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Dividends in the dual risk model

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Abstract

We consider the dual risk model with special dividend or tax payments: If an arriving gain finds the surplus above a barrier b or if it would bring the surplus above that level, a certain part of the gain is paid as dividends or taxes. We obtain expressions for the joint Laplace-Stieltjes transform of the time to ruin and the amount of dividends paid until ruin, and for the expected discounted dividend paid until ruin. We consider the case where the dividend paid from each gain is a general function of the gain. More explicit results are obtained when the dividend is a given percentage of the gain amount.
Keywords: dual risk model, time to ruin, dividend, $M/G/1$ queue, busy period

1 Introduction

This paper is devoted to the analysis of dividend policies in the compound Poisson dual risk model. Since the pioneering work of De Finetti, many types of dividend policies have been studied for the classical Cramér-Lundberg model, i.e., for risk processes modeled as a Lévy process without positive jumps. In the barrier dividend strategy all the premium inflow received while the process is above the barrier is paid as dividends. Under some conditions, this policy was proven to be optimal for the classical risk model. Another dividend policy is the threshold strategy, where dividends are paid at a fixed rate smaller than the premium rate, when the surplus is above a given horizontal threshold. This policy was proven to be optimal under some conditions when the dividend rate is bounded from above. For a comprehensive review on the dividends strategies, see [4].

Another process that has been studied in the actuarial literature is the dual risk model. This is a Lévy process without negative jumps. It describes the surplus of a company with fixed expense rate and occasional gains. Examples are pharmaceutical, petroleum, or R& D companies. Since Avanzi et al. [2] various performance measures of the dual risk model have been studied. Avanzi et al. [2, 4] considered the optimal dividend policy for the compound Poisson dual risk model with and without perturbation of Brownian motion and proved that the policy that maximizes the expected present value of the dividends is the barrier strategy, i.e., all the overflow above a barrier is paid as dividends. The barrier strategy is proven to be optimal for the more general spectrally positive

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Lévy process in [6] as well as for the dual risk process with terminal cost [12]. Under the threshold policy no dividends are paid when the surplus is below a given horizontal threshold and dividends are paid continuously at a constant rate when the surplus is above that barrier. [9] studied the threshold dividend strategy for the dual risk model. [13] proved that the threshold policy is optimal under the restriction that the dividends are absolutely continuous and the dividend rate is bounded from above. When dividends can be either absolutely continuous with bounded rate or given as a lump amount, the optimal policy is a combination of the threshold policy and the barrier strategy [5].

In this paper we consider the compound Poisson dual risk model with the following dividend policy: Above a given horizontal barrier a certain part of each gain (or from the part of a gain which brings the surplus above the barrier) is paid out as dividends or as taxes. Thus it differs from the barrier strategy since not all the overflow above the barrier is paid as dividends and it differs from the threshold policy since dividends are not paid continuously. As an example consider R&D, pharmaceutical or petroleum companies, where whenever there is a gain and the surplus of the company is high it gives some bonus or dividends to its employees or shareholders from each gain. Alternatively, especially in the petroleum industry, a company might pay tax from each gain to the regulator. Our main results are expressions for the joint Laplace-Stieltjes transform of the time to ruin and the amount of dividends, and for the expected discounted dividends until ruin.

The paper is organized as follows. Section 2 contains a model description and some results about scale functions that will be used later on. Section 3 is devoted to the case that a fixed proportion of the gain above a certain threshold is paid out as dividend, while Section 4 considers a general function of the gain. In both cases, we derive expressions for the joint Laplace-Stieltjes transform of the time to ruin and the total dividend until ruin; we also discuss the expected discounted dividends.

2 Preliminaries

2.1 The model

Let $U(t)$ be the surplus process of the dual risk model without dividend payments:

$$U(t) = U(0) - ct + \sum_{i=1}^{N(t)} Y_i, \tag{2.1}$$

where $c > 0$, $N(t)$ is a Poisson process with rate λ and Y_i, $i = 1, 2, \dots,$ are i.i.d. positive random variables with distribution G and Laplace-Stieltjes transform $\mathcal{L}_Y(\cdot)$. $U(t)$ can also represent the work in process in an M/G/1 queue or the inventory level in a storage model or dam model with occasional inflow and a constant demand rate. It is well-known that

$$\mathbb{E}[e^{-\theta(U(t)-U(0))}] = e^{t\psi(\theta)}, \tag{2.2}$$

where

$$\psi(\theta) = c\theta - \lambda + \lambda\mathcal{L}_Y(\theta); \tag{2.3}$$

$\psi(\theta)$ is the Laplace exponent of the process. Let $\Phi(\theta)$ be the biggest solution to

$$\psi(\theta) = \theta. \tag{2.4}$$

It is well-known that when $\psi'(0) \geq 0$ then $\Phi(0) = 0$, otherwise $\Phi(0) > 0$. When $\psi'(0) \geq 0$ the busy period in the M/G/1 queue is finite with probability 1, or in the dual risk model ruin occurs with probability 1. Otherwise the busy period can be infinite with positive probability or equivalently, the ruin probability is less than 1.

We consider a horizontal barrier $b > 0$. From each gain of size x that occurs while the process is above b or from each overshoot of size x of a gain that brings the surplus above b, a dividend of size $I(x)$ is paid. We assume that $0 < I(x) < x$.

Thus above b the process behaves as $\mathbb{U}$, where

$$d\mathbb{U}(t) = -cdt + dS(t), \tag{2.5}$$

where $S(t) = \sum_{i=1}^{N(t)}(Y_i - I(Y_i))$. Throughout the paper we assume that $c = 1$, and we denote by $\mathcal{L}_A$ the Laplace-Stieltjes transform of a random variable A.

The Laplace exponent of $\mathbb{U}(t)$ is $\psi_1(\theta)$, where

$$\psi_1(\theta) = \theta - \lambda + \lambda\mathcal{L}_{Y-I(Y)}(\theta). \tag{2.6}$$

Let Φ_1 be the inverse of ψ_1. We denote by X_1 the spectrally negative risk process with $dX_1(t) = -d\mathbb{U}(t)$. Let $\mathcal{U}$ be the net surplus process, i.e., after dividends payout. Our aim is to obtain the Laplace-Stieltjes transform of the total dividends and the expected discounted dividend paid until ruin.

2.2 Scale function

In this subsection we review some fundamental results for the exit times for spectrally negative Lévy processes in terms of scale functions; these results are used in the rest of the paper.

Let $X(t)$ be a spectrally negative Lévy process, and more specifically,

$$X(t) = X(0) + t - \sum_{i=1}^{N(t)} Y_i. \tag{2.7}$$

In our context $X(t) = b - U(t)$, thus $X(0) = b - U(0)$. Hence, cf. (2.2),

$$\mathbb{E}[e^{\theta(X(t)-X(0))}] = e^{\psi(\theta)t},$$

where ψ is as in Equation (2.3).

For the following notations and definitions we refer the reader to Chapter 8 in [8]. The q-scale function, $W^{(q)}(x)$, is the unique right-continuous function whose Laplace transform is given by

$$\int_0^\infty e^{-\beta x}W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}, \qquad \text{for } \beta > \Phi(q).$$

We also define the function $Z^{(q)}(x, \theta)$ by:

$$Z^{(q)}(x, \theta) = e^{\theta x}\left(1 - (\psi(\theta) - q)\int_0^x e^{-\theta y}W^{(q)}(y)dy\right), \tag{2.8}$$

(see also Eq. (4)-(5) in [1]). For $\theta = 0$ we get the function $Z^{(q)}(x)$,

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)dy. \tag{2.9}$$

Let $\tau_b^+ = \inf\{t : X(t) = b\}$ and let $\tau_a^- = \inf\{t : X(t) \le a\}$. Then for $0 \le x < b$ the following hold: (by $\mathbb{E}_x$ we denote the conditional expectation given that $X(0) = x$):

$$\mathbb{E}_0[e^{q\tau_b^+} 1_{\{\tau_b^+ < \infty\}}] = e^{-\Phi(q)b}, \tag{2.10}$$

$$\mathbb{E}_x[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}}] = \frac{W^{(q)}(x)}{W^{(q)}(b)}. \tag{2.11}$$

It is known (see e.g. (5) in [1] and (8) in [10]) that:

$$\mathbb{E}_x[e^{-q\tau_0^-} e^{\theta X(\tau_0^-)} 1_{\tau_0^- < \tau_b^+}] = Z^{(q)}(x,\theta) - \frac{W^{(q)}(x)}{W^q(b)} Z^{(q)}(b,\theta). \tag{2.12}$$

In the sequel we need also the derivative of $Z^{(q)}(x,\theta)$ with respect to θ:

$$\frac{d}{d\theta} Z^{(q)}(x,\theta) = xZ^{(q)}(x,\theta) + e^{\theta x}\left(-\psi'(\theta)\int_0^x e^{-\theta y} W^{(q)}(y)dy + (\psi(\theta) - q)\int_0^x y e^{-\theta y} W^{(q)}(y)dy\right). \tag{2.13}$$

Letting $\theta = 0$ in (2.8) we obtain that

$$\mathbb{E}_x[e^{-q\tau_0^-} 1_{\{\tau_0^- < \tau_b^+\}}] = Z^{(q)}(x) - Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)}. \tag{2.14}$$

We shall also use the q-resolvent measure, which has a density $u^{(q)}(x,y)$ where

$$\begin{aligned} u^{(q)}(x,y)dy &= \mathbb{E}_x \int_0^\infty e^{-qt} 1_{X_t \in dy, t < \tau_0^- \wedge \tau_b^+} dt \\ &= \left(\frac{W^{(q)}(x)W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y)\right) dy. \end{aligned} \tag{2.15}$$

3 The joint Laplace-Stieltjes transform of the total dividends and the time to ruin – the case $I(x) = (1-\alpha)x$

In this section we restrict ourselves to the case that a dividend of size $I(x) = (1-\alpha)x$ is paid from each gain of size x that occurs while the process is above b and from each overshoot of size x of a gain that brings the surplus above b. We devote a section to this special case because it is an important case, and because it allows us to introduce the key arguments which will, in Section 4, also enable us to handle the general $I(x)$ case. In Subsection 3.1 we determine the Laplace-Stieltjes transform of the total dividend in one cycle. Subsection 3.2 is devoted to the derivation of the joint Laplace-Stieltjes transform of the time to ruin and the dividends until ruin. In Subsection 3.3 we consider the expected discounted dividends for the case $I(x) = (1-\alpha)x$.

3.1 The Laplace-Stieltjes transform of the total dividends in one cycle

Assume that $\mathcal{U}(0) = b + y$, $y > 0$. Let $\mathcal{D}_{b+y}$ be the total dividends paid until the process reaches b. Let $\mathcal{D}$ be the total dividends paid until the process reaches b when $\mathcal{U}(0) = b + Y$.

Denote by $\mathcal{B}$ a busy period of an $M/G/1$ queue with workload described in (2.1), but with the ith gain equalling αY_i, so the service time equals αY_i.

We have the following equation for $\mathcal{L}_{\mathcal{D}_{b+y}}$ and $\mathcal{L}_{\mathcal{D}}$.

Proposition 3.1.

$$\mathcal{L}_{\mathcal{D}_{b+y}}(\theta) = \exp\left(-\{\theta(1-\alpha) + \lambda\alpha(1 - \mathcal{L}_{\mathcal{D}}(\theta))\}y\right). \tag{3.1}$$

Proof. Given that $\mathcal{U}(0) = b + y$ then $(1-\alpha)y$ is paid immediately as dividend and the net surplus is $b + \alpha y$. Assume that during time αy there are exactly n Poisson arrivals. Each one of the arrivals starts a regular busy period (which can also be infinite). Let $\mathcal{B}_i$ and $\mathcal{D}_i$ be, respectively, the ith busy period and the amount of dividends paid in the ith busy period. Those busy periods are independent, and distributed as the busy period in an M/G/1 queue with arrival rate λ and service time distributed as αY. Thus, given that $N(\alpha y) = n \geq 0$, using a well-known branching argument that has been developed for the busy period LST in an $M/G/1$ queue (cf. pp. 60-61 of [11]),

$$\mathbb{E}[\exp(-\theta\mathcal{D}_{b+y})1_{\mathcal{D}_{b+y}<\infty}|N(\alpha y) = n] = e^{-\theta(1-\alpha)y}\mathcal{L}^n_{\mathcal{D}}(\theta). \tag{3.2}$$

By unconditioning,

$$\begin{aligned}\mathcal{L}_{\mathcal{D}_{b+y}}(\theta) &= \sum_{n=0}^{\infty} e^{-\theta(1-\alpha)y}e^{-\lambda\alpha y}\frac{(\lambda\alpha y)^n}{n!}\mathcal{L}^n_{\mathcal{D}}(\theta)\\ &= \exp\left(-\{\theta(1-\alpha) + \lambda\alpha(1 - \mathcal{L}_{\mathcal{D}}(\theta))\}y\right).\end{aligned} \tag{3.3}$$

□

Proposition 3.2. *For* $\mathrm{Re}\ \theta \geq 0$*,* $\mathcal{L}_{\mathcal{D}}(\theta)$ *satisfies the following equation:*

$$\mathcal{L}_{\mathcal{D}}(\theta) = \mathcal{L}_Y\left(\theta(1-\alpha) + \lambda\alpha(1 - \mathcal{L}_{\mathcal{D}}(\theta))\right). \tag{3.4}$$

It is the unique solution of the equation $x = \mathcal{L}_Y(\theta(1-\alpha) + \lambda\alpha(1-x))$ *with* $|x| < 1$ *if* $\mathrm{Re}\ \theta > 0$ *and if* $\mathrm{Re}\ \theta \geq 0$ *and* $\lambda\alpha\mathbb{E}[Y] < 1$*, i.e., if* $\Phi_1(0) = 0$*; if* $\Phi_1(0) > 0$*, then* $\mathcal{L}_{\mathcal{D}}(0) = 1$ *is the unique solution of the equation with* $|x| \leq 1$*.*

Proof. To show that $\mathcal{L}_{\mathcal{D}}(\theta)$ satisfies (3.4), we replace y by the random variable Y in (3.3) and take expectations. The uniqueness follows by an application of Rouché's theorem (see, e.g., the proof of a very similar result for the LST of the busy period in the $M/G/1$ queue on pp. 47-48 of [11]). □

3.2 The joint Laplace-Stieltjes transform of the time to ruin and the dividends until ruin

Let $\mathcal{D}^{(u)}_{tot}$ be the total dividends paid until ruin when the initial surplus is u and let $\mathcal{P}^{(u)}_{tot}$ be the time until ruin when the initial surplus is u. In this section we obtain their joint Laplace-Stieltjes transform $\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot}-\beta\mathcal{P}^{(u)}_{tot}}]$, expressed in the function $Z^{(q)}(x,\theta)$ that was defined in (2.8).

Let C_u be the time until the $\mathcal{U}$ process upcrosses b given that the initial surplus is u, and let R_u be the overshoot when this event occurs. Let P_u be the period that starts when $\mathcal{U}$ overshoots b and ends when it again reaches b, given that $\mathcal{U}(0) = u$. Let H_u be the time the process reaches 0 starting at u. For future use we introduce

$$\nu(\theta) := \theta(1-\alpha) + \lambda\alpha(1 - \mathcal{L}_{\mathcal{D}}(\theta)). \tag{3.5}$$

Proposition 3.3. **(i)** *For $u \le b$,*

$$\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot} - \beta\mathcal{P}^{(u)}_{tot}}] = \frac{Z^{(\beta)}(b-u, \nu(\frac{\beta\alpha}{1-\alpha} + \theta))}{Z^{(\beta)}(b, \nu(\frac{\beta\alpha}{1-\alpha} + \theta))}. \tag{3.6}$$

(ii) *For $u > b$,*

$$\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot} - \beta\mathcal{P}^{(u)}_{tot}}] = \frac{e^{-\nu(\frac{\beta\alpha}{1-\alpha} + \theta)(u-b)}}{Z^{(\beta)}(b, \nu(\frac{\beta\alpha}{1-\alpha} + \theta))}. \tag{3.7}$$

Proof. **(i)** $\mathcal{U}(0) = u \le b$.

$$\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot} - \beta\mathcal{P}^{(u)}_{tot}}] = \mathbb{E}[e^{-\beta H_u} 1_{\{H_u < C_u\}}] + \mathbb{E}[e^{-\beta(C_u + P_u) - \theta\mathcal{D}_{b+R_u}} 1_{\{C_u < H_u\}}]\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot} - \beta\mathcal{P}^{(b)}_{tot}}].$$

Notice that

$$P_u = \frac{\alpha}{1-\alpha}\mathcal{D}_{b+R_u}, \tag{3.8}$$

thus, by (3.3) and (3.5),

$$\mathbb{E}[e^{-\beta(C_u+P_u) - \theta\mathcal{D}_{b+R_u}} 1_{\{C_u < H_u\}} | R_u = y] = \mathbb{E}[e^{-\beta C_u - \nu(\frac{\beta\alpha}{1-\alpha} + \theta)y} 1_{\{C_u < H_u\}}]. \tag{3.9}$$

Hence, by (3.9) and the strong Markov property:

$$\begin{aligned}
&\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot} - \beta\mathcal{P}^{(u)}_{tot}}] \\
&= \mathbb{E}[e^{-\beta H_u} 1_{\{H_u < C_u\}}] + \mathbb{E}[e^{-\beta C_u - (\frac{\beta\alpha}{1-\alpha} + \theta)\mathcal{D}_{b+R_u}} 1_{\{C_u < H_u\}}]\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot} - \beta\mathcal{P}^{(b)}_{tot}}] \\
&= \mathbb{E}[e^{-\beta H_u} 1_{\{H_u < C_u\}}] + \mathbb{E}[e^{-\beta C_u - \nu(\frac{\beta\alpha}{1-\alpha} + \theta)R_u} 1_{\{C_u < H_u\}}]\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot} - \beta\mathcal{P}^{(b)}_{tot}}].
\end{aligned} \tag{3.10}$$

C_u is the same as τ_0^- – the time to ruin in the classical risk model with initial surplus $b-u$, similarly, H_u is the same as τ_b^+ – the first time to reach b starting at $b-u$ in the classical risk model X (see below (2.9)). Thus,

$$\begin{aligned}
&\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot} - \beta\mathcal{P}^{(u)}_{tot}}] \\
&= \mathbb{E}_{b-u}[e^{-\beta\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}}] \\
&+ \mathbb{E}_{b-u}[e^{-\beta\tau_0^- + \nu(\frac{\beta\alpha}{1-\alpha} + \theta)X(\tau_0^-)} 1_{\{\tau_0^- < \tau_b^+\}}]\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot} - \beta\mathcal{P}^{(b)}_{tot}}] \\
&= \frac{W^{(\beta)}(b-u)}{W^{(\beta)}(b)} + (Z^{(\beta)}(b-u, \nu(\frac{\beta\alpha}{1-\alpha} + \theta)) \\
&- W^{(\beta)}(b-u)\frac{Z^{(\beta)}(b, \nu(\frac{\beta\alpha}{1-\alpha} + \theta))}{W^{(\beta)}(b)})\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot} - \beta\mathcal{P}^{(b)}_{tot}}].
\end{aligned} \tag{3.11}$$

$\mathbb{E}[e^{-\theta\mathcal{D}_{tot}^{(b)}-\beta\mathcal{P}_{tot}^{(b)}}]$ is obtained by substituting $u=b$ in (3.11):

$$\mathbb{E}[e^{-\theta\mathcal{D}_{tot}^{(b)}-\beta\mathcal{P}_{tot}^{(b)}}]=\frac{1}{Z^{(\beta)}(b,\nu(\frac{\beta\alpha}{1-\alpha}+\theta))}.$$

Thus (3.6) follows.

(ii) $\mathcal{U}(0)=u>b$.

In this case $C_u=0$ and $R_u=u-b$. By (3.3) and (3.8),

$$\mathbb{E}[e^{-\theta\mathcal{D}_{tot}^{(u)}-\beta\mathcal{P}_{tot}^{(u)}}]=e^{-\nu(\frac{\beta\alpha}{1-\alpha}+\theta)(u-b)}\mathbb{E}[e^{-\theta\mathcal{D}_{tot}^{(b)}-\beta P_{tot}^{(b)}}]=\frac{e^{-\nu(\frac{\beta\alpha}{1-\alpha}+\theta)(u-b)}}{Z^{(\beta)}(b,\nu(\frac{\beta\alpha}{1-\alpha}+\theta))}.$$

□

Remark 3.1. *The LST's of $\mathcal{P}_{tot}^{(u)}$ and of $\mathcal{D}_{tot}^{(u)}$ immediately follow from the theorem: For $\mathcal{U}(0)=u\le b$,*

$$\mathbb{E}[e^{-\beta\mathcal{P}_{tot}^{(u)}}]=\frac{Z^{(\beta)}(b-u,\nu(\frac{\beta\alpha}{1-\alpha}))}{Z^{(\beta)}(b,\nu(\frac{\beta\alpha}{1-\alpha}))},$$
$$\mathbb{E}[e^{-\theta\mathcal{D}_{tot}^{(u)}}]=\frac{Z^{(0)}(b-u,\nu(\theta))}{Z^{(0)}(b,\nu(\theta))},$$

and for $\mathcal{U}(0)=u>b$,

$$\mathbb{E}[e^{-\beta\mathcal{P}_{tot}^{(u)}}]=\frac{\mathrm{e}^{-\nu(\frac{\beta\alpha}{1-\alpha})(u-b)}}{Z^{(\beta)}(b,\nu(\frac{\beta\alpha}{1-\alpha}))},$$
$$\mathbb{E}[e^{-\theta\mathcal{D}_{tot}^{(u)}}]=\frac{\mathrm{e}^{-\nu(\theta)(u-b)}}{Z^{(0)}(b,\nu(\theta))}.$$

Next, we obtain the ruin probability for this model. Recall that above b the surplus process behaves as $\mathbb{U}$, with Laplace exponent ψ_1 with inverse Φ_1. The Laplace-Stieltjes transform of $\mathcal{B}$ can also be written as follows:

$$\mathcal{L}_{\mathcal{B}}(\theta)=\mathbb{E}[e^{-\Phi_1(\theta)\alpha Y}]=\mathcal{L}_{\alpha Y}(\Phi_1(\theta)). \tag{3.12}$$

Corollary 3.1. *The ruin probability for the dual model $\mathcal{U}$ with $I(x)=(1-\alpha)x$ is 1 if $\psi_1'(0)\ge 0$. Otherwise it equals*

$$\begin{cases}\frac{Z(b-u,\lambda\alpha(1-\mathcal{L}_{\alpha Y}(\Phi_1(0)))}{Z(b,\lambda\alpha(1-\mathcal{L}_{\alpha Y}(\Phi_1(0)))} & \text{if} \quad u\le b,\\[2ex] \frac{e^{-\lambda\alpha(1-\mathcal{L}_{\alpha Y}(\Phi_1(0)))}}{Z(b,\lambda\alpha(1-\mathcal{L}_{\alpha Y}(\Phi_1(0)))} & \text{if} \quad u>b.\end{cases} \tag{3.13}$$

Proof. Substitute $\theta=\beta=0$ and notice that

$$\nu(0)=\lambda\alpha(1-\mathcal{L}_{\mathcal{D}}(0))=\lambda\alpha(1-\mathcal{L}_{\mathcal{B}}(0))=\lambda\alpha(1-\mathcal{L}_{\alpha Y}(\Phi_1(0))). \tag{3.14}$$

Now observe that $\Phi_1(0)=0$ if $\psi_1'(0)\ge 0$ (i.e., if $\lambda\alpha\mathbb{E}[Y]\le 1$) and $\Phi_1(0)>0$ if $\psi_1'(0)<0$. □

3.3 The expected discounted dividends for $I(x) = (1-\alpha)x$

Let $\mathcal{V}^{(q)}(u,b)$ be the expected discounted dividends until the process $\mathcal{U}$ reaches b the first time from above.

Proposition 3.4. **(i)** *Let $\mathcal{U}(0) = u = b + y, y > 0$. Then*

$$\mathcal{V}^{(q)}(u,b) = (1-\alpha)y + V(q)\frac{\lambda}{\eta}(1 - \exp(-\alpha\eta y)), \tag{3.15}$$

where

$$V(q) = \mathbb{E}[\mathcal{V}^{(q)}(b+Y,b)] = (1-\alpha)\mathbb{E}[Y] + V(q)\frac{\lambda}{\eta}(1 - \mathcal{L}_Y(\alpha\eta)), \tag{3.16}$$

i.e.,

$$V(q) = \frac{(1-\alpha)\mathbb{E}[Y]}{1 - \frac{\lambda}{\eta}(1 - \mathcal{L}_Y(\alpha\eta))}, \tag{3.17}$$

and

$$\eta = q + \lambda(1 - \mathcal{L}_{\mathcal{B}}(q)). \tag{3.18}$$

(ii) *Let $\mathcal{U}(0) = u \leq b$ then*

$$\begin{aligned}
\mathcal{V}^{(q)}(u,b) &= -(1-\alpha)\Bigg[(b-u)Z^{(q)}(b-u) - (1-\lambda\mathbb{E}[Y])\bar{W}^{(q)}(b-u) - q\int_0^{b-u} yW^{(q)}(y)dy \\
&\quad - \frac{W^{(q)}(b-u)}{W^{(q)}(b)}\left(bZ^{(q)}(b) - (1-\lambda\mathbb{E}[Y])\bar{W}^{(q)}(b) - q\int_0^b yW^{(q)}(y)dy\right)\Bigg] \\
&\quad + V(q)\frac{\lambda}{\eta}\left(Z^{(q)}(b-u) - \frac{W^{(q)}(b-u)}{W^{(q)}(b)}Z^{(q)}(b)\right) \\
&\quad - V(q)\frac{\lambda}{\eta}\left(Z^{(q)}(b-u,\alpha\eta) - \frac{W^{(q)}(b-u)}{W^{(q)}(b)}Z^{(q)}(b,\alpha\eta)\right),
\end{aligned} \tag{3.19}$$

where $\bar{W}^{(q)}(b) = \int_0^b W^{(q)}(x)dx$.

Proof. **(i)** $\mathcal{U}(0) = u = b + y, y > 0$.

First we prove (3.15). $(1-\alpha)y$ is taken from the overshoot of size y. If there is no gain during the period αy then this is the amount of dividends. Assume that there are n gain arrivals during time αy. Each such arrival starts a busy period (which might be finite or infinite) of an M/G/1 queue with arrival rate λ and service time distributed as αY (recall that we assumed that $c = 1$). Let $N(\alpha y)$ be the number of arrivals during αy. The dividends in the different busy periods are i.i.d. and the expected discounted (to the beginning of the busy period) dividend in such a busy period is $V(q)$. Given that $N(\alpha y) = n$, let $T_{(i)}$ be the time of the ith arrival that starts such a busy period. Thus, we have to discount the dividends in the ith busy period by $e^{-q(T_{(i)} + \sum_{j=1}^{i-1}\mathcal{B}_j)}$. Given that $N(\alpha y) = n \geq 1$, the density function of $T_{(i)}$ is as the ith order statistic from a uniform distribution on $(0, \alpha y)$. Thus:

$$
\begin{aligned}
&\mathcal{V}^{(q)}(b+y,b) = (1-\alpha)y \\
&+\sum_{n=1}^{\infty}\frac{e^{-\alpha\lambda y}(\alpha\lambda y)^n}{n!}nV(q)\sum_{j=1}^{n}\binom{n-1}{j-1}\int_0^{\alpha y} e^{-qv}(\frac{v}{\alpha y})^{j-1}(1-\frac{v}{\alpha y})^{n-j}\mathcal{L}_{\mathcal{B}}^{j-1}(q)\frac{dv}{\alpha y} \\
&= (1-\alpha)y + V(q)\alpha\lambda y\int_{u=0}^{1}\left(\sum_{m=0}^{\infty}\frac{(\alpha\lambda y)^m e^{-\alpha\lambda y}}{m!}(u\mathcal{L}_{\mathcal{B}}(q)+1-u)^m e^{-qu\alpha y}\right)du \\
&= (1-\alpha)y + V(q)\alpha\lambda y\int_0^1 \exp(-u\alpha y q + \alpha\lambda y(u\mathcal{L}_{\mathcal{B}}(q)-u))du \\
&= (1-\alpha)y + V(q)\frac{\lambda}{\eta}(1-\exp(-\alpha\eta y)). \qquad (3.20)
\end{aligned}
$$

(3.16) is obtained by replacing y with Y and taking expectation.

(ii) $\mathcal{U}(0) = u \le b$.

Until the process $\mathcal{U}$ reaches b it behaves as U. In terms of the process $X = b-U$, the overshoot above b is the same as $-X(\tau_0^-)$, i.e., the absolute value of the deficit in the process X. Thus, using (3.15), the expected discounted dividends until $\mathcal{U}$ reaches b (or X reaches 0), $\mathcal{V}^{(q)}(u,b)$ is:

$$
\mathbb{E}_{b-u}\left[e^{-q\tau_0^-}1_{\{\tau_0^-<\tau_b^+\}}\left(-(1-\alpha)X(\tau_0^-)+V(q)\frac{\lambda}{\eta}(1-\exp(\alpha\eta X(\tau_0^-)))\right)\right]. \qquad (3.21)
$$

By (2.12),

$$
\mathbb{E}_{b-u}[e^{-q\tau_0^-}e^{\alpha\eta X(\tau_0^-)}1_{\{\tau_0^-<\tau_b^+\}}] = Z^{(q)}(b-u,\alpha\eta) - \frac{W^{(q)}(b-u)}{W^q(b)}Z^{(q)}(b,\alpha\eta). \qquad (3.22)
$$

By (2.14),

$$
\mathbb{E}_{b-u}[e^{-q\tau_0^-}1_{\{\tau_0^-<\tau_b^+\}}] = Z^{(q)}(b-u) - \frac{W^{(q)}(b-u)}{W^{(q)}(b)}Z^{(q)}(b). \qquad (3.23)
$$

By (2.12), (2.13), (2.9) and substituting $\psi'(0) = 1-\lambda\mathbb{E}[Y]$ (since $c=1$):

$$
\begin{aligned}
&\mathbb{E}_{b-u}[e^{-q\tau_0^-}X(\tau_0^-)1_{\tau_0^-<\tau_b^+}] = \frac{\partial}{\partial\theta}\left(Z^{(q)}(b-u,\theta) - \frac{W^{(q)}(b-u)}{W^q(b)}Z^{(q)}(b,\theta)\right)|_{\theta=0} \\
&= (b-u)Z^{(q)}(b-u) - (1-\lambda\mathbb{E}[Y])\bar{W}^{(q)}(b-u) - q\int_0^{b-u} yW^{(q)}(y)dy \\
&-\frac{W^{(q)}(b-u)}{W^{(q)}(b)}\left(bZ^{(q)}(b) - (1-\lambda\mathbb{E}[Y])\bar{W}^{(q)}(b) - q\int_0^b yW^{(q)}(y)dy\right). \qquad (3.24)
\end{aligned}
$$

(3.22)-(3.24) yield (3.19).

□

Substituting $u = b$ and $Z^{(q)}(0,\theta) = 1$ in (3.19) we obtain that:

$$\mathcal{V}^{(q)}(b,b)=(1-\alpha)\left[\frac{W^{(q)}(0)}{W^{(q)}(b)}\left(bZ^{(q)}(b)-(1-\lambda\mathbb{E}[Y])\bar{W}^{(q)}(b)-q\int_0^b yW^{(q)}(y)dy\right)\right]$$
$$+V(q)\frac{\lambda}{\eta}\left(1-\frac{W^{(q)}(0)}{W^{(q)}(b)}Z^{(q)}(b)\right)-V(q)\frac{\lambda}{\eta}\left(1-\frac{W^{(q)}(0)}{W^{(q)}(b)}Z^{(q)}(b,\alpha\eta)\right). \tag{3.25}$$

Next we derive $\mathcal{V}_{tot}^{(q)}(u)$ – the total discounted dividends given that the initial surplus is u.

Proposition 3.5. *For $u>b$,*

$$V_{tot}(u)=\mathcal{V}^{(q)}(u,b)+\frac{\delta_u\mathcal{V}^{(q)}(b,b)}{1-\delta_b}, \tag{3.26}$$

where

$$\delta_u=e^{-\Phi_1(q)\alpha(u-b)}. \tag{3.27}$$

For $u\le b$,

$$\delta_u=Z^{(q)}(b-u,\alpha\Phi_1(q))-\frac{W^{(q)}(\alpha(b-u))}{W^{(q)}(b)}Z^{(q)}(b,\alpha\Phi_1(q)). \tag{3.28}$$

Proof. δ_u is the discounted time until the process $\mathcal{U}$ reaches b from above. When $u>b$, $\mathcal{U}$ behaves as $\mathbb{U}$. Since $(1-\alpha)(u-b)$ is paid as dividends, and the time to reach b is distributed as the time that $X_1=\alpha(u-b)-\mathbb{U}$ reaches $u-b$ starting at 0. Thus, (3.27) follows from (2.10). For $u\le b$ the time until $\mathcal{U}$ reaches b is C_u+P_u. C_u is the same as the time to ruin τ_0^- for the risk process $X=b-U$. P_u is the same as the time to reach $-X(\tau_0^-)$ by the process X_1 which behaves as $-\mathbb{U}$ starting at 0. (2.12) and (2.10) yield that:

$$\begin{aligned}\delta_u&=\mathbb{E}[e^{-q(C_u+P_u)}1_{C_u<H_u}]=\mathbb{E}_{b-u}[e^{-q\tau_0^-}e^{\Phi_1(q)\alpha X(\tau_0^-)}]\\&=Z^{(q)}(b-u,\alpha\Phi_1(q))-\frac{W^{(q)}(b-u)}{W^{(q)}(b)}Z^{(q)}(b,\alpha\Phi_1(q)).\end{aligned} \tag{3.29}$$

Substituting $u=b$ and $W^{(q)}(0)=1$ yields that

$$\begin{aligned}\delta_b&=\mathbb{E}[e^{-q(C_{b,j}+P_{b,j})}]=\mathbb{E}_0[e^{-q\tau_0^-}e^{\Phi_1(q)\alpha X(\tau_0^-)}]\\&=1-\frac{1}{W^{(q)}(b)}Z^{(q)}(b,\alpha\Phi_1(q)).\end{aligned}$$

$\mathcal{V}_{tot}(u)$, the expected total discounted dividends until ruin, is given by

$$\mathcal{V}_{tot}(u)=\mathcal{V}^{(q)}(u,b)+\delta_uV^{(q)}(b,b)\sum_{n=0}^{\infty}\delta_b^n=\mathcal{V}^{(q)}(u,b)+\frac{\delta_u\mathcal{V}^{(q)}(b,b)}{1-\delta_b}. \tag{3.30}$$

□

4 General dividends

In this section we generalize the previous results, assuming that from each gain of size x which arrives when the surplus is above b or from each overshoot of size x of a gain that brings the surplus above b, the dividend paid is $I(x)$, where $0<I(x)<x$.

4.1 The Laplace-Stieltjes transform of the time to ruin and the total dividends paid

Assume $\mathcal{U}(0) = u + y$ (or that the overshoot is y). Let $\mathcal{B}_{b+y}, \mathcal{D}_{b+y}$, be respectively the time to reach b from above and the total dividends paid until this time, given that before paying dividends $\mathcal{U}(0) = b + y$. $\mathcal{B}$ and $\mathcal{D}$ are similarly defined with y replaced by Y. Notice that $\mathcal{B}$ is a busy period in an M/G/1 queue with Poisson arrivals at rate λ and service distributed as $Y - I(Y)$, and $\mathcal{D}$ is the dividends paid during this period. Let

$$\mathcal{L}_{\mathcal{B}_{b+y},\mathcal{D}_{b+y}}(\beta,\theta) = \mathbb{E}[e^{-\beta\mathcal{B}_{b+y}-\theta\mathcal{D}_{b+y}}]$$

be the joint Laplace-Stieltjes transform of $\mathcal{B}_{b+y}$ and $\mathcal{D}_{b+y}$. Then

Proposition 4.1. *For* Re $\beta, \theta \geq 0$, *the joint LST of $\mathcal{B}$ and $\mathcal{D}$ satisfies the following equation:*

$$\mathcal{L}_{\mathcal{B},\mathcal{D}}(\beta,\theta) = \mathbb{E}[e^{-(\beta+\lambda-\lambda\mathcal{L}_{\mathcal{B},\mathcal{D}}(\beta,\theta))Y-(\theta-\beta-\lambda+\lambda\mathcal{L}_{\mathcal{B},\mathcal{D}}(\beta,\theta))I(Y)}]. \tag{4.1}$$

It is the unique solution of the equation $x = \mathbb{E}[e^{-\theta I(Y)-(\beta+\lambda(1-x))(Y-I(Y))}]$ with $|x| < 1$ if Re $\beta > 0$, Re $\theta > 0$ *and if* Re $\beta \geq 0$, Re $\theta \geq 0$ *and* $\lambda\mathbb{E}(Y - I(Y)) < 1$, *i.e., if* $\Phi_1(0) = 0$*; if* $\Phi_1(0) > 0$, *then* $\mathcal{L}_{\mathcal{B},\mathcal{D}}(0) = 1$ *is the unique solution of the equation with* $|x| \leq 1$.

Proof. First assume that the overshoot over b is y. In this case the amount of dividends withdrawn immediately is $I(y)$, and the net jump is $y - I(y)$. The time until the process reaches b is $y - I(y) + \sum_{i=0}^{N(y-I(y))} \mathcal{B}_i$, where $\mathcal{B}_i$ are i.i.d. distributed as $\mathcal{B}$. Similarly, the amount of dividends paid until the process reaches b when the overshoot is y is $I(y) + \sum_{i=0}^{N(y-I(y))} \mathcal{D}_i$, where $\mathcal{D}_i$ are i.i.d. distributed as $\mathcal{D}$. Thus the joint Laplace-Stieltjes transform of the time until the process hits b and the dividends paid up to this time given that the overshoot is y, $\mathcal{L}_{\mathcal{B}_{b+y},\mathcal{D}_{b+y}}(\beta,\theta)$, is:

$$\begin{aligned}
&\mathcal{L}_{\mathcal{B}_{b+y},\mathcal{D}_{b+y}}(\beta,\theta) = \mathbb{E}[e^{-\beta(y-I(y))-\theta I(y)-\sum_{j=0}^{N(y-I(y))}(\beta\mathcal{B}_i+\theta\mathcal{D}_i)}] \\
&= e^{-\beta(y-I(y))-\theta I(y)} \sum_{j=0}^{\infty} e^{-\lambda(y-I(y))} \frac{(\lambda(y-I(y)))^j}{j!} (\mathcal{L}_{\mathcal{B},\mathcal{D}}(\beta,\theta))^j \\
&= e^{-\beta(y-I(y))-\theta I(y)-\lambda(y-I(y))(1-\mathcal{L}_{\mathcal{B},\mathcal{D}}(\beta,\theta))}.
\end{aligned} \tag{4.2}$$

Assuming now that the overshoot is a random variable Y and replacing y by Y in (4.2), yields (4.1). The uniqueness follows again by Rouché's theorem, cf. the proof of Proposition 3.2. □

Let $\mathcal{U}(0) = u$, $0 \leq u \leq b$, thus $X(0) = b - u$. First we consider the joint Laplace-Stieltjes transform of the time until the process reaches b when it upcrosses b before ruin, and the total dividends paid until this time.

Theorem 4.1. *Let $\mathcal{U}(0) = u \leq b$. Then*

$$\mathbb{E}[e^{-\beta(C_u+P_u)}1_{C_u<H_u}\mathcal{L}_{B_{\mathcal{U}(C_u)},\mathcal{D}_{\mathcal{U}(C_u)}}(\beta,\theta)]$$

$$= \int_0^b \int_{z=y}^{\infty} u^{(\beta)}(b-u,y)\lambda dF_Y(z)\mathcal{L}_{\mathcal{B}_{b+z-y},\mathcal{D}_{b+z-y}}(\beta,\theta)dy. \tag{4.3}$$

Proof. $X(t) = b - \mathcal{U}(t)$ is a spectrally negative Lévy process and for $\mathcal{U}(0) = u$, C_u and H_u are the same as the first time that $X = b - \mathcal{U}$ downcrosses 0 or hits b respectively. We apply the definition of the density $u^{(\beta)}(\cdot,\cdot)$ of the β-resolvent measure given in (2.15). $u^{(\beta)}(b-u,y)\lambda dF_y(z)$ is the Laplace transform of the time until the process X (starting at $b-u$) reaches y before hitting b or downcrossing 0, and then downcrosses 0 due to a claim of size $z > y$. It is the same as the Laplace transform of the time until U (starting at u) reaches $b - y$ before hitting 0 or before upcrossing b, and then upcrosses b due to a gain of size $z > y$. Multiplying it by $\mathcal{L}_{\mathcal{B}_{b+z-y},\mathcal{D}_{b+z-y}}(\beta,\theta)$ yields the result. □

To obtain the joint Laplace-Stieltjes transform of $\mathcal{D}^u_{tot}$ and $\mathcal{P}^{(u)}_{tot}$ we write similar equations as (3.8) and (3.10).

Proposition 4.2. **(i)** *Let $u \le b$ then*

$$\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot}-\beta\mathcal{P}^{(u)}_{tot}}] = \frac{W^{(\beta)}(b-u)}{W^{(\beta)}(b)}$$
$$+\int_0^b\int_{z=y}^{\infty} u^{(\beta)}(b-u,y)\lambda dF_Y(z)\mathcal{L}_{\mathcal{B}_{b+z-y},\mathcal{D}_{b+z-y}}(\beta,\theta)dy\ \mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot}-\beta\mathcal{P}^{(b)}_{tot}}], \tag{4.4}$$

where $\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot}-\beta\mathcal{P}^{(b)}_{tot}}]$ is obtained by substituting $u = b$ in (4.4).

(ii) *Let $u = b + y$, $y > 0$. Then*

$$\mathbb{E}[e^{-\theta\mathcal{D}^{(u)}_{tot}-\beta\mathcal{P}^{(u)}_{tot}}] = \mathcal{L}_{\mathcal{B}_u,\mathcal{D}_u}(\beta,\theta)\mathbb{E}[e^{-\theta\mathcal{D}^{(b)}_{tot}-\beta\mathcal{P}^{(b)}_{tot}}]. \tag{4.5}$$

Proof. **(i)** Similar arguments as in the proof of Proposition 3.3 yield that the Laplace-Stieltjes transform of the time until $\mathcal{U}$ reaches 0 before hitting b (and thus ruin occurs before dividends are paid) is $\frac{W^{(\beta)}(b-u)}{W^{(\beta}(b)}$. According to the definition (2.15), $u^{(\beta)}(b-u,y)\lambda dF_Y(z)dy$ is the Laplace-Stieltjes transform of the time until the process equals $b-y$, and a claim of size $z > y$ occurs. Multiplying it by $\mathcal{L}_{\mathcal{B}_{b+z-y},\mathcal{D}_{b+z-y}}(\beta,\theta)$ gives the joint Laplace-Stieltjes transform of the time until the process reaches b from above and the dividends paid until this time. Due to the strong Markov property, multiplying it by the joint Laplace-Stieltjes transform of the time to ruin and the dividends paid until ruin, starting at b, yields (4.4).

(ii) $\mathcal{L}_{\mathcal{B}_u,\mathcal{D}_u}(\beta,\theta)$ is the joint Laplace transform of the time until reaching b from above and the total dividends paid up to this time. Thus, (4.5) holds due to the strong Markov property.

□

Below we find the ruin probability for this case.

Corollary 4.1. *The ruin probability for the dual model $\mathcal{U}$ with general $I(x)$ is 1 if $\psi_1'(0) \ge 0$, i.e., if $\lambda\mathbb{E}(Y - I(Y)) \le 1$. Otherwise it equals*

$$\frac{W^{(0)}(b-u)}{W^{(0)}(b)} + \frac{\frac{1}{W^{(0)}(b)}\int_0^b\int_{z=y}^{\infty} u^{(0)}(b-u,y)\lambda dF_Y(z)e^{-\lambda(z-y-I(z-y))(1-\mathcal{L}_{Y-I(Y)}(\Phi_1(0)))}dy}{1-\int_0^b\int_{z=y}^{\infty} u^{(0)}(0,y)\lambda dF_Y(z)e^{-\lambda(z-y-I(z-y))(1-\mathcal{L}_{Y-I(Y)}(\Phi_1(0)))}dy} \tag{4.6}$$

if $u \le b$ *and*

$$\frac{\frac{1}{W^{(0)}(b)} e^{-\lambda(u-b-I(u-b))(1-\mathcal{L}_{Y-I(Y)}(\Phi_1(0)))} dy}{1-\int_0^b \int_{z=y}^{\infty} u^{(0)}(0,y)\lambda dF_Y(z) e^{-\lambda(z-y-I(z-y))(1-\mathcal{L}_{Y-I(Y)}(\Phi_1(0)))} dy} \tag{4.7}$$

if $u > b$.

Proof. Solving (4.1) for $\mathcal{L}_{\mathcal{B},\mathcal{D}}(0,0)$ yields that

$$\mathcal{L}_{\mathcal{B},\mathcal{D}}(0,0) = \mathcal{L}_{Y-I(Y)}(\Phi_1(0)). \tag{4.8}$$

As seen in Proposition 4.1, $\mathcal{L}_{\mathcal{B},\mathcal{D}}(0,0) = 1$ when $\psi_1'(0) \le 0$ and $\Phi_1(0) = 0$. In this case $\mathcal{L}_{\mathcal{B}_{b+y},\mathcal{D}_{b+y}}(0,0) = 1$. Notice that,

$$\begin{aligned}
&\int_0^b \int_{z=y}^{\infty} u^{(0)}(b-u,y)\lambda dF_Y(z) dy \\
&= Z^{(0)}(b-u) - \frac{Z^{(0)}(b)}{W^{(0)}(b)} W^{(0)}(b-u) = 1 - \frac{W^{(0)}(b-u)}{W^{(0)}(b)}.
\end{aligned} \tag{4.9}$$

The first line in (4.9) is the probability that X_1 dowcrosses 0 before reaching b starting at $b-u$. Due to (2.14) the first equality holds. The last equality holds since $Z^{(0)} = 1$.

Substituting $u = b$ and solving (4.4) for the ruin probability starting at b yields that the ruin probability starting at b is 1, and thus the ruin probability starting at u is 1.

In the case that $\psi_1'(0) > 0$, i.e., $\Phi_1(0) > 0$, (4.8) yields that $\mathcal{L}_{\mathcal{B},\mathcal{D}}(0,0) < 1$. Substituting (4.8) in (4.4) and solving for $u = b$ and noticing that $W^{(0)}(0) = 1$ (see Lemma 8.6 in [8]) yields that the ruin probability starting at b is

$$\frac{\frac{1}{W^{(0)}(b)}}{1-\int_0^b \int_{z=y}^{\infty} u^{(0)}(0,y)\lambda dF_Y(z) e^{-\lambda(z-y-I(z-y))(1-\mathcal{L}_{Y-I(Y)}(\Phi_1(0)))} dy}. \tag{4.10}$$

Substituting $\theta = \beta = 0$ and (4.10) in (4.4) and (4.5) gives (4.6) and (4.7). □

4.2 The expected discounted dividends

The analysis of the expected discounted dividends for general $I(x)$ is similar to the analysis for the case where $I(x) = (1-\alpha)x$. Let $\mathcal{V}^{(q)}(u,b)$ be the expected discounted dividends until $\mathcal{U}$ reaches b given that $\mathcal{U}(0) = u$ as defined in Section 3.3, but where the dividend that is taken from a gain of size x is $I(x)$, and $\mathcal{B}$ is the generic busy period in a M/G/1 queue with arrival rate λ and service time distributed as $Y - I(Y)$. Thus in the general case η is defined as in (3.18) with $\mathcal{B}$ as defined above.

Proposition 4.3. **(i)** *Let* $\mathcal{U}(0) = u = b + y, y > 0$. *Then*

$$\mathcal{V}^{(q)}(u,b) = I(y) + V(q)\frac{\lambda}{\eta}(1 - \exp(-(y - I(y))\eta), \tag{4.11}$$

where $V(q) = \mathbb{E}[\mathcal{V}^{(q)}(Y+b,b)]$ *is given by*

$$V(q) = \frac{\mathbb{E}[I(Y)]}{1 - \frac{\lambda}{\eta}(1 - \mathbb{E}[\exp(-\eta(Y - I(Y)))])}. \tag{4.12}$$

(ii) *Let* $\mathcal{U}(0) = u \leq b$ *then*

$$\mathcal{V}^{(q)}(u,b) = \int_{y=0}^{b} \int_{z=y}^{\infty} u^{(q)}(b-u,y)\mathcal{V}^{(q)}(b+z-y,b)\lambda dF(z)dy, \tag{4.13}$$

where $\mathcal{V}^{(q)}(b+z-y,b)$ *is given by (4.11).*

Proof. **(i)** The proof of (4.11) is the same as the proof of (3.15).

(ii) (4.13) is straightforward from the definition of the potential measure $u^{(q)}(\cdot,\cdot)$ in (2.15).

□

Now we obtain the expected discounted dividends for the general case. The proof of the following proposition is similar to the proof of Proposition 3.5.

Proposition 4.4.

$$\mathcal{V}_{tot}(u) = \mathcal{V}^{(q)}(u,b) + \frac{\delta_u \mathcal{V}^{(q)}(b,b)}{1-\delta_b}, \tag{4.14}$$

where

$$\delta_u = e^{-\Phi_1(q)(u-b-I(u-b))} \tag{4.15}$$

for $u > b$, *and*

$$\delta_u = \int_{y=0}^{b} \int_{z=y}^{\infty} u^{(q)}(b-u,y)e^{-\Phi_1(q)(z-y-I(z-y))}\lambda dF(z)dy \tag{4.16}$$

for $u \leq b$.

Proof. (4.14) is similar to (3.30). For $u > b$ the discount factor in (4.15) is obtained similarly to (3.27).

The discounted time until the process reaches b from above when $\mathcal{U}(0) = u < b$ is

$$\delta_u = [e^{-q(C_u+P_u)}1_{C_u<H_u}] = \mathbb{E}_{b-u}[e^{-q\tau_0^-}1_{\tau_0^-<\tau_b^+}e^{\Phi_1(q)(X(\tau_0^-)-I(X(\tau_0^-)))}]. \tag{4.17}$$

To evaluate (4.17) we use the q-potential measure again:

$$\delta_u = \int_{y=0}^{b} \int_{z=y}^{\infty} u^{(q)}(b-u,y)e^{-\Phi_1(q)(z-y-I(z-y))}\lambda dF(z)dy. \tag{4.18}$$

□

The mean of the total discounted dividends is obtained as in (3.30).

Remark 4.1. *Substituting $I(y) = (1-\alpha)y$ in (4.15) immediately yields (3.27). Substituting $I(y) = (1-\alpha)y$ in (4.16)gives*

$$\delta_u = \int_{y=0}^{b} \int_{z=y}^{\infty} u^{(q)}(b-u, y) e^{-\Phi_1(q)\alpha(z-y)} \lambda dF(z) dy. \tag{4.19}$$

The definition of $u^{(q)}$ in (2.15) yields that the expression in (4.19) equals $E_{b-u}[e^{-q\tau_0^-} 1_{\tau_0^- < \tau_b^+} e^{\Phi_1(q)\alpha X(\tau_0^-)}]$ as in (3.29).

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