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Analysis of $M^x/G/1$ queues with impatient customers

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Abstract This paper considers a batch arrival $M^x/G/1$ queue with impatient customers. We consider two different model variants. In the first variant, customers in the same batch are assumed to have the same impatience time, and impatience times associated with batches are i.i.d. according to a general distribution. In the second variant, impatience times of customers in the same batch are independent, and they follow a general distribution. Both variants are related to an M/G/1 queue in which the service time of a customer depends on its waiting time. Our main focus is on the virtual and actual waiting times, and on the loss probability of customers.

Keywords Batch-arrival queue · Impatient customers · Virtual waiting time · Actual waiting time · Loss probability · Busy period

Mathematics Subject Classification (2000) 60K25 · 60J25

1 Introduction

The performance analysis of queueing systems with impatience has recently experienced a surge of interest (cf., e.g., the special issue [15]). On the one hand, this is motivated by advances in approximating large queueing systems

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with abandonments via many-server asymptotics and related diffusion approximations. On the other hand, it is triggered by the strong connection between queueing systems with impatience and various application areas, ranging from call centers and health care to perishable inventories and organ transplantation systems.

We consider batch-arrival M/G/1 queues with impatient customers. Batches of customers arrive according to a Poisson process with rate λ, and the number of customers in each batch follows a general discrete probability distribution with probability function p_n $(n = 1, 2, \ldots)$ and mean $\mathrm{E}[B]$. Service times of customers are independent and identically distributed (i.i.d.) with probability distribution function (PDF) $G(x)$ $(x \geq 0)$ and mean $\mathrm{E}[G]$. We assume that the service time distribution has a probability density function (p.d.f.) $g(x)$ $(x \geq 0)$. Customers are served under the first-come first-served (FCFS) service discipline, unless otherwise mentioned.

Each customer has her own maximum allowable waiting time, referred to as the impatience time. If the elapsed waiting time of a customer reaches her impatience time, she leaves the system immediately without receiving her service. We assume that once a customer starts to receive her service, she remains in the system until the end of her service, even if her impatience time is expired. We consider two different models of batch-arrival impatient customers. In the first model, customers in the same batch are assumed to have the same impatience time, and impatience times associated with batches are i.i.d. according to a general distribution. In the second model, on the other hand, impatience times of customers in the same batch are independent, and they follow a general distribution. Throughout this paper, we denote the first model by $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$, and the second model by $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$. For both models, we denote the PDF of the impatience time by the same notation $H(x)$ $(x \geq 0)$. We assume that customers may have infinite impatience time (i.e., such customers are patient), so that $H(x)$ may be defective:

$$\lim_{x \to \infty} H(x) = 1 - h_{\infty}. \tag{1}$$

In the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$ queue, $h_{\infty} \in [0, 1)$ represents the probability that a randomly chosen batch consists of patient customers, while in the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queue, it represents the probability that a randomly chosen customer is patient.

Related literature. The case of batch arrivals in a queueing system with impatience has not yet received much attention. A key reference is [20], which considers the M^{x}/M/c+D queue, and which focuses on the loss probability, providing an exact expression for it in terms of the waiting time distribution of the ordinary M^{x}/M/c queue. We refer to p. 364 of [16] for an overview of the literature of the single server queue with impatience. See [1, 2, 14, 26] for GI/G/1+G, [12] for GI/G/1+D, [23] for M/G/1+M, [8, 17] for M/G/1+PH, [3, 12, 14] for M/G/1+D, [1, 2, 5] for M/G/1+G. All these references concern impatience w.r.t. waiting time. Impatience w.r.t. sojourn time is being considered in [8], and also in [5]; to distinguish between the two, one might use the notations $+\mathrm{G}^{\mathrm{w}}$ and $+\mathrm{G}^{\mathrm{s}}$ for impatience w.r.t. waiting and sojourn time, respectively. An exact analysis of queueing systems with impatience and multiple servers is not only given in [20], but also, e.g., in [4,7]. See, furthermore, [6] for bounds and approximations for the loss probability, [19] for waiting time approximations and, e.g., [30] for asymptotics and [28] for a diffusion control problem for a multiserver queue for which the tradeoff between blocking and abandonment is studied.

Main results. A key observation of the paper is, that the virtual waiting time (workload) in both the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$ and $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queue can be viewed as the workload in an M/G/1 queue in which the service time of a customer depends on its waiting time y in a particular way (PDF $G(w \mid y)$). We derive an expression for the workload density in that M/G/1 queue. We also express the loss probability in the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$ and $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queue into that workload density. In subsequent sections $G(w \mid y)$ is worked out in detail for the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$ and $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queue. We thus derive the distribution of the steady-state workload and waiting time, and also the loss probability, in those batch arrival models with impatience.

Notation. For convenience of the notation, we define $\overline{p}_n$ $(n = 1, 2, \ldots)$, $\overline{G}(x)$ $(x \geq 0)$, and $\overline{H}(x)$ $(x \geq 0)$ as the complementary PDFs given by

$$\overline{p}_n = \sum_{k=n}^{\infty} p_k, \tag{2}$$
$$\overline{G}(x) = 1 - G(x), \qquad \overline{H}(x) = 1 - H(x),$$

respectively. Note here that

$$\overline{p}_1 = 1. \tag{3}$$

We define $g^{(n)}(x)$ $(x \geq 0,\ n = 1, 2, \ldots)$ as the n-fold convolution of $g(x)$, and $G^{(n)}(x)$ $(x \geq 0,\ n = 1, 2, \ldots)$ as the corresponding PDF.

$$g^{(1)}(x) = g(x), \qquad g^{(n)}(x) = g^{(n-1)} * g(x), \quad n = 2, 3, \ldots,$$
$$G^{(n)}(x) = \int_0^x g^{(n)}(y)dy.$$

We also define ρ as the traffic intensity.

$$\rho = \lambda \mathrm{E}[B]\mathrm{E}[G].$$

To avoid inessential complications, we assume $G(0) = 0$ and $H(0) = 0$, i.e., there are no customers with zero service times or impatience times. Empty sum and empty product terms are defined as zero and one, respectively.

Organization of the paper. In Section 2, we discuss properties of the virtual waiting time and the loss probability which the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$ and $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queues have in common. In Section 3, we focus on the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$

queue. We consider the stability condition, the number of losses, and the actual waiting time distribution. The obtained expressions simplify considerably for the special cases (geometric batch size, exponential service times, exponential patience) which are treated in Subsection 3.2. The $\mathrm{M^x/G/1{+}G_{diff}}$ queue, which is considered in Section 4, is considerably more complicated than the $\mathrm{M^x/G/1{+}G_{same}}$ queue; we have to be satisfied with results for the $\mathrm{M^{geo}/G/1{+}G_{diff}}$ queue (virtual waiting time, number of losses) and for the $\mathrm{M^x/M/1{+}M_{diff}}$ queue. In the latter case, we can determine the queue length and virtual waiting time distribution, as well as the number of losses.

Section 5 is devoted to the $\mathrm{M^x/G/1{+}D}$ queue, i.e., a queue with constant impatience, which is a special case of both the $\mathrm{M^x/G/1{+}G_{same}}$ and the $\mathrm{M^x/G/1{+}G_{diff}}$ queue. We obtain the distributions of virtual and actual waiting time. In addition, we conduct an analysis of the busy period. Finally, we conclude this paper in Section 6.

2 Unified observation for virtual waiting time and loss probability

In this section, we briefly discuss some properties of the virtual waiting time (workload) and the loss probability which the $\mathrm{M^x/G/1{+}G_{same}}$ and $\mathrm{M^x/G/1{+}G_{diff}}$ queues have in common. Regarding each batch as a single customer, we obtain single-arrival queueing models whose virtual waiting time processes are identical to those of the original $\mathrm{M^x/G/1{+}G_{same}}$ and $\mathrm{M^x/G/1{+}G_{diff}}$ queues. It is readily verified that each of these single-arrival queueing models can be formulated as an M/G/1 queue with service times dependent on waiting times [22], which have the following features.

(i) Customers arrive according to a Poisson process with rate λ,
(ii) All customers are patient, i.e., they do not leave the system before their service completion, and
(iii) The service time of an arriving customer who finds y ($y \geq 0$) amount of work in system is distributed according to a PDF $G(w \mid y)$ ($w \geq 0$).

We define the mean service time of a customer who finds y amount of work in the system, $\beta(y)$, ($y \geq 0$) as

$$\beta(y) = \int_{w=0}^{\infty} w dG(w \mid y). \tag{4}$$

In the M/G/1 queue with service times dependent on waiting times, the system is stable if the following conditions are satisfied.

$$\lambda\beta(y) < \infty \quad \text{for all } y \geq 0, \qquad \limsup_{y\to\infty} \lambda\beta(y) < 1. \tag{5}$$

Remark 1 [22] states that this stability condition is proved in [11]. Intuitively, (5) implies that the mean amount of work brought into the system per time unit is less than one if the workload in system is sufficiently large. Note that a similar stability condition is derived in Theorem 10.2 of [21] for an M/G/1 queue where both the arrival rate and the service time distribution are dependent on the workload.

Once the two-variable function $G(w \mid y)$ $(w \geq 0,\ y \geq 0)$ is given, and if it satisfies (5), we can characterize the stationary virtual waiting time of this single-arrival model as follows. To make the notation simpler, we define $\overline{G}(w \mid y)$ $(w \geq 0,\ y \geq 0)$ as

$$\overline{G}(w \mid y) = 1 - G(w \mid y).$$

Let $v(x)$ $(x > 0)$ denote the PDF of the stationary virtual waiting time, and let π_0 denote the stationary probability that the system is empty. Note that

$$\pi_0 + \int_{0+}^{\infty} v(x)dx = 1. \tag{6}$$

Using the level-crossing argument [9, 13], we have

$$v(x) = \lambda\pi_0\overline{G}(x \mid 0) + \lambda\int_{0+}^{x} v(y)\overline{G}(x-y \mid y)dy, \qquad x > 0. \tag{7}$$

This is a Volterra integral equation of the second kind, whose solution is uniquely given by

$$v(x) = \pi_0\sum_{n=1}^{\infty}\phi_n(x), \qquad x > 0, \tag{8}$$

where $\{\phi_n(x), x > 0\}_{n=1,2,\ldots}$ is a sequence of functions defined as

$$\phi_1(x) = \lambda\overline{G}(x \mid 0), \qquad x > 0, \tag{9}$$

$$\phi_n(x) = \lambda\int_{0+}^{x}\phi_{n-1}(y)\overline{G}(x-y \mid y)dy, \qquad x > 0,\ n = 2, 3, \ldots. \tag{10}$$

Furthermore, π_0 is obtained from (6):

$$\pi_0 = \left[1 + \sum_{n=1}^{\infty}c_n\right]^{-1}, \tag{11}$$

where

$$c_n = \int_{0+}^{\infty}\phi_n(x)dx, \qquad n = 1, 2, \ldots.$$

Therefore, once $\overline{G}(w \mid y)$ $(w \geq 0,\ y \geq 0)$ of the $\mathrm{M^x/G/1{+}G_{same}}$ and $\mathrm{M^x/G/1{+}G_{diff}}$ queues are derived, π_0 and $v(x)$ $(x \geq 0)$ in these models are immediately obtained. In addition, we can also obtain the loss probability based on this. For each of the $\mathrm{M^x/G/1{+}G_{same}}$ and $\mathrm{M^x/G/1{+}G_{diff}}$ queues, let P_{loss} denote the stationary loss probability, i.e., the probability that a randomly chosen customer leaves the system without receiving service. The following theorem shows that P_{loss} is given in terms of π_0, as is the case for the single-arrival M/G/1+G queue [14].

Theorem 1 *In each of the* $\mathrm{M^x/G/1+G_{same}}$ *and* $\mathrm{M^x/G/1+G_{diff}}$ *queues,* P_{loss} *is given by*

$$P_{\mathrm{loss}} = 1 - \frac{1-\pi_0}{\rho}. \tag{12}$$

Proof Applying Little's law to the server, we obtain

$$1 - \pi_0 = \lambda \mathrm{E}[B] \cdot (1 - P_{\mathrm{loss}}) \cdot \mathrm{E}[G],$$

which implies (12). □

Theorem 1 can be intuitively understood by rewriting (12) into

$$1 - P_{\mathrm{loss}} = \frac{1-\pi_0}{\rho}.$$

The right-hand side of this equation represents the ratio of the amount per time unit of processed workload $1-\pi_0$ and that of offered workload ρ. When customers arrive one by one, i.e., in the M/G/1+G queue, $1 - P_{\mathrm{loss}}$ is also equal to

$$\pi_0 + \int_{0+}^{\infty} v(y)\overline{H}(y)dy. \tag{13}$$

The following corollary is an extension of this relation to the $\mathrm{M^x/G/1+G_{same}}$ and $\mathrm{M^x/G/1+G_{diff}}$ queues.

Corollary 1 P_{loss} *satisfies*

$$1 - P_{\mathrm{loss}} = \frac{\lambda\pi_0\beta(0) + \lambda \displaystyle\int_{y=0+}^{\infty} v(y)\beta(y)dy}{\rho}, \tag{14}$$

where $\beta(y)$ $(y \geq 0)$ *is defined as (4).*

Proof Note that (4) implies

$$\begin{aligned}\beta(y) &= \int_{w=0}^{\infty} dG(w \mid y) \int_{t=0}^{w} dt \\ &= \int_{t=0}^{\infty} dt \int_{w=t}^{\infty} dG(w \mid y) \\ &= \int_{t=0}^{\infty} \overline{G}(t \mid y)dt, \qquad y \geq 0.\end{aligned} \tag{15}$$

Taking the integral of both sides of (7) over $x \in (0, \infty)$, we obtain

$$\begin{aligned}1 - \pi_0 &= \lambda\pi_0\beta(0) + \lambda \int_{x=0+}^{\infty} dx \int_{y=0+}^{x} \overline{G}(x-y \mid y)v(y)dy \\ &= \lambda\pi_0\beta(0) + \lambda \int_{y=0+}^{\infty} v(y)dy \int_{x=y}^{\infty} \overline{G}(x-y \mid y)dx\end{aligned}$$

$$= \lambda \left(\pi_0 \beta(0) + \int_{y=0+}^{\infty} v(y)\beta(y)dy \right).$$

It then follows from Theorem 1 that

$$\lambda \pi_0 \beta(0) + \lambda \int_{y=0+}^{\infty} v(y)\beta(y)dy = \rho(1 - P_{\mathrm{loss}}),$$

which implies (14). □

In the M/G/1+G queue, $\lambda\beta(y) = \rho\overline{H}(y)$ holds for $y \geq 0$, so that the right-hand side of (14) reduces to (13). Also, it is easy to see that the numerator of the right-hand side of (14) can be interpreted as the amount of processed workload per time unit, which is also equal to $1 - \pi_0$ as mentioned above.

We derive $\overline{G}(w \mid y)$ $(w \geq 0, y \geq 0)$ for the M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ and M$^{\mathrm{x}}$/G/1+G$_{\mathrm{diff}}$ queues in Sections 3 and 4, respectively. Using the above results, we then derive stability conditions of the original models, and obtain $v(x)$ $(x > 0)$, π_0, P_{loss}, and other performance measures. We also discuss the M$^{\mathrm{x}}$/G/1+D queue in Section 5, which is a special case of both M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ and M$^{\mathrm{x}}$/G/1+G$_{\mathrm{diff}}$ queues.

3 The M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ queue

In this section, we consider the M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ queue. We assume that customers of the same batch have the same impatience time, which is distributed according to a general distribution with the PDF $H(x)$ $(x \geq 0)$.

3.1 General case

We first derive $\overline{G}(w \mid y)$ $(w \geq 0,\ y \geq 0)$ defined in Section 2. Suppose an arriving batch sees A amount of work in the system. Let N_{loss} (resp. N_{admit}) denote the number of lost (resp. admitted) customers in this batch. Further let W_i $(i = 1, 2, \ldots, N_{\mathrm{admit}})$ denote the service time of the ith admitted customer. We define the joint conditional probability $F(n, k; w_1, w_2, \ldots, w_k \mid y)$ $(y \geq 0,$ $w_1 > 0, w_2 > 0, \ldots, w_k > 0,\ n = 0, 1, \ldots,\ k = 1, 2, \ldots)$ as

$$\begin{aligned} &F(n, k; w_1, w_2, \ldots, w_k \mid y) \\ &\quad = \Pr(N_{\mathrm{loss}} = n, N_{\mathrm{admit}} = k, W_1 \leq w_1, W_2 \leq w_2, \ldots, W_k \leq w_k \mid A = y). \end{aligned}$$

We further define $f(n, k; w_1, w_2, \ldots, w_k \mid y)$ $(y \geq 0,\ w_1 > 0, w_2 > 0, \ldots, w_k > 0,\ n = 0, 1, \ldots,\ k = 1, 2, \ldots)$ as

$$f(n, k; w_1, w_2, \ldots, w_k \mid y) = \frac{\partial^k F(n, k; w_1, w_2, \ldots, w_k \mid y)}{\partial w_1 \partial w_2 \cdots \partial w_k}. \tag{16}$$

Lemma 1 *$f(n,k;w_1,w_2,\ldots,w_k \mid y)$ ($y \geq 0$, $w_1 > 0, w_2 > 0, \ldots, w_k > 0$, $n = 0,1,\ldots$, $k = 1,2,\ldots$) is given by*

$$\begin{aligned}
&f(0,k;w_1,w_2,\ldots,w_k \mid y) \\
&\quad = p_k \overline{H}\Big(y + \sum_{m=1}^{k-1} w_m\Big) \prod_{i=1}^{k} g(w_i), \\
&f(n,k;w_1,w_2,\ldots,w_k \mid y) \\
&\quad = p_{n+k} \left[\overline{H}\Big(y + \sum_{m=1}^{k-1} w_m\Big) - \overline{H}\Big(y + \sum_{m=1}^{k} w_m\Big)\right] \prod_{i=1}^{k} g(w_i), \quad n = 1,2,\ldots.
\end{aligned}$$

Proof Lemma 1 follows from the definition of $f(n,k;w_1,w_2,\ldots,w_k \mid y)$. □

Lemma 2 *In the M*$^{\mathrm{X}}$*/G/1+G*$_{\mathrm{same}}$ *queue, $\overline{G}(w \mid y)$ ($w \geq 0$, $y \geq 0$) and $\beta(y)$ ($y \geq 0$) are given by*

$$\overline{G}(w \mid y) = \overline{H}(y)\overline{G}(w) + \int_0^w \overline{H}(y+u) \sum_{k=2}^{\infty} \overline{p}_k g^{(k-1)}(u) \overline{G}(w-u) du, \tag{17}$$

$$\beta(y) = \mathrm{E}[G] \sum_{k=1}^{\infty} \overline{p}_k \mathrm{E}\left[\overline{H}\Big(y + \sum_{m=1}^{k-1} G_m\Big)\right], \tag{18}$$

where $\{G_n\}_{n=1,2,\ldots}$ denotes a sequence of i.i.d. random variables distributed according to the service time distribution $G(x)$.

The proof of Lemma 2 is provided in Appendix A.

Theorem 2 *The M*$^{\mathrm{X}}$*/G/1+G*$_{\mathrm{same}}$ *queue is stable if*

$$\rho < \infty, \qquad \rho h_\infty < 1. \tag{19}$$

Proof The mean batch size $\mathrm{E}[B]$ is given in terms of $\overline{p}_n$ ($n = 1,2,\ldots$) by

$$\mathrm{E}[B] = \sum_{n=1}^{\infty} n p_n = \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} 1 = \sum_{i=1}^{\infty} \overline{p}_i.$$

It then follows from (18) that

$$\mathrm{E}[G]\mathrm{E}[B] h_\infty \leq \beta(y) \leq \mathrm{E}[G]\mathrm{E}[B]\overline{H}(y), \qquad y \geq 0,$$

which implies

$$\rho h_\infty \leq \lambda\beta(y) \leq \rho\overline{H}(y), \qquad y \geq 0,$$

and, cf. (1),

$$\lim_{y\to\infty} \lambda\beta(y) = \rho h_\infty.$$

Therefore, if (19) holds, the stability condition (5) is satisfied. □

In the rest of this section, we assume that (19) holds, so that the system is stable. Recall that under this assumption, the stationary virtual waiting time distribution is immediately obtained from (8) and (11).

We next consider the number of losses. Let $P_{\mathrm{loss}}(n)$ $(n = 0, 1, \ldots)$ denote the probability that the number of lost customers in a randomly chosen batch is equal to n. Also, let $P_{\mathrm{loss}}(n \mid x)$ $(x \geq 0,\ n = 0, 1, \ldots)$ denote the conditional probability that the number of lost customers in a randomly chosen batch is equal to n, given that this batch finds x amount of work on arrival. Owing to PASTA, $P_{\mathrm{loss}}(n)$ is given in terms of $P_{\mathrm{loss}}(n \mid x)$ by

$$P_{\mathrm{loss}}(n) = \pi_0 P_{\mathrm{loss}}(n \mid 0) + \int_{0+}^{\infty} P_{\mathrm{loss}}(n \mid x) v(x) dx, \qquad n = 0, 1, \ldots. \tag{20}$$

Theorem 3 *$P_{\mathrm{loss}}(n \mid x)$ $(n = 0, 1, \ldots,\ x \geq 0)$ is given by*

$$P_{\mathrm{loss}}(0 \mid x) = \sum_{k=1}^{\infty} p_k \mathrm{E}\left[\overline{H}\Big(x + \sum_{m=1}^{k-1} G_m\Big)\right], \tag{21}$$

$$P_{\mathrm{loss}}(n \mid x) = p_n H(x) + \sum_{k=1}^{\infty} p_{n+k} \mathrm{E}\left[\overline{H}\Big(x + \sum_{m=1}^{k-1} G_m\Big) - \overline{H}\Big(x + \sum_{m=1}^{k} G_m\Big)\right], \quad n = 1, 2, \ldots. \tag{22}$$

Proof Theorem 3 immediately follows from Lemma 1 and the following relations.

$$P_{\mathrm{loss}}(0 \mid x) = \sum_{k=1}^{\infty} \int_{w_1=0}^{\infty} \int_{w_2=0}^{\infty} \cdots \int_{w_k=0}^{\infty} f(0, k; w_1, w_2, \ldots, w_k \mid x) \cdot dw_k dw_{k-1} \cdots dw_1,$$

$$P_{\mathrm{loss}}(n \mid x) = p_n H(x) + \sum_{k=1}^{\infty} \int_{w_1=0}^{\infty} \int_{w_2=0}^{\infty} \cdots \int_{w_k=0}^{\infty} f(n, k; w_1, w_2, \ldots, w_k \mid x) dw_k dw_{k-1} \cdots dw_1.$$

□

Finally, we consider the actual waiting time of each customer. For a randomly chosen customer, let H denote her impatience time, and let N_{front} denote the number of customers of the same batch who are in front of her. Let $\hat{A}$ denote the workload in system seen by the batch which contains the randomly chosen customer, and let $\hat{W}_i$ $(i = 1, 2, \ldots, N_{\mathrm{front}})$ denote the service time of the i-th customer of the batch. We then define $Q(0 \mid y)$ $(y \geq 0)$ and $Q(n; w_1, w_2, \ldots, w_n \mid y)$ $(y \geq 0,\ w_1 > 0,\ w_2 > 0,\ \ldots,\ w_n > 0,\ n = 1, 2, \ldots)$ as

$$Q(0 \mid y) = \Pr(N_{\mathrm{front}} = 0, H > y \mid \hat{A} = y), \tag{23}$$

and

$$Q(n; w_1, w_2, \ldots, w_n \mid y) = \Pr\Big(N_{\mathrm{front}} = n, H > y + \sum_{i=1}^{n} W_i,$$

$$\hat{W}_1 \le w_1, \hat{W}_2 \le w_2, \ldots, \hat{W}_n \le w_n \mid \hat{A} = y\Big). \tag{24}$$

We define $q(n; w_1, w_2, \ldots, w_n \mid y)$ ($y \ge 0$, $w_1 > 0$, $w_2 > 0$, …, $w_n > 0$, $n = 1, 2, \ldots$) as

$$q(n; w_1, w_2, \ldots, w_n \mid y) = \frac{\partial^n Q(n; w_1, w_2, \ldots, w_n \mid y)}{\partial w_1 \partial w_2 \cdots \partial w_n}.$$

We further define $q_{\mathrm{total}}(w \mid y)$ ($w > 0$, $y \ge 0$, $n = 1, 2, \ldots$) as

$$q_{\mathrm{total}}(w \mid y) = q(1; w \mid y) \tag{25}$$

$$+ \sum_{n=2}^{\infty} \int_{w_1=0+}^{w} \int_{w_2=0+}^{w-w_1} \cdots \int_{w_{n-1}=0+}^{w-w_1-w_2-\cdots-w_{n-2}} \cdot q\Big(n; w_1, w_2, \ldots, w_{n-1}, w - \sum_{i=1}^{n-1} w_i \mid y\Big) dw_{n-1} dw_{n-2} \cdots dw_1. \tag{26}$$

Lemma 3 *$Q(0 \mid y)$ ($y \ge 0$) and $q(n; w_1, w_2, \ldots, w_n \mid y)$ ($y \ge 0$, $w_1 > 0$, $w_2 > 0$, …, $w_n > 0$, $n = 1, 2, \ldots$) are given by*

$$Q(0 \mid y) = \frac{1}{\mathrm{E}[B]} \cdot \overline{H}(y), \tag{27}$$

$$q(n; w_1, w_2, \ldots, w_n \mid y) = \frac{\overline{p}_{n+1}}{\mathrm{E}[B]} \cdot \overline{H}\Big(y + \sum_{m=1}^{n} w_m\Big) \prod_{i=1}^{n} g(w_i). \tag{28}$$

Proof Because $\hat{A}$ and N_{front} are independent, (23) and (24) are rewritten to be

$$Q(0 \mid y) = \Pr(N_{\mathrm{front}} = 0) \Pr(H > y),$$

and

$$Q(n; w_1, w_2, \ldots, w_n \mid y) = \Pr(N_{\mathrm{front}} = n) \Pr\Big(H > y + \sum_{m=1}^{n} w_m, \hat{W}_1 \le w_1, \hat{W}_2 \le w_2, \ldots, \hat{W}_n \le w_n\Big).$$

We then obtain (27) and (28), using (3) and

$$\Pr(N_{\mathrm{front}} = n) = \sum_{k=n+1}^{\infty} \frac{k p_k}{\mathrm{E}[B]} \cdot \frac{1}{k} = \frac{\overline{p}_{n+1}}{\mathrm{E}[B]}, \qquad n = 0, 1, \ldots.$$

□

Corollary 2 *$q_{\mathrm{total}}(w \mid y)$ ($w > 0$, $y \ge 0$) is given by*

$$q_{\mathrm{total}}(w \mid y) = \frac{\overline{H}(y + w)}{\mathrm{E}[B]} \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(w). \tag{29}$$

Proof It follows from (26) and Lemma 3 that

$$\begin{aligned} q_{\text{total}}(w \mid y) = {} & \frac{\overline{p}_2}{\mathrm{E}[B]} \cdot \overline{H}(y+w) g(w) \\ & + \sum_{n=2}^{\infty} \frac{\overline{p}_{n+1}}{\mathrm{E}[B]} \cdot \overline{H}(y+w) \int_{w_1=0+}^{w} \int_{w_2=0+}^{w-w_1} \cdots \int_{w_{n-1}=0+}^{w-w_1-w_2-\cdots-w_{n-2}} \\ & \cdot \left\{ \prod_{i=1}^{n-1} g(w_i) \right\} g\Big(w - \sum_{i=1}^{n-1} w_i\Big) dw_{n-1} dw_{n-2} \cdots dw_1, \end{aligned}$$

which implies (29). □

Remark 2 It follows from (15), (17), (27), and (29) that

$$\overline{G}(w \mid y) = \mathrm{E}[B] \left(Q(0 \mid y) \overline{G}(w) + \int_{0+}^{w} q_{\text{total}}(u \mid y) \overline{G}(w-u) du \right), \qquad w \geq 0,\ y \geq 0, \quad (30)$$

and,

$$\begin{aligned} \beta(y) &= \mathrm{E}[B]\mathrm{E}[G] Q(0 \mid y) + \mathrm{E}[B] \int_{w=0}^{\infty} dw \int_{u=0+}^{w} q_{\text{total}}(u \mid y) \overline{G}(w-u) du \\ &= \frac{\rho}{\lambda} \cdot Q(0 \mid y) + \mathrm{E}[B] \int_{u=0+}^{\infty} q_{\text{total}}(u \mid y) du \int_{w=u}^{\infty} \overline{G}(w-u) dw \\ &= \frac{\rho}{\lambda} \left(Q(0 \mid y) + \int_{0+}^{\infty} q_{\text{total}}(u \mid y) du \right), \qquad y \geq 0. \end{aligned} \quad (31)$$

Let D denote the actual waiting time of a randomly chosen customer who is not lost. Let $D(x)$ $(x \geq 0)$ and $d(x)$ $(x > 0)$ denote the PDF and the p.d.f. of D, respectively. Note that

$$D(0) + \int_{0+}^{\infty} d(x) dx = 1. \quad (32)$$

The following theorem follows immediately from the definitions of $Q(0 \mid x)$ and $q_{\text{total}}(t \mid y)$.

Theorem 4 *$D(0)$ and $d(x)$ $(x > 0)$ are given by*

$$D(0) = \frac{1}{1 - P_{\text{loss}}} \cdot \pi_0 Q(0 \mid 0) = \frac{\pi_0}{(1 - P_{\text{loss}})\mathrm{E}[B]}, \quad (33)$$

$$d(x) = \frac{1}{1 - P_{\text{loss}}} \left[\pi_0 q_{\text{total}}(x \mid 0) + v(x) Q(0 \mid x) + \int_{0+}^{x} v(t) q_{\text{total}}(x - t \mid t) dt \right]. \quad (34)$$

Remark 3 Using (14) and (31), we can verify that

$$1 - P_{\mathrm{loss}} = \pi_0 Q(0 \mid 0) + \int_{0+}^{\infty} \Big[\pi_0 q_{\mathrm{total}}(x \mid 0) + v(x) Q(0 \mid x) + \int_{0+}^{x} v(t) q_{\mathrm{total}}(x - t \mid t) dt \Big] dx,$$

so that (33) and (34) satisfy the normalization condition (32).

The actual waiting time distribution is thus given in terms of the virtual waiting time distribution. In addition, it is directly given by the solution of a Volterra integral equation of the second kind as follows.

Lemma 4 *$d(x)$ $(x \geq 0)$ satisfies the following integral equation.*

$$d(x) = D(0)\overline{H}(x) \left(\lambda \overline{G}_{\mathrm{batch}}(x) + \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x) \right) + \lambda \overline{H}(x) \int_{0+}^{x} d(y) \overline{G}_{\mathrm{batch}}(x-y) dy, \quad x > 0, \tag{35}$$

where $\overline{G}_{\mathrm{batch}}(x)$ $(x \geq 0)$ is defined as

$$\overline{G}_{\mathrm{batch}}(x) = 1 - \sum_{n=1}^{\infty} p_n G^{(n)}(x). \tag{36}$$

The proof of Lemma 4 is provided in Appendix B.

Remark 4 If $p_1 = 1$ and $p_n = 0$ $(n = 2, 3, \ldots)$, i.e., in the M/G/1+G queue, we have

$$D(0) = \frac{\pi_0}{1 - P_{\mathrm{loss}}}, \qquad d(x) = \frac{v(x)\overline{H}(x)}{1 - P_{\mathrm{loss}}},$$

and (7) is rewritten to be

$$v(x) = \lambda \pi_0 \overline{G}(x) + \lambda \int_{0+}^{x} v(y) \overline{H}(y) \overline{G}(x-y) dy.$$

It then follows that

$$\begin{aligned} d(x) &= \frac{\overline{H}(x)}{1 - P_{\mathrm{loss}}} \left[\lambda \pi_0 \overline{G}(x) + \lambda (1 - P_{\mathrm{loss}}) \int_{0+}^{x} d(y) \overline{G}(x-y) dy \right] \\ &= \lambda D(0) \overline{H}(x) \overline{G}(x) + \lambda \overline{H}(x) \int_{0+}^{x} d(y) \overline{G}(x-y) dy, \end{aligned}$$

which is consistent with (35), because $\overline{p}_n = 0$ $(n = 2, 3, \ldots)$ and $\overline{G}_{\mathrm{batch}}(x) = \overline{G}(x)$ $(x \geq 0)$ in this case.

Theorem 5 *$d(x)$ $(x \ge 0)$ and $D(0)$ are given by*

$$d(x) = D(0) \sum_{n=1}^{\infty} \phi_{\mathrm{D},n}(x), \quad x > 0, \tag{37}$$

$$D(0) = \left[1 + \sum_{n=1}^{\infty} c_{\mathrm{D},n} \right]^{-1}, \tag{38}$$

where $\{\phi_{\mathrm{D},n}(x);\, x > 0\}_{n=1,2,\ldots}$ is a sequence of functions given by

$$\phi_{\mathrm{D},1}(x) = \overline{H}(x) \left(\lambda \overline{G}_{\mathrm{batch}}(x) + \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)}(x) \right), \tag{39}$$

$$\phi_{\mathrm{D},n}(x) = \lambda \overline{H}(x) \int_{0+}^{x} \phi_{\mathrm{D},n-1}(y) \overline{G}_{\mathrm{batch}}(x-y) dy, \qquad n = 2, 3, \ldots, \tag{40}$$

and $c_{\mathrm{D},n}$ $(n = 1, 2, \ldots)$ is given by

$$c_{\mathrm{D},n} = \int_{0+}^{\infty} \phi_{\mathrm{D},n}(x) dx.$$

Proof (37) and (38) is proved in the exactly same way as (8) and (11), using Lemma 4 and (32). □

Remark 5 Using (12) and (33), we can verify that

$$D(0) = \frac{\pi_0}{(1 - P_{\mathrm{loss}})\mathrm{E}[B]} = \frac{\pi_0 \rho}{(1 - \pi_0)\mathrm{E}[B]},$$

which implies

$$\pi_0 = \frac{\mathrm{E}[B] D(0)}{\rho + \mathrm{E}[B] D(0)}. \tag{41}$$

Therefore, π_0 is given in terms of $D(0)$. Furthermore, using (12) and (41), it is verified that P_{loss} is also given in terms of $D(0)$ by

$$P_{\mathrm{loss}} = 1 - \frac{1}{\rho + \mathrm{E}[B] D(0)}. \tag{42}$$

3.2 Special cases

In this subsection, we consider several special cases of the $\mathrm{M^x/G/1{+}G_{same}}$ queue. In Section 3.2.1, we show that the expression for the number of losses given in Theorem 3 is much simplified in the $\mathrm{M^{geo}/G/1{+}G_{same}}$ queue. Next, we derive a specialized formula for the actual waiting time distribution in the $\mathrm{M^{geo}/M/1{+}G_{same}}$ queue in Section 3.2.2. We consider the $\mathrm{M^x/M/1{+}M_{same}}$ queue with general batch sizes in Section 3.2.3, and obtain the Laplace-Stieltjes transform (LST) of the virtual waiting time. Finally in Section 3.2.4, we consider the $\mathrm{M^{geo}/M/1{+}M_{same}}$ queue, which is the intersection of the $\mathrm{M^{geo}/M/1{+}G_{same}}$ and $\mathrm{M^x/M/1{+}M_{same}}$ queues.

3.2.1 $M^{\mathrm{geo}}/G/1+G_{\mathrm{same}}$ queue

We assume that sizes of batches follow a geometric distribution, i.e.,

$$p_n = (1-\alpha)\alpha^{n-1}, \qquad n = 1, 2, \ldots, \tag{43}$$

where $\alpha \in [0, 1)$. In this case, the traffic intensity ρ is given by

$$\rho = \frac{\lambda \mathrm{E}[G]}{1-\alpha}. \tag{44}$$

Note that (2) is reduced to be

$$\overline{p}_n = \alpha^{n-1}, \qquad n = 1, 2, \ldots, \tag{45}$$

so that we have from (17) and (18),

$$\begin{aligned} \overline{G}(w \mid y) &= \overline{H}(y)\overline{G}(w) + \int_0^w \overline{H}(y+u) \sum_{k=1}^{\infty} \alpha^k g^{(k)}(u) \overline{G}(w-u) du, \\ \beta(x) &= \mathrm{E}[G] \sum_{k=1}^{\infty} \alpha^{k-1} \mathrm{E}\left[\overline{H}\Big(x + \sum_{m=1}^{k-1} G_m\Big)\right]. \end{aligned} \tag{46}$$

Theorem 6 *In the $M^{\mathrm{geo}}/G/1+G_{\mathrm{same}}$ queue,*

(i) $P_{\mathrm{loss}}(n \mid x)$ $(n = 0, 1, \ldots)$ is given by

$$P_{\mathrm{loss}}(0 \mid x) = \frac{\lambda \beta(x)}{\rho}, \qquad x \geq 0, \tag{47}$$

$$P_{\mathrm{loss}}(n \mid x) = [1 - P_{\mathrm{loss}}(0 \mid x)](1-\alpha)\alpha^{n-1}, \quad x \geq 0, n = 1, 2, \ldots, \tag{48}$$

and

(ii) $P_{\mathrm{loss}}(n)$ $(n = 0, 1, \ldots)$ is given in terms of the loss probability P_{loss} by

$$P_{\mathrm{loss}}(0) = 1 - P_{\mathrm{loss}}, \tag{49}$$

$$P_{\mathrm{loss}}(n) = P_{\mathrm{loss}}(1-\alpha)\alpha^{n-1}, \qquad n = 1, 2, \ldots. \tag{50}$$

Remark 6 (48) implies that the conditional number of lost customers in a batch $[N_{\mathrm{loss}} \mid N_{\mathrm{loss}} > 0]$ is distributed according to the geometric distribution with parameter $1 - \alpha$. This is almost obvious from the memoryless property of geometric distributions.

Remark 7 (49) can be understood intuitively as follows. Let M denote a generic random variable for sizes of batches. For a randomly chosen customer, let $\tilde{M}$ denote the number of customers in the same batch who are in front of her. $1 - P_{\mathrm{loss}}$ is then given by the probability that $A + \sum_{n=1}^{\tilde{M}} G_n$ is less than H, where H denotes a generic random variable for impatience times. On the other hand, $P_{\mathrm{loss}}(0)$ is given by the probability that $A + \sum_{n=1}^{M-1} G_n$ is less than H. If M is geometrically distributed, $\tilde{M}$ and $M-1$ have the same distribution, so that (49) holds.

Proof We first consider (i). (47) immediately follows from (21), (44), and (46). Similarly, (48) is obtained from (22) as follows.

$$
\begin{aligned}
P_{\mathrm{loss}}(n \mid x) &= (1-\alpha)\alpha^{n-1}H(x) \\
&\quad + (1-\alpha)\alpha^{n-1}\left\{\sum_{k=1}^{\infty}\alpha^k \mathrm{E}\left[\overline{H}\Big(x+\sum_{m=1}^{k-1}G_m\Big)\right]\right. \\
&\qquad\qquad \left. - \sum_{k=1}^{\infty}\alpha^k \mathrm{E}\left[\overline{H}\Big(x+\sum_{m=1}^{k}G_m\Big)\right]\right\} \\
&= (1-\alpha)\alpha^{n-1}\left\{H(x)+\alpha\overline{H}(x)\right. \\
&\qquad \left. -(1-\alpha)\sum_{k=2}^{\infty}\alpha^{k-1}\mathrm{E}\left[\overline{H}\Big(x+\sum_{m=1}^{k-1}G_m\Big)\right]\right\} \\
&= (1-\alpha)\alpha^{n-1}\left\{H(x)+\alpha\overline{H}(x)+(1-\alpha)\overline{H}(x)\right. \\
&\qquad \left. -(1-\alpha)\sum_{k=1}^{\infty}\alpha^{k-1}\mathrm{E}\left[\overline{H}\Big(x+\sum_{m=1}^{k-1}G_m\Big)\right]\right\} \\
&= (1-\alpha)\alpha^{n-1}\left\{H(x)+\alpha\overline{H}(x)+(1-\alpha)\overline{H}(x)-P_{\mathrm{loss}}(0 \mid x)\right\} \\
&= (1-\alpha)\alpha^{n-1}[1-P_{\mathrm{loss}}(0 \mid x)].
\end{aligned}
$$

Next, we consider (ii). Using (20) and (47), we have

$$
P_{\mathrm{loss}}(0) = \frac{\lambda\pi_0\beta(0)}{\rho} + \int_0^{\infty}\frac{\lambda\beta(x)}{\rho}\cdot v(x)dx.
$$

(49) thus follows from (14). Similarly, for $n = 1, 2, \ldots$, (20) and (48) imply

$$
\begin{aligned}
P_{\mathrm{loss}}(n) &= (1-\alpha)\alpha^{n-1}\left[\pi_0[1-P_{\mathrm{loss}}(0 \mid 0)] + \int_{0+}^{\infty}[1-P_{\mathrm{loss}}(0 \mid x)]v(x)dx\right] \\
&= (1-\alpha)\alpha^{n-1}(1-P_{\mathrm{loss}}(0)).
\end{aligned}
$$

We then obtain (50) from this equation and (49). □

3.2.2 M^{geo}/M/1+G_{same} queue

In this subsection, we assume (43) and

$$
g(x) = \mu\exp[-\mu x], \qquad x \ge 0, \tag{51}
$$

where $\mu > 0$. It follows from (44) that

$$
\rho = \frac{\lambda}{\mu(1-\alpha)}. \tag{52}
$$

In this case, we have

$$\begin{aligned}\sum_{k=2}^{\infty}\overline{p}_k g^{(k-1)}(x) &= \sum_{k=2}^{\infty}\alpha^{k-1}\cdot\frac{\mu\exp[-\mu x](\mu x)^{k-2}}{(k-2)!}\\ &= \alpha\mu\exp[-\mu x]\exp[\alpha\mu x]\\ &= \alpha\mu\exp[-(1-\alpha)\mu x]. \end{aligned} \tag{53}$$

(17) is then reduced to be

$$\begin{aligned}\overline{G}(w \mid y) &= \overline{H}(y)\exp[-\mu w]\\ &\quad + \alpha\mu\int_0^w \overline{H}(y+u)\exp[-(1-\alpha)\mu u]\exp[-\mu(w-u)]du\\ &= \exp[-\mu w]\left[\overline{H}(y) + \alpha\mu\int_0^w \overline{H}(y+u)\exp[\alpha\mu u]du\right]. \end{aligned} \tag{54}$$

Although $\overline{G}(w \mid y)$ in (54) takes a simpler form than the general case (17), it does not seem to substantially simplify the formulas (7) and (8) for the virtual waiting time.

On the other hand, $\overline{G}_{\text{batch}}(x)$ $(x \geq 0)$ given by (36) is reduced to be

$$\begin{aligned}\overline{G}_{\text{batch}}(x) &= 1 - \sum_{n=1}^{\infty}(1-\alpha)\alpha^{n-1}\left(1 - \sum_{m=0}^{n-1}\frac{\exp[-\mu x](\mu x)^m}{m!}\right)\\ &= \sum_{n=1}^{\infty}(1-\alpha)\alpha^{n-1}\sum_{m=0}^{n-1}\frac{\exp[-\mu x](\mu x)^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{\exp[-\mu x](\mu x)^m}{m!}\sum_{n=m+1}^{\infty}(1-\alpha)\alpha^{n-1}\\ &= \sum_{m=0}^{\infty}\frac{\exp[-\mu x](\alpha\mu x)^m}{m!}\\ &= \exp[-(1-\alpha)\mu x]. \end{aligned} \tag{55}$$

This leads to a specialized formula for the actual waiting time distribution, which is given in the following theorem.

Theorem 7 *In the M^{geo}/M/1+G_{same} queue, $D(0)$ and $d(x)$ are given by*

$$d(x) = (\lambda+\alpha\mu)D(0)\overline{H}(x)\exp[\lambda J(x) - (1-\alpha)\mu x], \qquad x > 0, \tag{56}$$

$$D(0) = \left[1 + \int_{0+}^{\infty}(\lambda+\alpha\mu)\overline{H}(x)\exp[\lambda J(x) - (1-\alpha)\mu x]dx\right]^{-1}, \tag{57}$$

where

$$J(x) = \int_0^x \overline{H}(y)dy, \qquad x \geq 0.$$

Proof Because (57) immediately follows from (56) and the normalization condition (32), we consider (56) below. Recall that $d(x)$ $(x > 0)$ is given in terms of $\phi_{\mathrm{D},n}(x)$ $(x > 0,\ n = 1, 2, \ldots)$ by (37). Using (53) and (55), we rewrite (39) and (40) as

$$\phi_{\mathrm{D},1}(x) = \overline{H}(x)\left(\lambda \exp[-(1-\alpha)\mu x] + \alpha\mu \exp[-(1-\alpha)\mu x]\right),$$
$$\phi_{\mathrm{D},n}(x) = \lambda \overline{H}(x) \int_{0+}^{x} \phi_{\mathrm{D},n-1}(y) \exp[-(1-\alpha)\mu(x-y)]dy, \qquad n = 2, 3, \ldots.$$

We then define $\psi_{\mathrm{D},n}(x)$ $(x > 0,\ n = 1, 2, \ldots)$ as

$$\psi_{\mathrm{D},n}(x) = \frac{\phi_{\mathrm{D},n}(x) \exp[(1-\alpha)\mu x]}{\lambda + \alpha\mu}. \tag{58}$$

It is easy to verify that $\psi_{\mathrm{D},n}(x)$ satisfies

$$\psi_{\mathrm{D},1}(x) = \overline{H}(x), \tag{59}$$
$$\psi_{\mathrm{D},n}(x) = \lambda \overline{H}(x) \int_{0+}^{x} \psi_{\mathrm{D},n-1}(y)dy, \qquad n = 2, 3, \ldots. \tag{60}$$

By induction, we prove

$$\psi_{\mathrm{D},n}(x) = \overline{H}(x) \cdot \frac{\big(\lambda J(x)\big)^{n-1}}{(n-1)!}, \qquad x > 0, n = 1, 2, \ldots. \tag{61}$$

For $n = 1$, (61) immediately follows from (59). We then assume that (61) holds for some $n = m$ $(m = 1, 2, \ldots)$. Under this assumption, it follows from (60) that

$$\begin{aligned}
\psi_{\mathrm{D},m+1}(x) &= \lambda \overline{H}(x) \int_{0+}^{x} \frac{\big(\lambda J(y)\big)^{m-1} \overline{H}(y)}{(m-1)!} dy \\
&= \overline{H}(x) \int_{0+}^{x} \frac{d}{dy}\left[\frac{\big(\lambda J(y)\big)^{m}}{m!}\right] dy \\
&= \overline{H}(x) \cdot \frac{\big(\lambda J(x)\big)^{m}}{m!},
\end{aligned}$$

so that (61) also holds for $n = m+1$. Therefore, we have (61) for $n = 1, 2, \ldots$.

It then follows from (58) and (61) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \phi_{\mathrm{D},n}(x) &= (\lambda + \alpha\mu) \sum_{n=1}^{\infty} \psi_{\mathrm{D},n}(x) \exp[-(1-\alpha)\mu x] \\
&= (\lambda + \alpha\mu) \overline{H}(x) \exp[\lambda J(x)] \exp[-(1-\alpha)\mu x].
\end{aligned}$$

(56) now follows from this equation and (37). □

As mentioned in Remark 5, π_0 and P_{loss} are given in terms of $D(0)$. Therefore, the number of losses $P_{\mathrm{loss}}(n)$ $(n = 0, 1, \ldots)$ in the M$^{\mathrm{geo}}$/M/1+G$_{\mathrm{same}}$ queue can be obtained from Theorems 6 and 7. Based on these results, we derive ordering relations of P_{loss} and $P_{\mathrm{loss}}(n)$ $(n = 0, 1, \ldots)$ with respect to the variability of impatience times. Let H denote a generic random variable for impatience times. Further let L denote a generic random variable for the number of losses in a randomly chosen batch, i.e., $\Pr(L = n) = P_{\mathrm{loss}}(n)$ $(n = 0, 1, \ldots)$.

Definition 1 ([24, Theorem 3.A.1])
Let X and Y denote two non-negative random variables with the same finite mean $\mathrm{E}[X] = \mathrm{E}[Y] < \infty$. X is said to be smaller than or equal to Y in the convex order (denoted by $X \leq_{\mathrm{cx}} Y$) if and only if

$$\int_x^\infty \Pr(X > u)du \leq \int_x^\infty \Pr(Y > u)du, \qquad \text{for all } x \geq 0.$$

Remark 8 The convex order compares the variability of random variables. Note that $X \leq_{\mathrm{cx}} Y \Rightarrow \mathrm{E}[X] = \mathrm{E}[Y]$ and $\mathrm{Cv}[X] \leq \mathrm{Cv}[Y]$, where $\mathrm{Cv}[\cdot]$ denotes the coefficient of variation [24, Eq. (3.A.4)].

Theorem 8 *Consider two stationary M$^{\mathrm{geo}}$/M/1+G$_{\mathrm{same}}$ queues with the same arrival rate λ, the same batch size distribution $p_n = (1-\alpha)\alpha^{n-1}$ $(n = 1, 2, \ldots)$, the same service time distribution $G(x)$ $(x \geq 0)$, and the same finite mean impatience time $\mathrm{E}[H] < \infty$. We denote quantities in the k-th $(k = 1, 2)$ queue with a superscript $\langle k \rangle$. It then follows that*

$$H^{\langle 1 \rangle} \leq_{\mathrm{cx}} H^{\langle 2 \rangle} \Rightarrow P_{\mathrm{loss}}^{\langle 1 \rangle} \leq P_{\mathrm{loss}}^{\langle 2 \rangle}, \tag{62}$$

and

$$H^{\langle 1 \rangle} \leq_{\mathrm{cx}} H^{\langle 2 \rangle} \Rightarrow L^{\langle 1 \rangle} \leq_{\mathrm{st}} L^{\langle 2 \rangle}, \tag{63}$$

where $\leq_{\mathrm{st}}$ denotes the usual stochastic order, which is defined as [24, Page 4]

$$L^{\langle 1 \rangle} \leq_{\mathrm{st}} L^{\langle 2 \rangle} \Leftrightarrow \sum_{n=i}^{\infty} P_{\mathrm{loss}}^{\langle 1 \rangle}(i) \leq \sum_{n=i}^{\infty} P_{\mathrm{loss}}^{\langle 2 \rangle}(i), \qquad \textit{for all } i = 1, 2, \ldots.$$

Remark 9 (62) is an extension of the ordering property of P_{loss} in the single-arrival M/M/1+G queue mentioned in [27]. Also, a related result in the single-arrival M/G/1+G queue can be found in [18].

Proof We first consider (62). Because of (42), P_{loss} increases with $D(0)$ when ρ and $\mathrm{E}[B]$ are fixed. Therefore, it is sufficient to show

$$H^{\langle 1 \rangle} \leq_{\mathrm{cx}} H^{\langle 2 \rangle} \Rightarrow D^{\langle 1 \rangle}(0) \leq D^{\langle 2 \rangle}(0). \tag{64}$$

We rewrite the integral in (57) by partial integration.

$$\int_{0+}^\infty (\lambda + \alpha\mu)\overline{H}(x) \exp[\lambda J(x) - (1 - \alpha)\mu x]dx$$

$$
\begin{aligned}
&= \frac{\lambda+\alpha\mu}{\lambda}\int_{0+}^{\infty}\exp[-(1-\alpha)\mu x]\cdot\frac{d}{dx}\Big[\exp[\lambda J(x)]\Big]dx \\
&= \frac{\lambda+\alpha\mu}{\lambda}\Big[\exp[-(1-\alpha)\mu x]\cdot\exp[\lambda J(x)]\Big]_0^{\infty} \qquad (65)\\
&\quad + \frac{(\lambda+\alpha\mu)(1-\alpha)\mu}{\lambda}\int_{0+}^{\infty}\exp[-(1-\alpha)\mu x]\cdot\exp[\lambda J(x)]dx \\
&= -\frac{\lambda+\alpha\mu}{\lambda}+\frac{\lambda+\alpha\mu}{\rho}\int_{0+}^{\infty}\exp[-(1-\alpha)\mu x]\cdot\exp[\lambda J(x)]dx. \qquad (66)
\end{aligned}
$$

Note here that $\lim_{x\to\infty}\exp[-(1-\alpha)\mu x+\lambda J(x)]=0$ follows from the stability condition

$$\rho h_\infty = \frac{\lambda h_\infty}{(1-\alpha)\mu} < 1.$$

Because the two queues have the same mean impatience time $\mathrm{E}[H]<\infty$, we have

$$J^{\langle k\rangle}(x) = \mathrm{E}[H] - \int_x^{\infty}\overline{H}^{\langle k\rangle}(y)dy, \qquad k=1,2,$$

and therefore,

$$H^{\langle 1\rangle} \le_{\mathrm{cx}} H^{\langle 2\rangle} \Rightarrow J^{\langle 1\rangle}(x) \ge J^{\langle 2\rangle}(x), \qquad \text{for all } x\ge 0.$$

(64) now follows from (57) and (66). We thus proved (62).

Furthermore, (63) immediately follows from (62) because (50) implies

$$\sum_{n=i}^{\infty}P_{\mathrm{loss}}^{\langle k\rangle}(i) = P_{\mathrm{loss}}^{\langle k\rangle}\sum_{n=i}^{\infty}(1-\alpha)\alpha^{n-1}, \qquad k=1,2,\ i=1,2,\ldots.$$

□

3.2.3 M$^{\mathrm{x}}$/M/1+M$_{\mathrm{same}}$ queue

Here, we assume that batch sizes are generally distributed. We assume (51) and

$$\overline{H}(x) = \exp[-\eta x], \qquad x\ge 0, \qquad (67)$$

where $\eta>0$. In this case, (17) is reduced to be

$$
\begin{aligned}
\overline{G}(w\mid y) &= \exp[-\eta y]\exp[-\mu w] + \int_0^{w}\exp[-\eta(y+u)] \\
&\qquad \cdot\sum_{k=2}^{\infty}\overline{p}_k\cdot\frac{\mu\exp[-\mu u](\mu u)^{k-2}}{(k-2)!}\exp[-\mu(w-u)]du \\
&= \exp[-\eta y-\mu w]\left[1+\sum_{k=2}^{\infty}\overline{p}_k(\mu/\eta)^{k-1}\int_0^{w}\frac{\eta\exp[-\eta u](\eta u)^{k-2}}{(k-2)!}du\right] \\
&= \exp[-\eta y-\mu w]\left[1+\sum_{k=2}^{\infty}\overline{p}_k(\mu/\eta)^{k-1}\sum_{j=k-1}^{\infty}\frac{\exp[-\eta w](\eta w)^j}{j!}\right]
\end{aligned}
$$

$$= \exp[-\eta y]\overline{R}(w), \tag{68}$$

where

$$\overline{R}(x) = \exp[-\mu x]\sum_{k=1}^{\infty}\overline{p}_k(\mu/\eta)^{k-1}\sum_{j=k-1}^{\infty}\frac{\exp[-\eta x](\eta x)^j}{j!}.$$

We define $\overline{R}^*(s)$ ($\mathrm{Re}(s) > 0$) as

$$\overline{R}^*(s) = \int_0^{\infty}\exp[-sx]\overline{R}(x)dx.$$

Also, we define $v^*(s)$ ($\mathrm{Re}(s) > 0$) as the LST of the stationary virtual waiting time.

$$v^*(s) = \pi_0 + \int_{0+}^{\infty}\exp[-sx]v(x)dx. \tag{69}$$

Lemma 5 *$\overline{R}^*(s)$ ($\mathrm{Re}(s) > 0$) is given by*

$$\overline{R}^*(s) = \frac{1}{\mu+s}\sum_{k=1}^{\infty}\overline{p}_k\left(\frac{\mu}{\mu+\eta+s}\right)^{k-1}. \tag{70}$$

Proof By definitions of $\overline{R}(x)$ and $\overline{R}^*(s)$, we have

$$\begin{aligned}\overline{R}^*(s) &= \sum_{k=1}^{\infty}\overline{p}_k(\mu/\eta)^{k-1}\frac{1}{s+\mu}\int_0^{\infty}(s+\mu)\exp[-(s+\mu)x] \\ &\qquad \cdot\sum_{j=k-1}^{\infty}\frac{\exp[-\eta x](\eta x)^j}{j!}dx \\ &= \sum_{k=1}^{\infty}\overline{p}_k(\mu/\eta)^{k-1}\frac{1}{s+\mu}\left(\frac{\eta}{s+\mu+\eta}\right)^{k-1},\end{aligned}$$

which implies (70). □

Remark 10 Let $p^*(z)$ ($|z| \le 1$) denote the probability generating function (PGF) of the batch size distribution.

$$p^*(z) = \sum_{n=1}^{\infty} z^n p_n.$$

We can rewrite (70) as

$$\overline{R}^*(s) = \frac{1}{\mu+s}\cdot\frac{1-p^*\left(\dfrac{\mu}{\mu+\eta+s}\right)}{1-\dfrac{\mu}{\mu+\eta+s}}.$$

Theorem 9 *In the M$^{\text{x}}$/M/1+M$_{\text{same}}$ queue, the LST $v^*(s)$ (*$\text{Re}(s) > 0$*) of the stationary virtual waiting time is given by*

$$v^*(s) = \pi_0 + \pi_0 \sum_{n=0}^{\infty} \prod_{k=0}^{n} \lambda \overline{R}^*(s + k\eta). \tag{71}$$

Furthermore, π_0 is given by

$$\pi_0 = \left[1 + \sum_{n=0}^{\infty} \prod_{k=0}^{n} \lambda \overline{R}^*(k\eta) \right]^{-1}. \tag{72}$$

Proof It follows from (7), (68), and (69) that

$$\begin{aligned}
v^*(s) &= \pi_0 + \lambda\pi_0 \overline{R}^*(s) + \lambda \int_{x=0+}^{\infty} \exp[-sx]dx \int_{y=0+}^{x} \overline{R}(x-y) \exp[-\eta y] v(y) dy \\
&= \pi_0 + \lambda\pi_0 \overline{R}^*(s) \\
&\quad + \lambda \int_{y=0+}^{\infty} \exp[-sy] \exp[-\eta y] v(y) dy \int_{x=y}^{\infty} \overline{R}(x-y) \exp[-s(x-y)] dx \\
&= \pi_0 + \lambda\pi_0 \overline{R}^*(s) + \lambda[v^*(s+\eta) - \pi_0] \overline{R}^*(s) \\
&= \pi_0 + \lambda \overline{R}^*(s) v^*(s+\eta).
\end{aligned} \tag{73}$$

(71) is then obtained from the iteration based on (73). Note that (72) immediately follows from (71) and $\lim_{s \to 0+} v^*(s) = 1$. Finally, it should be observed that the infinite sums in (71) and (72) converge; this follows using d'Alembert ratio principle, since $|\lambda \overline{R}^*(s + k\eta)| < 1$ for k sufficiently large. □

3.2.4 M$^{\text{geo}}$/M/1+M$_{\text{same}}$ queue

We consider the M$^{\text{geo}}$/M/1+M$_{\text{same}}$ queue, i.e., we assume that (43), (51), and (67) hold. Note that this model is a special case of both of the models discussed in Sections 3.2.2 and 3.2.3.

Corollary 3 *In the M$^{\text{geo}}$/M/1+M queue, π_0 and $v^*(s)$ (*$\text{Re}(s) > 0$*) are given by*

$$\pi_0 = \left[1 + \frac{\lambda}{\mu} + \frac{\lambda + \alpha\mu}{\mu} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} \frac{\lambda}{k\eta + (1-\alpha)\mu} \right]^{-1}, \tag{74}$$

$$v^*(s) = \pi_0 + \frac{\pi_0 \lambda}{s + \mu} + \frac{\pi_0(\lambda + \alpha\mu)}{s + \mu} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu}. \tag{75}$$

Proof Using (45), we rewrite (70) as

$$\overline{R}^*(s) = \frac{1}{\mu + s} \sum_{k=1}^{\infty} \alpha^{k-1} \left(\frac{\mu}{\mu + \eta + s} \right)^{k-1}$$

$$= \frac{1}{\mu + s} \cdot \frac{1}{1 - \dfrac{\alpha\mu}{\mu + \eta + s}}$$
$$= \frac{s + \eta + \mu}{(\mu + s)\{s + \eta + (1-\alpha)\mu\}}.$$

(71) is then reduced to be

$$\begin{aligned}
v^*(s) &= \pi_0 + \pi_0 \sum_{n=0}^{\infty} \prod_{k=0}^{n} \lambda \cdot \frac{s + (k+1)\eta + \mu}{(s + k\eta + \mu)\{s + (k+1)\eta + (1-\alpha)\mu\}} \\
&= \pi_0 + \pi_0 \sum_{n=0}^{\infty} \frac{s + (n+1)\eta + \mu}{s + \mu} \prod_{k=0}^{n} \frac{\lambda}{s + (k+1)\eta + (1-\alpha)\mu} \\
&= \pi_0 + \pi_0 \sum_{n=0}^{\infty} \frac{s + (n+1)\eta + (1-\alpha)\mu + \alpha\mu}{s + \mu} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu} \\
&= \pi_0 + \pi_0 \sum_{n=0}^{\infty} \frac{s + (n+1)\eta + (1-\alpha)\mu}{s + \mu} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu} \\
&\quad + \pi_0 \sum_{n=0}^{\infty} \frac{\alpha\mu}{s + \mu} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu} \\
&= \pi_0 + \frac{\pi_0 \lambda}{s + \mu} + \pi_0 \sum_{n=1}^{\infty} \frac{\lambda}{s + \mu} \prod_{k=1}^{n} \frac{\lambda}{s + k\eta + (1-\alpha)\mu} \\
&\quad + \pi_0 \sum_{n=0}^{\infty} \frac{\alpha\mu}{s + \mu} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu} \\
&= \pi_0 + \frac{\pi_0 \lambda}{s + \mu} + \pi_0 \sum_{n=0}^{\infty} \frac{\lambda + \alpha\mu}{s + \mu} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu}.
\end{aligned}$$

From this equation, (75) follows. (74) then follows from (75) and the normalization condition $\lim_{s \to 0+} v^*(s) = 1$. □

Let $d^*(s)$ ($\mathrm{Re}(s) > 0$) denote the LST of the actual waiting time.

$$d^*(s) = D(0) + \int_{0+}^{\infty} \exp[-sx] d(x) dx.$$

Corollary 4 *In the* $M^{\mathrm{geo}}/M/1{+}M$ *queue,* $D(0)$ *and* $d^*(s)$ *(*$\mathrm{Re}(s) > 0$*) are given by*

$$D(0) = \left[1 + \frac{\lambda + \alpha\mu}{\lambda} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} \frac{\lambda}{k\eta + (1-\alpha)\mu}\right]^{-1}, \tag{76}$$

$$d^*(s) = D(0) + D(0) \cdot \frac{\lambda + \alpha\mu}{\lambda} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} \frac{\lambda}{s + k\eta + (1-\alpha)\mu}. \tag{77}$$

Proof Using (67), we rewrite (56) as

$$
\begin{aligned}
d(x) &= (\lambda + \alpha\mu) D(0) \exp[-\eta x] \exp\big[(\lambda/\eta)(1 - \exp[-\eta x]) - (1-\alpha)\mu x\big] \\
&= (\lambda + \alpha\mu) D(0) \cdot \exp[-\{\eta + (1-\alpha)\mu\} x] \cdot \exp\big[(\lambda/\eta)(1 - \exp[-\eta x])\big] \\
&= (\lambda + \alpha\mu) D(0) \cdot \exp[-\{\eta + (1-\alpha)\mu\} x] \sum_{n=0}^{\infty} \frac{(\lambda/\eta)^n (1 - \exp[-\eta x])^n}{n!}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
d^*(s) &= D(0) + (\lambda + \alpha\mu) D(0) \sum_{n=0}^{\infty} \frac{(\lambda/\eta)^n}{n!} \cdot \frac{1}{s + \eta + (1-\alpha)\mu} \\
&\qquad \cdot \int_{0+}^{\infty} \{s + \eta + (1-\alpha)\mu\} \exp[-\{s + \eta + (1-\alpha)\mu\} x] \\
&\qquad\qquad \cdot (1 - \exp[-\eta x])^n dx \\
&= D(0) + D(0) \sum_{n=0}^{\infty} \frac{(\lambda/\eta)^n}{n!} \cdot \frac{\lambda + \alpha\mu}{s + \eta + (1-\alpha)\mu} \\
&\qquad \cdot \prod_{k=1}^{n} \frac{k\eta}{s + (k+1)\eta + (1-\alpha)\mu} \\
&= D(0) + D(0) \cdot \frac{\lambda + \alpha\mu}{\lambda} \sum_{n=0}^{\infty} \prod_{k=0}^{n} \frac{\lambda}{s + (k+1)\eta + (1-\alpha)\mu},
\end{aligned}
$$

which implies (77). (76) now follows from (77) and $\lim_{s \to 0+} d^*(s) = 1$. □

Remark 11 It is readily verified that (74) and (76) satisfy (41).

4 M$^{\mathrm{x}}$/G/1+G$_{\mathrm{diff}}$ queue

In this section, we consider the case that all individual customers of a batch have i.i.d. impatience times. We assume that impatience times of customers are distributed according to the PDF $H(x)$ ($x \geq 0$). This case is mathematically more complicated than the M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ case, and we have to restrict attention to some special cases: the M$^{\mathrm{geo}}$/G/1+G$_{\mathrm{diff}}$ and M$^{\mathrm{x}}$/M/1+M$_{\mathrm{diff}}$ queues. For the M$^{\mathrm{geo}}$/G/1+G$_{\mathrm{diff}}$ queue, we perform an analysis of the virtual waiting time and the number of losses, in a similar way to that for the M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ queue in the previous section. As we will see, results for the M$^{\mathrm{geo}}$/G/1+G$_{\mathrm{diff}}$ queue are more complicated than those of the M$^{\mathrm{x}}$/G/1+G$_{\mathrm{same}}$ queue, even with the assumption of geometric batch sizes. For the M$^{\mathrm{x}}$/M/1+M$_{\mathrm{diff}}$ queue, we take another approach based on the queue length process, which is formulated as a continuous-time Markov chain with the skip-free to the left property.

4.1 $\mathrm{M^{geo}/G/1{+}G_{diff}}$ queue

We assume that the batch size distribution is given by (43). Recall that once we derive the two-variable function $\overline{G}(w \mid y)$ ($w \geq 0$, $y \geq 0$), we can obtain the virtual waiting time distribution and the loss probability based on the results in Section 2. Let $f(n,k;w_1,w_2,\ldots,w_k \mid y)$ ($x \geq 0$, $w_1 > 0, w_2 > 0, \ldots, w_k > 0$, $n = 0,1,\ldots$, $k = 1,2,\ldots$) denote the joint conditional PDF defined in the same way as in (16) for the $\mathrm{M^x/G/1{+}G_{same}}$ queue. We further define $f^*(z,k;w_1,w_2,\ldots,w_k \mid y)$ ($x \geq 0$, $w_1 > 0, w_2 > 0, \ldots, w_k > 0$, $|z| \leq 1$, $k = 1,2,\ldots$) as the PGF of $f(n,k;w_1,w_2,\ldots,w_k \mid y)$ with respect to the number of losses n:

$$f^*(z,k;w_1,w_2,\ldots,w_k \mid y) = \sum_{n=0}^{\infty} z^n f(n,k;w_1,w_2,\ldots,w_k \mid y).$$

To obtain $f(n,k;w_1,w_2,\ldots,w_k \mid y)$, we introduce the following quantities. Let $N_{\mathrm{loss}}^{(k)}$ ($N_{\mathrm{loss}}^{(k)} = 0,1,\ldots,N_{\mathrm{loss}}$) denote the number of lost customers in a batch who are in front of the k-th ($k = 1,2,\ldots,N_{\mathrm{admit}}$) admitted customer. We define $\hat{F}^*(z,k;w_1,w_2,\ldots,w_k \mid y)$ ($y \geq 0$, $|z| < 1$, $w_1 > 0, w_2 > 0, \ldots, w_k > 0$, $k = 1,2,\ldots$) as

$$\begin{aligned}\hat{F}^*(z,k;w_1,w_2,\ldots,w_k \mid y) = \sum_{n=0}^{\infty} z^n \Pr(&N_{\mathrm{admit}} \geq k, N_{\mathrm{loss}}^{(k)} = n,\\ &W_1 \leq w_1, W_2 \leq w_2, \ldots, W_k \leq w_k \mid A = y).\end{aligned}$$

We further define $\hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y)$ ($y \geq 0$, $|z| < 1$, $w_1 > 0$, $w_2 > 0$, $\ldots$, $w_k > 0$, $k = 1,2,\ldots$) as

$$\hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y) = \frac{\partial^k \hat{F}^*(z,k;w_1,w_2,\ldots,w_k \mid y)}{\partial w_1 \partial w_2 \cdots \partial w_k}.$$

Let $H_\alpha^*(z,y)$ ($|z| \leq 1$, $y \geq 0$) denote a defective PGF of the number of lost customers when no customers are admitted in a batch, given that y amount of work is seen by the batch.

$$\begin{aligned}H_\alpha^*(z,y) &= \sum_{n=1}^{\infty} z^n \Pr(N_{\mathrm{loss}} = n, N_{\mathrm{admit}} = 0 \mid A = y)\\ &= \sum_{n=1}^{\infty} z^n (1-\alpha)\alpha^{n-1}\{H(y)\}^n\\ &= zH(y) \cdot \frac{1-\alpha}{1-z\alpha H(y)}.\end{aligned} \tag{78}$$

Also let $\overline{H}_\alpha^*(z,y)$ ($|z| \leq 1$, $y \geq 0$) denote a defective PGF of the number of (lost) customers in front of the first admitted customer in a batch when at least one customer is admitted, given that y amount of work is seen by the batch.

$$\overline{H}^*_\alpha(z,y) = \sum_{n=0}^{\infty} z^n \Pr(N_{\text{admit}} \geq 1, N^{(1)}_{\text{loss}} = n \mid A = y) \tag{79}$$

$$= \sum_{m=1}^{\infty} (1-\alpha)\alpha^{m-1} \sum_{n=0}^{m-1} z^n \{H(y)\}^n \overline{H}(y) = \frac{\overline{H}(y)}{1 - z\alpha H(y)}. \tag{80}$$

Note that we have

$$\overline{H}^*_\alpha(1,y) = 1 - H^*_\alpha(1,y).$$

Remark 12 It is readily verified that $H^*_\alpha(1,w)$ (resp. $\overline{H}^*_\alpha(1,w)$) denote the PDF (resp. complementary PDF) of the greatest impatience time among customers in the same batch.

Remark 13 Because sizes of batches follow the geometric distribution with parameter $1-\alpha$ and they are independent of the amount of work seen on arrival, it follows for any $y \geq 0$,

$$\frac{z(1-\alpha)}{1-z\alpha} = H^*_\alpha(z,y) + z\overline{H}^*_\alpha(z,y) \cdot \frac{1-\alpha}{1-z\alpha}.$$

Lemma 6 *$\hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y)$ ($y \geq 0$, $|z| \leq 1$, $w_1 > 0, w_2 > 0, \ldots, w_k > 0$, $k = 1,2,\ldots$) is given by*

$$\hat{f}^*(z,1;w_1 \mid y) = \overline{H}^*_\alpha(z,y) g(w_1), \tag{81}$$

$$\hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y) = \overline{H}^*_\alpha(z,y) g(w_1) \prod_{i=2}^{k} \alpha \overline{H}^*_\alpha\Big(z, y + \sum_{j=1}^{i-1} w_j\Big) g(w_i),$$
$$k = 2,3,\ldots. \tag{82}$$

In addition, $f^*(z,k;w_1,w_2,\ldots,w_k \mid y)$ is given in terms of the joint density $\hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y)$ by

$$f^*(z,k;w_1,w_2,\ldots,w_k \mid y) = \hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y)$$
$$\cdot \Big[1 - \alpha + \alpha H^*_\alpha\Big(z, y + \sum_{m=1}^{k} w_m\Big)\Big]. \tag{83}$$

Proof For $k = 1$, (81) immediately follows from the definition (80) of $\overline{H}^*_\alpha(z,y)$. We thus consider $k = 2,3,\ldots$. Let $N^{(k)}_{\text{rest}}$ ($k = 1,2,\ldots$) denote the number of customers in a batch behind the k-th admitted customer, given that $N_{\text{admit}} \geq k$. Owing to the memoryless property of the geometric distribution, we have

$$\Pr(N^{(k)}_{\text{rest}} = n) = (1-\alpha)\alpha^n, \qquad n = 0,1,\ldots. \tag{84}$$

Therefore, we obtain

$\hat{f}^*(z,k;w_1,w_2,\ldots,w_k \mid y)$

$$
\begin{aligned}
&= \hat{f}^*(z, k-1; w_1, w_2, \ldots, w_{k-1} \mid y) \cdot \sum_{n=1}^{\infty} \Pr(N_{\mathrm{rest}}^{(k-1)} = n) \\
&\qquad \cdot \sum_{i=0}^{n-1} z^i \Big\{ H\Big(y + \sum_{m=1}^{k-1} w_m\Big) \Big\}^i \overline{H}\Big(y + \sum_{m=1}^{k-1} w_m\Big) g(w_k) \\
&= \hat{f}^*(z, k-1; w_1, w_2, \ldots, w_{k-1} \mid y) \cdot \alpha \overline{H}_\alpha^*\Big(z, y + \sum_{m=1}^{k-1} w_m\Big) g(w_k), \\
&\qquad k = 2, 3, \ldots. \qquad (85)
\end{aligned}
$$

(82) now follows from (81) and (85).

Furthermore, it is easy to see that

$$
\begin{aligned}
&f^*(z, k; w_1, w_2, \ldots, w_k \mid y) \\
&\quad = \hat{f}^*(z, k; w_1, w_2, \ldots, w_k \mid y) \\
&\qquad \cdot \left(\Pr(N_{\mathrm{rest}}^{(k)} = 0) + \sum_{n=1}^{\infty} \Pr(N_{\mathrm{rest}}^{(k)} = n) z^n \{H(y)\}^n \right).
\end{aligned}
$$

(83) thus immediately follows from (78) and (84). □

To determine $\overline{G}(w \mid y)$, we consider the total amount of work brought into the system. We define $\hat{f}_{\mathrm{total}}^*(z, k; w \mid y)$ ($|z| < 1$, $y \geq 0$, $w > 0$, $k = 1, 2, \ldots$) as

$$
\hat{f}_{\mathrm{total}}^*(z, 1; w \mid y) = \hat{f}^*(z, 1; w \mid y),
$$

and for $k = 2, 3, \ldots$,

$$
\begin{aligned}
&\hat{f}_{\mathrm{total}}^*(z, k; w \mid y) \\
&\quad = \int_{w_1=0}^{w} \int_{w_2=0}^{w-w_1} \cdots \int_{w_{k-1}=0}^{w-w_1-w_2-\cdots-w_{k-2}} \\
&\qquad \cdot \hat{f}^*\Big(z, k; w_1, w_2, \ldots, w_{k-1}, w - \sum_{m=1}^{k-1} w_m \mid y\Big) dw_{k-1} dw_{k-2} \cdots dw_1.
\end{aligned}
$$

We further define $\hat{f}_{\mathrm{total}}^*(z; w \mid y)$ as

$$
\hat{f}_{\mathrm{total}}^*(z; w \mid y) = \sum_{k=1}^{\infty} \hat{f}_{\mathrm{total}}^*(z, k; w \mid y).
$$

Lemma 7 *For fixed z ($|z| \leq 1$) and y ($y \geq 0$), $\hat{f}_{\mathrm{total}}^*(z; w \mid y)$ is given by the solution of the following Volterra integral equation of the second kind.*

$$
\begin{aligned}
&\hat{f}_{\mathrm{total}}^*(z; w \mid y) \\
&\quad = \overline{H}_\alpha^*(z, y) g(w) + \int_{0+}^{w} \hat{f}_{\mathrm{total}}^*(z; u \mid y) \cdot \alpha \overline{H}_\alpha^*(z, y+u) g(w-u) du, \quad w > 0.
\end{aligned}
$$

Proof With (81) and (85), it is verified that $\hat{f}^*_{\text{total}}(z,k;w \mid y)$ is given by the following recursion.

$$\hat{f}^*_{\text{total}}(z,1;w \mid y) = \overline{H}^*_\alpha(z,y)g(w), \quad w>0, \tag{86}$$

$$\hat{f}^*_{\text{total}}(z,k;w \mid y) = \int_{0+}^{w} \hat{f}^*_{\text{total}}(z,k-1;u \mid y)\cdot \alpha\overline{H}^*_\alpha(z,y+u)g(w-u)du, \quad w>0, k=2,3,\ldots. \tag{87}$$

$\hat{f}^*_{\text{total}}(z;w \mid y)$ thus satisfies

$$\begin{aligned}
&\hat{f}^*_{\text{total}}(z;w \mid y)\\
&\quad= \hat{f}^*_{\text{total}}(z,1;w \mid y) + \sum_{k=2}^{\infty}\hat{f}^*_{\text{total}}(z,k;w \mid y)\\
&\quad= \overline{H}^*_\alpha(z,y)g(w) + \sum_{k=2}^{\infty}\int_{0+}^{w}\hat{f}^*_{\text{total}}(z,k-1;u \mid y)\cdot\alpha\overline{H}^*_\alpha(z,y+u)g(w-u)du\\
&\quad= \overline{H}^*_\alpha(z,y)g(w) + \int_{0+}^{w}\hat{f}^*_{\text{total}}(z;u \mid y)\cdot\alpha\overline{H}^*_\alpha(z,y+u)g(w-u)du.
\end{aligned}$$

□

Theorem 10 *In the $\mathrm{M}^{\mathrm{geo}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queue, $\overline{G}(w \mid y)$ ($w \ge 0$, $y \ge 0$) is given by*

$$\overline{G}(w \mid y) = \int_{w}^{\infty}\frac{1-\alpha}{1-\alpha H(y+t)}\cdot\hat{f}_{\text{total}}(t \mid y)dt, \tag{88}$$

where for fixed y ($y \ge 0$), $\hat{f}_{\text{total}}(w \mid y)$ is given by the solution of the following Volterra integral equation of the second kind.

$$\hat{f}_{\text{total}}(w \mid y) = \frac{\overline{H}(y)g(w)}{1-\alpha H(y)} + \int_{0+}^{w}\frac{\alpha\overline{H}(y+u)}{1-\alpha H(y+u)}\cdot\hat{f}_{\text{total}}(u \mid y)g(w-u)du, \quad w>0. \tag{89}$$

The proof of Theorem 10 is provided in Appendix C.

Theorem 11 *The $\mathrm{M}^{\mathrm{geo}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queue is stable if*

$$\rho < \infty, \qquad \rho h_\infty < 1. \tag{90}$$

The proof of Theorem 11 is provided in Appendix D.

Remark 14 Owing to Theorems 2 and 11, the $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{same}}$ and $\mathrm{M}^{\mathrm{geo}}/\mathrm{G}/1+\mathrm{G}_{\mathrm{diff}}$ queues have the same stability condition.

In the rest of this section, we assume (90). The stationary virtual waiting time distribution is thus immediately obtained from (8) and (11).

Next, we consider the number of losses. We define $P^*_{\text{loss}}(z,k \mid y)$ ($|z| \le 1$, $y \ge 0$, $k=1,2,\ldots$) as

$$P^*_{\text{loss}}(z,k \mid y) = \sum_{n=0}^{\infty} z^n \Pr(N_{\text{loss}}=n, N_{\text{admit}}=k \mid A=y).$$

Lemma 8 $P^*_{\mathrm{loss}}(z,k \mid y)$ *($|z| \le 1$, $y \ge 0$, $k = 1,2,\ldots$) is given by*

$$P^*_{\mathrm{loss}}(z,0 \mid y) = H^*_\alpha(z,y), \tag{91}$$

$$P^*_{\mathrm{loss}}(z,k \mid y) = \overline{H}^*_\alpha(z,y)\mathrm{E}\left[\left\{\prod_{i=2}^{k} \alpha\overline{H}^*_\alpha\Big(z, y+\sum_{j=1}^{i-1} G_j\Big)\right\} \cdot \left[1-\alpha+\alpha H^*_\alpha\Big(z, y+\sum_{m=1}^{k} G_m\Big)\right]\right], \tag{92}$$

where $\{G_n\}_{n=1,2,\ldots}$ denotes a sequence of i.i.d. random variables distributed according to the service time distribution $G(x)$.

Proof (91) is obvious by definition (78) of $H^*_\alpha(z,y)$. On the other hand, using Lemma 6, we have for $k = 1,2,\ldots,$

$$\begin{aligned}
&P^*_{\mathrm{loss}}(z,k \mid y) \\
&\quad= \int_{w_1=0+}^{\infty}\int_{w_2=0+}^{\infty}\cdots\int_{w_k=0+}^{\infty} f^*(z,k;w_1,w_2,\ldots,w_k \mid y)dw_k dw_{k-1}\cdots dw_1 \\
&\quad= \int_{w_1=0+}^{\infty}\int_{w_2=0+}^{\infty}\cdots\int_{w_k=0+}^{\infty} \overline{H}^*_\alpha(z,y)g(w_1) \\
&\qquad \cdot\left\{\prod_{i=2}^{k}\alpha\overline{H}^*_\alpha\Big(z,y+\sum_{j=1}^{i-1}w_j\Big)g(w_i)\right\}\left[1-\alpha+\alpha H^*_\alpha\Big(z,y+\sum_{m=1}^{k}w_m\Big)\right] \\
&\qquad \cdot dw_k dw_{k-1}\cdots dw_1,
\end{aligned}$$

which yields (92). □

Let $P^*_{\mathrm{loss}}(z \mid y)$ ($|z| \le 1$, $y \ge 0$) denote the PGF of the number of losses in a batch which finds y amount of work in the system on arrival (cf. Theorem 3).

$$P^*_{\mathrm{loss}}(z \mid y) = \sum_{n=0}^{\infty} P_{\mathrm{loss}}(n \mid y) z^n.$$

Furthermore, let $P^*_{\mathrm{loss}}(z)$ ($|z| \le 1$) denote the PGF of the number of losses.

$$P^*_{\mathrm{loss}}(z) = \sum_{n=0}^{\infty} z^n P_{\mathrm{loss}}(n).$$

Note that $P^*_{\mathrm{loss}}(z)$ is given in terms of $P^*_{\mathrm{loss}}(z \mid y)$ by (cf. (20))

$$P^*_{\mathrm{loss}}(z) = \pi_0 P^*_{\mathrm{loss}}(z \mid 0) + \int_{0+}^{\infty} v(y) P^*_{\mathrm{loss}}(z \mid y)dy.$$

Theorem 12 $P^*_{\mathrm{loss}}(z \mid y)$ *(*$|z| \le 1$, $y \ge 0$*) is given by*

$$P^*_{\mathrm{loss}}(z \mid y) = H^*_\alpha(z,y) + \overline{H}^*_\alpha(z,y) \sum_{k=1}^{\infty} \mathrm{E}\left[\left\{ \prod_{i=2}^{k} \alpha \overline{H}^*_\alpha\Bigl(z, y + \sum_{j=1}^{i-1} G_j\Bigr) \right\} \cdot \left[1 - \alpha + \alpha H^*_\alpha\Bigl(z, y + \sum_{m=1}^{k} G_m\Bigr)\right]\right].$$

Proof Theorem 12 immediately follows from Lemma 8 and

$$P^*_{\mathrm{loss}}(z \mid y) = P^*_{\mathrm{loss}}(z, 0 \mid y) + \sum_{k=1}^{\infty} P^*_{\mathrm{loss}}(z, k \mid y).$$

4.2 $\mathrm{M^x/M/1{+}M_{diff}}$ queue

In this subsection, we assume that service times and impatience times are exponentially distributed, i.e., (51) and (67) follow, while the batch size distribution p_n $(n = 1, 2, \ldots)$ is a general discrete distribution. Let $L(t)$ $(t \ge 0)$ denote the total number of customers in the system at time t. In the $\mathrm{M^x/M/1{+}M_{diff}}$ queue, it is readily verified that $L(t)$ is formulated as a continuous time Markov chain $\mathcal{M} = \{L(t) \in \{0, 1, \ldots\};\ t \ge 0\}$ with infinitesimal generator

$$\boldsymbol{\Gamma} = \begin{pmatrix} -\lambda & p_1\lambda & p_2\lambda & p_3\lambda & \cdots \\ \mu & -\lambda-\mu & p_1\lambda & p_2\lambda & \cdots \\ 0 & \mu+\eta & -\lambda-\mu-\eta & p_1\lambda & \cdots \\ 0 & 0 & \mu+2\eta & -\lambda-\mu-2\eta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{93}$$

Note that there always exists a non-negative integer K such that

$$\sum_{n=1}^{\infty} n \cdot p_n \lambda < \mu + k\eta \quad \text{for all } k \ge K,$$

and therefore $\mathcal{M}$ is irreducible and positive recurrent. Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ denote the stationary probability vector of $\mathcal{M}$.

$$\boldsymbol{\pi\Gamma} = \mathbf{0}, \qquad \boldsymbol{\pi e} = 1,$$

where $\boldsymbol{e}$ denotes a column vector whose elements are all equal to one.

Lemma 9 π_n *(*$n = 1, 2, \ldots$*) satisfies*

$$\pi_n = \frac{\lambda}{\mu + (n-1)\eta} \sum_{k=0}^{n-1} \pi_k \overline{p}_{n-k}, \qquad n = 1, 2, \ldots. \tag{94}$$

Proof We consider a censored Markov chain $\mathcal{M}^{(n)}$ $(n = 0, 1, \ldots)$ obtained by observing $\mathcal{M}$ only when $L(t) \in \{0, 1, \ldots, n\}$. It is easy to see that the infinitesimal generator $\boldsymbol{\Gamma}^{(n)}$ of $\mathcal{M}^{(n)}$ is given by

$$\boldsymbol{\Gamma}^{(n)} = \begin{pmatrix} -\lambda & p_1\lambda & p_2\lambda & \cdots & p_{n-1}\lambda & \overline{p}_n\lambda \\ \mu & -\lambda-\mu & p_1\lambda & \cdots & p_{n-2}\lambda & \overline{p}_{n-1}\lambda \\ 0 & \mu+\eta & -\lambda-\mu-\eta & \cdots & p_{n-3}\lambda & \overline{p}_{n-2}\lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda-\mu-(n-2)\eta & \overline{p}_1\lambda \\ 0 & 0 & 0 & \cdots & \mu+(n-1)\eta & -\mu-(n-1)\eta \end{pmatrix}, \tag{95}$$

where $\overline{p}_n$ $(n = 1, 2, \ldots)$ is defined as in (2). Note that $\overline{p}_1 = 1$. Let $\boldsymbol{\pi}^{(n)} = (\pi_0^{(n)}, \pi_1^{(n)}, \ldots, \pi_n^{(n)})$ denote the stationary probability vector of $\mathcal{M}^{(n)}$.

$$\boldsymbol{\pi}^{(n)}\boldsymbol{\Gamma}^{(n)} = 0, \qquad \boldsymbol{\pi}^{(n)}\boldsymbol{e} = 1.$$

It is readily verified that $\boldsymbol{\pi}^{(n)}$ and the stationary probability vector of the original Markov chain $\mathcal{M}$ satisfy

$$\zeta_n\boldsymbol{\pi}^{(n)} = (\pi_0, \pi_1, \ldots, \pi_n),$$

where $\zeta_n = \sum_{i=0}^{n} \pi_i$. Therefore, we have

$$(\pi_0, \pi_1, \ldots, \pi_n)\boldsymbol{\Gamma}^{(n)} = \boldsymbol{0},$$

and in particular,

$$\sum_{k=0}^{n-1} \pi_k\overline{p}_{n-k}\lambda - \pi_n\{\mu + (n-1)\eta\} = 0.$$

We thus obtain (94). □

Remark 15 $\pi_1, \pi_2, \ldots$ is recursively given in terms of π_0 with (94). Therefore, an approximation to π_n $(n = 0, 1, \ldots)$ can be computed using (94) and the normalization condition

$$\sum_{n=0}^{\infty} \pi_n = 1.$$

Let $\pi^*(z)$ $(|z| \leq 1)$ denote the PGF of the queue length distribution.

$$\pi^*(z) = \sum_{n=0}^{\infty} \pi_n z^n.$$

It follows from (94) that

$$\sum_{n=1}^{\infty} \pi_n\{\mu + (n-1)\eta\}z^n = \lambda \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n-1} \pi_k\overline{p}_{n-k}$$

$$
\begin{aligned}
&= \lambda \sum_{k=0}^{\infty} \pi_k z^k \sum_{n=k+1}^{\infty} z^{n-k} \overline{p}_{n-k} \\
&= z\lambda \mathrm{E}[B] \cdot \pi^*(z) \tilde{p}^*(z),
\end{aligned}
$$

where $\tilde{p}^*(z)$ denotes the PGF of the equilibrium distribution for batch sizes.

$$
\tilde{p}^*(z) = \frac{1 - \displaystyle\sum_{n=1}^{\infty} p_n z^n}{\mathrm{E}[B](1-z)} = \sum_{n=1}^{\infty} \frac{\overline{p}_n}{\mathrm{E}[B]} \cdot z^{n-1}.
$$

On the other hand, we have

$$
\sum_{n=1}^{\infty} \pi_n \{\mu + (n-1)\eta\} z^n = (\mu - \eta)(\pi^*(z) - \pi_0) + z\eta \cdot \frac{d\pi^*(z)}{dz}.
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{d\pi^*(z)}{dz} &= \pi^*(z) \cdot \frac{\lambda \mathrm{E}[B] \tilde{p}^*(z)}{\eta} - \frac{\mu - \eta}{\eta z} \cdot (\pi^*(z) - \pi_0) \\
&= \frac{\pi_0(\mu - \eta)}{\eta z} + \pi^*(z) \cdot \frac{1}{\eta} \left(\lambda \mathrm{E}[B] \tilde{p}^*(z) - \frac{\mu - \eta}{z} \right).
\end{aligned}
$$

Below, we consider a real-variable function $\phi_{\mathrm{L}} : (0, 1] \to \mathbb{R}$ given by

$$
\phi_{\mathrm{L}}(x) = \sum_{n=0}^{\infty} \pi_n x^n.
$$

Note that $\phi_{\mathrm{L}}(x)$ $(0 < x \leq 1)$ satisfies

$$
\frac{d\phi_{\mathrm{L}}(x)}{dx} = \frac{(\mu - \eta)\pi_0}{\eta x} + \phi_{\mathrm{L}}(x) \cdot \frac{1}{\eta} \left(\lambda \mathrm{E}[B] \tilde{p}^*(x) - \frac{\mu - \eta}{x} \right), \qquad 0 < x \leq 1, \tag{96}
$$

and

$$
\lim_{x \to 0+} \phi_{\mathrm{L}}(x) = \pi_0, \qquad \phi_{\mathrm{L}}(1) = 1.
$$

Theorem 13 *π_0 and $\phi_{\mathrm{L}}(x)$ $(0 < x \leq 1)$ are given by*

$$
\pi_0 = \left\{ 1 + \frac{\lambda \mathrm{E}[B]}{\eta} \int_0^1 t^{(\mu-\eta)/\eta} \tilde{p}^*(t) \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_t^1 \tilde{p}^*(u) du \right] dt \right\}^{-1}, \tag{97}
$$

$$
\phi_{\mathrm{L}}(x) = \pi_0 \left\{ 1 + \frac{\lambda \mathrm{E}[B] x}{\eta} \int_0^1 t^{(\mu-\eta)/\eta} \tilde{p}^*(tx) \cdot \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_{tx}^x \tilde{p}^*(u) du \right] dt \right\}. \tag{98}
$$

The proof of Theorem 13 is provided in Appendix E.

Remark 16 If $\mu < \eta$, $t^{(\mu-\eta)/\eta}$ on the right-hand side of (98) is singular at $t = 0$. However, we can verify that

$$\begin{aligned}
&\int_0^1 t^{(\mu-\eta)/\eta} \tilde{p}^*(tx) \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_{tx}^x \tilde{p}^*(u)du\right] dt \\
&\qquad = \lim_{a\to 0+} \int_a^1 t^{(\mu-\eta)/\eta} \tilde{p}^*(tx) \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_{tx}^x \tilde{p}^*(u)du\right] dt \\
&\qquad \le 1 \cdot \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_0^x \tilde{p}^*(u)du\right] \lim_{a\to 0+} \int_a^1 t^{(\mu-\eta)/\eta} dt \\
&\qquad = \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_0^x \tilde{p}^*(u)du\right] \lim_{a\to 0+} \frac{1 - a^{(\mu-\eta)/\eta+1}}{(\mu-\eta)/\eta + 1} \\
&\qquad = \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_0^x \tilde{p}^*(u)du\right] \cdot \frac{1}{\mu/\eta} < \infty.
\end{aligned}$$

Remark 17 If the batch sizes are geometrically distributed, i.e., (43) holds, we have

$$\lambda \mathrm{E}[B] \tilde{p}^*(z) = \frac{\lambda}{1 - \alpha z}.$$

In this case, $\phi_{\mathrm{L}}(x)$ reduces to

$$\begin{aligned}
\phi_{\mathrm{L}}(x) &= \pi_0 \left\{1 + \frac{x}{\eta} \int_0^1 t^{(\mu-\eta)/\eta} \frac{\lambda}{1-\alpha t x} \exp\left[\frac{\lambda}{\eta} \int_{tx}^x \frac{1}{1-\alpha u} du\right] dt\right\} \\
&= \pi_0 \left\{1 + \frac{x}{\eta} \int_0^1 t^{(\mu-\eta)/\eta} \frac{\lambda}{1-\alpha t x} \cdot \left(\frac{1-\alpha t x}{1-\alpha x}\right)^{\lambda/(\eta\alpha)} dt\right\} \\
&= \pi_0 \left\{1 + \frac{\lambda x}{\eta} \left(\frac{1}{1-\alpha x}\right)^{\lambda/(\eta\alpha)} \int_0^1 t^{(\mu-\eta)/\eta} (1-\alpha t x)^{\lambda/(\eta\alpha)-1} dt\right\} \\
&= \pi_0 \left\{1 + \frac{\lambda x}{\eta} \left(\frac{1}{1-\alpha x}\right)^{\lambda/(\eta\alpha)} \cdot (\alpha x)^{-(\mu-\eta)/\eta - 1} (\alpha x)^{(\mu-\eta)/\eta+1} \right. \\
&\qquad \left. \cdot \int_0^1 t^{(\mu-\eta)/\eta} (1-\alpha t x)^{\lambda/(\eta\alpha)-1} dt\right\} \\
&= \pi_0 \left\{1 + \frac{\lambda (\alpha x)^{-(\mu-\eta)/\eta}}{\alpha\eta} \left(\frac{1}{1-\alpha x}\right)^{\lambda/(\eta\alpha)} B\left(\alpha x, \frac{\mu}{\eta}, \frac{\lambda}{\eta\alpha}\right)\right\},
\end{aligned}$$

where $B(x; \beta_1, \beta_2)$ $(x > 0,\ \beta_1 > 0,\ \beta_2 > 0)$ denotes the incomplete Beta function.

$$B(x; \beta_1, \beta_2) = \int_0^x t^{\beta_1 - 1} (1-t)^{\beta_2 - 1} dt = x^{\beta_1} \int_0^1 t^{\beta_1 - 1} (1 - tx)^{\beta_2 - 1} dt, \qquad \beta_1 > 0, \beta_2 > 0.$$

Therefore, with $\phi_{\mathrm{L}}(1) = 1$, π_0 is given by

$$\pi_0 = \left\{1 + \frac{\lambda \alpha^{-\mu/\eta}}{\eta} \left(\frac{1}{1-\alpha}\right)^{\lambda/(\eta\alpha)} B\left(\alpha, \frac{\mu}{\eta}, \frac{\lambda}{\eta\alpha}\right)\right\}^{-1}.$$

4.2.1 Virtual waiting time and number of losses

We first consider the stationary virtual waiting time in the M^{x}/M/1+M$_{\text{diff}}$ queue. Obviously, if the number of customers in the system is equal to n ($n = 1, 2, \ldots$), then the virtual waiting time is given by the absorption time of an absorbing-state Markov chain with finite state-space $\{0, 1, \ldots, n\}$, the initial state equal to n, and infinitesimal generator given by

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \mu & -\mu & 0 & \cdots & 0 & 0 \\ 0 & \mu+\eta & -\mu-\eta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\mu-(n-2)\eta & 0 \\ 0 & 0 & 0 & \cdots & \mu+(n-1)\eta & -\mu-(n-1)\eta \end{pmatrix}.$$

The LST $v^*(s)$ ($\mathrm{Re}(s) > 0$) of the stationary virtual waiting time is then given in terms of the stationary queue length distribution π_n ($n = 0, 1, \ldots$).

$$f^*(s) = \pi_0 + \sum_{n=1}^{\infty} \pi_n \prod_{i=1}^{n} \frac{\mu + (i-1)\eta}{s + \mu + (i-1)\eta}.$$

Next, we consider the number of losses in each batch. Let $P(n, k)$ ($k = 0, 1, \ldots, n$, $n = 1, 2, \ldots$) denote the conditional probability that the number of lost customers in a randomly chosen batch is equal to k, given that this batch contains n customers on arrival.

$$P(n, k) = \Pr(N_{\text{loss}} = k \mid B = n),$$

where B denotes the number of customers in a batch, and N_{loss} denotes the number of lost customers out of B. Further let L_{A} denote the number of customers in the system this batch finds on arrival. We define $P_m(n, k)$ ($k = 0, 1, \ldots, n$, $n = 1, 2, \ldots$, $m = 0, 1, \ldots$) as

$$P_m(n, k) = \Pr(N_{\text{loss}} = k \mid L_{\text{A}} = m, B = n).$$

Because the size of an arriving batch is independent of the number of customers in the system seen on arrival, it follows from PASTA that

$$P(n, k) = \sum_{m=0}^{\infty} \pi_m P_m(n, k), \qquad k = 0, 1, \ldots, n, n = 1, 2, \ldots. \tag{99}$$

Owing to the memoryless property of the exponential distribution, $P_m(n, k)$ ($k = 0, 1, \ldots, n$, $n = 1, 2, \ldots$, $m = 0, 1, \ldots$) can be recursively computed as follows. For $m = 0$, $P_0(1, 0)$ and $P_0(1, 1)$ are given by

$$P_0(1, 0) = 1, \qquad P_0(1, 1) = 0.$$

$P_0(n,k)$ for $n = 2, 3, \ldots$ is then given recursively by

$$\begin{aligned}
P_0(n,0) &= \frac{\mu}{\mu + (n-1)\eta} \cdot P_0(n-1,0), \\
P_0(n,k) &= \frac{\mu}{\mu + (n-1)\eta} \cdot P_0(n-1,k) + \frac{(n-1)\eta}{\mu + (n-1)\eta} \cdot P_0(n-1,k-1), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \ldots, n-1, \\
P_0(n,n) &= \frac{(n-1)\eta}{\mu + (n-1)\eta} \cdot P_0(n-1,n-1).
\end{aligned}$$

Similarly, for $m = 1, 2, \ldots$, we have

$$\begin{aligned}
P_m(1,0) &= \frac{\mu + (m-1)\eta}{\mu + m\eta} \cdot P_{m-1}(1,0), \\
P_m(1,1) &= \frac{\mu + (m-1)\eta}{\mu + m\eta} \cdot P_{m-1}(1,1) + \frac{\eta}{\mu + m\eta},
\end{aligned}$$

and

$$\begin{aligned}
P_m(n,0) &= \frac{\mu + (m-1)\eta}{\mu + (m+n-1)\eta} \cdot P_{m-1}(n,0), \\
P_m(n,k) &= \frac{\mu + (m-1)\eta}{\mu + (m+n-1)\eta} \cdot P_{m-1}(n,k) \\
&\quad + \frac{(n-1)\eta}{\mu + (m+n-1)\eta} \cdot P_m(n-1,k-1), \qquad k = 1, 2, \ldots, n,
\end{aligned}$$

which determines $P_m(n,k)$ ($k = 0, 1, \ldots, n$, $n = 1, 2, \ldots$, $m = 1, 2, \ldots$).

Therefore, using (99), we can compute $P(n,k)$ ($k = 0, 1, \ldots, n$) for any n via an algorithmic approach. In addition, the probability function $P_{\mathrm{loss}}(k)$ ($k = 0, 1, \ldots$) of the number of losses in a randomly chosen batch is given by

$$P(k) = \sum_{n=k}^{\infty} p_n P(n,k).$$

5 $\mathrm{M^x/G/1{+}D}$ queue

In this section, we consider a special case of both $\mathrm{M^x/G/1{+}G_{same}}$ and $\mathrm{M^x/G/1{+}G_{diff}}$ queues, viz., the $\mathrm{M^x/G/1{+}D}$ queue. This is the case that impatience times of customers are constant and equal to τ ($\tau > 0$):

$$\overline{H}(x) = \begin{cases} 1, & x < \tau, \\ 0, & x \geq \tau. \end{cases} \tag{100}$$

Lemma 10 *In the $M^x/G/1+D$ queue, $\overline{G}(w \mid y)$ $(w \geq 0,\ y \geq 0)$ is given by*

$$\overline{G}(w \mid y) = \begin{cases} \overline{G}_{\mathrm{batch}}(w), & y < \tau, w < \tau - y, \\ \overline{G}(w) + \displaystyle\sum_{k=2}^{\infty} \overline{p}_k \int_0^{\tau - y} g^{(k-1)}(u)\overline{G}(w-u)du, & y < \tau, w \geq \tau - y, \\ 0, & y \geq \tau, \end{cases} \tag{101}$$

where $\overline{G}_{\mathrm{batch}}(x)$ $(x \geq 0)$ is defined in (36).

The proof of Lemma 10 is provided in Appendix F.

In the $\mathrm{M^x/G/1+D}$ queue, it is clear that the system is always stable. Therefore, we can determine $v(x)$ and π_0, using Lemma 10, (8), (11), and (101). Let $\phi(x)$ $(x > 0)$ denote a function defined as

$$\phi(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \rho^n \tilde{g}_{\mathrm{batch}}^{(n)}(x), & 0 < x < \tau, \\ \rho \overline{G}_{\mathrm{batch}}(x \mid 0) + \lambda \displaystyle\int_{0+}^{\tau} \phi(y)\overline{G}(x-y \mid y)dy, & x \geq \tau, \end{cases}$$

where $\tilde{g}_{\mathrm{batch}}^{(n)}(x)$ $(x \geq 0,\ n = 1, 2, \ldots)$ denotes a p.d.f. which is given by

$$\tilde{g}_{\mathrm{batch}}^{(1)}(x) = \frac{\overline{G}_{\mathrm{batch}}(x)}{\mathrm{E}[B]\mathrm{E}[G]}, \qquad x \geq 0,$$

$$\tilde{g}_{\mathrm{batch}}^{(n)}(x) = \tilde{g}_{\mathrm{batch}}^{(n-1)} * \tilde{g}_{\mathrm{batch}}^{(1)}(x), \qquad x \geq 0, n = 2, 3, \ldots.$$

Theorem 14 *π_0 and $v(x)$ in the $M^x/G/1+D$ queue are given by*

$$\pi_0 = \left[1 + \int_{0+}^{\infty} \phi(x)dx\right]^{-1}, \tag{102}$$

$$v(x) = \pi_0 \phi(x), \qquad x > 0. \tag{103}$$

Proof Using (101), it is easy to see that for $0 < x < \tau$, (9) and (10) are reduced to be

$$\phi_n(x) = \rho^n \tilde{g}_{\mathrm{batch}}^{(n)}(x), \qquad 0 < x < \tau.$$

Therefore, (8) implies

$$v(x) = \pi_0 \sum_{n=1}^{\infty} \rho^n \tilde{g}_{\mathrm{batch}}^{(n)}(x), \quad 0 < x < \tau,\ n = 1, 2, \ldots.$$

On the other hand, for $x \geq \tau$, it follows from (7) and (101) that

$$v(x) = \pi_0 \rho \overline{G}_{\mathrm{batch}}(x \mid 0) + \lambda \int_{0+}^{\tau} v(y)\overline{G}(x-y \mid y)dy, \qquad x \geq \tau.$$

From these equations, (103) immediately follows. Furthermore, (102) is obtained from the normalization condition (6). □

Using Theorem 14, we can verify that if the traffic intensity ρ is less than one,

$$v(x) = \frac{\pi_0}{1-\rho} \cdot v_{\mathrm{M^x/G/1}}(x), \qquad x < \tau, \tag{104}$$

where $v_{\mathrm{M^x/G/1}}(x)$ denotes the PDF of the stationary virtual waiting time $V_{\mathrm{M^x/G/1}}$ in the corresponding $\mathrm{M^x/G/1}$ queue without impatience, which is identical to that in the ordinary M/G/1 queue with the arrival rate λ and the complementary PDF of the service time distribution $\overline{G}_{\mathrm{batch}}(x)$. Note that (104) is valid only for $\rho < 1$ (otherwise the corresponding M/G/1 queue is not stable), whereas Theorem 14 still holds for $\rho \geq 1$.

Next, we consider the actual waiting time.

Theorem 15 *In the $M^x/G/1+D$ queue, $D(0)$ and $d(x)$ ($x \geq 0$) are given by*

$$D(0) = \left[1 + \int_{0+}^{\tau} \left\{ \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)}(x) + \sum_{n=1}^{\infty} \rho^n \left(\tilde{g}_{\mathrm{batch}}^{(n)}(x) + \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)} * \tilde{g}_{\mathrm{batch}}^{(n)}(x) \right) \right\} dx \right]^{-1}, \tag{105}$$

$$d(x) = \begin{cases} D(0) \left[\displaystyle\sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)}(x) + \sum_{n=1}^{\infty} \rho^n \left(\tilde{g}_{\mathrm{batch}}^{(n)}(x) + \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)} * \tilde{g}_{\mathrm{batch}}^{(n)}(x) \right) \right], & 0 < x < \tau, \\ 0, & x \geq \tau. \end{cases} \tag{106}$$

Proof For $x \geq \tau$, $d(x) = 0$ immediately follows from (35) and (100). On the other hand, for $0 < x < \tau$, it is verified that (39) and (40) reduce to

$$\phi_{\mathrm{D},1}(x) = \rho \tilde{g}_{\mathrm{batch}}^{(1)}(x) + \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)}(x), \qquad 0 < x < \tau,$$

$$\phi_{\mathrm{D},n}(x) = \rho^n \tilde{g}_{\mathrm{batch}}^{(n)}(x) + \rho^{n-1} \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)} * \tilde{g}_{\mathrm{batch}}^{(n-1)}(x), \qquad 0 < x < \tau,\ n = 2, 3, \ldots.$$

Therefore, using (37), we obtain

$$d(x) = D(0) \left[\rho \tilde{g}_{\mathrm{batch}}^{(1)}(x) + \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)}(x) + \sum_{n=2}^{\infty} \rho^n \tilde{g}_{\mathrm{batch}}^{(n)}(x) + \sum_{n=2}^{\infty} \rho^{n-1} \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)} * \tilde{g}_{\mathrm{batch}}^{(n-1)}(x) \right]$$

$$= D(0)\left[\sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)}(x) + \sum_{n=1}^{\infty} \rho^n \tilde{g}_{\text{batch}}^{(n)}(x) \right.$$
$$\left. + \sum_{n=1}^{\infty} \rho^n \sum_{m=1}^{\infty} \overline{p}_{m+1} g^{(m)} * \tilde{g}_{\text{batch}}^{(n)}(x) \right].$$

(106) then follows from this equation, and (105) follows from the normalization condition (32). □

For $0 < x < \tau$, we can rewrite (106) as

$$d(x) = \frac{D(0)\mathrm{E}[B]}{1-\rho}\left[(1-\rho)\sum_{m=1}^{\infty} \frac{\overline{p}_{m+1}}{\mathrm{E}[B]} \cdot g^{(m)}(x) \right.$$
$$\left. + \sum_{n=1}^{\infty}(1-\rho)\rho^n \left(\frac{\overline{p}_1}{\mathrm{E}[B]} \cdot \tilde{g}_{\text{batch}}^{(n)}(x) + \sum_{m=1}^{\infty} \frac{\overline{p}_{m+1}}{\mathrm{E}[B]} \cdot g^{(m)} * \tilde{g}_{\text{batch}}^{(n)}(x) \right) \right].$$

Therefore, if $\rho < 1$ (cf. (104)),

$$\begin{aligned} d(x) &= \frac{D(0)\mathrm{E}[B]}{1-\rho}\left[\Pr(V_{\mathrm{M^x/G/1}} = 0)\sum_{m=1}^{\infty} \frac{\overline{p}_{m+1}}{\mathrm{E}[B]} \cdot g^{(m)}(x) + \frac{\overline{p}_1}{\mathrm{E}[B]} \cdot v_{\mathrm{M^x/G/1}}(x) \right. \\ &\qquad \left. + \sum_{m=1}^{\infty} \frac{\overline{p}_{m+1}}{\mathrm{E}[B]} \cdot g^{(m)} * v_{\mathrm{M^x/G/1}}(x) \right] \\ &= \frac{D(0)\mathrm{E}[B]}{1-\rho} \cdot d_{\mathrm{M^x/G/1}}(x), \qquad x < \tau, \end{aligned} \tag{107}$$

where $d_{\mathrm{M^x/G/1}}(x)$ denotes the PDF of the stationary actual waiting time $D_{\mathrm{M^x/G/1}}$ in the corresponding M$^{\mathrm{x}}$/G/1 queue. Note that the normalization constant is determined from the relation

$$\begin{aligned} 1 - D(0) &= \int_{0+}^{\tau} d(x)dx \\ &= \frac{D(0)\mathrm{E}[B]}{1-\rho}\int_{0+}^{\tau} d_{\mathrm{M^x/G/1}}(x)dx \\ &= \frac{D(0)\mathrm{E}[B]}{1-\rho}\left(\Pr(D_{\mathrm{M^x/G/1}} \le \tau) - \Pr(D_{\mathrm{M^x/G/1}} = 0)\right) \\ &= \frac{D(0)\mathrm{E}[B]}{1-\rho}\left(\Pr(D_{\mathrm{M^x/G/1}} \le \tau) - \frac{1-\rho}{\mathrm{E}[B]}\right), \end{aligned}$$

which implies

$$D(0) = \frac{1-\rho}{\mathrm{E}[B]\Pr(D_{\mathrm{M^x/G/1}} \le \tau)}. \tag{108}$$

This leads to an alternative formula for the loss probability P_{loss} in the case of $\rho < 1$.

Corollary 5 *If $\rho < 1$, the stationary loss probability P_{loss} in the $M^{x}/G/1{+}D$ queue is given by*

$$P_{\mathrm{loss}} = \frac{(1-\rho)\Pr(D_{\mathrm{M^x/G/1}} > \tau)}{1-\rho\Pr(D_{\mathrm{M^x/G/1}} > \tau)}, \tag{109}$$

where $D_{\mathrm{M^x/G/1}}$ denotes a generic random variable for the actual waiting time in the ordinary $M^{x}/G/1$ queue.

Proof We obtain (109) with a straightforward calculations using (42) and (108). □

Remark 18 The formula (109) is stated in [10] without a proof. Kim and Kim [20] shows a similar result for the $\mathrm{M^x/M}/c{+}\mathrm{D}$ queue, where they state that the same approach does not seem to be applicable to the $\mathrm{M^x/G/1{+}D}$ queue.

Remark 19 When the batch size distribution is geometric, i.e., in the $\mathrm{M^{geo}/G/1{+}D}$ queue, the distribution of the number of losses in $P_{\mathrm{loss}}(n)$ $(n = 0, 1, \ldots)$ is given by (49), (50), and (109) if $\rho < 1$.

5.1 Busy period

In this subsection, we derive the LST of the busy period in the $\mathrm{M^{[x]}/G/1{+}D}$ queue. We define $g_{\mathrm{batch}}(x)$ $(x \geq 0)$ as the p.d.f. of the total amount of work required by a batch (cf. (36)).

$$g_{\mathrm{batch}}(x) = \sum_{n=1}^{\infty} p_n g^{(n)}(x).$$

We further define $g_{\mathrm{batch}}^{(n)}(x)$ $(x \geq 0)$ as

$$g_{\mathrm{batch}}^{(1)}(x) = g_{\mathrm{batch}}(x), \qquad g_{\mathrm{batch}}^{(n)}(x) = g_{\mathrm{batch}}^{(n-1)} * g_{\mathrm{batch}}(x), \quad n = 2, 3, \ldots.$$

Let $\{N(t);\, t \geq 0\}$ denote a Poisson counting process with rate λ. By definition,

$$\Pr(N(t) = k) = \frac{\exp[-\lambda t](\lambda t)^k}{k!}, \qquad t \geq 0,\ k = 0, 1, \ldots.$$

Also, let $\{Y(t);\, t \geq 0\}$ denote a compound Poisson process defined as

$$Y(t) = \sum_{n=1}^{N(t)} G_{\mathrm{batch},n}, \qquad t \geq 0,$$

where $\{G_{\mathrm{batch},n}\}_{n=1,2,\ldots}$ denotes a sequence of i.i.d. random variables that follow the p.d.f. $g_{\mathrm{batch}}(x)$, which represents the total required service time of arriving batches. We define $h(x,t)$ $(x > 0,\ t \geq 0)$ as the p.d.f. of $Y(t)$.

$$h(x,t) = \frac{d\Pr(Y(t) \leq x)}{dx} = \sum_{k=1}^{\infty} \frac{\exp[-\lambda t](\lambda t)^k}{k!} \cdot g_{\mathrm{batch}}^{(k)}(x).$$

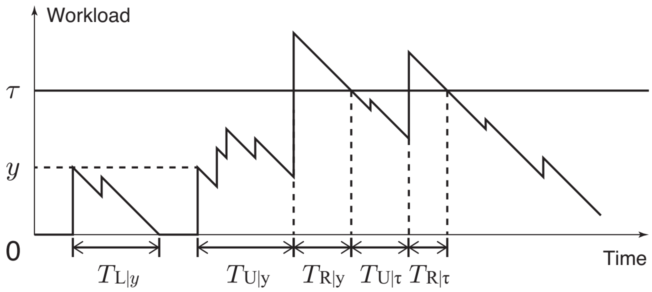

Fig. 1 Examples of busy periods with initial workload $y < \tau$.

Note that

$$\Pr(Y(t) = 0) + \int_{0+}^{\infty} h(x,t)dx = \exp[-\lambda t] + \int_{0+}^{\infty} h(x,t)dx = 1.$$

Let $V_{[t]}$ $(t \geq 0)$ denote the virtual waiting time at time t. For $0 < y \leq \tau$, we define $T_{L|y}$ and $T_{U|y}$ as stopping times given by

$$T_{L|y} = \inf\{t > 0; Y(t) = -y + t\}, \quad T_{U|y} = \inf\{t > 0; Y(t) \geq \tau - y + t\}.$$

Note that $T_{L|y}$ and $T_{U|y}$ are related to the lengths of first passage times to $V_{[t]} = 0$ and $V_{[t]} = \tau$, respectively, given that $V_{[0]} = y$ (see Figure 1). If $T_{L|y} < T_{U|y}$ then $V_{[T_{L|y}]} = 0$, and the busy period terminates. On the other hand, if $T_{U|y} < T_{L|y}$ then $V_{[T_{U|y}]} = \tau + T_{R|y}$, where $T_{R|y}$ denotes the overshoot above level τ. The system then does not admit new batches during the next $T_{R|y}$ time units. Subsequently a new cycle starts with $y = \tau$, until one of the boundaries is crossed. If the lower boundary is crossed then the busy period terminates, otherwise a new cycle starts with $y = \tau$. In all cycles, the random variables $T_{U|y}$ and $T_{R|y}$ are dependent (unless the service time distribution is exponential).

We then define LSTs $\Psi^*_{L|y}(s)$ and $\Psi^{**}_{U|y}(s,\omega)$ $(\mathrm{Re}(s) > 0)$ as

$$\begin{aligned}\Psi^*_{L|y}(s) &= \mathrm{E}\left[\mathbb{1}\{T_{L|y} < T_{U|y}\} \exp\left[-sT_{L|y}\right]\right], \\ \Psi^{**}_{U|y}(s,\omega) &= \mathrm{E}\left[\mathbb{1}\{T_{U|y} < T_{L|y}\} \exp\left[-sT_{U|y}\right] \exp\left[-\omega T_{R|y}\right]\right],\end{aligned}$$

where $\mathbb{1}\{\cdot\}$ denotes an indicator function. We also define $\psi_{L|y}(t)$ $(t > y)$ and $\zeta_{U|y}(t,x)$ $(t > 0,\ x > 0)$ as the p.d.f. and the joint p.d.f. corresponding to $\Psi^*_{L|y}(s)$ and $\Psi^{**}_{U|y}(s,\omega)$, respectively. Note that

$$\begin{aligned}\Psi^*_{L|y}(s) &= \exp[-\lambda y]\exp[-sy] + \int_{y+}^{\infty} \psi_{L|y}(t)\exp[-st]dt, \\ \Psi^{**}_{U|y}(s,\omega) &= \int_{x=0+}^{\infty}\int_{t=0+}^{\infty} \zeta_{U|y}(t,x)\exp[-st]\exp[-\omega x]dtdx.\end{aligned}$$

We first consider $T_{\mathrm{L}|y}$. For $0 < y \leq \tau$ and $t \geq 0$, we define $g_{\tau-y}(x,t)$ as a p.d.f. given by

$$\begin{aligned} g_{\tau-y}(x,t) &= \frac{d}{dx}\Pr(Y(t) \leq x, T_{\mathrm{U}|y} > t) \\ &= \frac{d}{dx}\Pr(Y(t) \leq x, Y(w) < \tau - y + w, w \leq t), \quad x < \tau - y + t. \end{aligned}$$

Lemma 11 *$g_{\tau-y}(x,t)$ ($x < \tau - y + t$) is given by*

$$g_0(x,t) = \frac{t-x}{t} \cdot h(x,t), \qquad y = \tau,$$

and

$$\begin{aligned} &g_{\tau-y}(x,t) \\ &= h(x,t) - \mathbb{1}\{x > \tau - y\}\biggl[h(x, x-\tau+y)\exp[-\lambda(t-x+\tau-y)] \\ &\qquad + (t-x+\tau-y)\int_0^{x-\tau+y} \frac{1}{t-u} \cdot h(u+\tau-y, u) \\ &\qquad \cdot h(x-\tau+y-u, t-u)du\biggr], \quad y < \tau. \end{aligned}$$

Proof Lemma 11 immediately follows from the results in [25]. □

Lemma 12 *$\Psi_{\mathrm{L}|y}(t)$ ($t > y$) is given by*

$$\Psi_{\mathrm{L}|y}(t) = \sum_{n=1}^{\infty}(-1)^n \Psi_{\mathrm{L}|y}(n;t),$$

where $\Psi_{\mathrm{L}|y}(n;t)$ ($n = 1, 2, \ldots$) is defined as

$$\begin{aligned} \Psi_{\mathrm{L}|y}(1;t) &= g_{\tau-y}(t-y,t) - \exp[-\lambda y] g_\tau(t-y, t-y), \\ \Psi_{\mathrm{L}|y}(n;t) &= \int_y^t \Psi_{\mathrm{L}|y}(n-1;w) g_\tau(t-w, t-w)dw, \quad n = 2, 3, \ldots. \end{aligned}$$

Proof It is readily to see that $\Psi_{\mathrm{L}|y}(t)$ ($t > y$) satisfies a Volterra integral equation of the second kind:

$$\begin{aligned} \Psi_{\mathrm{L}|y}(t) = g_{\tau-y}(t-y,t) &- \exp[-\lambda y] g_\tau(t-y, t-y) \\ &- \int_y^t \Psi_{\mathrm{L}|y}(w) g_\tau(t-w, t-w)dw. \end{aligned}$$

This concludes the proof. □

We next consider $T_{\mathrm{U}|y}$ and $T_{\mathrm{R}|y}$. For $0 < y \le \tau$, we define a p.d.f. $g_{y,\tau-y}(x,t)$ $(0 < x < \tau,\ t \ge 0)$ as

$$g_{y,\tau-y}(x,t) = \frac{d}{dx} \Pr\big(Y(t) \le x, \min(T_{\mathrm{L}|y}, T_{\mathrm{U}|y}) > t\big) .$$

By definition, we have

$$\begin{aligned} g_{y,\tau-y}(x,t) = g_{\tau-y}(x,t) - \mathbb{1}\{t > y\}\Big[&\exp[-\lambda y]h(x, t-y) \\ &+ \int_y^t \psi_{\mathrm{L}|y}(w)h(x-w+y, t-w)dw\Big]. \end{aligned} \tag{110}$$

Furthermore, we define $g(w \mid y)$ $(w > 0,\ 0 \le y < \tau)$ as the p.d.f. corresponding to the complementary PDF $\overline{G}(w \mid y)$. From (101), we have

$$g(w \mid y) = \begin{cases} g_{\mathrm{batch}}(w), & w < \tau - y, \\ g(w) + \displaystyle\sum_{k=2}^{\infty} \overline{p}_k \int_0^{\tau-y} g^{(k-1)}(u) g(w-u)du, & w \ge \tau - y. \end{cases} \tag{111}$$

Lemma 13 *The joint density $\zeta_{\mathrm{U}|y}(t,x)$ of $T_{\mathrm{U}|y}$ and $T_{\mathrm{R}|y}$ is given by*

$$\begin{aligned} \zeta_{\mathrm{U}|y}(t,x) &= \int_{\min(0,t-y)}^{t+\tau-y} g_{y,\tau-y}(u,t) \cdot \lambda g(t+\tau-y-u+x \mid u)du \\ &= \begin{cases} \displaystyle\int_0^{t+\tau-y} g_{\tau-y}(u,t) \cdot \lambda g(t+\tau-y-u+x \mid u)du, & t \le y, \\ \displaystyle\int_{t-y}^{t+\tau-y} g_{y,\tau-y}(u,t) \cdot \lambda g(t+\tau-y-u+x \mid u)du, & t > y. \end{cases} \end{aligned}$$

Proof Lemma 13 immediately follows from (110) and the definitions of the p.d.f.s $\zeta_{(T_{U_y},R_y)}(t,r)$ and $g_{y,\tau-y}(u,t)$. □

We thus obtain the LSTs $\Psi^*_{\mathrm{L}|y}(s)$ and $\Psi^{**}_{\mathrm{U}|y}(s,\omega)$ from Lemmas 12 and 13.

Finally, we consider the length T_{BP} of a busy period. We define $\Psi^*_{\mathrm{BP}}(s)$ $(\mathrm{Re}(s) > 0)$ as the LST of T_{BP}.

Theorem 16 *$\Psi^*_{\mathrm{BP}}(s)$ $(\mathrm{Re}(s) > 0)$ is given by*

$$\begin{aligned} \Psi^*_{\mathrm{BP}}(s) = &\int_0^{\tau} \left(\Psi^*_{\mathrm{L}|w}(s) + \Psi^{**}_{\mathrm{U}|w}(s,s) \cdot \frac{\Psi^*_{\mathrm{L}|\tau}(s)}{1 - \Psi^{**}_{\mathrm{U}|\tau}(s,s)} \right) g(w \mid 0)dw \\ &+ \int_0^{\infty} \exp[-sw] \cdot \frac{\Psi^*_{\mathrm{L}|\tau}(s)}{1 - \Psi^{**}_{\mathrm{U}|\tau}(s,s)} \cdot g(\tau + w \mid 0)dw. \end{aligned} \tag{112}$$

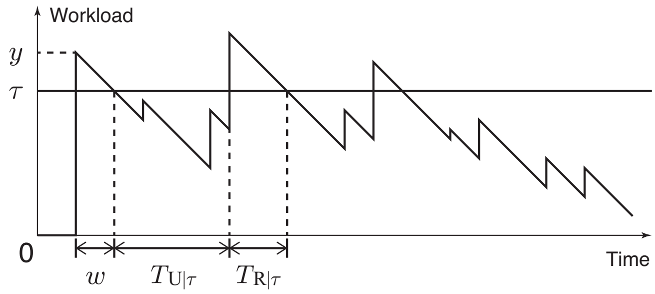

Fig. 2 An example of a busy period with initial workload $y > \tau$.

Proof We distinguish between two cases: (I) the total required service time of the first batch in the busy period process equals some value $y < \tau$, and (II) this service requirement exceeds τ. See Figures 1 and 2.

For case (I) we further distinguish two possibilities: (I-i) the workload will not exceed level τ before the end of the busy period, and (I-ii) the workload *does* exceed that level before the end of the busy period. By definition, the busy period LST in case (I-i) is given by $\Psi^*_{L_y}(s)$.

In case (I-ii), on the other hand, $T_{\mathrm{U}|y} + T_{\mathrm{R}|y}$ forms the first part of the busy period. The remaining part of the busy period is independent of the previous part. It consists of two components. The first component is a geometrically distributed number of intervals that start when the workload process downcrosses level τ and end when, after the next upcrossing of level τ, that level is downcrossed once more. The second component is an interval that also starts when the workload downcrosses level τ, but ends when the busy period ends – this is an interval in which level τ is not reached anymore. It has LST $\Psi^*_{L_\tau}(s)$. The lengths of those intervals that occur geometrically often are all i.i.d., and distributed as $T_{\mathrm{U}|\tau} + T_{\mathrm{R}|\tau}$. Therefore, the LST of the total length of the busy period in case (I) is given by the first term on the right-hand side of (112).

Similarly, we can readily verify that the LST of the busy period in case (II) is given by the second term on the right-hand side of (112). $\square$

6 Conclusion

In this paper we have considered a batch arrival $\mathrm{M}^{\mathrm{x}}/\mathrm{G}/1$ queue with impatient customers. We have considered both the case in which customers in the same batch have the same impatience time, and the case in which these impatience times may differ. Observing that the workload in both variants can be viewed as the workload in an M/G/1 queue in which the service time of a customer depends on its waiting time y in a particular way (PDF $G(w \mid y)$), we have first focussed on deriving an expresssion for that workload density. We have subsequently expressed the actual waiting time distribution (for the first variant) and the loss probability (for both model variants) into that workload density. We have also considered several special cases. One of those is the

$M^{[X]}/G/1+D$ queue, for which both model variants coincide. Our results for the latter model include the busy period distribution.

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Appendices

A Proof of Lemma 2

By definition, $\overline{G}(w \mid y)$ $(w \geq 0,\ y \geq 0)$ is given by

$$\overline{G}(w \mid y) = \int_w^\infty \sum_{n=0}^\infty \sum_{k=1}^\infty f_{\text{total}}(n,k;t \mid y)dt, \tag{113}$$

where $f_{\text{total}}(n,k;w \mid y)$ $(w > 0,\ y \geq 0,\ n = 0,1,\ldots,\ k = 1,2,\ldots)$ is defined as

$$f_{\text{total}}(n,1;w \mid y) = f(n,1;w \mid y),$$

and

$$\begin{aligned} &f_{\text{total}}(n,k;w \mid y) \\ &= \int_{w_1=0}^{w} \int_{w_2=0}^{w-w_1} \cdots \int_{w_{k-1}=0}^{w-w_1-w_2-\cdots-w_{k-2}} f\Big(n,k;w_1,w_2,\ldots,w_{k-1},w-\sum_{m=1}^{k-1} w_m \mid y\Big) \\ &\qquad \cdot dw_{k-1}dw_{k-2}\cdots dw_1, \quad k = 2,3,\ldots. \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} f_{\text{total}}(0,1;w \mid y) &= p_1 \overline{H}(y) g(w), \\ f_{\text{total}}(n,1;w \mid y) &= p_{n+1} \left[\overline{H}(y) - \overline{H}(y+w)\right] g(w), \quad n = 1,2,\ldots, \end{aligned}$$

and for $k = 2,3,\ldots$,

$$\begin{aligned} &f_{\text{total}}(0,k;w \mid y) \\ &= p_k \int_{w_1=0}^{w} \int_{w_2=0}^{w-w_1} \cdots \int_{w_{k-1}=0}^{w-w_1-w_2-\cdots-w_{k-2}} \overline{H}\Big(y + \sum_{m=1}^{k-1} w_m\Big) \\ &\qquad \cdot \left\{\prod_{i=1}^{k-1} g(w_i)\right\} g\left(w - \sum_{m=1}^{k-1} w_m\right) dw_{k-1}dw_{k-2}\cdots dw_1, \\ &f_{\text{total}}(n,k;w \mid y) \\ &= p_{n+k} \int_{w_1=0}^{w} \int_{w_2=0}^{w-w_1} \cdots \int_{w_{k-1}=0}^{w-w_1-w_2-\cdots-w_{k-2}} \left[\overline{H}\Big(y + \sum_{m=1}^{k-1} w_m\Big) - \overline{H}(y+w)\right] \end{aligned}$$

$$\cdot \left\{ \prod_{i=1}^{k-1} g(w_i) \right\} g\left(w - \sum_{m=1}^{k-1} w_m \right) dw_{k-1} dw_{k-2} \cdots dw_1, \quad n = 1, 2, \ldots.$$

With a straightforward calculation, we can verify that for $k = 2, 3, \ldots$,

$$f_{\mathrm{total}}(0, k; w \mid y) = p_k \int_0^w \overline{H}(y+u) g^{(k-1)}(u) g(w-u) du,$$

$$f_{\mathrm{total}}(n, k; w \mid y) = p_{n+k} \int_0^w \left[\overline{H}(y+u) - \overline{H}(y+w) \right] g^{(k-1)}(u) g(w-u) du, \quad n = 1, 2, \ldots.$$

We then define $f_{\mathrm{total}}(w \mid y)$ ($y \geq 0$, $w > 0$, $n = 0, 1, \ldots$) as

$$f_{\mathrm{total}}(w \mid y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} f_{\mathrm{total}}(n, k; w \mid y).$$

It follows that

$$\begin{aligned}
f_{\mathrm{total}}(w \mid y) &= p_1 \overline{H}(y) g(w) + \sum_{n=1}^{\infty} p_{n+1} \left[\overline{H}(y) - \overline{H}(y+w) \right] g(w) \\
&\quad + \sum_{k=2}^{\infty} p_k \int_0^w \overline{H}(y+u) g^{(k-1)}(u) g(w-u) du \\
&\quad + \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} p_{n+k} \int_0^w \left[\overline{H}(y+u) - \overline{H}(y+w) \right] g^{(k-1)}(u) g(w-u) du \\
&= p_1 \overline{H}(y) g(w) + \overline{p}_2 \left[\overline{H}(y) - \overline{H}(y+w) \right] g(w) \\
&\quad + \sum_{k=2}^{\infty} p_k \int_0^w \overline{H}(y+u) g^{(k-1)}(u) g(w-u) du \\
&\quad + \sum_{k=2}^{\infty} \overline{p}_{k+1} \int_0^w \left[\overline{H}(y+u) - \overline{H}(y+w) \right] g^{(k-1)}(u) g(w-u) du \\
&= p_1 \overline{H}(y) g(w) + \overline{p}_2 \left[\overline{H}(y) - \overline{H}(y+w) \right] g(w) \\
&\quad + \sum_{k=2}^{\infty} \overline{p}_k \int_0^w \overline{H}(y+u) g^{(k-1)}(u) g(w-u) du \\
&\quad - \sum_{k=2}^{\infty} \overline{p}_{k+1} \overline{H}(y+w) g^{(k)}(w) \\
&= \overline{H}(y) g(w) - \sum_{k=1}^{\infty} \overline{p}_{k+1} \overline{H}(y+w) g^{(k)}(w) \\
&\quad + \sum_{k=2}^{\infty} \overline{p}_k \int_0^w \overline{H}(y+u) g^{(k-1)}(u) g(w-u) du
\end{aligned}$$

Using (113), we have

$$\begin{aligned}
\overline{G}(w \mid y) &= \int_w^{\infty} f_{\mathrm{total}}(t \mid y) dt \\
&= \overline{H}(y) \overline{G}(w) - \sum_{k=1}^{\infty} \overline{p}_{k+1} \int_w^{\infty} \overline{H}(y+t) g^{(k)}(t) dt \\
&\quad + \sum_{k=2}^{\infty} \overline{p}_k \int_{t=w}^{\infty} dt \int_{u=0}^{t} \overline{H}(y+u) g^{(k-1)}(u) g(t-u) du.
\end{aligned}$$

Therefore, we obtain (17) noting that

$$\begin{aligned}
&\sum_{k=2}^{\infty} \overline{p}_k \int_{t=w}^{\infty} dt \int_{u=0}^{t} \overline{H}(y+u) g^{(k-1)}(u) g(t-u) du \\
&\quad = \sum_{k=2}^{\infty} \overline{p}_k \int_{u=w}^{\infty} \overline{H}(y+u) g^{(k-1)}(u) du \int_{t=u}^{\infty} g(t-u) dt \\
&\qquad + \sum_{k=2}^{\infty} \overline{p}_k \int_{u=0}^{w} \overline{H}(y+u) g^{(k-1)}(u) du \int_{t=w}^{\infty} g(t-u) dt \\
&\quad = \sum_{k=1}^{\infty} \overline{p}_{k+1} \int_{w}^{\infty} \overline{H}(y+u) g^{(k)}(u) du + \sum_{k=2}^{\infty} \overline{p}_k \int_{u=0}^{w} \overline{H}(y+u) g^{(k-1)}(u) \overline{G}(w-u) du.
\end{aligned}$$

Next we consider (18). Using (15) and (17), we have

$$\begin{aligned}
\beta(y) &= \mathrm{E}[G]\overline{H}(y) + \int_{w=0}^{\infty} dw \int_{u=0}^{w} \overline{H}(y+u) \sum_{k=2}^{\infty} \overline{p}_k g^{(k-1)}(u) \overline{G}(w-u) du \\
&= \mathrm{E}[G]\overline{H}(y) + \sum_{k=2}^{\infty} \overline{p}_k \int_{u=0}^{\infty} \overline{H}(y+u) g^{(k-1)}(u) du \int_{w=u}^{\infty} \overline{G}(w-u) dw \\
&= \mathrm{E}[G] \left[\overline{H}(y) + \sum_{k=2}^{\infty} \overline{p}_k \int_{u=0}^{\infty} \overline{H}(y+u) g^{(k-1)}(u) du \right].
\end{aligned}$$

(18) then follows from this equation. □

B Proof of Lemma 4

It follows from (30), (33), and (34) that

$$\begin{aligned}
&(1 - P_{\text{loss}}) \left[D(0)\overline{G}(w) + \int_{0+}^{w} d(u) \overline{G}(w-u) du \right] \\
&\quad = \pi_0 Q(0 \mid 0) \overline{G}(w) \\
&\qquad + \int_{0+}^{w} \left[\pi_0 q_{\text{total}}(u \mid 0) + v(u) Q(0 \mid u) + \int_{0+}^{u} v(t) q_{\text{total}}(u-t \mid t) dt \right] \overline{G}(w-u) du \\
&\quad = \pi_0 \left[Q(0 \mid 0) \overline{G}(w) + \int_{0+}^{w} q_{\text{total}}(u \mid 0) \overline{G}(w-u) du \right] \\
&\qquad + \int_{0+}^{w} v(u) Q(0 \mid u) \overline{G}(w-u) du + \int_{t=0+}^{w} v(t) dt \int_{u=t}^{w} q_{\text{total}}(u-t \mid t) \overline{G}(w-u) du \\
&\quad = \frac{\pi_0}{\mathrm{E}[B]} \cdot \overline{G}(w \mid 0) + \int_{0+}^{w} v(t) Q(0 \mid t) \overline{G}(w-t) dt \\
&\qquad + \int_{t=0+}^{w} v(t) dt \int_{u=0}^{w-t} q_{\text{total}}(u \mid t) \overline{G}(w-t-u) du \\
&\quad = \frac{1}{\mathrm{E}[B]} \left[\pi_0 \overline{G}(w \mid 0) + \int_{0+}^{w} v(t) \overline{G}(w-t \mid t) dt \right].
\end{aligned}$$

From this formula and (7), we obtain

$$v(x) = (1 - P_{\text{loss}}) \lambda \mathrm{E}[B] \left[D(0) \overline{G}(x) + \int_{0+}^{x} d(u) \overline{G}(x-u) du \right].$$

Using (27), (29), (33), (34), and this equation, we have

$$
\begin{aligned}
d(x) &= \frac{\pi_0 \overline{H}(x)}{\mathrm{E}[B](1-P_{\mathrm{loss}})} \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x) + \frac{\overline{H}(x)}{\mathrm{E}[B]} \cdot \frac{v(x)}{1-P_{\mathrm{loss}}} \\
&\quad + \frac{\overline{H}(x)}{\mathrm{E}[B]} \int_{0+}^{x} \frac{v(t)}{1-P_{\mathrm{loss}}} \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-t) dt \\
&= D(0)\overline{H}(x) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x) + \lambda \overline{H}(x) \left[D(0)\overline{G}(x) + \int_{0+}^{x} d(u)\overline{G}(x-u) du \right] \\
&\quad + \lambda \overline{H}(x) \int_{0+}^{x} \left[D(0)\overline{G}(t) + \int_{0+}^{t} d(u)\overline{G}(t-u) du \right] \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-t) dt \\
&= D(0)\overline{H}(x) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x) + \lambda \overline{H}(x) \int_{0+}^{x} d(u)\overline{G}(x-u) du \\
&\quad + \lambda D(0)\overline{H}(x) \left[\overline{G}(x) + \int_{0+}^{x} \overline{G}(t) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-t) dt \right] \\
&\quad + \lambda \overline{H}(x) \int_{u=0+}^{x} d(u) du \int_{t=u}^{x} \overline{G}(t-u) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-t) dt \\
&= D(0)\overline{H}(x) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x) + \lambda \overline{H}(x) \int_{0+}^{x} d(u)\overline{G}(x-u) du \\
&\quad + \lambda D(0)\overline{H}(x) \left[\overline{G}(x) + \int_{0+}^{x} \overline{G}(t) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-t) dt \right] \\
&\quad + \lambda \overline{H}(x) \int_{u=0+}^{x} d(u) du \int_{t=0}^{x-u} \overline{G}(t) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-u-t) dt \\
&= D(0)\overline{H}(x) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x) \\
&\quad + \lambda D(0)\overline{H}(x) \left[\overline{G}(x) + \int_{0+}^{x} \overline{G}(t) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-t) dt \right] \\
&\quad + \lambda \overline{H}(x) \int_{u=0+}^{x} d(u) \left[\overline{G}(x-u) + \int_{t=0}^{x-u} \overline{G}(t) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(x-u-t) dt \right] du.
\end{aligned}
$$

We then obtain (35) noting that

$$
\begin{aligned}
&\overline{G}(w) + \int_{0}^{w} \overline{G}(t) \sum_{n=1}^{\infty} \overline{p}_{n+1} g^{(n)}(w-t) dt \\
&\qquad = \overline{G}(w) + \sum_{n=1}^{\infty} \overline{p}_{n+1} \int_{0}^{w} (1-G(t)) g^{(n)}(w-t) dt \\
&\qquad = 1 - G^{(1)}(w) + \sum_{n=1}^{\infty} \overline{p}_{n+1} \left(G^{(n)}(w) - G^{(n+1)}(w) \right) \\
&\qquad = 1 + \sum_{n=0}^{\infty} \overline{p}_{n+2} G^{(n+1)}(w) - \sum_{n=0}^{\infty} \overline{p}_{n+1} G^{(n+1)}(w) \\
&\qquad = 1 - \sum_{n=0}^{\infty} p_{n+1} G^{(n+1)}(w) \\
&\qquad = \overline{G}_{\mathrm{batch}}(w).
\end{aligned} \tag{114}
$$

□

C Proof of Theorem 10

We define $f^*_{\mathrm{total}}(z,k;w \mid y)$ ($|z| \le 1$, $y \ge 0$, $w > 0$, $k = 1,2,\dots$) as

$$f^*_{\mathrm{total}}(z,1;w \mid y) = f^*(z,1;w \mid y),$$

and for $k = 2,3,\dots,$

$$\begin{aligned} &f^*_{\mathrm{total}}(z,k;w \mid y) \\ &= \int_{w_1=0}^{w} \int_{w_2=0}^{w-w_1} \cdots \int_{w_{k-1}=0}^{w-w_1-w_2-\cdots-w_{k-2}} f^*\Big(z,k;w_1,w_2,\dots,w_{k-1},w-\sum_{m=1}^{k-1} w_m\Big) \\ &\qquad \cdot dw_{k-1}dw_{k-2}\cdots dw_1. \end{aligned}$$

Also, we define $f^*_{\mathrm{total}}(z;w \mid y)$ as

$$f^*_{\mathrm{total}}(z;w \mid y) = \sum_{k=1}^{\infty} f^*_{\mathrm{total}}(z,k;w \mid y).$$

Note that (83) implies

$$f^*_{\mathrm{total}}(z,k;w \mid y) = \hat{f}^*_{\mathrm{total}}(z,k;w \mid y) \cdot \big(1-\alpha+\alpha H^*_{\alpha}(z,y+w)\big),$$

and therefore

$$f^*_{\mathrm{total}}(z;w \mid y) = \hat{f}^*_{\mathrm{total}}(z;w \mid y) \cdot \big(1-\alpha+\alpha H^*_{\alpha}(z,y+w)\big).$$

By definition, $\overline{G}(w \mid y)$ is given by

$$\overline{G}(w \mid y) = \int_w^{\infty} f^*_{\mathrm{total}}(1;t \mid y)dt = \int_w^{\infty} \hat{f}^*_{\mathrm{total}}(1;t \mid y) \cdot \big(1-\alpha+\alpha H^*_{\alpha}(1,y+t)\big)dt.$$

Owing to Lemma 7, we can then verify that Theorem 10 follows noting (78), (80), and

$$\hat{f}_{\mathrm{total}}(w \mid y) = \hat{f}^*_{\mathrm{total}}(1;w \mid y).$$

□

D Proof of Corollary 11

We show that (5) holds under the assumption (90). We define $Z(y)$ ($y \ge 0$) as

$$Z(y) = \frac{\alpha\overline{H}(y)}{1-\alpha H(y)} = 1 - \frac{1-\alpha}{1-\alpha H(y)}.$$

Because of the assumption $H(0) = 0$, it follows that

$$Z(y) \le Z(0) = \alpha < 1, \qquad y \ge 0. \tag{115}$$

In addition, it is readily verified that $Z(y)$ is non-increasing, and

$$\lim_{y\to\infty} Z(y) = \frac{\alpha h_\infty}{1-\alpha+\alpha h_\infty}. \tag{116}$$

With $Z(y)$, (89) is rewritten to be

$$\hat{f}_{\mathrm{total}}(w \mid y) = \frac{Z(y)g(w)}{\alpha} + \int_{0+}^{w} Z(y+u)g(w-u)\hat{f}_{\mathrm{total}}(u \mid y)du,$$

and its solution is represented as

$$\hat{f}_{\mathrm{total}}(w \mid y) = \sum_{n=1}^{\infty} K_n(w,y),$$

where

$$K_1(w,y) = \frac{Z(y)g(w)}{\alpha}, \tag{117}$$

$$K_n(w,y) = \int_{0+}^{w} Z(y+u)g(w-u)K_{n-1}(u,y)du. \tag{118}$$

Because of (115), $K_n(w,y)$ $(n = 1, 2, \ldots)$ is bounded above by

$$K_n(w,y) \le \alpha^{n-1} g^{(n)}(w), \qquad w \ge 0,\ y \ge 0,\ n = 1, 2, \ldots,$$

so that

$$\hat{f}_{\mathrm{total}}(w \mid y) \le \sum_{n=1}^{\infty} \alpha^{n-1} g^{(n)}(w).$$

It then follows from (4), (88), and this inequality that

$$\begin{aligned}
\lambda\beta(y) &= \lambda \int_{w=0}^{\infty} w \cdot \frac{1-\alpha}{1-\alpha H(y+w)} \cdot \hat{f}_{\mathrm{total}}(w \mid y)dw \\
&\le \lambda \int_{w=0}^{\infty} w \cdot \frac{1-\alpha}{1-\alpha+\alpha h_\infty} \sum_{n=1}^{\infty} \alpha^{n-1} g^{(n)}(w)dw \\
&= \lambda \sum_{n=1}^{\infty} \frac{1-\alpha}{1-\alpha+\alpha h_\infty} \cdot \alpha^{n-1} n\mathrm{E}[G] \\
&= \frac{\lambda \mathrm{E}[G]}{1-\alpha+\alpha h_\infty} \cdot \frac{1}{1-\alpha} \qquad (119) \\
&= \rho \cdot \frac{1}{1-\alpha+\alpha h_\infty} < \infty. \qquad (120)
\end{aligned}$$

Furthermore, owing to the dominated convergence theorem,

$$\begin{aligned}
\lim_{y\to\infty} \lambda\beta(y) &= \lambda \int_{w=0}^{\infty} w \lim_{y\to\infty} \frac{1-\alpha}{1-\alpha H(y+w)} \cdot \hat{f}_{\mathrm{total}}(w \mid y)dw \\
&= \lambda \int_{w=0}^{\infty} w \cdot \frac{1-\alpha}{1-\alpha+\alpha h_\infty} \left(\sum_{n=1}^{\infty} \lim_{y\to\infty} K_n(w,y) \right) dw.
\end{aligned}$$

Note here that (116), (117), and (118) imply

$$\lim_{y\to\infty} K_n(w,y) = \frac{1}{\alpha} \left(\frac{\alpha h_\infty}{1-\alpha+\alpha h_\infty} \right)^n g^{(n)}(w), \qquad n = 1, 2, \ldots.$$

We thus obtain

$$\lim_{y\to\infty} \lambda\beta(y) = \frac{\lambda}{\alpha} \sum_{n=1}^{\infty} \frac{1-\alpha}{1-\alpha+\alpha h_\infty} \left(\frac{\alpha h_\infty}{1-\alpha+\alpha h_\infty} \right)^n \int_{w=0}^{\infty} w g^{(n)}(w)dw$$

$$
\begin{aligned}
&= \frac{\lambda}{\alpha} \sum_{n=1}^{\infty} \frac{1-\alpha}{1-\alpha+\alpha h_{\infty}} \left(\frac{\alpha h_{\infty}}{1-\alpha+\alpha h_{\infty}} \right)^n n\mathrm{E}[G] \\
&= \frac{\lambda h_{\infty} \mathrm{E}[G]}{1-\alpha} \\
&= \rho h_{\infty} < 1,
\end{aligned}
$$

which completes the proof. □

E Proof of Theorem 13

We define $\psi_{\mathrm{L}}(x)$ $(0 < x \le 1)$ as

$$
\psi_{\mathrm{L}}(x) = \phi_{\mathrm{L}}(x) \exp\left[\frac{1}{\eta} \int_x^1 \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right].
$$

It follows from (96) that

$$
\begin{aligned}
\frac{d\psi_{\mathrm{L}}(x)}{dx} &= \frac{d\phi_{\mathrm{L}}(x)}{dx} \cdot \exp\left[\frac{1}{\eta} \int_x^1 \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right] \\
&\quad - \phi_{\mathrm{L}}(x) \cdot \frac{1}{\eta} \left(\lambda \mathrm{E}[B] \tilde{p}^*(x) - \frac{\mu-\eta}{x} \right) \exp\left[\frac{1}{\eta} \int_x^1 \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right] \\
&= \frac{(\mu-\eta)\pi_0}{\eta x} \exp\left[\frac{1}{\eta} \int_x^1 \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right], \qquad 0 < x \le 1.
\end{aligned}
$$

We thus have

$$
\psi_{\mathrm{L}}(x) = \psi_{\mathrm{L}}(a) + \int_a^x \frac{(\mu-\eta)\pi_0}{\eta y} \exp\left[\frac{1}{\eta} \int_y^1 \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right] dy,
$$

where $a > 0$ denotes a positive real number. Therefore, $\phi_{\mathrm{L}}(x)$ is given by

$$
\begin{aligned}
\phi_{\mathrm{L}}(x) &= \psi_{\mathrm{L}}(x) \exp\left[-\frac{1}{\eta} \int_x^1 \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right] \\
&= \phi_{\mathrm{L}}(a) \exp\left[\frac{1}{\eta} \int_a^x \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right] \\
&\quad + \int_a^x \frac{(\mu-\eta)\pi_0}{\eta y} \exp\left[\frac{1}{\eta} \int_y^x \left(\lambda \mathrm{E}[B] \tilde{p}^*(u) - \frac{\mu-\eta}{u} \right) du \right] dy \\
&= \phi_{\mathrm{L}}(a) \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u) du + \frac{\mu-\eta}{\eta} (\log a - \log x) \right] \\
&\quad + \int_a^x \frac{(\mu-\eta)\pi_0}{\eta y} \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_y^x \tilde{p}^*(u) du + \frac{\mu-\eta}{\eta} (\log y - \log x) \right] dy \\
&= \phi_{\mathrm{L}}(a) x^{-(\mu-\eta)/\eta} a^{(\mu-\eta)/\eta} \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u) du \right] \\
&\quad + \pi_0 x^{-(\mu-\eta)/\eta} \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u) du \right] \qquad (121) \\
&\qquad \cdot \frac{\mu-\eta}{\eta} \int_a^x y^{(\mu-\eta)/\eta - 1} \exp\left[-\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^y \tilde{p}^*(u) du \right] dy \\
&= x^{-(\mu-\eta)/\eta} \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u) du \right] \\
&\quad \cdot \left\{ \phi_{\mathrm{L}}(a) a^{(\mu-\eta)/\eta} + \pi_0 \frac{\mu-\eta}{\eta} \int_a^x y^{(\mu-\eta)/\eta - 1} \exp\left[-\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^y \tilde{p}^*(u) du \right] dy \right\},
\end{aligned}
$$

$$0 < x \leq 1. \quad (122)$$

Next, we consider the limit $a \to 0+$. Note that if $\mu < \eta$, the right-hand side of (122) takes the indeterminate form $\infty - \infty$ when $a \to 0+$. To remove this singularity, we perform a partial integration.

$$\begin{aligned}
&\pi_0 \frac{\mu-\eta}{\eta} \int_a^x y^{(\mu-\eta)/\eta-1} \exp\left[-\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^y \tilde{p}^*(u)du\right] dy \\
&= \pi_0 \int_a^x \frac{d}{dy}\left[y^{(\mu-\eta)/\eta}\right] \cdot \exp\left[-\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^y \tilde{p}^*(u)du\right] dy \\
&= \pi_0 x^{(\mu-\eta)/\eta} \exp\left[-\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u)du\right] - \pi_0 a^{(\mu-\eta)/\eta} \\
&\quad + \pi_0 \frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x y^{(\mu-\eta)/\eta} \tilde{p}^*(y) \exp\left[-\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^y \tilde{p}^*(u)du\right] dy.
\end{aligned}$$

From this equation, we obtain

$$\begin{aligned}
\phi_{\mathrm{L}}(x) &= x^{-(\mu-\eta)/\eta} \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u)du\right] \cdot (\phi_{\mathrm{L}}(a) - \pi_0) a^{(\mu-\eta)/\eta} \\
&\quad + \pi_0 \left\{1 + \frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \left(\frac{y}{x}\right)^{(\mu-\eta)/\eta} \tilde{p}^*(y) \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_y^x \tilde{p}^*(u)du\right] dy\right\} \\
&= x^{-(\mu-\eta)/\eta} \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_a^x \tilde{p}^*(u)du\right] \cdot (\phi_{\mathrm{L}}(a) - \pi_0) a^{(\mu-\eta)/\eta} \\
&\quad + \pi_0 \left\{1 + \frac{\lambda \mathrm{E}[B] x}{\eta} \int_{a/x}^1 t^{(\mu-\eta)/\eta} \tilde{p}^*(tx) \exp\left[\frac{\lambda \mathrm{E}[B]}{\eta} \int_{tx}^x \tilde{p}^*(u)du\right] dt\right\}, \quad (123)
\end{aligned}$$

where $t = y/x$. Note that

$$\begin{aligned}
\lim_{a \to 0+} (\phi_{\mathrm{L}}(a) - \pi_0) a^{(\mu-\eta)/\eta} &= \lim_{a \to 0+} a^{(\mu-\eta)/\eta+1} \sum_{n=1}^{\infty} \pi_n a^{n-1} \\
&= \lim_{a \to 0+} a^{\mu/\eta} \sum_{n=1}^{\infty} \pi_n a^{n-1} \\
&= 0.
\end{aligned}$$

Therefore, taking the limit $a \to 0+$ on the right-hand side of (123), we obtain (98). Finally, (97) follows from (98) and $\phi_{\mathrm{L}}(1) = 1$. □

F Proof of Lemma 10

It is readily verified that using (100), we can rewrite (17) to be

$$\overline{G}(w \mid y) = \begin{cases}
\overline{G}(w) + \displaystyle\int_0^w \sum_{k=2}^{\infty} \overline{p}_k g^{(k-1)}(u) \overline{G}(w-u)du, & y < \tau, w < \tau - y, \\
\overline{G}(w) + \displaystyle\sum_{k=2}^{\infty} \overline{p}_k \int_0^{\tau-y} g^{(k-1)}(u) \overline{G}(w-u)du, & y < \tau, w \geq \tau - y, \\
0, & y \geq \tau.
\end{cases}$$

In addition, it follows from (114) that

$$\overline{G}_{\mathrm{batch}}(w) = \overline{G}(w) + \int_0^w \sum_{k=2}^{\infty} \overline{p}_k g^{(k-1)}(u) \overline{G}(w-u)du.$$

We thus obtain (101). □

References

1. Baccelli, F., Hebuterne, G.: On queues with impatient customers. In: Proc. of Performance '81 (F.J. Kylstra, ed.), North-Holland Publ. Cy., 159–179 (1981).
2. Baccelli, F., Boyer, P., Hebuterne, G.: Single-server queues with impatient customers. Adv. Appl. Prob. 16, 887–905 (1984).
3. Bae, J., Kim, S., Lee, E.Y.: The virtual waiting time of the M/G/1 queue with impatient customers. Queueing Syst. 38, 485–494 (2001).
4. Barrer, D.Y.: Queuing with impatient customers and ordered service. Oper. Res. 5, 650–656 (1957).
5. Boxma, O.J., Perry, D., Stadje, W.: The M/G/1+G queue revisited. Queueing Syst. 67, 207–220 (2011).
6. Boxma, O.J., de Waal, P.R.: Multiserver queues with impatient customers. In: Proc. ITC 14 (J. Labetoulle and J.W. Roberts, eds.), North-Holland Publ. Cy., 743–756 (1994).
7. Brandt, A., Brandt, M.: On the $M(n)/M(n)/s$ queue with impatient calls. Perform. Eval. 35, 1–18 (1999).
8. Brandt, A., Brandt, M.: Workload and busy period for M/G/1 with a general impatience mechanism. Queueing Syst. 75, 189–209 (2013).
9. Brill, P.H., Posner, M.J.M.: Level crossings in point processes applied to queues: Single-server case. Oper. Res. 25, 662–674 (1977).
10. Boots, N.K., Tijms, H.: A multiserver queueing system with impatient customers. Manag. Sci. 45, 444–448 (1999).
11. Callahan, J.R.: The nothing hot delay problem in the production of steel. Ph.D. Thesis, Department of Industrial Engineering, University of Toronto (1971).
12. Cohen, J.W.: Single server queues with restricted accessibility. J. Eng. Math. 3, 265–284 (1969).
13. Cohen, J.W.: On up- and downcrossings. J. Appl. Prob. 14, 405–410 (1977).
14. Daley, D.J.: General customer impatience in the queue GI/G/1. J. Appl. Prob. 2, 186–205 (1965).
15. Hasenbein, J., Perry, D. (eds.): Special issue on Queueing Systems with abandonment. Queueing Syst. 75, 111–384 (2013).
16. Inoue, Y., Takine, T.: Analysis of the loss probability in the M/G/1+G queue. Queueing Syst. 80, 363–386 (2015).
17. Inoue, Y., Takine, T.: A computational algorithm for the loss probability in the M/G/1+PH queue. Stoch. Models 33, 116–148 (2017).
18. Inoue, Y., Takine, T.: The M/D/1+D queue has the minimum loss probability among M/G/1+G queues. Oper. Res. Letters 43, 629–632 (2015).
19. Iravani, F., Balcioglu, B.: Approximations for the $M/GI/N+G$ type call center. Queueing Syst. 58, 137–153 (2008).
20. Kim, B., Kim, J.: A batch arrival $M^X/M/c$ queue with impatient customers. Oper. Res. Letters 42, 180–185 (2014).
21. Meyn, S.P., Tweedie, R.L.: Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. Adv. Appl. Prob. 25, 518–548 (1993).
22. Posner, M.: Single-server queues with service time dependent on waiting time. Oper. Res. 21, 610–616 (1973).
23. Rao, S.S.: Queueing with balking and reneging in M/G/1 systems. Metrika 12, 173–188 (1967).
24. Shaked, M., Shanthikumar, J.G.: Stochastic Orders, Springer, New York, NY (2007).
25. Stadje, W., Zacks, S.: Upper first-exit times of compound Poisson processes revisited. Prob. Eng. Inform. Sci. 17, 459–465 (2003).
26. Stanford, R.E.: Reneging phenomena in single channel queues. Math. Oper. Res. 4, 162–178 (1979).
27. van Velthoven, J., van Houdt, B., Blondia, C.: On the probability of abandonment in queues with limited sojourn and waiting times. Oper. Res. Letters 34, 333–338 (2006).
28. Weerasinghe, A., Mandelbaum, A.: Abandonment versus blocking in many-server queues: Asymptotic optimality in the QED regime. Queueing Syst. 75, 279–337 (2013).
29. Zacks, S., Perry, D., Bshouty, D., Bar-Lev, S.: Distributions of stopping times for compound Poisson processes with positive jumps and linear boundaries. Stoch. Models 15, 89–101 (1999).

30. Zeltyn, S., Mandelbaum, A.: Call centers with impatient customers: Many-server asymptotics of the M/M/n+G queue. Queueing Syst. 51, 361–402 (2005).