i}{\gamma_i + l_i\nu_i}\Phi_i^{(\boldsymbol{l},m-1)} + \sum_{j\neq i}\frac{\mathbb{1}_{\{l_j\geq 1\}}\lambda_j}{\gamma_i + l_i\nu_i}\Phi_i^{(\boldsymbol{l}-\boldsymbol{1}_j,m)} \\
&+ \frac{\mathbb{1}_{\{m=0\}}\gamma_{i-1}}{\gamma_i + l_i\nu_i}\sum_{\boldsymbol{l}'\leq \boldsymbol{l}}\sum_{k=0}^{|\boldsymbol{l}-\boldsymbol{l}'|-1}\Phi_{i-1}^{(\boldsymbol{l}',k)}\Gamma_{i-1,k}^{(\boldsymbol{l}-\boldsymbol{l}')} + \frac{\mathbb{1}_{\{m\geq 1\}}(l_i+1)\nu_i}{\gamma_i + l_i\nu_i}\Phi_i^{(\boldsymbol{l}+\boldsymbol{1}_i,m-1)}(n-1) \\
&+ \frac{\mathbb{1}_{\{m=0\}}\gamma_{i-1}}{\gamma_i + l_i\nu_i}\sum_{\boldsymbol{l}'\leq \boldsymbol{l}}\Phi_{i-1}^{(\boldsymbol{l}',|\boldsymbol{l}-\boldsymbol{l}'|)}(n-1)\Gamma_{i-1,|\boldsymbol{l}-\boldsymbol{l}'|}^{(\boldsymbol{l}-\boldsymbol{l}')}, \quad n \geq 1.
\end{aligned}
$$

Then we have that

(i) $\Phi_i^{(\boldsymbol{l},m)}(n)$ *is nondecreasing in* n.

(ii) $\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) = \Phi_i^{(\boldsymbol{l},m)}$.

Proof. By induction on n, we have that $\Phi_i^{(\boldsymbol{l},m)}(n)$ is increasing in n and $\Phi_i^{(\boldsymbol{l},m)}(n) \leq \Phi_i^{(\boldsymbol{l},m)}$ for all n. Thus (i) is proved. Moreover, $\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n)$ exists and $\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) \leq \Phi_i^{(\boldsymbol{l},m)}$. Suppose that $\{\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) : i = 1, \cdots, N, |\boldsymbol{l}| + m = k\}$ and $\{\Phi_i^{(\boldsymbol{l},m)} : i = 1, \cdots, N, |\boldsymbol{l}| + m = k\}$ are different solutions of the system of equations (3.14). Then $\{a\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) + (1-a)\Phi_i^{(\boldsymbol{l},m)} : i = 1, \cdots, N, |\boldsymbol{l}| + m = k\}$ is a solution for any $a \in \mathbb{R}$. Since $\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) \leq \Phi_i^{(\boldsymbol{l},m)}$, there exists a such that

$$a\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) + (1-a)\Phi_i^{(\boldsymbol{l},m)} \geq 0$$

for all $i = 1, \cdots, N$ and $(\boldsymbol{l}, m)$ with $|\boldsymbol{l}| + m = k$ and

$$a\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) + (1-a)\Phi_i^{(\boldsymbol{l},m)} = 0$$

for some $i = 1, \cdots, N$ and $(\boldsymbol{l}, m)$ with $|\boldsymbol{l}| + m = k$. Hence there exists a nonnegative (vector) solution of (3.14) with a zero component, which is a contradiction, because $\frac{\mathbb{1}_{\{m\geq 1\}}\lambda_i}{\gamma_i+l_i\nu_i}\Phi_i^{(\boldsymbol{l},m-1)} + \sum_{j\neq i}\frac{\mathbb{1}_{\{l_j\geq 1\}}\lambda_j}{\gamma_i+l_i\nu_i}\Phi_i^{(\boldsymbol{l}-\boldsymbol{1}_j,m)} + \frac{\mathbb{1}_{\{m=0\}}\gamma_{i-1}}{\gamma_i+l_i\nu_i}\sum_{\boldsymbol{l}'\leq\boldsymbol{l}}\sum_{k=0}^{|\boldsymbol{l}-\boldsymbol{l}'|-1}\Phi_{i-1}^{(\boldsymbol{l}',k)}\Gamma_{i-1,k}^{(\boldsymbol{l}-\boldsymbol{l}')}$ is positive for all $i = 1, \cdots, N$ and $(\boldsymbol{l}, m)$ with $|\boldsymbol{l}|+m = k$. Therefore, $\lim_{n\to\infty}\Phi_i^{(\boldsymbol{l},m)}(n) = \Phi_i^{(\boldsymbol{l},m)}$ for all $i = 1, \cdots, N$ and $(\boldsymbol{l}, m)$ with $|\boldsymbol{l}| + m = k$. □

3.2 Station size analysis at arbitrary time points

In the previous subsection we have found the generating functions of the number of customers at the beginning of glue periods, visit periods, and switchover periods in terms of $\phi_i(\mathbf{z}, w)$. We now represent the generating function of the number of customers at arbitrary time points in terms of $\phi_i(\mathbf{z}, w)$, as shown below in Theorem 4. This will allow us to obtain the moments of the station size distribution at arbitrary time points.

Theorem 4. *(a) The joint generating function, $R_s^{(i)}(\mathbf{z})$, of the number of customers in the orbit at an arbitrary time point in a switchover period from station i is given by*

$$R_s^{(i)}(\mathbf{z}) = \frac{\gamma_i}{\mathbb{E}[S_i]}\phi_i(\mathbf{z}, \beta_i(\mathbf{z}))\frac{1-\sigma_i(\mathbf{z})}{\sum_{j=1}^N \lambda_j(1-z_j)}. \tag{3.16}$$

(b) The joint generating function, $R_g^{(i)}(\mathbf{z}, w)$, of the number of customers in the queue and in the orbit at an arbitrary time point in a glue period of station i is given by

$$R_g^{(i)}(\mathbf{z}, w) = \gamma_i\phi_i(\mathbf{z}, w). \tag{3.17}$$

(c) The joint generating function, $R_v^{(i)}(\mathbf{z}, w)$, of the number of customers in the queue and in the orbit at an arbitrary time point in a visit period of station i is given by

$$R_v^{(i)}(\mathbf{z}, w) \quad = \quad \frac{\gamma_i}{\rho_i\mathbb{E}[C]}\frac{\phi_i(\mathbf{z}, w)-\phi_i(\mathbf{z}, \beta_i(\mathbf{z}))}{w-\beta_i(\mathbf{z})}\frac{1-\beta_i(\mathbf{z})}{\sum_{j=1}^N \lambda_j(1-z_j)}. \tag{3.18}$$

Proof. (a) Notice that the number of customers in the orbit at an arbitrary time point in a switchover period from station i is the sum of two independent terms: the number of customers at the beginning of the switchover period and the number of customers who arrived during the elapsed switchover period. The generating function of the former is $\tilde{R}_s^{(i)}(\mathbf{z})$ and the generating function of the latter is given by $\frac{1-\sigma_i(\mathbf{z})}{\mathbb{E}[S_i]\left(\sum_{j=1}^N \lambda_j(1-z_j)\right)}$. Thus

$$R_s^{(i)}(\mathbf{z}) = \tilde{R}_s^{(i)}(\mathbf{z})\frac{1-\sigma_i(\mathbf{z})}{\mathbb{E}[S_i]\left(\sum_{j=1}^N \lambda_j(1-z_j)\right)},$$

from which and (3.2) we get (3.16).

(b) By the theory of Markov regenerative processes,

$$R_g^{(i)}(\mathbf{z}, w) = \gamma_i \int_0^\infty \phi_i(\mathbf{z}; w; t)dt.$$

This yields (3.17).

(c) Notice that the number of customers in the system at an arbitrary time point in a visit period consists of two parts: the number of customers in the system at the beginning of the service of the customer currently in service and the number of customers who arrived during the elapsed time of the current service. The generating function of the former is given by (see Remark 3 of [1] for a detailed proof)

$$\frac{\tilde{R}_v^{(i)}(\mathbf{z}, w) - \tilde{R}_v^{(i)}(\mathbf{z}, \beta_i(\mathbf{z}))}{\mathbb{E}[Y_i^{(iq)}](w - \beta_i(\mathbf{z}))} = \frac{\gamma_i}{\mathbb{E}[Y_i^{(iq)}]} \frac{\phi_i(\mathbf{z}, w) - \phi_i(\mathbf{z}, \beta_i(\mathbf{z}))}{w - \beta_i(\mathbf{z})}, \tag{3.19}$$

and the generating function of the latter is given by

$$\frac{1 - \beta_i(\mathbf{z})}{\mathbb{E}[B_i]\left(\sum_{j=1}^N \lambda_j(1 - z_j)\right)}. \tag{3.20}$$

From (3.19) and (3.20) we have

$$R_v^{(i)}(\mathbf{z}, w) = \frac{\gamma_i}{\mathbb{E}[Y_i^{(iq)}]\mathbb{E}[B_i]} \frac{\phi_i(\mathbf{z}, w) - \phi_i(\mathbf{z}, \beta_i(\mathbf{z}))}{w - \beta_i(\mathbf{z})} \frac{1 - \beta_i(\mathbf{z})}{\sum_{j=1}^N \lambda_j(1 - z_j)}.$$

Since $\rho_i = \frac{\mathbb{E}[Y_i^{(iq)}]\mathbb{E}[B_i]}{\mathbb{E}[C]}$, (3.18) follows from the above equation. □

We introduce the following scaled moments:

$$\Psi_{g,i}^{(\boldsymbol{l},m)} = \frac{1}{\boldsymbol{l}!m!} \frac{\partial^{|\boldsymbol{l}|+m}}{\partial \mathbf{z}^{\boldsymbol{l}} \partial w^m} R_g^{(i)}(\mathbf{z}, w)\Big|_{\mathbf{z}=\mathbf{1}-, w=1-},$$

$$\Psi_{v,i}^{(\boldsymbol{l},m)} = \frac{1}{\boldsymbol{l}!m!} \frac{\partial^{|\boldsymbol{l}|+m}}{\partial \mathbf{z}^{\boldsymbol{l}} \partial w^m} R_v^{(i)}(\mathbf{z}, w)\Big|_{\mathbf{z}=\mathbf{1}-, w=1-},$$

$$\Psi_{s,i}^{(\boldsymbol{l})} = \frac{1}{\boldsymbol{l}!} \frac{\partial^{|\boldsymbol{l}|}}{\partial \mathbf{z}^{\boldsymbol{l}}} R_s^{(i)}(\mathbf{z})\Big|_{\mathbf{z}=\mathbf{1}-}.$$

These moments satisfy the following theorem, which can be derived by using Equations (3.16), (3.17) and (3.18).

Theorem 5. *We have that*

(i) $\Psi_{g,i}^{(\boldsymbol{l},m)} = \gamma_i \Phi_i^{(\boldsymbol{l},m)}$,

(ii) $\Psi_{v,i}^{(\boldsymbol{l},m)} = \frac{\gamma_i}{\rho_i \mathbb{E}[C]} \sum_{\boldsymbol{l}' \leq \boldsymbol{l}} \sum_{k=0}^{|\boldsymbol{l}-\boldsymbol{l}'|} \Phi_i^{(\boldsymbol{l}',m+k+1)} \eta_{i,k}^{(\boldsymbol{l}-\boldsymbol{l}')}$, *where* $\eta_{i,m}^{(\boldsymbol{l})} = \frac{1}{\boldsymbol{l}!} \frac{\partial^{|\boldsymbol{l}|}}{\partial \mathbf{z}^{\boldsymbol{l}}} \left(\frac{-(\beta_i(\mathbf{z})-1)^{m+1}}{\sum_{j=1}^N \lambda_j (1-z_j)} \right)\Big|_{\mathbf{z}=\mathbf{1}-}$.

(iii) $\Psi_{s,i}^{(\boldsymbol{l})} = \frac{\gamma_i}{\mathbb{E}[S_i]} \sum_{\boldsymbol{l}' \leq \boldsymbol{l}} \Theta_i^{(\boldsymbol{l}')} \zeta_i^{(\boldsymbol{l}-\boldsymbol{l}')}$, *where* $\zeta_i^{(\boldsymbol{l})} = \frac{1}{\boldsymbol{l}!} \frac{\partial^{|\boldsymbol{l}|}}{\partial \mathbf{z}^{\boldsymbol{l}}} \left(\frac{1-\sigma_i(\mathbf{z})}{\sum_{j=1}^N \lambda_j (1-z_j)} \right)\Big|_{\mathbf{z}=\mathbf{1}-}$

and $\Theta_i^{(\boldsymbol{l})} = \frac{1}{\boldsymbol{l}!} \frac{\partial^{|\boldsymbol{l}|}}{\partial \mathbf{z}^{\boldsymbol{l}}} \phi_i(\mathbf{z}, \beta_i(\mathbf{z}))\big|_{\mathbf{z}=\mathbf{1}-}$. *Moreover,* $\Theta_i^{(\boldsymbol{l})}$ *is given by*

$$\Theta_i^{(\boldsymbol{l})} = \sum_{\boldsymbol{l}' \leq \boldsymbol{l}} \sum_{k=0}^{|\boldsymbol{l}-\boldsymbol{l}'|} \Phi_i^{(\boldsymbol{l}',k)} \Delta_{i,k}^{(\boldsymbol{l}-\boldsymbol{l}')},$$

where $\Delta_{i,m}^{(\boldsymbol{l})} = \frac{1}{\boldsymbol{l}!} \frac{\partial^{|\boldsymbol{l}|}}{\partial \mathbf{z}^{\boldsymbol{l}}} (\beta_i(\mathbf{z}) - 1)^m \big|_{\mathbf{z}=\mathbf{1}-}$.

From now on we obtain the first and second moments of the station sizes of each type of customers in steady state. Let M_i^o and Υ be the steady state random variables corresponding to $M_i^o(t)$ and $\Upsilon(t)$, respectively. That is, M_i^o is the number of customers in the orbit of station i in steady state and Υ is the number of glued customers in steady state. Let M_i^{oq} be the number of customers in the orbit of station i plus the glued customers in the queue of station i in steady state, and M_i be the number of customers in station i (including the customer in service at station i) in steady state. Moreover, we define the following indicator random variables: for $i = 1, \ldots, N$,

$$I_{v,i} = \begin{cases} 1 & \text{if the server is serving at station } i \text{ in steady state,} \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{g,i} = \begin{cases} 1 & \text{if the server is in the glue period of station } i \text{ in steady state,} \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{s,i} = \begin{cases} 1 & \text{if the server is switching from station } i \text{ to station } i+1 \text{ in steady state,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that for $i = 1, \ldots, N$,

$$M_i^o = \sum_{k=1}^N M_i^o (I_{v,k} + I_{g,k} + I_{s,k}),$$

$$M_i^{oq} = M_i^o + \Upsilon (I_{v,i} + I_{g,i}),$$

$$M_i = M_i^{oq} + I_{v,i}.$$

Therefore, the mean station sizes, $\mathbb{E}[M_i^o], \mathbb{E}[M_i^{oq}]$, and $\mathbb{E}[M_i]$, $i = 1, \dots, N$, are given by

$$\mathbb{E}[M_i^o] = \sum_{k=1}^{N} \Big(\rho_k \Psi_{v,k}^{(\mathbf{1}_i,0)} + \frac{\mathbb{E}[G_k]}{\mathbb{E}[C]} \Psi_{g,k}^{(\mathbf{1}_i,0)} + \frac{\mathbb{E}[S_k]}{\mathbb{E}[C]} \Psi_{s,k}^{(\mathbf{1}_i)} \Big), \tag{3.21}$$

$$\mathbb{E}[M_i^{oq}] = \mathbb{E}[M_i^o] + \rho_i \Psi_{v,i}^{(\mathbf{0},1)} + \frac{\mathbb{E}[G_i]}{\mathbb{E}[C]} \Psi_{g,i}^{(\mathbf{0},1)}, \tag{3.22}$$

$$\mathbb{E}[M_i] = \mathbb{E}[M_i^{oq}] + \rho_i. \tag{3.23}$$

Now, in order to obtain the second moments of the station sizes, $\mathbb{E}[M_i^o M_j^o], \mathbb{E}[M_i^{oq} M_j^{oq}]$, and $\mathbb{E}[M_i M_j]$, $i, j = 1, \dots, N$, note that

$$M_i^o M_j^o = \sum_{k=1}^{N} M_i^o M_j^o (I_{v,k} + I_{g,k} + I_{s,k}),$$

$$M_i^{oq} M_j^{oq} = M_i^o M_j^o + M_i^o \Upsilon (I_{v,j} + I_{g,j}) + M_j^o \Upsilon (I_{v,i} + I_{g,i}) + \Upsilon^2 (I_{v,i} + I_{g,i}) \mathbb{1}_{\{i=j\}},$$

$$M_i M_j = M_i^{oq} M_j^{oq} + M_i^{oq} I_{v,j} + M_j^{oq} I_{v,i} + I_{v,i} \mathbb{1}_{\{i=j\}}.$$

Therefore, the second moments of the station sizes are given by

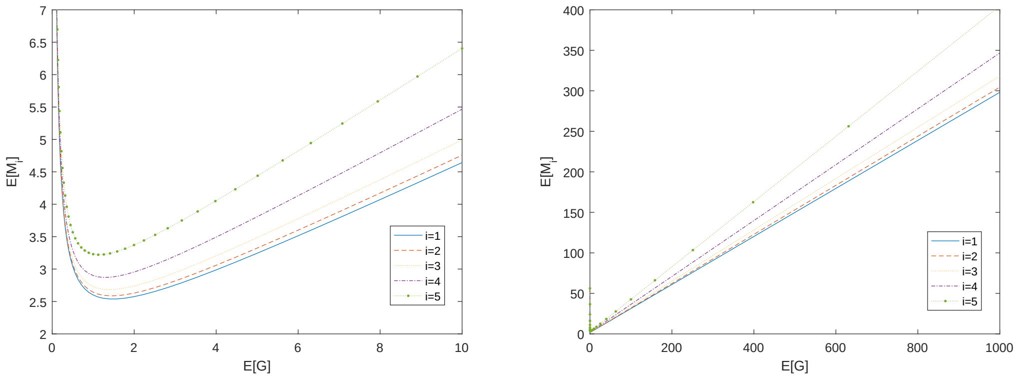
$$\mathbb{E}[M_i^o M_j^o] = \begin{cases} \sum_{k=1}^{N} \Big(\rho_k \Psi_{v,k}^{(\mathbf{1}_i+\mathbf{1}_j,0)} + \frac{\mathbb{E}[G_k]}{\mathbb{E}[C]} \Psi_{g,k}^{(\mathbf{1}_i+\mathbf{1}_j,0)} + \frac{\mathbb{E}[S_k]}{\mathbb{E}[C]} \Psi_{s,k}^{(\mathbf{1}_i+\mathbf{1}_j)} \Big) & \text{if } i \neq j, \\ 2 \sum_{k=1}^{N} \Big(\rho_k \Psi_{v,k}^{(2\mathbf{1}_i,0)} + \frac{\mathbb{E}[G_k]}{\mathbb{E}[C]} \Psi_{g,k}^{(2\mathbf{1}_i,0)} + \frac{\mathbb{E}[S_k]}{\mathbb{E}[C]} \Psi_{s,k}^{(2\mathbf{1}_i)} \Big) & \text{if } i = j, \end{cases} \tag{3.24}$$

$$\mathbb{E}[M_i^{oq} M_j^{oq}] = \begin{cases} \mathbb{E}[M_i^o M_j^o] + \rho_i \Psi_{v,i}^{(\mathbf{1}_j,1)} + \rho_j \Psi_{v,j}^{(\mathbf{1}_i,1)} + \frac{\mathbb{E}[G_i]}{\mathbb{E}[C]} \Psi_{g,i}^{(\mathbf{1}_j,1)} + \frac{\mathbb{E}[G_j]}{\mathbb{E}[C]} \Psi_{g,j}^{(\mathbf{1}_i,1)} & \text{if } i \neq j, \\ \mathbb{E}[(M_i^o)^2] + 2 \Big(\rho_i \Psi_{v,i}^{(\mathbf{1}_i,1)} + \frac{\mathbb{E}[G_i]}{\mathbb{E}[C]} \Psi_{g,i}^{(\mathbf{1}_i,1)} + \rho_i \Psi_{v,i}^{(\mathbf{0},2)} + \frac{\mathbb{E}[G_i]}{\mathbb{E}[C]} \Psi_{g,i}^{(\mathbf{0},2)} \Big) & \text{if } i = j, \end{cases} \tag{3.25}$$

$$\mathbb{E}[M_i M_j] = \begin{cases} \mathbb{E}[M_i^{oq} M_j^{oq}] + \rho_i \Psi_{v,i}^{(\mathbf{1}_j,0)} + \rho_j \Psi_{v,j}^{(\mathbf{1}_i,0)} & \text{if } i \neq j, \\ \mathbb{E}[(M_i^{oq})^2] + 2\rho_i \Psi_{v,i}^{(\mathbf{1}_i,0)} + 2\rho_i \Psi_{v,i}^{(\mathbf{0},1)} + \rho_i & \text{if } i = j. \end{cases} \tag{3.26}$$

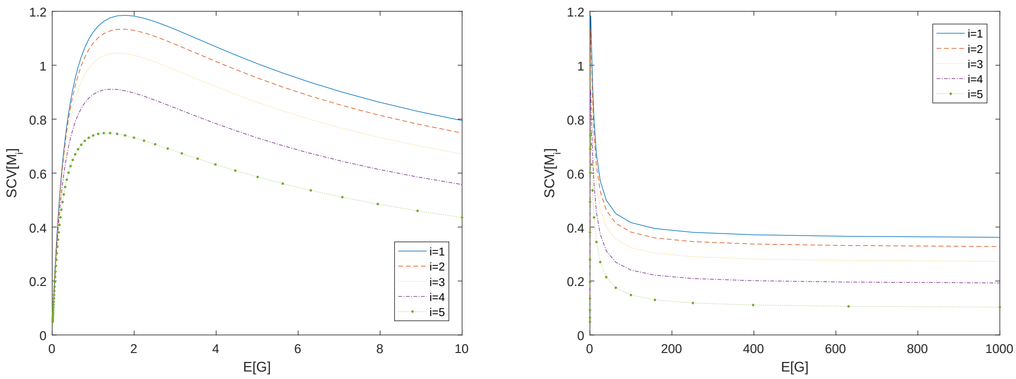
3.3 A numerical example

In this subsection we present numerical results for the first and second moments of the number of customers in each station. The expression for the mean number of customers in each station is given by (3.23), together with (3.21) and (3.22). By using the formulas (3.24)-(3.26), we can obtain an expression for the variance of the number of customers in each station and an expression for the covariance of the numbers of customers in two different stations. Note that these moments are expressed in terms of $\Phi_i^{(\boldsymbol{l},m)}$. Therefore, these moments can be obtained by using Theorems 2 and 3. In the following numerical example we consider a single server polling model with five stations (i.e., $N = 5$).



(a) $0 \leq \mathbb{E}[G] \leq 10$. (b) $0 \leq \mathbb{E}[G] \leq 1000$.

Figure 1: The mean number of customers in station i, $\mathbb{E}[M_i]$, $i = 1, \ldots, 5$, varying $\mathbb{E}[G]$.

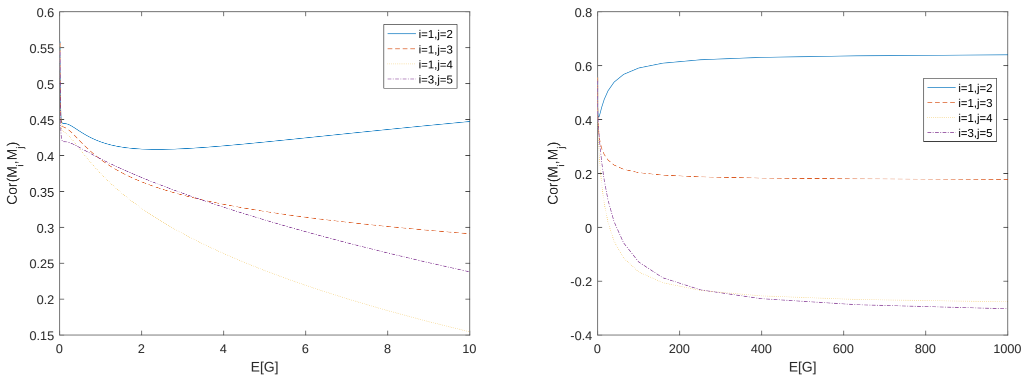


(a) $0 \leq \mathbb{E}[G] \leq 10$. (b) $0 \leq \mathbb{E}[G] \leq 1000$.

Figure 2: The squared coefficient of variation for the number of customers in station i, $\text{SCV}[M_i]$, $i = 1, \ldots, 5$, varying $\mathbb{E}[G]$.

Example 1. We assume that the arrival rate of type-i customers is $\lambda_i = 0.025$ for all i, $i = 1, \ldots, 5$. The service times of type-i customers are exponentially distributed with means $\mathbb{E}[B_1] = 1, \mathbb{E}[B_2] = 2, \mathbb{E}[B_3] = 4, \mathbb{E}[B_4] = 8$ and $\mathbb{E}[B_5] = 16$, respectively. Hence the total utilization of the server is $\rho = \sum_{i=1}^{5} \rho_i = 0.775 < 1$. The switchover times from station i to station $i+1$ are deterministic with $\mathbb{E}[S_i] = 1$ for all i, $i = 1, \ldots, 5$. The retrial rate of customers in the orbit of station i is $\nu_i = 1$ for all i, $i = 1, \ldots, 5$. The glue periods at station i are exponentially distributed with parameters γ_i, $i = 1, \ldots, 5$. We assume that γ_i is the same for all i, i.e., $\mathbb{E}[G_i] = \mathbb{E}[G]$ for all i, $i = 1, \ldots, 5$.

In Figure 1 we plot the mean number of customers in station i, $\mathbb{E}[M_i]$, $i = 1, \ldots, 5$, varying the



(a) $0 \le \mathbb{E}[G] \le 10$.

(b) $0 \le \mathbb{E}[G] \le 1000$.

Figure 3: The correlation coefficient of the numbers of customers in station i and station j, $\mathrm{Cor}(M_i, M_j)$, $(i,j) = (1,2)$, $(i,j) = (1,3)$, $(i,j) = (1,4)$ and $(i,j) = (3,5)$, varying $\mathbb{E}[G]$.

mean glue period $\mathbb{E}[G]$. In Figure 2 we plot the squared coefficient of variation (SCV) for the number of customers in station i, $\mathrm{SCV}[M_i]$, $i = 1, \ldots, 5$, varying $\mathbb{E}[G]$. In Figure 3 we plot the correlation coefficient of the numbers of customers in two different stations, $\mathrm{Cor}[M_1, M_2]$, $\mathrm{Cor}[M_1, M_3]$, $\mathrm{Cor}[M_1, M_4]$ and $\mathrm{Cor}[M_3, M_5]$, varying $\mathbb{E}[G]$. We vary $\mathbb{E}[G]$ from 0 to 10 in Figures 1(a), 2(a) and 3(a), in order to better reveal the behavior of the system for small $\mathbb{E}[G]$. On the other hand, we vary $\mathbb{E}[G]$ from 0 to 1000 in Figures 1(b), 2(b) and 3(b), in order to examine the behavior of the system for large $\mathbb{E}[G]$.

We can draw the following conclusions from these plots:

- For small glue period lengths, the chances for a customer to retry are very low, hence the station size is large.
- If the glue period is very large, the customers face a long delay before getting served.
- There exists an optimal glue length at which each station has a minimum mean station size.
- The figures suggest that the following happens when the mean glue period grows large:
  - (i) The mean numbers of customers grow linearly in $\mathbb{E}[G]$.
  - (ii) The squared coefficient of variation tends to a limit when $\mathbb{E}[G] \to \infty$.

(iii) The correlation coefficients between the numbers of customers in different stations tend to some limit when $\mathbb{E}[G] \to \infty$.

In [21] the author considers classical polling systems with a branching-type service discipline like exhaustive or gated service, and without glue periods, for the case that switchover times become large. It is readily seen that our polling model starts to behave very similarly as such a polling model, when the glue periods grow large; indeed, every type-i customer will now almost surely become glued during the first glue period of station i that it experiences during its stay in the system, and hence will be served during the first visit period of station i after its arrival to the system - just as in an ordinary gated polling system. However, we cannot immediately apply the asymptotic results of [21] where switchover times become large, because it considers deterministic switchover times, while the focus is on the waiting time distribution. In a future paper we intend to study the asymptotic behaviour of polling systems with large switchover times, thus also obtaining the asymptotic behaviour of polling systems with large glue periods. We shall, among others, derive asymptotic expressions for the kth moment of the station size. Our preliminary findings are in agreement with the limiting behaviours of the mean station size, the squared coefficient of variation of the station sizes, and the correlation coefficient of the station sizes. The mean station size is asymptotically linear in the mean switchover times, and the squared coefficient of variation and the correlation coefficient of the station sizes converge as the mean switchover times go to infinity (with the ratios of the swichover times being constant), as displayed in the figures.

4 The polling system with retrials and general glue periods

In [1] and in Section 3 of the current paper we have presented the distribution and mean of the number of customers at different time epochs for a gated polling model with retrials and glue periods, where the glue periods are deterministic and exponentially distributed, respectively. In this section, we assume that glue periods have general distributions. We first consider the distribution of the total workload in the system and present a workload decomposition. Subsequently we use this to obtain a pseudo conservation law, i.e., an exact expression for a weighted sum of the mean waiting times. In its turn, the pseudo conservation law is used to obtain an approximation for the mean waiting times of all customer types. We present numerical results that indicate that the approximation is very accurate. Finally we use this approximation to optimize a weighted sum of the mean waiting times, $\sum_{i=1}^{N} c_i\mathbb{E}[W_i]$, where c_i, $i = 1, \dots, N$ are positive constants and $\mathbb{E}[W_i]$ is the mean waiting time of a type-i customer until the start of its service, by choosing the glue period lengths, given the total glue period in a cycle.

4.1 Workload distribution and decomposition

Define V as the amount of work in the system in steady state. Furthermore, let $\tilde{B}(s) = \sum_{i=1}^{N} \lambda_i(1-\tilde{B}_i(s))$. The LST of the amount of work at an arbitrary time can be written as

$$\mathbb{E}[e^{-sV}] = \frac{1}{\mathbb{E}[C]} \sum_{i=1}^{N} \left(\mathbb{E}[S_i]\mathbb{E}[e^{-sV_i^{(S)}}] + \mathbb{E}[G_i]\mathbb{E}[e^{-sV_i^{(G)}}] + \rho_i\mathbb{E}[C]\mathbb{E}[e^{-sV_i^{(D)}}]\right), \tag{4.1}$$

where $V_i^{(S)}$, $V_i^{(G)}$ and $V_i^{(D)}$ are the amount of work in the system during the switchover time from station i, glue period of station i and visit period of station i, respectively.

Let $V_i^{(X)}$, $V_i^{(Y)}$ and $V_i^{(Z)}$ be the work in the system at the *start* of glue period of station i, visit period of station i and switchover period from station i, respectively. We know that

$$\mathbb{E}[e^{-sV_i^{(S)}}] = \mathbb{E}[e^{-sV_i^{(Z)}}]\frac{1-\tilde{S}_i(\tilde{B}(s))}{\mathbb{E}[S_i]\tilde{B}(s)},$$
$$\mathbb{E}[e^{-sV_i^{(G)}}] = \mathbb{E}[e^{-sV_i^{(X)}}]\frac{1-\tilde{G}_i(\tilde{B}(s))}{\mathbb{E}[G_i]\tilde{B}(s)}.$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{N} \left(\mathbb{E}[e^{-sV_i^{(S)}}]\mathbb{E}[S_i] + \mathbb{E}[e^{-sV_i^{(G)}}]\mathbb{E}[G_i]\right) \\
&= \sum_{i=1}^{N} \left(\mathbb{E}[e^{-sV_i^{(Z)}}]\frac{1-\tilde{S}_i(\tilde{B}(s))}{\tilde{B}(s)} + \mathbb{E}[e^{-sV_i^{(X)}}]\frac{1-\tilde{G}_i(\tilde{B}(s))}{\tilde{B}(s)}\right) \\
&= \sum_{i=1}^{N} \left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_i^{(Z)}}]\tilde{S}_i(\tilde{B}(s)) + \mathbb{E}[e^{-sV_i^{(X)}}] - \mathbb{E}[e^{-sV_i^{(X)}}]\tilde{G}_i(\tilde{B}(s))}{\tilde{B}(s)}\right) \\
&= \sum_{i=1}^{N} \left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_{i+1}^{(X)}}] + \mathbb{E}[e^{-sV_i^{(X)}}] - \mathbb{E}[e^{-sV_i^{(Y)}}]}{\tilde{B}(s)}\right) \\
&= \sum_{i=1}^{N} \left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_i^{(Y)}}]}{\tilde{B}(s)}\right).
\end{aligned} \tag{4.2}$$

Furthermore, using the last formula of the proof of Theorem 2 in Boxma et al. [6], but with our notations, we have

$$\rho_i \mathbb{E}[C]\mathbb{E}[e^{-sV_i^{(D)}}] = \frac{\mathbb{E}[e^{-sV_i^{(Y)}}] - \mathbb{E}[e^{-sV_i^{(Z)}}]}{\tilde{B}(s) - s}. \tag{4.3}$$

Substituting (4.2) and (4.3) in (4.1), we have

$$\begin{aligned}\mathbb{E}[e^{-sV}] &= \frac{1}{\mathbb{E}[C]}\sum_{i=1}^{N}\left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_i^{(Y)}}]}{\tilde{B}(s)} + \frac{\mathbb{E}[e^{-sV_i^{(Y)}}] - \mathbb{E}[e^{-sV_i^{(Z)}}]}{\tilde{B}(s) - s}\right)\\ &= \frac{s}{\mathbb{E}[C](s - \tilde{B}(s))}\sum_{i=1}^{N}\left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_i^{(Y)}}]}{\tilde{B}(s)}\right). \end{aligned} \tag{4.4}$$

Define the idle time as the time the server is not serving customers (i.e., the sum of all the switchover and glue periods). Let $V^{(Idle)}$ be the amount of work in the system at an arbitrary moment in the idle time. We have

$$\begin{aligned}\mathbb{E}[e^{-sV^{(Idle)}}] &= \frac{1}{\mathbb{E}\big[\sum_{i=1}^{N}(S_i + G_i)\big]}\sum_{i=1}^{N}\left(\mathbb{E}[e^{-sV_i^{(S)}}]\mathbb{E}[S_i] + \mathbb{E}[e^{-sV_i^{(G)}}]\mathbb{E}[G_i]\right)\\ &= \frac{1}{\mathbb{E}\big[\sum_{i=1}^{N}(S_i + G_i)\big]}\sum_{i=1}^{N}\left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_i^{(Y)}}]}{\tilde{B}(s)}\right)\\ &= \frac{1}{(1-\rho)\mathbb{E}[C]}\sum_{i=1}^{N}\left(\frac{\mathbb{E}[e^{-sV_i^{(Z)}}] - \mathbb{E}[e^{-sV_i^{(Y)}}]}{\tilde{B}(s)}\right). \end{aligned} \tag{4.5}$$

We know that the LST of the amount of work at steady state, $V_{M/G/1}$, in the standard $M/G/1$ queue where the arrival rate is $\sum_{i=1}^{N}\lambda_i$ and the LST of the service time distribution is $\sum_{i=1}^{N}\frac{\lambda_i}{\sum_{j=1}^{N}\lambda_j}\tilde{B}_i(s)$, is given by

$$\mathbb{E}[e^{-sV_{M/G/1}}] = \frac{(1-\rho)s}{s - \tilde{B}(s)}. \tag{4.6}$$

From Equations (4.4), (4.5) and (4.6) we have

$$\mathbb{E}[e^{-sV}] = \mathbb{E}[e^{-sV_{M/G/1}}]\mathbb{E}[e^{-sV^{(Idle)}}].$$

In Theorem 2.1 of [5], a workload decomposition property has been proved for a large class of single-server multi-class queueing systems with service interruptions (like switchover periods or breakdowns). It amounts to the statement that, under certain conditions, the steady-state workload is in distribution equal to the sum of two independent quantities: (i) the steady-state workload in the corresponding queueing model without those interruptions, and (ii) the steady-state workload at an arbitrary interruption epoch.

The gated polling model with glue periods and retrials of the present paper satisfies all the assumptions of Theorem 2.1 of [5], and hence, in agreement with what we have seen above, the workload decomposition indeed holds.

4.2 Pseudo conservation law

By the workload decomposition, it is shown in [5] that

$$\sum_{i=1}^{N} \rho_i \mathbb{E}[W_i] = \rho \frac{\sum_{i=1}^{N} \lambda_i \mathbb{E}[B_i^2]}{2(1-\rho)} + \rho \frac{\mathbb{E}\big[\big(\sum_{i=1}^{N}(S_i+G_i)\big)^2\big]}{2\mathbb{E}\big[\sum_{i=1}^{N}(S_i+G_i)\big]} + \frac{\mathbb{E}\big[\sum_{i=1}^{N}(S_i+G_i)\big]}{2(1-\rho)}\big(\rho^2 - \sum_{i=1}^{N}\rho_i^2\big) + \sum_{i=1}^{N}\mathbb{E}[F_i], \tag{4.7}$$

where F_i is the work left in station i at the end of a visit period of station i (and hence at the start of a switchover from station i). Other than $\mathbb{E}[F_i]$, Equation (4.7) is independent of the service discipline. Note that $\mathbb{E}[F_i] = \mathbb{E}[Z_i^{(i)}]\mathbb{E}[B_i]$. To find $\mathbb{E}[Z_i^{(i)}]$ we will derive a relation between $\mathbb{E}[Z_i^{(i)}]$ and $\mathbb{E}[Y_i^{(iq)}]$. $\mathbb{E}[Y_i^{(iq)}]$ consists of the following three parts:

(i) Mean number of type-i customers who were already present at the end of the previous visit to station i and who are glued during the glue period just before the current visit to station i.

(ii) Mean number of type-i customers who have arrived during the time interval from the end of the previous visit to station i to the start of the glue period of station i just before the visit to station i, and who are glued during that glue period.

(iii) Mean number of type-i customers who arrive during the glue period of station i just before the visit to station i.

Note that (i) equals $(1-\tilde{G}_i(\nu_i))\mathbb{E}[Z_i^{(i)}]$ because the mean number of type-i customers who were present at the end of the previous visit to station i is $\mathbb{E}[Z_i^{(i)}]$, and the probability that a customer who was present at the end of the previous visit to station i is glued during the glue period just before the current visit to station i, is $1-\tilde{G}_i(\nu_i)$. (ii) equals $(1-\tilde{G}_i(\nu_i))\lambda_i\big((1-\rho_i)\mathbb{E}[C]-\mathbb{E}[G_i]\big)$. Here $\lambda_i\big((1-\rho_i)\mathbb{E}[C]-\mathbb{E}[G_i]\big)$ is the mean number of type-i customers who have arrived during the time interval from the end of the previous visit to station i to the start of the glue period of station i just before the visit to station i.

Finally, (iii) equals $\lambda_i \mathbb{E}[G_i]$. Therefore,

$$\mathbb{E}[Y_i^{(iq)}] = (1 - \tilde{G}_i(\nu_i))\mathbb{E}[Z_i^{(i)}] + (1 - \tilde{G}_i(\nu_i))(1 - \rho_i)\lambda_i \mathbb{E}[C] + \tilde{G}_i(\nu_i)\mathbb{E}[G_i]\lambda_i. \tag{4.8}$$

Since $\rho_i = \frac{\mathbb{E}[Y_i^{(iq)}]\mathbb{E}[B_i]}{\mathbb{E}[C]}$, we have $\mathbb{E}[Y_i^{(iq)}] = \lambda_i \mathbb{E}[C]$. Hence, by (4.8), we get

$$\mathbb{E}[Z_i^{(i)}] = \lambda_i \rho_i \mathbb{E}[C] + \frac{\lambda_i \tilde{G}_i(\nu_i)}{1 - \tilde{G}_i(\nu_i)} \left(\mathbb{E}[C] - \mathbb{E}[G_i]\right).$$

Therefore, $\mathbb{E}[F_i]$ is given by

$$\mathbb{E}[F_i] = \rho_i^2 \mathbb{E}[C] + \frac{\rho_i \tilde{G}_i(\nu_i)}{1 - \tilde{G}_i(\nu_i)} \left(\mathbb{E}[C] - \mathbb{E}[G_i]\right). \tag{4.9}$$

The first term on the right-hand side equals the mean amount of work for type-i customers who arrived at station i during a visit period of station i. The second term is interpreted as follows: Since $\lambda_i(\mathbb{E}[C]-\mathbb{E}[G_i])$ is the mean number of type-i customers who arrive during one cycle excluding the glue period of station i in that cycle, $(\tilde{G}_i(\nu))^k \rho_i(\mathbb{E}[C]-\mathbb{E}[G_i])$ is the mean amount of work for type-i customers who arrive during the kth previous cycle excluding the glue period of station i in that cycle, and who are present at the end of the current visit period of station i. Hence, the second term, which is $\sum_{k=1}^{\infty}(\tilde{G}_i(\nu))^k \rho_i(\mathbb{E}[C]-\mathbb{E}[G_i])$, is the mean amount of work for type-i customers who were present in the orbit of station i at the beginning of the visit period of station i.

From Equations (4.7) and (4.9) together with (2.1), we obtain the following pseudo conservation law:

$$\begin{aligned}\sum_{i=1}^{N} \rho_i \mathbb{E} W_i =& \rho\left(\frac{\sum_{i=1}^{N} \lambda_i \mathbb{E}[B_i^2]}{2(1-\rho)} + \frac{\mathbb{E}\left[\left(\sum_{i=1}^{N}(S_i + G_i)\right)^2\right]}{2\mathbb{E}\left[\sum_{i=1}^{N}(S_i + G_i)\right]}\right) + \left(\rho^2 + \sum_{i=1}^{N}\rho_i^2\right)\frac{\mathbb{E}\left[\sum_{i=1}^{N}(S_i + G_i)\right]}{2(1-\rho)} \\ &+ \sum_{i=1}^{N} \frac{\rho_i \tilde{G}_i(\nu_i)}{1 - \tilde{G}_i(\nu_i)}\left(\frac{\mathbb{E}\left[\sum_{j=1}^{N}(S_j + G_j)\right]}{1-\rho} - \mathbb{E}[G_i]\right).\end{aligned} \tag{4.10}$$

4.3 Approximation of the mean waiting times

We now use the pseudo conservation law to find an approximation for the mean waiting times of all customer types. Below we briefly sketch the idea behind the approximation. Everitt [8] has developed a method to approximate the mean waiting times in an ordinary gated polling system (without retrials and glue periods). The idea in this approximation is that an arriving customer first has to wait for the residual cycle time, until the server begins a new visit to its station. Subsequently, it has to wait for the service times of all customers of the same type, who arrived before it, in the elapsed cycle time. This leads to $\mathbb{E}[W_j] = (1+\rho_j)\mathbb{E}[R_{c_j}]$, where $\mathbb{E}[R_{c_j}]$ is the mean of the residual time of a cycle starting with a visit to station j, which is the same as the mean of the elapsed time of a cycle starting with a visit to station j. Next, Everitt assumed that for all j, this mean residual cycle time is independent of j, i.e., $\mathbb{E}[R_{c_j}] \approx \mathbb{E}[R_c]$, leading to the approximation $\mathbb{E}[W_j] \approx (1+\rho_j)\mathbb{E}[R_c]$ for the model without retrials and glue periods.

In this paper, we introduce a similar type of approximation, including one extra term, for the mean waiting times in the model with retrials and glue periods:

$$\mathbb{E}[W_j] \approx (1+\rho_j)\mathbb{E}[R_c] + \frac{\tilde{G}_j(\nu_j)}{1-\tilde{G}_j(\nu_j)} \left(\mathbb{E}[C] - \mathbb{E}[G_j]\right). \tag{4.11}$$

In the appendix we provide a detailed derivation of (4.11). The first term on the right-hand side of (4.11) is the same as the term in [8]. The second term on the right-hand side is added because not every customer who arrives in a particular cycle receives service in that cycle. The type-j customers arriving during any period other than the glue period of station j receive service in the following visit period with probability $1-\tilde{G}_j(\nu_j)$. Furthermore, the type-j customers arriving during any period other than a glue period of station j, have to wait for a geometric number (with parameter $1-\tilde{G}_j(\nu_j)$) of cycles before receiving service. Since a type-j customer arrives during a period other than a glue period of station j with probability $\frac{\mathbb{E}[C]-\mathbb{E}[G_j]}{\mathbb{E}[C]}$, the mean number of cycles until an arbitrary type-j customer receives its service, is $\frac{\mathbb{E}[C]-\mathbb{E}[G_j]}{\mathbb{E}[C]} \times \frac{\tilde{G}_j(\nu_j)}{1-\tilde{G}_j(\nu_j)}$. The second term is obtained by multiplying this mean number of cycles and the mean cycle time.

It should be noted that, in reality, the mean residual cycle times for station i and station j ($j \neq i$) are not equal. A key element of our approximation is to assume that they *are* equal. We can now use the pseudo conservation law to determine the one unknown term $\mathbb{E}[R_c]$: Substituting (4.11) in (4.10) yields

$$\begin{aligned}
&\sum_{i=1}^N \rho_i(1+\rho_i)\mathbb{E}[R_c] + \sum_{i=1}^N \frac{\rho_i\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)} \left(\mathbb{E}[C] - \mathbb{E}[G_i]\right) \\
&\approx \rho\left(\frac{\sum_{i=1}^N \lambda_i \mathbb{E}[B_i^2]}{2(1-\rho)} + \frac{\mathbb{E}\big[\big(\sum_{i=1}^N (S_i+G_i)\big)^2\big]}{2\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}\right) + \Big(\rho^2 + \sum_{i=1}^N \rho_i^2\Big)\frac{\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}{2(1-\rho)} \\
&\quad + \sum_{i=1}^N \frac{\rho_i\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}\left(\frac{\mathbb{E}\big[\sum_{j=1}^N (S_j+G_j)\big]}{1-\rho} - \mathbb{E}[G_i]\right).
\end{aligned}$$

By (2.1), we have

$$\mathbb{E}[R_c] \approx \frac{\rho}{\rho + \sum_{i=1}^N \rho_i^2}\left(\sum_{i=1}^N \frac{\rho_i \mathbb{E}[B_i^2]}{2(1-\rho)\mathbb{E}[B_i]} + \frac{\mathbb{E}\big[\big(\sum_{i=1}^N (S_i+G_i)\big)^2\big]}{2\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]} + \rho\frac{\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}{2(1-\rho)}\right)$$
$$+ \frac{\sum_{i=1}^N \rho_i^2}{\rho + \sum_{i=1}^N \rho_i^2}\left(\frac{\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}{2(1-\rho)}\right). \tag{4.12}$$

Substituting (4.12) in (4.11) and using (2.1), we get

$$\mathbb{E}[W_j] \approx \frac{1+\rho_j}{\rho + \sum_{i=1}^N \rho_i^2}\left\{\rho\left(\sum_{i=1}^N \frac{\rho_i \mathbb{E}[B_i^2]}{2(1-\rho)\mathbb{E}[B_i]} + \frac{\mathbb{E}\big[\big(\sum_{i=1}^N (S_i+G_i)\big)^2\big]}{2\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]} + \rho\frac{\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}{2(1-\rho)}\right)\right.$$
$$\left. + \Big(\sum_{i=1}^N \rho_i^2\Big)\left(\frac{\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}{2(1-\rho)}\right)\right\} + \frac{\tilde{G}_j(\nu_j)}{1-\tilde{G}_j(\nu_j)}\left(\frac{\mathbb{E}\big[\sum_{i=1}^N (S_i+G_i)\big]}{(1-\rho)} - \mathbb{E}[G_j]\right). \tag{4.13}$$

We now consider various examples to compare the above approximation results with the exact analysis from [1] for deterministic glue periods and from Section 3 of the present paper for exponentially distributed glue periods. Further, we will compare the results of this approximation with simulation results for the case that the glue periods follow a gamma distribution.

Deterministic glue periods

In the numerical example of Table 1 we consider a two-station polling system. The switchover times and service times are exponentially distributed. We keep the parameters of station 1 fixed, $\lambda_1 = 1, \mathbb{E}[B_1] = 0.45, \mathbb{E}[S_1] = 1, G_1 = 0.5, \nu_1 = 1$, and vary the parameters of station 2.

Exponential glue periods

In the numerical example of Table 2 we consider a three-station polling system. The switchover times are deterministic, and the service times are exponentially distributed. We keep the parameters of station 1 fixed, $\lambda_1 = 1, \mathbb{E}[B_1] = 0.45, \mathbb{E}[G_1] = 0.5$. Further, the switchover times and exponential retrial rates of all three stations are fixed, $S_1 = S_2 = S_3 = 1$ and $\nu_1 = \nu_2 = \nu_3 = 1$.

Gamma distributed glue periods

In the above two examples we can get the exact mean waiting times using the method in [1] and Section 3 of this paper. In the numerical examples of Table 3 we compare the approximate mean waiting times

λ_2	$\mathbb{E}[B_2]$	$\mathbb{E}[S_2]$	G_2	ν_2	Exact $(\mathbb{E}[W_1], \mathbb{E}[W_2])$	Approx $(\mathbb{E}[W_1], \mathbb{E}[W_2])$
1	0.45	1	0.5	1	(71.61, 71.61)	(71.61, 71.61)
0.5	0.45	1	0.5	1	(21.44, 20.34)	(21.49, 20.24)
0.5	0.2	1	0.5	1	(15.18, 13.96)	(15.21, 13.83)
0.5	0.2	2	0.5	1	(20.52, 18.82)	(20.55, 18.71)
0.5	0.2	2	1	1	(23.01, 11.48)	(22.99, 11.67)
0.5	0.2	2	1	0.5	(22.97, 20.31)	(22.99, 20.20)

Table 1: Comparison of exact and approximate mean waiting times for a polling system with deterministic glue periods.

λ_2	$\mathbb{E}[B_2]$	$\mathbb{E}[G_2]$	λ_3	$\mathbb{E}[B_3]$	$\mathbb{E}[G_3]$	Exact $(\mathbb{E}[W_1], \mathbb{E}[W_2], \mathbb{E}[W_3])$	Approx $(\mathbb{E}[W_1], \mathbb{E}[W_2], \mathbb{E}[W_3])$
1	0.3	0.5	1	0.3	0.5	(121.0, 121.0, 121.0)	(121.0, 121.0, 121.0)
1	0.3	0.5	0.5	0.3	0.5	(47.59, 47.58, 46.74)	(47.71, 47.71, 46.24)
1	0.3	0.5	0.5	0.1	0.5	(33.65, 33.64, 32.54)	(33.69, 33.69, 31.97)
2	0.3	0.5	0.5	0.1	0.5	(246.8, 246.6, 242.3)	(242.4, 257.1, 230.2)
2	0.15	0.5	0.5	0.1	0.5	(33.52, 33.51, 32.42)	(33.56, 33.56, 31.86)
2	0.15	2	0.5	0.1	0.5	(44.88, 19.71, 43.64)	(45.22, 19.50, 42.92)
2	0.15	2	0.5	0.1	1	(48.66, 21.42, 28.75)	(49.03, 21.17, 27.98)

Table 2: Comparison of exact and approximate mean waiting times for a polling system with exponentially distributed glue periods.

with simulation results, for a polling system where the lengths of glue periods are gamma distributed.

We consider a five-station polling system in which the glue periods, switchover times and service times are all gamma distributed. We simulate such a system to find the mean waiting times. We also give a 95% confidence interval for the mean waiting times obtained using simulations. We have generated one million cycles, splitting this into ten periods of 10^5 cycles, and using the results of these ten periods to obtain confidence intervals. Then we compare the simulation results with the results obtained using the approximation formula. Here, k and θ are, respectively, the shape and the scale parameters of the gamma distribution with probability density function $\frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$.

The values of the parameters are listed in Table 3(a). Table 3(b) shows the mean waiting times by simulation, along with 95% lower and upper confidence bounds. Table 3(c) shows the approximate mean waiting times. We can draw the following conclusions about the mean waiting time approximation.

Parameters	(i)	(ii)	(iii)	(iv)
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	(0.1, 0.1, 0.1, 0.1, 0.1)	(0.1, 0.1, 0.1, 0.1, 0.2)	(0.1, 0.1, 0.1, 0.3, 0.2)	(0.1, 0.2, 0.1, 0.2, 0.2)
$(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$	(2.0, 2.0, 2.0, 2.0, 2.0)	(2.0, 2.0, 2.0, 2.0, 3.0)	(2.0, 2.0, 2.0, 1.0, 3.0)	(2.0, 5.0, 2.0, 4.0, 3.0)
$(k_{B_1}, k_{B_2}, k_{B_3}, k_{B_4}, k_{B_5})$	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 0.8)	(1.0, 1.0, 1.0, 0.5, 0.8)	(1.0, 0.5, 1.5, 0.5, 0.8)
$(\theta_{B_1}, \theta_{B_2}, \theta_{B_3}, \theta_{B_4}, \theta_{B_5})$	(1.5, 1.5, 1.5, 1.5, 1.5)	(1.5, 1.5, 1.5, 1.5, 1.5)	(1.5, 1.5, 1.5, 1.5, 1.5)	(1.5, 1.5, 1.5, 1.5, 1.5)
$(k_{S_1}, k_{S_2}, k_{S_3}, k_{S_4}, k_{S_5})$	(2.0, 2.0, 2.0, 2.0, 2.0)	(2.0, 2.0, 2.0, 2.0, 1.0)	(2.0, 2.0, 2.0, 3.0, 1.0)	(2.0, 5.0, 2.0, 3.0, 1.0)
$(\theta_{S_1}, \theta_{S_2}, \theta_{S_3}, \theta_{S_4}, \theta_{S_5})$	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 1.0)
$(k_{G_1}, k_{G_2}, k_{G_3}, k_{G_4}, k_{G_5})$	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 2.0)	(1.0, 1.0, 1.0, 0.5, 2.0)	(1.0, 3.0, 1.0, 0.5, 2.0)
$(\theta_{G_1}, \theta_{G_2}, \theta_{G_3}, \theta_{G_4}, \theta_{G_5})$	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0, 1.0)

(a) The parameter values for the four cases.

Cases	$(\mathbb{E}[W_1], \mathbb{E}[W_2], \mathbb{E}[W_3], \mathbb{E}[W_4], \mathbb{E}[W_5])$	95% lower confidence bound	95% upper confidence bound
(i)	(68.94, 68.92, 68.87, 68.91, 68.84)	(68.65, 68.64, 68.60, 68.58, 68.54)	(69.23, 69.21, 69.14, 69.24, 69.15)
(ii)	(108.59, 108.47, 108.40, 108.28, 72.43)	(107.86, 107.66, 107.58, 107.50, 71.99)	(109.33, 109.27, 109.23, 109.06, 72.87)
(iii)	(217.54, 218.44, 219.51, 548.58, 144.61)	(216.04, 217.11, 218.01, 544.87, 143.68)	(219.04, 219.77, 221.01, 552.28, 145.54)
(iv)	(276.39, 158.43, 283.36, 343.52, 183.40)	(275.30, 157.84, 282.15, 342.07, 182.71)	(277.47, 159.01, 284.57, 344.97, 184.09)

(b) Simulation results.

Cases	Approx $(\mathbb{E}[W_1], \mathbb{E}[W_2], \mathbb{E}[W_3], \mathbb{E}[W_4], \mathbb{E}[W_5])$
(i)	(69.00, 69.00, 69.00, 69.00, 69.00)
(ii)	(108.25, 108.25, 108.25, 108.25, 72.84)
(iii)	(210.22, 210.22, 210.22, 566.37, 140.93)
(iv)	(274.10, 155.12, 284.14, 348.72, 182.01)

(c) Approximation results.

Table 3: Comparison of mean waiting times from simulations and approximate mean waiting times for a polling system with gamma distributed glue periods.

- The mean waiting time approximation is very accurate. In only two cases (the fourth case in Table 2 and the case (iii) in Table 3) the error is in the order of 5%; in all other cases, we find errors which typically are less than 2%.
- The mean waiting time approximation at one station is independent of the change in retrial rates of other stations, which is not true in reality.
- The mean waiting time approximations for two totally symmetric stations are the same, independent of their order in the system; but this is also not quite true in reality.

4.4 Optimal choice of the glue period distributions

In this subsection we discuss an optimization problem for the choice of the distributions of the glue periods, G_i, $i = 1, \ldots, N$, to minimize the weighted sum of the mean waiting times $\sum_{i=1}^N c_i \mathbb{E}[W_i]$, subject to the constraint $\sum_{i=1}^N \mathbb{E}[G_i] = L$, where c_i, $i = 1, \ldots, N$, and L are positive constants. Because we do not have an explicit formula for the mean waiting time, it is difficult to solve exactly the constrained minimization problem. Instead of finding the exact solution of the constrained minimization problem, we will find the optimal choice of the distributions of G_i, $i = 1, \ldots, N$, to

$$\begin{aligned} &\text{minimize} \qquad \sum_{i=1}^N c_i U_i \\ &\text{subject to} \qquad \sum_{i=1}^N \mathbb{E}[G_i] = L, \end{aligned}$$

where U_i is the approximation of $\mathbb{E}[W_i]$ given by the right-hand side of (4.13). Note that under the constraint $\sum_{i=1}^N \mathbb{E}[G_i] = L$, the objective function of the minimization problem becomes

$$\begin{aligned} \sum_{i=1}^N c_i U_i =& \sum_{i=1}^N \frac{c_i(1+\rho_i)}{\rho + \sum_{j=1}^N \rho_j^2} \left[\rho \left(\frac{\sum_{j=1}^N \lambda_j \mathbb{E}[B_j^2]}{2(1-\rho)} + \frac{\mathbb{E}[(\sum_{j=1}^N S_j)^2] + 2L \sum_{j=1}^N \mathbb{E}[S_j] + \mathbb{E}[(\sum_{j=1}^N G_j)^2]}{2\mathbb{E}[\sum_{j=1}^N S_j + L]} \right) \right. \\ &\left. + \frac{\mathbb{E}[\sum_{j=1}^N S_j + L]}{2(1-\rho)} \left(\rho^2 + \sum_{j=1}^N \rho_j^2 \right) \right] + \sum_{i=1}^N \frac{c_i \mathbb{E}[e^{-\nu_i G_i}]}{1 - \mathbb{E}[e^{-\nu_i G_i}]} \left(\frac{\mathbb{E}\left[\sum_{j=1}^N S_j + L\right]}{1-\rho} - \mathbb{E}[G_i] \right). \quad (4.14) \end{aligned}$$

By Jensen's inequality, it can be shown that if the nondeterministic glue period distributions with means g_i, are changed to the degenerate (deterministic) ones with the same means g_i, $i = 1, \ldots, N$, then the right-hand side of (4.14) becomes strictly smaller. Therefore, the above optimization problem becomes as follows:

$$\begin{aligned} &\text{minimize} \qquad U(g_1, \ldots, g_N) \\ &\text{subject to} \end{aligned}$$

$$g_i > 0, \quad i = 1, \ldots, N, \quad (4.15)$$

$$\sum_{i=1}^N g_i = L, \quad (4.16)$$

where

$$
\begin{aligned}
&U(g_1,\ldots,g_N)\\
&=\sum_{i=1}^{N}\frac{c_i(1+\rho_i)}{\rho+\sum_{j=1}^{N}\rho_j^2}\Bigg[\rho\Bigg(\frac{\sum_{j=1}^{N}\lambda_j\mathbb{E}[B_j^2]}{2(1-\rho)}+\frac{\mathbb{E}[(\sum_{j=1}^{N}S_j)^2]+2L\sum_{j=1}^{N}\mathbb{E}[S_j]+L^2]}{2\mathbb{E}[\sum_{j=1}^{N}S_j+L]}\Bigg)\\
&\quad+\frac{\sum_{j=1}^{N}\mathbb{E}[S_j]+L}{2(1-\rho)}\Big(\rho^2+\sum_{j=1}^{N}\rho_j^2\Big)\Bigg]+\sum_{i=1}^{N}c_i\Big(-1+\frac{1}{1-e^{-\nu_i g_i}}\Big)\Bigg(\frac{\sum_{j=1}^{N}\mathbb{E}[S_j]+L}{1-\rho}-g_i\Bigg).
\end{aligned}
\tag{4.17}
$$

Since $U(g_1,\ldots,g_N)$ is continuous on $\mathcal{D}\equiv\{(g_1,\ldots,g_N): g_1>0,\ldots,g_N>0, g_1+\cdots+g_N=L\}$ and $U(g_1,\ldots,g_N)\to\infty$ as $\min\{g_1,\ldots,g_N\}\to 0+$, $U(g_1,\ldots,g_N)$ takes a minimum at a point in $\mathcal{D}$. At a minimum point $(g_1,\ldots,g_N)$, there exists a Lagrange multiplier κ satisfying

$$
f_i(g_i)=\kappa,\quad i=1,\ldots,N, \tag{4.18}
$$

where

$$
f_i(g_i)\equiv c_i-\frac{c_i}{1-e^{-\nu_i g_i}}-\frac{c_i\nu_i e^{-\nu_i g_i}}{(1-e^{-\nu_i g_i})^2}\Bigg(\frac{\sum_{j=1}^{N}\mathbb{E}S_j+L}{1-\rho}-g_i\Bigg),\quad i=1,\ldots,N.
$$

For each $i=1,\ldots,N$, the function $f_i:(0,L)\to(-\infty,f_i(L))$ is bijective, continuous and strictly increasing. Therefore, it has the inverse function $h_i:(-\infty,f_i(L))\to(0,L)$, which is also continuous and strictly increasing. Therefore, Equation (4.18) and the constraints (4.15) and (4.16) can be written as

$$
\sum_{j=1}^{N}h_j(\kappa)=L,\quad -\infty<\kappa<\min\{f_1(L),\ldots,f_N(L)\}, \tag{4.19}
$$

$$
g_i=h_i(\kappa),\quad i=1,\ldots,N. \tag{4.20}
$$

Since $\lim_{\kappa\to-\infty}\sum_{j=1}^{N}h_j(\kappa)=0$, $\lim_{\kappa\to(\min\{f_1(L),\ldots,f_N(L)\})-}\sum_{j=1}^{N}h_j(\kappa)>L$ and $\sum_{j=1}^{N}h_j(\kappa)$ is strictly increasing in κ, (4.19) has a unique solution, say κ^*. Therefore, from (4.20), the optimal solution $(g_1^*,\ldots,g_N^*)$ is given by

$$
g_i^*=h_i(\kappa^*),\quad i=1,\ldots,N.
$$

We will now consider a few numerical examples to look at the dependency of different system characteristics and the respective optimal glue periods. In [2] a similar system was studied with a focus on optical switches, where the revenue of the system depended on distributing glue periods optimally to each station. In these examples we will look at the problem of minimizing $\sum_{i=1}^{N} c_i U_i$, that is the weighted waiting cost of the system given that the sum of expected values of glue periods is fixed. Since the optimization problem showed that the system performs best when the glue periods are deterministic, we will only consider models with deterministic glue periods.

We consider a three-station model and in each case vary one parameter to study how the system performs under certain changes. In all the cases the sum of the lengths of deterministic glue periods is fixed, $L = 3$, and the service times and the switchover times are exponentially distributed. The switchover times are symmetric and fixed for all three stations, i.e. $\mathbb{E}[S_i] = 2$ for all $i = 1, 2, 3$.

(i) Case 1: In this case we keep all system parameters symmetric except the arrival rate λ_i of each station. Let $\nu_i = 1$, $\mathbb{E}[B_i] = 1$ and $c_i = 1$ for all $i = 1, 2, 3$. In Table 4 we show the optimal values of g_1, g_2, g_3 and $\sum_{i=1}^{N} c_i U_i$ for different values of λ_i.

(ii) Case 2: In this case we keep all system parameters symmetric except the mean service time $\mathbb{E}[B_i]$ of each station. Let $\lambda_i = 1$, $\nu_i = 1$, and $c_i = 1$ for all $i = 1, 2, 3$. In Table 5 we show the optimal values of g_1, g_2, g_3 and $\sum_{i=1}^{N} c_i U_i$ for different values of $\mathbb{E}[B_i]$.

(iii) Case 3: In this case we keep all system parameters symmetric except the retrial rate ν_i of each station. Let $\lambda_i = 0.25$, $\mathbb{E}[B_i] = 1$ and $c_i = \rho_i$ for all $i = 1, 2, 3$. In Table 6 we show the optimal values of g_1, g_2, g_3 and $\sum_{i=1}^{N} c_i U_i$ for different values of ν_i. Note that in this case $\sum_{i=1}^{N} c_i U_i = \sum_{i=1}^{N} \rho_i \mathbb{E}[W_i]$.

(iv) Case 4: In this case we keep all system parameters symmetric except the weight c_i of each station. Let $\lambda_i = 0.25$, $\mathbb{E}[B_i] = 1$ and $\nu_i = 1$ for all $i = 1, 2, 3$. In Table 7 we show the optimal values of g_1, g_2, g_3 and $\sum_{i=1}^{N} c_i U_i$ for different values of c_i.

We can draw the following conclusions about the optimal allocation of glue periods using the above method.

- The allocation doesn't depend on the arrival rate or mean service time of a station. This is due to the following observation: The first term on the right-hand side of Equation (4.17) is independent of g_i and the second term is independent of arrival rates and mean service times. This might not be the case in exact analysis.

- The higher the retrial rate, the shorter the length of the glue period assigned to the station.
- The higher the weight allocated to a station, the bigger the length of the glue period assigned to the station. This helps us in scenarios when a waiting cost is associated with stations.

λ_1	λ_2	λ_3	g_1	g_2	g_3	$\sum_{i=1}^{N} c_i U_i$
0.3	0.3	0.3	1	1	1	359.898
0.3	0.2	0.2	1	1	1	115.063
0.3	0.2	0.1	1	1	1	84.362

Table 4: Optimal length of glue periods for different arrival rates.

$\mathbb{E}[B_1]$	$\mathbb{E}[B_2]$	$\mathbb{E}[B_3]$	g_1	g_2	g_3	$\sum_{i=1}^{N} c_i U_i$
0.3	0.3	0.3	1	1	1	422.888
0.3	0.2	0.2	1	1	1	137.887
0.3	0.2	0.1	1	1	1	101.876

Table 5: Optimal length of glue periods for different mean service times.

ν_1	ν_2	ν_3	g_1	g_2	g_3	$\sum_{i=1}^{N} c_i U_i$
3	3	3	1.0000	1.0000	1.0000	84.001
3	2	2	0.8340	1.0830	1.0830	90.680
3	2	1	0.7134	0.9157	1.3710	101.679

Table 6: Optimal length of glue periods for different retrial rates.

c_1	c_2	c_3	g_1	g_2	g_3	$\sum_{i=1}^{N} c_i U_i$
3	3	3	1.0000	1.0000	1.0000	418.823
3	2	2	1.1268	0.9366	0.9366	323.736
3	2	1	1.2311	1.0263	0.7426	271.086

Table 7: Optimal length of glue periods for different weights.

5 Suggestions for further research

In this paper we have studied a gated polling model with the special features of retrials and glue, or reservation periods. For the case of exponentially distributed glue periods, we have presented an algorithm to obtain the moments of the number of customers in each station. We would like to point out that phase-type glue periods can in principle be handled by the same method.

For generally distributed glue periods, we have obtained an expression for the steady-state distribution of the total workload in the system, and we have used it to derive a pseudo conservation law for a weighted sum of the mean waiting times, which in turn led us to an accurate approximation of the individual mean waiting times. A topic for further research is to analyze the exact waiting time distribution, for exponentially distributed glue periods and for constant glue periods.

The introduction of the concept of glue period was motivated by the wish to obtain insight into the performance of certain switches in optical communication systems. We have considered the optimal choice of the glue period lengths, under the constraint that the total glue period length per cycle is fixed. A topic for further study is the unconstrained counterpart to this optimization problem; a complication one then faces is that the objective function for the optimization can be nonconvex. In fact, it is possible that the Hessian of the objective function is not positive semi-definite even for the two-station system. However it still seems to be intuitively natural that there will exist a unique solution for the optimization problem.

Not restricting ourselves to optical communications, one can also interpret a glue period as a reservation period - a window of opportunity for claiming service at the next visit of the server to a station.

It would be interesting to study reservation periods in more detail, and in particular to consider the problem of choosing reservation periods in such a way that some objective function is optimized.

Appendix: Approximation of the mean waiting times

Below we outline a method to approximate the mean waiting times of all customer types. The arrival of a type-i customer occurs either during a glue period of station i or during any other period. At the start of the visit period the customers which will be served in the current visit period are fixed. The mean length of the visit period is now the same irrespective of the order in which these customers are served. Without loss of generality, we will assume that the customers who arrive during a glue period of station i are served first and then customers who retry are served.

Let $\bar{W}_i$ and $\tilde{W}_i$ denote the waiting time of type-i customers who arrive during a glue period of station i and any other period, respectively. Further, $G_{i_{res}}$ denotes the residual time of a glue period of station i. Finally $C_{i_{res}}$ denotes the residual time of a non-glue period of station i. A type-i customer arriving during a glue period of station i has to wait for the residual glue period. Further, it has to wait for all the customers who arrived before it during the glue period. Therefore

$$\mathbb{E}[\bar{W}_i] = \mathbb{E}[G_{i_{res}}] + \rho_i \mathbb{E}[G_{i_{res}}] = (1 + \rho_i)\mathbb{E}[G_{i_{res}}].$$

A type-i customer arriving during a non-glue period of station i has to wait for the residual non-glue period, and the glue period. Then it either gets in the queue for service or it remains in the orbit. With probability $\tilde{G}_i(\nu_i)$ it remains in the orbit and has to wait until the next visit to get served, and this repeats. Hence, on average, it has to wait for $\tilde{G}_i(\nu_i)/(1 - \tilde{G}_i(\nu_i))$ cycles before it gets into the queue for service. When it gets in the queue it has to wait for all the type-i customers who have arrived during the glue period to get served, and then the customers who arrived before it and who will be served in the current visit period (on average this number is approximately equal to the number of customers who arrived during the residual non-glue period before the arrival of the tagged customer). Therefore

$$\begin{aligned}\mathbb{E}[\tilde{W}_i] &\approx \mathbb{E}[C_{i_{res}}] + \mathbb{E}[G_i] + \frac{\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}\mathbb{E}[C] + \rho_i\mathbb{E}[G_i] + \rho_i\mathbb{E}[C_{i_{res}}] \\ &= (1+\rho_i)\left(\mathbb{E}[C_{i_{res}}] + \mathbb{E}[G_i]\right) + \frac{\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}\mathbb{E}[C].\end{aligned}$$

The probability that a type-i customer arrives during a glue period of station i is $\mathbb{E}[G_i]/\mathbb{E}[C]$, and the probability that it arrives during a non-glue period equals $1 - (\mathbb{E}[G_i]/\mathbb{E}[C])$. Therefore

$$\begin{aligned}\mathbb{E}[W_i] &= \frac{\mathbb{E}[G_i]}{E[C]}\mathbb{E}[\bar{W}_i] + \frac{\mathbb{E}[C]-\mathbb{E}[G_i]}{\mathbb{E}[C]}\mathbb{E}[\tilde{W}_i] \\ &\approx (1+\rho_i)\left(\frac{\mathbb{E}[G_i]}{E[C]}\mathbb{E}[G_{i_{res}}] + \frac{\mathbb{E}[C]-\mathbb{E}[G_i]}{\mathbb{E}[C]}\left(\mathbb{E}[C_{i_{res}}] + \mathbb{E}[G_i]\right)\right) + \frac{\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}\left(\mathbb{E}[C] - \mathbb{E}[G_i]\right).\end{aligned}$$

Let R_{c_i} be the residual cycle time of the system with respect to station i. Then

$$\mathbb{E}[R_{c_i}] = \frac{\mathbb{E}[G_i]}{E[C]}\mathbb{E}[G_{i_{res}}] + \frac{\mathbb{E}[C]-\mathbb{E}[G_i]}{\mathbb{E}[C]}\left(\mathbb{E}[C_{i_{res}}] + \mathbb{E}[G_i]\right), \quad i = 1, \cdots, N.$$

We assume that $\mathbb{E}[R_{c_i}] = \mathbb{E}[R_c]$ for all $i = 1, \ldots, N$. We thus obtain (4.11):

$$\mathbb{E}[W_i] \approx (1+\rho_i)\mathbb{E}[R_c] + \frac{\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}\left(\mathbb{E}[C] - \mathbb{E}[G_i]\right).$$

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